ESSAYS ON MACROECONOMIC POLICY & AFFINE TERM STRUCTURE MODELS

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To Ines, that took care of the kids and supported me so much during the journey and to my two beloved daughters Anne & Lucia, for whom I will need to catch up with in order to compensate for the time spent writing this thesis.

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Table of Contents

List of Tables

List of Figures

1. Introduction

This thesis is focused mainly in the application and discussion of term structure models in order to link macroeconomic theory, debt policy and debt management theory with term structure dynamics. Throughout the thesis term structure models are discussed and calibrated using real European data. It will be shown how results are in line with economic theory and bring some light to latest European developments.

One of many motivations for addressing macroeconomics to term structure dynamics is that every time governments or the private sector issue new debt or roll over existing debt, they are faced with questions such as: how much debt should be issued and for which tenors or maturities? The answer should be: depending on their macroeconomic risks. Thus, debt re-structuring decisions should take into account the sensitivities of the yield curve as a result of macroeconomic shocks and be capable to foresee (or at least be aware of) the effects that certain –and also some rather extreme- macroeconomic scenarios would have on their costs of financing. Affine term structure models should be used as a finance tool to assist strategic decision making. The maturity structure of debt taking into account observable macroeconomic risks cannot be ignored, as seen on recent developments on the European sovereign debt crises, which resulted in calls for a unifying and transparent debt management framework.

In this introductory chapter, which was published in Jakas (2012a), some developments on affine terms structure models are discussed with focus –amongst others– on debt policy. The intention is to discuss how affine terms structure models can help shape debt policy and improve the issuers' debt costs of financing and as a result –hopefully- improve their credit worthiness.
The novelty of this work is twofold: Firstly, to establish the link between affine term structure models with debt management theory. Secondly, as debt management theory still lacks of a generalised framework for analysing different debt management

strategies, this work intends to answer some questions by focusing on the link that can be established by applying affine term structure models and show how these models contribute to two main streams of debt management theory: I) the theory of the price level as seen in the works of Leeper (1991), Sims (1994), Woodford (1995, 1996), Dupor (1997) and Cochrane (2001) and, II) the optimal taxation approach as seen in the works of Angeletos (2001), Faraglia, Marcet and Scott (2008) and Missale (1997). Some attention is given to sovereign debt where markets move under certain set of events from complete to incomplete information, generating information asymmetries, and in the process of doing so they change from perfect to imperfect market conditions and hence, exhibit multiple equilibria; the "good" and the "bad" equilibrium. The work makes a few references about market frictions which exhibit widening bid-offer spreads resulting in illiquidity, increases in the governments cost of financing, roll-over risks, financial distress and even sovereign debt crises. This work shows that the yield curve behaviour is a reflection of a government's macroeconomic risks and the risk of debt roll-over due to unfavourable markets expectations on government's policies concerning their future surpluses or fiscal imbalances.

It is also shown how the existence of multiple equilibria can transform investors' risk perception on governments' issuances from risk-free into risky and that this is because they expect government to default, either via a regular default within the roll-over process or via an increase in the price level. When this occurs, investors run on government issuances in a desperate attempt to recuperate their investment, due to the existence of information asymmetries they do not know where they are standing in the queue to convert their investments into cash when credit markets crunch.

This thesis is organised as follows. The introduction would provide the reader a high-level review of the literature outlining some of the major discussions about the theoretical underpinnings and empirics seen on affine term structure models and debt management. In addition, this chapter also makes reference to the latest developments on the European sovereign crisis, the theory of the fragility of the Eurozone and links to market dynamics. Though we cannot answer of all the questions we raise in this chapter we will try to address some of these in more detail in each of the preceding chapters accordingly.

The chapter "Theory of a Term Structure Model Applied to European Data" published in Jakas (2011) begins with the basic asset pricing equation which is slightly adapted in order to account for various European macroeconomic variables such as Unemployment, Consumer Confidence Index, Production Price Index and ECB M3. In this chapter we aim to identify evidence of a term structure violating the no-arbitrage condition and compare results to a hypothetical no-arbitrage model. Subsequently we discuss the extent to which the theory is in line with the empirical results. This chapter will present slight adjustments to the classical consumption based asset pricing model by introducing in the utility function unemployment data and survey data such as consumers' confidence index. The incorporation of a monetary aggregate will also be extensively discussed in this chapter and its inclusion in the model appears to us to be robust enough. This will show that a few state space variables such as European unemployment rate, the European Consumers' Confidence Index, European Production Price Index (PPI) and a monetary aggregate such as ECB M3 for Europe, it is possible to explain yield curve movements with strikingly very good results. For instance, the unemployment rate and the consumer confidence index exhibit a shift and a slope effect on the yield curve, thus the front-end yields moving faster than the long-end resulting in steepening or flattening effects. The production price index has a twist effect on the yield curve (flattening or steepening of the curve) which results in lower-end yields shifting in opposite directions to the long-end. The empirical work in this chapter shows that yields are negatively correlated to money supply, as expected in classical IS-LM models. And that money supply exhibits a slope effect, with the lower end of the curve shifting faster than the longer end.

The chapter "Introductory Notes on Affine Term Structure Models: Continuous Time Approach" provides a review of the foundations on affine term structure models, comparing some of the analytical results from various authors particularly from

Cochrane (2005) and Piazzesi (2010). The intention is to present some basic algebra and show the reader on a step by step basis how to get to the no-arbitrage approach specification. This chapter is of interest to academics and practitioners involved in teaching or applying affine term structure models under the continuous time approach, as most of the literature has proven to be very difficult to follow. The affine term structure models developed in this chapter start with the basic pricing equation, the pricing equation for asset returns and the holding period returns. The chapter also explains the relationship of bond prices and Ito's lemma and finally the fundamental pricing equation for fixed income securities is obtained, as this expression is later required for the definition of an affine term structure model. Discussions are linked to prominent research such as Piazzesi (2010) and results are compared to those of Cochrane (2005) and Duffie and Kan (1996). Here, we have learned that there are still some conceptual differences in the literature which requires further attention which goes beyond the scope of this thesis.

The chapter "Affine Term Structure Models applied to a Macro-Finance Model" show how a macro-finance model can be used as the theoretical foundations for calibrating the state space vector of an affine term structure model. Thus, this chapter explores the use of macro-finance models in order to explain the links of macroeconomics and the state vector used for calibrating a term structure model. It will be shown that some of the findings in Chapter 4 can be applied to the models discussed in chapter 5. Here it is worth mentioning that in order to link macroeconomic policy to term structure dynamics the use of a macro-finance model would help the researcher select state variables that are supported by economic theory.

The chapter "Discrete Approach to Affine Term Structure Models Applied to German and Greek Government Yields" published in Jakas (2012b) provides valuable evidence from a European perspective of discrete affine models for risky and risk-free assets. It is a first attempt to apply an affine model to compare two government bonds which are denominated in a common currency but that at the same time enjoy different risk profiles. In this chapter some of the algebra and concepts seen in the continuous time affine term structure literature are plugged into the discrete time approach. Starting point for this paper has been the celebrated papers from Backus, Foresi, Telmer (1996-98). In addition, some of the developments seen on the continuous approach as documented in Piazzesi (2010), Singleton (2006) and Duffie and Kan (1996) have been explored and adapted to the discrete time approach. This chapter focused mainly on the multifactor cases of affine term structure models. The results are encouraging and the models fit the observed yields as well as give evidence of a reasonable predictive-ability. This chapter will show that governments that exhibit a positive correlation of their yields to aggregate consumption growth need to ensure low deficits and debt burdens during booming periods so that they can still have capacity to issue new debt for the rainy days. From the Greek case this essay shows that a deterioration of government’s deficit-to-GDP ratio results in a fall in yields. This is mainly because the increase in spending helps to boom the economy. However, this is more than offset by the deterioration of its debt-to-GDP ratio, thus a deterioration of the latter ratio will more than offset any positive effects stemming from any budget-deficit-induced counter-cyclical policies. This chapter will also show that debt and deficit ratios can play a role in times of financial distress.

The chapter “The Yield Curve and Economic Fundamentals: A Continuous Time Model” published in Jakas and Jakas (2013) is an attempt to explore the continuous time approach to affine term structure models and calibrates the German government yields with the same macroeconomic variables used in the discrete approach in previous chapters. The intention is to show that affine term structure models can also be used in order to test the extent to which yields are in line with fundamental macroeconomic data and hence are part of a network of macroeconomic variables.

The chapter “Discrete Affine Term Structure Models Applied to the Government Debt and Fiscal Imbalances” published in Jakas (2013a) is an attempt to use the affine term structure models to the understanding of governments’ debt and fiscal imbalances, providing a valuable toolkit for policy and decision makers and contributing to the fiscal theory of the price level, the optimum taxation approach, debt management and fixed income portfolio strategies. It will be shown that it does not matter if the theory of the price level is at work or not. Mainly because either ways any deterioration of the cumulated surplus or deficit leads to higher financing costs either via inflation or via increases in the credit spreads to the benchmark curves. This chapter will also show that the front-end of the yield curve is where the risk lies as seen for the Greek yield curve. Hence, if for a given government its yields are not seen as a market benchmark and therefore exhibit increasing yields in times when aggregate marginal utility is high and consumption growth is low, then the government must avoid deficits and fund counter-cyclical policies with liquidity reserves for this purpose. Reserves must be financed via the long-end of the curve, for instance, 20- to 30-year maturity bonds. It will be shown that only those governments who enjoy of the so-called risk-free status are "allowed" to issue in the front-end, as an increase in the financing costs due to increases in total debt outstanding are compensated –at least partially– by falls in the short-term rate. The novelty of this essay is that will show that the mathematics underlying affine term structure models can be very useful for the analysis of governments' fiscal imbalances. Governments which are more sensitive to macroeconomic risks affecting their fiscal imbalances are likely to exhibit higher spreads if they run deficits, thus limiting their ability to counter-cyclical policies in times when consumption growth is low and aggregate marginal utility is high.

The chapter "The Term Structure, Latent Factors and Macroeconomic Data: A Local Linear Level Model" published in Jakas (2013b) is a contribution to some of the term structure literature seen in Ang and Piazzesi (2003) and Diebold, Rudebusch and Aruoba (2006) this research combines 1) the use of state space techniques via a Kalman (1960) filter in order to identify an unobserved state or latent factors depicting the slope, the level and a seasonal component to an observation equation linking those state factors to observable yields and, 2) the use of affine model calibrated with latent factors obtained in 1). In addition, this section also shows how macro-variables influence these unobserved components via OLS regressions linking latent factors and macroeconomic data. It will be shown that results confirm the views of Diebold, Rudebusch and Aruoba (2006) and, provide strong evidence of macroeconomic effects on yields however and, weaker evidence of yield curve effects on the macroeconomy. This essay will also explore the possibility of breaking the yield curve in two: the money market and the capital market yield curves. It will be shown that by doing so results improve significantly compared to those seen in the no-arbitrage experiences. The essay show that both, the local level model as well as the no-arbitrage term structure model perform quite well in explaining yield curve movements. However, similar to most of current literature and in line with previous chapters the explanatory power diminishes under the no-arbitrage approach as maturities become larger.

Finally a conclusion is presented under chapter “Conclusion and Final Remarks”. The reader should know that this thesis is manly a set of papers or essays which have been published either in an international, peer-reviewed journal or handbooks, so that each of the essays or chapters provide their own conclusions, and towards the end of the thesis the main conclusions of the thesis is presented, which summarises the most important findings of preceding chapters. Throughout the thesis, it would appear that the author is reproducing specifications which have been already mentioned in previous chapters. This is mostly for the readers’ convenience, so that the reader does not need to navigate back and forth in the thesis to follow the notation or let the reader the possibility skip chapters and go to the chapter of interest.

With respect to this introductory chapter, it has been organised as follows; Section I reviews affine term structure model literature, with reference to its main theoretical contributions and summarising some of the main academic research streams. In section II, evidence on affine term structure models is presented and some of its main findings are being briefly discussed. Section III, outlines the relevance of understanding the yield curve as part of a network of macroeconomic variables, focused is given on its contribution to debt management and strategic financial management. Section IV introduces main two streams of literature within the debt management theory; 1) the theory of the price level as seen in the works of Leeper

(1991), Sims (1994), Woodford (1995, 1996), Dupor (1997) and Cochrane (2001). And 2) the theory of the optimal taxation approach as seen in the works of Angeletos (2001), Faraglia, Marcet and Scott (2008) and Missale (1997). Finally, section V introduces to the concepts of "good" and "bad" equilibrium and links them with markets imperfections.

Section I. Affine term structure models a brief introduction

So far the literature has exhibit at least two groups that have increasingly diverged: the mathematical finance and the macroeconomic literature.

With respect to the mathematical finance literature, it could be said that most of it has been mainly concentrated on a different objective: to construct a consistent theory for the valuation of interest rate dependent derivatives or assets, which ignores that financial markets are a part in a network of dynamic interactions on an economy's macroeconomic variables. An example of this can be seen in Heath-Jarrow-Morton (1992), Vasicek (1977) or Cox-Ingersol-Ross (1985 a, b).

With respect to macroeconomic literature, traditional classical theories on the term structure such as in Meiselman (1962), Malkiel (1966), Hicks (1946), Lutz (1940), Culbertson (1957) have not suffice to link macroeconomic theory and the term structure of interests. An attempt to improve this can be seen on Michaelsen (1963) and -not so recently- in Turnovsky (1989).

None of the theoretical approaches mentioned above, have developed a generalised explanation of the term structure and its dynamics due to changes in macroeconomic variables. In fact, so far the economic theory still fails to unify the theoretical underpinnings that explain –under a general framework– the existence of a term structure, its shape, and its shape movements, the steepening, flattening, parallel shifts and twists.

Even though there is still no general theory of macroeconomics and the term structure of interests, the literature has made dramatic progress which have its foundations in the so-called consumption based asset pricing theory as seen on seminal papers from Rubenstein (1976), Lucas (1978) and Breeden (1979) which in its advanced version has crystallised into the celebrated affine term structure models, which has been extensively documented in Piazzesi (2010).

Consumption based asset pricing models despite being theoretically robust, they have performed poorly empirically as seen in the controversial works of Mehra and Prescott (1985) and Hansen and Singleton (1983) therefore, most of the literature has been mainly focused in the equity and bond premium puzzles. In addition, the literature has also been primarily confined to the study of utility functions e.g. as in Epstein and Zin (1989), which introduces the so-called recursive preferences to the classical constant relative risk aversion (CRRA) utility functions, this is extensively documented in e.g. Guvenen and Lustig (2007).

Consumption based asset pricing models have developed the theoretical property that asset returns are a function of aggregate marginal utility growth and therefore, can be easily linked to observable macroeconomic variables or indicators, and so have given birth to the so-called factor pricing as well as the affine term structure models. This introductory chapter will focus on affine term structure models and will review a few research developments on this topic.

It would be useful to start by recalling the basic asset pricing equation as in, for instance, Cochrane (2001), which shows that asset prices are a function of expected future aggregate marginal utility growth, which has been reproduced below following Cochrane's (2001) notation,

$$P_t = E_t\left[\left(\beta \frac{u^{'}(c_{t+1})}{u^{'}(c_t)}\right)x_{t+1}\right] \tag{1}$$

For P_t being the present value at time t, of an asset paying off x_{t+1}, E_t is the expectations operator. β the subjective discount factor and $u'(c_{t+1}) / u'(c_t)$ being the aggregate marginal utility growth.

Recalling the basic pricing equation and assuming a 1 year zero coupon bond which pays out 1 monetary unit at maturity, the investor's first order condition for E_t conditional to an information set at time t, would yield something like:

$$E\left[m_{t+1} | I_t\right] = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}\right] = e^{-y_t^{(N)}}$$

Which for convenience it will be transformed to:

$$y_t^{(N)} = -\log E\left[m_{t+1} | I_t\right] \qquad (2)$$

Equation (2) basically shows that the stochastic discount factor is the inverse of the observable yields. The stochastic discount factor is subject to a set of information I_t comprising the state of the economy at time t.

Assuming a log utility function $u(c_t)$ with a constant relative risk aversion holds as shown in (3) below;

$$u(c_t) = \frac{c_t^{(1-\gamma)}}{(1-\gamma)} \qquad (3)$$

For c_t being the level of consumption in time t. Now we need to adapt this by taking into account that yields and consumption are part of a network of macroeconomic variables, e.g. by saying that consumption is a function of I_t which comprises the state of the economy at time t such that I_t belongs to a set of information I so that

$$E\left[c_{t+1} | I_t\right] = f(x_t) \qquad (4)$$

For which x_t will exhibit the following behaviour

$$x_t = \rho x_{t-1} + u_t \qquad (5)$$

x_t is a state space vector with macroeconomic data which could include information such as unemployment, consumer confidence index, production price index and monetary aggregate, etc.

So recalling (1) for logarithmic utility function and log prices $p_t^{(N)}$, nominal yields can be specified as:

$$y_t^{(N)} = -\frac{1}{N} p_t^{(N)} = -\frac{1}{N} \ln E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \tag{6}$$

Transforming (6) for all into their natural logarithms, and assuming normality we obtain:

$$y_t^{(N)} = -\frac{\ln \beta}{N} + \frac{1}{N} \gamma E_t \left[\Delta c_{t+1} \right] - \frac{1}{N} \frac{\gamma^2}{2} Var\left[\Delta c_{t+1} \right] \tag{7}$$

Now from a recursive perspective we need to analyse the *lambda* coefficients that we expect for different maturities. Hence, let us assume there is a sequence of weights λ_i assumed to converge to a constant value as maturity increases. The only point of this is to adjust the values to observable data.

We present a recursion, including the behaviour of coefficients λ_i above and assuming a recursion that starts for $N = 1$ to N under normality,

$$y_t^{(N)} = -\frac{\ln \beta}{N} + \frac{\gamma E_t \left(\sum_{i=0}^{N=T-t} \lambda_{t,1+i}^{(N)} \left[\Delta c_{t+1+i} \right] \right)}{N} - \frac{\gamma^2 Var \sum_{i=0}^{N=T-t} \lambda_{t,1+i}^{(N)} \left[\Delta c_{t+1+i} \right]}{2N} \tag{8}$$

Finally, concerning the affine term structure model, most of the recent literature however, has been too focused on the “continues-versus-discrete” debate, or discussions on the estimation methods, which has been mainly confined to maximum likelihood (ML) versus generalised method of moments (GMM) debate.

"There are situations in which it is better to trade some small efficiency gains for the robustness of simpler procedures or more easily interpretable moments; OLS can be better than GLS" [Cochrane, (2001) page 293].

It is not scope of this thesis to enter in this debate and the use of the estimation procedures will depend rather on practicalities depending on the tools and methods readily available. Said all that, for empirical analysis, equations (1) to (8) can be summarised together into a space state system similar to Piazzesi (2010). These specifications are put together with an observation equation which links observable yields to the state space vector and a state equation which describes the dynamics of the state as follows:

$$y_t^{(N)} = A(N) + B(N)' x_t + \varepsilon_t^{(N)} \tag{9}$$

$$x_t = \rho x_{t-1} + u_t \tag{10}$$

For x_t being the state space vector with macroeconomic data used in (4) and already explained in (5). Coefficients $A(N)$ and $B(N)'$ depend on yields' maturities N. The reader should notice that " ' " denotes for the transpose of a vector or matrix. $\varepsilon_t^{(N)}$ and u_t are measurement errors and for the sake of simplicity are assumed to be normally distributed and i.i.d. Equations (9) and (10) if set under the violation of the no-arbitrage condition, would imply that parameters $A(N)$ and $B(N)'$ can be estimated by means of OLS, for $A(N)$ being a vector of intercepts and $B(N)'$ a coefficient matrix. However, under the no-arbitrage condition this is less straight forward.

A representative investor will try to anticipate changes in bond yields and this will be done by observing interdependencies of these bond yields with macroeconomic variables. Moreover, macroeconomic risks are the drivers of changes in yields because yields are part of a network of economic variables and because macroeconomic risks also affect governments' current and future surpluses.

Macroeconomic risks also affect markets expectations on yields because they also reflect markets expectations on governments expected future primary surpluses.

Affine term structure models refer to models that explain changes in asset values due to changes in macroeconomic variables, see for example Piazzesi (2010). Hence, if changes in macroeconomic variables are the foundations for changes in asset prices, the resulting sensitivities of these explanatory variables reflect a kind of "beta" coefficient to a certain macroeconomic risk. For example, a change in unemployment can result in more than proportional changes in government yields, thus reflecting the risk that the holder of a government issuance has as a result of a shock to the unemployment variable of the economy of concern. Equation (6) shows a simple affine term structure model for a yield curve with N different maturities. For instance, a government bond which should act as a hedge in times of low consumption growth will exhibit a negative $B(N)$ with respect to changes in unemployment rate. Hence, an increase in unemployment rate should exhibit a fall in government bond yields because an increase in unemployment will result in a fall in future consumption growth, as described in equation (1).

However, the above only works as long as the government security is seen as a risk-free asset. If the government fails to address the macroeconomic risks affecting its surplus, it will result in reverting from its issuance from risk-free to risky. This is because it will imply a change in the sign of the $B(N)'$ term. Thus using the example above, the negative $B(N)'$ term will become positive, as an increase in unemployment will result in a deterioration of government's deficit and thus an increase in the size of its debt with even further deterioration of its deficit. Hence, an increase in the unemployment rate would instead result in an increase in the bond yields with the subsequent deterioration in the value of the asset. Unless the government is able to address the macroeconomic risks affecting its deficits/surpluses, there is no reason to believe that the investors will still be willing to lend for a low return. Government securities which change from the "good" equilibrium to the "bad" equilibrium will experience a change in the sign of their $B(N)'$ term. In fact, if investors believe that Government securities will change from "good" equilibrium to the "bad" equilibrium, they will run on the government's debt, as investors will believe that the market will do so, and they do not know where they are standing on the queue to convert their bonds into cash. Thus, this further increase yields even above their fundamental values and attracts speculative attacks on weaker budgetary sovereigns. Moreover, those investors not being able –or not willing to sell at a low price– to avoid realising losses will engage in shorting similar sovereigns which exhibit similar behaviour in order to hedge their investment. In the process of doing to they spread contagion to other sovereign issuances, even if these actions are fundamentally not justified. All this gives rise to self-fulfilling prophecies and herding behaviour or bandwagon effects typical of bank runs or credit crises as documented in Jakas (2010). Empirical evidence of multiple equilibria in the European sovereign debt crisis is found in Bierne and Fratscher (2013). These authors show that a deterioration of in countries' fundamentals and fundamentals contagion are the main explanation on the rise of spreads during the crisis and they also show that empirical models with economic fundamentals do a poor job in explaining sovereign risk in the pre-crisis period for European economies, which they suggest that the market pricing of sovereign risk may not have been fully reflecting fundamentals prior to the crisis. Empirical evidence of self-fulfilling liquidity crises in the Eurozone can be found in De Grauwe and Ji (2013). These authors show that government bond markets in the Eurozone are more fragile and more susceptible to self-fulfilling prophecies than in stand-alone countries that are able to monetise their budgetary deficits. This empirical analysis is based on the so-called theory of fragility of the Eurozone, published in De Grauwe (2011a,b). One of the main outcomes of this theory is that the mere absence of a guarantee that all sovereign issuers will always have liquidity to attend their debt servicing obligations creates fragility in a monetary union, as members are susceptible to movements of distrust and they conclude arguing that if investors fear some payment difficulty, they will have an incentive to sell sovereign issuances.

Section II. Empirical Evidence on Affine Term Structure Models

Empirical evidence on affine term structure models is extensively documented in Piazzesi (2010). Most of the evidence is confined to the use of US and CRSP data.

Little or virtually no research has been performed based on real data used by real market participants such as Reuters or Bloomberg sources, which are the primary sources used by traders and fund managers. In addition, most of the evidence has been limited to US data only and little reference to the Euro-Zone. Looking into the works from Hördahl and Oreste (2010) they mainly confine their work to a joint model of macroeconomic and term structure dynamics to estimate inflation risk premia. While it is true that they look not only to US data but also to euro data, their focus is not to model the European benchmark term structure but the inflation risk premia. Other celebrated ECB working papers such as in Hördahl, Tristani and Vestin (2007) are confined to show that micro founded dynamic stochastic general equilibrium models with nominal rigidities can be successful in replicating features of bond yield data, however the work is based all on US data from the Federal Bank of Saint Louis. e.g. to be more precise they use PCECC96 for consumption and PCECTPI for prices. Amisano and Tristani (2007) focus on inflation not on term structure. Hördahl, Tristani and Vestin (2004) confines the work solely to German data and does not calibrate with European aggregated macroeconomic data in order to analyse from a euro-zone perspective. And there are many more previous to the creation of the euro which are not worth mentioning here, as we confine our work to a Eurozone perspective using real data from the euro-area.

Empirical evidence is mainly confined to discussion what short or long yields are to be used as proxies. However, little is said about what factors should be used to enter into as proxies for the state variables describing the economy at each moment in time. Good results have been observed in cross-sectional fits. However, the yield coefficients in this prediction do not include the no-arbitrage restrictions.

Empirical results for unconditional first moments have shown that yields of bonds with longer maturities are on average higher than those of bonds with shorter maturities, which shows that the yield curve is on average upward sloping which is easy to generate with an affine term structure model. However, the factors which show persistence, such as inflation, contribute to one of the main problems associated to affine models.

Section III. Why is relevant to understand the yield curve as part of a network of macroeconomic variables?

First, understanding the yield curve is important from a practitioners' perspective, as the yield curve is used as benchmark for pricing a universe of assets, derivatives, structures or plain vanilla assets. Investors and banks would apply a spread on a particular benchmark curve, e.g. German government yield curve, to value an asset. At the same time the mere spreads they are applying reflect their expectations on the variability of the assets' earnings as a function of the economy's expected future performance. For example, a corporate bond issued by a company which has its earnings more or less correlated to an economy's consumption growth, is expected to exhibit widening spreads in times of economic distress, hence when aggregate marginal utility growth high. Authors such as Piazzesi (2010), Fama and Bliss (1987), Campbell and Schiller (1991), Cochrane and Piazzesi (2002) assert that understanding the yield curve is important for forecasting, as long-maturity yields are expected values of average risk-adjusted future short yields and as a consequence they would contain information about economic agents' expectations about the future of the economy. Moreover, yields will contain information about real variables as seen in the works of Harvey (1988), Estrella and Hadouvelis (1991), Hamilton and Kim (2002), Ang, Piazzesi and Wei (2002) as well as nominal variables such as inflation as seen in the works of Mishkin (1990) and Fama (1990).

Second, financial intermediaries or investors analyse the sensitivities of their portfolio of assets as a function of movements in the yield curve. Thus, they are interested in understanding how much profit or loss could result from a particular portfolio of assets as a result of a downward shift in the benchmark curve. This analysis gives them a quick review of the interest rate risk exposure they have on a portfolio. They would possibly run a certain number of scenarios using historical data to determine sensitivities. However, the yield curve is part of a network of macroeconomic variables and affine term-structure models help rephrase this: *how much profit or loss results from a portfolio of assets as a result of a change in a predefined set of macroeconomic variables, such as unemployment, production price index, consumer confidence index and/or a particular monetary aggregate?* Thus, if the economy is heading towards recession, management can observe how portfolios are adapting their strategy to avoid financial distress. Affine term structure models can help monitor strategy and shape strategic investment decision making.

Third, understanding the yield curve is important for governments and firms, as yield curve behaviour will determine their debt financing costs (DFC) and their ability to access additional financing in times of financial distress.

To try and depict this in a better way, let us assume that for particular market segment it is possible to derive a market yield curve by applying a an observable spread above an observable benchmark curve, which for the sake of simplicity in this example it would be the government yield curve. If spreads are constant with respect to the maturities and with respect to time, an increase in the government yield curve would, ceteris paribus, increase the segment's cost of financing in the same proportion. Hence, an increase in the government yield curve in 1 basis point (bsp), all other variables being equal, would increase the firm's cost of financing by 1 bsp.

However, the above depicted relationship could be observed differently, it could well be the case that corporate yields on certain market segments exhibit a more-than-proportional relationship to changes in the benchmark curve. In fact, this could actually vary depending on the maturities, possibly being the front end more volatile than the longer end and the longer the maturities the greater the magnifying effect.

Therefore, if the yield curve is a dependent variable part of a network of macroeconomic variables, it is of interest for both private as well as public debt management, policy makers or business leaders to understand these relationships, as macroeconomic variables have an effect on private and public debt cost of financing.

Fourth, another point to take into account is that governments or central banks could actively manage the yield curve to achieve some kind of optimum results in terms of financial resource allocation to the wider economy or improve budget deficits. The outcome of this depends largely in how changes in the total government debt outstanding for different maturities can influence movements in the yield curve. What are possible policy prescriptions? Is it necessary for a government to have a Debt Policy in place or is it irrelevant? Is a transparent government debt management policy relevant to investors? and if so could it be used as part of a stabilisation strategy?

Fifth, can active yield curve management result in some kind of collateral damage? Thus, governments, by undertaking an active yield curve management strategy in order to optimise their cost of financing could however result in undesired effects to the wider economy, particularly if important economic sectors are very sensitive to government yield curve movements. In other words, active yield curve management can have beneficial effects from government's cost of financing perspective however it could result in improving certain sectors in detriment of others or simply creating financial incentives which might result being not optimum for the economy as a whole in the long run.

For example, an increase in money supply in the lower end of the curve with the subsequent fall in interest rates could result in a non-optimal reallocation of financial resources and subsequent reduction in the diversification of the economy's production factors. Thus a subsequent fall in interest rates could result in reallocating financial resources to banks, or to boost the construction, real estate and housing sector, as more individuals are able to have access to funding at a lower interest rate. This could possibly result in a so-called bandwagon effect or herding behaviour, as soaring house prices attract even more funding for speculative purposes which in turn increase house prices even further and so on until the bubble bursts.

Sixth, another aspect of understanding the yield curve from a macroeconomic perspective is that gives the practitioner the chance to analyse the sensitivity or elasticity of the macroeconomic variables on government cost of financing. The beauty of this approach is that it can also be applied to corporate debt management.

To draw this more precisely, let us assume that unemployment data exhibits a more than proportional effect on government yields in the lower end of the curve and no effect in the long end. This would mean that governments, whose yield curves are very sensitive to unemployment data and exhibit a very volatile unemployment rate, should analyse their refinancing costs when issuing new debt with longer term maturities. As in times when bad news arrives for future consumption growth, the effects that this would have on their re-financing costs –should they need to roll over debt- could result in being rather unfavourable if they issue short term debt instead.
Governments face the risk of a decrease in their tax revenues due to output shocks. Thus an unexpected fall in GDP could result in a decrease in government tax revenues, as businesses and individuals experience a fall in their taxable income. For example, an increase in unemployment results in a decrease in expected aggregate consumption and expected aggregate investment. As a result there is not only a decrease in expected tax revenues but also an increase in expected government spending, which further deteriorates government finances.
This moves us to an interesting area: what are the macroeconomic risks associated with a particular government curve compare to other government curves. For example, is the German government curve more sensitive to, e.g. unemployment data or consumer confidence index, compare to the US or the UK, and for which maturities? How should a Government's debt policy look like under a certain set of macroeconomic risks? What are the macroeconomic factors that should be tackled by policy makers in order to prevent certain macroeconomic risks influencing negatively on their expected future re-financing requirements?

Seventh, Governments face time inconsistencies in the process of issuing debt. In an optimal scenario, governments would not need to increase total debt outstanding and the roll of new debt would only be used to pay back the old maturing one. However, governments issue debt as a consequence of an increase in budget deficits in order to compensate for the fall in private consumption in times of economic crises. In other words, governments engage in counter-cyclical interventions in order to smooth the decrease in aggregate consumption growth. In the process of doing so they avoid increasing taxes and alternatively increase long term debt. The increase of long term debt has the property of acting as a hedge for distortionary taxation and innovations in aggregate consumption growth. Another benefit is that allows governments to trade current inflation for future inflation and spread the effects stemming from surplus shocks across maturities. However, the theory of the price level shows that if the government engages in such a policy will need to convince investors that current deficit is only temporary and that the government still has the ability to generate funds in order to repay debt servicing.

This is simple, why would you lend someone any money if you don't believe that repayment is a rather likely scenario? The less likely it becomes to investors that government actions signal a probable repayment of the debt –in real terms- the less keen they will be on investing in government debt, unless they are compensated for this risk.

The problem is that in times of financial distress, hence when financial resources are more valuable the stronger become these conditions to countries exhibiting less credible financial capacity. Is in these very moments when governments need to repay maturing debt and re-issue new one at a higher cost, as the financial markets do not believe that the governments will have the ability or political will of engaging in sometimes unpopular measures, particularly for those whose macroeconomic risks have been a pending subject. Here again is where the papers of De Grauwe (2011a,b), De Grauwe and Ji (2013) and Beirne and Fratzscher (2013) become relevant.

Generally, in times of financial distress, those governments which have kept an acceptable budgetary surplus will be able to benefit from low yields, as in times of low consumption growth government bonds are valued most and thus exhibit low interest rate levels. However, for the case of risky assets the opposite occurs, as in times of economic distress there is no appetite for investors for risky assets as they pay-off poorly. Only under a "compensation" for the risk-taking, are investors going to be willing to take this risk into their balance sheets. In order to be successful, government ministers of finance should asked themselves: what if financial crises took place in this very moment, with the subsequent fall in consumption growth with a severe increase in unemployment rates? What is the outlook for the government expected future surplus? Has the government the capacity to smooth consumption whilst maintain low costs of financing? Or has the government not done the homework all these years and will now face financial distress? How are tax-revenues going to be affected by a severe fall in GDP growth? How are yields going to move as a consequence of innovations on these macroeconomic variables? Are the new issuances still going to be attractive to the markets at a low price and for how long? The answer to all this is: depending on how macroeconomic risks affect their expected future surpluses and their cost of financing.

Eighth, in times of financial distress if investors perceive that governments are prone to fail to address macroeconomic risks affecting their surplus, they are left with no other choice but to adjust their portfolios accordingly. The reason is that the country's inherent macroeconomic risks affect the yield curve because they affect also governments' expected future surpluses. The deterioration of sovereign spreads is a signal of the market so that governments address these issues in order to restore confidence. In the process of doing so, they will need to cut down on government spending with the subsequent negative effects on growth. For example, Government spending comprises in Europe way more than 40% of total GDP and in times when bad news for aggregate consumption growth arrives, private expenditure is unlikely to grow enough to compensate for the fall in government spending.

The problem of not understanding this link is magnified by the fact that government securities have the property of acting as hedging instruments for times of low consumption growth, hence in times of financial distress. This is because they perform better in times when aggregate marginal utility is valued most. When governments fail to address their macroeconomic risks they give way to the possibility of a run on their issuances, which has a magnifying effect in times when bad news for consumption growth arrives.

This is because under the "good" equilibrium Government securities are only optimal when inflation is low and the probability of deterioration in its present value is virtually zero. However, if there is asymmetric information and there are doubts about the capacity or ability of the government to manage its inherent macroeconomic risks and hence ensure its ability to generate funds via either tax revenues, issuing new debt at a low cost or adjust deficits in the future, will result in investors categorising the debt as risky. An investment that was once seen as a risk-free asset and that – unexpectedly– turns out to be riskier, forces the representative investor to re-assess portfolio and adjust accordingly, with the subsequent effects on the yield curve which are prone to be long-lasting. If governments issue in a currency they do not control, a deterioration of their fiscal imbalances has a consequence on their spreads. Mostly due to liquidity issues, as the government cannot guarantee payment in a currency not controlled by its central bank.

Section IV. Public Debt Policy: a brief introduction

Some recent work on term structure and debt management policy can be seen in Cochrane (2001), Angeletos (2001), Faraglia, Marcet and Scott (2008) and Missale (1997).

Cochrane (2001) analyses the effects of long-term debt and optimal policy in the fiscal theory. In his work, the maturity of the debt matters, as it determines how news about current deficit implies current inflation or future inflation. In fact, governments could trade-off between current and future inflation by issuing long term debt. Thus, if primary surplus remains constant and there is an increase in total debt outstanding, an increase in the price level is expected. In fact, in this model the effect of debt on the price level is attenuated for longer maturities and as the maturity shortens. An important contribution is that surpluses are strongly negatively correlated with changes in the total value of the debt and that long term debt allows governments to offset surplus shocks as they come. The entire analysis in the fiscal theory stems from the idea that primary surplus must equal bond redemptions plus net repurchases and that the real value of outstanding debt equals the present value of real surpluses. In other words, there is a sequence of price levels for a given sequence of debt policy and surpluses.

Governments face the risk of a decrease in their tax revenues due to output shocks. Thus an unexpected fall in GDP could result in a decrease in government tax revenues, as businesses and individuals experience a fall in their taxable income. For example, an increase in unemployment results in a decrease in expected aggregate consumption and expected aggregate investment. As a result there is not only a decrease in expected tax revenues but also an increase in expected government spending, which further deteriorates expectations on government finances.

The fiscal theory of the price level was first developed by Leeper (1991), Sims (1994), Woodford (1995, 1996) and Dupor (1997) and says that the price level is determined by the ratio of nominal debt to the present value of real primary surpluses, in simple terms, the present value of outstanding debt equals the present value of real surpluses. The theory also reads that current surpluses must equal redemptions plus net repurchases, thus as in Cochrane (1998) review of the theory suggested

The present value identity:

$$\frac{B}{\pi} = \frac{s}{(1+y)}$$

B is total debt outstanding, s government primary surplus, for π is the price level and y the real interest rate. Assuming that the real interest rate remains constant, an increase in the government surplus would, ceteris paribus, result in a decrease in the price level.

The flow identity:

$$\frac{B_{t-1}}{\pi_t} - \frac{B_t}{\pi_t} = s_t$$

For B_{t-1} being a bond outstanding at the end of the period *t-1* and redeemed in *t*. B_t is the net repurchases or net new issues depending on the surplus/deficit at t. A negative surplus (or a deficit) would imply an increase in new debt outstanding and a positive surplus would imply a decrease in total debt outstanding.

The present value identity

Starting with the present value identity, making it formally and accounting for several periods yields:

$$\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t} + \sum_{i=0}^{N=T-t} \beta^{(N)} E_t\left(\frac{1}{\pi_{t+N}}\right) B_{t-1,1+i}^{(t+N)} = E_t \sum_{i=0}^{N=T-t} \beta^{(N)} s_{t+N} \tag{11}$$

For $B_{t-1}^{(N=T-t=0)}$ denote a zero coupon bond outstanding at the end of period *t-1* that matures in N. Notice that in this case $\beta^{(N)}$ is the discount factor and $\beta^{(N)} = \frac{1}{(1+y_t^{(N)})}$. Now by combining this with the affine term structure literature it can be said that $y_t^{(N)}$ is affine as derived from equations (1) to (10). π_t denotes the price level and s_t the real primary surplus, and the real primary surplus can also be said to be affine:

$$s_t^{(N)} = A(N) + B(N)' x_t + \varepsilon_t^{(N)} \tag{12}$$

From equations (1) to (12) it has been determined how macroeconomic variables depicted in the state-space vector x_t affect, both the yield curve and expected future primary surpluses, as shown in Jakas (2013a). In fact (11) shows that an expected deterioration in government finances as a result of macroeconomic shocks would, ceteris paribus, increase the price level. The increase in the price level is expected to result in a subsequent increase in nominal yields to compensate investors for their loss in purchasing power due to a decrease in the discount factor. If the adverse effects of macroeconomic shocks are expected to remain permanent, it would also have the effect of further increasing government's cost of financing, which will further deteriorate the net present value of future surpluses.

By including what we have already learned in section 1, equation (11) can be rearranged by replacing $\beta^{(N)}$ from a consumption asset pricing framework as follows:

$$\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t} + \sum_{i=0}^{N=T-t} E_t\left[m_{t+N} \left| I_t \right.\right] \times E_t\left[\frac{1}{\pi_{t+N}}\right] B_{t-1,1+i}^{(t+N)} = E_t \sum_{i=0}^{N=T-t} m_{t+N} s_{t+N} \qquad (13)$$

Notice that in (13) inflation has been detached from the stochastic discount factor m_{t+N} but only for the purpose of analysing the price level in isolation. In fact, from an affine term structure model I_t reflects all macroeconomic information at that point in time including inflation.

The flow identity

Cochrane's review of the theory of the price level suggests an analytical framework which focuses mostly on the effects of the flow identity against shocks which can be dissipated by issuing long term debt. However, his work does not link the flow identity to a well-known risk: the-so called "debt-roll over" risk. The debt roll over risk refers to the risk of an increase in the debt cost of financing during the process of rolling over maturing debt, hence repaying maturing debt by issuing new one. The increase in the cost of financing is usually as a consequence of unfavourable market movements at a particular moment in time. This will be analysed in detail later. For the time being it will be enough just to modify the present value identity in equation (11) a bit, which boils down to:

$$\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t} - \sum_{i=0}^{N=T-t} E_t\left[m_{t+N} \left| I_t \right.\right] \times E_t\left(\frac{1}{\pi_{t+N}}\right)\left(B_{t,1+i}^{(t+N)} - B_{t-1,1+i}^{(t+N)}\right) = s_t \qquad (14)$$

Now, from (14) current surpluses must equal redemptions plus net repurchases. From (11) and (14) it is possible to read that unfavourable surplus shocks can force governments to increase debt issuance in order to pay redemptions. Intuitively, from (11) we read that unfavourable scenarios affecting (14) are likely to increase the cost of financing via increases in the price level. Therefore, governments, in order to be able to smooth innovations on their surpluses need to have resources for times when consumption growth is low, so that they can compensate for the fall in private sector consumption and still be able to attend budgetary obligations without resulting in a deterioration in the country's welfare. Failure to do so the question is not if default is probable or not, the question is how is default to occur.

Analysing these results by means of the classical consumption asset pricing framework it is necessary to determine tow premises; A) the assumption that governments would target a certain size of surplus/deficit and B) central banks follow an inflation targeting framework restricted to output growth. The consumption asset pricing framework yields that: under the normality assumption, an expected deterioration in government finances would result in a decrease in expected future consumption growth and thus an increase in aggregate marginal utility growth with the subsequent increase in the stochastic discount factor and hence a fall in yields for risk free assets, which are negatively correlated with consumption growth. The negative effects are likely to be via fiscal adjustments, as governments would need to reduce spending in the future because current deficits have to be compensated –ideally- with future surpluses. This is rather a timing issue, as fiscal consolidation should take part during economic booms. The premise is that government securities for non-core countries are risk free only if remained within certain degree of surplus/deficit fiscal discipline. The other aspect is that central banks will be inflation focused. These two theories can be easily brought together under certain premises.

For example, by denoting G_t as government spending and τ_t as the tax rate which depends on output Y_t, and including the expectations operator it could be possible to rearrange (14) so that the flow identity becomes:

$$E_t\left[\tau_t(Y_t)\right]-E_t\left[G_t\right]=\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t}-\sum_{i=0}^{N=T-t}E_t\left[m_{t+N}\left|I_t\right.\right]\times E_t\left(\frac{1}{\pi_{t+N}}\right)\left(B_{t,1+i}^{(t+N)}-B_{t-1,1+i}^{(t+N)}\right) \qquad (15)$$

And the present value identity accounting for several periods is now:

$$E_t\sum_{i=0}^{N=T-t}E_t\left[m_{t+N}\left|I_t\right.\right]\times\left[\tau_{t+N}\left(Y_{t+N}\right)-G_{t+N}\right]=\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t}+\sum_{i=0}^{N=T-t}E_t\left[m_{t+N}\left|I_t\right.\right]\times E_t\left(\frac{1}{\pi_{t+N}}\right)B_{t-1,1+i}^{(t+N)} \qquad (16)$$

Now we need to differentiate the effects in the front end versus the effects in the long end of the yield curve. This can be done by recalling once more the inflation targeting framework, because we know that an increase in the price level due to a deterioration of the primary surplus is expected to increase yields in the front end of the curve due to central bank anchoring inflation expectations but also due to an increase in debt-roll over risk. As those issues from governments under current financial distress will observe an increase in their yields simply because investors are not sure if the government will be able to roll debt over successfully and hence be able to repay the maturing bonds by issuing new ones. This is mainly because countries central bank are solely inflation focused and do not act as lender of last resort for governments. This gives place to a so-called “fragility” of the Eurozone.

There has been a growing literature trying to link between the debt management and debt policy with fiscal theory. An example of these are seen in Faraglia, Marcet and Scott (2008), as well as Angeletos (2001) and Buera and Nicolini (2004). These works show the argument that the composition of government debt should be chosen to that fluctuations in the market value of debt is offset by changes in expected future deficits. Most of the discussion is centralised on the so-called complete versus incomplete market approach to debt management. Discussions here are that the incomplete market approach needs to be incorporated into the theory, allowing for important frictions such as transaction costs and liquidity effects to determine price levels and optimal fiscal policy prescriptions.

Angeletos (2001) explores the optimal debt policy under the assumption of non-state-contingent debt for the cases of incomplete as well as for complete markets. This was motivated by the fact that the literature until then had explored only under the assumption of state-contingent debt, as seen on seminal work such as Lucas and Stokey (1983) who assumes that debt is state-contingent in Arrow-Debreu complete markets. This is also seen in the works of Chari, Christiano & Kehoe (1991 and 1994) as well as in Chari and Kehoe (1999). Angeletos argument is on the basis that state-contingent debt assumption is rather unrealistic. In Angeletos work the complete markets paradigm of optimal fiscal policy holds even for non-state-contingent debt and that the maturity structure should be managed in advance, in order to hedge against fiscal innovations, as this type of shocks result in an increase in real interest rates and the maturity structure could be used to smooth this type of shocks.

Faraglia, Marcet and Scott (2008) revised these works including some recent research such as in Buera and Nicolini (2004). Their conclusion is that the Angeletos (2001) and Buera and Nicolini (2004) complete market approach to debt management results in implausible and unstable recommendations for optimal debt portfolios. The theory of debt management has to focus not only in providing insurance against fiscal innovations but also supplement with recognitions of capital market imperfections.

Debt management from an optimal taxation approach is reviewed by Missale (1997), in this review of the literature is shown that the aim of this approach is to issue low yielding debt instruments when times of low consumption growth and hence in times of recession when tax revenues are lower and government spending higher than expected. This approach shows that governments are not forced to increase taxes on labour income when innovations on consumption and government spending occur. Once more going back to the seminal paper of Lucas and Stokey (1983) in a model of state-contingent securities they show that negative indexation of debt returns to government spending shocks supports an efficient tax allocation from high to low spending states and vice versa. However this approach does not address the effects that the required risk premia will have on the governments' cost of financing. In state-contingent securities, an investor will expect now yields to pay less in times of low consumption growth and pay more in times when aggregate marginal utility growth is low with the subsequent increase in the risk premia.

An interesting view is the one taken by the so-called overlapping generations' model, as seen in Stiglitz (1983), Fischer (1983), Peled (1985), Pagano (1988) and Gale (1990). Main findings are that long term debt enables intergenerational risk-sharing. This works when innovations to consumption growth result in a negative productivity shock which is insured by the elder generations.

The need to link affine term structure models to fiscal theory of the price level and the optimal taxation approach is because governments need to place new debt when bad times arrive and the gains from surprising the market with systematic increases in their budget deficits, unexpected tax increases or increasing the price level is likely to be short lived and the costs of disappointing the representative investor long-lasting. Barro (1995) shows how focused should be placed in the maturity of the debt structure with the intention to match surpluses to repayments in order to reduce the "roll-over" risk. Hence, in times when yields are high the government's inter-temporal budget deteriorates if new debt needs to be rolled-over or issued. This is because macroeconomic shocks affecting the yield curve also have an effect on government revenues and spending.

Section V. In progress for a novel theory

This section studies an optimal debt policy for a stochastic representative agent economy, where government surplus is endogenous and stochastic function of a set of macroeconomic variables, and financed either by taxes or public debt. This section explores the case of non-contingent debt similar to Angeletos (2001), however it differs from Angeletos' research as it includes the possibility for governments to trade current inflation for future inflation by issuing long term debt as in Cochrane (2001). Therefore, the important contribution of this section to current research is that the analysis is based for debt markets exhibiting multiple equilibria and thus being able to move under certain states/events from a state of complete with complete information, to a state of incomplete information with information asymmetries.

The intention is to give possibility for both, a "good" equilibria and a "bad" equilibria under which for a given set of states or events it is possible to account for market frictions such as transaction costs, widening and tightening of bid-offer spreads and hence account for illiquidity but also account for a complete halt in trading known as sovereign debt crisis. This happens when those who are long bonds do not wish to sell so that they can avoid realising a loss and those who are short are only willing to buy at prices which are too low for those holding long positions. This situation can stay as long as those who are long are still able to access funding and those who are short are still able to borrow from other market participants at an optimum rate (repo rate or bond borrow fee rate). However, these unfavourable market scenarios are expected to result in an increase in the yields, particularly in the front end of the curve, with the subsequent increase in the cost of financing, and this is magnified in the times of debt roll-over and expected low consumption growth as well as expected future surplus shocks.

It will be shown how government securities which are originally risk-free can become risky assets when some of the equations shown from (11) to (16) become inequalities due to market imperfections and information asymmetries.

To be more precise, let us assume that a representative investor does not know what the surplus at time t is going to be and makes its expectations on a set of information I_t. In addition, let $\Omega(I_t)$ be a number of possible states in any single period; it could be denoted $\omega_0 = 0$ for the initial state and $\Omega = \{1, \dots, \Omega\}$, and a transition probability function $\Phi_{\omega,\omega'}$ for moving from state ω to state ω'. Finally it can be denoted $\omega^{t-1} \equiv (\omega_0, \dots, \omega_{t-1})$ and subset of Ω^{t-1} as the history at date t.

It could be possible to define the "good" and "bad" equilibrium by adapting (14). E.g. in the good equilibrium, markets' expectations are that governments will solvent enough in order to pay redemptions:

$$\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t} - \sum_{i=0}^{N=T-t} E_t\left[m_{t+N}(\omega^t | \omega^{t-1})\right] \times E_t\left(\frac{1}{\pi_{t+N}(\omega^t | \omega^{t-1})}\right)\left(B_{t,1+i}^{(t+N)} - B_{t-1,1+i}^{(t+N)}\right) \le E_t\left[s_t(\omega^t | \omega^{t-1})\right] \quad (17)$$

The "bad" equilibrium otherwise.

In addition, we could denote two types of prices P_{it}^{Bid} and P_{jt}^{Ask}, for "bid" being the prices at which the i-th investor would be willing to buy and "ask" the prices at which the j-th investor would be willing to sell. For $i \neq j$ always. Notice that $P_{it}^{Bid} \le P_{jt}^{Ask}$ can be assumed to hold always for a single investor but does not hold across investors hence there is a probability that $P^{Bid} = P^{Ask}$ at which a transaction will take place. There will be no trading if the probability $p(P^{Bid} = P^{Ask}) = 0$. An if the bid-offer spread defined as $Z_t(P_t^{Ask} - P_t^{Bid})$ increases, then the probability $p(P^{Bid} = P^{Ask}) \to 0$.

Also notice that as bid-offer spread widens yields will irremediably increase with the subsequent increase not only in the debt cost of financing but also increase in the debt roll-over risk. Unequivocally, the bid-offer spread is a measure of transaction costs as well as illiquidity.

From a present value identity point of view the "good" equilibrium will be if

$$\frac{B_{t-1}^{(N=T-t=0)}}{\pi_t(\omega^t|\omega^{t-1})} + \sum_{i=0}^{N=T-t} E_t\left[m_{t+N}|(\omega^t|\omega^{t-1})\right] \times E_t\left(\frac{1}{\pi_{t+N}(\omega^t|\omega^{t-1})}\right) B_{t-1,1+i}^{(t+N)} \leq E_t \sum_{i=0}^{N=T-t} E_t\left[s_t(\omega^t|\omega^{t-1})\right] \tag{18}$$

The "bad" equilibrium would be otherwise.

The theoretical analysis shown in (18) can be summarised under the affine term structure model framework and be used as a general framework for debt policy and debt management strategies. The aim is to integrate the latest developments on the affine term structure literature into the theory of debt management for a model of optimal fiscal policy. This work can be of interest to academics, economists, policy makers as well as for practitioners particularly for the investment banking, insurance and fund management industry. The intention is to contribute to a unifying theory of the term structure of interest in relation to macroeconomic policy, and to present empirical evidence using financial markets and macroeconomic data.

Section VI. Links to The Theory of Fragility of the Euro from De Grauwe (2013a,b)

In this thesis will link equations (15), (16), (17) and (18) to De Grauwe (2013a,b) theory of the fragility of the Euro. This theory will be discussed in more detail in chapter 4 under proposition 4. The theory of the fragility shows that shocks to the above mentioned equations result in multiple equilibria mainly because of the absence of a guarantee that enough funds will be available from the central bank to member countries. The assumption here is that there is a cost and benefit of defaulting on the debt and that investors account for this risk. In De Grauwe's theory of the fragility shows that the benefit of defaulting in that the government can reduce the interest burden on total debt outstanding and by doing so it will have to apply less unpopular austerity, hence will have to reduce spending or increase taxes by less than without default. This means that the state variables to calibrate the bonds will change, depending on the outcome of this theory. If the government's cost of defaulting is larger than the benefit of defaulting, as the case of Germany bonds, the state space vector will contain state variables which describe activity (unemployment and consumer confidence indices), monetary aggregates (M3) and the price level (e.g. PPI). However, if the government's benefit of a default is larger than the cost of defaulting on the debt then the state space vector will contain state variables which describe the level of solvency of the country, hence the state variables used for calibrating risky government bond assets will be the debt-to-GDP ratio, surplus-to-GDP ratio and unemployment rates.

Figure 1-1 below show that there are mainly three groups of yields in the Euro-Zone: 1) one group is Greece, which exhibit yields which are clearly very sensitive to movements in the Debt-to-GDP ratio; 2) Ireland, Portugal, Spain and Italy, which have suffered from intermediate solvency shocks which lead to spreads widening during periods of multiple equilibria. Taking the Spanish yields as a representative member of this group and comparing it with the Greek case below, it can be seen that markets have had very different views of the two set of countries. 3) The last group would be of those countries which have experienced rather small solvency shocks, e.g. Germany, Netherland, France, Luxembourg and Belgium.

In general, the reader should notice that these shocks generate a structural break from 2008 and onwards, embracing the beginning of a new era for the Eurozone.

Figure 1-1. Yields and debt ratios from GIIPS Countries and Germany.

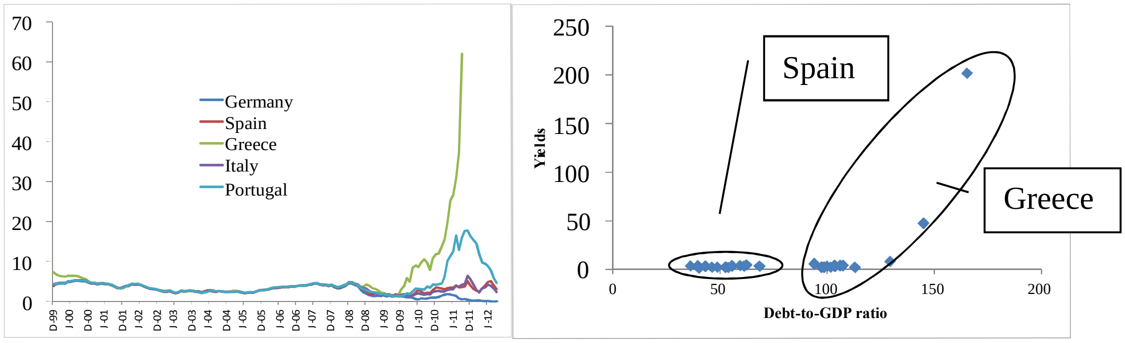
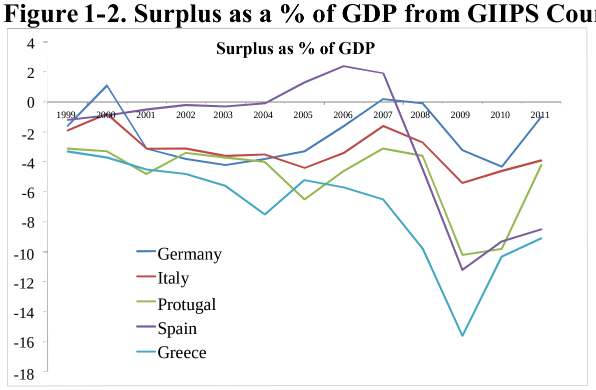


Figure 1-2 depicts the Surplus-to-GDP ratio of GIIPS countries and Germany. Here again, it can be seen that the levels observed for Greece are significant. However, Spain gets very close to Greek levels towards the end of the period, what makes the difference for the markets is that Spain has a much smaller Debt-to-GDP ratio compared to Greece, as shown in figures 1-3.

Figure 1-2. Surplus as a % of GDP from GIIPS Countries and Germany.



This confirms the views of De Grauwe and Ji (2013) and De Grauwe (2011a,b), as it can be seen in figure 1-3, how the sensitivity of the yields to solvency shocks will be different depending on the initial debt level. Spain for instance, remains within the 50-70 per cent of Debt-to-GDP ratio, whereas Greece starts with debt levels between 80-90 per cent and grows to levels way above the 150 per cent of Debt-to-GDP ratio. These results suggest that there is some incentive for governments to pursue a target of Debt-to-GDP ratio, as failure to do so it could result in an intermediate or large solvency shock, which in turn could further increase the cost of financing (see figure 1-4).

Figure 1-3. The Case of Spain versus Greece Problem.

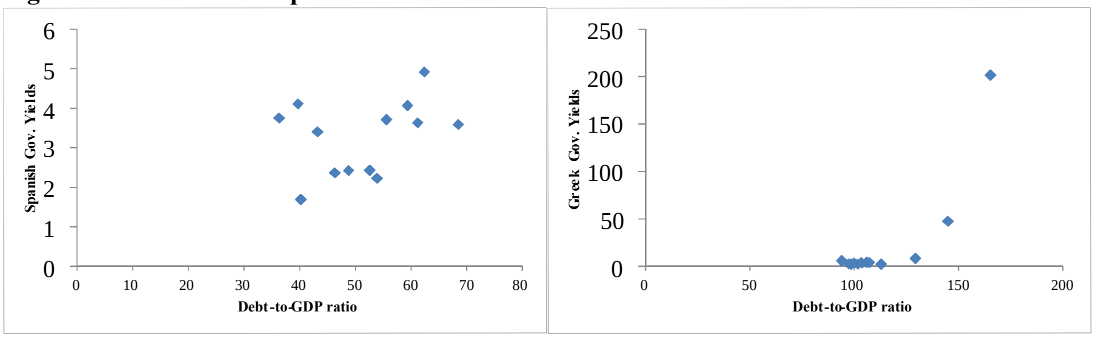
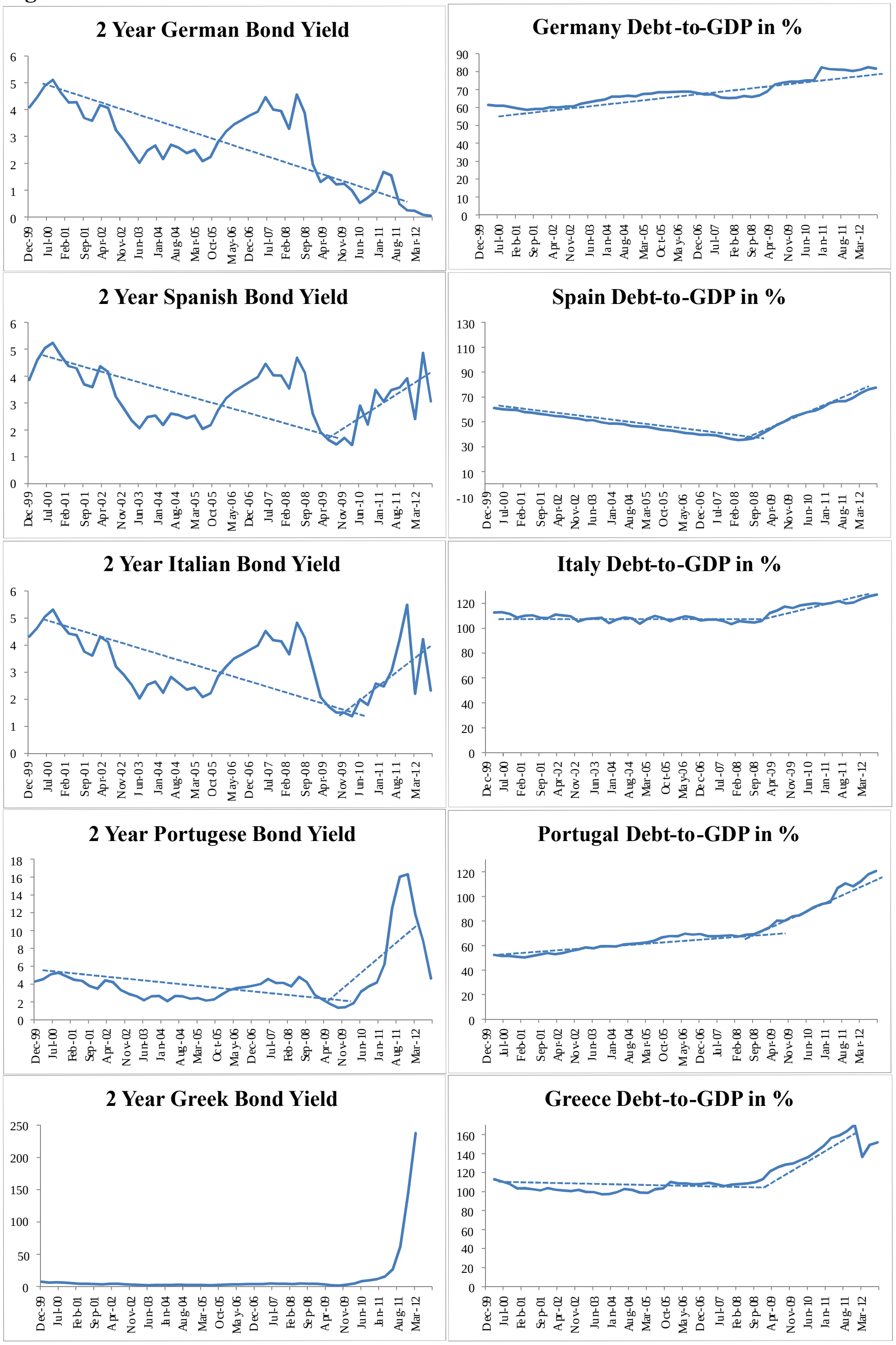


Figure 1-4 shows the existence of a structural change which appears to become more evident since 2008. Since 2008 all GIIPSs countries exhibit a deterioration of the Debt-to-GDP ratio and an increase in yields. A completely different picture is seen in Germany; 1) the debt ratio grows at a steady level with no apparent structural break; 2) Yields decrease and, 3) debt ratio remains at 55-75 per cent range, thus remains at low levels. Chapter 5 will calibrate German and Greek government bonds according to discussions in this chapter and in chapter 4.

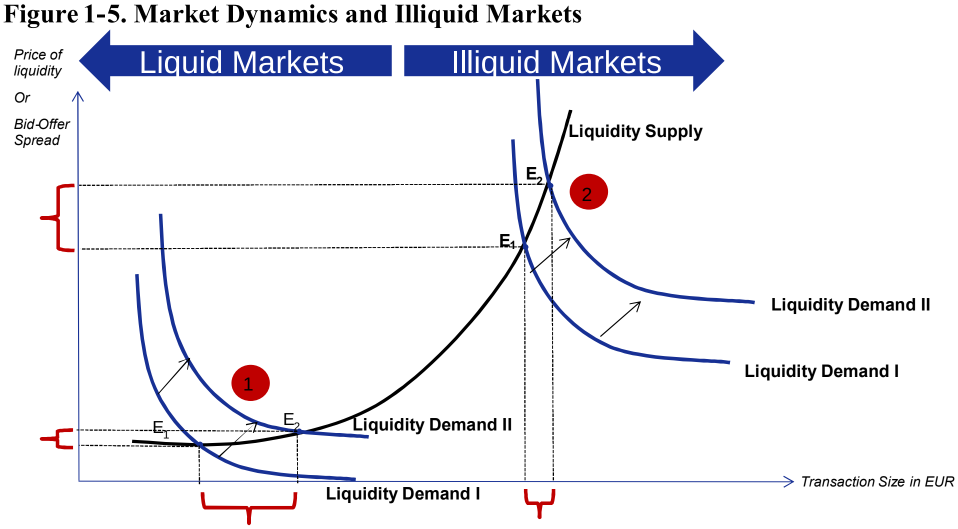
Figure 1-4. 2Year Bond Yields versus Debt-to-GDP Ratios GIIP countries and Germany.



Section VII. Links of the Fragility of the Eurozone to Illiquid Markets

This section shows how a liquid bond can become illiquid as a consequence of solvency shocks on the government issuer. The liquidity supply curve depicts the combination of prices and volumes at which a financial intermediate or an investor is willing to exchange liquid for illiquid assets. Notice that by doing so the investor is taking on the balance sheet a risky asset in exchange for liquidity. The model also assumes that the price for liquidity is the bid-offer spread. If supply remains constant and liquidity demand increases, the bid-offer spreads become wider, as shown in in quadrants 1 and 2 (red highlighted). Figure 1-5 explains how shocks in the liquidity demand will be different depending on the initial liquidity level of the market. For instance, liquidity shocks in quadrant 1 have rather low effects on the bid-offer spread, in contrast to liquidity shock in quadrant 2, where small liquidity shocks have more than proportional increases in the bid-offer spread.

Figure 1-5. Market Dynamics and Illiquid Markets

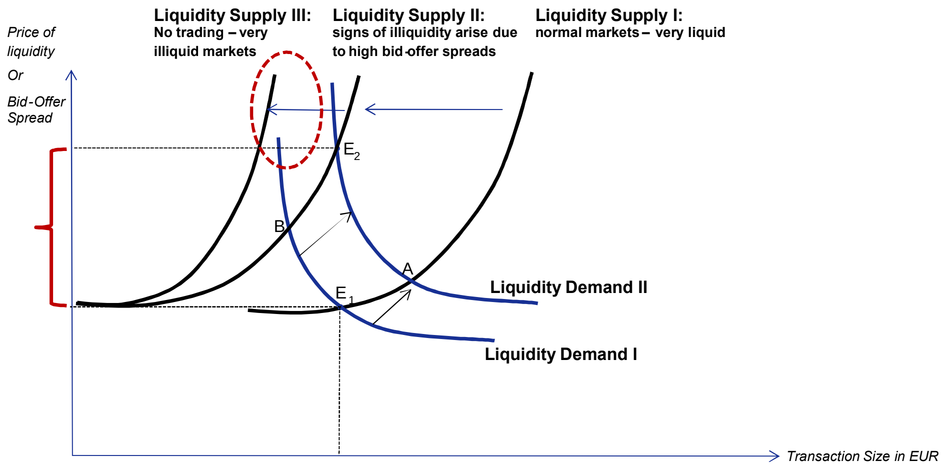


1 In normal market conditions, when liquidity supply is elastic, an increase in demand would, ceteris paribus, have a less than proportional increase in the bid offer spread.

2 However and, as liquidity supply becomes inelastic, any increase in the demand for liquidity can only be fulfilled at a price which increases more than proportionally.

Figure 1-6 show the effects of simultaneous shocks: 1) in the liquidity supply and 2) in the liquidity demand. Here, this can result in a halt in trading with markets breaking down as those who wish to sell cannot find any buyers and those willing to buy cannot find any sellers. This stems mostly out of discussion from equations (17) and (18) of this chapter.

Figure 1-6. Contraction in the Supply of Liquidity.

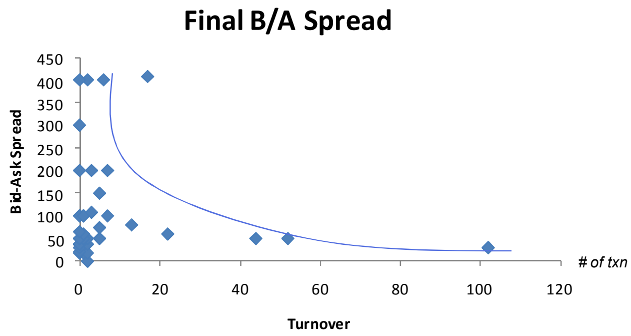


As liquidity supply contracts, the price for liquid assets becomes more expensive. Once liquidity contracts further, there is a halt in trading, as supply does not meet demand any more, hence markets are not perfect and positions become stale.

Those being long liquidity are reluctant to invest, as risk aversion arises and those with liquidity shortages decide not to borrow as the price is too high.

Figure 1-7 shows that there is an apparent relationship between turnover and bid-offer spreads. Here we analyse 50 bonds found in Bloomberg. It can be seen that the greater the turnover the tighter the bid-offer spread. However, when turnover is low the number of transactions exhibiting wider bid-offer spreads is higher. As governments observe a solvency shock that is perceived by the markets to be intermediate or large, it is more likely that there will be a fall in the liquidity supply resulting in higher yields and in wider bid-offer spreads, as in discussion in figures 1-5 and 1-6. The logical consequence is a decrease in turnover and eventually a halt in trading as liquidity deteriorates.

Figure 1-7. Final or closing bid-ask (bid-offer) spreads.



There is a statistical relationship between bid-ask spreads and turnover:

→ The lower the turnover the more likely it is to find higher bid-ask spreads.

As turnover falls, bid-ask spreads are likely to become larger, which possibly contributes to further falls in liquidity.

This thesis will not undertake research on bid-offer spreads and yields. However, this is an area of interest and until now the use of bid-offer spreads as state variables for calibrating an affine term structure model has not been performed. Here we would like to indicate that this is a topic for future research which can have significant follow-up potential.

In the next chapter, we will regress via OLS European yields using a state vector of macroeconomic variables.

2. Theory of a Term Structure Model Applied to European Data (violating the no-arbitrage condition)

Section I. Introduction

This essay is motivated by the fact that consumption based asset pricing models despite being robust theoretically, they perform poorly empirically.

To overcome this, the model has been rearranged slightly replacing the usual consumption data by unemployment data and a consumer confidence index to better account for expectations. By doing so the model remains as a consumption based model with all its theoretical properties. Another aspect is that the price level has been added into the aggregate marginal utility function. Finally, the model also includes the effects of changes in monetary aggregates on asset prices. An no-arbitrage model is derived in this paper, using a state space system with an observation equation which links observable yields to these macroeconomic variables and a state equation which describes the dynamics of these variables. Expected aggregate marginal utility growth is modelled using unemployment figures, consumer confidence index and production price index. Money supply is modelled using monetary aggregate M3. This essay is published in Jakas (2011).

The rest of the paper is organised as follows: In section II we recall the basic prising equation to remind the reader how this looks like from a yield perspective. In section III we discuss our model and show how unemployment, consumer confidence index and the price level enter the aggregate marginal utility function and how money supply is added to the basic pricing equation. Section IV outlines the empirical process and we show how we go about testing our model. Sections V and VI we discuss the data and present the results. Section VII we present some of the related literature and in section VIII we conclude and present our final remarks.

Section II. The Basic Asset Pricing Equation

The literature extensively documented the poor empirical results on consumption based asset pricing models. A useful summary can be seen on Guvenen and Lustig (2007) and Cochrane (2001).

It would be useful to start by recalling the basic asset pricing equation presented in (1), which for the reader's convenience will be reproduced below

$$P_t = E_t\left[\left(\beta \frac{u'(c_{t+1})}{u'(c_t)}\right)x_{t+1}\right] \tag{1}$$

For P_t being the present value at time t, of an asset paying off x_{t+1}, E_t is the expectations operator. β the subjective discount factor and $u'(c_{t+1}) / u'(c_t)$ being the aggregate marginal utility growth which encapsulates investor's first order condition with respect to consumption and thus together with β comprises the inter-temporal marginal rate of substitution (Rubinstein, 1976; Lucas, 1978; Breeden, 1979).

According to the evidence, it appears that aggregate marginal utility growth using solely consumption data appears to have poor explanatory power on asset prices. For example, the arithmetical expectation of consumption growth as in (1) does not necessarily mean that it encapsulates all information about consumers' expectations and their behaviour.

The intention is to deal with this problem, so it will be assumed that expected aggregate marginal utility growth is a function of current unemployment plus another variable which encapsulates consumers' expectations. In addition, the new aggregate marginal utility growth will be adjusted for the price level and for changes in monetary aggregates.

Recalling the basic pricing equation and assuming a 1 year zero coupon bond which pays out 1 monetary unit at maturity, the investor's first order condition for E_t conditional to an information set at time t, would yield:

$$E\left[m_{t+1}\middle| I_t\right] = E_t\left[\beta \frac{u^{'}(c_{t+1})}{u^{'}(c_t)}\right] = e^{-y_t^{(N)}}$$

For $y_t^{(N)}$ representing the bond yields with maturity N at time t and I_t being agents' information set at time t. Which for convenience it will be transformed to equation (2) from chapter 1 which for the reader's convenience it is reproduced again below:

$$-\log E\left[m_{t+1}\middle| I_t\right] = y_t^{(N)} \tag{2}$$

Equation (2) basically shows that the stochastic discount factor is the inverse of the observable yields.

Section III. The Model

In an attempt to depict the idea that aggregate marginal utility growth is determined by consumers' confidence, unemployment, the price level and a monetary aggregate, we would then propose the following identity:

$$y_t^{(N)} = -\log E_t\left[\beta \frac{u^{'}(c_{t+1})}{u^{'}(c_t)} \frac{M_{t+1}^s}{M_t^s}\right] \tag{3}$$

For consumption being:

$$c_t = \frac{(1-U_t)wC_t^e}{\Pi_t} \tag{4}$$

For U_t representing the unemployment rate, which should not get confused with the utility function which is noted in this essay as $u(c_t)$. w is the nominal wage level, which for the sake of simplicity it will be assumed to be constant but it does not have to be that way. C_t^e represents consumers' expectations on their future consumption growth determined by the consumer confidence index, Π_t is the price level and M_t^s represents a monetary aggregate to account for money supply, for this case M3 (M4 for the UK equivalent). For our analysis it will be assumed that preferences follow a log utility function with constant relative risk aversion as specified in the introductory chapter in equation (3), which again for the reader's convenience we reproduced below as follows:

$$u(c_t) = \frac{c_t^{(1-\gamma)}}{(1-\gamma)} \tag{5}$$

Conditions (3) to (5) have the following implications:

1. We will start describing what we believe the effects of U_t – unemployment – have on the yield curve. Unemployment is expected to have a negative relationship to risk-free assets' returns. And this is because an increase in unemployment results in a decrease in expected future consumption growth and therefore, results in an increase in aggregate marginal utility with the subsequent increase in the discount factor for assets which are uncorrelated with consumption growth. From (3) we can see that an increase in aggregate marginal utility results in an increase in the discount factor for risk-free assets, and as a consequence, yields are expected to fall.
2. The improvements in consumers' expectations about the future of the economy should have a positive relationship to government yields. An improvement in consumers' expectations about the future results in an increase in expected future consumption growth and therefore, results in a decrease in aggregate marginal utility with the subsequent decrease in the discount factor for assets which are uncorrelated with consumption growth such as government bonds. An improvement in consumers' expectations results in a decrease in the discount factor for risk-free assets and as a consequence risk-free assets' yields are expected to rise.

3. Points (1) and (2) have been relatively straight forward. However, the effects of the price level on the yield curve are rather ambiguous. Hence, these effects are governed by: a) two effects which move in opposite directions. Thus, the effect of the price level on expected aggregate marginal utility growth and the effect of central banks reaction as a consequence of increases in the price level. And b) these effects will be determined by the segment or maturity we are at in the curve, as central banks have only greater influence on the lower end of the curve, thus the money market curve and lesser effects in the longer end, the capital markets curve. To describe this idea better: an increase in the price level is expected to have a negative effect in expected future consumption growth and is expected to increase future aggregate marginal utility with the subsequent increase in the discount factor for assets which are uncorrelated with consumption growth such as government bonds. As in Piazzesi (2006), it is assumed throughout the paper that unfavourable change in the price level is always bad news for consumption. Therefore an increase in the price level is expected to result in a fall in e.g. government bond yields. However if the central bank responses to increases in the price level is to increase interest rates, then yields are expected to be positively correlated with the price level but only in the lower end of the curve. Simply because central banks are more efficient in influencing the money market curve than the capital markets. The effect throughout the curve will depend on the elasticity of substitution between money and capital market instruments. Thus, for those parts of the curve where money markets are perfect substitutes of capital markets, i.e. presumably up to the 2 to 3 year maturities, changes in the yield curve are expected to be more influenced by monetary policy reaction to changes in the price level rather than changes in expected aggregate marginal utility growth. The elasticity of substitution is expected to dissipate as maturity increases and money market instruments elasticity of substitution become inelastic. As elasticity of substitution between money markets and capital markets become more inelastic the longer the maturities, the yield curve is governed rather by changes in expected aggregate marginal utility growth than by central banks policy reaction to expected changes in the price level.

4. By adjusting the stochastic discount factor to a monetary aggregate indicator we intend to account for these effects described in Turnovsky (1989). Assuming that the price level remains unchanged, an expansionary monetary policy will result in increases in asset prices with the subsequent fall in yields, as a consequence of an increase in real money balances. For central banks, to be able to apply an expansionary monetary policy they need to either, relax reserves requirements, increase money supply via open market operations, acquire government debt securities and therefore inject additional quantity of money in the financial system, all this drives to asset price increases particularly government bonds. An increase in real money balances increases the quantity of money available for speculative purposes, which results in rising asset prices. Risk-free assets such as government bonds are expected to be affected up to the level that the elasticity of substitution between money markets and capital markets allows it to. On the longer end of the government curve, movements in monetary aggregates are expected to dissipate. This is because in our model monetary aggregates do not enter into the aggregate marginal utility function and therefore, have little effects on risk-free assets which are expected to act as hedges during bad times, when consumption growth is expected to be low. In this model, if an expansionary monetary policy results in an increase in consumption, due to a rise in real money balances, this would have no effect on asset prices, as these increases are assumed to be offset by increases in the consumer price levels. In fact, with this model we are able to show that if the price level is independent to increases in monetary aggregates, it would mean that an expansionary monetary policy affect asset prices.

   The model now asserts that if the price level is expected to remain unchanged and there is an expected increase in money supply, asset prices are expected to rise. This is because there is an increase in real money balances which results in a decrease in yields. However this could be seen from a different perspective: if the quantity of money increases, and this increase results in increases in real money balances for speculative purposes rather than for increases in consumption growth, money could be seen as to have lost value against other assets (particularly risky assets) as the quantity of money has increased at a greater rate than the quantity of risky assets. The paradigm here is that the price level will not always necessarily increase due to increases in the quantity of money and as long as the increase in the quantity of money has no effect in consumption growth, there is no reason to believe that could have an effect in the price level measured by consumer price indices. Therefore, money has lost value against other assets, whilst maintaining its purchasing power against consumption goods.

The idea that money supply is a determinant of asset prices is not new, see for example Kindleberger (1978) and Allen and Gale (2000). In recent cases, asset prices have risen due to for what it appears to have been an expansion in credit, e.g. following financial liberalisation. Another example is the rise in real estate and stock prices that occurred in Japan in the late 1980's. Once the BoJ decided to tackle inflation with sharp increases in interest rates – with the subsequent reduction in money supply by imposing reserves requirements –, asset prices fell dramatically. Similar effects were observed in Scandinavian countries.

These examples suggest a relationship between credit or money supply and the rise in asset prices.

Therefore, in the wake of these events it seems plausible to take monetary aggregates into account. So recalling (1) and (5) and accounting for a logarithmic utility function and log prices, nominal yields can be specified as:

$$y_t^{(N)} = -\frac{1}{N} p_t^{(N)} = -\frac{1}{N} \ln E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \frac{M_{t+1}^s}{M_t^s} \right] \tag{6}$$

Transforming (6) for all state space components discussed from (3) to (5) into their natural logarithms, and assuming normality we obtain:

$$y_t^{(N)} = -\frac{\ln\beta}{N} - \frac{1}{N}E_t\left[\Delta m_{t+1}^s\right] + \frac{1}{N}\gamma E_t\left[-\Delta u_{t+1} + \Delta c_{t+1}^e \pm \Delta\pi_{t+1}\right]$$
$$-\frac{1}{N}\frac{\gamma^2}{2}Var\left[-\Delta u_{t+1} + \Delta c_{t+1}^e - \Delta\pi_{t+1}\right] \quad (7)$$

For Δm_{t+1}^s, Δu_{t+1}, Δc_{t+1}^e and $\Delta\pi_{t+1}$ being the rate of change in monetary supply, unemployment, consumer expectations and inflation for the period *t* to *t*+1 or more precisely: $\ln(M_{t+1}^s / M_t^s)$, $\ln(U_{t+1} / U_t)$, $\ln(C_{t+1}^e / C_t^e)$ and $\ln(\Pi_{t+1} / \Pi_t)$.

Equation (7) shows how an increase in the volatility results in falling yields for a risk-free asset, which is supposed to hedge in times of low consumption growth. This can be seen in the last term. Hence, if any components comprising the marginal utility growth become more volatile and result in a higher volatility for expected future consumption growth, risk-free assets are expected to experience a fall in yields as a result of the precautionary savings effect seen on the variance term from (7).

Notice that changes in the price level can be positive or negative depending on the level of substitution between money market and capital market instruments. So adapting equation (7) to discussions in the introductory chapter 1 equation (8) and accounting for an elasticity of substitution our model would boil down to,

$$y_t^{(N)} = -\frac{\ln\beta}{N} - \frac{E_t\left(\sum_{i=0}^{N=T-t}\lambda_{t,1+i}^{(N)}\Delta m_{t+1+i}^s\right)}{N}$$
$$+\frac{\gamma E_t\left(\sum_{i=0}^{N=T-t}\lambda_{t,1+i}^{(N)}\left[-\Delta u_{t+1+i} + \Delta c_{t+1+i}^e - \left(1-\varepsilon_{t+1+i}^{(N)}\right)\Delta\pi_{t+1+i}\right]\right)}{N} \quad (8)$$
$$-\frac{\gamma^2 Var\sum_{i=0}^{N=T-t}\lambda_{t,1+i}^{(N)}\left[-\Delta u_{t+1+i} + \Delta c_{t+1+i}^e - \left(1-\varepsilon_{t+1+i}^{(N)}\right)\Delta\pi_{t+1+i}\right]}{2N}$$

The elasticity of substitution ε_{t+i} who affects the price level coefficient term under (1 - $\varepsilon^{(N)}$) measures the degree of substitution between two instrument types: the money market instruments and capital market instruments (e.g. 2 year maturity repos and 2 year government securities). As maturities become longer, the elasticity of substitution becomes inelastic so that the sign changes as ε_{t+i} gets closer to 0. If elasticity equals 1, instruments become perfect substitutes (perfectly elastic) and the price level is neutral because the effect of the central bank reaction to changes in the price level is exactly off-set by the effect resulting from shocks from the aggregate utility function. If elasticity is greater than 1, elasticity is said to be very elastic and therefore, the curve movements will be mainly govern by central bank policy and less by aggregate marginal utility growth.

Why are we differentiating only for the case of the price level? Because of central banks open market operations. For the cases of unemployment and consumer confidence we assume that there is rather no effective intervention adding extra shocks into the model. Now for the case of the price level it is virtually impossible to ignore money market shocks resulting from these interventions and spreading throughout the curve.

Clearly, as maturity increases, the variance of the discount factor is expected to fall by a rate of $1/(2N)$ as shown in (8). Another aspect from (7) and (8) is that a volatile monetary policy is expected to have negative effect on risk-free-assets' yields. Recalling that risk-free assets act as a hedge for consumption growth, it could be understood from this that a volatile monetary policy could have a negative effect on consumption growth, and as a result risk-free asset yields are expected to fall with the subsequent increase in risk-free asset prices.

One of the shortcomings seen in (7) and (8) is the assumption of investors' homoscedastic expectations. This is only for convenience, as it does not necessarily have to be this way.

Section IV. The Empirics

For empirical analysis, equations (3) to (8) can be summarised together into a state space system similar to Piazzesi (2010). These specifications are put together with an observation equation which links observable yields to the state vector and a state equation which describes the dynamics of the state as discussed in equations (9) and (10) of preceding chapter section 1. However, in this essay we violate the no-arbitrage condition as yields are separately regressed to a vector of state variables. A no-arbitrage version of an affine model will be estimated at a later point.

Section V. The Data

This empirical work is based on monthly European macroeconomic data particularly from the ECB and Eurostat available in Bloomberg. EONIA, Euribor and German government yields have been obtained from Bloomberg. Most of the data series is only available since 1998. This makes this analysis difficult, hence for lack of longer time series. The data points for the macroeconomic data are assumed to be released at every end of month. The day of the month at which the data is released is not relevant on a monthly basis analysis. We have tested this and results remain unaffected. The period considered is from January 1999 until July 2008. This results in 114 observations and 9 regressions for each of the yields comprising EONIA for the overnight rate, Euribor 3 months, Euribor 6 months and the German government securities for 2, 5, 10, 15, 20 and 30 years.

Section VI. Empirical Results

Table 2-1 below shows the regression results from chapter 1 equation [9]. Strikingly, almost all coefficients are significantly different from zero and the model also appears to perform well in predicting interest rates and yields. R-Squared falls as maturities increases. On average, we observed an upward sloping yield curve starting with an average overnight rate of the period of 3.06% up to 4.66% for a 30 year German government bond. Results also show that the lower end of the curve is far more volatile than the longer end. The lower end exhibiting a standard deviation of 1.20 compare to 0.61 and 0.63 for 15 and 30 year government bonds. Most importantly, the majority of the coefficients confirm our expectations as in [7] and [8] as well as discussions in section 3 points 1) to 4).

Figure 2-1 shows the observed versus the estimated values for EONIA rate, the Euribor three months, as well as German Government bonds 5 years and 30 years respectively (GBRD 5Yr and GBRD 10Yr). The model predicts fairly well the lower end of the curve and, with less accuracy but still with success, the longer end. Notice, that we are not using the lagged value of any endogenous variable to obtain these yields hence, we are inferring these values purely from our macro data via OLS. This is important, because most of the models until now have included and AR(1) with a lag of the endogenous variable, this gave these authors good results for obvious reasons.

Figure 2-2 shows the coefficients from the explanatory variables plotted as a function of the maturities. Notice how the coefficients for the consumer confidence index and monetary aggregate M3 dissipate away as maturity increases. In fact, coefficients become less significant for the longer end of the curve, as seen in table 1. Thus, as the time horizon increases, the effects of changes in consumer confidence and the quantity of money have lower predictive power. This makes sense, as not many of us make consumption decisions under an investment horizon greater than 10 years. Similarly, monetary shocks fall dramatically during the first 5 years maturities confirming that monetary aggregates will have only effects on the lower end of the curve.

The price level, which is in this case measured by the Production Price Index, exhibits a positive relationship to yields in the lower end of the curve. Not surprisingly, this relationship reverts as maturities become longer. This is in line with our discussion as in [7] and [8] as well as in section 3 points 3) to 4). Notice how the five year maturity is not affected by the price level at all. Again, as maturities greater than the 5 years are less affected by monetary policy shocks and more influenced by movements in aggregate marginal utility growth. Below the 5 years maturities, changes in the price level are positively correlated to yields. This is because asset prices are here rather influenced by monetary policy shocks than changes in aggregate marginal utility growth.

Table 2-1. **EONIA, Euribor Rates, and the German Government Yield Curve as Proxies for the Term Structure of Interest Rates.**

	EONIA Coefficient Standard Error T-Ratio [Prob]	Euribor 3M Coefficient Standard Error T-Ratio [Prob]	Euribor 6M Coefficient Standard Error T-Ratio [Prob]	2 Years Coefficient Standard Error T-Ratio [Prob]	5 Years Coefficient Standard Error T-Ratio [Prob]	10 Years Coefficient Standard Error T-Ratio [Prob]	15 Years Coefficient Standard Error T-Ratio [Prob]	20 Years Coefficient Standard Error T-Ratio [Prob]	30 Years Coefficient Standard Error T-Ratio [Prob]
Intercept	15.61	0.70	-14.46	-11.90	6.85	26.11	32.90	39.96	40.10
	4.64	5.15	5.91	5.45	5.70	5.30	5.93	5.74	5.69
	3.36[.001]	0.14[.892]	-2.45[.016]	-2.18[.031]	1.23[.221]	4.92[.000]	5.55[.000]	6.96[.000]	7.05[.000]
Log Confidence Index	1.02	0.58	1.39	4.79	3.31	1.42	0.01	-0.51	-0.02
	0.44	0.48	0.56	0.51	0.52	0.50	0.56	0.54	0.54
	2.34[.021]	1.20[.231]	2.49[.014]	9.34[.000]	6.31[.000]	2.84[.005]	-0.010[.988]	-0.95[.346]	-0.03[.977]
Log Unemployment	-9.72	-9.63	-8.05	-5.50	-4.30	-3.59	-3.66	-3.77	-3.45
	0.52	0.58	0.66	0.61	0.63	0.60	0.67	0.64	0.64
	-18.65[.000]	-16.66[.000]	-12.14[.000]	-8.98[.000]	-6.87[.000]	-6.04[.000]	-5.50[.000]	-5.84[.000]	-5.41[.000]
Log Euro-Zone M3	-7.25	-8.36	-8.98	-4.94	-2.38	-1.15	-1.28	-0.37	-0.60
	0.66	0.74	0.84	0.78	0.80	0.76	0.85	0.82	0.81
	-10.93[.000]	-11.37[.000]	-10.65[.000]	-6.34[.000]	-2.99[.003]	-1.52[.131]	-1.52[.132]	-0.46[.648]	-0.74[.463]
Log Production Prices	14.65	20.45	23.43	10.63	2.63	-2.28	-2.04	-4.72	-4.94
	1.85	2.05	2.35	2.17	2.22	2.11	2.36	2.28	2.26
	7.93[.000]	9.98[.000]	9.97[.000]	4.90[.000]	1.19[.237]	-1.08[.282]	-0.87[.338]	-2.07[.041]	-2.18[.031]
R-Squared	0.94	0.93	0.91	0.90	0.82	0.76	0.64	0.61	0.69
R-Bar-Squared	0.94	0.93	0.91	0.89	0.82	0.75	0.63	0.60	0.68
S. E. of Regression	0.29	0.32	0.37	0.34	0.35	0.33	0.37	0.36	0.36
F-Stat: F(4; 117)	485.06[.000]	398.17[.000]	290.10[.000]	271.61[.000]	135.48[.000]	90.70[.000]	52.10[.000]	45.48[.000]	65.36[.000]
Average Yields	3.06	3.19	3.27	3.21	3.68	4.16	4.42	4.60	4.66
Standard Deviation	1.20	1.21	1.20	1.07	0.81	0.66	0.61	0.56	0.63
Residual Sum of Squares	9.85	12.13	15.94	13.59	14.18	12.86	16.05	15.08	14.78
Durbin Watson Statistic	1.22	0.59	0.67	0.47	0.41	0.33	0.32	0.31	0.27
A: Serial Correlation	18.03[.000]	59.79[.000]	53.25[.000]	70.47[.000]	75.88[.000]	82.93[.000]	84.88[.000]	85.03[.000]	88.369[.000]
B: Functional Form	3.46[.019]	9.45[.000]	8.41[.000]	11.92[.000]	12.73[.000]	16.75[.000]	37.95[.000]	33.63[.000]	31.74[.000]
C: Normality	2.57[.277]	5.29[.071]	18.93[.000]	1.28[.528]	1.29[.524]	1.21[.545]	2.67[.263]	4.41[.110]	2.224[.328]
D: Heteroscedasticity	0.34[.853]	6.61[.010]	5.44[.020]	24.04[.000]	45.63[.000]	59.93[.000]	69.38[.000]	77.76[.000]	78.02[.000]

A: Breusch-Godfrey, Lagrange multiplier test of residual serial correlation, hence to test the null that the disturbances follow an AR(p). B: Ramsey's RESET test using the square of the fitted values, hence to test the null that the model has no omitted variables. C: Based on Jarque-Bera test of skewness and kurtosis of residuals to test the null hypothesis of normally distributed disturbances. D: Based on LM test for ARCH effects, test the null of homoscedasticity. Note: OLS regression computed with observed yields and observed state space variables. Variables being the Eurostat Consumer Confidence Index, Unemployment Rate, Monetary Aggregate M3, Production Prices for the Euro-Zone. (Source: Eurostat, ECB and Bloomberg).

This could also mean simply that interest rates or yields in the lower end of the curve are rather influenced by central bank policy responses to this particular index (PPI). Yields or interest rates on the longer end exhibit a negative relationship to current price level data, because the market knows that central bank will increase interest rates and as result expected future changes in the price level will fall with the subsequent decrease in yields. This is important, because contrary to the controversial results presented by Cook and Hahn (1989) this evidence predicts that increases in the central bank policy rates as a consequence of a contractionary monetary policy would immediately lower long-term nominal interest rates. In general, the effects of the price level on the term structure of interest rates appears to us, at least for the period analysed, to be in line with discussions outlined under section 3, points 3 and 4.

The monetary aggregate M3, the price level and unemployment account for the main drivers moving the lower end of the curve. However, only the price level and unemployment are persistent and their effects do not die away with longer maturities. A reason why these variables are persistent can be explained by unemployment and shocks in the price level having a persistent effect on aggregate consumption and investors being averse to persistence, as Piazzesi (2006) asserts, aversion to persistence generates concerns on future consumption when bad news arrives.

In general, our model predictions become poorer as maturity increases as seen in lower correlation coefficients as well as by increasing residual sum of squares, despite the fact that the standard deviation of the dependent variables decreases with longer maturities. The model, would on average, predict an upward sloping yield curve, as shown for higher observed mean of dependent variables. This is because the price level and unemployment coefficients are persistent throughout the yield curve and under normality these two effects account for an upward sloping yield curve, whereas consumer confidence and monetary aggregate shocks die away as maturities become longer.[1]

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[1] Due to the existence of autocorrelation and heteroscedasticity issues (see Table 1) autocorrelation and heteroscedasticity consistent regressions were performed. Results do not differ to those already presented in Table I, therefore have been omitted here.

Figure 2-1. **Comparison of benchmark model using OLS versus observed values.**

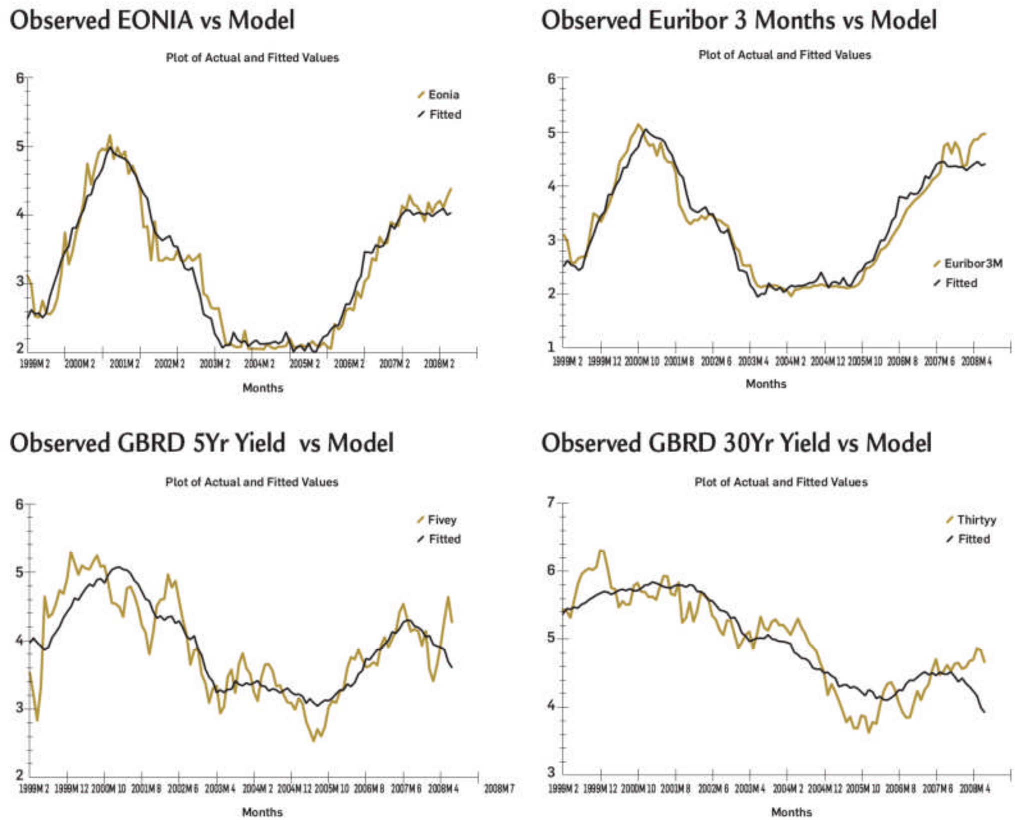
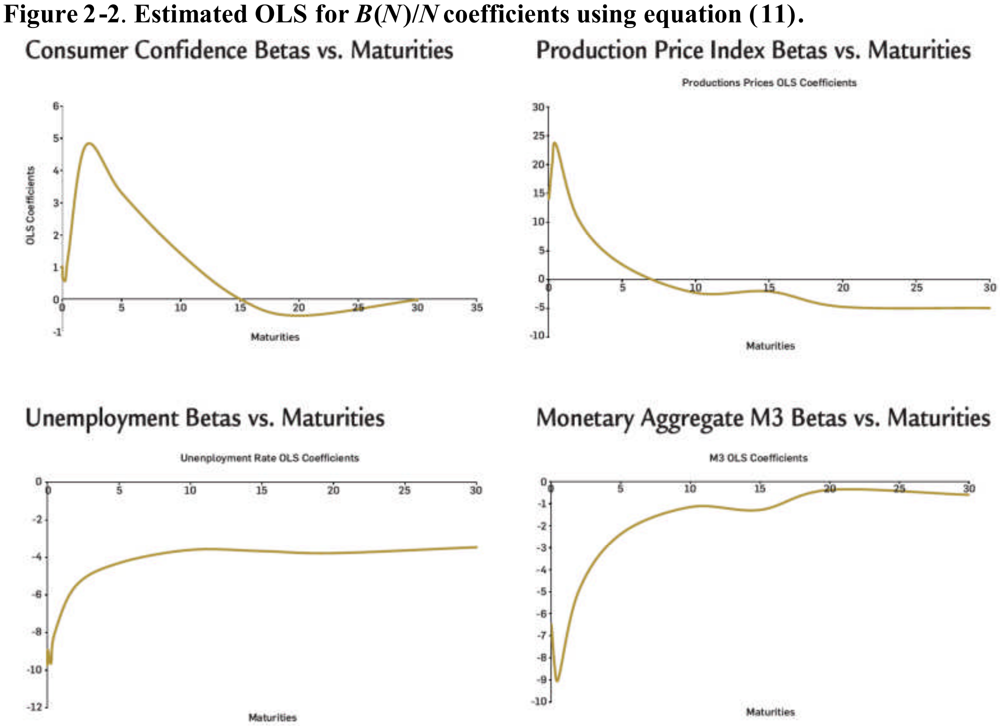


Figure 2-2. **Estimated OLS for** ***B*(*N*)/*N*** **coefficients using equation (11).**



Figures 2-3 and 2-4 we analyse the impulse response function after running a VAR using 2 lags and 25 steps (each step represents an interval of 1 month) on the EONIA and the 5 Yr German Government bond yields. For example, it can we seen that an increase in the orthogonalised shock to the exogenous variable production prices (i.e. the proxy for the price level shown in the graph under the variable name D.pptxemu_ix) causes a short series of increases in the EONIA (shown in the graph under the variable name D.eonia_ix) that dies out in less than 5 months. Not surprisingly, the same increase in the orthogonalised shock to the price level –instead- generates a short series of decreases in the 5year Government bond yields that, similarly, dies out in less than 5 months. It can also be seen that ECB money supply M3 (D.Lecmsm3_ix) and EU unemployment have greater effect on the 5year German Government bond yields than on the EONIA rate. This is very important because it means that now we know that for a given macroeconomic environment, the yield curve is either expected to flatten, steepen or to twist which hence, has implications from a public debt policy as well as from a trading strategy point of view. As this can help reduce governments' debt roll-over risk, as well as shape sovereign debt portfolio trading or investment strategies.

Figure 2-3 Impulse response analysis on the EONIA rate

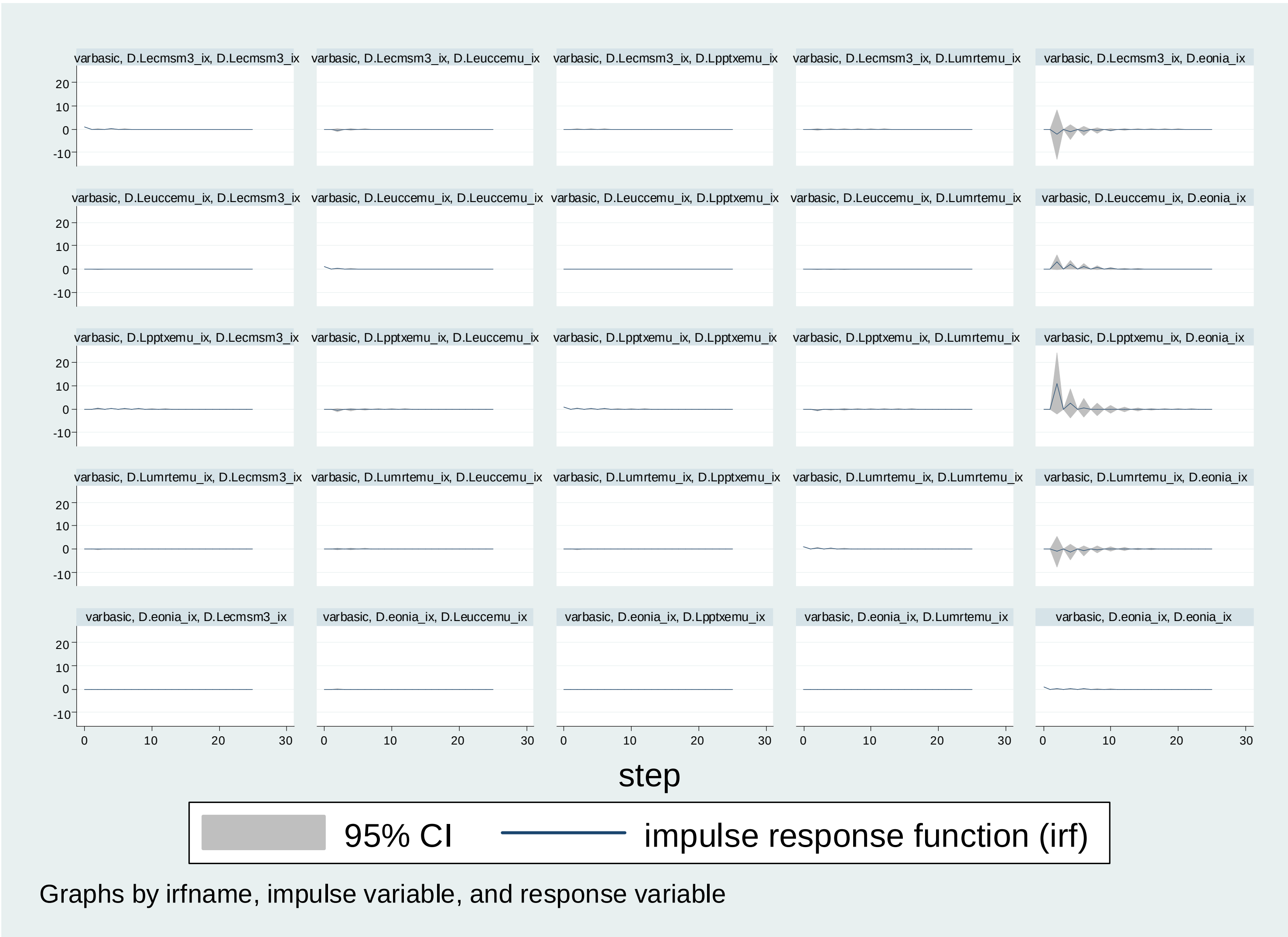
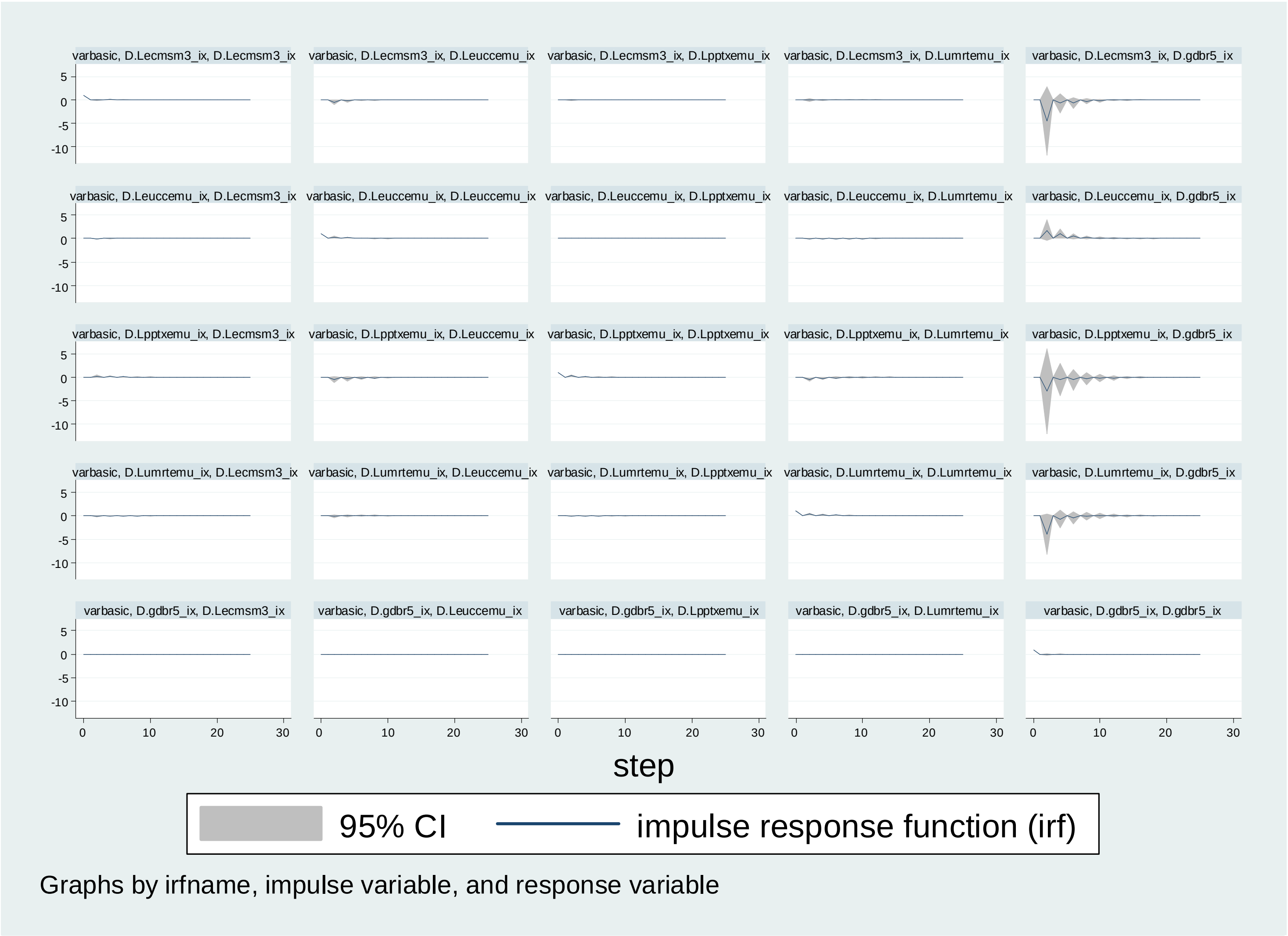


Figure 2-4. Impulse response analysis on the 5 year German government bond



So far most of the empirical literature has been focused on testing Taylor-Rules for the US Federal Reserve, as for example in Hamalainen (2004). See also Battini and Haldane (1999) and Clarida *et al* (1999, 2002).

Figures 2-5 and 2-6 repeats the regressions with robust standard errors and we update our data until September 2012, hence increasing our sample to 154 months. We also test these macro-factors with other European government bond yields. All countries, with the exception of Greece exhibit upwards sloping yields curves. In all cases yield volatility falls as maturities increase. Greece and Portugal exhibit the largest sensitivities to these macro-factors. For the cases of Spain and Italy, the variance is mostly explained by natural logs of PPI and M3, whereas Germany instead is more sensitive to unemployment. Spanish and Italian government bond yields are positively correlated to unemployment for the long maturities only, indicating that these are risky assets, as they perform better when unemployment is low. Greek and Portuguese government bond yields are all risky, as all these yields exhibit a positive correlation to unemployment rate. Coefficients on PPI are positive for all bond yields, with the exception of the medium and long term German yields. R^2 shows that macro-factors for GIIPSs countries have less predictive ability compared to German and money market yields.

Table 2-2. OLS model: Yields versus Macro-Factors for Germany, Spain, Greece, Italy and Portugal.

Germany	2 Y	5 Y	10 Y	15 Y	20 Y	30 Y
LnU_t	-8.2575 (.3662)	-6.3314 (.3792)	-4.2146 (.3244)	-4.0705 (.3213)	-3.7366 (.3134)	-3.3468 (.3285)
$LnPPI_t$	4.517 (1.354)	-2.8580 (1.447)	-4.2146 (1.2668)	-4.6346 (1.3310)	-5.9063 (1.2731)	-5.9705 (1.3056)
$LnM3_t$	-3.5281 (.5592)	-.9371 (.5821)	-.20975 (.5022)	-.3995 (.5388)	.1215 (.5128)	.1572 (.5259)
$LnCC_t$	3.4493 (.4797)	2.7454 (.4978)	1.6208 (.4365)	0.1131 (.4553)	-.09345 (.4375)	0.3061 (.4484)
Intercept	15.804 (2.658)	26.4275 (2.8097)	31.5144 (2.5159)	37.5344 (2.6241)	39.1804 (2.5907)	39.3726 (2.5481)
R^2	0.9427	0.9125	.9451	.8408	.8292	.8465
$\hat{\mu}$	2.6872	3.2254	3.8008	4.0759	4.2675	4.3236
$\hat{\sigma}$	1.3998	1.1973	0.9721	0.8916	0.8510	0.9029

Spain	2 Y	5 Y	10 Y	15 Y	20 Y	30 Y
LnU_t	-2.375 (.4454)	0.3867 (.4300)	-2.3333 (.3488)	3.1515 (.3455)	n.a.	3.3694 (.3106)
$LnPPI_t$	24.656 (1.196)	22.137 (1.8489)	18.9491 (1.2668)	19.1706 (1.4037)	n.a.	16.735 (1.4037)
$LnM3_t$	-9.5350 (.7769)	-8.6828 (.7264)	-7.6973 (.4919)	-7.6908 (.5623)	n.a.	-6.9484 (.5013)
$LnCC_t$	3.0376 (.5756)	1.678 (.6276)	.2907 (.5342)	0.055 (.5417)	n.a.	-0.0332 (.4475)
Intercept	-34.662 (3.947)	-29.8792 (4.1756)	-21.3006 (3.6651)	-22.3675 (3.7099)	n.a.	-18.007 (3.1046)
R^2	0.7635	0.5976	0.4963	.4836	n.a.	.4926
$\hat{\mu}$	3.255	3.9208	4.5482	4.8146	n.a.	5.0596
$\hat{\sigma}$	1.0045	.8392	.7699	0.8258	n.a.	0.7652
Greece	**2 Y**	**5 Y**	**10 Y**	**15 Y**	**20 Y**	**30 Y**
LnU_t	149.63 (35.06)	52.68 (7.95)	34.46 (2.9212)	29.71 (2.4887)	25.2688 (2.531)	23.1213 (2.0416)
$LnPPI_t$	829.58 (189.5)	253.92 (42.32)	147.264 (14.35)	126.7891 (11.60)	109.234(1 0.283)	98.2778 (8.7094)
$LnM3_t$	-299.21 (71.06)	-87.85 (15.96)	-49.71 (5.5155)	-43.108 (4.4520)	-37.73 (3.9321)	-34.45 (3.3385)
$LnCC_t$	-92.746 (34.24)	-17.561 (8.2475)	-6.7507 (3.579)	-7.1850 (3.0282)	-7.066 (2.6487)	-7.0496 (2.3771)
Intercept	-1085.5 (247.8)	-425.21 (52.875)	-278.81 (19.906)	-230.42 (18.0696)	-189.12 (16.114)	-161.57 (15.2501)
R^2	0.5367	0.6749	0.7956	0.79583	0.7817	0.7815
$\hat{\mu}$	11.569	7.209	7.4403	7.2641	7.105	6.9819
$\hat{\sigma}$	32.394	9.5734	6.5938	6.6595	4.8946	4.3406
Italy	**2 Y**	**5 Y**	**10 Y**	**15 Y**	**20 Y**	**30 Y**
LnU_t	-3.5628 (.5012)	-.6426 (.4467)	1.0464 (.3287)	1.7208 (.3242)	1.8713 (.3360)	1.8974 (.2885)
$LnPPI_t$	26.663 (2.469)	24.06 (2.184)	18.084 (1.621)	17.2779 (1.6415)	16.8346 (1.7081)	14.5199 (1.4644)
$LnM3_t$	-10.69 (.9519)	-9.8192 (.8587)	-7.6051 (.6619)	-6.9058 (.6461)	-7.0617 (.6764)	-6.3079 (.5859)
$LnCC_t$	1.0679 (.7132)	-0.3194 (.6959)	-1.1171 (.5377)	-1.0678 (.5504)	-1.7275 (.5659)	-1.5602 (.4856)
Intercept	-22.26 (4.591)	-17.4717 (4.394)	-8.9168 (3.436)	-8.1509 (3.468)	-6.5630 (3.5236)	-3.2141 (3.184)
R^2	0.7153	0.5515	0.4434	0.4008	0.3857	0.3823
$\hat{\mu}$	3.2746	3.9702	4.635	4.9386	5.1059	5.2279
$\hat{\sigma}$	1.0723	.8756	0.6979	0.6934	0.7239	0.6559
Portugal	**2 Y**	**5 Y**	**10 Y**	**15 Y**	**20 Y**	**30 Y**
LnU_t	9.5601 (1.965)	12.889 (1.6710)	11.209 (1.0046)	11.3535 (1.0494)	n.a.	7.9871 (0.6038)
$LnPPI_t$	84.443 (10.47)	81.004 (8.778)	58.49 (5.0182)	57.47 (5.3640)	n.a.	39.91 (2.771)
$LnM3_t$	-29.166 (.6301)	-27.97 (3.386)	-20.47 (1.9418)	-19.93 (1.9418)	n.a.	-14.06 (1.0667)
$LnCC_t$	.6301 (2.682)	-.6086 (2.3436)	-.9122 (1.3952)	-.7776 (1.4371)	n.a.	-.001 (0.8017)
Intercept	-151.62 (17.36)	-147.41 (14.02)	-104.29 (8.4566)	-105.16 (8.5927)	n.a.	-72.151 (5.322)
R^2	0.5991	0.7004	0.7498	0.7378	n.a.	0.7388
$\hat{\mu}$	4.401	5.055	5.4058	5.5496	n.a.	5.3793
$\hat{\sigma}$	3.2836	3.174	2.3348	2.3620	n.a.	1.6126

Table 2-3. OLS, Robust for Money Market Yields vs. Macroeconomic Data (Dec 1999 to Sep 2012)

Money Market	Eonia	1 Week	2 Week	Eur3M	Eur6M
LnU_t	-10.003 (.1642)	-8.6905 (.1957)	-8.6454 (.1872)	-10.7142 (.2805)	-10.0962 (.3048)
$LnPPI_t$	14.418 (.6887)	13.991 (.9589)	13.909 (.9435)	16.941 (.9933)	16.964 (1.1167)
$LnM3_t$	-7.5101 (.2913)	-6.7280 (.4129)	-6.6755 (.4071)	-7.6538 (.4242)	-7.3139 (.47845)
$LnCC_t$	-.1352 (.2660)	-.2691 (.3424)	-.0534 (.3274)	-.5882 (.4454)	.04398 (.4955)
Intercept	24.881 (1.874)	17.741 (2.3910)	16.5903 (2.3351)	18.3189 (2.5269)	11.1226 (2.7609)
R^2	0.9756	0.9427	.9451	.9529	.9388
$\hat{\mu}$	2.4389	2.5534	2.5515	2.7140	2.8233
$\hat{\sigma}$	1.434	1.2155	1.2176	1.3983	1.3330

Section VII. Related literature

So far most of the empirical literature has been focused on testing Taylor-Rules for the US Federal Reserve, as for example in Hamalainen (2004). See also Battini and Haldane (1999) and Clarida *et al* (1999, 2002).

Bernanke (2002) analyses the use broader set of information instead of the usual Taylor-Rules framework for analysing monetary policy, as most of the empirical literature has been mainly confined to a limited amount of information, merely output and the price level.

Studies on empirical reaction functions of the ECB can be seen in Hayo and Hofmann (2003), Gerdesmeier and Roffia (2003), Gerlach-Kirsten (2003) and Gerlach and Schnabel (1999). Most of these works lack of investigation of *real* ECB policy reaction function. Fendel (2007) studies the ECB reaction function using monthly data based merely on a Taylor-Rule framework. In his work, the ECB reaction function is confined to the European Overnight Index Average (EONIA) representing the overnight policy rate, the Harmonised Index for Consumer Prices (HICP) as the inflation for Euro-Zone and output proxy taken from the Industrial Production Index for the Euro Area. The EONIA is not the Policy Rate (PR) however it would be expected to fluctuate around the

PR. Furthermore, it is reasonable to believe that it would not be optimum otherwise and hence, the ECB will have enough incentives to ensure that the EONIA fluctuates around its PR and ensure effectiveness of its monetary policy.

Hence, resent studies have been focused in analysing monetary policy by including the central bank reaction function in a small empirical macro model of inflation and output, most of it US data (see, for example also, Rudebusch, 2000, Rudebusch and Svensson, 1999 as well as Mc Callum, 19949). Ang and Piazzesi (2003) show how macro-variables add to the understanding of yield curve movements. In fact, from a European perspective, the ECB (as well as the majority of central banks) is likely to analyse a significant number of time series not only confined to the above mentioned ones. This essay analyses a variety of macroeconomic data releases for the Euro-Zone and examines their effect on the EONIA rate via a reaction function for the ECB. The macroeconomic data analysed here is not confined only to the inflation and output. We also include additional variables such as unemployment as well as consumer confidence indices, the price level and central bank monetary aggregates as suggested by Christiano and Rostagno (2001), King (2002) and Nelson (2003)

Section VIII. Conclusions and Final Remarks

This paper presents slight adjustments to the classical consumption based asset pricing models by introducing unemployment data and survey data such as consumers' confidence index. The incorporation of a monetary aggregate has also been extensively discussed in this paper and its inclusion in the model appears to us to be robust enough. This paper has also tested empirically an affine term structure model which has been adapted to the above mentioned data. We have shown that using current theoretical developments and a few state space variables such as European unemployment data, the European Consumers' Confidence Index, European Production Price Index (PPI) and a monetary aggregate such as ECB M3 for Europe, it is possible to explain yield curve movements with strikingly very good results, particularly for the German bond yields.

With respect to the German bond yields, unemployment and consumer confidence index have exhibited a shift and a slope effect on the yield curve, for front-end yields moving faster than in the long end. Production price index has a twist effect on the yield curve (flattening or steepening of the curve) which results in lower-end yields shifting in opposite directions to the long end. This empirical work shows that yields are negatively correlated to money supply, as expected in classical IS-LM models. And that money supply exhibits a slope effect, with the lower end of the curve shifting faster than the longer end. In the light of these results, we suggest that further analysis is needed to understand what other variables or mechanisms are governing the longer end of the curve, as for the lower end our results show that the variables used have very high predictive ability already. In addition, we also suggest further research on cross-sectional data across non-EU countries. In this essay we have used macroeconomic data to explain yield curve movements and we show that GIIPS yields are more sensitive to macroeconomic shocks compared to the German benchmark. In fact, we see that most of the GIIPS bond yields –in contrast to the German benchmark- exhibit positive coefficients for the unemployment rate. Indicating that governments' cost of financing will be penalised when unemployment increases. This means that these governments cannot undertake counter-cyclical fiscal stimulus by issuing new debt in times of low consumption growth and high unemployment.

3. Introductory Notes on Affine Term Structure Models: Continuous Time Approach

Section I. Introduction and Motivation

The intention of this essay is mainly to detail some theoretical aspects as well as some of the algebra discussed in affine term structure models which are usually missing in most of the literature. This chapter can be of particular interest for those requiring an introductory session on affine term structure models or as lecture notes for an introductory course in continuous time affine term structure literature. Thus authors take for granted that the reader is already familiar with the notation and most of the literature gives the impression that it has been written only for a particular audience. Here, the basic asset pricing equation is discussed, as well as the marginal rate of substitution, the notation of the constant relative risk aversion, the holding period returns and the relationship between consumption asset pricing models and affine term structure models are all discussed and hence, this essay links these with detail algebra for the analytical solution under Vasicek (1976) and Cox, Ingersoll and Ross (1985). Also results are compared to those seen in Cochrane (2005, pp. 368-379), Duffie and Kan (1996) Singleton (2006, pp. 311-334) and Piazzesi (2010, pp.691-758), as these are examples of authoritative literature in this topic. This essay comprises the foundation work from the paper published in Jakas and Jakas (2013) which will presented in chapter 6.

This paper is organised as follows: section II discusses the basic asset pricing equation, in section III it is shown how the pricing equation for returns is obtained from the stochastic discount factor, section IV links discussions from sections II and III to the holding period return. Section V we show the solution under the Vasicek (1977) process and section VI links asset returns to a state space system with an observation equation. In a similar fashion section VII develops the pricing equations under the Cox *et al* (1985) process. Section VIII a more generalised version under a multifactor setup is specified which accounts for a *k*-vector of state variables and accounts for the possibility of combining the two processes explained in sections V and VII.

Section II. The Basic Asset Pricing Equation

The aim of this section is to discuss the relationship between the basic asset pricing equation, the marginal rate of substitution, the stochastic discount factor and yields.

It would be useful to start by recalling the basic asset pricing equation as, e.g. in Cochrane (2005, page 4), which shows that asset prices are a function of expected future aggregate marginal utility growth, which boils down to:

$$P_t = E_t\left[\left(\beta \frac{u_c(c_{t+1})}{u_c(c_t)}\right)R_{t+1}\right] \qquad (1)$$

For P_t being the present value at time t, of an asset paying-off R_{t+1}, E_t is the expectations operator. B the subjective discount factor and $u_c(c_{t+1})/u_c(c_t)$ being the aggregate marginal utility growth, defined as the quotient of the first derivatives of the utility function $u(c_t)$ with respect to c_{t+1} and c_t, which in (1) the $u_c(c_{t+1})$ is used in order to notate for $\partial u(c_{t+1})/\partial c_{t+1}$.

Assuming a utility function with constant relative risk aversion (CRRA)

$$u(c_t) = c_t^{(1-\gamma)} /(1-\gamma)$$

it would result in

$$u_c(c_t) = c_t^{-\gamma} \text{ so that } u_c(c_{t+1})/u_c(c_t) = (c_{t+1} / c_t)^{-\gamma}.$$

The literature often identifies from (1) two main components, the stochastic pay-offs R_{t+1} and the so-called stochastic discount factor m_{t+1} which is usually presented as follows

$$m_{t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} \tag{2}$$

Notice that (2) could be rewritten under expectations and define the discount factor as

$$E[m_{t+1}] = E_t\left[\beta \frac{u_c(c_{t+1})}{u_c(c_t)} \right] = e^{-y_t^{(N)} N} \tag{3}$$

For $y_t^{(N)}$ the yield for a zero coupon bond with maturity N in time t. For the time being and without loss of generality it will be assumed that N = 1 and hence, for convenience (3) will be transformed to:

$$y_t^{(N)} = -\log E[m_{t+1}] \tag{4}$$

Now following Rubinstein (1976) and Lucas (1978) (4) could be rewritten as

$$y_t^{(N)} = -\log E[m_{t+1} | I_t] \tag{4.a}$$

Where I_t denotes the representative agent's information set on time t upon which expectations are conditioned in determining the yields. In other words, (4) can be interpreted as a set of yields that are contingent on the realizations of the variables in the information set I_t observed at time t. Now all is needed is just to specify the functional form for (4).

A note on continuous time perspective of the stochastic discount factor:

The stochastic discount factor from a continuous time perspective will be tracked from a *level* perspective instead of a *growth* perspective. This is only because $u_c(c_{t+1})/u_c(c_t)$ does not behave well when dt is small.

So tracking the level instead of the growth of the aggregate marginal utility would mean that (2) will rather be like in (2.a) below

$$\Lambda_t = \beta u_c(c_t) \qquad (2.a)$$

This is simply because it is easier to integrate

$$P_t \Lambda_t = \int_{s=0}^{\infty} \beta u_c(c_{t+s}) ds$$

Than to integrate

$$P_t = \int_{s=0}^{\infty} \frac{u_c(c_{t+s})}{u_c(c_t)} ds$$

Equation (2.a) will be revisited later, as it will be needed for the analytical solution on the Vasicek affine term structure model.

Section III. The Basic Pricing Equation for Asset Returns

In this section it is shown how the basic pricing equation for asset returns is obtained, for more details the reader can refer either to Cochrane's Asset Pricing (2005), pages 25 or Singleton's Empirical Dynamic Asset Pricing (2006), page 8.
Equation (4) basically shows that the log-stochastic discount factor for a bond paying off 1 unit at maturity is the inverse of the observable yields.

The stochastic discount factor is not observable, however the yields are. So if the above holds true, the stochastic discount factor can be estimated from observable yields.

Now summarising by substituting (2) in (1) yields

$$P_t = E_t[m_{t+1} R_{t+1}] \qquad (5)$$

Using the definition of covariance[2] on (5) and assuming that the asset pays-off 1 monetary unit at maturity would imply that

$$1 = \text{cov}_t\left(m_{t+1}R_{t+1}\right) + E_t(m_{t+1})E_t(R_{t+1}) \tag{6}$$

Knowing that $R^f_{t+1} = 1 / E_t\,[m_{t+1}]$,[3] replacing on (6) and rearranging yields the expected return of an asset which is the risk-free rate plus a risk adjustment.

$$E_t(R_{t+1}) = R^f - R^f\,\text{cov}_t\left(m_{t+1}R_{t+1}\right) \tag{7}$$

Rearranging (7) into continuous time perspective can be presented something like that[4]:

$$E_t\left(\frac{dP_t}{P_t}\right) = r_t^f dt - E_t\left(\frac{dP_t}{P_t}\frac{d\Lambda_t}{\Lambda_t}\right) \tag{8}$$

For r_t^f being $\log(R^f_{t+1})$. Notice that here is noted $d\Lambda_t$ instead but simply because of the comments made on the note for equation (2.a). Equation (8) will be revisited later but before it is still necessary to understand the relationship between (8) and the holding period return. This is because (8) neither says anything about the data generating process or how returns behave as a stochastic process and it neither explains how returns change as a consequence of changes from one maturity to the next.

Section IV. The Holding Period Return (hpr)

In this section main concern focuses on the relationship between the basic equation for asset returns and the so-called holding period return. The aim is to introduce a stochastic process and apply Ito's Lemma to obtain the fundamental differential equation for bonds.

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[2] $\text{cov}(xy) = E(xy) - E(x)E(y)$: $\text{cov}\left(m_{t+1}R_{t+1}\right) = E\left(m_{t+1}R_{t+1}\right) - E\left(m_{t+1}\right)E\left(R_{t+1}\right)$

[3] This can be easily seen in (3) by taking R_{t+1} out of the expectations operator and replacing it for R^f_{t+1} and by assuming that $P_t = 1$.

[4] See J. Cochrane, Asset Pricing, Revised Edition, Chapter I, pages 25-29 for more details.

First the holding period return is defined. Secondly, the *hpr* is combined with the basic equation for asset returns already discussed in section III. Finally, the fundamental differential equation for bonds is derived by introducing Ito's Lemma. The fundamental differential equation for bonds is needed in order to solve via ordinary differential equation the yields from (4.a) for different maturities. This will be presented in section V where this is applied to Vasicek's stochastic process and in section VII for the Cox-Ingersoll-Ross (1985a,b) case.

The concept of the holding period return *hpr* can be defined as the change in the price of a bond being bought at $t = 0$ with maturity, let's say $N = 1$, and held to $t = 1$ which is when $N = 0$. So that *hpr* can be specified as

$$hpr = \frac{P(N - \Delta N, t + \Delta t) - P(N, t)}{P(N, t)} \tag{9}$$

This is solved via a series for a function of two variables so that

$$P(N - \Delta N, t + \Delta t) = P(N, t) - \frac{\partial P(N,t)}{\partial N}\Delta N + \frac{\partial P(N,t)}{\partial t}\Delta t \tag{10}$$

Substituting (10) into (9) results

$$hpr = -\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N}\Delta N + \frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial t}\Delta t \tag{11}$$

Now because in the limit the assumption is that Δt, ∂t and $\Delta N \rightarrow dt$, and, if *hpr* is represented as a function of changes in t, so that substituting accordingly, hence ΔN, Δt and ∂t by dt gives the equation for the holding period return as follows:

$$hpr = -\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N}dt + \frac{\partial P(N,t)}{P(N,t)} \tag{12}$$

So now that the holding period return specified in (12) as well as the basic pricing equation (8) have been obtained all is need is to put them together which results in the fundamental pricing equation applied to fixed income securities, which satisfies

$$E_t\left(\frac{dP(N,t)}{P(N,t)}\right)-\left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N}+r_t^{\,f}\right)dt=-E_t\left(\frac{dP(N,t)}{P(N,t)}\frac{d\Lambda_t}{\Lambda_t}\right) \tag{13}$$

From what has been learned from (12), and adapted to (8) the reader should notice –for notation's sake– that P_t is now noted as $P(N,t)$ from here onwards.

Introducing a stochastic process and Ito's Lemma to the fundamental pricing equation for fixed income securities

Now (13) will be rearranged by making an assumption for a stochastic process for $dP(N,t)$ and, Itos' Lemma.

It will be assumed that all time dependence stems for state variable x and assume that x behaves according to the following stochastic process, this is the univariate
case under Vasicek thus,

$$dx=\mu_x dt+\sigma_x dz \tag{14}$$

And for μ_x being a drift of a mean, σ_x is the standard deviation and dz a Brownian motion.

The intention is to specify a stochastic process for $dP(N,t)$ so that revisiting Ito's Lemma, and for the reader not familiar with the detail, it has been summarised as:

$$dP=\frac{\partial P}{\partial x}dx+\frac{1}{2}\frac{\partial^2 P}{\partial x^2}(dx)^2 \tag{15}$$

And replacing dx for a Vasicek process discussed in (14)

$$dP=\frac{\partial P}{\partial x}(\mu_x dt+\sigma_x dz)+\frac{1}{2}\frac{\partial^2 P}{\partial x^2}(\mu_x dt+\sigma_x dz)^2 \tag{16}$$

And applying Ito to (16), hence dt, dz and $dz^2 = dt$ and that $dt \cdot dz$ and $dt^2 = 0$, so that rearranging and resolving yields

$$dP = \left(\frac{\partial P}{\partial x} \mu_x + \frac{1}{2} \frac{\partial^2 P}{\partial x^2} \sigma_x^2 \right) dt + \frac{\partial P}{\partial x} \sigma_x dz \tag{17}$$

Substituting (17) into the fundamental pricing equation applied to fixed income securities as in (13), but before doing that it is convenient to rearrange (13) by collecting the term on the right with the first term on the left hand side of the equation, so that:

$$E_t \left(\frac{dP(N,t)}{P(N,t)} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP(N,t)}{P(N,t)} \right) - \left(\frac{1}{P(N,t)} \frac{\partial P(N,t)}{\partial N} + r_t^f \right) dt = 0 \tag{18}$$

And now here comes again another assumption, because it is not known how the stochastic discount factor behaves, thus it will be assumed in a similar fashion to Duffie and Kan (1996), that:

$$\frac{d\Lambda}{\Lambda} = -xdt - \sigma_\Lambda dz \tag{19}$$

Rearranging (18) a bit and substituting as in (19) boils down to:

$$E_t \left(\frac{dP(N,t)}{P(N,t)} (1 - xdt - \sigma_\Lambda dz) \right) - \left(\frac{1}{P(N,t)} \frac{\partial P(N,t)}{\partial N} + r_t^f \right) dt = 0 \tag{20}$$

Now substituting as per Ito see from (17) into (20) results in

$$E_t\left[\frac{1}{P(N,t)}\begin{pmatrix}\left(\frac{\partial P(N,t)}{\partial x}\mu_x+\frac{1}{2}\frac{\partial^2 P(N,t)}{\partial x^2}\sigma_x^2\right)dt\\ +\frac{\partial P(N,t)}{\partial x}\sigma_x dz\end{pmatrix}(1-xdt-\sigma_\Lambda dz)\right] \\ -\left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N}+r_t^f\right)dt=0 \tag{21}$$

Once more if the reader remembers that dt, dz and $dz^2 = dt$ and that $E[dz^2]$, $dt \cdot dz$ and dt^2 = 0, so that resolving yields

$$E_t\left[\frac{1}{P(N,t)}\left(\left(\frac{\partial P(N,t)}{\partial x}\mu_x+\frac{1}{2}\frac{\partial^2 P(N,t)}{\partial x^2}\sigma_x^2\right)dt+\frac{\partial P(N,t)}{\partial x}\sigma_x\sigma_\Lambda\right)\right]dt \\ -\left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N}+r_t^f\right)dt=0 \tag{22}$$

Notice that now the term $\partial P(N,t)/\partial x * \sigma_x dz$ seen in (21) disappears in (22) because $E_t[dz^2]$ = 0.

Multiplying (22) by $P(N,t)$ and eliminating dt the fundamental differential equation for bonds is obtained:

$$\frac{\partial P(N,t)}{\partial x}\mu_x+\frac{1}{2}\frac{\partial^2 P(N,t)}{\partial x^2}\sigma_x^2-\frac{\partial P(N,t)}{\partial N}-r_t^f P(N,t)=\frac{\partial P(N,t)}{\partial x}\sigma_x\sigma_\Lambda \tag{23}$$

Now all is needed is to guess a functional form for $P(N, t)$ and solve the above partial differential equation.

Section V. Univariate Case Under Vasicek

Similar to Piazzesi (2010), Cochrane (2001) and Singleton (2006) it is assumed

$$P(N, x_t) = e^{A(N)+B(N)x_t} \qquad (24)$$

Which can also be specified as:

$$\ln[P(N, x_t)] = A(N) + B(N)x_t \qquad (24.a)$$

Given the guess (24), the derivatives that appear in (23) are as shown below, notice that this essay follows a similar notation to that seen in Cochrane (2005);

$$\frac{1}{P(N,x_t)}\frac{\partial P(N,x_t)}{\partial x_t} = B(N)$$

$$\frac{1}{P(N,x_t)}\frac{\partial^2 P(N,x_t)}{\partial x_t^2} = B(N)^2$$

$$\frac{1}{P(N,x_t)}\frac{\partial P(N,x_t)}{\partial N} = \frac{\partial A(N)}{\partial N} + \frac{\partial B(N)}{\partial N}x_t$$

Substituting the above derivatives into (24) yields:

$$B(N)\mu_x + \frac{1}{2}B(N)^2\sigma_x^2 - \frac{\partial A(N)}{\partial N} - \frac{\partial B(N)}{\partial N}x_t - r_t^f P(N,t) = B(N)\sigma_x\sigma_\Lambda \qquad (25)$$

Algebraically, is not nice to have r_t^f and x_t at the same time. However, by leveraging from Taylor (1993), hence that due to either market or policy responses r_t^f is also a function of macroeconomic variables it will be assumed, again as in Piazzesi (2010), Cochrane (2005) or Singleton (2006) that the risk-free rate of return is a function of macroeconomic variables, thus;

$$r_t^f = \gamma_0 + \gamma_1 x_t \qquad (26)$$

Notice that so far it has been assumed a one factor model here, so in this case there is 1 factor that encapsulates all macroeconomic behaviour influencing the yield curve. A multi factor will be analysed later in section VIII.

So now that is known (26) this can be plugged into (25) so that

$$\begin{aligned} & B(N)\mu_x + \frac{1}{2}B(N)^2\sigma_x^2 - \frac{\partial A(N)}{\partial N} - \frac{\partial B(N)}{\partial N}x_t \\ & -(\gamma_0 + \gamma_1 x_t)P(N,t) = B(N)\sigma_x\sigma_\Lambda \end{aligned} \tag{27}$$

Setting the boundary condition $P(0, x) = 1$ which is when $A(0) = 0$ and $B(0) = 0$, so that $e^{A(0)+B(0)x} = 1$ thus,

$$B(N)\mu_x + \frac{1}{2}B(N)^2\sigma_x^2 - \frac{\partial A(N)}{\partial N} - \frac{\partial B(N)}{\partial N}x_t - \gamma_0 - \gamma_1 x_t = B(N)\sigma_x\sigma_\Lambda \tag{28}$$

Now, in (14) it has been shown how dx is expected to behave stochastically, however it has not been specified further on how μ_x is represented following Vasicek, which the process would rather look like

$$dx = \phi(\bar{x} - x_t)dt + \sigma_x dz \tag{29}$$

Notice that: $\mu_x = E[dx]$. [5]

In which case, substituting (29) into (28) and rearranging boils down to

$$\begin{aligned} & B(N)\phi(\bar{x} - x_t) + \frac{1}{2}B(N)^2\sigma_x^2 - \frac{\partial A(N)}{\partial N} - \\ & \frac{\partial B(N)}{\partial N}x_t - \gamma_0 - \gamma_1 x_t = B(N)\sigma_x\sigma_\Lambda \end{aligned} \tag{30}$$

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[5] $E[dx] = E[\phi(\bar{x} - x_t)dt] + E[\sigma_x dz] \rightarrow E[dx] = \phi(\bar{x} - x_t)$

This has to hold for all x_t so that the terms multiplying x_t and the constant terms are zero.

$$\frac{\partial A(N)}{\partial N} = \frac{1}{2} B(N)^2 \sigma_x^2 + B(N)(\phi\bar{x} - \sigma_x \sigma_\Lambda) - \gamma_0 \tag{31}$$

$$\frac{\partial B(N)}{\partial N} = -B(N)\phi - \gamma_1 \tag{32}$$

Equations (31) and (32) can be solved by simple integration. The reader should notice that a change of variable is required for $v = -B(N)\phi - \gamma_1$. Starting with (32) and then substituting when solving (31).

So for simplicity's sake it will be started with (32),

$$\int \frac{1}{B(N)\phi + \gamma_1} \partial B(N) = -N \tag{33}$$

Which solving the integral yields

$$\ln\left(\frac{B(N)\phi + \gamma_1}{B(0)\phi + \gamma_1}\right) = -\phi N \tag{34}$$

Hence,

$$B(N) = \frac{[B(0)\phi + \gamma_1]e^{-\phi N} - \gamma_1}{\phi} \tag{35}$$

Since $B(0) = 0$, therefore, the solution is

$$B(N) = \frac{\gamma_1}{\phi}\left(e^{-\phi N} - 1\right) \tag{36}$$

Assuming the reader wishes to obtain Cochrane's (2001) page 370 solution, all is needed is to multiply by (-1), as in his model the term multiplying $B(N)$ is negative, you also need to remember that $\gamma_1 = 1$ and $-\gamma_0 = 0$, as for Cochrane's solution, $r^f = x_t$, so that (36) would be;

$$B(N) = \frac{1}{\phi}\left(1 - e^{-\phi N}\right).$$

Which is exactly Cochrane's solution.

Now integrating equation (31), but first substituting from (36) accordingly, hence:

$$\frac{\partial A(N)}{\partial N} = \frac{1}{2}\left[\frac{\gamma_1}{\phi}\left(e^{-\phi N} - 1\right)\right]^2 \sigma_x^2 + \left[\frac{\gamma_1}{\phi}\left(e^{-\phi N} - 1\right)\right]\left(\phi\bar{x} - \sigma_x \sigma_\Lambda\right) - \gamma_0 \tag{37}$$

And rearranging:

$$\frac{\partial A(N)}{\partial N} = \frac{\sigma_x^2 \gamma_1^2}{2\phi^2}\left(e^{-2\phi N} + 1 - 2e^{-\phi N}\right) + \frac{\gamma_1\left(\phi\bar{x} - \sigma_x \sigma_\Lambda\right)}{\phi}\left(e^{-\phi N} - 1\right) - \gamma_0 \tag{38}$$

Which after grouping terms of equal power it can be written as

$$\begin{aligned} \frac{\partial A(N)}{\partial N} &= \frac{\sigma_x^2 \gamma_1^2}{2\phi^2} - \frac{\gamma_1\left(\phi\bar{x} - \sigma_x \sigma_\Lambda\right)}{\phi} - \gamma_0 \\ &+ \left(\frac{\gamma_1\left(\phi\bar{x} - \sigma_x \sigma_\Lambda\right)}{\phi} - \frac{\sigma_x^2 \gamma_1^2}{\phi^2}\right)e^{-\phi N} + \frac{\sigma_x^2 \gamma_1^2}{2\phi^2} e^{-2\phi N} \end{aligned} \tag{39}$$

Since $A(0) = 0$, this equation can be easily integrated thus the following has been obtained;

$$A(N)=\left(\frac{\sigma_x^2\gamma_1^2}{2\phi^2}-\frac{\gamma_1(\phi\bar{x}-\sigma_x\sigma_\Lambda)}{\phi}-\gamma_0\right)N-\left(\frac{\gamma_1(\phi\bar{x}-\sigma_x\sigma_\Lambda)}{\phi^2}-\frac{\sigma_x^2\gamma_1^2}{\phi^3}\right)\left(e^{-\phi N}-1\right)$$
$$-\frac{\sigma_x^2\gamma_1^2}{4\phi^3}\left(e^{-2\phi N}-1\right) \tag{40}$$

I will rearrange the last term on (40) and rework first as follows:

$$[B(N)]^2=\left[\frac{\gamma_1}{\phi}\left(e^{-\phi N}-1\right)\right]^2$$

Which rearranging and collecting terms, reads

$$[B(N)]^2+2\frac{\gamma_1}{\phi}B(N)=\frac{\gamma_1^2}{\phi^2}\left(e^{-2\phi N}-1\right)$$

And substituting in (40) yields

$$A(N)=\left(\frac{\sigma_x^2\gamma_1^2}{2\phi^2}-\frac{\gamma_1(\phi\bar{x}-\sigma_x\sigma_\Lambda)}{\phi}-\gamma_0\right)N$$
$$-\left(\frac{(\phi\bar{x}-\sigma_x\sigma_\Lambda)}{\phi}-\frac{\sigma_x^2\gamma_1}{\phi^2}\right)B(N)-\frac{\sigma_x^2}{4\phi}\left[B(N)^2+2\frac{\gamma_1}{\phi}B(N)\right] \tag{41}$$

Now if the reader is aiming to Cochrane's (2001) solution, this can be obtained by substituting $\gamma_1 = 1$ and $-\gamma_0 = 0$, as for Cochrane, $r^f = x_t$ and re-arranging and collecting terms, results in:

$$A(N)=\left(\frac{\sigma_x^2}{2\phi^2}-\frac{(\phi\bar{x}-\sigma_x\sigma_\Lambda)}{\phi}\right)N-\left(\frac{(\phi\bar{x}-\sigma_x\sigma_\Lambda)}{\phi}-\frac{\sigma_x^2}{\phi^2}\right)B(N)$$
$$-\frac{\sigma_x^2}{4\phi}\left[B(N)^2+2\frac{\gamma_1}{\phi}B(N)\right] \tag{42}$$

$$A(N) = \frac{\sigma_x^2}{2\phi^2} N - \frac{(\phi\bar{x} - \sigma_x \sigma_\Lambda)}{\phi}(N + B(N)) + \frac{\sigma_x^2}{\phi^2} B(N) - \frac{\sigma_x^2}{4\phi}\left[B(N)^2 + 2\frac{1}{\phi} B(N)\right] \quad (43)$$

$$A(N) = \frac{\sigma_x^2}{2\phi^2} N - \frac{(\phi\bar{x} - \sigma_x \sigma_\Lambda)}{\phi}(N + B(N)) + \frac{\sigma_x^2}{\phi^2} B(N) - \frac{\sigma_x^2}{4\phi} B(N)^2 - \frac{\sigma_x^2}{2\phi^2} B(N) \quad (44)$$

And step by step:

$$A(N) = \frac{\sigma_x^2}{2\phi^2} N - \frac{(\phi\bar{x} - \sigma_x \sigma_\Lambda)}{\phi}(N + B(N)) + \frac{\sigma_x^2}{\phi^2} B(N) - \frac{\sigma_x^2}{4\phi} B(N)^2 - \frac{\sigma_x^2}{2\phi^2} B(N)$$

$$A(N) = \frac{\sigma_x^2}{2\phi^2} N - \frac{(\phi\bar{x} - \sigma_x \sigma_\Lambda)}{\phi}(N + B(N)) - \frac{\sigma_x^2}{4\phi} B(N)^2 + \frac{\sigma_x^2}{2\phi^2} B(N)$$

$$A(N) = \frac{\sigma_x^2}{2\phi^2}(N + B(N)) - \frac{(\phi\bar{x} - \sigma_x \sigma_\Lambda)}{\phi}(N + B(N)) - \frac{\sigma_x^2}{4\phi} B(N)^2$$

results in

$$A(N) = \left[\frac{\sigma_x^2}{2\phi^2} - \frac{(\phi\bar{x} - \sigma_x \sigma_\Lambda)}{\phi}\right](N + B(N)) - \frac{\sigma_x^2}{4\phi} B(N)^2 \quad (45)$$

Now the reader must remember again that in Cochrane's presentation the second term shown in (24) is instead negative, hence *B*(*N*) has a "-", so taking this into account and rearranging slightly yields:

$$A(N) = -\frac{\sigma_x^2}{4\phi} B(N)^2 - \left[\bar{x} - \frac{\sigma_x \sigma_\Lambda}{\phi} - \frac{\sigma_x^2}{2\phi^2}\right](N - B(N)) \quad (46)$$

Which is exactly as in Cochrane, page 371.

Section VI. Linking Now to a State Space System with an Observation Equation

Now recalling (3) and applying logarithms to obtain the yields,

$$E_t\left[m_{t+1}\right] = e^{-Ny_t^{(N)}} \tag{47}$$

Which also means that the present value of a bond is

$$P(N,t) = e^{-Ny_t^{(N)}} \tag{48}$$

Now recalling the guess in (24) and substituting in (48)

$$P(N,x) = e^{A(N)+B(N)x_t} \tag{49}$$

Applying logarithms to (48) and (49) and equating both terms and rearranging boils down to

$$y_t^{(N)} = -\frac{A(N)}{N} - \frac{B(N)}{N}x_t \tag{50}$$

For

$$a(N) = -\frac{A(N)}{N} \tag{51}$$

$$b(N) = -\frac{B(N)}{N} \tag{52}$$

Recalling our assumption in (29) means that yields follow an affine model which is a state space system with an observation equation linking observable yields to the state vector and a state equation describing the dynamics of the observation and the state:

$$y_t^{(N)} = a(N) + b(N)x_t \tag{53}$$

$$x_t = x_{t-1} + \phi\left(\bar{x} - x_{t-1}\right) + \varepsilon_t \tag{54}$$

As discussed, equation (54) is simply the discrete version of (29), e.g. rearranged is a AR(1) would yield,

$$\Delta x_t = \phi(\bar{x} - x_{t-1}) + \varepsilon_t$$

Section VII. Univariate Case under Cox-Ingersoll-Ross (1985) model

Same as with Vasicek, it will start with the fundamental pricing equation for bonds (13) but instead of plugging process (29) which is the Vasicek process, the CIR process will be plugged, which are defined as

$$dx = \mu_x dt + \sigma_x \sqrt{x_t}\, dz \tag{55}$$

$$\frac{d\Lambda}{\Lambda} = -x dt - \sigma_\Lambda \sqrt{x_t}\, dz \tag{56}$$

But first as what it has been learned from Ito,

$$dP = \frac{\partial P}{\partial x}\left(\mu_x dt + \sigma_x \sqrt{x_t}\, dz\right) + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\left(\mu_x dt + \sigma_x \sqrt{x_t}\, dz\right)^2 \tag{57}$$

And because of what is known from Ito, hence that dt, dz and $dz^2 = dt$ and that that $E[dz^2]$, $dt \cdot dz$ and $dt^2 = 0$, rearranging (57) and resolving yields

$$dP = \left(\frac{\partial P}{\partial x}\mu_x + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\sigma_x^2 x_t\right)dt + \frac{\partial P}{\partial x}\sigma_x \sqrt{x_t}\, dz \tag{58}$$

Substituting (56) and (58) into (13) and rearranging yields;

$$E_t\left[\frac{1}{P(N,t)}\left(\left(\frac{\partial P(N,t)}{\partial x}\mu_x + \frac{1}{2}\frac{\partial^2 P(N,t)}{\partial x^2}\sigma_x^2 x_t\right)dt + \frac{\partial P(N,t)}{\partial x}\sigma_x \sqrt{x}_t dz\right)\left(1 - x_t dt - \sigma_\Lambda \sqrt{x}_t dz\right)\right]$$
$$-\left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N} + r_t^f\right)dt = 0$$

(59)

Once more if the reader remember that dt, dz and $dz^2 = dt$ and that $E[dz^2]$, $dt \cdot dz$ and $dt^2 =$ 0, resolving yields the fundamental equation for bonds under CIR model, and this is how is done

$$E_t\left[\frac{1}{P(N,t)}\left(\left(\frac{\partial P(N,t)}{\partial x}\mu_x + \frac{1}{2}\frac{\partial^2 P(N,t)}{\partial x^2}\sigma_x^2\right)dt + \frac{\partial P(N,t)}{\partial x}\sigma_x\sigma_\Lambda\right)\right]dt$$
$$-\left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N} + r_t^f\right)dt = 0 \qquad (60)$$

Notice that now the term $\partial P/\partial x * \sigma_x \sqrt{x_t} dz$ disappears because $E_t\,[dz^2] = 0$.
Multiplying (60) by $P(N,t)$ and eliminating dt it is possible to obtain the fundamental differential equation for bonds under CIR

$$\frac{\partial P}{\partial x}\mu_x + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\sigma_x^2 x_t - \frac{\partial P(N,t)}{\partial N} - r_t^f P(N,t) = \frac{\partial P}{\partial x}\sigma_x\sigma_\Lambda x_t \qquad (61)$$

Equation (61) differs from (23) on that two terms are now multiplied by x_t.
Now all is needed to do here is guess a functional form for $P(N, t)$ and solve the above partial differential equation.

Similar to what we did in Vasicek model we do the following guess:

$$P(N, x_t) = e^{A(N) + B(N)x_t} \qquad (62)$$

Which can also be as

$$\ln[P(N, x_t)] = A(N) + B(N)x_t \tag{62.a}$$

Given the guess (62), the derivatives that appear in (61) are as follows:

$$\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial x} = B(N)$$

$$\frac{1}{P(N,t)}\frac{\partial^2 P(N,t)}{\partial x^2} = B(N)^2$$

$$\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N} = \frac{\partial A(N)}{\partial N} + \frac{\partial B(N)}{\partial N}x_t$$

Substituting the above derivatives into (61) yields:

$$B(N)\mu_x + \frac{1}{2}B(N)^2\sigma_x^2 x_t - B(N)\sigma_x\sigma_\Lambda x_t - \frac{\partial A(N)}{\partial N} - \frac{\partial B(N)}{\partial N}x_t - r_t^f P(N,t) = 0 \tag{63}$$

And for the risk free rate being:

$$r_t^f = \gamma_0 + \gamma_1 x_t \tag{64}$$

Same as in previous section, the reader should remember that so far it is assumed a one factor model here, so in this case there is 1 factor that encapsulates all macroeconomic behaviour influencing the yield curve. A multi factor will be analysed in the next section.

So now that specification (64) is known, it can be plugged into (63) so that

$$\begin{aligned} &B(N)\mu_x + \frac{1}{2}B(N)^2\sigma_x^2 x_t - B(N)\sigma_x\sigma_\Lambda x_t \\ &- \frac{\partial A(N)}{\partial N} - \frac{\partial B(N)}{\partial N}x_t - (\gamma_0 + \gamma_1 x_t)P(N,t) = 0 \end{aligned} \tag{65}$$

Setting the boundary condition $P(0, x) = 1$ which is when

$A(0) = 0$ and $B(0) = 0$, so that $e^{A(0)+B(0)x} = 1$

Now, in (55) it has been specified how dx is expected to behave stochastically, however it has not been specified further on how the mean μ_x is supposed to be represented following CIR, thus the process would rather look like

$$dx = \phi(\bar{x} - x)dt + \sigma_x \sqrt{x_t}\, dz \tag{66}$$

In which case, substituting (66) into (65) and rearranging boils down to

$$\begin{aligned} &B(N)\phi(\bar{x} - x_t) + \frac{1}{2}B(N)^2 \sigma_x^2 x_t \\ &- B(N)\sigma_x \sigma_\Lambda x_t - \frac{\partial A(N)}{\partial N} - \frac{\partial B(N)}{\partial N} x_t - \gamma_0 - \gamma_1 x_t = 0 \end{aligned} \tag{67}$$

This has to hold for all x so that the terms multiplying x and the constant terms are zero.

$$\frac{\partial B(N)}{\partial N} = -\gamma_1 - (\phi + \sigma_x \sigma_\Lambda)B(N) + \frac{1}{2}\sigma_x^2 B(N)^2 \tag{68}$$

$$\frac{\partial A(N)}{\partial N} = \phi \bar{x} B(N) - \gamma_0 \tag{69}$$

Equations (68) and (69) will be solved by simple integration. A hint for the reader: (68) is quite bombastic to solve. So it will start with (68) and then substitute to solve (69).

So here it goes first with equation (68) which rearranging and integrating yields;

$$\int \frac{1}{-\gamma_1 - (\phi + \sigma_x \sigma_\Lambda)B(N) + \frac{1}{2}\sigma_x^2 B(N)^2} \partial B(N) = N \tag{70}$$

Which could be resolve as follows

$$\int \frac{1}{-a - bB(N) + cB(N)^2} \partial B(N) = N \tag{71}$$

For a = 1; $b = (\varnothing + \sigma_x \, \sigma_\Lambda)$ and $c = 1/2 \; \sigma_x^2$, which crystallises to:

$$-\frac{2}{\sqrt{-4ac - b^2}} \tan^{-1}\left(\frac{2cB(N) + b}{\sqrt{-4ac - b^2}}\right) = N \tag{72}$$

(72) is not helpful because $\sqrt{-4ac - b^2}$ is a complex number, and can only be resolve by understanding that the square root is imaginary so that

$$\sqrt{-4ac - b^2} = i\sqrt{4ac + b^2} = i\zeta \tag{73}$$

And this makes that if

$$\tan^{-1}(ix) = \frac{1}{2} i \ln\left(\frac{1 - ix}{1 + ix}\right) \tag{74}$$

Hence,

$$\tan^{-1}(ix) = \frac{1}{2} i \ln\left(\frac{1 - i\,\dfrac{2cB(N) + b}{i\zeta}}{1 + i\,\dfrac{2cB(N) + b}{i\zeta}}\right) \tag{75}$$

So now substituting (75) and (73) in (72), eliminating i and substituting back $b = (\varnothing + \sigma_x \, \sigma_\Lambda)$ and $c = 1/2 \; \sigma_x^2$, yields

$$N = -\frac{1}{\varsigma}\ln\left(\frac{1-\dfrac{\sigma_x^2 B(N)+(\phi+\sigma_x\sigma_\Lambda)}{\zeta}}{1+\dfrac{\sigma_x^2 B(N)+(\phi+\sigma_x\sigma_\Lambda)}{\zeta}}\right) = -\frac{1}{\varsigma}\ln\left(\frac{\zeta-\sigma_x^2 B(N)+(\phi+\sigma_x\sigma_\Lambda)}{\zeta+\sigma_x^2 B(N)+(\phi+\sigma_x\sigma_\Lambda)}\right) \tag{76}$$

for

$$\varsigma = \sqrt{2\sigma_x^2 + (\phi+\sigma_x\sigma_\Lambda)^2}$$

Taking logarithms and collecting terms for B(N), and going step by step boils down to

$$e^{-\varsigma N}\left[\zeta+\sigma_x^2 B(N)+(\phi+\sigma_x\sigma_\Lambda)\right] = \zeta-\sigma_x^2 B(N)+(\phi+\sigma_x\sigma_\Lambda) \tag{77}$$

$$e^{-\varsigma N}\zeta+e^{-\varsigma N}\sigma_x^2 B(N)+e^{-\varsigma N}(\phi+\sigma_x\sigma_\Lambda)-\zeta+\sigma_x^2 B(N)-(\phi+\sigma_x\sigma_\Lambda)=0 \tag{78}$$

$$e^{-\varsigma N}\zeta-\zeta+e^{-\varsigma N}\sigma_x^2 B(N)+\sigma_x^2 B(N)+e^{-\varsigma N}(\phi+\sigma_x\sigma_\Lambda)-(\phi+\sigma_x\sigma_\Lambda)=0 \tag{79}$$

$$\left(e^{-\varsigma N}-1\right)\zeta+\left(e^{-\varsigma N}+1\right)\sigma_x^2 B(N)+\left(e^{-\varsigma N}-1\right)(\phi+\sigma_x\sigma_\Lambda)=0 \tag{80}$$

$$\left(e^{-\varsigma N}-1\right)\zeta+\left(e^{-\varsigma N}-1\right)(\phi+\sigma_x\sigma_\Lambda)=-\left(e^{-\varsigma N}+1\right)\sigma_x^2 B(N) \tag{81}$$

$$\frac{\left(e^{-\varsigma N}-1\right)\zeta+\left(e^{-\varsigma N}-1\right)(\phi+\sigma_x\sigma_\Lambda)}{-\left(e^{-\varsigma N}+1\right)\sigma_x^2}=B(N) \tag{82}$$

$$-\frac{\left(e^{-\varsigma N}-1\right)\zeta+\left(e^{-\varsigma N}-1\right)(\phi+\sigma_x\sigma_\Lambda)}{\left(e^{-\varsigma N}+1\right)\sigma_x^2}=B(N) \tag{83}$$

$$\frac{\left(1-e^{-\varsigma N}\right)\zeta+\left(1-e^{-\varsigma N}\right)(\phi+\sigma_x\sigma_\Lambda)}{\left(e^{-\varsigma N}+1\right)\sigma_x^2}=B(N) \tag{84}$$

$$\frac{\left(1-e^{-\varsigma N}\right)\left[\zeta+(\phi+\sigma_x\sigma_\Lambda)\right]}{\left(e^{-\varsigma N}+1\right)\sigma_x^2}=B(N) \tag{85}$$

Remember that the solution will have to be for $B(N=0) = 0$, so we do not need a constant because at N=0; the term 1 - $e^{-\zeta N}$ equals zero. Now it will integrate (69), by first substituting results from (85) and rearranging, thus

$$A(N) = \frac{\phi\bar{x}\left[\zeta + (\phi + \sigma_x \sigma_\Lambda)\right]}{\sigma_x^2} \int \left[\frac{(1 - e^{-\varsigma N})}{(e^{-\varsigma N} + 1)} - \gamma_0 \right] \partial N \tag{85}$$

To make it easier to operate it will denoted

$$a = \frac{\phi\bar{x}\left[\zeta + (\phi + \sigma_x \sigma_\Lambda)\right]}{\sigma_x^2} \tag{86}$$

$$A(N) = a \int \left[\frac{(1 - e^{-\varsigma N})}{(e^{-\varsigma N} + 1)} - \gamma_0 \right] \partial N \tag{87}$$

It will resolve by parts, hence[6]

$$A(N) = a \left\{ \frac{2\log\left[-2(e^{\varsigma N} + 1) - \varsigma N\right]}{\varsigma} - \gamma_0 N \right\} \tag{88}$$

Remember that the solution will have to be for $A(N=0) = 0$, so this can be achieved by subtracting 1 from the term $e^{-\varsigma N}$ +1, in addition, as I did with Vasicek if the one factor is the risk-free rate of return r^f than γ_0 and γ_1 are 0 and 1 respectively.

$$A(N) = a\, \frac{2\log\left[-2(e^{\varsigma N}) - \varsigma N\right]}{\varsigma}. \tag{89}$$

[6] I resolved this one by entering (1-exp(-bx))*(exp(-bx)+1)^-1 in the http://www.freemathhelp.com integral calculator.

Section VIII. Multifactor Affine Models

A more generalised setting of affine class of term structure models would be to allow for multiple factors and to account for the Vasicek as well as for the CIR stochastic processes explained in the previous chapters. Here, it will follow similar notation to Cochrane (2005) which was initially developed by Duffie and Kan (1996) and Dai and Singleton (2000). It will also compare their results with those of Piazzesi (2010). Which they differ in 1 thing: Piazzesi does not account for the discount factor.

Due to the fact that it deals with multiple factors setup, the notation falls under the usual matrix algebra. It recalls then equations (19), (26) and (29) and transforms them into matrices in order to account for more than one variable as follows:

$$dx = \phi(\bar{x} - x_t)dt + \sigma_x dz \tag{90}$$

$$r_t^f = \gamma_0 + \gamma' x_t \tag{91}$$

$$\frac{d\Lambda}{\Lambda} = -r^f dt - \sigma_\Lambda' dz \tag{92}$$

$$\sigma_x = \Sigma s(x_t) \tag{93}$$

$$s_i(x_t) = \sqrt{s_{0i} + s_{1i}' x_t} \tag{94}$$

Equation (90) describes the stochastic process of the state variables. This is the usual mean reversing process whereby dx is likely to be negative if x_t is above its mean and, is likely to be positive if x is below its mean. x_t and its mean are both k-dimensional vectors. ϕ is a $k \times k$ matrix of diagonal elements ϕ_i which represent the speed of adjustment at which each of x_i elements reverse to their means. σ_x is a diagonal $k \times k$ matrix comprising the volatility of the state variables. dz is a k-vector of shocks moving x_t away from its mean and with dz_i elements being normally distributed with mean zero and variance dt.

Equation [91] describes the short rate as a function of the state variables. Notice that it has been denoted by prime " " for the transpose of the k-dimensional vector γ. The sign of each of the γ_i elements should depend on the theory and should be estimated empirically. The same accounts for the γ_0 constant term. Piazzesi (2010) sets these to 0 and 1 for γ_0 and γ_i respectively.

Equation (92) is the discount factor as a function of the short rate and hence as a function of the state variables and a random error term $\sigma_\Lambda dz$. Notice that the sign here are negative but this is because a similar notation as in Cochrane (2005) has been used, because at the end the sign will depend on the empirical result from (91), where σ_Λ is a k-dimensional vector describing the sensitivities of the discount factor responding to k-shocks. Same as in (90) dz is a k-vector of shocks with dz_i elements being normally distributed with mean zero and variance dt.

Equation (93) and (94) describe the volatilities of the state variables. $s(x)$ is a diagonal $k \times k$ matrix with elements $s_i(x)$. Notice that by doing so we can generalise for both Vasicek and the CIR cases. Because the Vasicek is a *Gaussian* process and CIR is a *square root* process. With (94) it can accounted for both cases, thus $s_{1i} = 0$ and $s_{0i} = 1$ for the Vasicek case, whereby the variance parameters in Σ are free. Alternatively, if the intention is to account for the CIR case, then it is possible to set $s_{1i} = 1$ and $s_{0i} = 0$. Piazzesi (2010) and the celebrated paper from Duffie and Kan (1996) as well as Dai and Singleton (2000) remember us of the conditions required to obtain a unique solution to the stochastic differential equations, and these comprise the *Feller* and the *Lipschitz* conditions, for which the author suggests the reader to refer to Piazzesi (2010) page 706 for some examples on how this works.

Procedure will be exactly as for the univariate Vasicek and CIR. First it will take the basic asset pricing equation from (13) and rewrite it in (95). Then Ito's lemma to our guess in (24) will be applied, which for convenience is now (96), thus that prices are linear functions of the state variables

$$E_t\left(\frac{dP(N,t)}{P(N,t)}\right) = \left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N} + r_t^f\right)dt - E_t\left(\frac{dP(N,t)}{P(N,t)}\frac{d\Lambda_t}{\Lambda_t}\right) \tag{95}$$

$$P(N,x) = e^{A(N)+B(N)'x_t} \tag{96}$$

Applying Ito's lemma to obtain $dP(N,t)/\,P(N,t)$.

$$\frac{dP(N,t)}{P(N,t)} = \frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial x}dx + \frac{1}{P(N,t)}\frac{1}{2}dx'\frac{\partial^2 P(N,t)}{\partial x^2}dx \tag{97}$$

Given the guess (96), the derivatives that appear in (97) are as follows:

$$\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial x} = B(N)$$

$$\frac{1}{P(N,t)}\frac{\partial^2 P}{\partial x^2} = B(N)B(N)'$$

$$\frac{1}{P(N,t)}\frac{\partial P}{\partial N} = \frac{\partial A(N)}{\partial N} + \frac{\partial B(N)'}{\partial N}x_t$$

Breaking (95) into three parts when substituting for the basic pricing equation for bonds, so starting with left term on (95) so that

$$E_t\left(\frac{dP(N,t)}{P(N,t)}\right) = E_t\left(B(N)'dx + \frac{1}{2}dx'B(N)B(N)'dx\right) \tag{98}$$

Substituting dx for (90)

$$E_t\left(\frac{dP(N,t)}{P(N,t)}\right) = E_t\begin{pmatrix} B(N)'(\phi(\bar{x}-x)dt + \sigma_x dz) \\ +\frac{1}{2}(\phi(\bar{x}-x)dt + \sigma_x dz)'B(N)B(N)'(\phi(\bar{x}-x)dt + \sigma_x dz) \end{pmatrix} \tag{99}$$

Once more if we remember that dt, dz and $dz^2 = dt$ and that $E[dz^2]$, $dt \cdot dz$ and $dt^2 = 0$, resolving yields

$$E_t\left(\frac{dP(N,t)}{P(N,t)}\right) = B(N)'\phi(\bar{x}-x)dt + \frac{1}{2}E_t[dz'\sigma_x{}'B(N)B(N)'\sigma_x dz] \tag{100}$$

Now to organise the last term on (100) a bit nicer, it will be used the fact that $Tr[AB]$ = $Tr[BA]$ for square matrices and because the last term is a scalar, thus

$$E_t\left[(\sigma_x dz)' B(N)B(N)' (\sigma_x dz)\right] = Tr\left[E_t\left(dz'\sigma_x' B(N)B(N)'\sigma_x dz\right)\right]$$

$$= Tr\left[E_t\left(B(N)'\sigma_x dzdz'\sigma_x' B(N)\right)\right]$$

$$= Tr\left[B(N)'\sigma_x E_t\left[dzdz'\right]\sigma_x' B(N)\right]$$

$$= \sum_i^N \left[\sigma_x' B(N)\right]_i^2 E\left(dz_i^2\right) \tag{101}$$

A not so elegant way of resolving this one would be if instead: $E_t(dz_i dz_j) = 0$ and $E_t(\sigma_i\sigma_j) = 0$ implying,

$$\left(dz'\sigma_x' B(N)B(N)'\sigma_x dz\right) =$$

$$\begin{pmatrix} dz_{i=1} & \dots & dz_{i=N} \end{pmatrix} \times \left[\begin{pmatrix} \sigma_{x:1} & \dots & \sigma_{x:N} \end{pmatrix} \times \begin{pmatrix} B(N)_{11} B(N)_{11} & \dots & B(N)_{11} B(N)_{1N} \\ \dots & \dots & \dots \\ B(N)_{N1} B(N)_{11} & \dots & B(N)_{N1} B(N)_{1N} \end{pmatrix} \times \begin{pmatrix} \sigma_{x:1} \\ \dots \\ \sigma_{x:N} \end{pmatrix}\right] \times \begin{pmatrix} dz_{i=1} \\ \dots \\ \sigma_{i=N} \end{pmatrix}$$

Which would give us the same as (101). By plugging (93) and (94) into (101) results in equation (102) below. Notice that the intention is to isolate Σ from $s(x_t)$ in order to obtain a specification which is closer to that of Piazzesi (2010) rather than to that of Cochrane's (2001), which boils down to:

$$E_t\left[(\sigma_x dz)' B(N)B(N)' (\sigma_x dz)\right] = \sum_i^N \left[\Sigma' B(N)\right]_i^2 \left(s_{0i} + s'_{1i}\, x\right) dt \tag{102}$$

Plugging it back (102) into (100) yields

$$E_t\left(\frac{dP_t}{P_t}\right)=B(N)'\phi(\bar{x}-x)dt+\frac{1}{2}\sum_{i}^{N}\left[\Sigma' B(N)\right]_i^2\left(s_{0i}+s'_{1i}\,x\right)dt \qquad (103)$$

Solving now for the first term on the right of equation (95) which it has been rewritten below for convenience,

$$\left(\frac{1}{P(N,t)}\frac{\partial P(N,t)}{\partial N}+r_t^f\right)dt$$

and by plugging the corresponding derivatives and replace for the risk free rate of return as follows;

The derivatives:

$$\frac{\partial A(N)}{\partial N}+\frac{\partial B(N)'}{\partial N}x_t$$

And the equation describing the risk free rate

$$r_t^f=\gamma_0+\gamma' x_t$$

So that (95) now will look something like:

$$\left(\frac{\partial A(N)}{\partial N}+\frac{\partial B(N)'}{\partial N}x_t+\gamma_0+\gamma' x_t\right)dt \qquad (104)$$

So now going for the last term on equation (95), for which in order not to re-invent the wheel it will be combined with results from (99) with (91) and (92) yields

$$- E_t\left(\frac{dP_t}{P_t}\frac{d\Lambda_t}{\Lambda_t}\right) =$$

$$E_t\left(\begin{aligned}&\left[B(N)'(\phi(\bar{x}-x)dt+\sigma_x dz)+\frac{1}{2}(\phi(\bar{x}-x)dt+\sigma_x dz)' B(N)B(N)'(\phi(\bar{x}-x)dt+\sigma_x dz)\right]\\&\times\left[-(\gamma_0+\gamma' x_t)dt-\sigma_\Lambda' dz\right]\end{aligned}\right)$$

Above almost every term gets eliminated as, once more, if we remember that dt, dz and $dz^2 = dt$ and that $E[dz^2]$, $dt \cdot dz$ and $dt^2 = 0$, resolving yields

$$- E_t\left(\frac{dP_t}{P_t}\frac{d\Lambda_t}{\Lambda_t}\right) = B(N)'\sigma_x dzdz'\sigma_\Lambda \tag{105}$$

Now organising (105) knowing that $dzdz' \rightarrow dt$ and σ_x is a diagonal $k \times k$ matrix comprising the volatility of the state variables with elements $\Sigma_i(s_{0i}+s_{1i}'x_t)$. So that by isolating $(s_{0i}+s_{1i}'x_t)$ from Σ_i and applying the fact that $E_t(\sigma_i\sigma_j) = 0$ (105) boils down to

$$- E_t\left(\frac{dP_t}{P_t}\frac{d\Lambda_t}{\Lambda_t}\right) = \sum_i^N [B(N)'\Sigma]_i \sigma_\Lambda (s_{0i} + s'_{1i} x_t)dt \tag{106}$$

By gathering all the parts together, thus equation (103), (104) and (106), and eliminating dt term yields

$$\begin{aligned}&B(N)'\phi(\bar{x}-x_t)+\frac{1}{2}\sum_i^N [\Sigma' B(N)]_i^2 (s_{0i}+s'_{1i} x_t)\\&=\left(\frac{\partial A(N)}{\partial N}+\frac{\partial B(N)'}{\partial N}x_t+\gamma_0+\gamma' x_t\right)+\sum_i^N [B(N)'\Sigma]_i \sigma_\Lambda (s_{0i}+s'_{1i} x_t)\end{aligned}$$

And again as performed for the univariate Vasicek and CIR cases, the terms on the constant and each x_i must separately be zero. Here it will be aiming first for the $\partial A(N)/\partial N$ which results in

$$B(N)'\phi\bar{x} + \frac{1}{2}\sum_{i}^{N}\left[\Sigma' B(N)\right]_i^2 s_{0i} = \frac{\partial A(N)}{\partial N} + \gamma_0 + \sum_{i}^{N}\left[B(N)'\Sigma\right]_i \sigma_\Lambda s_{0i}$$

Now rearranging for $\partial A(N)/\partial N$ term results in,

$$\frac{\partial A(N)}{\partial N} = -\gamma_0 + B(N)'\phi\bar{x} - \sum_{i}^{N}\left[B(N)'\Sigma\right]_i \sigma_\Lambda s_{0i} + \frac{1}{2}\sum_{i}^{N}\left[\Sigma' B(N)\right]_i^2 s_{0i} \qquad (107)$$

And now going for $\partial B(N)/\partial N$, simply taking the terms which are multiplying x_t and the transpose, then solving:

$$B(N)'\phi x_t + \frac{1}{2}\sum_{i}^{N}\left[\Sigma' B(N)\right]_i^2 s'_{1i}\, x_t = \frac{\partial B(N)'}{\partial N}x_t + \gamma' x_t + \sum_{i}^{N}\left[B(N)'\Sigma\right]_i \sigma_\Lambda s'_{1i}\, x_t$$

Taking the transpose and dividing by x_t,

$$\phi' B(N) + \frac{1}{2}\sum_{i}^{N}\left[\Sigma' B(N)\right]_i^2 s'_{1i} = \frac{\partial B(N)'}{\partial N} + \gamma + \sum_{i}^{N}\left[B(N)\Sigma\right]_i \sigma_\Lambda s'_{1i}$$

And rearranging to obtain $\partial B(N)/\partial N$ boils down to

$$\frac{\partial B(N)}{\partial N} = -\gamma + \phi' B(N) - \sum_{i}^{N}\left[B(N)'\Sigma\right]_i \sigma_\Lambda s_{1i} + \frac{1}{2}\sum_{i}^{N}\left[\Sigma' B(N)\right]_i^2 s_{1i} \qquad (108)$$

To solve the above ODE a MATLAB "ode45" is used as in Piazzesi (2010).

Comparing (107) and (108) with Piazzesi (2010) we see that her model does not account for the terms:

$$-\sum_{i}^{N}\left[B(N)'\Sigma\right]_i \sigma_\Lambda s_{0i}$$

$$-\sum_{i}^{N}\left[B(N)'\Sigma\right]_i \sigma_\Lambda s_{1i}$$

Why? Simply because Piazzesi (2010) does not account for the discount factor $d\Lambda/\Lambda$ and the reason why this is not accounted for lies on the fact that her specification of (106) is zero, and therefore the term containing $d\Lambda/\Lambda$ is instantaneously eliminated and therefore only (103) and (104) feed into $\partial A(N)/\partial N$ and $\partial B(N)/\partial N$. Finally, the signs shown on the terms for computing $\partial A(N)/\partial N$ and $\partial B(N)/\partial N$ in Piazzesi (2010) differ to those from Cochrane (2005), but this is mainly due to the guess on (24) and (95) where in Cochrane the term $B(N)$ is negative because his guess is $P(N,x_t) = \exp[A(N) - B(N)x_t]$.

Section IX. Conclusions and final remarks

We have documented some algebra and shown step by step how we get to the affine term structure models starting with the basic pricing equation, the pricing equation for asset returns, the holding period returns, Ito's lemma and finally obtained the fundamental pricing equation for fixed income securities which is required under the affine term structure model. We have also documented some of the differences in the notation between authoritative literature such as Cochrane (2005) and Piazzesi (2010). Our findings show that Piazzesi (2010) does not account for the discount factor $d\Lambda/\Lambda$ in contrast to Cochrane (2005). In addition, Piazzesi (2010) sets γ_0 and γ_1 to 0 and 1 respectively but by doing so coefficients of state variables on the short rate are lost and this is what we are actually trying to capture. The differences between Piazzesi (2010) and Cochrane (2005) result in different ways of facing numerical solutions, which go beyond the scope of this thesis.

4. Affine Term Structure Models applied to a Macro-Finance Model

Section I. Introduction

This essay explores the use of discrete time affine term structure models applied to a macro-finance model[7] which links asset pricing theory, Taylor rules and debt dynamics in order to study the optimal term structure, and hence contribute to fiscal stabilisation policies and the optimal taxation approach. This chapter makes use of affine term structure models in a similar set up as seen in the celebrated papers from Backus, Foresi and Telmer (1998 and 1996) and Backus, Telmer and Wu (1999). The model used applies the multifactor cases under Vasicek (1977) taking into account some of the developments seen in the latest affine term structure research such as Duffie and Kan (1996), Piazzesi (2010) and Singleton (2006).

With respect to the optimal taxation approach, this paper is a contribution to some of the developments achieved in Missale (1997), Faraglia, Marcet and Scott (2008), Angeletos (2002) and Buera and Nicolini (2004). This essay comprises the foundations for the works published in Jakas (2012) and Jakas and Jakas (2013).

This paper is organised as follows: Section II, presents some introductory notes on discrete multi-factor affine terms structure models; section III introduces the main features of the present macro-finance model; section IV introduces the consumer's preferences; section V presents the central bank policy; section VI introduces the government problem and; section VII presents some final remarks linking with next chapter which calibrates the macro-finance model discussed.

[7] This section was motivated by remarks given by Prof. Dr. Kenneth Singleton, for which the author of this essay is very grateful.

Section II. Some introductory notes

This section is an introductory piece for those not familiar with discrete affine term structure models and some of the features have already been discussed in previous chapters, however for convenience to the reader we have rewritten them again below .

As seen in most recent affine term structure literature log prices can be specified as a linear function of a state vector x_{t+1} as follows:

$$-\ln\left[P_{t+1}^{(N)}\right] = A(N) + B(N)' x_{t+1} \tag{1}$$

For $A(N)$ being a scalar, $B(N)'$ a $1\times k$ vector of coefficients and x_{t+1} a $k\times 1$ vector of state variables. Note that the transpose of a vector or matrix is specified with a " ' ". Equation (1) is only a guess, as the functional form is not known. However, the literature appears to have generally accepted this as seen in Piazzesi (2010), Singleton (2006), Cochrane (2005), as well as in Backus-Foresi-Telmer (1996) and (1998) and seminal papers of Duffie and Kan (1996).

There is a mathematical link between log prices and yields. A possible specification could be

$$y_t^{(N)} = -\frac{\ln E\left[P_t^N\right]}{N} \tag{2}$$

From the guess shown in (1) it is possible to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$. These parameters are obtained by linking observable yields to an observation equation describing the behaviour of a space state vector. This can be done by combining equations (2) at t+1 with (1) which boils down to

$$y_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{B(N)'}{N} x_{t+1} \tag{3}$$

Thus the short rate could be specified as follows:

$$y_{t+1}^{(1)} = A(N=1) + B(N=1)' x_{t+1} \quad (4)$$

Empirically, equation (4) would look like

$$y_{t+1}^{(1)} = \gamma_0 + \gamma_1 ' x_{t+1} \quad (5)$$

The stochastic discount factor m_{t+1} is related to one-period bond yields (or the short rate) inversely as follows,

$$y_t^{(1)} = -\ln E[m_{t+1}] \quad (6)$$

Now it is necessary to specify the stochastic process for x_{t+1} as well as for the stochastic discount factor shown in (6). A good starting point is to use the pricing kernel à la Backus-Foresi-Telmer (1998) which here is combined with the Vasicek (1977). A possible specification would be like:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1} \quad (7)$$

$$-\ln[m_{t+1}] = \delta + y_t^{(1)} + \lambda' \varepsilon_{t+1} \quad (8)$$

Equation (7) describes the stochastic process of the independent state variables. This is the usual mean reversing process whereby Δx_{t+1} is likely to be negative if x_t is above $\bar{x}$ and, is likely to be positive if x_t is below its mean $\bar{x}$. x_t and $\bar{x}$ are both *k*-dimensional vectors. Φ is a $k \times k$ matrix of diagonal elements Φi which represent the speed of adjustment at which each of $x_{i,t}$ elements reverse to their means. σ_x is a diagonal $k \times k$ matrix comprising the volatility of the state variables. ε_{t+1} is a *k*-vector of shocks moving x_t away from $\bar{x}$ and with $\varepsilon_{i,t+1}$ elements being normally distributed with mean zero and variance 1.

Equation (8) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (8) was originally the univariate

Vasicek (1977) case. In this essay we transform this specification and adapt it for the multifactor case of a *k*-dimensional vector of state variables as in Jakas (2012). Same as in Backus-Foresi-Telmer (1998) δ is specified as follows:

$$\delta = \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 \qquad (9)$$

Clearly, specification (9) is fortuitous, the only aim is to normalise the stochastic discount factor so that it becomes the inverse of the short rate. Notice that by substituting [9] in [8] results in the following conditional means and variance:

$$E\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - y_t^{(1)} - \lambda'\varepsilon_{t+1}\right] = -\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - y_t^{(1)}$$

$$Var\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - y_t^{(1)} - \lambda'\varepsilon_{t+1}\right] = \sum_{i=1}^{k}\lambda_i^2$$

And assuming $E[\ln x] = \mu(x) + {}^1\!/_2\sigma^2(x)$, which yields

$$E[\ln m_{t+1}] = -y_t^{(1)}$$

Section III. A Macro-Finance Dynamic Model

In this section, a macro-finance model is introduced in order to link consumption based asset pricing theory as seen in the celebrated papers from Rubinstein (1976), Lucas (1978) and Breeden (1979), with inflation targeting Taylor (1993) rules partially inspired by some of the developments seen in Rudebusch and Wu (2008) and Sims (2000 and 2005). In addition, the intention is also to link with some of the recent achievements seen in the fiscal theory of debt sustainability mostly from Biacchiocchi and Missale (2005), Missale (1997) and, Missale, Giavazzi and Benigno (1997), and Bohn (1988, 1990 and 1998). In this section the starting point will be to first describe consumers' preferences; secondly, focus will be given to a model on central bank policy reaction function with inflation targeting rules however this essay will not depart from the so called "New Keynesian Phillips curve" proposed by Calvo (1983) and Roberts (1995) and assume that the central bank intends to enforce an upper bound on the price level (Sims, 2000); thirdly, the chapter will also present link to debt dynamics and fiscal sustainability foundations to yield curve movements; finally, it will be shown that these relationships can be summarised in a set of measurement equations were the different variables entering these equations are stacked in a state vector to calibrate the observed yield curves and obtain parameters $A(N)/N$ and $B(N)'/N$ as discussed in (3).

Section IV. Consumer preferences

Proposition 1 *If consumer preferences follow a log utility function with constant relative risk aversion where consumption is a function of unemployment and consumer confidence index, any increase in unemployment or decrease in consumer confidence will result in an increase in aggregate marginal utility growth with the subsequent increase in the discount factor and hence a fall in the short-rate* $y_t^{(1)}$ *as follows;*

$$y_t^{(1)} = -\gamma_0 + \begin{pmatrix} -\gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} \ln U_t \\ \ln C_t^e \end{pmatrix} \tag{10}$$

Where $\gamma_1 < 0$*, as shown in (10) and* $\gamma_2 > 0$*,* γ_2 *and* γ_1 *denote the reaction parameters of the investors on changes in the unemployment rate and consumer confidence index.*

Equation (10) is derived as follows; starting with the assumption that consumers follow a log utility function with constant relative risk aversion as in (11) and (12) below

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \tag{11}$$

$$\frac{\partial u(c_t)}{\partial c_t} = c_t^{-\gamma} \tag{12}$$

Where natural log consumption is negatively correlated to the unemployment rate and positively correlated to consumer confidence, so that

$$\ln c_t = a_0 - b_1 \ln U_t + b_2 \ln C_t^e + \sigma_c \varepsilon_t \tag{13}$$

Whereby a_0, b_1 and b_2 are parameters and $\ln U_t$ and $\ln C_t^e$ denote the natural logarithms of unemployment rate and consumers' expectations via a consumer confidence index.
It has also been assumed that investors first order condition holds so that

$$E[m_{t+1}] = E\left[\beta\left(\frac{u_c(c_{t+1})}{u_c(c_t)}\right)^{-\gamma}\right] \tag{14}$$

Recalling that the stochastic discount factor is related to the short rate and hence would imply (14) can be expressed as follows:

$$\ln m_{t+1} = \ln\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right) = -y_t^{(1)} \tag{15}$$

Operating under (15) and distributing the natural logarithms results in

$$\ln m_{t+1} = \ln \beta - \gamma \ln c_{t+1} + \gamma \ln c_t \tag{16}$$

For simplicity's sake and without loss of generality initial consumption is normalised so that $c_t = 1$ and hence $\ln c_t = 0$, as for all initial value of $c_{t=0}$ is known at time t, thus (16) would boil down to

$$\ln m_{t+1} = \ln \beta - \gamma \ln c_{t+1} \tag{17}$$

Substituting $\ln c_t$ with the guess shown in (13), results in

$$\ln m_{t+1} = \ln \beta - \gamma\left(a_0 - b_1 \ln U_t + b_2 \ln C_t^e + \sigma_c \varepsilon_t\right) \tag{18}$$

With following conditional mean and variance

$$E[\ln m_{t+1}] = \ln \beta - \gamma a_0 + \gamma b_1 \ln U_t - \gamma b_2 \ln C_t^e \tag{19}$$

$$Var[\ln m_{t+1}] = [\gamma \sigma_c]^2 \tag{20}$$

So that under log-normality

$$E[\ln m_{t+1}] = \ln \beta - \gamma a_0 + \gamma b_1 \ln U_t - \gamma b_2 \ln C_t^e + \frac{1}{2}[\gamma \sigma_c]^2 \tag{21}$$

Grouping the constant terms and the terms multiplying the variables yields

$$E[\ln m_{t+1}] = \ln \beta - \gamma a_0 + \frac{1}{2}[\gamma \sigma_c]^2 + \gamma b_1 \ln U_t - \gamma b_2 \ln C_t^e \tag{22}$$

In order to bring this to a simple regression term it could be specified as follows

$$\gamma_0 = \ln \beta - \gamma a_0 + \frac{1}{2}[\gamma \sigma_c]^2$$
$$\gamma_1 = -\gamma b_1$$
$$\gamma_2 = \gamma b_2$$

Notice that the terms $\ln \beta - \gamma a_0 + \frac{1}{2}[\gamma \sigma_c]^2$, $-\gamma b_1$ and γb_2 are assumed to be constant. So this means that combining (12) with (13) and (21) it is possible to derive the following measurement or observation equation:

$$y_t^{(1)} = -\gamma_0 - \gamma_1 \ln U_t + \gamma_2 \ln C_t^e$$

The above measurement equation suggests, for example, that an increase in unemployment is expected to result in a decrease in the short rate or risk free rate, as any increase in unemployment results in an increase in aggregate marginal utility growth with the subsequent decrease in short-term yields, as long as the underlying assets are risk free. On the contrary, increases in consumer confidence result in an increase in $y_t^{(1)}$.

Section V. Central Bank Policy

Proposition 2 *If the central bank applies a pure Taylor targeting, the policy rule can be of the form*

$$y_t^{(1)} = \alpha_0 - \alpha_1 \ln U_t + \alpha_2 \ln \pi_t \qquad (23)$$

Where $\alpha_1<0$ and $\alpha_2>0$, as described in (23) and denotes the reaction parameters of the central bank as a consequence of changes in the unemployment and the inflation rates.

In this work the author follows here some of the ideas seen in Giannoni and Woodford (2002a) combining with the works of Benhabib, Schmitt-Grohé and Uribe (2001) which is an interest rate rule model variant of one published in Sims (2000).
However, the intention is not to depart much from the so called "New Keynesian Phillips curve" proposed by Calvo (1983) and Roberts (1995), and assume that the central bank intends to enforce an upper bound on the inflation (Sims, 2000) as well as the unemployment rates, but at the same time need to ensure that the net worth of the reserves –which back the value of its money issues– do not become negative. The constraint is that if reserves net worth becomes negative the central bank will only be able to overcome this with fiscal backing, thus via distortionary taxation. Therefore, assuming that fiscal backing is not an option and that the central bank need to ensure the value of its reserves are not worth less than the money issued we retain a so-called "deflationary demonetisation". Thus, current inflation π_t will look something like the so-called "New Keynesian Phillips Curve" using unemployment rate as a measure of output gap,

$$\pi_t = \psi_0 - \psi_1 \ln U_t + \psi_2 E_t[\pi_{t+1}] + \sigma_\pi u_t \qquad (24)$$

π_t being the inflation rate in t, $\ln U_t$ and $E_t[\pi_{t+1}]$ being the natural logarithms of unemployment rate and expected inflation in time t for $t+1$. Finally, σ_π being the deviation and u_t would include the usual properties of normal behaviour mean zero and variance equal to 1.

It will be assumed that the objective of monetary policy is to minimise the expected value of a loss function of the form

$$W = E\left\{\sum_{t=0}^{n} \beta^t L_t\right\}, \tag{25}$$

Where β^t is the discount factor in period t so that the loss L for each period is

$$L_t = \phi_1\left(E[\ln U_{t+1}] - \ln U^*\right)^2 + \phi_2\left(E[\ln \pi_{t+1}] - \ln \pi^*\right)^2 \tag{26}$$

For $\varnothing_1$ and $\varnothing_2$ being the sensitivities of central bank losses due to deviations of expected unemployment and expected inflation to their targets $\ln U^*$ and $\ln\pi^*$. In addition, the model argues for a central bank which has the dual objective of unemployment and inflation targeting. The model incorporates unemployment as target rather than the usual output gap.

Any shocks governing (25) will require optimal responses for state-contingent paths depending on $E[\ln U_{t+1}]$ and $E[\ln\pi_{t+1}]$ that minimise (26) subject to the constraint described in (24). Therefore, the Lagrangian for this problem, looking forward for any date, would result in

$$l_{t=0} = E_{t=o}\sum_{t=0}^{n} \beta^{t-t_0}\left\{\begin{array}{l}\frac{1}{2}\left(\phi_1\left(E[\ln U_{t+1}] - \ln U^*\right)^2 + \phi_2\left(E[\ln \pi_{t+1}] - \ln \pi^*\right)^2\right) \\ + \lambda_t\left(\pi_t - \psi_0 + \psi_1 \ln U_t - \psi_2 E_t[\pi_{t+1}] - \sigma_\pi u_t\right)\end{array}\right\} \tag{27}$$

Differentiating (27) with respect to inflation and unemployment result in the following pairs of first order conditions:

$$\phi_1\left(E[\ln U_{t+1}] - \ln U^*\right) + \lambda_t \psi_1 = 0 \tag{28}$$

$$\phi_2\left(E[\ln \pi_{t+1}] - \ln \pi^*\right) + \lambda_t - \lambda_{t-1} = 0 \tag{29}$$

Assuming that all λ_t are constant throughout time, so that on the long run $\lambda_t \rightarrow \lambda$. The model also assumes that the central bank has been –on average– successful in targeting unemployment and inflation, implying that $\ln U^* = \mu_{\ln U}$ and $\ln \pi^* = \mu_{\ln \pi}$ so that the long run value for unemployment and inflation targets have remained constant for the period and equal their means. In addition, and for simplicity's sake and without loss of generality it could be assumed that $\lambda\psi = \varnothing_0$ which would imply the following central bank policy rule;

$$r_t^f = \phi_0 - \phi_1\left(E_t\left[\ln U_{t+1}\right] - \mu_{\ln U}\right) + \phi_2\left(E_t\left[\ln \pi_{t+1}\right] - \mu_{\ln \pi}\right) + \sigma_{r^f}\varepsilon_t \qquad (30)$$

Assuming that the above target variables follow a stochastic behaviour:

$$\ln U_{t+1} = \ln U_t + \varphi\left(\mu_{\ln U} - \ln U_t\right) + \sigma_{\ln U}\varepsilon_{t+1} \qquad (31)$$

$$\ln \pi_{t+1} = \ln \pi_t + \varphi\left(\mu_{\ln \pi} - \ln \pi_t\right) + \sigma_{\ln \pi}\varepsilon_{t+1} \qquad (32)$$

For $\sigma_{\ln U}$ and $\sigma_{\ln \pi}$ being the deviation for the natural logs of inflation and unemployment and ε_{t+1} would include the usual properties of normal behaviour mean zero and variance equal to 1.

Assuming normality (31) and (32) would look like:

$$E\left[\ln U_{t+1}\right] = \ln U_t + \varphi\left(\mu_{\ln U} - \ln U_t\right) + \frac{1}{2}\left(\varphi\mu_{\ln U} + \sigma_{\ln U}\right)^2 \qquad (33)$$

$$E\left[\ln \pi_{t+1}\right] = \ln \pi_t + \varphi\left(\mu_{\ln \pi} - \ln \pi_t\right) + \frac{1}{2}\left(\varphi\mu_{\ln \pi} + \sigma_{\ln \pi}\right)^2 \qquad (34)$$

Replacing (33) and (34) in (30)

$$\begin{aligned} r_t^f = \phi_0 - \phi_1&\left(\left(\ln U_t + \varphi_{\ln U}\left(\mu_{\ln U} - \ln U_t\right) + \frac{1}{2}\left(\varphi_{\ln U}\mu_{\ln U} + \sigma_{\ln U}\right)^2\right) - \mu_{\ln U}\right) \\ + \phi_2&\left(\left(\ln \pi_t + \varphi\left(\mu_{\ln \pi} - \ln \pi_t\right) + \frac{1}{2}\left(\varphi\mu_{\ln \pi} + \sigma_{\ln \pi}\right)^2\right) - \mu_{\ln \pi}\right) + \sigma_{r^f}\varepsilon_t \end{aligned} \qquad (35)$$

Rearranging the constant terms and the terms multiplying $\ln U_t$ and $\ln \pi_t$ yields

$$
\begin{aligned}
r_t^f &= \phi_0 + \phi_1(1-\varphi_{\ln U})\mu_{\ln U} - \phi_2(1-\varphi_{\ln \pi})\mu_{\ln \pi} \\
&+ \frac{1}{2}\left(\phi_2(\varphi\mu_{\ln \pi} + \sigma_{\ln \pi})^2 - \phi_1(\varphi_{\ln U}\mu_{\ln U} + \sigma_{\ln U})^2\right) \\
&- \phi_1(1-\varphi)\ln U_t + \phi_2(1-\varphi_{\ln \pi})\ln \pi_t + \sigma_{r^f}\varepsilon_t
\end{aligned} \tag{36}
$$

For which it will denoted as follows, by rearranging the constant terms with the terms multiplying the variables

$$
\begin{aligned}
\alpha_0 &= \phi_0 + \phi_1(1-\varphi_{\ln U})\mu_{\ln U} - \phi_2(1-\varphi_{\ln \pi})\mu_{\ln \pi} \\
&+ \frac{1}{2}\left(\phi_2(\varphi\mu_{\ln \pi} + \sigma_{\ln \pi})^2 - \phi_1(\varphi_{\ln U}\mu_{\ln U} + \sigma_{\ln U})^2\right)
\end{aligned} \tag{37.a}
$$

$$
-\alpha_1 = -\phi_1(1-\varphi) \tag{37.b}
$$

$$
\alpha_2 = \phi_2(1-\varphi_{\ln \pi}) \tag{37.c}
$$

So this means that combining (30) with (36) and according to what has been learned from (7), it is possible to reproduce the following measurement or observation equation:

$$
y_t^{(1)} = \alpha_0 - \alpha_1 \ln U_t + \alpha_2 \ln \pi_t \tag{38}
$$

Section VI. The Government Problem

In this section some of the ideas seen in Bartolini and Cottarelli (1994) and Blanchard and Weil (1992) are presented and combined some of the research seen in the celebrated papers from Bohn (1998), Giavazzi and Missale (2004), Bacchiocchi and Missale (2005) and inspired by the research from Escolano (2010).

This section will describe how government's public debt dynamics, fiscal sustainability, and cyclical adjustments of budgetary aggregates play a role in yield curve dynamics.

Proposition 3 *If the government chooses a composition of debt and budget to stabilise the debt and overall balance to GDP ratios, constraint to a chosen level of unemployment, in order to minimise a government loss function, so that any increases in the government's loss results in a deterioration of private sector's expected future consumption growth, so that the short rate would depict the following relationship*

$$y_t^{(1)} = \beta_0 + \begin{pmatrix}\beta_1 & \beta_2 & \beta_3\end{pmatrix}\begin{pmatrix}\ln d_t \\ \ln s_t \\ \ln U_t\end{pmatrix} \tag{39}$$

For $\ln d_t$ *being the government's natural log of total debt outstanding to GDP ratio,* $\ln s_t$ *being the natural log of government's overall surplus to GDP ratio and* $\ln U_t$ *being the natural logarithm of unemployment rate. Where* $\beta_1 > 0$ *,* $\beta_2 < 0$ *and* $\beta_3 > 0$ *denote the coefficients of changes in the short rate as a consequence of shocks in the debt and balance ratios to GDP, as well as shocks in the unemployment rate.*

We will start defining the debt and the overall government balance as a ratio to GDP, so that

$$\ln d_t = \ln\left(\frac{B_t}{GDP_t}\right) \tag{40}$$

$$\ln s_t = \ln\left(\frac{S_t}{GDP_t}\right) \tag{41}$$

For $\ln d_t$ and $\ln s_t$ are assumed to be governed by the following dynamics

$$\ln d_t = \left(1 + y_t^{(1)}\right)\ln d_{t-1} - \ln s_t \tag{42}$$

Notice that (42) is intuitive, hence if surplus s_t decreases and GDP remains constant, new debt is issued in order to repay debt servicing plus finance overall deficits. Notice that because it is not possible to estimate natural logarithms for negative numbers the following rule applies: the lns_t term is positive when s_t< 0 (for the case of deficits) and is negative for s_t< 0 (for the case of surplus). So that if s_t< 0 then the lns_t is positive and hence the lnd_t increases; and if s_t> 0 then the lns_t is negative and hence the lnd_t decreases.

Rearranging terms so that we can depict this as a function of the 1 period yield or short rate boils down to

$$y_t^{(1)} = -1 + \ln s_t + \frac{1}{\ln d_{t-1}} \ln d_t \quad (43)$$

As in Bacchiocchi and Missale (2005) it will assumed that the government minimises a quadratic loss function, whereby the government chooses the composition of the debt to stabilise the debt and the overall government's surplus ratios however with the purpose of reducing the financing costs described in (43). If a deterioration in surplus to GDP ratio (hence, deficit to GDP ratio increases) is more than compensated by an increase in the debt to GDP ratio, then yields are expected in rise. In addition, we include the unemployment rate as a government target, as any stabilisation policy on (43) could have an effect on unemployment. Notice that if the deviations are above the target these become increasingly costly.

$$\ln L_t = m' E_{t-1} \begin{pmatrix} y_t^{(1)} \\ \ln u_t \end{pmatrix} + \frac{1}{2} w E_{t-1} \begin{pmatrix} y_t^{(1)} \\ \ln U_t \end{pmatrix}^2 \quad (44)$$

For m' being a 1×2 vector of parameters of elements m_i and w being a 2×1 vector of parameters of elements w_i. Minimising (44) yields:

$$\ln L_{t, E[y_t^{(1)} \quad u_t]'} = m' I + w' E_{t-1} \begin{pmatrix} y_t^{(1)} \\ \ln U_t \end{pmatrix} \quad (45)$$

Notice that in (45) I made I equal a 2×1 vector of elements equal to 1. Substituting (43) in (45) and operating the constant terms boils down to

$$\ln L_{t, E[y_t^{(1)} \quad u_t]'} = m + w' E_{t-1} \begin{pmatrix} -1 + \frac{1}{\ln d_{t-1}} \ln d_t - \ln s_t \\ \ln U_t \end{pmatrix} \quad (46)$$

In equation (46) m is a scalar equal to m_i + m_j resulting from multiplying $m' \times I$. Rearranging (46) and collecting the constant terms, and the terms multiplying the variables and assuming that any variable at t-1 is known so it can grouped as a constant term so that proposed changes in (46) are as follows

$$\ln L_t = m - w_1 + \begin{pmatrix} \frac{w_1}{\ln d_{t-1}} & -w_1 & w_2 \end{pmatrix} \begin{pmatrix} \ln d_t \\ \ln s_t \\ \ln U_t \end{pmatrix} \tag{47}$$

It will be depicted that the short rate will be positively related to (47). This is plausible as a deterioration of government's financial position would result in higher yields in order to compensate investors for higher risks. This is based on the idea that any increase in government losses will be only compensated by additional increases in distortionary taxation or inflation, thus lower expected future consumption.

The link between equation (47) and government short rate will be as follows:

$$y_t^{(1)} = \alpha_0 + \alpha_1 \ln L_t \tag{48}$$

Equation (48) assumes that a deterioration of government losses results in an increase in yields.

Combining equation (47) with (48) boils down to

$$y_t^{(1)} = \alpha_0 + [\alpha_1 (m - w_1)] + \alpha_1 \frac{w_1}{\ln d_{t-1}} \ln d_t - (\alpha_1 w_1) \ln s_t + (\alpha_1 w_2) \ln U_t \tag{49}$$

In order to link the theoretical discussion from above to an empirical formulae, we will denote as follows by rearranging the constant terms including d_{t-1} (as this is a known value at time t), so that a possible approximation could be specified as follows,

$$
\begin{aligned}
\beta_0 &= \alpha_0 + \left[\alpha_1 \left(m - w_1\right)\right] \\
\beta_1 &= \alpha_1 \frac{w_1}{\ln d_{t-1}} \\
\beta_2 &= -\alpha_1 w_1 \\
\beta_3 &= \alpha_1 w_2
\end{aligned}
\tag{58}
$$

So this means that combining (43) with (44) we can reproduce the following measurement equation:

$$
y_t^{(1)} = \beta_0 + \begin{pmatrix}\beta_1 & \beta_2 & \beta_3\end{pmatrix} \begin{pmatrix}\ln d_t \\ \ln s_t \\ \ln u_t\end{pmatrix} \tag{59}
$$

<u>*Section VII. The Theory of the Fragility of the Eurozone*</u>

Proposition 4 *If the theory of the fragility of the Eurozone from De Grauwe (2011a,b) and Kopf(2011) holds, investors will not expect the sovereign to default, if and only if government solvency shocks are small and the cost of default is larger than the benefit of default. In which case equations (10) and (23) are relevant and equation (51) is left out of the model. In contrast, if a government solvency shock is large, there are two equally possible equilibria in which case the government bond becomes risky and equation (39) will have an effect on yields and equations (10) and (23) are subdued.*

Here we link the theory of the fragility of the Eurozone from De Grauwe (2011a,b) and Kopf (2011) with discussions from chapter 1.

Figure 4-1. The Theory of the Fragility of the Eurozone from De Grauwe (2011a,b) and Kopf (2011)

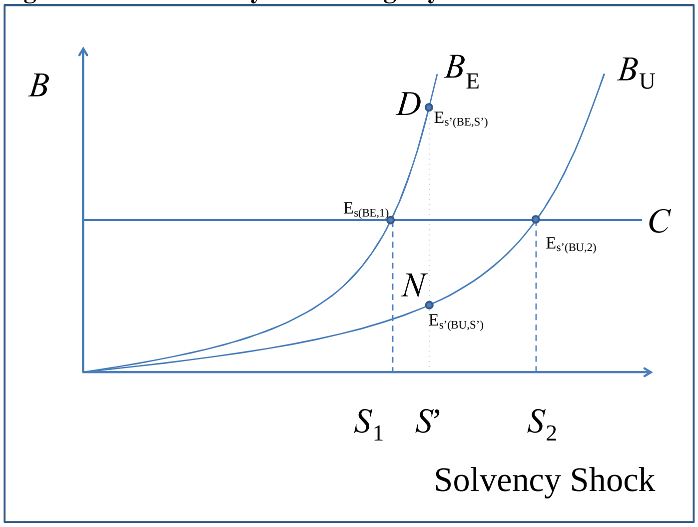


Figure 4-1 above describes the theory of the fragility of the Eurozone, which is a model of multiple equilibria. According to De Grauwe (2011a,b), Kopf (2011), Gros (2012) and De Grauwe and Ji (2013) there are three possible equilibria. The starting point of their analysis is that governments have a cost *C* and benefit *B* of defaulting and that a representative investor takes this into account when pricing the debt. The idea is that if governments experience a surplus shock as a consequence of an economic recession, this shock will translate in a solvency shock *S*'. A government that experiences a solvency shock and is not able to monetise deficits is usually forced to undertake unpopular austerity measures in order attend debt servicing. Along these lines the greater the shock the greater the austerity required. Another aspect of the model is that the benefit of default on whether the default is expected or not, is depicted in two benefit curves B_U and B_E. B_U represents the benefit of a default that investors do not expect to happen and, B_E represents the benefit of a default that investors do expect to happen. The benefit curves are upward sloping, as when the shocks become larger, the benefit of default increases. De Grauwe and Ji (2013) show that though they depict a non-linear relationship between solvency shocks and benefits, they also argue that this does not have to be that way. Another aspect of the model is that the steepness of the curve will largely depend on: the initial debt, the efficiency of the tax system and the size of the external debt. The greater the initial debt level the steeper the curve will be. In fact increases in the debt level will result in the curve becoming steeper and the government benefits of default will become more sensitive to solvency shocks. Likewise, if the tax system becomes inefficient, the curve will become steeper, and hence more sensitive to solvency shocks. In a similar way, as the external debt becomes larger, the curve not only becomes steeper due to increases in the debt burden but also because there will be less domestic resistance against default, making default even more attractive.

In a nutshell, the model can be summarised as follows:

1) **The solvency shock is small**. This implies that $S' < S_1$, where both curves $B_U < C$ and $B_E < C$. In this case, the cost of default is always greater than the benefit of default. When the cost of default is greater than the benefit of default investors do not expect the government to default. De Grauwe and Ji (2013) and De Grauwe (2013a, b) argue that his would be the cases for Germany or the Netherlands.
2) **The solvency shock is intermediate**. This implies that $S_1 < S' \leq S_2$. In this case, the model shows that there are two possible equilibriums, hence *D* and *N* (to default and not to default). According to De Grauwe (2011a, b) this will largely depend on which of the two equilibriums prevails, which will mostly depend on investors' expectations, as both equilibriums are equally possible. In this case curves $B_U < C$ and $B_E > C$. De Grauwe and Ji (2013) and De Grauwe (2013a, b) argue that his would be the cases for Spain, Italy, Ireland and Portugal.
3) **The solvency shock is large**. This implies that $S_2 < S'$. In this case, the cost of default is smaller than the benefit of default. When the cost of default is lower than the benefit of default investors expect the government to default where both curves $B_U > C$ and $B_E > C$. De Grauwe and Ji (2013) and De Grauwe (2013a, b) argue that his would be the case of Greece.

We propose that when the solvency shock is small, state space variables entering the model are those obtained from propositions 1 and 2, as the coefficients of the state variables described in propositions 3 will be insignificant or significantly close to zero. If, on the contrary, solvency shocks are intermediate or large, the state space variables entering the model should be those described in propositions 3, as the coefficients of the state variables described under propositions 1 and 2 will be subdued and we assume that they will either be insignificant or significantly close to zero.

This could be specified as follows:

$$x_t = \begin{cases} B_U, B_E < C \text{ and } S' < S_1 \text{ then the state vector } x_t : \{\ln C_t^e; \ln \pi_t; \ln U_t\} \\ B_E > C; B_U < C \text{ and } S_1 < S' < S_2 \text{ then vector } x_t : \{\ln C_t^e; \ln \pi_t; \ln U_t; \ln d_t; \ln s_t\} \\ B_U, B_E > C \text{ and } S' > S_2 \text{ then the state vector } x_t : \{\ln d_t; \ln s_t; \ln U_t\} \end{cases} \quad (60)$$

Equation (60) specifies that when solvency shocks are small, the state space vector will be governed by the factors described in propositions 1 and 3. If the solvency shock is intermediate, the state space vector will be governed by variables which include propositions 1, 2 and 3. However, when the solvency shock is large, as for the case of Greece, the state space vector will be calibrated with the factors discussed in proposition 3 only.

Section VIII. Conclusion and Final Remarks

Equations (10) to (51) constitute a small macroeconomic model with its own dynamics. Equations (10), (23) and (39) are the measurement equations which can be stacked in a state space vector x_t assuming it follows a Vasicek (1977) process, already discussed in (7) and (8). In chapter five German and Greek government yields are calibrated using some of the theory discussed in this macro-finance model. Thus, to be more precise affine models in German government securities are calibrated as in proposition 1 and taking some modifications to proposition 2 arguing that central banks would influence monetary aggregates such as M3 as a consequence of open market operations and reacting to movements in the price level instead of inflation as discussed in Sims (2000 and 2005) and works of Benhabib, Schmitt-Grohé and Uribe (2001). This is mostly for convenience as by using these state variables results improve significantly. Greek Government bonds will be calibrated using proposition 3. This is mostly in order to remain in line with our results in the survey conducted in chapter five. Another reason for doing so is that current developments show that when an asset becomes riskier, investors are looking into the solvency profile of the issuer rather than inflation and monetary policy, as the Greek government cannot finance its self by influencing the path of inflation and central bank monetary aggregates. So it will be concluded that if the bond is risk-free (solvency shocks for issuer are small), such as money market instruments and German government bonds the state vector will comprise the natural logarithms of unemployment rate, natural logarithms of consumer confidence index (as in proposition 1); natural logarithms of monetary aggregate levels and natural logarithms of the price level (modified proposition 2). If the bond is risky (solvency shocks for issuer are large), the model will calibrate with the natural logarithms of unemployment rate, natural logarithms of government surplus to GDP ratio and natural logarithms of total government debt to GDP ratio (proposition 3).

5. Discrete Approach to Affine Term Structure Models Applied to German and Greek Government Yields

Section I. Introduction

This essay documents some algebra and concepts seen in the continuous time affine term structure literature and plugs them into the discrete time approach. The paper's starting point is the celebrated papers from Backus, Foresi, Telmer (1996-98) and incorporates the developments seen on the continuous time approach as documented in Piazzesi (2010), Cochrane (2005), Singleton (2006) and Le, Singleton and Dai (2010). Subsequently, the models are calibrated using the Interbank as well as German and Greek government yields.

This research concentrates on the multifactor cases of affine term structure models, as the weaknesses seen on the one factor models under Vasicek (1977) and CIR (1985) are already sufficiently documented in Backus, Foresi and Telmer (1998).

Most of the empirical evidence on affine term structure literature has been mainly confined to US data. This research fits a discrete time affine term structure model using European macroeconomic data for German and Greek yields. Other macroeconomic data used are Greek unemployment as well as Greek debt and deficit to GDP ratios for the Greek yield curve. Focus is also given to discussion of results with special attention to economic policy, as well as portfolio management implications. This is in line with discussions in chapter 4, propositions 3 and 4 as well as with latest research seen in De Grauwe (2011a, b), De Grauwe and Ji (2013), Gros (2012) and Beirne and Fratzscher (2013). We show that for a government that enjoys a risk-free status, hence a government that experiences small liquidity shocks, such as the German government, the state variables for calibrating the bond yields can be the unemployment rate, consumer confidence index, the price level and the monetary aggregate M3. On the contrary, if the government does not enjoy the risk-free status, such as the Greek government, hence the government suffers from large liquidity shocks, the state variables for calibrating the bond yields can be the debt-to-GDP ratio, the government surplus-to-GDP ratio and the unemployment rate.

This essay has been published under Jakas (2012) and is organised as follows, section 2 introduces some of the notation with reference to latest developments seen in Piazzesi (2010) and Ang and Piazzesi (2003), Cochrane (2005), Singleton (2006) and Duffie and Kan (1996). In section 3, the Vasicek (1977) model is discussed under the multifactor setup. Section 4 presents the CIR (1985) which is adapted to fit the affine model. Section 5 presents a generalised version of affine term structure models a la Duffie and Kan (1996) but on a discrete version. Section 6 calibrates the models and main results are discussed and presented using interbank and German government yields. In section 7 we calibrate the Greek bonds and discuss some of the results. Finally, section 8 conclusion and final remarks are summarised.

Section II. Recalling some basic concepts and introducing new ones

Recalling from previous chapters that it is denoted $y_t^{(N)}$ for the yield of a zero coupon bond with maturity N in time t. For the time being and without loss of generality it will be assumed that $N = 1$ and hence, for convenience, the 1 period yields can be specified as a function of the stochastic discount factor as follows:

$$y_t^{(N)} = -\ln E[m_{t+1}]. \tag{1}$$

The right hand side of (1) refers to the stochastic discount factor. Specifications on equation (1) are referred to as *pricing kernel* by the dynamic asset pricing literature.

The present value of a bond is specified as follows:

$$E\left[P_t^{(N+1)}\right] = E\left[m_{t+1} P_{t+1}^{(N)}\right]$$

For which the natural log notation will be used, thus implying

$$\ln\left[P_t^{(N+1)}\right] = \ln\left[m_{t+1}\right] + \ln\left[P_{t+1}^{(N)}\right] \tag{2}$$

For $\ln[P_t^{(N+1)}]$ being the natural log present value of a bond in time t with maturity N+1, which will equate the addition between of the log stochastic discount factor and the redemption value of the bond in t+1.

A notation common seen in Piazzesi (2010) as well as in Singleton (2006) and Cochrane (2005) is that the short rate is a linear function of state variables, thus,

$$r_t^f = \gamma_0 + \gamma_1' x_t \tag{3}$$

Equation (3) is not accounted for in any of the Backus, Foresi and Telmer (1998) and (1996) and Backus, Telmer and Wu (1999). For those not familiar with the notation r_t^f denote the short rate, γ_0 is a scalar constant term, γ_1' is a $1 \times k$ vector of coefficients describing how the short rate responds to shocks on independent state variables x_t. Finally x_t is a $k \times 1$ vector. Notice that " ' " is used to denote for the transpose of a vector or a matrix.

The results obtained from the multifactor version documented in Backus, Foresi and Telmer (1998) work very well for an average yield curve but require some changes, should the researcher wish to understand movements in the yield curve, i.e. steepening, flattening and/or twists as a result of changes in the state variables, simply because under their settings the state variables are on average zero so that at the end the yield curve depends on parameter $A(N)$ only.

Another aspect which is accounted for in the literature is the behaviour of the state variables. This has two components: 1) the specification of the mean reversing process and 2) the specification of the random error term. The novelty of this work also lies in plugging Piazzesi (2010) and Cochrane (2005) into the Backus-Foresi-Telmer (1996) and (1998). Results differ mainly because authors have different specifications and different assumptions about the mean reversing process as well as the specification of the random error term and the stochastic discount factor.

Another common aspect seen in the affine term structure literature is that log prices are linear functions of state variables. A possible specification could be:

$$-\ln\left[P_{t+1}^{(N)}\right] = A(N) + B(N)' x_{t+1}. \qquad (4)$$

This is only a *guess*, as the functional form of (4) is not known. However, the literature appears to agree on this, as seen on Piazzesi (2010), Singleton (2006), Cochrane (2005), as well as in Backus-Foresi-Telmer (1996) and (1998) and seminal papers of Duffie and Kan (1996).

From our guess in (4) we wish to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$. Once we have these parameters all we need to do is to plug them into the following yield curve and taking into account for several maturities (4) would now boil down to

$$y_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{B(N)'}{N} x_{t+1}. \qquad (5)$$

In the next sections the multi-factor models are discussed, for different stochastic processes governing the behaviour of the state variables and the discount factor as accounted under the Vasicek (1977), Cox-Ingersoll-Ross (1985) and generalized affine term structure models a la Duffie-Kan (1996) asset classes.

Section III. Multifactor Affine Term Structure under Vasicek

A good starting point is to use the pricing kernel a la Backus-Foresi-Telmer (1998) which here is combined with the Vasicek (1977) process a la Piazzesi (2010). A possible specification would be like:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1} \tag{6}$$

$$-\ln[m_{t+1}] = \delta + r_t^f + \lambda' \varepsilon_{t+1} \tag{7}$$

Equation (6) is the classical mean reversing process whereby x_{t+1} and its mean being both a $k \times 1$ vector of independent state variables and x_t being its 1 period lag. σ_x is a diagonal $k \times k$ matrix of standard deviations of the state variables. ε_{t+1} is a $k \times 1$ vector of random error terms with classical normal assumptions of mean zero and variance 1.

Equation (7) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (7) was originally the univariate Vasicek (1977) case. In this essay we transform this specification and adapt it for the multifactor case of a k-dimensional vector of state variables. In addition the short rate r_t^f is replaced by (3) which will bring this closer to the Cochrane (2005) and Piazzesi (2010) results. Same as in Backus-Foresi-Telmer (1998) δ is specified as follows:

$$\delta = \frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 \tag{8}$$

Specification for (8) is fortuitous, the only aim is to normalise the stochastic discount factor so that it becomes the inverse of the short rate. Notice that with (8), now (7) has the following conditional means and variance:

$$E\left[-\frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 - r_t^f - \lambda' \varepsilon_{t+1}\right] = -\frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 - r_t^f$$

$$Var\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - r_t^f - \lambda'\varepsilon_{t+1}\right] = \sum_{i=1}^{k}\lambda_i^2$$

And assuming $E[\ln x] = \mu(x) + \frac{1}{2}\sigma^2(x)$, which yields

$$E[\ln m_{t+1}] = -r_t^f$$

Backus-Foresi-Telmer (1998) multifactor under the Vasicek (1977) case set the x_t to zero and δ is replaced by the mean of the short rate. Here it will not be required to do this because the short rate follows as described in (3). This will make possible to generate any yield curve at any point in time, whereas Backus-Foresi-Telmer (1998) could only produce an average yield curve, instead our model can generate any yield curve and study the risk premium λ_i in time series fashion, should we wish to do so.

Here it is shown how to get there. Starting first with equation (2) and substituting the right hand term for (7) and (4) which boils down to:

$$\ln\left[P_t^{(N+1)}\right] = -\delta - r_t^f - \lambda'\varepsilon_{t+1} - A(N) - B(N)'x_{t+1} \tag{9}$$

The intention is to compute the present value recursively using what it is known from (2) for some guess of coefficients from (4). Since $P^{(N)}_{t+1} = 1$ and $A(N=0) = B(N=0)'=0$, which means this can be solve recursively, as for 1 period would imply $A(N=1) = \gamma_0$ and $B(N=1)' = \gamma_1'$ which means that equals the short rate as described in (3). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (2) to compute the present value of an $N+1$ maturity bond.

As discussed earlier modifications to the Backus-Foresi-Telmer (1998) version are added by replacing δ for (8), r_t^f for (3) and replacing x_{t+1} for the Vasicek (1977) process described in (6).

$$\ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - \gamma_0 - \gamma_1' x_t - \lambda'\varepsilon_{t+1}$$
$$- A(N) - B(N)'\left[x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1}\right] \quad (10)$$

The constant terms and the terms multiplying x_t and ε_{t+1} are grouped, so that at the end it would look something like this

$$\ln\left[P_t^{(N+1)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \gamma_0 + A(N) + B(N)'\Phi\bar{x}\right) - ,$$
$$(\gamma_1' + B(N)'(I - \Phi))x_t - (\lambda' + B(N)'\sigma_x)\varepsilon_{t+1} \quad (11)$$

The reader should remember what is known from (2), so that the conditional moments on (11) can satisfy,

$$E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \gamma_0 + A(N) + B(N)'\Phi\bar{x}\right)$$
$$- (\gamma_1' + B(N)'(I - \Phi))x_t \quad (12)$$

and

$$Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = (\lambda' + B(N)'\sigma_x)^2 \quad (13)$$

Recalling that the implied present value of a fixed income security yields

$$-E\left[\ln P_t^{(N+1)}\right] = -E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] - \frac{1}{2}Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] \quad (14)$$

Substituting (12) and (13) into (14) yields

$$
\begin{aligned}
-E\left[\ln P_t^{(N+1)}\right] &= \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \gamma_0 + A(N) + B(N)'\Phi\bar{x} \\
&+ \left(\gamma_1' + B(N)'(I-\Phi)\right)x_t - \frac{1}{2}\left(\lambda' + B(N)'\sigma_x\right)^2
\end{aligned} \tag{15}
$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (4) yields,

$$
A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - \left(\lambda' + B(N)'\sigma_x\right)^2\right) \tag{16}
$$

$$
B(N+1)' = \left(\gamma_1' + B(N)'(I-\Phi)\right) \tag{17}
$$

All is needed is to replace (16) and (17) into (5) and solve numerically by fitting the curve to the observed yields by adjusting λ's for a given choice of maturities. All other parameters are obtained from observations. Backus-Foresi-Telmer (1998) estimated the λ's for two factor model and fit the mean yields for maturities 60 and 120 months. For each of the λ's it is possible to adjust the parameters to a desired maturity, the greater the number of λ's, the better the fit will be, as it would be possible to fit for more maturities resulting in a better fit of the parameters A(N) and $B(N)$ to the observed curvature. This choice is rather arbitrary, as there is no more rule than the size of the autocorrelation coefficients. Hence, λ's from state variables which show greater persistence –thus with a greater degree of autocorrelation- are used to fit longer maturities and λ's linked to variables with a low autocorrelation coefficient are used for fitting shorter maturities, as they exhibit less persistence.

An important difference is that under the Backus-Foresi-Telmer (1998) setup (17) was equated to zero, as the means of x_t were equal to zero. Intuitively, parameters γ_0 and γ_{1i} under Backus-Foresi-Telmer (1998) are 0 and 1 respectively. Here these parameters are free and obtained empirically for which it will be shown that parameters $\gamma_0 \neq 0$ and $\gamma_{1i} \neq 1$ and the signs for parameters $B(N)_i$ from (5) depend on γ_{1i}.

As in previous chapter the starting point is the pricing kernel a la Backus-Foresi-Telmer (1998) which combined with the CIR (1985) process a la Piazzesi (2010) yields:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \sqrt{x_t}\,\varepsilon_{t+1} \tag{18}$$

$$-\ln[m_{t+1}] = \left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) r_t^f + \lambda' \sqrt{x_t}\,\varepsilon_{t+1} \tag{19}$$

Equation (18) is the CIR mean reversing process whereby x_{t+1} and its mean being both a $k \times 1$ vector of independent state variables and x_t being its 1 period lag. σ_x is a diagonal $k \times k$ matrix of standard deviations of the state variables. ε_{t+1} is a $k \times 1$ vector of random error terms with classical normal assumptions of mean zero and variance 1.

Equation (19) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (19) was originally a univariate CIR (1985) case and here this specification is adapted for the multifactor case, as similar to the Vasicek (1977) discussed in previous section. Again, the short rate r_t^f is replaced by (3) which will bring this closer to the Duffie and Kan (1996), Piazzesi (2010), Cochrane (2005) and Singleton (2006) results.

The selection of the coefficient in (19) obeys the only purpose of normalising the stochastic discount factor so that it equates the inverse of the short rate, so that equation (19) results with the following conditional means and variance:

$$E\left[-\left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) r_t^f - \lambda' \sqrt{x_t}\,\varepsilon_{t+1}\right] = -\left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) r_t^f$$

$$Var\left[-\left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) r_t^f - \lambda' \sqrt{x_t}\,\varepsilon_{t+1}\right] = \left(\sum_{i=1}^{k}\lambda_i^2\right) r_t^f$$

And assuming $E[\ln x] = \mu(x) + \frac{1}{2}\sigma^2(x)$, which yields

$$E[\ln m_{t+1}] = -r_t^f$$

Backus-Foresi-Telmer (1998) documented the discrete multifactor case for the CIR under a different setup. Their example is mainly limited to a two factor under Longstaff and Schwartz (1992) setup. Here, the aim is confined to a generalised version of a multifactor model under the CIR with a *k*-dimensional vector of state variables. As in previous Vasicek example, the generalised CIR multifactor case is specified as follows.

Starting with equation (2) and substituting the right hand term for (19) as well as (4) which boils down to

$$\ln\left[P_t^{(N+1)}\right] = -\left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)r_t^f - \lambda'\sqrt{x_t}\,\varepsilon_{t+1} - A(N) - B(N)'x_{t+1} \qquad (20)$$

Same as for the Vasicek model the intention is to compute the present value recursively using what is known from (2) for some guess of coefficients from (4). Since $P^{(N)}_{t+1} = 1$ and $A(N{=}0) = B(N{=}0)' = 0$, which means it can be solve recursively, as for 1 period it will imply $A(N{=}1) = \gamma_0$ and $B(N{=}1)' = \gamma_1'$ and thus equating the short rate as described in (3). Unfortunately this does not work because for it to work it would require $\gamma_0 = 0$ and γ_1' to be a $k \times 1$ elements equal to 1, which is not true empirically. So it is necessary to modify the CIR case. To be more precise it will be necessary to sacrifice normality in order to be able to let a parameter λ's account for the discrepancies and thus enable the CIR model fit the observed values.

As discussed earlier modifications to the Backus-Foresi-Telmer (1998) version are added by replacing, r_t^f for (3) and replacing x_{t+1} for the CIR (1985) process already discussed in (18).

$$\begin{aligned}\ln\left[P_t^{(N+1)}\right] = &-\left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\left(\gamma_0 + \gamma_1' x_t\right) - \lambda'\sqrt{x_t}\,\varepsilon_{t+1} \\ &- A(N) - B(N)'\left[x_t + \Phi(\bar{x} - x_t) + \sigma_x\sqrt{x_t}\,\varepsilon_{t+1}\right]\end{aligned} \qquad (21)$$

Rearranging and collecting terms so that the constant terms and the terms multiplying x_t and ε_{t+1} are grouped, which would look something like this:

$$\ln\left[P_t^{(N+1)}\right] = -\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) - A(N) - B(N)'\Phi\bar{x}$$
$$-\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1' + B(N)'(I-\Phi)\right)x_t - (\lambda' + B(N)'\sigma_x)\sqrt{x_t}\,\varepsilon_{t+1} \tag{22}$$

Recalling (2), equation (22) has the following conditional moments,

$$E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = -\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) - A(N) - B(N)'\Phi\bar{x}$$
$$-\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1' + B(N)'(I-\Phi)\right)x_t \tag{23}$$

and

$$Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = (\lambda' + B(N)'\sigma_x)^2 x_t \tag{24}$$

Recalling (14), which is reproduced in (25) below,

$$-E\left[\ln P_t^{(N+1)}\right] = -E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] - \frac{1}{2}Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] \tag{25}$$

And now substituting (23) and (24) into (25) yields,

$$E\left[\ln P_t^{(N+1)}\right]=\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)+A(N)+B(N)'\Phi\bar{x}$$
$$+\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1'+B(N)'(I-\Phi)\right)x_t \tag{26}$$
$$-\frac{1}{2}\left(\lambda'+B(N)'\sigma_x\right)^2 x_t$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (4) yields,

$$A(N+1)=\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)+A(N)+B(N)'\Phi\bar{x} \tag{27.a}$$

$$B(N+1)'=\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1'+B(N)'(I-\Phi)\right)-\frac{1}{2}\left(\lambda'+B(N)'\sigma_x\right)^2 \tag{28.a}$$

Same as in previous Vasicek model, all is needed to do now is to replace (27) and (28) into (5) and solve numerically by fitting the curve to the observed values by adjusting λ's for a given choice of maturities.

Backus-Foresi-Telmer (1998) do not account for (27.a) and (28.a). However, from their univariate case it is possible to intuit that $\gamma_0 = 0$ and γ_1' is a 1 × k elements equal to 1 which is not realistic. The CIR process described in equation (18) is slightly different whereby in their version $(I - \Phi)$ would be first order auto-regression coefficient φ. Under this paper's settings the use of the $(I - \Phi)$ brings it closer to the continuous time version described under the Duffie and Kan (1996), Cochrane (2005) and Piazzesi (2010) class of affine models.

However, notice that (27.a) and (28.a) only work if, and only if, $\gamma_0 = 0$ and γ_1' equals a 1 × k elements equal to 1. The problem stems from how (19) has been specified because γ_0 is not zero and elements in γ_1' are not one. So it will be necessary to change (19) and sacrifice the possibility of making (19) equal the short rate under normality. Thus, (19) will now look more like:

$$-\ln\left[m_{t+1}\right]=r_t^f+\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2+\lambda'\sqrt{x_t}\,\varepsilon_{t+1}$$

Now if what was shown in (20) to (28) is re-performed under the above setup, (27.a) and (28.a) would look more like

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} \tag{27.b}$$

$$B(N+1)' = \gamma_1' + B(N)'(I - \Phi) - \frac{1}{2}\left[\sum_{i=1}^{k}\lambda_i^2 - (\lambda' + B(N)'\sigma_x)^2\right] \qquad [28.b]$$

Notice that now (27.b) and (28.b) does give the opportunity to solve by applying the recursion as in (2). Since $P^{(N)}_{t+1} = 1$ and $A(N=0) = B(N=0)' = 0$, and now $A(N=1) = \gamma_0$ and $B(N=1)' = \gamma_1'$ which means that under (27.b) and (28.b) the model equals the short rate as described in (3) for $N = 1$. This paper will use (27.b) and (28.b) when calibrating the CIR model because the original (27.a) and (28.a) do not work for the reasons explained above.

Section V. Generalised Multifactor Affine Term Structure Duffie and Kan

The Vasicek (1977) and the Cox-Ingersoll-Ross (1985) are special cases of the generalised multifactor affine term structure models which were first developed by Duffie and Kan (1996) and translated into discrete time by Backus, Foresi and Telmer (1996). The intention will, as for the previous models, include some of the developments documented by Piazzesi (2010), Cochrane (2005), Singleton (2006) and Le, Singleton and Dai (2010).

Under the generalised affine term structure state variables and the stochastic discount factor are specified as follows.

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x\varepsilon_{t+1} \tag{29}$$

$$-\ln[m_{t+1}] = \delta + r_t^f + \lambda'\sigma_x\varepsilon_{t+1} \tag{30}$$

$$\delta = \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 \tag{31}$$

$$r_t^f = \gamma_0 + \gamma_1' x_t . \quad (32)$$

$$\sigma_x = \Sigma s(x_t) \quad (33)$$

$$s_i(x_t) = \sqrt{s_{0i} + s_{1i}' x_t} \quad (34)$$

Equation (29) describes the stochastic process of the independent state variables. This is the usual mean reversing process whereby Δx_{t+1} is likely to be negative if x_t is above its mean and, is likely to be positive if x_t is below its mean. x_t and its mean are both k-dimensional vectors. Φ is a $k \times k$ matrix of diagonal elements Φ_i which represent the speed of adjustment at which each of x_{it} elements reverse to their means. σ_x is a diagonal $k \times k$ matrix comprising the volatility of the state variables. ε_{t+1} is a k-vector of shocks moving x_t away from its mean and with $\varepsilon_{i,t+1}$ elements being normally distributed with mean zero and variance 1.

Equations (30) and (31) describe the stochastic discount factor as seen in Backus-Foresi-Telmer (1999) which introduces some changes to the already discussed version shown in the Vasicek case equations (7) and (8), thus here with somehow a different setting, as (30) now includes a σ_x term.

Equation (32) which was already discussed in the introduction in (3) will be replaced by the short rate r_t^f so that it would get closer to the Piazzesi (2010) results.

The selection of (31) obeys the only purpose of normalising the stochastic discount factor so that it equates the inverse of the short rate.

Equation (33) and (34) describe the volatilities of the state variables. $s(x)$ is a diagonal $k \times k$ matrix with elements $s_i(x)$. Notice that by doing so it is possible to generalise for both Vasicek and the CIR cases. Because the Vasicek is a *Gaussian* process and CIR is a *square root* process. With (34) enabling for both cases, thus $s_{1i} = 0$ and $s_{0i} = 1$ for the Vasicek case, whereby the variance parameters in Σ are free. Alternatively, if it is wished to account for the CIR case, then set $s_{1i} = 1$ and $s_{0i} = 0$. Piazzesi (2010) and the celebrated paper from Duffie and Kan (1996) as well as Dai and Singleton (2000) remember us of the conditions required to obtain a unique solution to the stochastic differential equations, and these comprise the *Feller* and the *Lipschitz* conditions, for which the reader is encouraged to refer to Piazzesi (2010) page 706 for some examples on how this works. As for our discussion this is not of crucial relevance to this research.

Replacing (31) into (30) and adjusting the signs accordingly yields

$$\ln[m_{t+1}] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 - r_t^f - \lambda'\sigma_x\varepsilon_{t+1} \tag{35}$$

Equation (35) has the following conditional means and variance:

$$E\left[-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 - r_t^f - \lambda'\sigma_x\varepsilon_{t+1}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 - r_t^f$$

$$Var\left[-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 - r_t^f - \lambda'\sigma_x\varepsilon_{t+1}\right] = \left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2$$

And assuming $E[\ln x] = \mu(x) + \frac{1}{2}\sigma^2(x)$, which yields

$$E[\ln m_{t+1}] = -r_t^f \tag{36}$$

Backus-Telmer-Wu (1999) documented an affine case for two variables under a slightly different setup. Now, similar to the previous Vasicek and CIR examples, the generalised version follows.

This starts again with equation (2) and substitute the right hand term for (35) and (4) which boils down to

$$\ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 - r_t^f - \lambda'\sigma_x\varepsilon_{t+1} - A(N) - B(N)'x_{t+1} \tag{37}$$

Same as for the Vasicek and CIR models the intention is to compute the present value recursively using what is known from (2) for some guess of coefficients from [4]. Since

$P^{(N)}_{t+1} = 1$ and $A(N=0) = B(N=0)' = 0$, which means this can be solved recursively, as for 1 period would imply $A(N=1) = \gamma_0$ and $B(N=1)' = \gamma_1'$ and by doing so it the short rate is obtained as described in (3).

As discussed earlier modifications to the Backus-Foresi-Telmer (1998) version will be added by replacing, r_t^f for (3) and replacing x_{t+1} for the general affine version process described in (29).

$$\ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 - \left(\gamma_0 + \gamma_1' x_t\right) - \lambda'\sigma_x\varepsilon_{t+1} - A(N) - B(N)'\left[x_t + \Phi(\bar{x} - x_t) + \sigma_x\varepsilon_{t+1}\right] \tag{38}$$

Accounting now for [33] and [34], [38] would now look more like

$$\ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i} - \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i}x_t - \gamma_0 - \gamma_1' x_t - \lambda'\Sigma s(x)\varepsilon_{t+1} - A(N) - B(N)' x_t - B(N)'\Phi\bar{x} + B(N)'\Phi x_t - B(N)'\Sigma s(x)\varepsilon_{t+1} \tag{39}$$

Rearranging and collecting terms so that the constant terms and the terms multiplying x_t and ε_{t+1} are grouped, resulting in (39) being

$$\ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i} - \gamma_0 - A(N) - B(N)'\Phi\bar{x} - \left(\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i} + \gamma_1' + B(N)'(I - \Phi)\right)x_t - \left[\lambda' + B(N)'\right]\Sigma s(x)\varepsilon_{t+1} \tag{40}$$

And has conditional moments,

$$E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i} - \gamma_0 - A(N) - B(N)'\Phi\bar{x} - \left(\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i} + \gamma_1' + B(N)'(I - \Phi)\right)x_t \tag{41}$$

and

$$Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = \left[\lambda' + B(N)'\right]^2 \Sigma' \Sigma s_{0i} + \left[\lambda' + B(N)'\right]^2 \Sigma' \Sigma s_{1i} x_t \tag{42}$$

The implied present value of a fixed income security being

$$-E\left[\ln P_t^{(N+1)}\right] = -E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] - \frac{1}{2} Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] \tag{43}$$

So that substituting (41) and (42) into (43) yields

$$\begin{aligned} -E\left[\ln P_t^{(N+1)}\right] &= \frac{1}{2}\left(\sum_{i=1}^{k} \lambda_i^2\right) \Sigma' \Sigma s_{0i} + \gamma_0 + A(N) + B(N)' \Phi \bar{x} \\ &+ \left(\frac{1}{2}\left(\sum_{i=1}^{k} \lambda_i^2\right) \Sigma' \Sigma s_{1i} + \gamma_1' + B(N)'(I - \Phi)\right) x_t \\ &- \frac{1}{2}\left[\lambda' + B(N)'\right]^2 \Sigma' \Sigma s_{0i} - \frac{1}{2}\left[\lambda' + B(N)'\right]^2 \Sigma' \Sigma s_{1i} x \end{aligned} \tag{44}$$

Rearranging and collecting terms, with the constant terms and the terms multiplying x_t being grouped. Finally, lining up with (4) results in

$$A(N+1) = \gamma_0 + A(N) + B(N)' \Phi \bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k} \lambda_i^2 - \left[\lambda' + B(N)'\right]^2\right) \Sigma' \Sigma s_{0i} \tag{45}$$

$$B(N+1)' = \gamma_1' + B(N)'(I - \Phi) + \frac{1}{2}\left(\sum_{i=1}^{k} \lambda_i^2 - \left[\lambda' + B(N)'\right]^2\right) \Sigma' \Sigma s_{1i} \tag{46}$$

Same as in previous Vasicek and CIR model, all is needed now is to replace (45) and (46) into (5) and solve numerically by fitting the curve to the observed values by adjusting λ's for a given choice of maturities.

Backus-Telmer-Wu (1999) and Backus-Foresi-Telmer (1998) and (1996) document (45) and (46) in somehow different setup. The pricing kernel described in (30) is normalised, so that under log normal conditions the stochastic discount factor equates the short rate, this is not so obvious in their case. A matrix Σ of free parameters is also included in the model similar to Duffie and Kan (1996), Cochrane (2005) and Piazzesi (2010), which is not included in Backus-Telmer-Wu (1999) and Backus-Foresi-Telmer (1998) and (1996). As in the previous cases, Backus-Telmer-Wu (1999) and Backus-Foresi-Telmer (1998) their versions show that $\gamma_0 = 0$ and γ_1' is vector of $1 \times k$ elements equal to 1. Finally, also the process described in equation (29) is slightly different whereby in their version $(I - \Phi)$ would be first order auto-regression coefficient φ. Under our setting the use of the $(I - \Phi)$ brings us closer to the continuous time version described under the Duffie and Kan (1996), Cochrane (2005) and Piazzesi (2010) class of affine models. However, Piazzesi (2010) does not account neither for λ's nor the volatility of the stochastic discount factor in a way that allows fitting the curve to observed yields.

Ideally, (45) and (46) would allow to identify back (16) and (17) as well as (27) and (28), but this does not quite match because of the $\Sigma'\Sigma$ multiplying both λ' and $B(N)'$ terms in (45) and (46). This is because of how the stochastic discount factor has been specified in (30) and (31). Under this setup we differ to the stochastic discount factor under Vasicek (1977) and CIR (1985) documented earlier in (7) and (19) mainly because in these specifications it did not account for the volatility σ_x as part of the stochastic discount factor, as depicted in (30) and (31). This has been mainly for convenience only. Thus, without loss of generality (45) and (46) are adapted slightly, thus yielding

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - [\lambda' + B(N)'\sigma_x]^2\right)s_{0i} \tag{47}$$

$$B(N+1)' = \gamma_1' + B(N)'(I - \Phi) + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - [\lambda' + B(N)'\sigma_x]^2\right)s_{1i} \tag{48}$$

Notice that now when s_{0i} is 1 and s_{1i} is zero the model accounts for the Vasicek (1977) case discussed in Section 3 and when s_{0i} is zero and s_{1i} is 1 the model accounts for the CIR (1985) as discussed in section 4.

Section VI. Calibrating Under the Discrete Approach

This paper calibrates the Vasicek (1977) and the CIR (1985) using macroeconomic data. The models discussed in sections 3 and 4, shown in (16) and (17) for the Vasicek process, and (27.b) and (28.b) under the CIR approach are fitted using monthly Euro-Zone Unemployment Rate, Euro-Zone M3, Euro-Zone Production Price Index and European Commission Consumer Confidence Index, as per our discussion chapter 4 propositions 1, 2 and 4. Results are compared for estimated values of $a(N) = A(N)/N$ and $b(N) = B(N)/N$ with those observed via OLS published in Jakas (2011). This empirical work is based on monthly observations. EONIA, Euribor and German government yields have been obtained from Bloomberg. Most of the data series is only available since 1999. The period considered is from December 1999 until January 2010. This results in 122 observations. The yields are estimated under the restrictions (16) and (17) as well as (27.b) and (28.b) mentioned above and compared with the observed data.

Figure 5-1 below shows the coefficients $b(N)_i$ for OLS, Vasicek and CIR for different maturities. Vasicek and CIR show more persistence than OLS. Under OLS, coefficients fall faster and die away as maturity increases. Notice under OLS only the unemployment and producer prices show some persistence. In addition, they also exhibit a “humped” shape which this paper is not capable to reproduce under Vasicek or CIR and confirm the results seen on Backus, Telmer and Wu (1999). Not surprisingly, under the Vasicek as well as CIR approach, the coefficients are almost identical both models estimate virtually the same values. This is possible thanks to the use of more than one variable and the use of λ’s to fit for the same maturities. The sign of the $b(N)_i$ coefficients are primarily governed by the estimated parameters γ_{1i}, which are -in turn- are estimated via OLS by regressing the EONIA with the four factors (Unemployment, PPI, M3 and Consumer Confidence Index). Discussions on these results are shown in Jakas (2011).

Figure 5-2 shows that this behaviour is also observed for the coefficients $a(N)$. In general, it could be said that affine models exhibit coefficients which have a smoother behaviour across maturities than those seen under the OLS approach. $a(N)$ increases as maturities become longer. Under the OLS approach $a(N)$ is much steeper than those estimated under the Vasicek and CIR, and becomes negative for the Euribor 3 months, 6 months and the 2 year German government.

Economically, results are interpreted as follows: An increase in unemployment results in an increase in expected aggregate marginal utility with a the subsequent decrease in risk free assets' yields, as these are dear most in times of low consumption growth. An increase in consumer confidence results in a decrease in expected aggregate marginal utility growth and therefore, risk free yields are expected to increase as these assets act as a hedge in times when consumer confidence is low. An increase production prices has two implications, 1) an increase in production prices means that aggregate consumption growth is high, and in times when aggregate consumption growth is high risk free assets are dear less as these are negatively correlated with consumption growth, and investors are less risk averse and prefer riskier assets. 2) Taylor rules show that central bank policy will be to increase interest rates if production prices are expected to drift expected inflation away from target levels. Subsequently, an increase in interest rates results in an increase in yields, as financing becomes more expensive and margins between interest income and funding expense are tighter. An increase in monetary aggregates is expected to results in a fall in interest rates, as central banks conduct quantitative easing, yields fall mainly because the cost of financing generate higher margins and competition in the capital markets push bond prices up with the subsequent fall in yields. This behaviour is explained mainly by an increase in margins which in turn increases demand in the bond markets pushing prices up.

Figure 5-1 Analysis of the $b(N)_i$ coefficients under OLS, Vasicek and CIR models.

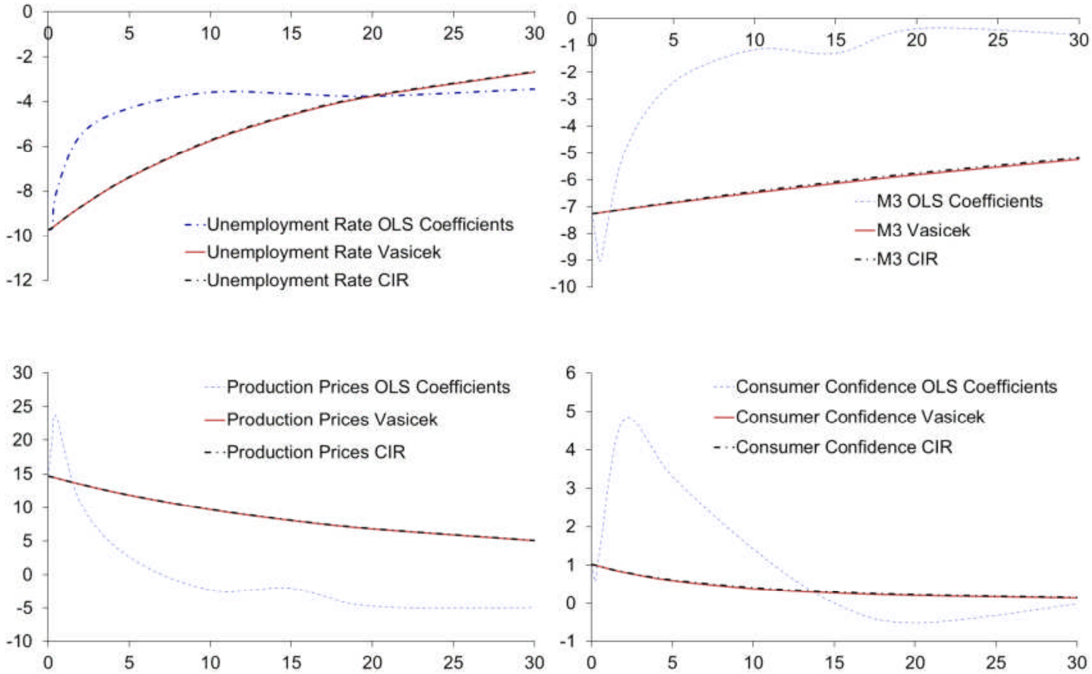
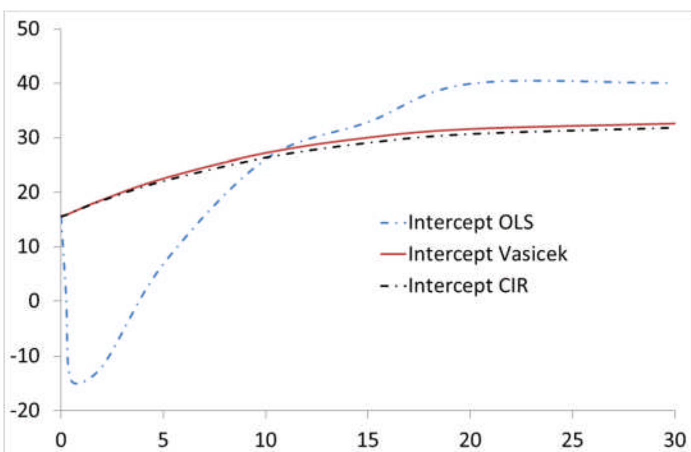


Figure 5-2 Analysis of the Intercept $a(N)$ coefficient under OLS, Vasicek and CIR models.



In order to analyse the models' ability to fit the curves with various shapes this research shows in figure 5-3 how the fitted curves look like at their flattest, mean or average, and steepest levels. In order to do this first the EONIA's historical highs, average and lows are identified and, subsequently the corresponding values observed for the four factors (unemployment, PPI, M3 and Consumer Confidence index). The model is capable of reproducing the observed sample data. Results have interesting policy implications which are outlined as follows: 1) the yield curve is at its flattest level when the overnight rate is at its highest value, unemployment is at its lowest level, production prices are at their highest levels, money supply (M3) is tight and consumer confidence levels is at its highs. And 2), when the yield curve is at its steepest the overnight rate is at its lowest, unemployment rate is at its highest levels, production prices are low, money supply is lax and consumer confidence level is the lowest.

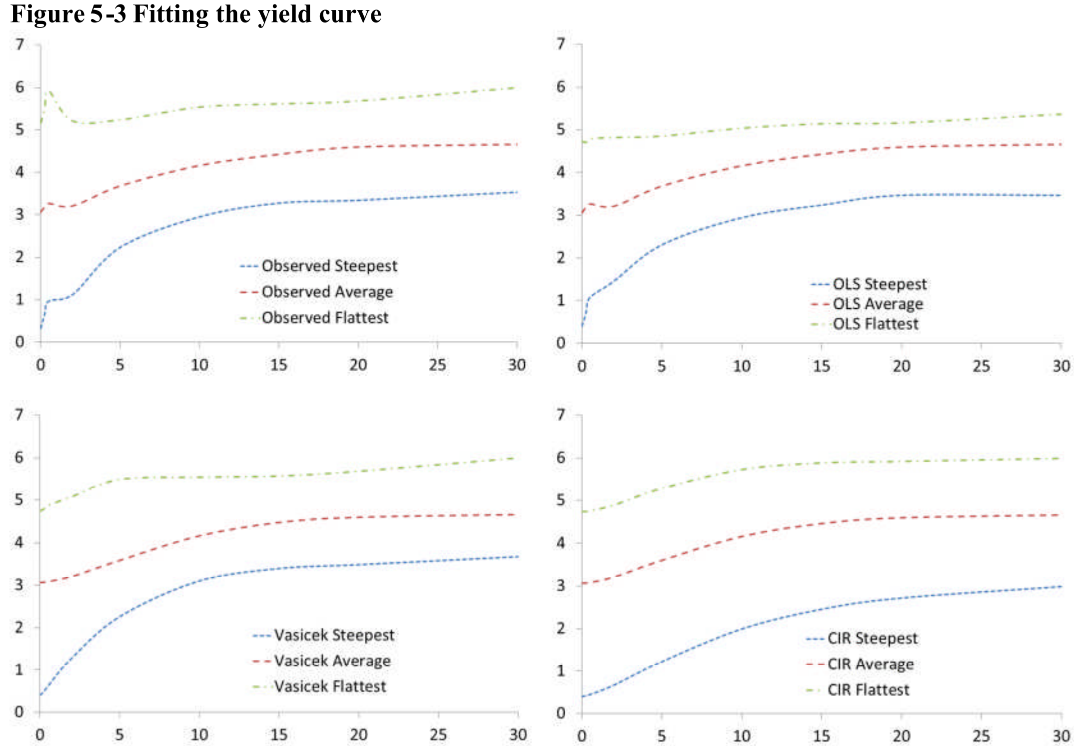
Notice that an economy exhibiting the above mention behaviour of the yield curve would suggest that the government could take advantage by issuing new debt at low financing costs in times of low consumption growth in order to undertake countercyclical fiscal policies without increasing distortionary taxation. In times of low consumption growth risk free assets exhibit low yields and hence low financing costs. Alternatively, in times of high consumption growth, when risk free assets exhibit high yields and thus prices are low, governments should reduce debt growth outstanding via buy-back programmes and thus reduce current refinancing costs. In fact, governments should under such scenario take advantage of buying back at a low redemption price.

From a portfolio management perspective, in times when the curve is at its steepest, a representative investor will have the incentive to undertake a flattening strategy such as shorting the 2 year maturity bonds and long the 10 to 30 years, as the losses generated in the 10 to 30 years will be more than offset by the profits on the front end. In times when the curve is at its flattest, a representative investor will have the incentive to long the 2 year maturities and short the long end (10 to 30 years).

The front end of the curve is relatively similar across models. The different models show discrepancies mainly on the long end. Hence, from the 10 year maturities onwards, OLS fails to describe the movement of the curve when yields are above their average. The observed, as well as the estimated via Vasicek and CIR describe a rather parallel shift whereas OLS shows more a flattening movement once yields have reached their means and are moving towards their flattest level.

In all cases it is shown that when the curve is at its flattest, yields are higher across the yield curve and when the yield curve is at its steepest the yields are at their lowest levels. All models agree with the observed yield curves that the front end is clearly more volatile and has a greater range of values than the long end. When yields are increasing, the yield curve is expected to become flatter if yields are currently below their means and, alternatively, when yields are falling, the yield curve is expected to become steeper if yields are above their mean values. This is mainly because the volatility in the yield curve is expected to fall as maturity increases. A possible way of describing this would be if it is assumed that the volatility at any point of the yield curve somehow obeys that $\sigma^{(N)} = \sigma^{(1)} / N$, thus the volatility and macroeconomic shock effects decrease as maturity increases.

Figure 5-3 Fitting the yield curve



The predictive-ability of the models is of special interest to this research. The intention is not only to explain economically the behaviour of the yield curve from a macroeconomic perspective but also understand how macroeconomic variables contribute to predict changes in the yield curve. Figures 5-4 and 5-5 show how the four factor Vasicek as well as the CIR models are capable of fitting the yield curve behaviour as a consequence of innovations in macroeconomic variables. The models are calibrated for the period starting from December 1999 to January 2010 and compared the estimated yields to those observed in the sample data. Results are encouraging, as fitted values are in line with the observable trends. Yields in the longer end are less predictable, compared to those seen in the front end however they still exhibit a high explanatory power and forecast quite well the underlying trends.

As maturities become longer, forecasts appear to be less convincing towards end of 2009 and beginning of 2010. However, even though not presented in figures 4-5, current data shows that yields have indeed fallen during 2011 to historic lows, Schatz (2 years German treasuries), Bobls (5 years German government) and Bunds (10 years German government bonds) reached historic lows below 2% for Bunds and even negative yields on less than 1 year Germany treasuries, thus showing that these models predicted in advance lower yield levels.

Figure 5-4 Vasicek fitted versus observed yields

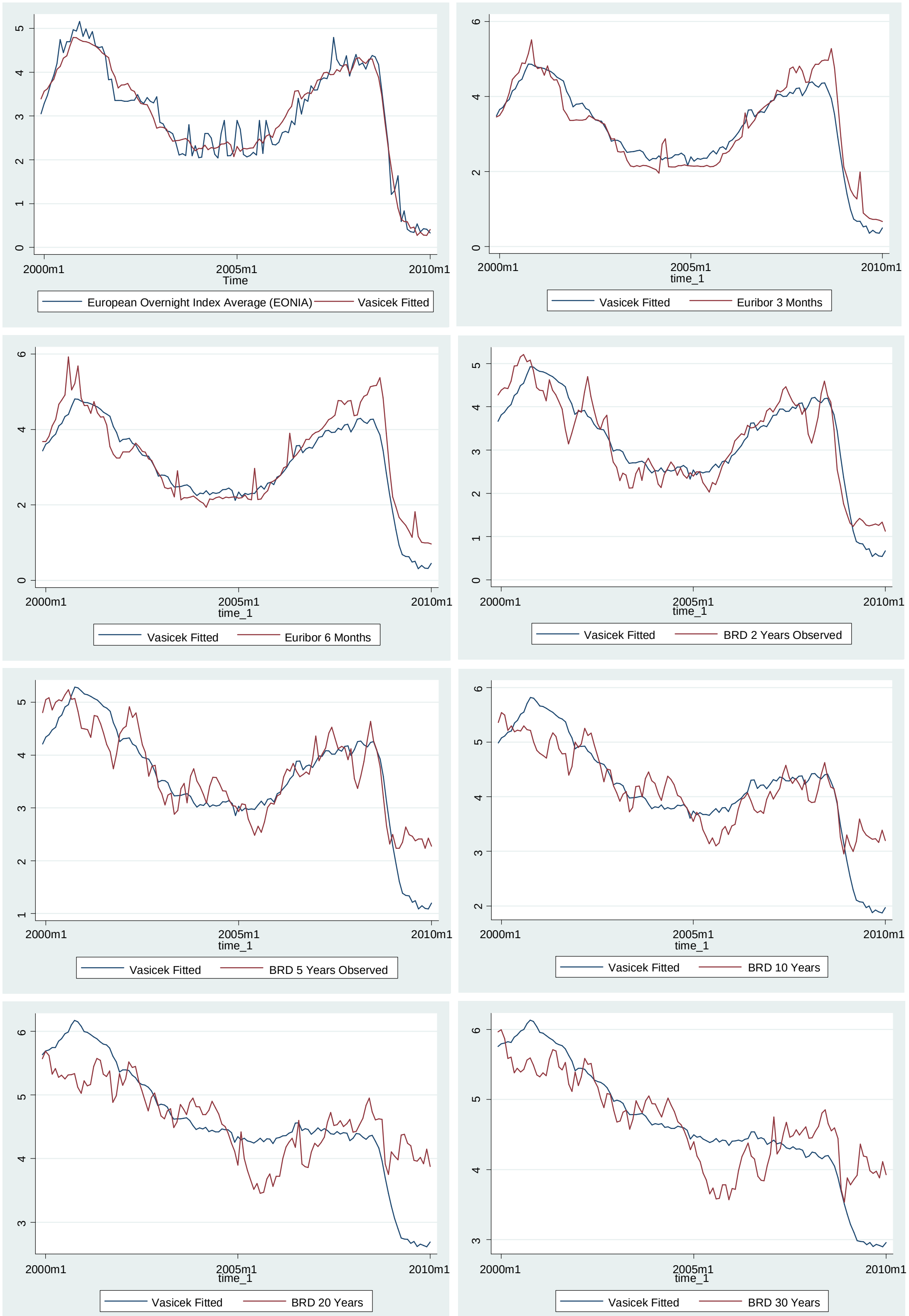
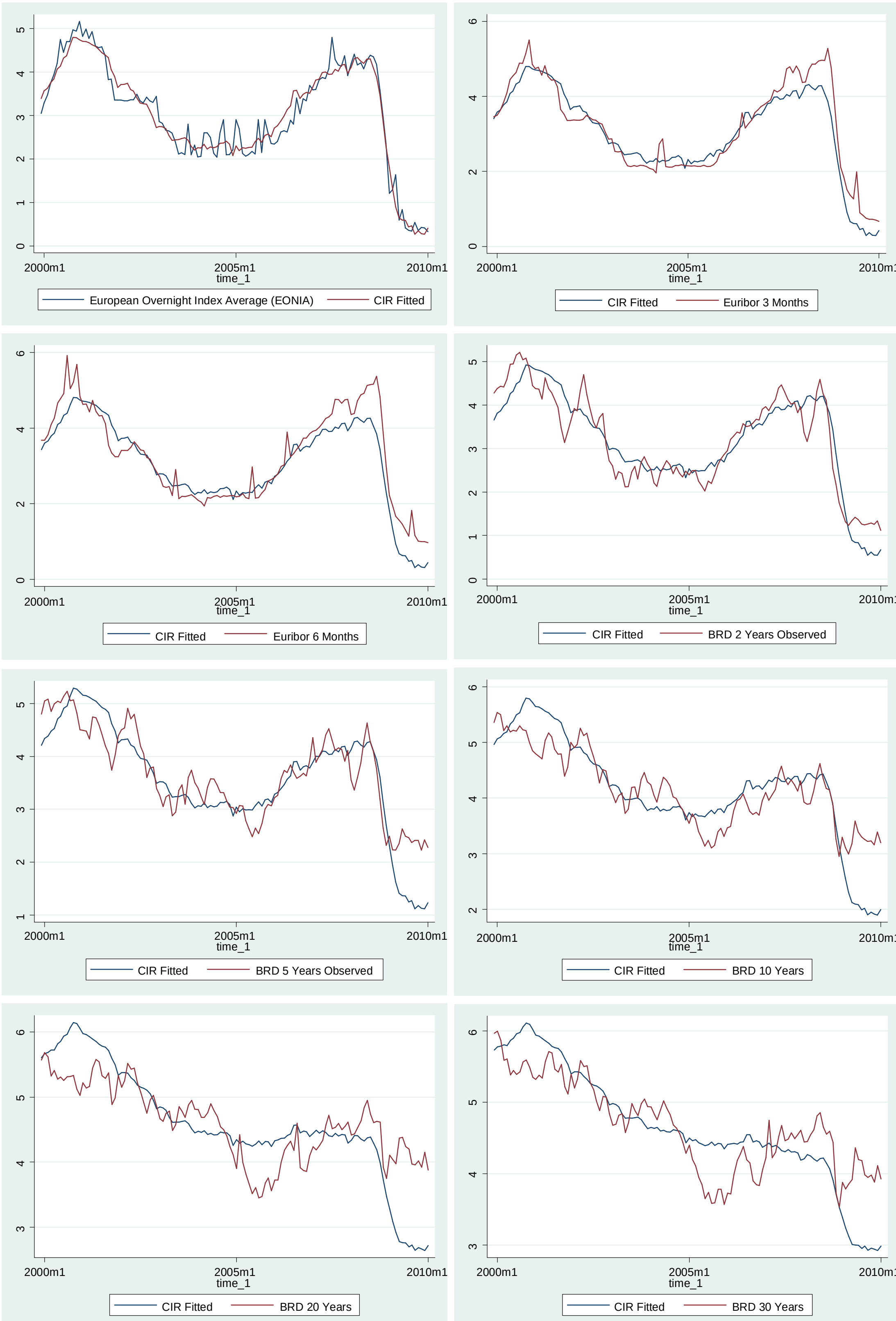


Figure 5-5 CIR fitted versus observed yields



Section VII. Calibrating with Greek Government Bonds

In this section, we calibrate an affine term structure model under CIR with Greek government bonds. The study was mainly dependent on the availability of yield data for the period during Greek's financial collapse as well as previous periods where Greece enjoyed some stability. The model should be capable of accounting for all states of the economy. Full statistics were only available for 2, 5, 10, 15 year bonds available in Bloomberg historical data. The period used varies depending on the maturity, but overall we considered the period June 2001 to April 2012. The macroeconomic variables have been picked after surveying 50 market participants from various market leading financial institutions. We picked the top three: Greek government *budget deficit* to GDP ratio, Greek unemployment rate and Greek government *debt* to GDP ratio, as discussions in chapter 4 propositions 3 and 4. Regression results show that all these factors were highly significant and that OLS as well as affine models perform very well, despite the volatility seen during the last two years.

Figures 5-6 and 5-7 below show the affine-CIR fitted versus the observed Greek Government yields, which are quite encouraging despite market conditions. Figure 5-7 shows that observed values exhibit more volatility than the fitted ones the closer we get to Greece's default.

Figure 5-8 shows the OLS-fitted versus observed. The reason for doing this comparison is mainly to show the differences stemming from a no-arbitrage approach with those using the OLS-approach, which violates the no-arbitrage condition depicted in an affine term structure model.

Figures 5-7 and 5-8 show that the affine-CIR as well as the OLS models can reproduce the dramatic rise in yields shortly before Greece's collapse. Not so lucky appear to be the results seen for the 2 year yields which exhibits a slower growth rate compared to the observed data. However, this is not so disappointing, as the model has been able to account for a yield movement from 4% up to almost 50% levels. Remarkably, the 5, 10 and 15 years exhibit surprisingly good results, mainly because these have had a far lower impact compared to the 2 year yields. This research shows that the macroeconomic variables used for calibrating the model explain very well yield dynamics.

In times of financial distress and when government bonds become risky assets, markets focus their attention more to the ability of governments to repay in the future and ratios such as debt-to-GDP as well as government deficit-to-GDP exhibit high explanatory power, in contrast to the German bonds, where markets look here more into unemployment as well as expected future consumption growth and less to debt-to-GDP or deficit-to-GDP ratios, because here German bonds act as a hedge for times where aggregate marginal utility growth is high and expected future consumption growth is low.

Figure 5-6 Greek government bond yield-curve CIR-fitted versus observed.

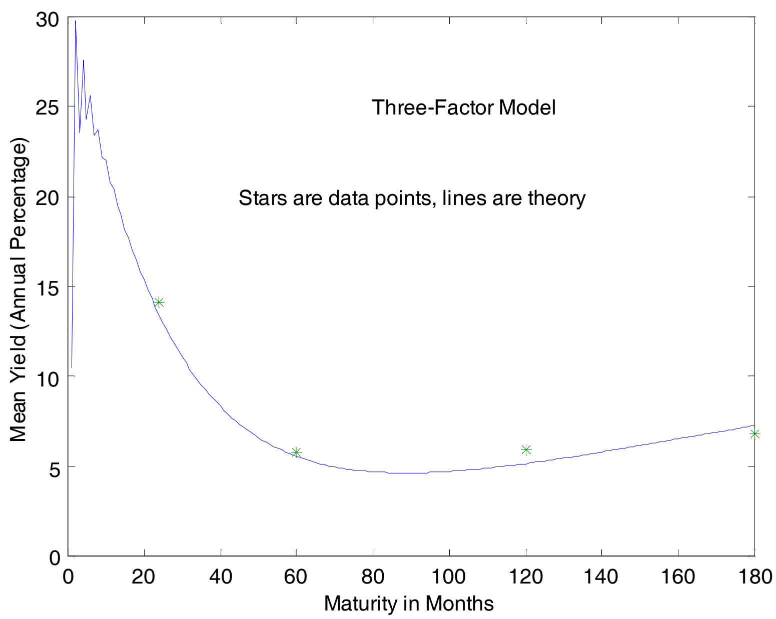


Figure 5-7 Greek government bond yields, CIR-fitted versus observed time series.

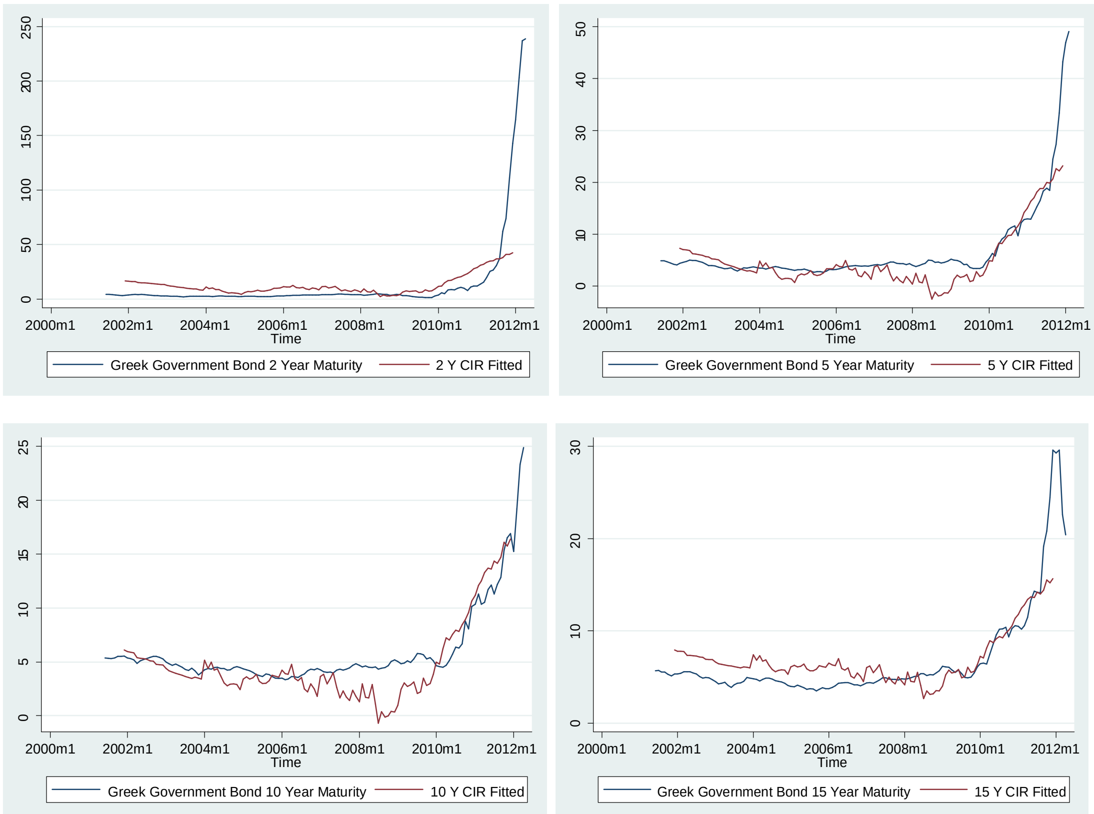
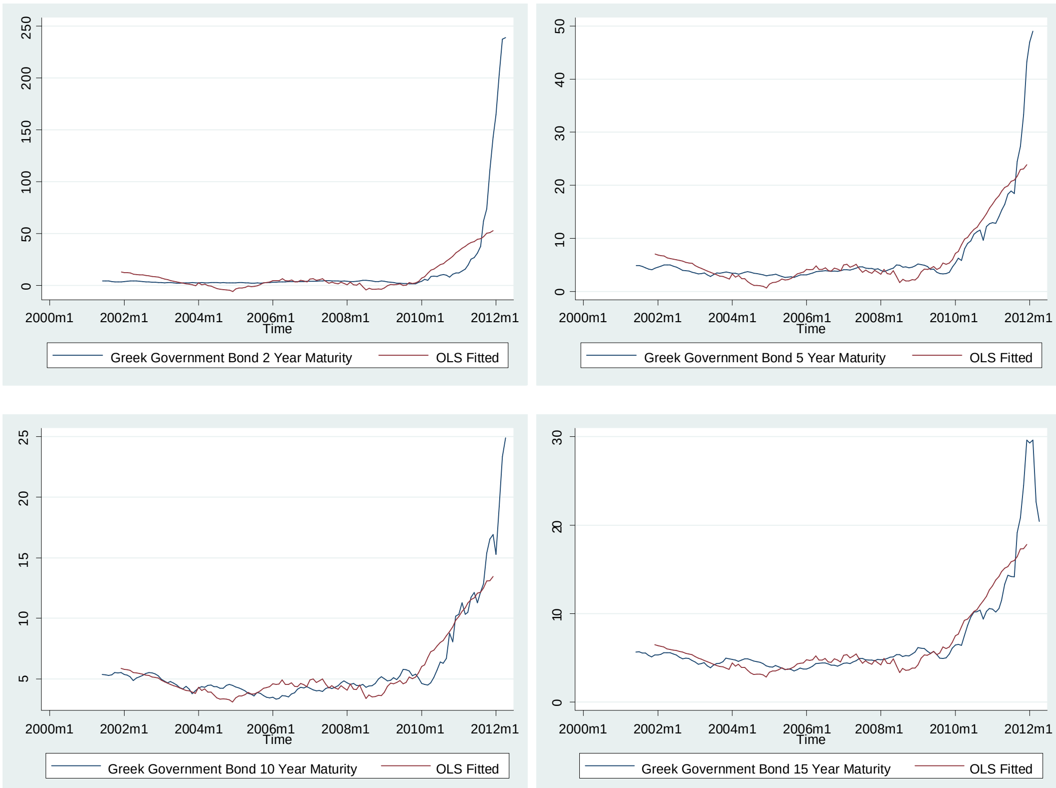


Figure 5-8 Greek Government Bonds OLS fitted versus observed time series.



In order to focus on the limitations of the model we analyse the ability of the model to generate an average yield curve in times when the short yields are at their *lowest* and at their *highest* levels. Figure 5-9 below shows different coefficients for the parameters $A(N)$ and $B(N)_i$ which have been estimated calibrating a space state vector observed when the 2 year Greek Government yield was at its lowest level, at its mean and at its highest level. We see that the model struggles when yields are at their highest levels. This should not surprise the reader, as these levels of yields are observed shortly before the Greek sovereign collapse and its subsequent default. Moreover, actually what is really struggling here is the Broydn function in Matlab rather than the model. In this paper we use this Matlab function to solve numerically equation (5) to adjust to observed yields (and to restrictions documented in 27.b and 28.b) by adjusting the vector λ' explained in previous sections. Parameters $A(N)$ and $B(N)_i$ exhibit virtually identical patterns and differ significantly only when yields are at their highest levels. Coefficients for Greek government budget deficit to GDP are negative which means that an increase in this ratio, hence a deterioration of its finances with respect to GDP, results in a fall in yields. A priori this might be seen as odd, but it makes sense if this is analysed together with the other two coefficients, thus looking into the size and the sign of the coefficients for unemployment and debt-to-GDP ratio. Unemployment and debt-to-GDP ratio show that these coefficients would more than offset any positive effect from the fall in yields as a result of an increase in the government deficit-to-GDP ratio.

Figure 5-9 Greek Government Bonds Affine-CIR estimated.

Coefficients $A(N)/N$ and $B(N)_i/N$ fitting equation (5) to observed yields, restricted to (27.b) and (28.b) and when state vectors are $x_{i;min}$; $x_{i;mean}$; and $x_{i;max}$.

Min Average Max

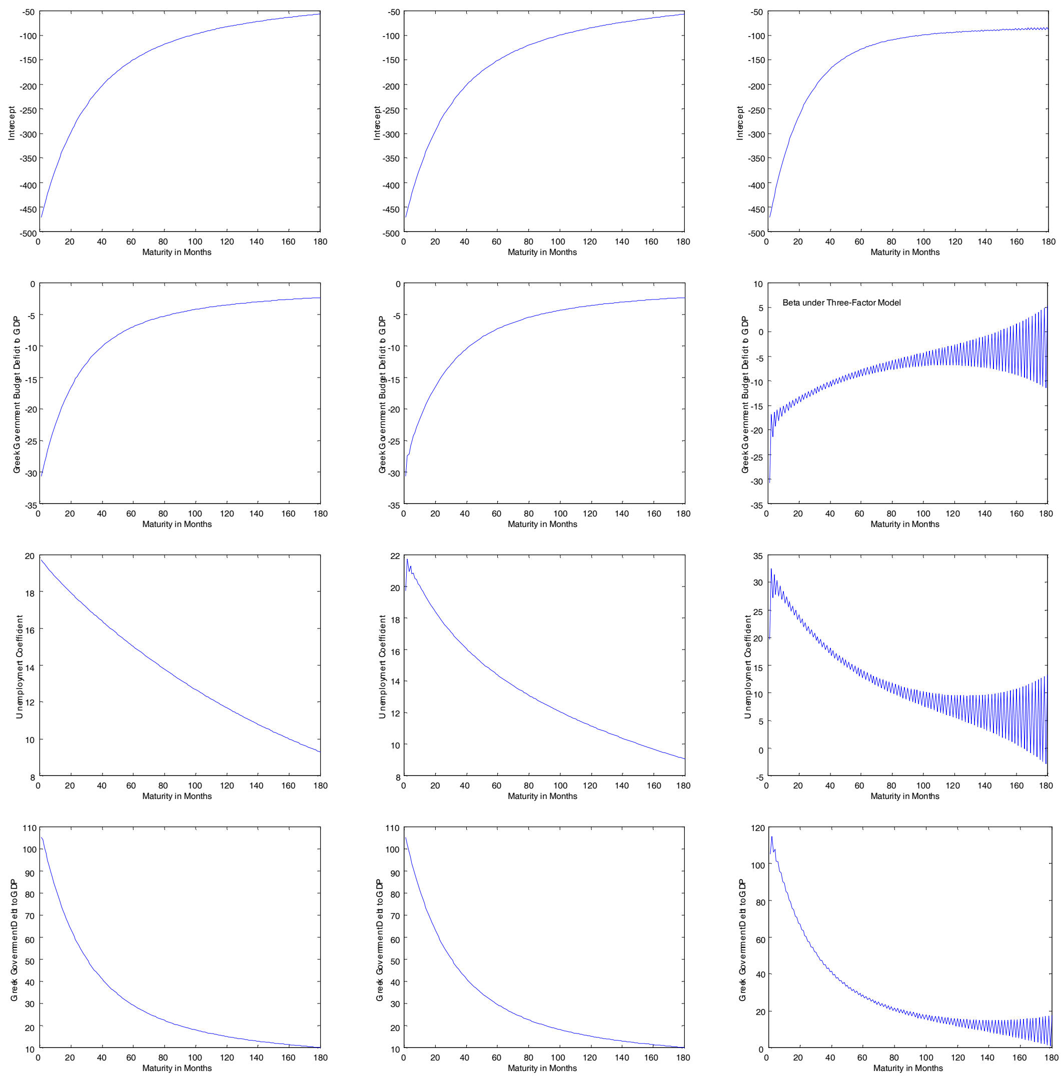


Table 5-1 shows some regression results. All variables used are very significant. OLS confirms our observation under the affine-CIR, thus if governments similar to the Greek case, engage in counter-cyclical fiscal policies that result in a deterioration of their government deficit with respect to GDP might still observe a fall in yields however, this is expected to be more than offset by an increase in yields as a consequence from a deterioration in their debt-to-GDP ratio. This implies that governments can run deficits to reduce unemployment only if the deterioration of its deficit does not result in a significant deterioration of their Debt to GDP ratio, as the deterioration of this ratio will more than offset any positive effect stemming from their countercyclical fiscal policies. The reader should notice that the signs of the coefficients (with the exception of the intercept) are in line with the theoretical discussions in from Chapter 4, proposition 3.

Table 5-1 OLS Regression Results and Selected Diagnostics

State Variables	**Greek Government Sovereign Yields**			
State Variables	***2 Years***	***5 Years***	***10 Years***	***15 Years***
Log Greek Gov. Deficit to GDP Ratio	-23.53547 (*t-stat*: -3.68) (*P*: 0.000)	-7.338331 (*t-stat*: -5.30) (*P*: 0.000)	-3.093097 (*t-stat*: -6.25) (*P*: 0.000)	-3.776979 (*t-stat*: -4.30) (*P*: 0.000)
Log Greek Unemployment Rate	18.56715 (*t-stat*: 3.04) (*P*: 0.003)	6.672775 (*t-stat*: 4.50) (*P*: 0.000)	3.40174 (*t-stat*: 5.99) (*P*: 0.000)	4.858906 (*t-stat*: 4.98) (*P*: 0.000)
Log Greek Government Debt to GDP Ratio	100.2863 (*t-stat*: 4.15) (*P*: 0.000)	39.68417 (*t-stat*: 7.90) (*P*: 0.000)	17.13523 (*t-stat*: 12.02) (*P*: 0.000)	24.58802 (*t-stat*: 7.59) (*P*: 0.000)
Intercept	-461.3932 (*t-stat*: -4.18) (*P*: 0.000)	-182.0015 (*t-stat*: -8.11) (*P*: 0.000)	-77.10492 (*t-stat*: - 13.32) (*P*: 0.000)	-113.5292 (*t-stat*: -7.81) (*P*: 0.000)
Number of observations	121	121	121	121
R-squared	0.5460	0.8158	0.8712	0.8183

<u>Section VIII. Conclusions and Final Remarks</u>

This paper documented some of the algebra and concepts seen in the continuous time affine term structure literature and plugged them into the discrete time approach. Starting point for this paper has been the celebrated papers from Backus, Foresi, Telmer (1996-98). In addition, some of the developments seen on the continuous approach as documented in Piazzesi (2010), Singleton (2006) and Duffie and Kan (1996) have been explored and adapted to the discrete time approach.

This research focused mainly on the multifactor cases of affine term structure models, as the weaknesses seen on the one factor models under Vasicek (1977) and CIR (1985) have been very well documented already in Backus, Foresi and Telmer (1998). Novelty of this research is that the multifactor affine term structure models under the Vasicek (1977) and the CIR (1985) process were calibrated using observed Interbank and German sovereign yields and European macroeconomic data as well as Greek sovereign yields. For the European and German yield curve, we calibrate macroeconomic data such as Euro-Zone Unemployment rate, Euro-Zone Production Price Index, Euro-Zone monetary aggregates M3 and Euro-Zone Consumer Confidence Index. For the Greek yields curve we use Greece's sovereign budget deficit-to-GDP ratio, Greek unemployment rate and Greek sovereign debt-to-GDP ratio. The results are encouraging and the models fit the observed yields as well as give evidence of a reasonable predictive-ability. These results has been supported by recent research as seen in the works of De Grauwe (2011a,b), Gros (2012), Kopf (2011), De Grauwe and Ji (2013) as well as Beirne and Fratzscher (2013). We show that for a government that enjoys a risk-free status, hence a government that experiences small liquidity shocks, such as the German government, the state variables for calibrating the bond yields can be the unemployment rate, consumer confidence index, the price level and the monetary aggregate M3. On the contrary, if the government does not enjoy the risk-free status, such as the Greek government, hence the government suffers from large liquidity shocks, the state variables for calibrating the bond yields can be the debt-to-GDP ratio, the government surplus-to-GDP ratio and the unemployment rate.

Main findings can be summarised as follows: In the case of the interbank rates and German sovereigns, an increase in unemployment results in a fall in yields on risk free assets and the curve is expected to steepen with front end yields falling faster than the long end. An increase in production prices are expected to result in yield curve flattening, with yields in the front end increasing at a faster pace than the long end. An increase in monetary aggregate M3 is expected to result in yield curve steepening, with yields in the front end falling faster than the long end. Finally, an increase in the consumer confidence index is expected to result in yields flattening, with front end yields increasing faster than the long end. This means that when the economy is booming risk free assets' yields are expected to flatten and when the economy is under recession risk free assets' yields are expected to steepen. From a portfolio management perspective, a representative investor would have incentives to short risk free assets in times when yields are at their steepest levels and set a curve flattening strategy shorting the front end allocating greater weight than to the long end. A more conservative strategy would be to short 2 years versus long the 10 and 20 years onwards, as the profits from the front end are expected to outweigh the losses on the long end.

For the case of Greek government bonds the model shows that if governments engage in counter cyclical fiscal policies when unemployment is high, this will only be possible if these policies do not result in a significant deterioration of the debt-to-GDP ratio. Governments that exhibit a positive correlation of their yields to aggregate consumption growth need to ensure low deficits and debt burdens during booming periods so that they can still have capacity to issue new debt for the rainy days.

From the Greek case this paper shows that a deterioration of government's deficit-to-GDP ratio results in a fall in yields. This is mainly because the increase in spending helps to boom the economy. However, this is more than offset by the deterioration of its debt-to-GDP ratio, thus a deterioration of the latter ratio will more than offset any positive effects stemming from any budget-deficit-induced counter-cyclical policies.

We have learned that debt and deficit ratios can play a role in times of financial distress. However, more research is needed in order to understand what can be done once it's already too late, thus once governments have not done their homework and run unprecedented and unsustainable deficits, thus the question we should try to answer is: what can be done in order to avoid distortionary taxations and aggressive fiscal discipline that lead to more social unrest and further financial markets nervousness.

6. The Yield Curve and Economic Fundamentals: A Continuous Time Model

Section I. Introduction

Fear, as any other human emotion, is part of the market economy. However, this does not mean that in even during such apparently irrational times, market behaviour cannot be accounted for by econometric models. In fact, according to the results in this essay, current, admittedly low yields seem to be explained by euro-zone macroeconomic variables fairly well. It turns out, therefore, that observed yields are still being described by a state space vector of macroeconomic variables. In the present study which was published in Jakas and Jakas (2013), we apply the continuous time affine term structure (CT-ATS) model to analyse euro-zone yield benchmark data. A novel approach, indeed, since most empirical works so far have limited themselves to US data, test Fed policy and Taylor's (1993) rules, as in Christiano et al. (1999), Cogley and Sargent (2001, 2002) Sims (1999) and Sims and Zha (2002), Piazzesi (2001), Cochrane and Piazzesi (2005), and Evans and Marshall (1998, 2001). In addition, we show that it is possible to depart from the usual maximum likelihood approach and show that other numerical methods such as those published in Press et al. (1996) usually used in physics are capable of compelling results.

Similarly, other authors have concentrated in testing expectations hypothesis as in Mankiw and Miron (1986) or used yield curve models to identify central bank latent targets as in Piazzesi (2001, 2002). Likewise Barr and Campbell (1997) and Campbell and Viceira (2001) also apply the ATS models to determine the correlation of real, short term yields with inflation and risk premium. Similar papers on inflation and risk premium with index linked bonds were seen in Buraschi and Jilsov (2005), and Campbell and Shiller (1991) with U.K. data mostly with two factor models.

A different perspective is seen in Ang and Piazzesi (2003), where they address whether macro variables add to our understanding of yields by looking at out-of-sample forecasts of yields. The works from Aït-Sahalia (2002) and Aït-Sahalia and Kimmel (2002) estimate multifactor models using closed form likelihood expansions and emphasis is given more to the methodology rather than to the state variables used. Ang et al. (2006) estimate a three-factor model based on a short rate, term spread, and GDP growth, but again mostly confined to US data whereas our paper the short rate is treated endogenously. Entering the short rate as an explanatory variable usually improves goodness of fit, however we still need to know what influences the yield curve, and the short rate is part of the yield. Backus, Foresi, and Telmer (1998 and 1996) present a discrete approach of bond pricing mostly on US data and applied to the understanding of the forward premium anomaly in foreign exchange prices.

In general, little has been done on Euro-Zone data with affine term structure models apart from the papers seen in Jakas (2011 and 2012) confined to ordinary last square (OLS) regressions and an affine discrete approach for testing Vasicek (1977) and Cox-Ingersoll-Ross (1985) stochastic processes under a multifactor setup. Most of the publications are either concentrated from an inflation or inflation risk premium perspective or the data is confined only to a particular European country instead of a euro-zone or Europe level. Looking into the works from Hördahl and Oreste (2010) they mainly confine their work to a joint model of macroeconomic and term structure dynamics to estimate inflation risk premia. While it is true that they look not only to US data but also to euro data, their focus is not to model the European benchmark term structure but the inflation risk premia. Other celebrated ECB working papers such as in Hördahl, Tristani and Vestin (2007) are confined to show that micro founded dynamic stochastic general equilibrium models with nominal rigidities can be successful in replicating features of bond yield data, however the work is based all on US data from the Federal Bank of Saint Louis. e.g. to be more precise they use PCECC96 for consumption and PCECTPI for prices. Amisano and Tristani (2007) focus on inflation not on term structure. Hördahl, Tristani and Vestin (2004) confines the work solely to German data and does not calibrate with European aggregated macroeconomic data in order to analyse from a euro-zone perspective. And there are many more previous to the creation of the euro which are not worth mentioning here, as we confine our work to a euro zone perspective using real data from the euro-area.

Our intention is to use the ATS model to link the fields of macroeconomics and mathematical finance from a practitioner's view, as we believe that the yield curve is an integral part of a network of macroeconomic variables. Moreover, most of the empirical work does not provide much discussion about the state variables used in those models. In fact, most of the empirical work shows a rather poor performance from a practical standpoint, mainly because even if the models are robust, theoretically speaking, they performed poorly on the empirical arena, and not all state variables fit the models so well. In contrast, in this work we have identified state variables which show that ATS models can perform better and thus this paper opens the opportunity for better forecasts.

We confined our work to the assumption that risk-free government bonds satisfy the Duffie-Kan (1996) class of affine models and we stick strictly to the methodology published in Cochrane (2005) and Piazzesi (2010). Our purpose is not to develop a dynamic stochastic general equilibrium model, but to simply calibrate the benchmark curve to observed macroeconomic data and show that low yields are explain by the data using when applying this model.

We define the European Yield curve benchmark as the European Overnight Index Average (EONIA) for the short rate, and the rest of the curve comprises the Euribor 3 and 6 months, and the 2, 5, 10, 15, 20 and 30 year German government yields. Furthermore, we use the following state vectors: (i) unemployment rate, (ii) consumer confidence, (iii) money aggregate ECB M3, and (iv) the production price index. We show that these state variables reproduce the above mentioned benchmark yield curves remarkably well.

Section II. The Model

In this section[8] we will briefly outline the basics equations used in the present paper. As was already mentioned, we will closely follow the approach in Piazzesi (2010) and Cochrane (2005). Therefore, we assume that the bond price is given by the approximation,

$$P(N,t) = exp\left[A(N) - \boldsymbol{B}^T(N) \cdot \boldsymbol{x}\right], \tag{1}$$

where t denotes time, N the bond maturity and $\boldsymbol{x}$ is vector of the *state variables.* Similarly, A and $\boldsymbol{B}$ are functions of maturity obeying the boundary conditions $A(0) = \boldsymbol{B}(0) = [0 \ldots 0]$.

As is customary, the bond yield, i.e. $y(N,t)$, is obtained from Eq.(1) using the equality,

$$y(N,t) = -\frac{ln P(N,t)}{N} = \frac{\sum_j B_j(N) x_j - A(N)}{N}, \tag{2}$$

where ln denotes natural logarithm. Similarly, the risk-free interest or so-called short rate -which in our case is the EONIA- is assumed to be a liner function of the state variables,

$$r = \delta_0 + \sum_j \delta_j x_j \ . \tag{3}$$

where δ_0 and δ_j are parameters in the model which, as is explained in the Appendix, are obtained from a linear regression of state variables to the observed risk-free interest data.

As proposed by Piazzesi (2010) and Cochrane (2005), functions A and $\boldsymbol{B}$ can be obtained using the so-called *expectation approach*. Accordingly, the expected change of the bond price with an infinitesimal change of time dt is given by the equation,

[8] This section has been written with the intellectual contribution of Prof. Dr. Mario Jakas and adapted to some of the remarks given by Prof. Dr. Kenneth Singleton.

$$E_t\left(\frac{dP|_N}{P}\right)-\left(\frac{1}{P}\frac{\partial P}{\partial N}+r\right)dt=-E_t\left(\frac{dP|_N}{P}\frac{d\Lambda}{\Lambda}\right), \tag{4}$$

where Λ is the *discount factor*, $dP|_N$ denotes differentiation at constant maturity and $E_t(x)$ represents the expectation operator (see Cochrane, 2005). Furthermore, the equations governing the time evolution of the state variables and that of the discount factor are given by the expressions,

$$dx_i=\sum_j \phi_{i,j}\left(\bar{x}_j-x_j\right)dt+\sum_j S_{i,j}\left|\alpha_j+\boldsymbol{\beta}_j^T\boldsymbol{x}\right|^{1/2}dz_j \ , \tag{5}$$

and,

$$\frac{d\Lambda}{\Lambda}=-r\,dt-\sum_j b_{\Lambda,j}\left|\alpha_j+\boldsymbol{\beta}_j^T\boldsymbol{x}\right|^{1/2}dz_j \ , \tag{6}$$

where $\bar{x}_j$ is the average value of the state variable x_j over the pertinent period of time and, dz_j is a $(0,1)$-normal distributed random variable. Similarly, $\boldsymbol{\beta}_j^T\boldsymbol{x}=\sum_k \beta_{j,k}x_k$, and $\phi_{i,j}$, $S_{i,j}$, α_j and $\beta_{j,k}$ are additional parameters entering the model.

By introducing expression (1) into Eq.(4) and using equations (5,6) as well as well-known properties of the random variables (see the Appendix), one may arrive to equations,

$$\frac{dA}{dN}=-\delta_0-\sum_{k,j}B_k\phi_{k,j}\,\bar{x}_j+\sum_j\alpha_j\left(b_{\Lambda,j}+\frac{1}{2}\sum_k B_kS_{k,j}\right)\sum_k B_kS_{k,j} \ , \tag{7a}$$

and,

$$\frac{dB_i}{dN}=\delta_i-\sum_j\left[B_j\phi_{j,i}+\beta_{j,i}\left(b_{\Lambda,j}+\frac{1}{2}\sum_k B_kS_{k,j}\right)\sum_m B_mS_{k,j}\right] . \tag{7b}$$

This system of ordinary differential equations can be solved by numerical means and so, using Eq.(2), the bond yields are readily obtained.

Although a much more detailed description of the procedure used to calculate the bond yields are produced in the Appendix, for the time being, suffices it to say that, in the first place, the values of δ_0, δ_j, $\phi_{i,j}$, $S_{i,j}$, α_j and $\beta_{j,k}$ are obtained by least-square-error fittings of predictions in Eqs.(3,5) to available data. Secondly, the values of $b_{\Lambda,j}$ are calculated by resorting, again, to last-square-fitting of Eq.(2) to observed yields.

It must be noticed that, contrary to the assumptions in the previous paper by Vasicek (1977) and Cox-Ingersoll-Ross (1985), where either α, or $\boldsymbol{\beta}$ are assumed to be zero, in the present paper α and $\boldsymbol{\beta}$ are treated as free, fitting parameters and therefore both can be different from zero. In addition to that, we have opted for assuming that $S_{i,j} = \delta_{i,j}$, where $\delta_{i,j}$ is the Kroenecker's delta function.

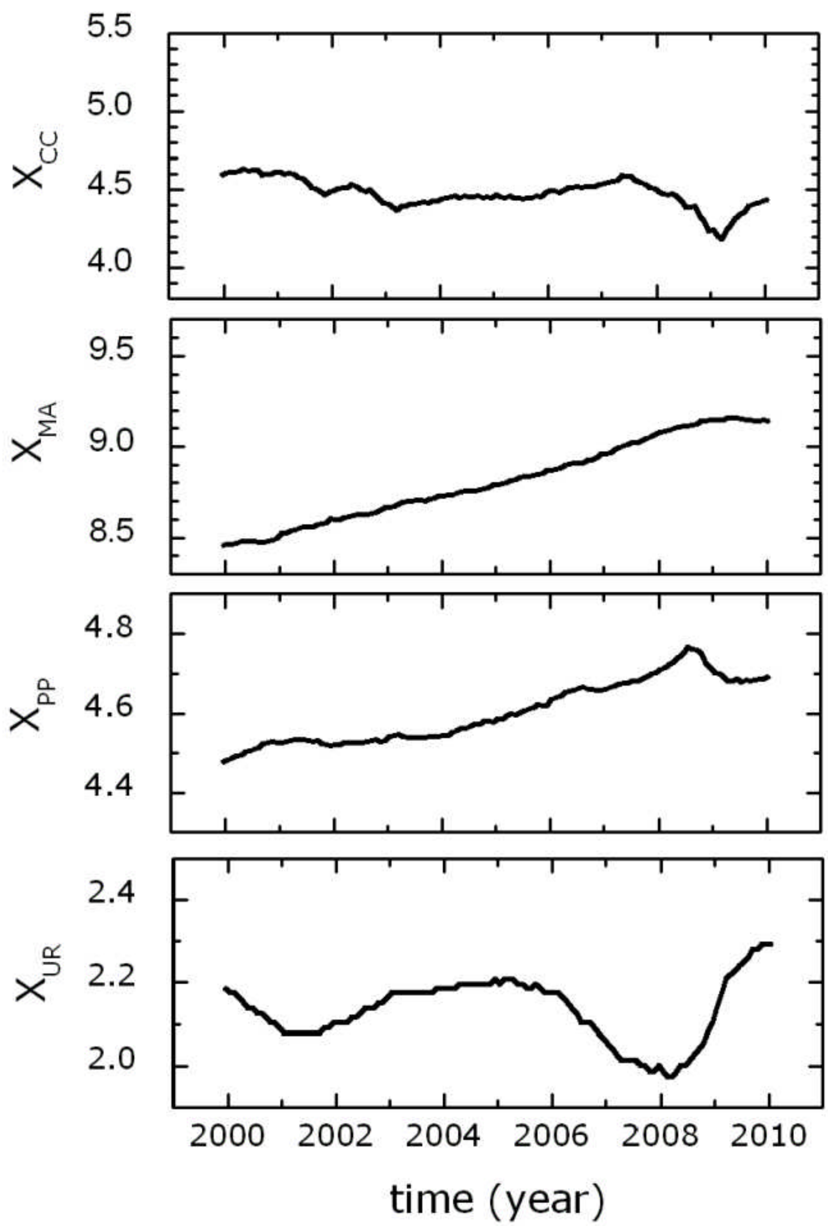
The results of numerically calculating the various expressions derived above are presented and discussed in the following section.

Section III. Results and discussion

Yield calculations.

As was already mentioned, in this paper we analyse the behaviour of the EONIA, 3-and 6-month Euribor rate as well as those of the 2, 5, 10, 15, 20 and 30-year German government bonds. Likewise, as state variables we used the natural logarithm of the euro-zone unemployment rate, the euro zone production price index, the ECB monetary aggregate M3 index, and the EU consumer confidence index, which will be denoted as x_{UR}, x_{PP}, x_{MA} and x_{CC}, respectively. Notice that the EU consumer confidence index exhibits negative values which complicate the use of natural logarithms. In order to overcome this we have simply added 100 to the observed values before taking logarithm. The data, which are published on a monthly basis, span a period of time ranging from December 1999 to January 2010. The aspects exhibited by the state variables are plotted in Figure 6-1.

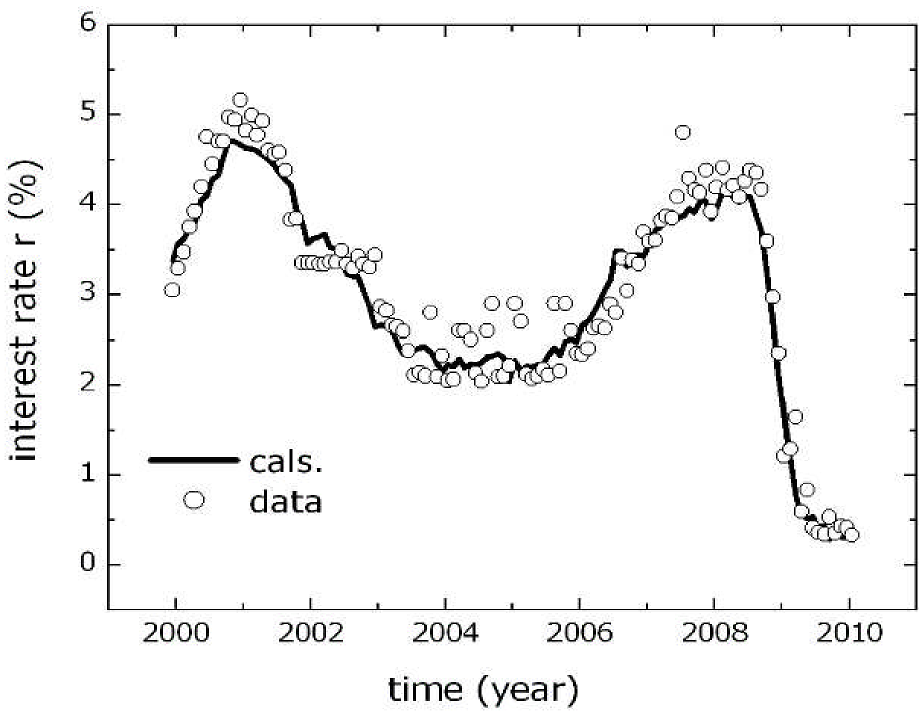
Figure 6-1 State variables used in this essay.
Namely, the natural logarithms of the unemployment rate (x_{UR}), production price index (x_{PP}), monetary aggregate (x_{MA}), and consumer confidence (x_{CC}).



In the first place, we calculate the parameters connecting the EONIA rate r and the aforementioned state variables as indicated in Eq.(3). After minimizing expression

(A1) we find that the δ_i-coefficient so obtained, reproduce the observed free-risk data with an accuracy of the order of 4% relative error, per point.

Figure 6-2 Risk-free interest or short rate (EONIA) as a function of time (starting on Dec. 1999). Open circles denote data, whereas the continuous line stands for the approximation in Eq.(3) using four states variables (see text).



Such an agreement becomes evident in Figure 6-2, where expression (3) appears to reproduce the main features of the EONIA rate remarkably well all over the entire range of time. Only a handful of points however deviate from theoretical curve, indicating that the majority of shocks in the EONIA are fairly well described by the chosen state variables, which confirms the results published in Jakas (2011 and 2012). The values of the coefficients in Eq.(3) that were used to plot the theoretical results in Figure 6-2 are listed in Table 6-1. Notice that the linear coefficients are multiplied by the mean-value of the corresponding state variable in order to compensate for the differences arising from the different absolute values of the state variables.

Table 6-1 Fitting coefficients in Eq.(3).
Linear coefficients, i.e. $\delta_i \ i = 1,\ldots,4$, multiplied by the mean-values of the state variables, i.e. $\bar{x}_{UR}$, $\bar{x}_{PP}$, $\bar{x}_{MA}$ and $\bar{x}_{CC}$.

δ_0	$\delta_1 \bar{x}_{UR}$	$\delta_2 \bar{x}_{PP}$	$\delta_3 \bar{x}_{MA}$	$\delta_4 \bar{x}_{CC}$
0.13	*-0.20*	*0.63*	*-0.61*	*0.068*

According to the results in Table 6-1, it turns out that the risk-free rate is more sensitive to euro zone production price and ECB monetary aggregate M3 indices rather than to unemployment rate, and shows nearly no sensitiveness to consumer confidence. Similarly, r appears to be negatively correlated with the monetary aggregate M3 whereas, r and the production price index is positively correlated. While r and unemployment show a slightly negative correlation.

Table 6-2 Value of the parameters entering the present calculations.
Excepting the means, which are directly obtained from data, all other figures are the result of the minimizing procedure described in text. Indices relate to the state variables as follows: 1: EU unemployment, 2: EU production price index, 3: ECB M3 and 4: EU consumer confidence.

	j \ i	1	2	3	4
$\bar{x}_i$	-	2.13	4.60	8.81	4.48
$\phi_{j,i}\,\bar{x}_i$	1	1.9	2.0	0.95	2.2
	2	0.94	4.7	0.67	0.53
	3	0.23	0.35	1.2	0.66
	4	1.1	0.54	1.3	3.3
$\sqrt{\left\lvert \alpha_i + \boldsymbol{\beta}_i^T \bar{\boldsymbol{x}} \right\rvert}$	-	0.035	0.029	0.026	0.058
$\beta_{j,i}\bar{x}_i \times 10^4$	1	0.95	1.4	6.1	1.8
	2	2.4	0.95	1.9	0.42
	3	4.6	0.85	0.81	0.16
	4	12.0	4.7	7.7	6.1
$\alpha_i \times 10^4$	-	1.9	2.6	0.46	3.4
$b_{\Lambda,i}\sqrt{\left\lvert \alpha_i + \boldsymbol{\beta}_i\,\bar{\boldsymbol{x}} \right\rvert}$	-	-0.022	-1.9	1.9	1.3

In Table 6-2 we list the values of the most relevant parameters obtained after performing minimization of expressions (A1-A4) and calculating the coefficients A and $\boldsymbol{B}$ according to the procedure described in the Appendix. As we have already done in Table 6-1, the differences arising from the different absolute values of the state variables are somehow removed by multiplying $\phi_{i,j}$ by $\bar{x}_j$, $\beta_{i,j}$ by $\bar{x}_j$, and $b_{\Lambda,i}$ by $\sqrt{\left\lvert \alpha_i + \boldsymbol{\beta}_i^T \bar{\boldsymbol{x}} \right\rvert}$.

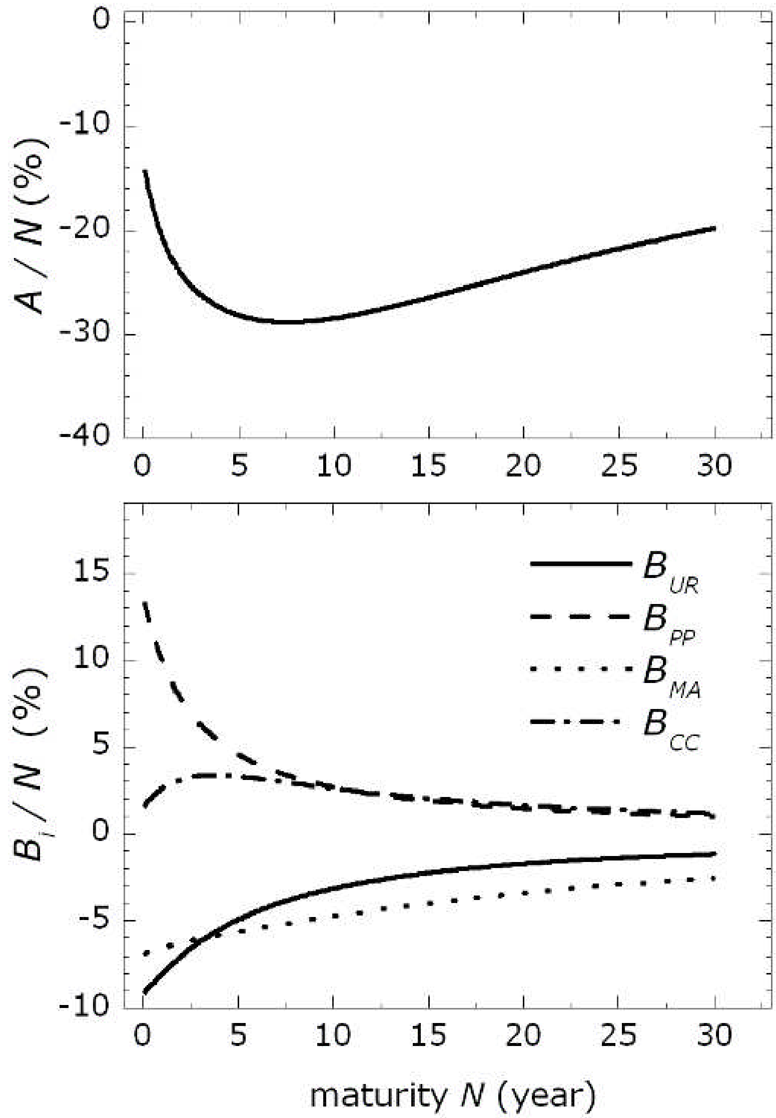
One can observe that the off-diagonal elements in matrix "$\phi_{i,j}\bar{x}_j$" are not small compared with those in the diagonal. This indicates that state variables are certainly interrelated. By observing that the values exhibited by $\beta_{i,j}\bar{x}_j$ and α_i are all comparable, we can unambiguously conclude that the case analysed in this paper does not seem to fit on any of the cases described in Vasicek (1977) and Cox-Ingersoll-Ross (1985).

Similarly, the results for $b_{\Lambda,i}\sqrt{\left|\alpha_i + \boldsymbol{\beta}_i^T \boldsymbol{x}\right|}$ clearly show that the random shocks in the unemployment rate does have little or nearly no significant impact on shocking the bond yields.

It must be mentioned however that the results in Table 6-2 are presented for the sole purpose of recording the results of the present calculations. However, one should be cautious about interpreting them since, according to our own experience some of these results may change, depending on the initial guess used in the minimization procedure.

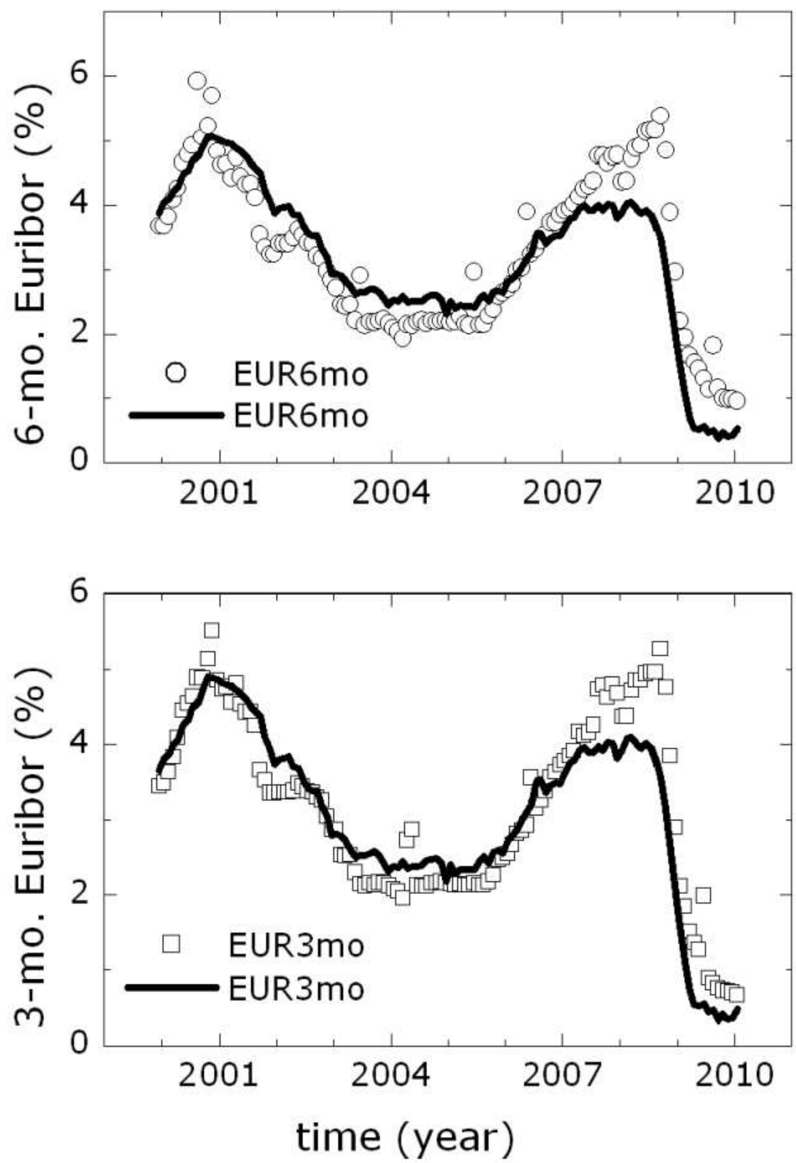
Figure 6-3 Coefficients $A(N)/N$ and $B(N)/N$ in Eq.(2), obtained by fitting to Euribor and German Government bonds.

It must be noticed that, in the legend of the lower figure, indices of coefficients B_i were replaced by abbreviation used for the corresponding state variables.



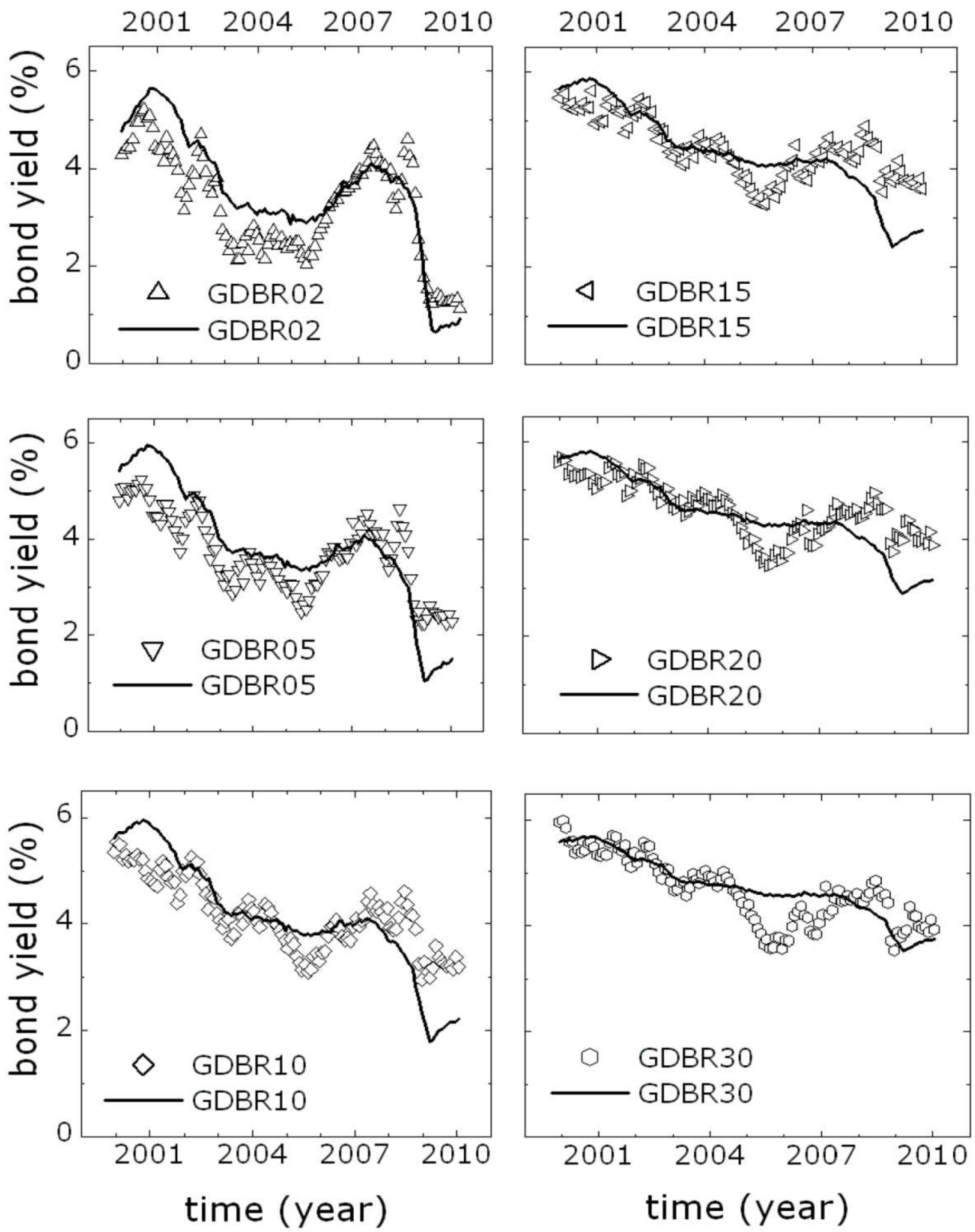
Once all the previous parameters are calculated one could obtain the bonds yields using the expression in Eq.(2). We start by plotting the results of calculating the three-and six-month Euribor rate. The results, which are plotted in Figure 6-4, show a remarkable agreement with data all over the whole range of time used in this paper. Our numerical calculations deviate from data with a mean relative error of the order of fifteen per cent. It must be mentioned though, that this is not at all an unexpected result since, in the light of the good agreement already found for the EONIA rate in Figure 6-2, these two, short-maturity bonds should be also well-described by the present model. This is so, because at small N-values, A / N and B_i / N are strongly dominated by the EONIA terms, namely δ_i.

Figure 6-4 Three- and six-month Euribor rate.
Data appear as open symbols, whereas continuous lines denote theoretical approximation given by Eq.(2).



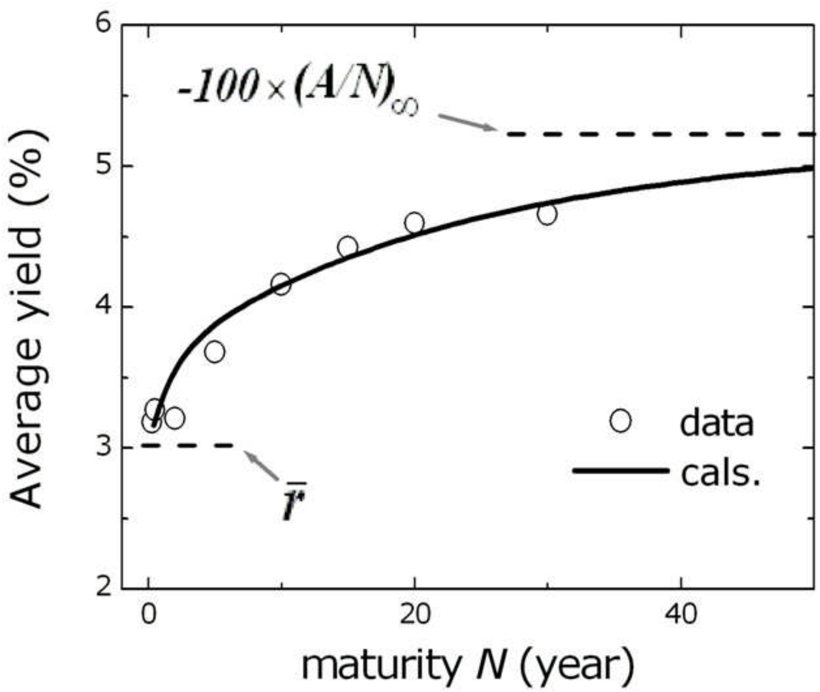
Remarkably however, a fairly good agreement is found between data and calculated yields for longer maturity bonds as those in Figure 6-5. There, one can see that two-, five-, ten, fifteen, twenty and thirty-years maturity German Government bonds appear to compare with calculations fairly well. According to the results of our numerical calculations, these bonds are reproduced within relative errors of the order of ten per cent.

Figure 6-5 Two-, five-, ten-, fifteen-, twenty- and thirty-year German Government Bonds. Data are denoted as open symbols, and the results of calculating Eq.(2) appear as continuous lines.



Another interesting aspect of the present calculations is depicted in Figure 6-6. It shows the mean-value of the Euribor and German Bonds taken over the period of time spanned by the used data, i.e. form Dec. 1999 to Jan 2010, and then plotted as a function of maturity N. The results show that, as is expected, the yield should increase with increasing maturity though, at rates which are larger for short N's and becoming nearly constant as maturity reaches values larger than, say, fifteen years.

Figure 6-6 Average bonds yield as a function of maturity.
Data appear as open circles, and calculations are denoted as a continuous line. Theoretical limiting values (see text), i.e. $\bar{r}$ (3.1%) and $-100 \times (A/N)_{\infty}$ (4.9%) are indicated by dashed lines.



Interestingly enough, these average yields should have limiting values which can be readily obtained from Eqs.(7a,b). For small maturities, one can readily verify that $dA/dN \cong -\delta_0$ and $dB_i/dN \cong \delta_i$, therefore, $A(N) \cong -\delta_0 N$ and $B_i(N) \cong \delta_i N$. As a consequence, according to Eq.(2) one has $y(0,t_n) = \sum_j \delta_j x_j^{(n)} + \delta_0$, which after taking average over time and taking into account Eq.(3) we may write $\bar{y}(0) = \sum_j \delta_j \bar{x}_j + \delta_0 = \bar{r}_{CAL}$, where $\bar{r}_{CAL}$ stands for the mean-value of the EONIA rate calculated using Eq.(3).

For a sufficiently large N, one has $dB_i/dN \cong 0$. This implies that $B_i(N) = k_1$ and, as a consequence, $dA/dN \cong k_2$, where k_1 and k_2 are two constant. Therefore $A(N) \propto k_2 N$ and, accordingly, as N approaches infinity, the leading term in calculating the average bond yield will be $-A(N)/N$, which can be denoted as $-(A/N)_\infty$.

As one can see in Figure 6-6, the two limiting values compare remarkably well with the average yields of the bonds used in this paper. This result also agrees with the known fact that bond yields approach asymptotically a constant value with an increase of maturity. Similarly, since $-A(N)/N >> B_i(N)/N$ for a large N, it means that, within the approximations in this paper, large maturity bonds must be less sensitive to the state variables an therefore, they will look nearly constant over time as one can observe in Figure 6-5.

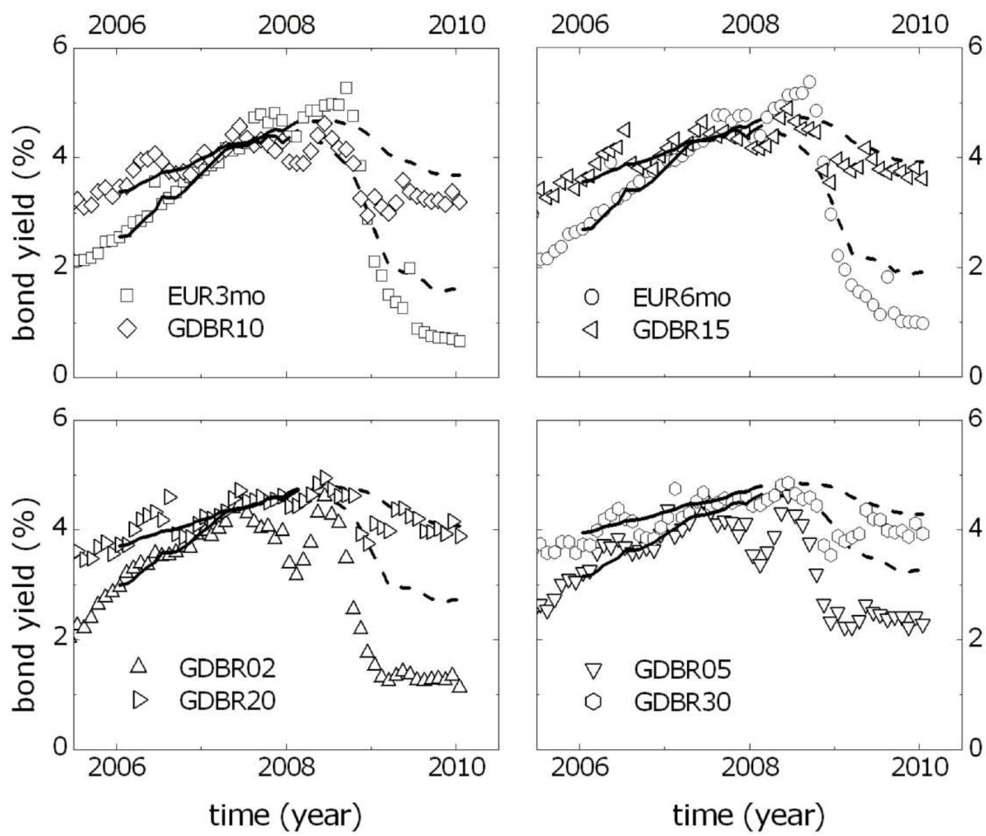
The Predictive power of the ATS-model

According to the approximation given by expression in Eq.(2), the bond yields depend on the coefficients A and $\boldsymbol{B}$ and the state variable $\boldsymbol{x}$. As we have already seen, $\boldsymbol{x}$ must be evaluated at the time bonds are evaluated, i.e. t_n, whereas A and $\boldsymbol{B}$ are calculated from our knowledge of $\boldsymbol{x}$ over a certain range of time within the near past. In other words, $\boldsymbol{x}$ provides all information relative to current state of the economy whereas A and $\boldsymbol{B}$ carry information about the state variables averaged out over the near past. Having said that, one may legitimately ask oneself: how actual must be the data used in calculating A and $\boldsymbol{B}$? In this subsection, we calculate the bond yields by using A and $\boldsymbol{B}$

obtained with previous state variables (excluding most recent ones), and only $\boldsymbol{x}$ assumes the present value.

The results of calculating the bonds yields using this approach is plotted in Figure 6-7. We call them *predicted values,* although these are not strictly speaking predictions since state variables are still the real ones, but the coefficients A and $\boldsymbol{B}$ are calculated using data ranging from 48 to 24 months older than the most recent ones, i.e. those of the 122nd month.

Figure 6-7 Predicting Euribor and German bonds yields values using current state variables However, coefficients A and $\mathbf{B}$ coefficients are calculated using data ranging from 48 to 24 months older than newest. Data are indicated by open symbols, predicted values are denoted as dashed lines and calculated yields within fitting range appear as continuous lines.



As one can see, the so-called predicted values compare remarkably well with data. Perhaps the worst agreement is observed for the 3 and 6 months Euribor rates and that of the two-year German bond. To some an extend, this is not at all unexpected since, as we have already stated, small-maturities bonds are more sensitive to current state variables than larger ones.

Section IV. Policy Implications

From an asset pricing perspective, the present results are perfectly in line with the theory. For example looking at parameter $\mathbf{B}_i / N$ from Figure 6-3, implies that an increase in unemployment results in a decrease in the short rate with a rather steepening effect along the yield curve, as the front end of the yield curve decreases faster than the long end. This is mainly explained by the Taylor's (1993) central bank policy rules as well as by modern asset pricing theory. From a Taylor's rule perspective, an increase in unemployment represents a fall in output, and a fall in inflationary risks which would lead to a fall in the policy rate. From the asset pricing theory side, an increase in unemployment produces an increase in the expected future aggregate marginal utility growth and hence it will cause a fall in the risk-free rate. In a similar fashion, it is possible to explain the effects of changes in the EU consumer confidence index in the yield curve. Since a decrease in this index represents an increase in aggregate marginal utility growth with the subsequent fall in the risk-free rate and, from a Taylor's rule perspective, it can be interpreted as a decrease in inflationary risk. Likewise, an increase in the production price index results in an increase in the risk free rate, as Taylor rules would expect central bank policy reaction as a consequence of an increase in inflation expectations. Finally, the monetary aggregate shows that an increase in the monetary aggregate would result in a fall in the short rate, in line with the classical IS-LM framework.

From a portfolio management perspective, the results in this paper show that EONIA, Euribor and the German government yields are at their lowest levels when unemployment is high, consumer confidence is low, production price index is low and lending is tight. A scenario that can be compensated by a central bank policy aimed at increasing the monetary aggregate to stimulate growth. On the contrary, yields are at their highest levels when unemployment is low, consumer confidence is high, production price index is high and monetary policy is lax, and partially compensated by central bank policy which tightens monetary aggregates and so, controlling inflation. When yields are at their highest levels the yield curves are flat and the representative investor will have the incentive to short the front end of the curve and take long positions in the long end of the yield curve. Alternatively, in times when yields are at their lowest levels the curve is at its steepest and a representative investor will have the incentive to take long positions in short maturity risk-free assets and short the long end. Either way, the gains in the front end are expected to more than offset losses in the long end. In addition, from a risk management perspective the positions in the long end act as a hedge against downside risks stemming from unfavourable and unexpected yield curve movements.

From a sovereign debt policy perspective, the previous analysis works differently, in a way that largely depends on the debt-roll over schedule. If yields are at their lowest levels, governments which enjoy the risk-free status should roll-over maturing bonds with longer maturities in order to ensure that the roll-over of debt does not happen in times when yields are high. In addition, in times when yields are high, which will coincide with a booming economy, governments –whose issuances enjoy a risk-free status– will have the chance to redeem short term issuances at lower prices and hence reduce the size of their total debt outstanding. By doing so they would generate capacity to increase indebtedness for the rainy days, thus in times when consumption growth is low and government's fiscal countercyclical engagement is desired.

Section V. Conclusions and Final Remarks

This paper shows that the affine term structure model performs very well as long as the state space variables selected have significant explanatory power over the short rate, which in this case is the European Overnight Index Average (EONIA). Specifically, this paper also shows that the state variables that can be used for this purpose are: (i) the EU unemployment, (ii) EU production price index, (iii) monetary aggregate ECB M3, and (iv) the EU consumer confidence index. These variables account for the EONIA rate over a period of time ranging from Dec. 1999 to Jan 2011 remarkably well. In addition to that, the proposed states variables not only are observed to work well during times of financial stability, but they also perform very well during periods of extended financial distress. We see that our calculations perform better on the front end of the yield curve rather than on the long end. This seems to be the case because front end yields are more sensitive to the state variables, whereas this dissipates as maturities become larger.

We have seen that yields are high in times when unemployment is low, consumer confidence, M3 and the price levels are high. During times of boom this yield curve exhibits a flat shape, with front end yields almost as high as the 30 year bonds and during times of recessions, the yield curve shows a steeper shape with long term yields exhibiting greater spreads versus short maturity bond yields. Our findings are thus in line with modern asset pricing theory, provide evidence of Taylor's (1993) central bank policy rules as well as of classical IS-LM models.

Finally, results in this paper lead us to conclude that current yield curve levels are indeed explained by current economic fundamentals and that its behaviour is in line with economic theory. This refutes press headlines pointing to speculative effects of market participants acting irrationally.

Appendix: Bond yields calculations

The scheme used in this paper to obtain the bond yields can be summarised as follows, [9]

i – Following Eq.(3), δ_0 and δ_j are obtained by minimizing the mean square error (MSE),

$$T_1 = \sum_{n,} \left(r^{(n)} - \delta_0 - \sum_j \delta_j x_j^{(n)} \right)^2 . \tag{A1}$$

This is carried out by the *amoeba* routine from Press et al. (1996), which can minimize an *N*-dimension function without using derivatives. It could be interesting to explore the use of so-called variance reduction techniques for bond pricing in affine models, as seen in for example in Rostan and Rostan (2012).

ii – Next, we calculate $\phi_{i,j}$'s by using Eq.(5) and assuming $dz_i^{(n)} = 0$. Therefore, we have to minimize the expression,

$$T_2 = \sum_{n,i} \left[dx_i^{(n)} - \sum_j \phi_{i,j} \left(\bar{x}_j - x_j^{(n)} \right) dt \right]^2 . \tag{A2}$$

iii – Having obtained $\phi_{i,j}$, we calculate α_j and $\beta_{j,k}$ by using Winner's condition $\sum_n \left(dz_i^{(n)} \right)^2 \cong N\, dt$, where N denotes the number of time-steps spanned by used data. Furthermore, since $S_{i,j} = \delta_{i,j}$, α_j and $\beta_{j,k}$ resulted from minimizing the expression,

$$T_3 = \sum_{n,i} \left\{ \frac{\left[dx_i^{(n)} - \sum_j \phi_{i,j} \left(\bar{x}_j - x_j^{(n)} \right) dt \right]^2}{\left| \alpha_i + \boldsymbol{\beta}_i\, \boldsymbol{x}^{(n)} \right|} - N\, dt \right\}^2 . \tag{A3}$$

[9] This section was possible thanks to the intellectual contribution from Prof. Dr. Mario Jakas.

iii – Finally, the $b_{\Lambda,j}$'s are calculated by finding the minimum of the expression,

$$T_4 = \sum_{k,n} \left(y^{(n)}(N_k) - \sum_j \frac{B_j(N_k)}{N_k} x_j^{(n)} - \frac{A(N_k)}{N_k} \right)^2 . \qquad \text{(A4)}$$

To this end, Eqs.(7a,b) are integrated along maturity by using the routine ODEINT of the Numerical Recipes package in Press et al. (1996). Obviously, such integration has to be carried out several times during the minimization of T_4, which is performed by the *amoeba* routine.

7. Discrete Affine Term Structure Models Applied to the Government Debt and Fiscal Imbalances

Section I. Introduction

This essay explores the use of discrete time affine term structure models applied to the theory of the price level and debt management in order to study the optimal term structure, and hence contribute to fiscal stabilisation policies and the optimal taxation approach. This essay makes use of affine term structure models in a similar set up as seen in the celebrated papers from Backus, Foresi and Telmer (1998 and 1996) and Backus, Telmer and Wu (1999). The essay applies the single and multifactor cases under Vasicek (1977) taking into account some of the developments seen in the latest affine term structure research such as Duffie and Kan (1996), Piazzesi (2010), Le, Singleton and Dai (2010) and Singleton (2006).

Starting point in this essay is the flow identity under the fiscal theory of the price level as seen in Cochrane (2001), Leeper (1995), Sims (1994), Woodford (1995, 1996) and Dupor (1997). With respect to the optimal taxation approach, this essay is a contribution to some of the developments achieved in Missale (1997), Faraglia, Marcet and Scott (2008), Angeletos (2002) and Buera and Nicolini (2004). The novelty of this essay is that uses affine term structure models to describe the path at which surplus change and hence affect the price level with the ultimately effect on yields. Hence, affine terms structure models can be used to link the theory of the price level, debt management and optimal taxation approach, in order to identify the optimal term structure.

An important critic to this essay is that no reference is being made to the vicious circle existing between the increase in unemployment and the deterioration of government surplus. Thus an increase in unemployment can lead to a subsequent adjustment in government expenditure and in turn also result in a deterioration of economic agents' expectations with further falls in agents' revenues. Hence, no reference is made to the so-called Keynes' paradox of thrift (or paradox of saving, see: John Maynard Keynes, The General Theory of Employment, Interest and Money, Chapter 7, p. 84). However, the intention of this essay is not to specifically address this theory but rather to show that affine term structure models can be used as an additional tool-kit for the analysis of government's structural deficits and their sensitivities to macroeconomic variables. Hence, under this set-up ATSM is more a tool for diagnosing rather than for forecasting.

In addition, it is shown that though it works well for some countries such as Germany, Italy, France, Spain, Ireland and Portugal, it does not work as well for Greece, as shocks in unemployment are mean reversing with a low speed of adjustment and hence have – statistically speaking– a low impact on Greece's government surplus. However, this is not enough reason for dropping the validity of the model. Instead, more research is needed in order to determine what other variables apart from unemployment have an impact on government expenditure, thus the use of ATSM on the analysis of government deficits has a generous follow-up potential and the use of the unemployment rate variable for this case is more exemplary than anything else.

Finally, the model assumes that macroeconomic variables follow a mean reversing stochastic process which taking into account the latest developments in the European sovereign debt crises, could question the validity of this proposition. In this essay, the analysis has been limited to the mean reversing proposition mainly for convenience and simplicity's sake however, the affine term structure literature accounts for a rich variety of stochastic processes which can include jump diffusions or links to extreme event theory, which are rather part of a different paper.

The use of data since 2008 is for convenience only, as we link this data to inflation linked financial instruments of various maturities and these are only available on its full since 2008 onwards. Before 2008, the financial markets were of the opinion that EMU would guarantee all EU government bonds without distinction. For instance, before 2008, the credit spreads of Greek bonds versus German treasuries were minimal. However, as agents learn that this is not the case, GIIPS spreads widen and hence only then this essay becomes relevant, thus only after this "structural" change. Therefore, is worth differentiating between before and after 2008. This approach is in line with latest research as seen in De Grauwe (2011a,b), De Grauwe and Ji (2013) as well as Beirne and Fratzscher (2013).

This essay is organised as follows: Section 2 basic and new concepts are introduced and discussed; section 3 an affine terms structure model is introduced; section 4 a recursive solution is presented for two possible scenarios: when the theory of the price level is at work and when the theory is not at work; in section 5 and subsequent subsections the model is extended in order to show the path of surplus shocks on total notional debt outstanding; the path of shocks from total notional debt outstanding on the price level and credit spreads and; the path of shocks stemming from government revenues and government expenditure and its effect on primary surplus. Section 6 calibrates some of the most relevant models already discussed in previous section with real data and present results, section 7 discusses the policy implications of our findings and outlines main conclusions and final remarks.

Section II. Recalling some basic concepts and introducing new ones

The flow identity depicts that surplus equals redemptions minus net new issuances.

$$S_t^N = P_t^N B_t^N - P_t^{N+1} B_t^{N+1} \tag{1}$$

For $S_t^{(N)}$ being the net primary surplus in t cumulated in period N, $P_t^{(N)}$ being the redemption price in t for a zero coupon bond with maturity N and remaining time to maturity N= 0. $B_t^{(N)}$ depicts the notional amount of a bond in time t with maturity N and remaining time to maturity N=0. Analogously, $P_t^{(N+1)}$ being the price of a new zero bond issued in t with remaining time to maturity N+1 and $B_t^{(N+1)}$ depicts the new bond's notional amount with remaining maturity N+1 in t. Notice that it is assumed that at origination all bonds have same maturity profile, it is denoted with N and with N+1 in order to differentiate when a bond is maturing or when a bond is a new issue, as for N=0 implies that $P_t^{(N)} B_t^{(N)}$ are the maturing amounts and for N+1=1 implying that the new issue amount of $P_t^{(N+1)}B_t^{(N+1)}$ will matured in t+1 and, at that point, the maturity of the bond will be N=0. The analysis should not be limited to total notional debt outstanding and surplus, but also include all other assets in the economy for which the government acts as a guarantor. This is because as these assets deteriorate together with surplus, the government is also force to increase its issuance in order to support asset prices, particularly those from the banking system. For the sake of simplicity the analysis here is limited to surplus shocks, but the reader can also apply it to shocks to assets in the banking system for which the government acts as a guarantor.

Equation (1) shows that if surplus $S_t^{(N)} > 0$ implies that $P_t^{(N)}B_t^{(N)} > P_t^{(N+1)}B_t^{(N+1)}$, hence the government is reducing total debt outstanding, as redemptions $P_t^{(N)}B_t^{(N)}$ are greater than the new bond issuances $P_t^{(N+1)}B_t^{(N+1)}$. Alternatively, if surplus $S_t^{(N)} < 0$ (thus is a deficit) implies that $P_t^{(N)}B_t^{(N)} < P_t^{(N+1)}B_t^{(N+1)}$ which means the government is increasing its total debt outstanding, as the new bond issuances $P_t^{(N+1)}B_t^{(N+1)}$ required to be greater than redemptions $P_t^{(N)}B_t^{(N)}$ in order to have enough funding to cover deficits.

Thus, a deterioration of the net primary surplus – hence an increase in the government's deficit – would require an increase in net new issues. An increase in net new issues is necessary in order to roll over the maturing debt whilst still be able to cover the increase in current financing requirements. Should the new issue price $P_t^{(N+1)}$ deteriorate, then the government will be forced to increase the new-issue notional amount $B_t^{(N+1)}$ in order to compensate for the fall in the price and thus be able to gather enough funds to pay back redemptions $P_t^{(N)}B_t^{(N)}$ and finance its deficit $S_t^{(N)}$.

It is assumed that the government only increases debt if it is strictly necessary, hence (1) would imply that should $S_t^{(N)}$ improve by exhibiting an increase in surplus, the government would reduce total debt outstanding as a consequence of a decrease in its funding requirements. If we think about investors' expectations for a 1 period forward at t+1, from equation (1) it is possible to intuit that the redemption price and redemption amount are known values at time t. However and, what the market participants do not know is the new issue cash equivalent of next debt roll-over in period t+1 of $P_{t+1}^{(N+1)}B_{t+1}^{(N+1)}$.

In fact, for the case of governments under financial distress investors are wary about their ability to issue new debt in times of low consumption growth and thus might not believe that they would obtain access to funds enough to redeem the maturing debt of $P_t^{(N)}B_t^{(N)}$ and finance their deficits $S_t^{(N)}$. The idea is that new issuances need to be sufficient so that (1) equates without forcing the government to issue at unfavourable prices $P_t^{(N+1)}$ and hence at a higher yield. Notice that the government avoids default by accepting lower prices and increasing debt outstanding if necessary. In order to depict this more precisely, (1) can be re-arranged, by moving $S_t^{(N)}$ as an explanatory variable to the right hand side and $P_t^{(N+1)}B_t^{(N+1)}$ to the left as endogenous, which can be specified as follows

$$P_t^{N+1}B_t^{N+1} = P_t^N B_t^N - S_t^N \tag{2}$$

Which means that if $\Delta S_t^{(N)} > 0$, then inevitably $\Delta\left(P_t^{N+1}B_t^{N+1}\right) < 0$, as a result of a fall in financing requirements.

The nominal price of a zero coupon bond will contain information about the price level as well as information about the real interest rates. Assuming that the real interest rates remain constant, it could be said that an increase in the price level will result in an increase in yields with the subsequent fall in the bond price. A possible specification could be

$$y_t^{(N+1)} = -\frac{\ln E\left[P_t^{N+1}\right]}{N+1} \tag{3}$$

Taking into account that yields contain information about real interests and expected inflation implies:

$$y_t^{(N+1)} = y_t^r + \ln\left(\frac{\Pi_{t+1}}{\Pi_t}\right) \tag{4}$$

For $y_t^{(N+1)}$ being the nominal yield of the zero coupon bond with maturity N+1 comprising the sum of the real interest rate y_t^r and the rate of growth of the price level or inflation being $\ln(\Pi_{t+1}/\Pi_t)$. Notice that by normalising current price level Π_t to 1 it makes no difference if the level or change in the level is used. The relationship resulting from (4) and (3) says that an increase in the price level would subsequently result in an increase in nominal yields with the subsequent fall in the present value of a new issue and hence increase government's costs of financing.

However, governments cannot always influence monetary policy which means they cannot determine the path of inflation, for instance, when central bank acts independently. In this case (4.1) would require a modification, whereby yields are obtained from a benchmark curve or short rate usually a risk free reference plus a credit spread. This could be specified as follows:

$$y_t^{(N+1)} = r_t^f + \theta_t^{(N+1)} \tag{5}$$

For $r_t^f = y_t^{(r)} + E\left[\ln \dfrac{\Pi_t}{\Pi_{t-1}}\right]$

Equation (5) describes the relationship between yields $y_t^{(N+1)}$, the short rate r_t^f , real yields $y_t^{(r)}$ and the spreads $\theta_t^{(N+1)}$. Now, we know that the fiscal theory suggests that governments' choice of how to finance its debt play an important role on the determination of the time path of the inflation rate. However, if government debt is issued in a foreign currency or in a currency which governments' have no or little control, then the theory of the price level is less likely to be at work. Instead, governments' choice of how to finance its debt will have a subdued role on the determination of the time path of the inflation rate but rather an important role in the determination of the time path of the credit spread. This means that when the theory of the price level is at work, then (4) describes best yields behaviour as a function of log price level changes and when governments issue in a currency which they cannot control, equation (5) will best describe yields behaviour.

The ability of the government to issue new debt in t with maturity N+1without incurring in a significant deterioration of its financing costs will depend largely on the market's view about government's ability to raise new funds in a future date, let us say t+1, which will also depend on the size of government's deficit at t+1. Why? Because the market's appetite to lend today will depend on their view about getting their investment redeemed in the future. To formulate this more precisely, I will adapt (2), and show that current issue price depends on the market's view about government's ability to issue new debt or to roll over the maturing one which will largely depend on the market's expected future government yields and thus government's surplus, hence

$$E\left[P_t^{N+1}\right]B_t^{N+1} = E\left[m_{t+1}P_{t+1}^{N+1}\right]B_{t+1}^{N+1} - E\left[m_{t+1}S_{t+1}^{N+1}\right] \tag{6}$$

Equation (6) shows that net present value of total debt outstanding in t with maturity N+1 and hence maturing in time t+1 should equal the net present value of the new bonds to be issued in t+1 minus the present value of expected future surplus. This is because in this model governments issue new debt in order to repay maturing one. For m_{t+1} being the stochastic discount factor which is related to one-period bond yields (or the short rate) inversely as follows,

$$y_t^{(1)} = -\ln E\left[m_{t+1}\right] \tag{7}$$

Between equations (2) and (6) there are important theoretical differences which are worth mentioning. Equation (2) says that the net present value of total debt outstanding depend on current surpluses and current redemptions. If surplus deteriorates, the government is required to rise more funding. However, (6) is instead saying that current bond prices depend not only on the expected future surplus, but also on markets' expectation on government's ability to issue new debt in the future, as the ability to issue new debt in the future will determine the size of redemptions of maturing bonds. This is an important difference, thus we are not saying anymore that the present value of total current debt outstanding depends on the present value of expected future surpluses but that also depends on expected future new bond prices. This means that in the hypothetical case that despite that surplus is expected to deteriorate in the future, current bond prices can still remain unaffected as long as the market continues to believe that the government will still be capable of issuing new debt, and hence be able to roll over maturing debt. I will show however, that this happens only if government's deficit remains within sustainable levels above which the bond becomes a risky asset and hence a Ponzi scheme.

Without loss of generality I will change (6) slightly as follows:

$$E\left[P_t^{N+1}\right] = E\left[m_{t+1}P_{t+1}^{N+1}\right]B_{t+1}^{N+1} - E\left[m_{t+1}S_{t+1}^{N+1}\right] \quad (8)$$

In equation (8) above, $B_t^{(N+1)}$ equals 1 so that for simplicity's sake only $B_{t+1}^{(N+1)}$ is left in the expression. (8) shows that if $E\left[m_{t+1}S_{t+1}^{N+1}\right] = 0$ then current present value $E\left[P_t^{N+1}\right]$ will equal the net present value of total new issuances rolling over in t+1, which has been specified as $E\left[m_{t+1}P_{t+1}^{N+1}\right]B_{t+1}^{N+1}$ and would imply that $B_{t+1}^{N+1} = 1$ so that $E\left[P_t^{N+1}\right] = E\left[m_{t+1}P_{t+1}^{N+1}\right]$. Alternatively, if $E\left[m_{t+1}S_{t+1}^{N+1}\right] < 0$, thus if the government incurs a deficit, then it would necessarily need to be that $B_{t+1}^{N+1} > 1$.

Finally and recalling some basics, we know that the expected price at t with maturity N+1 of a bond that redeems at t+1 is usually specified as follows:

$$E\left[P_t^{N+1}\right] = E\left[m_{t+1}P_{t+1}^{N}\right] \quad (9)$$

And by applying natural logarithms yields:

$$\ln\left[P_t^{(N+1)}\right] = \ln\left[m_{t+1}\right] + \ln\left[P_{t+1}^{(N)}\right]. \quad (10)$$

Equations (8) and (9) show that there are two ways of solving this, as the right hand side of (8) also equals the right hand side of (9). This paper will start solution for equation (9) and workout (8) thereafter. Solution for (9) would be rather straight forward, as solution for (8) requires a definition of the time path for each of the components such as surplus, revenues, expenditure, total notional debt outstanding and the price level.

<u>Section III. The theory of the price level and risk-free assets: the model</u>

Equation (9) will be solved using an affine term structure model as in Backus, Foresi and Telmer (1998) and a Vasicek (1977) stochastic process for the state variables.
As seen in most recent affine term structure literature log prices can be specified as a linear function of a state vector x_{t+1} as follows:

$$-\ln\left[P_{t+1}^{(N)}\right] = A(N) + B(N)' x_{t+1} \tag{11}$$

For $A(N)$ being a scalar, $B(N)'$ a $1\times k$ vector of coefficients and x_{t+1} a $k\times 1$ vector of state variables. Note that the transpose of a vector or matrix is specified with a " ' ". Equation (11) is only a guess, as the functional form is not known. However, the literature appears to have generally accepted this as seen in Piazzesi (2010), Singleton (2006), Cochrane (2005), as well as in Backus-Foresi-Telmer (1996) and (1998) and seminal papers of Duffie and Kan (1996).

From our guess shown in (11) we wish to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$. These parameters are obtained by linking observable yields to an observation equation describing the behaviour of a state space vector. This can be done by combining equations (3) at t+1 with (11) which boils down to

$$y_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{B(N)'}{N} x_{t+1} \tag{12}$$

Thus the short rate could be specified as follows:

$$y_{t+1}^{(1)} = A(N=1) + B(N=1)' x_{t+1} \tag{13}$$

Empirically, equation (13) would look like

$$y_{t+1}^{(1)} = \gamma_0 + \gamma_1 ' x_{t+1} \tag{14}$$

It is also needed to specify the stochastic process for x_{t+1} as well as for the stochastic discount factor shown in (7). A good starting point is to use the pricing kernel à la Backus-Foresi-Telmer (1998) which here is combined with the Vasicek (1977) process. A possible specification would be like:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1} \tag{15}$$

$$-\ln[m_{t+1}] = \delta + y_t^{(1)} + \lambda' \varepsilon_{t+1} \tag{16}$$

Equation (15) describes the stochastic process of the independent state variables. This is the usual mean reversing process whereby Δx_{t+1} is likely to be negative if x_t is above $\bar{x}$ and, is likely to be positive if x_t is below its mean $\bar{x}$. x_t and $\bar{x}$ are both k-dimensional vectors. Φ is a $k \times k$ matrix of diagonal elements Φ_i which represent the speed of adjustment at which each of $x_{i,t}$ elements reverse to their means. σ_x is a diagonal $k \times k$ matrix comprising the volatility of the state variables. ε_{t+1} is a k-vector of shocks moving x_t away from $\bar{x}$ and with $\varepsilon_{i,t+1}$ elements being normally distributed with mean zero and variance 1.

Equation (16) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (15) was originally the univariate Vasicek (1977) case. In this essay we transform this specification and adapt it for the multifactor case of a k-dimensional vector of state variables as in Jakas (2012). Same as in Backus-Foresi-Telmer (1998) δ is specified as follows:

$$\delta = \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 \qquad (17)$$

Clearly, specification (17) is fortuitous, the only aim is to normalise the stochastic discount factor so that it becomes the inverse of the short rate. Notice that with (17), now (16) has the following conditional means and variance:

$$E\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - y_t^{(1)} - \lambda'\varepsilon_{t+1}\right] = -\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - y_t^{(1)} \qquad (18)$$

$$Var\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - y_t^{(1)} - \lambda'\varepsilon_{t+1}\right] = \sum_{i=1}^{k}\lambda_i^2 \qquad (19)$$

And assuming $E[\ln x] = \mu(x) + \frac{1}{2}\sigma^2(x)$, which yields

$$E[\ln m_{t+1}] = -y_t^{(1)} \qquad (20)$$

It would be assumed that the price level Π_t is a function of total notional debt outstanding $B_t^{(N)}$ and total debt outstanding increases as $S_t^{(N)}$ deteriorates, hence as surplus turns into deficit. Finally, $S_t^{(N)}$ depends on the state vector x_{t+1}, so that macroeconomic shocks affecting government surplus will have an effect on the price level only if surplus shocks increase total debt outstanding and ultimately affecting the price level. These relationships could be specified as follows:

$$y_t^{(1)} = \alpha_0 + \alpha_1 y_t^r + \alpha_2 E\left[\ln\left(\frac{\Pi_t}{\Pi_{t-1}}\right)\right] \qquad (21)$$

$$\Pi_t\left(B_t^N\right) = \eta_0 + \eta_1 B_t^N\left(S_t^N, S_t^{N*}\right) \qquad (22)$$

$$B_t^N\left(S_t^N, S_t^{N*}\right) = \varphi_0 + \varphi_1\left[S_t^{N*} - S_t^N\left(x_t\right)\right] \qquad (23)$$

$$S_t^N\left(x_t\right) = \beta_0 + \beta_1' x_t \qquad (24)$$

Equation (21) is nothing but a way of estimating the unobservable parameters specified in (4), thus the coefficient for the real interest rate y_t^r and the coefficient for the expected inflation are estimated via observed inflation-linked swap data. Equations (22) and (23) are our empirical interpretation of the theory of the price level linked to the short rate. Hence, here we describe that a deterioration of government's budget deficit beyond a certain unobservable limit results in a systematic increase in governments total debt outstanding and hence in an increase in the price level Π_t, as this theory is at work when central bank is not fully independent and governments are able to monetise their deficits.

Notice that for the case where the theory of the price level is not at work because the government does not have control over the monetary policy, it would imply that (21) and (22) need to be adapted to account for the credit spreads. In this case, (21) and (22) are transformed to (25) and (26) as specified below. Notice that the use of this numbering for the equations is fortuitous, as the intention is to call the reader's attention to the idea that (25) and (26) are a derivation of (21) and (22). If the theory of the price level is not at work it is because governments cannot decide over the path of inflation and instead any increase in deficits result in a deterioration of the credit spreads, instead of an increase in the price level.

$$\theta_t^{(1)}\left(B_t^N\right) = \eta_0 + \eta_1 B_t^N\left(S_t^N, S_t^{N*}\right) \tag{25}$$

$$y_t^{(1)} = \alpha_0 + \alpha_1 y_t^r + \alpha_2 E_{t-1}[\pi_t] + \alpha_2 \theta_t^{(1)}\left(B_t^N\right) \tag{26}$$

For $E_{t-1}[\pi_t] = E\left[\ln\left(\Pi_t / \Pi_{t-1}\right)\right]$ being the expected inflation obtained from inflation-linked (IL) swaps.

Notice that (26) proposes a possible way of estimating (5), under the assumption that the true risk free is observable from IL-Swap data. Substituting (22) in (21) (or (25) in (26) and then continue the complete chain of substitutions through (22) to (24) it is possible to show that yields are a function of a state space which I summarised below

$$y_t^{(1)} = \psi_0 + \psi_1' x_t \tag{27}$$

Equations (21) to (27) will depend on the difference between the level of $S_t^{(N)}$ and an unobservable *sustainable* $S_t^{(N)*}$ shown in (23). Thus for any level of government deficit where $E_{t-1}\left[S_t^{(N)}\right] < 0$ and $S_t^{(N)*} < 0$ restricted to $E_{t-1}\left[S_t^{(N)}\right] < S_t^{(N)*}$ it would result in $\varphi_1 < 0$ and $\eta_1 < 0$, which implies that the fiscal theory of the price level is at work and thus any shocks in aggregate demand that results in a deterioration of government finances as a consequence of such surplus shocks will affect the price level and result in an increase in yields. Therefore, the short rate is governed by (27) and not by (14). If the price level is not at work and if surplus shocks have an effect in the total debt outstanding, the same logic applies but instead (21) and (22) are replaced by (25) and (26) because government's choice of how to finance its debt has an effect on the determination of the path of credit spreads and not on the determination of the path of price level.

Notice that for any $E_{t-1}\left[S_t^{(N)}\right] \geq S_t^{(N)*}$ would result in $\varphi_1 = 0$ and $\eta_1 = 0$, which means that any expected deficit within a given level of sustainability will have no effect on the price level and hence the fiscal theory of the price level would be subdued so that the short rate will be governed by (14) instead of (27). When the short rate is governed by (14) instead of (27) only then it could be said that the government bond acts as a hedge for times when aggregate marginal utility growth is high and government bond prices are negatively correlated to aggregate consumption growth. On the contrary, if the theory of the price level rules, government bond prices will be positively correlated to aggregate consumption growth mainly because a deficit that is perceived as unsustainable is expected to have an effect on the price level. If governments cannot monetise their deficits any deterioration of their surplus will only be financed with increases in total debt outstanding which will not generate inflation but increases on the credit spread.

When the fiscal theory of the price level is at work, the government bond is considered a risky asset and hence there would be no difference between contingent and non-contingent bond payoffs, as they would all be contingent to the price level or the credit spreads. However, if the fiscal theory of the price level is not at work, the government bond is risk-free and the difference between contingent and non-contingent bonds does mater.

Section IV. Solving when the theory of the price level is at work and when is not

When the theory of the price level is at work the signs of the coefficients depicted in (27) will rule. However, when the theory of the price level is subdued, the signs of the coefficients will be governed by (14). For which the solution assuming that (27) holds for a set of coefficients specified in (12) boils down to:

$$A(N+1)=\psi_0 + A(N)+B(N)'\Phi\bar{x}+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 -\left(\lambda'+B(N)'\sigma_x\right)^2\right) \tag{28}$$

$$B(N+1)'=\left(\psi_1'+B(N)'(I-\Phi)\right) \tag{29}$$

If the reader is interested in the algebra on how to obtain equations (28) and (29), please refer to Appendix, Section I.

The solution is obtained by computing the present value recursively using (10) for some guess of coefficients from (11). Since $P_{t+1}^{(N)}=1$ and $A(N=0) = B(N=0)' = 0$, which means this can be solved recursively, as for 1 period would imply $A(N=1) = \psi_0$ and $B(N=1)' = \psi_1'$ which means that equals the short rate as described in (27). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (10) to compute the present value of an N+1 maturity bond.

All is needed is to line up (28) and (29) into (12) and solve numerically by fitting the curve to the observed yields by adjusting λ's for a given choice of maturities. Parameters ψ_0 and ψ_{1i} are free and obtained empirically via OLS and the signs for parameters $B(N)_i$ in (12) depend on ψ_{1i}.

When the theory of the price level is not at work, equations (28) and (29) change, and this is because equation (14) is at work instead of equation (27) and by applying the same algebra discussed in Appendix Section I it would yield:

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - (\lambda' + B(N)'\sigma_x)^2\right) \quad (30)$$

$$B(N+1)' = (\gamma_1' + B(N)'(I - \Phi)) \quad (31)$$

When governments issue in a foreign currency or in a currency in which they have no or little control the theory of the price level is less likely to occur, however, the same forces which have an effect on surplus shocks will result instead in an increase in the credit spreads. So that in a similar fashion to equations (21) to (27) to obtain (28) and (29) replacing the price level for the credit spread as endogenous variable would give us similar analytical results. All is needed is to account for (25) and (26).

Thus, macroeconomic shocks that result in a deterioration of government's net primary surplus beyond a sustainable level can result in increases in total notional debt outstanding which ultimately either result in increases in yields due to increases in the price level or results in increases in yields due to deterioration of the credit spreads. Either ways result in higher yields and hence in a further deterioration of government finances.

Section V. Solving by analysing fiscal shocks on surplus debt rollover risk

In order to solve equation (8) it is required to have a model describing the behaviour of the effects of: 1) surplus shocks on total notional outstanding; 2) the effects of innovations in total notional debt outstanding on the price level or credit spreads and; 3) modelling the path of government surplus by analysing macroeconomic innovations on government revenues and government expenditure. So this section will be organised in three subsections, outlining the above mentioned points.

Section V.I. The effects of surplus shocks on total notional debt outstanding

This chapter would make use of affine terms structure models to describe the path of surplus shocks and its effect on total notional debt growth.

$$E\left[B_t^{(N)} g_{t+1}\right] = E\left[B_{t+1}^{(N)}\right] \tag{32}$$

Where $B_t^{(N)}$ being the current notional debt outstanding in time t which is expected to grow at g_{t+1} resulting in a total nominal debt at t+1 equal to $E\left[B_{t+1}^{(N)}\right]$. Equation (32) describes the path at which $B_t^{(N)}$ would need to grow from t to t+1, for which a *growth kernel* will be used similar to a pricing kernel as used in the affine term structure literature.

Thus g_{t+1} will be assumed that is a sort of growth kernel at time t+1 for period N = t+1 so that if $g_{t+1} > 1$ it would imply that nominal debt increases during period t to t+1, if $g_{t+1} < 1$ it would imply that nominal debt decreases between t to t+1. Rearranging (32) as a function of current total notional debt outstanding would yield

$$E\left[B_t^N\right] = E\left[\left(g_{t+1}\right)^{-1} B_{t+1}^N\right] \tag{33}$$

Using the natural logarithm notation (33) would look like

$$\ln\left[B_t^N\right] = \ln\left[B_{t+1}^N\right] - \ln\left[g_{t+1}\right] \tag{34}$$

Now we need to specify $\ln\left[B_{t+1}^{(N)}\right]$ and $-\ln\left[g_{t+1}\right]$ for which a similar expression will be used as in Backus, Foresi and Telmer (1998) and which for simplicity's sake and without loss of generality has been specified as follows

$$-\ln\left[g_{t+1}\right] = \frac{1}{2}\lambda^2 + \Delta b_t^{(1)} + \lambda\varepsilon_{t+1} \tag{35}$$

$$\Delta b_t^{(1)} = \varphi_0 + \varphi_1 \Delta s_t^{(1)} \tag{36}$$

For λ depicting the sensitivities at which $-\ln\left[g_{t+1}\right]$ changes due to shocks in ε_{t+1}. Equation (35) is fortuitous, the notation is on purpose so that under normality $E\left[\ln g_{t+1}\right]$ equals $-\Delta b_t^{(1)}$. In equation (36) $-\Delta b_t^{(1)}$ represents the debt growth rate in 1 year as a consequence of surplus shocks, φ_0 and φ_1 are coefficients and $\Delta s_t^{(1)}$ depicts the one period government's cumulated total primary surplus in time *t*.

Under these assumptions a government's total notional debt outstanding is expected to grow as follows

$$B_{t+1}^{(N)} = B_t^{(N)} \exp\left[\Delta b_{t+1}^{(1)} \times N\right] \tag{37}$$

For simplicity's sake and without loss of generality it is assumed $B_{t+1}^{(N)} = 1$. Applying natural logarithms to (37) and rearranging yields

$$\Delta b_{t+1}^{(N)} = \frac{1}{N} \ln\left[B_{t+1}^{N}\right] \tag{38}$$

It is also assumed that

$$-\ln\left[B_{t+1}^{(N)}\right] = A(N) + B(N)\Delta s_{t+1}^{(1)} \tag{39}$$

Notice that the functional form shown in (39) is only a guess that works quite well when plugging (39) in (38), which results in

$$\Delta b_{t+1}^{(N)} = -\frac{A(N)}{N} - \frac{B(N)}{N}\Delta s_{t+1}^{(1)} \tag{40}$$

The final assumption here is that surplus $\Delta s_{t+1}^{(1)}$ follows a Vasicek (1977) process as follows

$$\Delta s_{t+1}^{(1)} = \Delta s_t^{(1)} + \phi\left(\bar{s}_{\Delta s}^{(1)} - \Delta s_t^{(1)}\right) + \sigma_s \varepsilon_{t+1} \tag{41}$$

Equation (41) says that surplus will have a mean reversing AR(1) behaviour. Equation (41) describes the stochastic process of the government primary surplus. This is the usual mean reversing process whereby $\Delta s_{t+1}^{(1)}$ is likely to be negative if $\Delta s_{t}^{(1)}$ is above its mean and, is likely to be positive if $\Delta s_{t}^{(1)}$ is below its mean. If on average surplus is zero, the adjustment will depend fully on $\phi \Delta s_{t}^{(1)}$. For which the solution for the coefficients in (40) would look like:

$$A(N+1) = -\varphi_0 - A(N) - B(N)\phi \bar{s}_{\Delta s}^{(1)} + \frac{1}{2}\left[B(N)\sigma_{\Delta s}\right]^2 \tag{42}$$

$$B(N+1) = \left[B(N)(1-\phi) - \varphi_1\right] \tag{43}$$

For a detail algebra on how to get to (42) and (43) the reader should refer to Appendix Section II.

Section V.II. The effect of changes in total notional debt outstanding on the price level and credit spread

In a similar fashion to previous chapter, this part of the paper will make use of affine terms structure models to describe the path of shocks on notional debt outstanding and its effects on the price level. The path of the price level can be specified as follows:

$$E\left[\Pi_t^{(N)} \pi_{t+1}^{(N)}\right] = E\left[\Pi_{t+1}^{(N)}\right] \tag{44}$$

Where $E\left[\Pi_t^{(N)} \pi_{t+1}^{(N)}\right]$ being the price level in time t which is expected to grow at $\pi_{t+1}^{(N)}$ resulting in an expected future price level at t+1 equal to $E\left[\Pi_{t+1}^{(N)}\right]$. Equation (44) describes the path at which $\Pi_t^{(N)}$ would need to grow from t to t+1. For $\pi_{t+1}^{(N)}$ being the growth kernel at t+1 for $N = t+1$ so that if $\pi_{t+1}^{(N)} > 1$ it would imply that the price level increases during period t to t+1, if $\pi_{t+1}^{(N)} < 1$ it would imply that the price level decreases between t to t+1.

Rearranging (44) as a function of current price level yields

$$E\left[\Pi_t^N\right] = E\left[\left(\pi_{t+1}^N\right)^{-1}\Pi_{t+1}^N\right] \tag{45}$$

Using the natural logarithm notation (45) would look like

$$\ln\left[\Pi_t^{(N)}\right] = \ln\left[\Pi_{t+1}^{(N)}\right] - \ln\left[\pi_{t+1}^{(N)}\right] \tag{46}$$

Now we need to specify $\ln\left[\Pi_{t+1}^{(N)}\right]$ and $-\ln\left[\pi_{t+1}^{(N)}\right]$ for which a similar expression will be used as in previous chapter which for this case could be specified as follows

$$-\ln\left[\pi_{t+1}^N\right] = \frac{1}{2}\lambda^2 + \Delta\pi_t^{(1)} + \lambda\varepsilon_{t+1} \tag{47}$$

$$\Delta\pi_t^{(1)} = \gamma_0 + \gamma_1\Delta b_t^{(1)} \tag{48}$$

For λ depicting the sensitivities at which $-\ln\left[\pi_{t+1}^{(N)}\right]$ changes due to shocks in ε_{t+1}. $\Delta\pi_t^{(1)}$ represents the price level growth rate in 1 year as a consequence of changes in total notional debt outstanding, γ_0 and γ_1 are coefficients and $\Delta b_t^{(1)}$ depicts the government's one year growth on total notional debt at time t. And again, as in previous chapter, the notation is on purpose so that under normality $E\left[-\ln\pi_{t+1}^{(N)}\right]$ equals $-\Delta\pi_t^{(1)}$.

$$\Pi_{t+1}^{(N)} = \Pi_t^{(N)}\exp\left[\Delta\pi_{t+1}^{(1)} \times N\right] \tag{49}$$

For simplicity's sake it is assumed $\Pi_{t+1}^{(N)} = 1$. Applying natural logarithms to (49) and rearranging yields

$$\Delta \pi_{t+1}^{(1)} = -\frac{1}{N} \ln\left[\Pi_t^{(N)}\right] \tag{50}$$

If $\Pi_{t+1}^{(N)} > \Pi_t^{(N)}$ implies that the price level for period N is expected to increase which means that $\Pi_t^{(N)} < 1$ and $\Delta\pi_{t+1}^{(1)} > 0$. If on the contrary $\Pi_{t+1}^{(N)} < \Pi_t^{(N)}$ implies that the price level for period N is expected to decrease which means that $\Pi_t^{(N)} > 1$ and $\Delta\pi_{t+1}^{(1)} < 0$.

It is also assumed as in similar fashion to previous sections:

$$\ln\left[\Pi_{t+1}^{(N)}\right] = A(N) + B(N)\Delta b_{t+1}^{(1)} \tag{51}$$

Plugging (51) in (50) results in the inflation curve as a function of short term (e.g. one year) growth in total debt outstanding:

$$\Delta \pi_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{B(N)}{N} \Delta b_{t+1}^{(1)} \tag{52}$$

The final assumption here is that total notional debt outstanding $\Delta b_t^{(1)}$ follows a Vasicek (1977) process as follows

$$\Delta b_{t+1}^{(1)} = \Delta b_t^{(1)} + \phi\left(\bar{b}_{\Delta b}^{(1)} - \Delta b_t^{(1)}\right) + \sigma_{\Delta b} \varepsilon_{t+1} \tag{53}$$

Equation (53) says that changes in total notional debt outstanding will have a mean reversing AR(1) behaviour. This is a Vasicek (1977) stochastic process whereby the term $\left(\bar{b}_{\Delta b}^{(1)} - \Delta b_t^{(1)}\right)$ is likely to be negative if $\Delta b_t^{(1)}$ is above its mean $\bar{b}_{\Delta b}^{(1)}$ and, is likely to be positive if $\Delta b_t^{(1)}$ is below $\bar{b}_{\Delta b}^{(1)}$. If on average the change in total notional debt outstanding is zero, the adjustment will depend fully on $\phi\Delta b_t^{(1)}$. For which the solution for the coefficients in (52) would look like:

$$A(N+1) = \gamma_0 + A(N) + B(N)\phi\bar{b}_{\Delta b} + \frac{1}{2}\left[B(N)\sigma_{\Delta b}\right]^2 \quad (54)$$

$$(N+1) = \left[B(N)(1-\phi) + \gamma_1\right] \quad (55)$$

Details refer to Appendix Section III.

Section V.III. Modelling Government's net primary surplus as a function of state variables

In a similar fashion to previous chapters, this section deals with the use of affine terms structure models to describe the path of macroeconomic shocks on government revenues and government expenditure. This is crucial, as macroeconomic shocks can have different paths for revenues as well as for expenditure and need to be studied separately. Structural deficits can result from problems arising from these differences.

$$S_t^{(N)} = \tau_t^{(N)}(x_t) - G_t^{(N)}(x_t) \quad (56)$$

Equation (56) says that total cumulated surplus during holding period N at time t denoted $S_t^{(N)}$ equals total government revenues cumulated during N at time t which is denoted as $\tau_t^{(N)}(x_t)$ less government expenditure $G_t^{(N)}(x_t)$. Notice that we denote that government revenues as well as government expenditure are both a function of a $k \times 1$ state space vector x_t of macroeconomic variables. For simplicity's sake and to save some effort in the notation we will omit this in the following equations however, the reader will see that these variables will continue to be a function of vector x_t.

Cumulated government revenues and government expenditures path from t to t+1 can be specified as follows:

$$E\left[\tau_t^{(N)}\delta_{t+1}^{(N)}\right] = E\left[\tau_{t+1}^{(N)}\right] \quad (57)$$

$$E\left[G_t^{(N)}g_{t+1}^{N}\right] = E\left[G_{t+1}^{(N)}\right] \quad (58)$$

Equation (57) describes $E\left[\tau_t^{(N)}\delta_{t+1}^{(N)}\right]$ as being the government revenue in time t which is expected to grow at $\delta_{t+1}^{(N)}$ resulting in an expected future revenue level at t+1 equal to $E\left[\tau_{t+1}^{(N)}\right]$. Equation (57) describes the path at which $E\left[\tau_t^{(N)}\right]$ would need to grow from t to t+1. In similar fashion, equation (58) describes the growth path for government expenditure at which G_t^N would need to grow from t to t+1. For $\delta_{t+1}^{(N)}$ and $g_{t+1}^{(N)}$ being the growth kernels at t+1 for $N = t+1$ so that if e.g. $\delta_{t+1}^{(N)} > 1$ it would imply that the revenue level increases during period t to t+1, if $\delta_{t+1}^{(N)} < 1$ it would imply that the revenue growth decreases between t to t+1. The same would apply for the growth in government expenditure $g_{t+1}^{(N)}$. Rearranging (57) and (58)

$$E\left[\tau_t^{(N)}\right] = E\left[\left(\delta_{t+1}^{(N)}\right)^{-1}\tau_{t+1}^{(N)}\right] \tag{59}$$

$$E\left[G_t^{(N)}\right] = E\left[\left(g_{t+1}^{(N)}\right)^{-1}G_{t+1}^{(N)}\right] \tag{60}$$

Using the natural logarithm for (59) and (60) yields

$$\ln\left[\tau_t^{(N)}\right] = \ln\left[\tau_{t+1}^{(N)}\right] - \ln\left[\delta_{t+1}^{(N)}\right] \tag{61}$$

$$\ln\left[G_t^{(N)}\right] = \ln\left[G_{t+1}^{(N)}\right] - \ln\left[g_{t+1}^{(N)}\right] \tag{62}$$

Now we need to specify $\ln\left[\tau_{t+1}^{(N)}\right], -\ln\left[\delta_{t+1}^{(N)}\right]$, $\ln\left[G_{t+1}^{(N)}\right]$ and $-\ln\left[g_{t+1}^{(N)}\right]$ for which same notation will be used yielding

$$-\ln\left[\delta_{t+1}^{(N)}\right] = \frac{1}{2}\lambda_\delta^2 + \Delta\delta_t^{(1)} + \lambda_\delta \varepsilon_{t+1} \tag{63}$$

$$\Delta\delta_t^{(1)} = \alpha_0 + \alpha_1 x_t \tag{64}$$

$$-\ln\left[g_{t+1}^{(N)}\right] = \frac{1}{2}\lambda_g^2 + \Delta g_t^{(1)} + \lambda_g \varepsilon_{t+1} \tag{65}$$

$$\Delta g_t^{(1)} = \beta_0 + \beta_1 x_t \tag{66}$$

For λ_δ and λ_g depicting the sensitivities at which $\delta_{t+1}^{(N)}$ or $g_{t+1}^{(N)}$ change due to shocks in ε_{t+1}. $\Delta\delta_{t+1}^{(1)}$ represents the revenue growth rate in 1 year explained by innovations in macroeconomic state vector x_t and the same for $\Delta g_t^{(1)}$ which in this case describes the government expenditure growth in 1 year which is also a function of macroeconomic state vector x_t. And for α_0, α_1, β_0 and β_1 being coefficients obtained empirically, e.g. via OLS.

Under these assumptions the revenues and expenditure are expected to grow as follows

$$\tau_{t+1}^{(N)} = \tau_t^{(N)} \exp\left[\Delta\delta_{t+1}^{(1)} \times N\right] \quad (67)$$

$$G_{t+1}^{(N)} = G_t^{(N)} \exp\left[\Delta g_{t+1}^{(1)} \times N\right] \quad (68)$$

For simplicity's sake it is assumed $\tau_{t+1}^{(N)} = 1$ and $G_{t+1}^{(N)} = 1$. Applying natural logarithms to (67) and (68), and rearranging yields

$$\Delta\delta_{t+1}^{(N)} = -\frac{1}{N}\ln\left[\tau_t^{(N)}\right] \quad (69)$$

$$\Delta g_{t+1}^{(N)} = -\frac{1}{N}\ln\left[G_t^{(N)}\right] \quad (70)$$

It is also assumed that

$$\ln\left[\tau_{t+1}^{(N)}\right] = A(N)_\tau + B(N)_\tau' \, x_{t+1} \quad (71)$$

$$\ln\left[G_{t+1}^{(N)}\right] = A(N)_G + B(N)_G' x_{t+1} \quad (72)$$

For $A(N)_\tau$ and $A(N)_g$ being scalars, $B(N)'_\tau$ and $B(N)'_g$ being $1 \times k$ vectors of coefficients and x_{t+1} a $k \times 1$ vector of state variables.

From our guess shown in (71) and (72) we wish to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$, and hence by plugging (71) in (69) and (72) in (70) results in:

$$\Delta\delta_{t+1}^{(N)} = \frac{A(N)_\tau}{N} + \frac{B(N)_\tau'}{N} x_{t+1} \tag{73}$$

$$\Delta g_{t+1}^{(N)} = \frac{A(N)_G}{N} + \frac{B(N)_G'}{N} x_{t+1} \tag{74}$$

From (73) and (74), and recalling (56) boils down to

$$\Delta s_{t+1}^{(N)} = \Delta\delta_{t+1}^{(N)} - \Delta g_{t+1}^{(N)} \tag{75}$$

$$y_{t+1}^{(N)} = \kappa_0 + \kappa_1 \Delta s_{t+1}^{(N)} \tag{76}$$

The final assumption here is that the state space vector x_t follows a Vasicek (1977) process already discussed in (15) and now reproduced below for convenience as follows

$$x_{t+1} = x_t + \Phi\left(\bar{x} - x_t\right) + \sigma_x \varepsilon_{t+1} \tag{77}$$

Equation (77) says that the state space vector with elements $x_{i,t+1}$ will have a mean reversing AR(1) behaviour. For which the solution for the coefficients in (73) and (74) would look like:

$$A(N+1)_\tau = \alpha_0 + A(N)_\tau + B(N)_\tau' \Phi\bar{x} + \frac{1}{2}\left[B(N)_\tau' \sigma_x\right]^2 \tag{78}$$

$$B(N+1)_\tau' = \left[B(N)_\tau' (I - \Phi) + \alpha_1'\right] \tag{79}$$

$$A(N+1)_G = \beta_0 + A(N)_G + B(N)_G' \Phi\bar{x} + \frac{1}{2}\left[B(N)_G' \sigma_x\right]^2 \tag{80}$$

$$B(N+1)_G' = \left[B(N)_G' (I - \Phi) + \beta_1'\right] \tag{81}$$

For details on how to obtain (78) to (81) the reader should refer to Appendix, Section IV.

In order to depict how this works let us assume an economy where the only source of income is from employment and if the individuals are not employed they receive a transfer from the government. For simplicity's sake, we will assume that unemployment is a valid state variable which describes well shocks on government revenues and on government expenditure. For instance, if unemployment decreases, government revenues increase, but also expenditure decreases, as there are less unemployed and hence less payment transfers from the government and therefore surplus improves. However, if unemployment deteriorates, government revenues decrease too, as there is less taxable income. At the same time, expenses increase, as there are more unemployed and hence more transfer payments from the government to households.

If, let's say the constant terms in (73) and (74) are $A(N)_G / N = A(N)_\tau / N = 0$, $B(N)_G / N > 0$ $B(N)_\tau / N < 0$ for all N, and x_t being the unemployment rate, this will imply that an increase in unemployment has results in a deterioration of government surplus and that this might have an effect for various periods. The size and the sign (positive or negative) of the 1-period shock are dependent on the coefficients estimated as described in (64) and (66) However, because the term is being divided by *N*, this means that shocks dissipate as *N* becomes greater. With this model it is possible to study persistence of these shocks along *N* which is largely dependent on Φ which is nothing else but the speed of adjustment of a state variable such as unemployment towards its mean from x_t to $\bar{x}$. The closer Φ is to 1 the faster the adjustment, but the closer Φ is to zero the greater the persistence of these shocks in affecting government surplus for various periods. Here, it is possible to have state variables which have offsetting effects in period 1. However, if they differ in persistence, hence on the size of Φ it could result in cumulating deficits mainly because on the long run one variable causes persisting negative effects on surplus.

Section VI. Empirical Analysis

We analyse Bloomberg and Eurostat monthly data mostly from the period August 2008 to September 2012. As we are trying to link macroeconomic data with financial markets data such as inflation-linked swap yields, we are very much dependent on its availability. For the case of European inflation-linked swaps we only observe a complete inflation curve for all maturities in the Euro-Zone since August 2008 which leaves us only with 50 data points. We use the data since 2008 in order to take into account the conclusions from Beirne and Fratzscher (2013), who suggest that there is a structural change, as a consequence of what they refer to as a “wake-up call” or De Grauwe and Ji (2013) idea of investors neglecting some of the economic fundamentals of GIIPSs countries, previous to 2008.

Before the affine term structure models are calibrated, this section will start first by plotting the EONIA, the inflation as well as the real interest rates from 1 year inflation-linked swaps. The intention is to study how movements in inflation and real interest rates explain movements in the short rate.

In a second stage, we will regress via OLS, the EONIA rate to the inflation as well as the real interest rates from 1 year inflation-linked swaps following discussions on (4). Subsequently we will repeat this exercise for European and German government yields of various maturities. We will show that results are consistent across maturities.

In a third stage, we will repeat this exercise for Greek and Spanish government yields and show that though the fitted values appear to follow some of the movements in the yields, results are not as encouraging as for the European and German yields. However, we will show that we can remediate this by regressing (26), hence by incorporating the spread of these yields to the German benchmark in the regression as a third state variable. By doing so we can show the reader that inflation and real interest rates do not suffice as explanatory variables for non-benchmark yields, as there is a credit risk component which appears that needs to be taken into account when modelling Spanish and Greek yields.

In a fourth stage, we attempt to replicate the behaviour of governments total debt outstanding by using the price level, which for simplicity’s sake we use as proxy the production price index (PPI) for the respective countries. Here, we regress (22) as the intention is to analyse how the stochastic path of governments' total debt outstanding can be used as a state variable when modelling the time path of the price level. In this section we also show an example of fitting for the two year Spanish government yield using Spanish cumulated government deficit and Spanish unemployment rate as state variables.

The final piece of the analysis introduces the use of affine term structure models for: 1) to generate an inflation curve (or so-called the "breakeven inflation" curve) using as state variable government's total debt outstanding; 2) likewise we generate the average spread curves for Spanish and Greek government yields and analyse the coefficients $B(N_i)N_i$ for various maturities using as state variables Spanish and Greek governments' total debt outstanding and; 3) we present and example where we will calibrate an affine model as specified in (73) and (74) using as a state variable the country's unemployment rate and analyse the coefficients $B(N_i)N_i$ to show how affine term structure models can help understand how innovations in the unemployment rate are useful for determining the time path of governments' revenues and expenses and hence ultimately contribute to the understanding of the effects that innovations in macroeconomic variables have on governments' deficits.

Figure 7-1 below shows the EONIA, the inflation and the real interest rate from a 1 year inflation-linked swap. Only by looking at the data it is possible to see the effects of expected inflation and real interest rate movements on the short rate (EONIA). These components can be used as proxies, in order to replicate the identity described in equation (4).

Figure 7-1 Development of the EONIA, breakeven inflation and real interest rate observed from Euro 1 year inflation swap.

This is a possible observation of the various variables discussed in equation (4).

Figure 7-2, replicates the EONIA which has been obtained by regressing equation (4). The EONIA is the proxy for the nominal short rate and proxies for the real interest rate as well as for the inflation rate are the 1-year real yield and the breakeven inflation observed from the 1-year Euro inflation-linked (CPI) swaps. The reader can appreciate that the fitted values fit remarkably well the observed data.

Figure 7-2 Replication of EONIA using as state variables the real interest rates and the breakeven inflation rates from observed inflations swaps data.

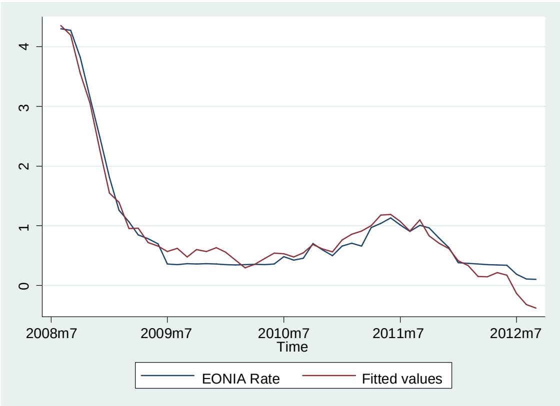


Figure 7-3 replicates the European government benchmark yields which have been obtained by regressing equation (4) for different maturities. The European government benchmark yields are here used as proxy for the nominal risk-free rates for various maturities. In addition, the proxies for the real interest rates as well as for the inflation rates are the corresponding 2, 5, 10, 15, 20 and 30-year real inflation-linked swap yields and the breakeven inflation rates which are observed from these same swaps. The reader can appreciate that the fitted values replicate remarkably well the observed data. This is the beauty of the availability of inflation-linked financial data, as it gives the possibility to segregate inflation expectations and expectations on real yields.

Similar to figure 7-3, figure 7-4 replicates the same regression for the German government benchmark yields and results are just as encouraging. These results are notably better than those seen in Jakas (2011, 2012, 2013b) and Jakas and Jakas (2013), as these empirical results show that inflation swap data performs better than the usual monthly survey data.

Figure 7-3Figure 3. Bloomberg EU Government Yields (Bloomberg Index), observed vs. fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.

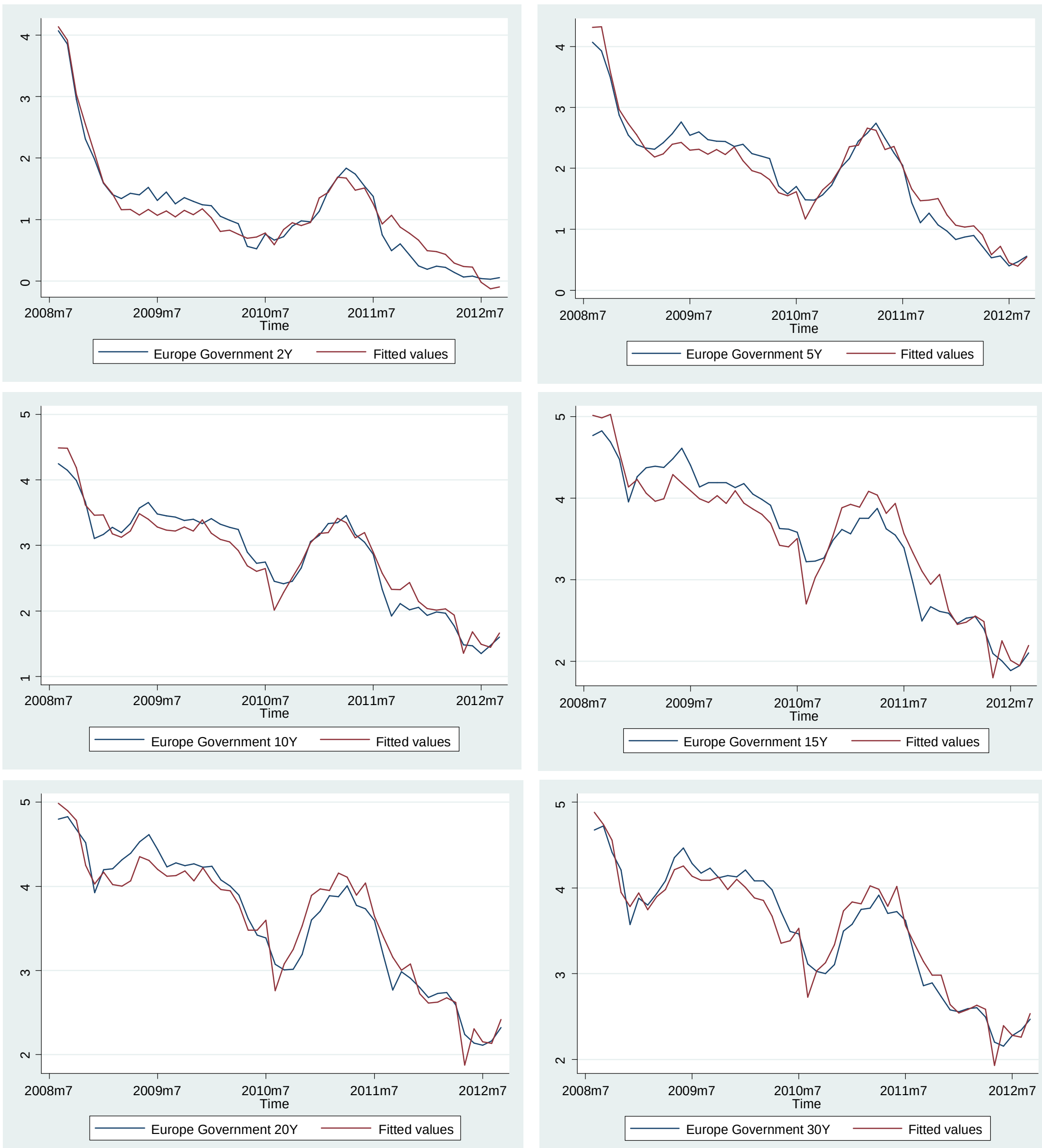


Figure 7-4 Bloomberg German Government Yields (Bloomberg Index), observed vs. fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.

Figures 7-5 and 7-6 replicate the Greek and Spanish government benchmark yields which have been obtained by regressing equation (4) in a same fashion as we did for figures 7-3 and 7-4. Not surprisingly, real yields and inflation have less predictive power on Greek and Spanish benchmark yields compared to the European or the German government benchmark yields. The reason is the existence of a credit spread as these are risky assets.

Figure 7-5 Bloomberg Greek Government Yields, Observed vs. Fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.

The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis.

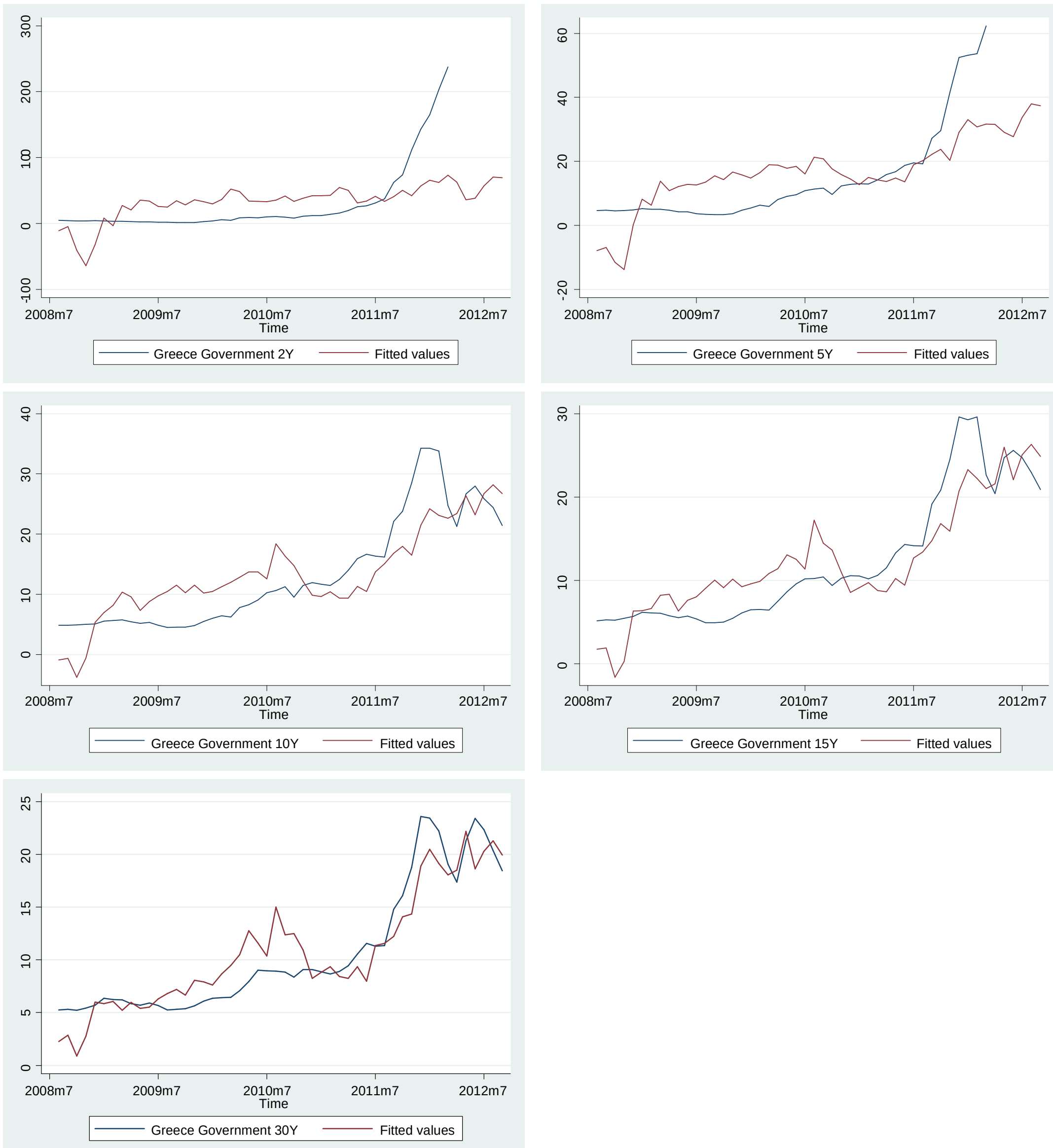
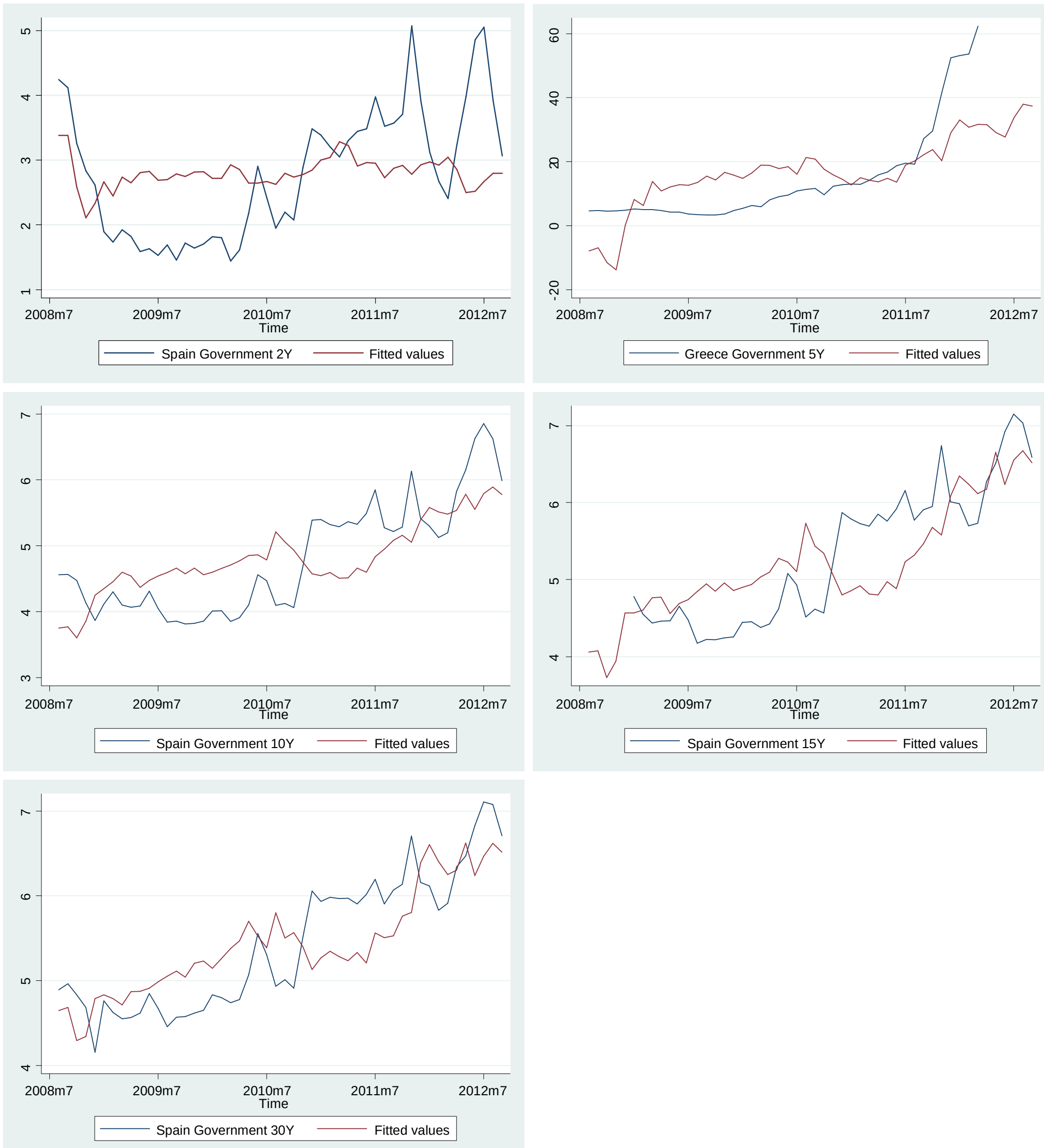


Figure 7-6 Bloomberg Spanish Government Yields, Observed vs. Fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.

The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis.



We will include the credit spread to equation (4) as discussed in (5) and as specified in (26), by doing so the fitted values show remarkable improvements as shown in Figures 7-7 and 7-8. Figures 7-7 and 7-8 replicate the Greek and Spanish government benchmark curves which have been obtained by regressing nominal yields to a state space vector comprising real yields, breakeven inflation and the spreads to the German benchmark. These results are obvious, as Spanish and Greek government bonds –amongst other peripherals have had a credit risk component due to liquidity constraints in line with Beirne and Fratzscher (2013) and De Grauwe and Ji (2013).

Figure 7-7 Fitting by incorporating the spread as in equation (26). Bloomberg Greek Government Curves, Observed vs. fitted using the observed credit spreads, breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.

The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis. Credit spreads are calculated as the difference to the German Benchmark.

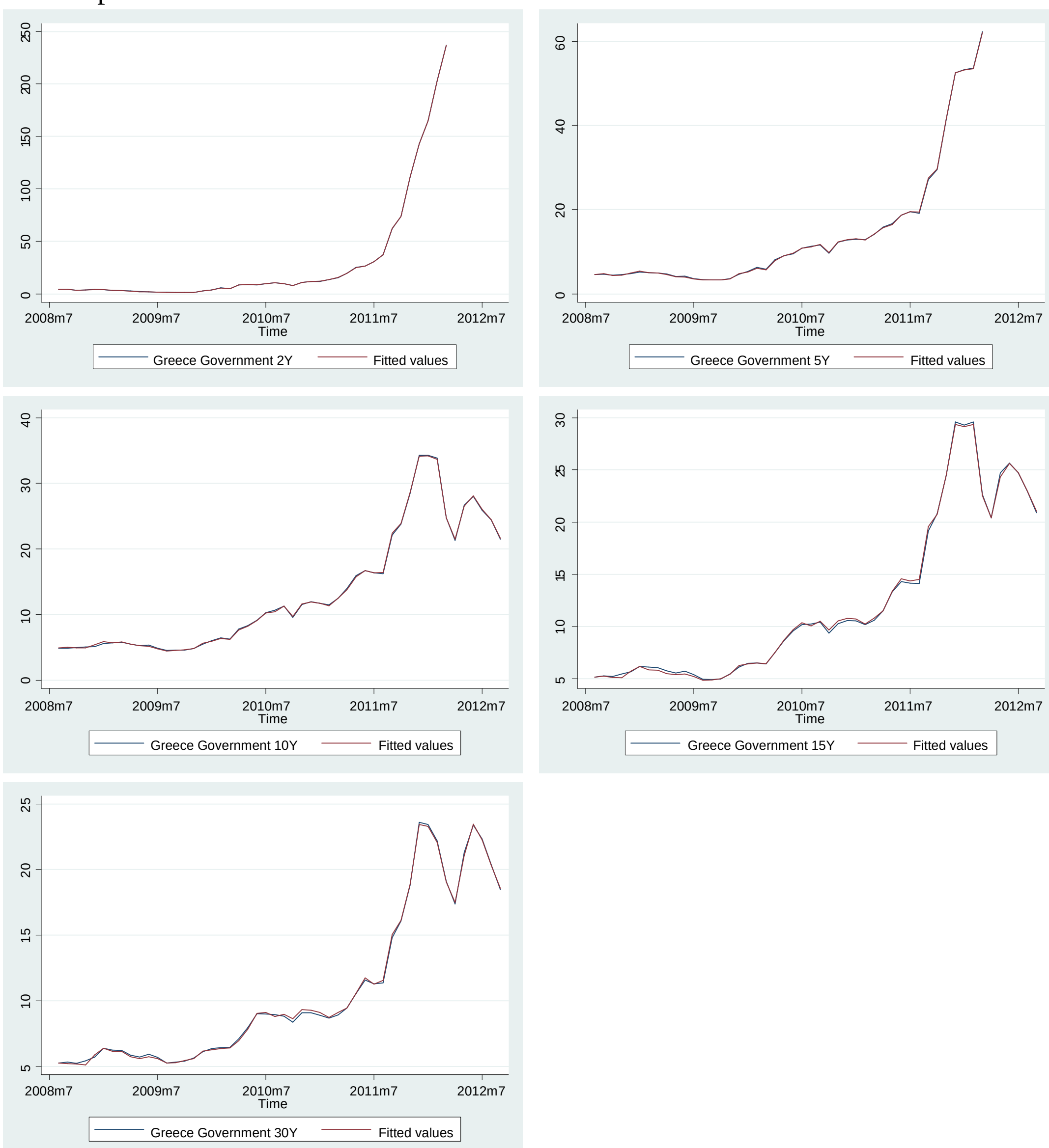
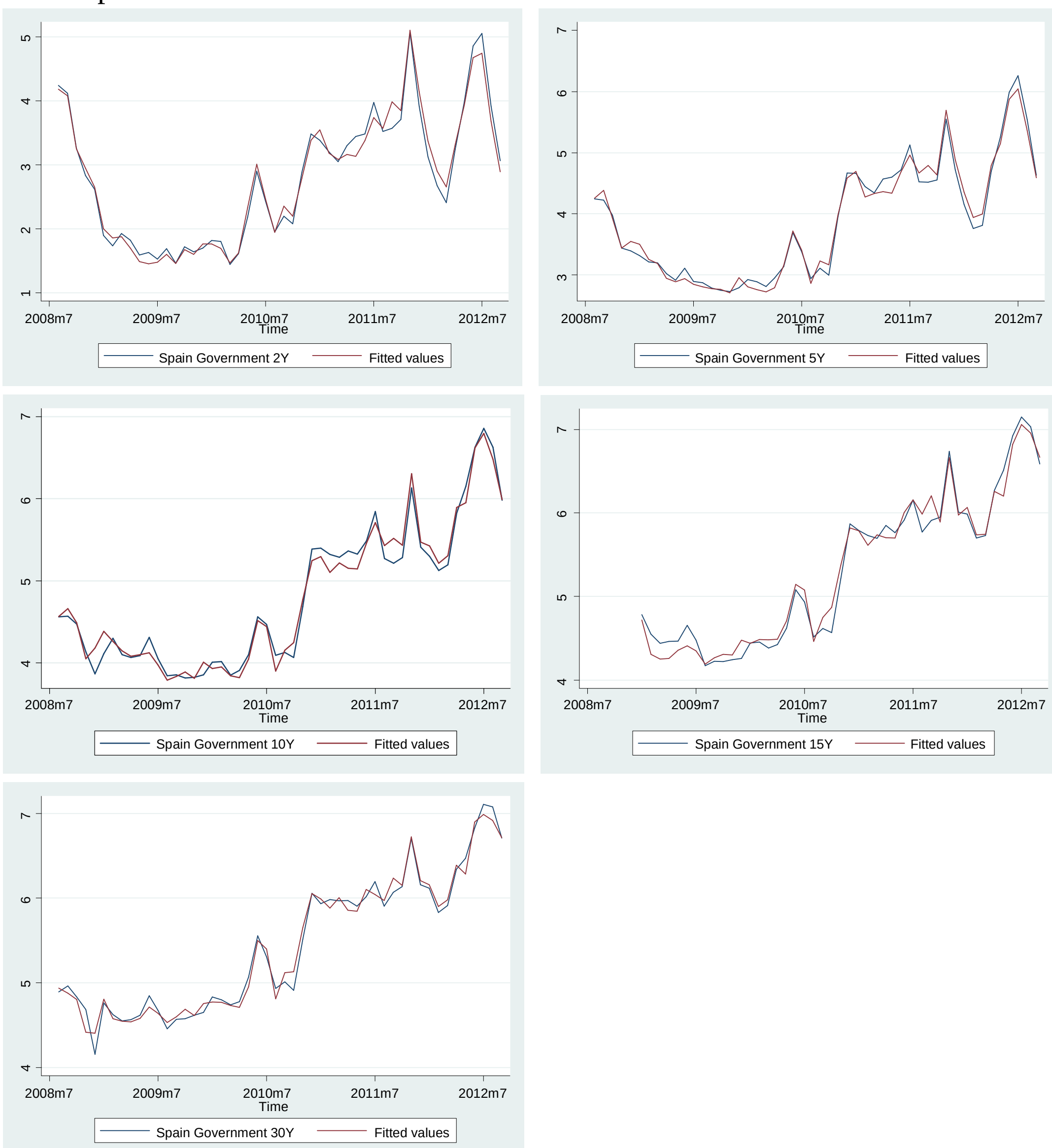


Figure 7-8 Fitting equation (26). Bloomberg Spanish Government Curves, Observed vs. Fitted using the observed credit spreads, breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.

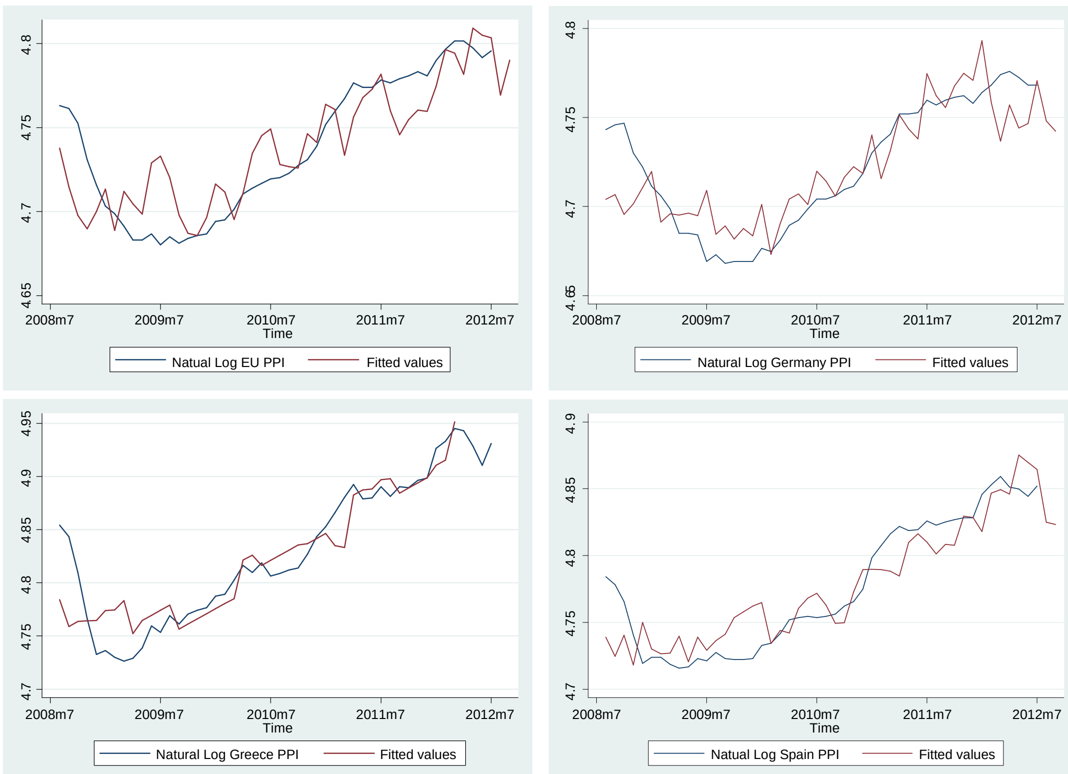
The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis. Credit spreads are calculated as the difference to the German Benchmark.



The theory of the price level suggests that an increase in total debt outstanding can lead to increases in the price level. Figure 7-9 shows PPI from EU, Germany, Spain and Greece, observed versus fitted values obtained by regressing equation (22). These time series show that levels in governments' total debt outstanding can be used as explanatory variables in order to explain changes in the price level, as increases in the price level is –

statistically speaking– positively correlated to government's total notional debt outstanding.

Figure 7-9 Fitting equation (22). Replication of Production Price Indices (PPI) for the Euro-Zone, Germany, Spain and Greece. Observed vs. Fitted values using the observed total government debt outstanding available in Bloomberg.



Deficits and unemployment can result in deterioration in government yields. The theory of the price level suggests that the net present value of a bond or the yield must reflect the net present value of expected future surpluses. If unemployment and surplus (deficits) deteriorate, then this should be observed in the short term yields. Figure 7-10 tries to replicate this relationship. Though it is possible to replicate the trend, we are not capable of replicating the volatility observed which appears to increase as yields become greater. Now, it is worth being cautious in making conclusions with figure 7-10, as the size of the sample is rather small and government deficits and unemployment are interrelated variables.

Figure 7-10 Replication of 2y Spanish Government yield applying two factors, Spanish cumulated government deficit and unemployment rates: Observed vs. Fitted values.

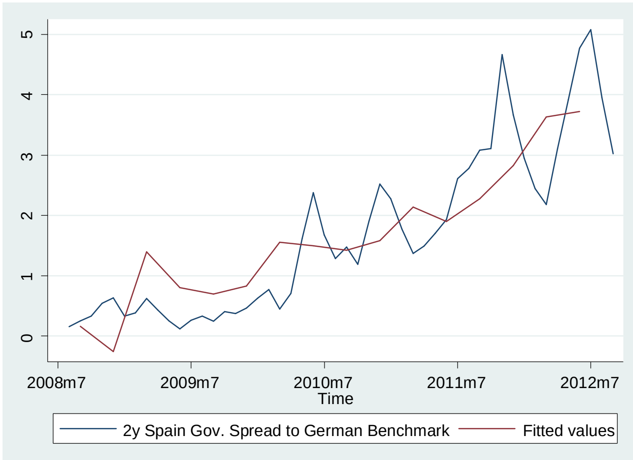


Figure 7-11 replicates the average Euro Breakeven Inflation curve by calibrating equation (52). It is possible to produce the average inflation curve (left quadrant) but it is not possible to replicate a reasonable time series that fits well the observed data (right quadrant). The fitted data is much less volatile which shows this requires further research. However, and despite this deficiency, it should be mentioned that the model is still capable of fitting the underlying upward trend.

Figure 7-12 replicates the spreads for Spanish and Greek government bonds. On average, we observe that the Spanish spreads show a "normal" upward-sloping, except the front end (2 years) exhibiting lower spreads. Spreads are constant between 5-years and 30-year maturities. The average Greek government spread curve gives a completely different view. The downward-sloping shows that the yields in the front end are more risky than the rest of the curve. This is because the market's view is that the Greek government will not be capable or raising new funds in order to pay the 2-year maturing bonds. The issuance is a Ponzi scheme and speculation is focused solely on the ability of the Greek government to convince core European partners on a bail out. In this case the theory of the price level is not at work, as aggregated European inflation is influencing very little the deterioration in the underlying yields and is rather a credit risk element, as a consequence of liquidity innovations resulting from solvency shocks.

Figure 7-11 Euro Breakeven Inflation observed vs. affine fitted values.
(Left) Replication of the Euro Breakeven Inflation observed vs. affine fitted values, using equation (52). (Right) 1Year Euro Breakeven Inflation observed vs. fitted.

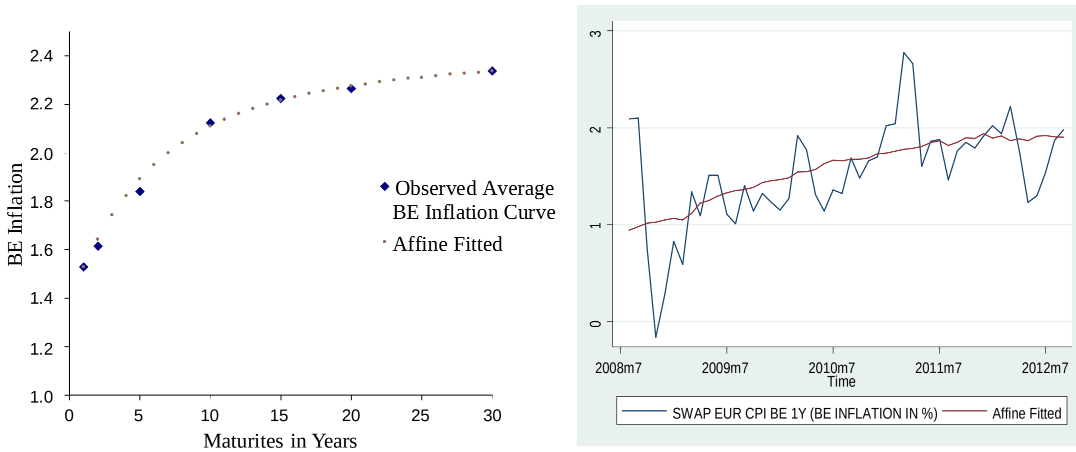


Figure 7-12 Spanish and Greek Government Spreads affine fitted.
Replication of the Spanish and Greek Government Spreads for each maturity bucket vs. Affine Fitted values, using equation (52) and replacing $\Delta\pi_{t+1}^{(N)}$(inflation) for $\theta_{t+1}^{(N)}$(credit spread).

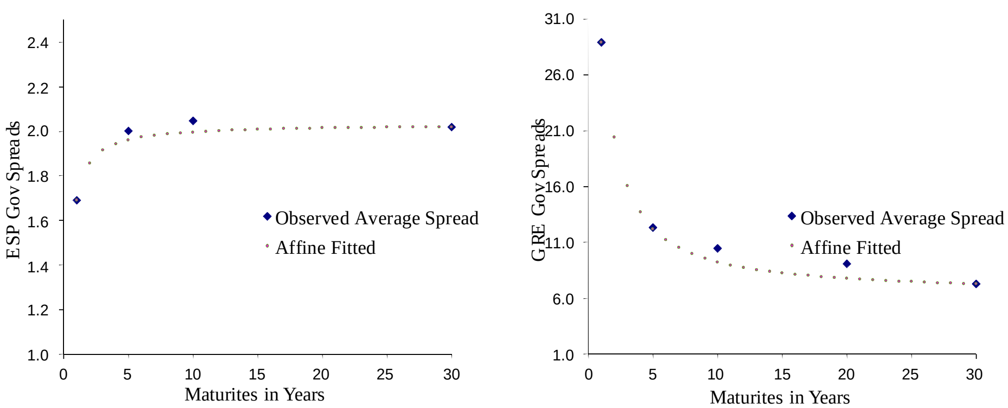


Figure 7-13. Shows how the $B(N)/N$ coefficients which depict the sensitivity of changes in the credit spread $\theta_{t+1}^{(N)}$ as a consequence of changes in total debt outstanding and the effect is diluted as the maturity increases, which makes sense because in times when default risk is high, the markets believe that the issuer will default first on the issuance that is just about to mature, thus where debt-roll-over risk is high. Interestingly,

comparing the parameters $B(N)/N$ estimated for Spain with those estimated for Greece it is possible to observe the magnitude of how sensitive is Greece to changes in total debt outstanding.

Figure 7-13 Affine term structure sensitivities of the Spanish and Greek Government Spreads
Affine term structure sensitivities of the Spanish and Greek Government Spreads for each maturity bucket vs. Affine Fitted values, using equation (52) and replacing $\Delta\pi_{t+1}^{(N)}$ (inflation) for $\theta_{t+1}^{(N)}$ (credit spread).

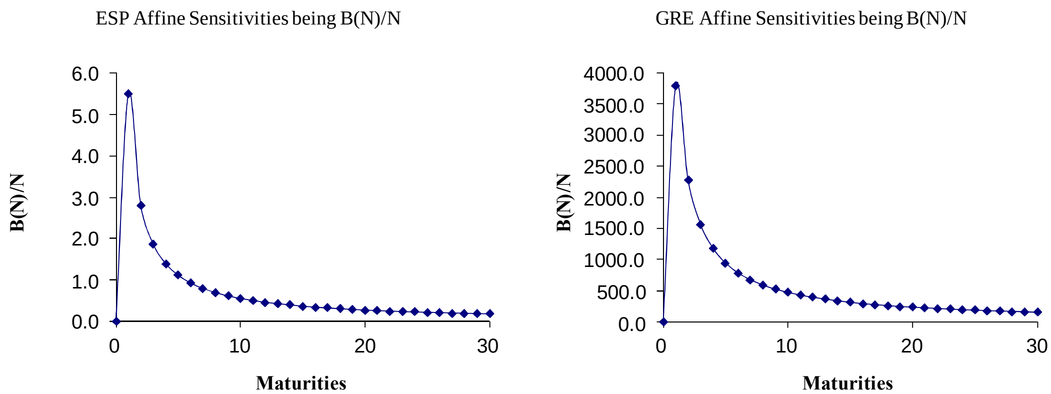


Figure 14 and 15 we show a cumulated expense versus income ratio for German, Spanish and Greek governments and the cumulated expense and income levels. Here, we try to capture discussion seen in equations (75) and (76). This idea is taken from the popular *cost-income ratio* used in the corporate finance literature, however we are particularly interested in its cumulated value as an indication of debt growth. For figure 7-14 and 7-15 we cumulate for values starting since 2001 (chart on the left) and 2008 (chart on the right). The cumulated expense to cumulated income ratio shown in figure 7-14 can be specified as follows $ratio_{t+i} = \sum_{t=1}^{t+i} Exp_t \Big/ \sum_{t=1}^{t+i} Inc_t$

For the case of Spain and Greece, there is a clear deterioration of this ratio particularly since 2008, as GDP falls, with the subsequent fall in tax revenues, increase in unemployment and thus further increase in Government expenses. We can see here we move from a relatively stable ratio and then moving towards higher levels since 2008. The case of Spain is interesting as the ratio has been in better shape compared to Germany throughout the decade (chart on the left) however, the data also show that if we instead cumulated from 2008 onwards (chart on the right), the ratio exhibits similar levels as those seen in the Greek case. No doubt we see that in case of macroeconomic shocks influencing surplus result in a deterioration of this ratio and unless this trend is reversed the gap remains thereafter for several periods. The developments since 2008 onwards is in line with the fact that cumulated deficits are in line with a deterioration of the credit spreads. Figure 7-15 is not less impressive, as the gap between cumulated expenses to cumulated government revenues or income increases at a faster pace for Greece and Spain compared to the German case.

Figure 7-14. Cumulated government expenses to cumulated government revenues ratios (cost-income-ratios) for Germany, Spain and Greece.

IMF Quarterly data, cumulative since 2001 (left) and 2008 (right) respectively.

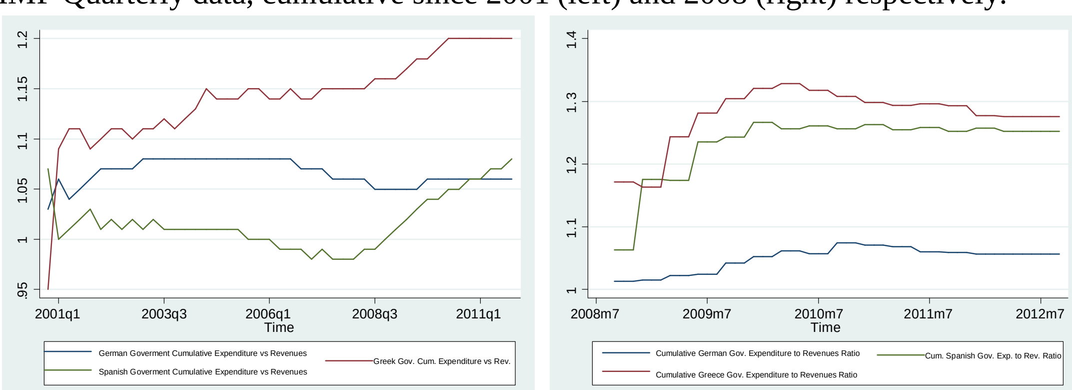
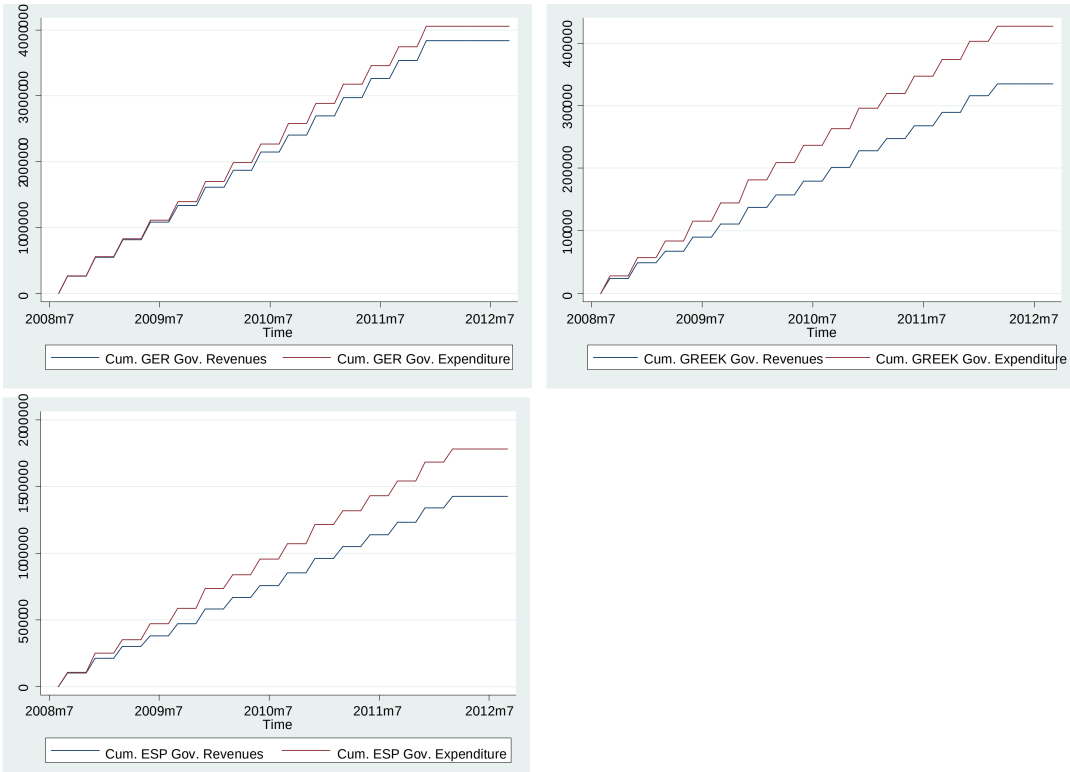


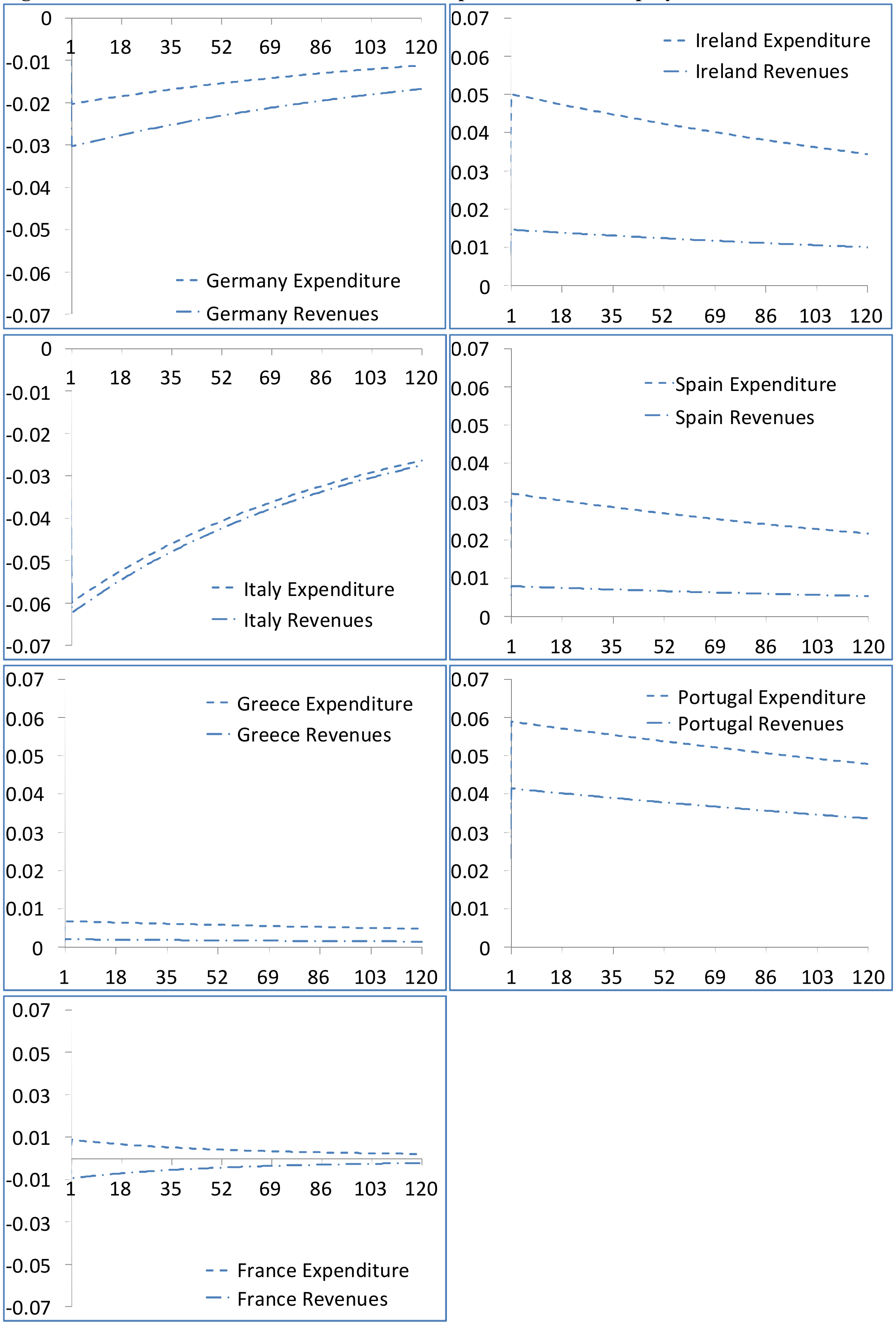
Figure 7-15. Cumulated government expense and cumulated government revenues for Germany, Spain and Greece.

IMF Quarterly data, cumulative since 2008.



We calibrate the model discussed in section V.III by using unemployment rate as the state variable x_t influencing government revenue and expenditure for Germany, Ireland, Greece, Spain, Italy, Portugal and France. Hence, using one factor version of equations (78) to (81) and restricted to (64), (66), (73) and (74). Figure 7-15 summarises the results obtained for coefficients $B(N)_\tau / N$ and $B(N)_G / N$ which describes the growth path in government's tax revenues and government's expenditure for various quarters as a consequence of shocks in the unemployment rate. In all countries we observe that an increase in unemployment results in a deterioration of government's surplus. For all cases in figure 7-15, unemployment exhibits persistence, thus shocks in unemployment remain for long periods of time and therefore has long lasting effects on governments' revenues and expenses. This is attributed to the autocorrelation coefficients of the lagged term in equation (77) specified as Φ which are very close to zero. There are only subtle differences between countries mostly when it comes to the size of the sensitivity of government's deficits to innovations in unemployment. For the cases of Ireland, Spain, Portugal and Greece, and increase in unemployment results in government expenditure increasing at a faster pace than revenues. Thus we can see that increases in unemployment result in fiscal imbalances that last various periods. It is possible to see that this gap is significantly large for Ireland, Spain and Portugal. Unemployment appears to have a lesser role for Greece's budgetary constraints, as Greece is a rather special case with respect to unemployment, as the initial debt level is likely to have played a more important role, as discussed in Chapter 1, Section VI. Alternatively, in the case of Germany, increases in unemployment result in government revenues falling at a faster pace than government expenditure thus, unemployment also has a negative effect on its surplus but mainly because the decrease in revenues has greater damaging effect on its fiscal imbalances than the increase in expenditure due to higher transfer payments to those unemployed. It is also possible to observe that the government reduces on expenditure when unemployment grows, presumably, in order to adjust fiscal imbalances however this adjustment appears only to partially offset the decrease in tax revenues, hence resulting in an overall deterioration of the German government surplus. Italy exhibits a similar result to Germany however the difference here is that the gap is tighter and that the unemployment rate despite exhibiting less persisting effects on Italy's government surplus the initial shock is larger than in for the Germany government. France exhibits a so-called "normal" case, thus an increase in unemployment results in a decrease in revenues and an increase in expenditure, closer to discussions in section V.III.

Figure 7-16 Sensitivities of Tax Revenues and Expenditure to Unemployment rates.



Note on Figure 7-16: $B(N)_\tau/N$ and $B(N)_G/N$ coefficients calibrated with unemployment rate figures for Germany, Ireland, Greece, Spain, Italy, Portugal and France using one factor version of equations (78) to (81) and restricted to (64), (66), (73) and (74) using quarterly data for tax revenues and unemployment from Eurostat and ECB Data warehouse.

Section VII. Policy Implications, Conclusions and Final Remarks

This paper is an attempt to link term structure models, the theory of the price level to debt dynamics and fiscal imbalances. This paper starts calibrating a state space vector for breakeven inflation and real yields observed from inflation linked swap data and results were encouraging. We can observe that they do a poorer job for Spanish and Greek bonds however we remediated this by incorporating the credit spreads. It shows that inflation has lower predictive ability on these yields and most of the movement is captured in the credit risk component, as fitted values improved significantly when incorporating the spread to German benchmark yields in the model.

We understand that if governments need to provide counter-cyclical policies in times when aggregate marginal utility growth is high and hence, when consumption growth is low, this will largely depend on their ability to issue new debt without exhibiting a deterioration of their financing costs. We have seen that increases in total debt outstanding can lead to higher inflation and higher yields, but for the governments where this appears not to be significant, such as for instance, Spain or Greece, the increases in higher levels of total debt outstanding translate in higher spreads rather in higher inflation. We see that either ways financing becomes more expensive. The German case is different, because greater levels of total debt outstanding do not translate in a deterioration of their cumulated surplus. This, we believe, can be attributed to the idea that in times of low consumption growth German yields are indeed low so that the Government can issue more debt without incurring higher financing costs and undertake counter-cyclical policies. This assertion has testable implications and hence could be part of future research or follow-ups. However, as Germany has a greater stake in European institutions than their periphery counterparts, it is natural that investors might speculate with the idea that German bonds are more likely to be backed by accommodative ECB policies than the periphery or GIIPSs bonds.

From our analysis using an affine term structure model, we observed that increases in unemployment, for instance, generated a gap between revenues and expenses that was significantly tighter for the case of Germany compared to the cases of Ireland, Spain and Portugal. Therefore, it appears that the German government budget is less sensitive to innovations in unemployment and hence cumulated government surplus remains relatively unaffected when controlling for this state variable compared to Ireland, Spain and Portugal. We understand that for these countries is more difficult to undertake countercyclical policies in the same as Germany can, mostly because we observe that during periods of financial distress issuing new debt becomes more expensive, as surplus deteriorates most.

In addition, we have observed that the front end is very sensitive to innovations in increases on government's total debt outstanding, possibly because investors perceive that that is the first tranche that the issuer is likely to default. It should be mentioned that few governments exhibit low financing costs in times when aggregate marginal utility growth is high. In fact this demonstrates that governments should run stress tests to their budgets and secure liquidity reserves by running orthodox fiscal policies during economic booms and issuing long term debt enough to cover eventualities in case of a stressed scenario that would have extraordinary deteriorating effects on their budgetary deficits. This implies ensuring long periods of enough liquidity reserves for the case when several short term tranches mature and hence avoid debt roll-over risk, thus the risk of not being able to issue enough new debt at low financing costs in order to redeem the maturing one.

<u>*Appendix Chapter 7*</u>

I. *Solving when the theory of the price level is at work and when is not*

Here it is shown how to get to the solution. Starting first with equation (10) and substituting the right hand term for (16) and (11) which boils down to:

$$\ln\left[P_t^{(N+1)}\right] = -\delta - y_t^{(1)} - \lambda' \varepsilon_{t+1} - A(N) - B(N)' x_{t+1} \tag{I.1}$$

In order to solve recursively δ is replaced by (17), and to be able to account for the theory of the price level $y_t^{(1)}$ is replaced by (27). In addition, x_{t+1} is also replaced for (15) to account for the Vasicek (1977) process, which all boils down to:

$$\ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - \psi_0 - \psi_1' x_t - \lambda' \varepsilon_{t+1} - A(N) - B(N)'\left[x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1}\right] \tag{I.2}$$

The constant terms and the terms multiplying x_t and ε_{t+1} are grouped, so that at the end it would look something like this

$$\ln\left[P_t^{(N+1)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \psi_0 + A(N) + B(N)'\Phi\bar{x}\right) - \left(\psi_1' + B(N)'(I - \Phi)\right)x_t - \left(\lambda' + B(N)'\sigma_x\right)\varepsilon_{t+1} \tag{I.3}$$

The right hand side of equation (10) which has now developed to (I.3) has the following conditional moments,

$$E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \psi_0 + A(N) + B(N)'\Phi\bar{x}\right) - \left(\psi_1' + B(N)'(I - \Phi)\right)x_t \tag{I.4}$$

and

$$Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] = \left(\lambda' + B(N)'\sigma_x\right) \tag{I.5}$$

Recalling that the implied present value of a fixed income security yields

$$-E\left[\ln P_t^{(N+1)}\right] = -E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] - \frac{1}{2}Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] \tag{I.6}$$

Substituting (I.4) and (I.5) into (I.6) yields

$$- E\left[\ln P_t^{(N+1)}\right] = \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \psi_0 + A(N) + B(N)'\Phi\bar{x}$$
$$+ \left(\psi_1' + B(N)'(I - \Phi)\right)x_t - \frac{1}{2}\left(\lambda' + B(N)'\sigma_x\right)^2 \qquad \text{(I.7)}$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (11) yields,

$$A(N+1) = \psi_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - \left(\lambda' + B(N)'\sigma_x\right)^2\right) \qquad \text{(I.8)}$$

$$B(N+1)' = \left(\psi_1' + B(N)'(I - \Phi)\right) \qquad \text{(I.9)}$$

The solution is obtained by computing the present value recursively using (10) for some guess of coefficients from (11). Since $P_{t+1}^{(N)} = 1$ and $A(N=0) = B(N=0)' = 0$, which means this can be solved recursively, as for 1 period would imply $A(N=1) = \psi_0$ and $B(N=1)' = \psi_1'$ which means that equals the short rate as described in (27). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (10) to compute the present value of an N+1 maturity bond.

All is needed is to line up (I.8) and (I.9) into (12) and solve numerically by fitting the curve to the observed yields by adjusting λ's for a given choice of maturities. Parameters ψ_0 and ψ_{1i} are free and obtained empirically and the signs for parameters $B(N)_i$ in (12) depend on ψ_{1i}.

When the theory of the price level is not at work, equations (I.8) and (I.9) change, and this is because equation (14) is at work instead of equation (27) and by applying the same algebra discussed in (I.1) to (I.7) it would now yield

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - \left(\lambda' + B(N)'\sigma_x\right)^2\right) \qquad \text{(I.10)}$$

$$B(N+1)' = \left(\gamma_1' + B(N)'(I - \Phi)\right) \qquad \text{(I.11)}$$

II. Solving the coefficients A(N+1) and B(N+1) assuming innovations in government surplus influence changes in total notional debt outstanding

I will solve (33) recursively starting with equation (34) and substituting terms for (35), (36), (39) and (41), and operating results in

$$\ln\left[B_t^N\right] = -A(N) - B(N)\Delta s_t^{(1)} - B(N)\phi\bar{s}_{\Delta s}^{(1)} + B(N)\phi\Delta s_t^{(1)} - B(N)\sigma_{\Delta s}\varepsilon_{t+1} - \frac{1}{2}\lambda^2 - \varphi_0 - \varphi_1\Delta s_t^{(1)} - \lambda\varepsilon_{t+1} \qquad \text{(II.1)}$$

Rearranging the constant terms and the terms multiplying $\Delta s_t^{(1)}$ and those multiplying ε_{t+1} boils down to

$$\ln\left[B_t^N\right] = -\varphi_0 - A(N) - B(N)\phi\bar{s}_{\Delta s}^{(1)} - \frac{1}{2}\lambda^2 + \left[B(N)(1-\phi) - \varphi_1\right]\Delta s_t^{(1)} - \left[B(N)\sigma_{\Delta s} + \lambda\right]\varepsilon_{t+1} \qquad \text{(II.2)}$$

Equation (II.2) has the following conditional mean and variance:

$$E\left[\ln B_t^N\right] = -\varphi_0 - A(N) - B(N)\phi\bar{s}_{\Delta s}^{(1)} - \frac{1}{2}\lambda^2 + \left[B(N)(1-\phi) - \varphi_1\right]\Delta s_t^{(1)}$$

$$Var\left[\ln B_t^N\right] = \left[B(N)\sigma_{\Delta s} + \lambda\right]^2$$

Recalling normality as in (37) and substituting

$$E\left[\ln B_t^N\right] = E\left[\ln B_{t+1}^N - \ln g_{t+1}\right] + \frac{1}{2}Var\left[\ln B_{t+1}^N - \ln g_{t+1}^N\right] \qquad \text{(II.3)}$$

Substituting the above conditional moments in (II.3) yields,

$$E\left[\ln B_t^N\right] = -\varphi_0 - A(N) - B(N)\phi\bar{s}_{\Delta s}^{(1)} - \frac{1}{2}\lambda^2 + \left[B(N)(1-\phi) - \varphi_1\right]\Delta s_t^{(1)} + \frac{1}{2}\left[B(N)\sigma_{\Delta s} + \lambda\right]^2 \qquad \text{(II.4)}$$

Rearranging and grouping the constant terms and the terms multiplying $\Delta s_t^{(1)}$ as well as lining up with (39) yields,

$$A(N+1) = -\varphi_0 - A(N) - B(N)\phi\bar{s}_{\Delta s}^{(1)} + \frac{1}{2}\left(\left[B(N)\sigma_{\Delta s} + \lambda\right]^2 - \lambda^2\right) \qquad \text{(II.5)}$$

$$B(N+1) = \left[B(N)(1-\phi) - \varphi_1\right] \qquad \text{(II.6)}$$

Equations (II.5) and (II.6) are resolved recursively using what is known from (34) restricted to (39) and (40). Since it has been assumed that $B_{t+1}^{(N)} = 1$, and $A(N=0) = B(N=0) = 0$, which means this can be solve recursively, as for 1 period would imply $A(N=1) = -\varphi_0$ and $B(N=1) = -\varphi_1$ which means that equals the 1 year debt growth rate $\Delta b_t^{(1)}$ described in (36). Now for any set of surplus shocks the resulting nominal debt outstanding can be computed.

Equations (II.5) and (II.6) also contain the parameter λ which is quite handy for adjusting to observable data. If this is not desired, still the researcher can set λ to zero for which (II.5) and (II.6) would be

$$A(N+1) = -\varphi_0 - A(N) - B(N)\phi\bar{s}_{\Delta s}^{(1)} + \frac{1}{2}\left[B(N)\sigma_{\Delta s}\right]^2 \tag{II.7}$$

$$B(N+1) = \left[B(N)(1-\phi) - \varphi_1\right] \tag{II.8}$$

III. Solving the coefficients A(N+1) and B(N+1) assuming that changes in total notional debt outstanding influence the price level or credit spreads

I will solve (44) recursively starting with equation (46) and substituting terms for (47), (48), (51) and (53), and operating results in

$$\ln\left[\Pi_t^{(N)}\right] = A(N) + B(N)\Delta b_t^{(1)} + B(N)\phi\bar{b}_{\Delta b} - B(N)\phi\Delta b_t^{(1)} + B(N)\sigma_{\Delta b}\varepsilon_{t+1} + \frac{1}{2}\lambda^2 + \gamma_0 + \gamma_1\Delta b_t^{(1)} + \lambda\varepsilon_{t+1} \tag{III.1}$$

Rearranging the constant terms and the terms multiplying $\Delta b_t^{(1)}$ and those multiplying ε_{t+1} boils down to

$$\ln\left[\Pi_t^{(N)}\right] = \gamma_0 + A(N) + B(N)\phi\bar{b}_{\Delta b} + \frac{1}{2}\lambda^2 + \left[B(N)(1-\phi) + \gamma_1\right]\Delta b_t^{(1)} + \left[B(N)\sigma_{\Delta b} + \lambda\right]\varepsilon_{t+1}$$

(III.2)

Equation (III.2) has the following conditional mean and variance:

$$E\left[\ln \Pi_t^{(N)}\right] = \gamma_0 + A(N) + B(N)\phi\bar{b}_{\Delta b} + \frac{1}{2}\lambda^2 + \left[B(N)(1-\phi) + \gamma_1\right]\Delta b_t^{(1)} \tag{III.3}$$

$$Var\left[\ln \Pi_t^{(N)}\right] = \left[B(N)\sigma_{\Delta b} + \lambda\right]^2 \qquad \text{(III.4)}$$

Recalling normality for (46)

$$E\left[\ln \Pi_t^{(N)}\right] = E\left[\ln \Pi_{t+1}^{(N)} - \ln \pi_{t+1}^{(N)}\right] + \frac{1}{2}Var\left[\ln \Pi_{t+1}^{(N)} - \ln \pi_{t+1}^{(N)}\right] \qquad \text{(III.5)}$$

Substituting (III.3) and (III.4) in (III.5) yields,

$$E\left[\ln \Pi_t^{(N)}\right] = \gamma_0 + A(N) + B(N)\phi\bar{b}_{\Delta b} + \frac{1}{2}\lambda^2 + \left[B(N)(1-\phi) + \gamma_1\right]\Delta b_t^{(1)} + \frac{1}{2}\left[B(N)\sigma_{\Delta b} + \lambda\right]^2 \qquad \text{(III.6)}$$

Rearranging and grouping the constant terms and the terms multiplying as well as lining up with (51) yields,

$$A(N+1) = \gamma_0 + A(N) + B(N)\phi\Delta b_t^{(1)} + \frac{1}{2}\left(\left[B(N)\sigma_{\Delta b} + \lambda\right]^2 + \lambda^2\right) \qquad \text{(III.7)}$$

$$B(N+1) = \left[B(N)(1-\phi) + \gamma_1\right] \qquad \text{(III.8)}$$

Equations (III.7) and (III.8) are resolved recursively. Since it has been assumed that $\Pi_{t+1}^{(N)} = 1$. and $A(N=0) = B(N=0) = 0$, which means this can be solve recursively, as for 1 period would imply $A(N=1) = \gamma_0$ and B(N=1) = γ_1 which means that equals the 1 year price level growth rate $\Delta\pi_t^{(1)}$ described in (48). Now for any set of notional debt outstanding the resulting price level can be computed and by doing so it is possible to study inflationary shocks or credit spreads – for the case where the government does not control monetary policy.

Equations (III.7) and (III.8) also contain the parameter λ which, as already mentioned, is quite handy for adjusting to observable data. If this is not desired, still the researcher can set λ to zero for which (III.7) and (III.8) would be

$$A(N+1) = \gamma_0 + A(N) + B(N)\phi\bar{b}_{\Delta b} + \frac{1}{2}\left[B(N)\sigma_{\Delta b}\right]^2 \qquad \text{(III.9)}$$

$$(N+1) = \left[B(N)(1-\phi) + \gamma_1\right] \qquad \text{(III.10)}$$

IV. Solving the coefficients A(N+1) and B(N+1) assuming net primary surplus is a function of a state vector of macroeconomic variables

I will solve (59) and (60) recursively starting with equation (61) and (62), and substituting terms for (63) to (66) as well as (71), (72) and (77) and which results in

$$\ln\left[\tau_t^{(N)}\right] = A(N) + B(N)_\tau^{'} x_t + B(N)_\tau^{'}\Phi\bar{x} - B(N)_\tau^{'}\Phi x_t + B(N)_\tau^{'}\sigma_x\varepsilon_{t+1} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2 + \alpha_0 + \alpha_1^{'} x_t + \lambda_\delta^{'}\varepsilon_{t+1} \quad \text{(IV.1)}$$

$$\ln\left[G_t^{(N)}\right] = A(N) + B(N)_G^{'} x_t + B(N)_G^{'}\Phi\bar{x} - B(N)_G^{'}\Phi x_t + B(N)_G^{'}\sigma_x\varepsilon_{t+1} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2 + \beta_0 + \beta_1^{'} x_t + \lambda_g^{'}\varepsilon_{t+1} \quad \text{(IV.2)}$$

Rearranging the constant terms and the terms multiplying x_t and those multiplying ε_{t+1} boils down to

$$\ln\left[\tau_t^{(N)}\right] = \alpha_0 + A(N) + B(N)_\tau^{'}\Phi\bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2 + \left[B(N)_\tau^{'}(I-\Phi) + \alpha_1^{'}\right]x_t + \left[B(N)_\tau^{'}\sigma_x + \lambda_\delta^{'}\right]\varepsilon_{t+1} \quad \text{(IV.3)}$$

$$\ln\left[G_t^{(N)}\right] = \beta_0 + A(N) + B(N)_G^{'}\Phi\bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2 + \left[B(N)_G^{'}(I-\Phi) + \beta_1^{'}\right]x_t + \left[B(N)_G^{'}\sigma_x + \lambda_g^{'}\right]\varepsilon_{t+1} \quad \text{(IV.4)}$$

Equations (IV.3) and (IV.4) have the following conditional mean and variance:

$$E\left[\ln\tau_t^{(N)}\right] = \alpha_0 + A(N) + B(N)_\tau^{'}\Phi\bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2 + \left[B(N)_\tau^{'}(I-\Phi) + \alpha_1^{'}\right]x_t \quad \text{(IV.5)}$$

$$E\left[\ln G_t^{(N)}\right] = \beta_0 + A(N) + B(N)_G^{'}\Phi\bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2 + \left[B(N)_G^{'}(I-\Phi) + \beta_1^{'}\right]x_t \quad \text{(IV.6)}$$

$$Var\left[\ln\tau_t^{(N)}\right] = \left[B(N)_\tau^{'}\sigma_x + \lambda_\delta^{'}\right]^2 \quad \text{(IV.7)}$$

$$Var\left[\ln G_t^{(N)}\right] = \left[B(N)_G^{'}\sigma_x + \lambda_g^{'}\right]^2 \quad \text{(IV.8)}$$

Recalling normality for (61) and (62) yields:

$$E\left[\ln \tau_t^{(N)}\right] = E\left[\ln \tau_{t+1}^{(N)} - \ln \delta_{t+1}^{(N)}\right] + \frac{1}{2} Var\left[\ln \tau_{t+1}^{(N)} - \ln \delta_{t+1}^{(N)}\right] \qquad \text{(IV.9)}$$

$$E\left[\ln G_t^{(N)}\right] = E\left[\ln G_{t+1}^{(N)} - \ln g_{t+1}^{(N)}\right] + \frac{1}{2} Var\left[\ln G_{t+1}^{(N)} - \ln g_{t+1}^{(N)}\right] \qquad \text{(IV.10)}$$

Substituting (IV.5), (IV.6), (IV.7) and (IV.8) in (IV.9) and (IV.10) respectively yields,

$$\begin{aligned} E\left[\ln \tau_t^{(N)}\right] &= \alpha_0 + A(N)_\tau + B(N)_\tau^{'} \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k} \lambda_{i\delta}^2 \\ &+ \left[B(N)_\tau^{'}(I-\Phi) + \alpha_1^{'}\right] x_t + \frac{1}{2}\left[B(N)_\tau^{'} \sigma_x + \lambda_\delta^{'}\right]^2 \end{aligned} \qquad \text{(IV.11)}$$

$$\begin{aligned} E\left[\ln G_t^{(N)}\right] &= \beta_0 + A(N)_G + B(N)_G^{'} \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k} \lambda_{ig}^2 \\ &+ \left[B(N)_G^{'}(I-\Phi) + \beta_1^{'}\right] x_t + \frac{1}{2}\left[B(N)_\tau^{'} \sigma_x + \lambda_\delta^{'}\right]^2 \end{aligned} \qquad \text{(IV.12)}$$

Rearranging and grouping the constant terms and the terms multiplying x_t as well as lining up with (71) and (72) yields,

$$A(N+1)_\tau = \alpha_0 + A(N)_\tau + B(N)_\tau^{'} \Phi \bar{x} + \frac{1}{2}\left(\left[B(N)_\tau^{'} \sigma_x + \lambda_\delta^{'}\right]^2 + \lambda_{i\delta}^2\right) \qquad \text{(IV.13)}$$

$$B(N+1)_\tau = \left[B(N)_\tau^{'}(I-\Phi) + \alpha_1^{'}\right] \qquad \text{(IV.14)}$$

$$A(N+1)_G = \beta_0 + A(N)_G + B(N)_G^{'} \Phi \bar{x} + \frac{1}{2}\left(\left[B(N)_G^{'} \sigma_x + \lambda_g^{'}\right]^2 + \lambda_{ig}^2\right) \qquad \text{(IV.15)}$$

$$B(N+1)_G = \left[B(N)_G^{'}(I-\Phi) + \beta_1^{'}\right] \qquad \text{(IV.16)}$$

Equations (IV.13) to (IV.16) are resolved recursively. Since it has been assumed that $\tau_{t+1}^{(N)} = 1$ and $G_{t+1}^{(N)} = 1$, and $A(N{=}0)_{\tau,G} = 0$ and for $B(N{=}0)'_{\tau,G}$ would result in a vector with all elements being equal to 0, which means this can be solve recursively, as for 1 period would imply $A(N=1)_{\tau,G}$ = [α_0, β_0] and $B(N=1)'_{\tau,G}$ = [α_1, β_1'] which means that equals the 1-year revenue growth rate $\Delta\delta_t^{(1)}$ and 1-year expenditure growth rate $\Delta g_t^{(1)}$, restricted to (73) and (74). Now for any set of macroeconomic state variables the resulting surplus can be computed and by doing so it is possible to study surplus shocks resulting from innovations in macroeconomic variables.

Equations (IV.13) to (IV.16) also contain the parameters λ_δ and λ_g which, as already mentioned, is quite handy for adjusting to observable data. If this is not desired, still the researcher can set λ_δ and λ_g to zero for which (IV.13) to (IV.16) would be

$$A(N+1)_\tau = \alpha_0 + A(N)_\tau + B(N)'_\tau \Phi\bar{x} + \frac{1}{2}\left[B(N)'_\tau \sigma_x\right]^2 \quad \text{(IV.17)}$$

$$B(N+1)'_\tau = \left[B(N)'_\tau (I-\Phi) + \alpha'_1\right] \quad \text{(IV.18)}$$

$$A(N+1)_G = \beta_0 + A(N)_G + B(N)'_G \Phi\bar{x} + \frac{1}{2}\left[B(N)'_G \sigma_x\right]^2 \quad \text{(IV.19)}$$

$$B(N+1)'_G = \left[B(N)'_G (I-\Phi) + \beta'_1\right] \quad \text{(IV.20)}$$

8. The Term Structure, Latent Factors and Macroeconomic Data: A Local Linear Level Model

Section I. Introduction

This paper is an attempt to apply a local level model as seen in Commandeur, Koopman and Ooms (2011) to yield curve dynamics in a similar fashion to the latent factor approach described in the paper by Diebold, Rudebusch and Aruoba (2006) and following the contributions from Diebold and Li (2006). The first stage of this analysis is to use a local level model -with other unobserved components- in order to identify latent factors such as the level, the slope and a seasonal factor. Subsequently, on a second stage, the model links macroeconomic data to these latent factors. Here, the intention is to model the latent factors using the same macroeconomic data as those in Jakas (2011 and 2012) and Jakas and Jakas (2013), and trying to understand which unobserved components are influenced most by the macroeconomy. This model differs from that of Diebold, Rudebusch and Aruoba (2006), as they use a state space model which nests a VAR in order to identify the latent factors such as level, slope and curvature. They then expand the model by incorporating three macroeconomic variables to the state vector. In contrast, in this essay the local level approach is used to identify the latent factors in a state space model and, in a second stage, these latent factors are modelled using macroeconomic data. In addition, since the local level model is used, we incorporate the seasonal component in lieu of the curvature. It could be said that the approach used here is closer to the works of Ang and Piazzesi (2003), Hördahl, Tristani and Vestin (2002) however, this research uses different European data and focused on European yields instead of US data to calibrate the models. In a first step this research departs from the no-arbitrage approach, as our intention is to estimate the latent factors *via* a state space model with an observation equation depicting the level, slope and seasonal components. In a second step an affine term structure model is calibrated with the unobserved states or latent factors in a similar set up as in Jakas (2012). In contrast to Jakas (2012) and Jakas and Jakas (2013) the affine model is calibrated with the latent factors instead of macroeconomic data. It should be said that some of the empirical literature in no-arbitrage such as in Backus, Foresi and Telmer (1998), Duffie and Kan (1996) and Dai and Singleton (2000), do not link latent variables to macroeconomic data or when they do so, empirics have been mostly limited to the short rate. In addition, this paper segregates money markets from capital markets and by doing so performance improves significantly for longer term maturities. This is a reasonable approach, as it could be considered that there are two markets governing the yield curve, somehow contradicting the no-arbitrage approach. The reason for taking this approach is mainly because traders and fixed income strategists make a difference between the two, as liquidity risks and market conventions are different. Notwithstanding, an affine term structure model is calibrated with latent factors and despite results are encouraging for money market yields they are observed to be less impressive for long term maturities.

The chapter is organised as follows: section II presents the local linear level model and the no-arbitrage approach; section III presents discussion of results; section IV an affine term structure model is calibrated using the latent factors estimated in previous section; section V presents some discussion on policy implications; and VI outlines main conclusions and final remarks.

Section II. Yields' unobserved components

In this section we specify yields as a state space model with an unobserved state or transition equation which is linked to an observation or measurement equation. This state space model nests a local level model with a stochastic slope and a stochastic seasonal component. We define the following state and observation equations following the notation from Commandeur, Koopman and Ooms (2011),

$$u_t = u_{t-1} + v_{t-1} + \xi_t \ , \tag{1}$$

$$v_t = v_{t-1} + \zeta_t \ , \tag{2}$$

$$\gamma_{1,t+1} = -\gamma_{1,t} - \gamma_{2,t} - \gamma_{3,t} + \omega_t \ , \tag{3}$$

$$\gamma_{2,t} = \gamma_{1,t-1} , \tag{4}$$

$$\gamma_{3,t} = \gamma_{2,t-1} \ , \tag{5}$$

$$y_t = \beta_1 u_t + \beta_2 v_t + \beta_3 \gamma_{1,t} + \varepsilon_t , \tag{6}$$

where $\xi_t = NID(0, \sigma_\xi^2)$, $\zeta_t = NID(0, \sigma_\varsigma^2)$, $\omega_t = NID(0, \sigma_\omega^2)$, and $\varepsilon_t = NID(0, \sigma_\varepsilon^2)$, $NID(x,\sigma^2)$ being a normal independent-distributed variable with mean x and variance σ^2 (>0). Equation (6) is the observation or measurement equation for y_t, namely, the yield of a zero coupon bond with a given maturity at time *t*. As shown below for the multivariate case, yields will be assumed to be a function of these latent variables or unobserved states, which comprise: (i) the linear trend or level u_t, (ii) the stochastic slope v_t and (iii) the stochastic seasonal component $\gamma_{1,t}$.

We estimate the unobserved states and parameters by maximum likelihood using a Kalman (1960) filter, which can be specified as follows,

$$\boldsymbol{z}_t = \boldsymbol{\alpha}\, \boldsymbol{z}_{t-1} + \boldsymbol{\theta}\, \boldsymbol{\varepsilon}_t , \tag{7}$$

$$\boldsymbol{y}_t = \boldsymbol{\beta}\, \boldsymbol{z}_{t-1} + \boldsymbol{\varphi}\, \boldsymbol{\eta}_t , \tag{8}$$

where,

$\boldsymbol{z}_t$: 5×1 vector of unobserved state variables;

$\boldsymbol{\varepsilon}_t$: 5×1 vector of state-error terms;

$\boldsymbol{y}_t$: $n\times1$ vector of observed endogenous variables depicting the yields;

$\boldsymbol{\eta}_t$: $n\times1$ vector of observation-error terms and,

$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$: $n\times n$ parameter matrices.

Combining equations (1) to (8) in matrix form yields

$$\boldsymbol{z}_t = \begin{pmatrix} u_t \\ v_t \\ \gamma_{1,t} \\ \gamma_{2,t} \\ \gamma_{3,t} \end{pmatrix}, \ \boldsymbol{\alpha} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{9}$$

$$z_{t-1} = \begin{pmatrix} u_{t-1} \\ v_{t-1} \\ \gamma_{1,t-1} \\ \gamma_{2,t-1} \\ \gamma_{3,t-1} \end{pmatrix}, \ \boldsymbol{\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \boldsymbol{\varepsilon}_t = \begin{pmatrix} \xi_t \\ \zeta_t \\ \omega_t \\ 0 \\ 0 \end{pmatrix}, \tag{10}$$

$$\boldsymbol{y}_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{n,t} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n,1} & \beta_{n,2} & \beta_{n,3} & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\varphi}\,\boldsymbol{\eta}_t = \begin{pmatrix} \varepsilon_{y_1} \\ \varepsilon_{y_2} \\ \vdots \\ \varepsilon_{y_n} \end{pmatrix}, \tag{11}$$

Notice that in the matrix $\boldsymbol{\alpha}$ we applied the usual constraints for a local level model with a stochastic slope and a stochastic seasonal component as seen in Commandeur, Koopman and Ooms (2011). Despite that we use monthly data, we set the number of season s to 4. This will not be an issue mainly because we let the seasonal component to remain flexible thanks to the random error term $\boldsymbol{\omega}_t$. In addition, $\boldsymbol{\theta}$ is diagonal as this ensures that random error terms remain uncorrelated, notice that the diagonal elements in $\boldsymbol{\theta}$ are set to one in order to allow the components of $\boldsymbol{\varepsilon}_t$ in Eq. (10) to be free parameters. This is a standard assumption as seen in the no-arbitrage literature (see Piazzesi, 2010; Dai and Singleton, 2000; or Duffie and Kan, 1996; or Backus, Foresi and Telmer, 1998). For simplicity, and without loss of generality, we can also assume that all components in vector $\boldsymbol{\varphi\eta}_t$ are free parameters in the model. By doing so, we follow the local level model as in Commandeur, Koopman and Ooms (2011) and depart from the local level model presented in Drukker and Gates (2011). Finally, we apply a constraint to the coefficients in the measurement equation $y_{1,t}$, however we let all other parameters in matrix $\boldsymbol{\beta}$ free, this does not necessary have to be the case, however it does not affect our analysis and simplifies the estimation. In addition, by letting $\boldsymbol{\beta}$ free for the rest of the maturities it is possible to observe or account for the existence of a term structure effect.

We compute maximum likelihood using the diffuse Kalman filter with the De Jong (1988, 1991) method for estimating the initial values, as our model is non-stationary. For convenience, we have also applied the optimization algorithm Newton-Raphson technique instead of the Marquart and Berndt-Hall-Hall-Hausman, as seen in the works from Diebold, Rudebusch and Aruoba (2006).

In a second stage we estimate via OLS, the effects of macroeconomic data on the latent factors. The macroeconomic data used are the natural logarithms of Euro-Zone Unemployment, Euro-Zone Consumer Confidence Index, ECB M3 levels and Euro-Zone Production Price Index, thus a possible specification could be.

$$\begin{pmatrix} u_t \\ v_t \\ \gamma_{1,t} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{11} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{11} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{11} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}, \tag{12}$$

where $\varepsilon_{1,t} = NID\left(0, \sigma_u^2\right)$, $\varepsilon_{2,t} = NID\left(0, \sigma_v^2\right)$ and $\varepsilon_{3,t} = NID\left(0, \sigma_{\gamma_1}^2\right)$.

We will perform estimations (1-12) twice. Firstly, for Money Market yields: EONIA; Euribor 3M, Euribor 6M, 2 and 5 year German Government Benchmark and, secondly, for the Capital Markets yield curve comprising the 5, 10, 15, 20 and 30 year German Government Benchmarks. The yields and the macroeconomic data are on a monthly basis available in Bloomberg and as of end of month. The period considered is from December 1999 until January 2010, hence resulting in 122 observations.

Linking Latent Factors with a no-arbitrage term structure model

The expected price at t with maturity N+1 of a bond that redeems at t+1 is usually specified as follows,

$$P_t^{N+1} = E\left[m_{t+1} P_{t+1}^N\right], \tag{13}$$

where P_t^{N+1} is the price of a zero coupon bond of maturity N+1 at time t, m_{t+1} being the stochastic discount factor and P_{t+1}^N being the price of the same bond at t+1. By applying natural logarithms one has,

$$\ln\left[P_t^{(N+1)}\right] = \ln\left[m_{t+1}\right] + \ln\left[P_{t+1}^{(N)}\right]. \tag{14}$$

Whereby log prices are related to yields and this can be described as follows,

$$y_{t+1}^{(N)} = -\frac{ln\left[P_{t+1}^{(N)}\right]}{N} \ . \tag{15}$$

As seen in most recent no-arbitrage affine term structure literature log prices can be specified as a linear function of a state vector $\boldsymbol{x}_{t+1}$ as follows:

$$-ln\left[P_{t+1}^{(N)}\right] = A(N) + \boldsymbol{B}(N)'\boldsymbol{x}_{t+1} \ \ , \tag{16}$$

where $A(N)$ is a scalar, $\boldsymbol{B}(N)'$ a $1\times k$ vector of coefficients and $\boldsymbol{x}_{t+1}$ a $k\times 1$ vector of state variables, which for this case $k=3$ for the level, slope and seasonal components. Note that the transpose of a vector or matrix is specified with a " ' ".

From (16) it is possible to find a closed solution and estimate the parameters $A(N)$ and $\boldsymbol{B}(N)'$. These parameters are obtained by linking observable yields to an observation equation describing the behaviour of a space state vector. This can be done by combining equations (15) and (16) at $t+1$ for any maturity, thus yielding,

$$y_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{\boldsymbol{B}(N)'}{N}\boldsymbol{x}_{t+1} \ . \tag{17}$$

Intuitively, the short rate could be specified as follows,

$$y_{t+1}^{(1)} = A(1) + \boldsymbol{B}(1)'\boldsymbol{x}_{t+1} \ . \tag{18}$$

Empirically, equation (18) looks like,

$$y_{t+1}^{(1)} = a_0 + \boldsymbol{a}_1'\boldsymbol{x}_{t+1} \ \ . \tag{19}$$

However, from the restrictions in (11) it is possible to set $a_0 = 0$ and $\boldsymbol{a}_1' = \begin{pmatrix}1 & 1 & 1\end{pmatrix}$. In addition, the state space vector x_t is calibrated as follows,

$$\boldsymbol{x}_t = \begin{pmatrix} u_t \\ v_t \\ \gamma_{1,t} \end{pmatrix} . \tag{20}$$

The stochastic processes for $\boldsymbol{x}_{t+1}$ and for the stochastic discount factor shown in (13) can be specified similarly to the pricing kernel à la Backus-Foresi-Telmer (1998) which here is combined with the Vasicek (1977), for which a possible specification would be like,

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \boldsymbol{\Phi}\left(\bar{\boldsymbol{x}} - \boldsymbol{x}_t\right) + \boldsymbol{\sigma}_x\boldsymbol{\varepsilon}_{t+1} \ \ , \tag{21}$$

$$-ln\left[m_{t+1}\right] = \delta + y_t^{(1)} + \boldsymbol{\lambda}'\boldsymbol{\varepsilon}_{t+1} \ \ \ . \tag{22}$$

Equation (21) describes the stochastic process of the independent state variables. Where $\boldsymbol{x}_t$ and $\bar{\boldsymbol{x}}$ are both *3*-dimensional vectors. $\boldsymbol{\Phi}$ is a 3×3 diagonal matrix, i.e. $\Phi_{i,i} = \Phi_i$, which represent the speed of adjustment at which each of $x_{i,t}$ elements reverse to their means. $\boldsymbol{\sigma}_x$ is a diagonal 3×3 matrix comprising the volatility of the state variables. $\boldsymbol{\varepsilon}_{t+1}$ is a (3×1)-vector of shocks moving $\boldsymbol{x}_t$ away from $\bar{\boldsymbol{x}}$ and with $\varepsilon_{i,t+1}$ elements being normally distributed with mean zero and variance unity.

Equation (22) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (22) was originally the univariate Vasicek (1977) case and this paper calibrates using latent factors instead of the short rate. In this paper and similar to Jakas (2012), the multifactor case of a 3-dimension state variable is used. Furthermore, same as in Backus-Foresi-Telmer (1998), δ is specified as follows,

$$\delta = \frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 \ . \tag{23}$$

Clearly, specification (23) is fortuitous, the only aim is to normalise the stochastic discount factor so that it becomes the inverse of the short rate. Notice that with (23), now (22) has the following conditional means and variance,

$$E\left[-\frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 - y_t^{(1)} - \boldsymbol{\lambda}'\boldsymbol{\varepsilon}_{t+1}\right] = -\frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 - y_t^{(1)},$$

$$Var\left[-\frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 - y_t^{(1)} - \boldsymbol{\lambda}'\boldsymbol{\varepsilon}_{t+1}\right] = \sum_{i=1}^{3}\lambda_i^2 \ ,$$

where $\boldsymbol{\lambda}' = (\lambda_1 \quad \lambda_2 \quad \lambda_3)$. Therefore, assuming $E[\ln x] = \mu(x) + {}^1\!/_2\sigma^2(x)$ it yields,

$E[\ln m_{t+1}] = -y_t^{(1)}$.

Here it is shown how to get to the solution. Starting first with equation (14) and substituting the right hand term for (22) and (16) one obtains,

$$ln\left[P_t^{(N+1)}\right] = -\delta - y_t^{(1)} - \boldsymbol{\lambda}'\boldsymbol{\varepsilon}_{t+1} - A(N) - \boldsymbol{B}(N)'\boldsymbol{x}_{t+1} \quad . \tag{24}$$

In order to solve recursively δ is replaced by (23) and $y_t^{(1)}$ is replaced by (19). In addition, x_{t+1} is also replaced for (21) to account for the Vasicek (1977) process. In sum one has,

$$ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 - \boldsymbol{\gamma}_1{}'\boldsymbol{x}_t - \boldsymbol{\lambda}'\boldsymbol{\varepsilon}_{t+1} - A(N) - \boldsymbol{B}(N)'[\boldsymbol{x}_t + \boldsymbol{\Phi}(\bar{\boldsymbol{x}} - \boldsymbol{x}_t) + \boldsymbol{\sigma}_x\boldsymbol{\varepsilon}_{t+1}]\ . \qquad (25)$$

Notice that a_0 does not appear in Eq. (25) because $a_0 = 0$. The constant terms and the terms multiplying x_t and ε_{t+1} are grouped, thus yielding,

$$ln\left[P_t^{(N+1)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 + A(N) + \boldsymbol{B}(N)'\boldsymbol{\Phi}\bar{\boldsymbol{x}}\right) - [\boldsymbol{a}_1' + \boldsymbol{B}(N)'(\boldsymbol{I} - \boldsymbol{\Phi})]\boldsymbol{x}_t - [\boldsymbol{\lambda}' + \boldsymbol{B}(N)'\boldsymbol{\sigma}_x]\boldsymbol{\varepsilon}_{t+1}\ , \qquad (26)$$

where $\boldsymbol{I}$ denotes the (3×3)-identity matrix. The right hand side of equation (25), which has now developed into (26), has the following conditional moments,

$$E\left[ln\, m_{t+1} + ln\, P_{t+1}^{(N)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 + A(N) + \boldsymbol{B}(N)'\boldsymbol{\Phi}\bar{\boldsymbol{x}}\right) - [\boldsymbol{a}_1' + \boldsymbol{B}(N)'(\boldsymbol{I} - \boldsymbol{\Phi})]\boldsymbol{x}_t\ , \qquad (27)$$

and,

$$Var\left[ln\, m_{t+1} + ln\, P_{t+1}^{(N)}\right] = (\boldsymbol{\lambda}' + \boldsymbol{B}(N)'\boldsymbol{\sigma}_x)^2\ . \qquad (28)$$

Bearing in mind that the implied present-value of a fixed income security yields,

$$-E\left[\ln P_t^{(N+1)}\right] = -E\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right] - \frac{1}{2}Var\left[\ln m_{t+1} + \ln P_{t+1}^{(N)}\right]\ . \qquad (29)$$

By substituting [27] and [28] into [29] one obtains,

$$-E\left[ln\, P_t^{(N+1)}\right] = \frac{1}{2}\sum_{i=1}^{3}\lambda_i^2 + A(N) + \boldsymbol{B}(N)'\boldsymbol{\Phi}\bar{\boldsymbol{x}} + [\boldsymbol{a}_1' + \boldsymbol{B}(N)'(\boldsymbol{I} - \boldsymbol{\Phi})]\boldsymbol{x}_t - \frac{1}{2}(\boldsymbol{\lambda}' + \boldsymbol{B}(N)'\boldsymbol{\sigma}_x)^2. \qquad (30)$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (16) yields,

$$A(N+1) = A(N) + \boldsymbol{B}(N)'\boldsymbol{\Phi}\bar{\boldsymbol{x}} + \frac{1}{2}\left(\sum_{i=1}^{3}\lambda_i^2 - [\boldsymbol{\lambda}' + \boldsymbol{B}(N)'\boldsymbol{\sigma}_x]^2\right), \qquad (31)$$

$$\boldsymbol{B}(N+1)' = \boldsymbol{a}_1' + \boldsymbol{B}(N)'(\boldsymbol{I} - \boldsymbol{\Phi}). \qquad (32)$$

The solution is obtained by computing the present value recursively using (14) for some guess of coefficients from (17). Since $P_{t+1}^{(0)} = 1$, $A(0) = 0$ and $\boldsymbol{B}(0)' = \boldsymbol{0}$, which means this can be solved recursively, as for one period would imply $A(1) = 0$ and $\boldsymbol{B}(1)' = \boldsymbol{a}_1'$ which means that equals the short rate as described in (19). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (17) to compute the present value of an N+1 maturity bond. So all we have to do is to replace (31) and (32) into (17) and solve numerically by fitting the curve to the observed yields by adjusting $\boldsymbol{\lambda}$ for a given choice of maturities, recalling that parameters a_0 and $\boldsymbol{a}_1'$ are restricted to discussion in (11) and (19).

Section III. Discussion of Results

Tables 8-1 and 8-2 show the coefficients and standard errors obtained from the state space model discussed in equations (1) to (11) for Money Market yields (comprising the maturities ranging from EONIA to 5 years) and for Capital Market yields (hence, the maturities ranging from 5 years to 30 years). Notice that it is reasonable to let an overlapping between 2 years and 5 years, as it is generally accepted that between these maturities often Capital and Money Market instruments act as substitutes. Both tables show that the coefficients are very significant with the exception of the seasonal component for the case of the Money Markets, as the seasonal component appears to be significant only for the case of the Capital Markets curve.

Table 8-1 Money Market Curve: EONIA, Euribor 3M, Euribor 6M, 2Yr and 5 Yr German Govy Bond.

	Level u_t	Slope v_t	Season γ_{1t}
EONIA	1	1	1
Euribor 3M	1.042 (.0108)	1.7072 (.2309)	.1453 (.1585)
Euribor 6M	1.0636 (.0107)	1.4985 (.2346)	.0375 (.1644)
BRD 2 Years	1.018 (.0143)	-.9528 (.2650)	.0005 (.1608)
BRD 5 Years	1.1148 (.0233)	-2.9520 (.4203)	-.0945 (.2291)
$\sigma^2_{u;v;\gamma}$ (state)	.0333 (.0048)	-	-
σ^2_y (observation)	.0655 (.0049)	-	-
$\hat{\mu}$	3.0781	-.0460	.0023
$\hat{\sigma}$	1.1236	.1691	.1935

Note: With the exception of the seasonal component all other parameters are very significant with p-values below 0.05, thus P>|z| = 0.

Tables 8-1 and 8-2 also show that for both yield curves –hence the Money Market as well as the Capital Market yield curves- the coefficients for the Level u_t increases with the maturity. Interestingly, the coefficients for the Slope factor v_t in the Money Market curve exhibit different behaviour to increasing maturities with respect to those seen for the Capital Market curve. The coefficients for v_t in the Money Market curve starts with a positive value and becomes negative for the 2 years onwards thus is decreasing. However, the coefficients for v_t in the capital market curve increase as maturities become longer.

Table 8-2 Capital Market Curve: 5, 10, 15, 20 and 30 year German Govy Bonds.

	Level u_t	Slope v_t	Season γ_{1t}
BRD 5 Years	1 -	1 -	1
BRD 10 Years	1.1242 (.0098)	3.3261 (.2345)	1.0536 (.1971)
BRD 15 Years	1.1921 (.0143)	4.4612 (.3414)	1.1864 (.2864)
BRD 20 Years	1.2360 (.0177)	5.3056 (.4228)	1.1558 (.3334)
BRD 30 Years	1.2541 (.0174)	5.257 (.4159)	1.0366 (.3343)
$\sigma^2_{u;v;\gamma}$ (state)	.0305 (.0045)	-	-
σ^2_y (observation)	.0065 (.0006)	-	-
$\hat{\mu}$	3.6729	.0129	.0197
$\hat{\sigma}$	.6855	.1493	.1717

Note: All estimates very significant with p-values P>|z| = 0.

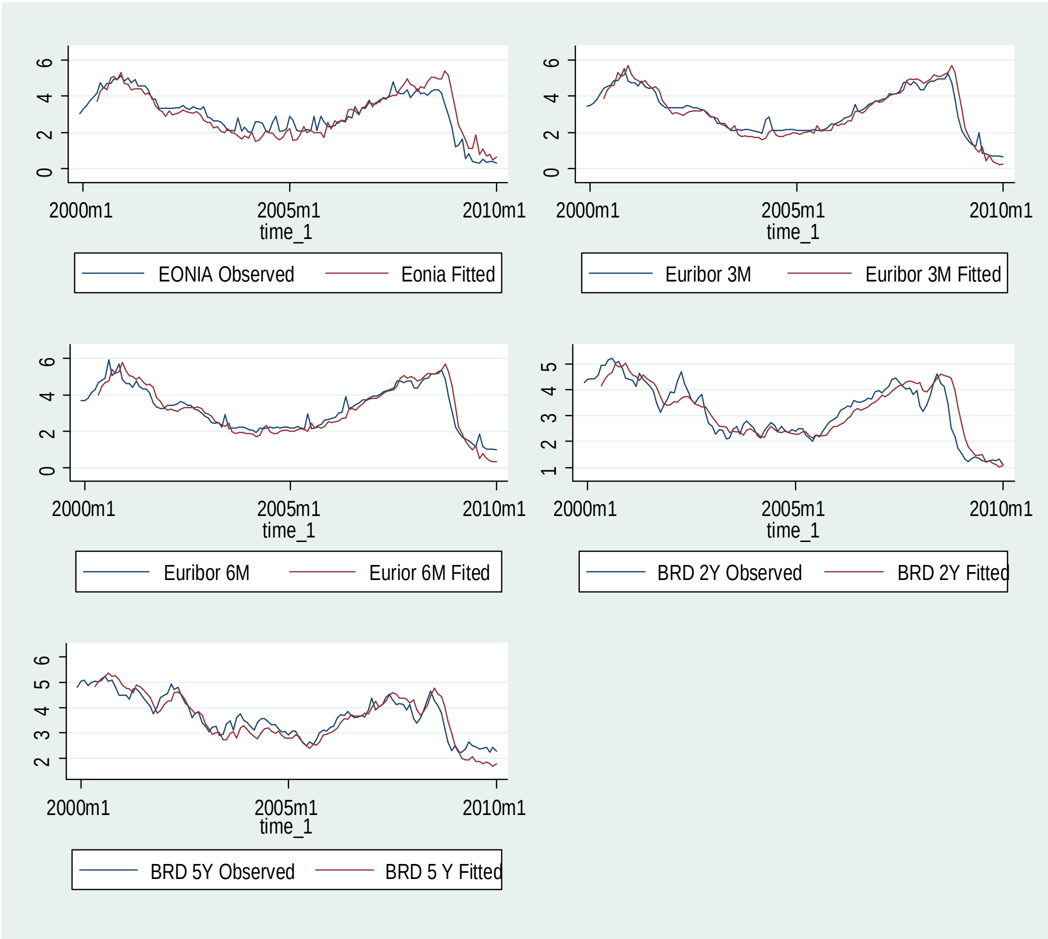
The seasonal component $\gamma_{1,t}$ for the Money Market curve is insignificantly close to zero, and for the capital markets curve is significantly close to one.

Figure 8-1 shows the latent factors (level, slope and seasonal components) as well as the EONIA and 5 year German Government bond. By comparing the top left hand chart which shows the levels for the money and capital markets with the bottom right hand chart which shows the EONIA and the 5 year German Government yield, it is possible to recognise that the level is possibly the most relevant parameter as it appears to follow almost the same stochastic path. The top right hand chart showing the time path for both slopes which vary mostly between -0.25 and +0.25 and breach these boundaries towards around pre and post Lehman's collapse. The capital markets slope appears to lag the money market slope at the beginning of the series and exhibits rather smoother turnarounds. The seasonality component for both time series seems stationary with no apparent trend.

Figure 8-1 Latent Factors Level, Slope, Curvature and rates for Money Markets and Capital Markets rates.

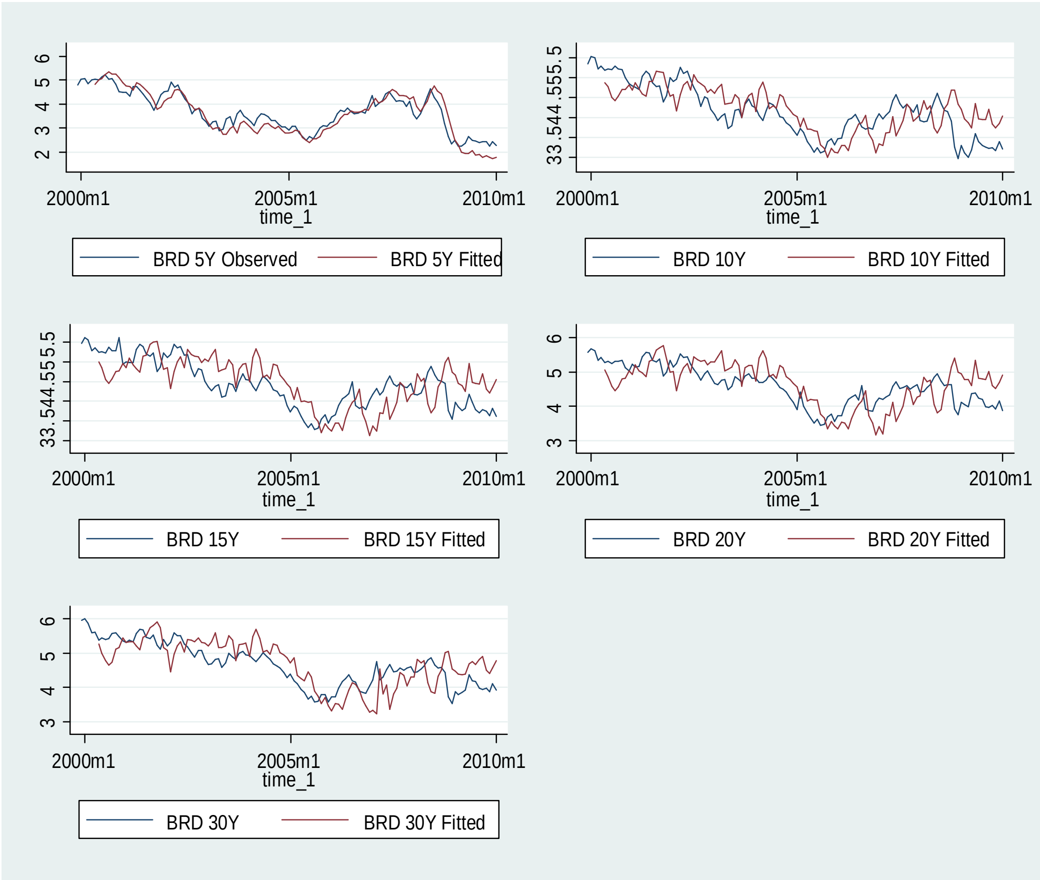
Figures 8-2 and 8-3 show fitted versus observed values obtained by running the state space model described in (1) to (11). The fitted values seem to follow quite close the observed yields. These results are encouraging, as they are very similar to those seen in Jakas (2011, 2012) and Jakas and Jakas (2013). Not surprisingly, this stems from the fact that the latent factors estimated do a good job in replicating the yields, as most of the effect comes from the Level, which shows a very similar behaviour to the EONIA rate. In addition, it can be seen that as maturities become larger the model performs poorer, but still better than the results seen in Jakas (2012) and Jakas and Jakas (2013), where solely macro data were used for calibrating the model. As maturities become longer, the Capital Market Level is likely to be more influential than the Money Market Level, thus suggesting that there are long term components evidencing a different structure between the front and the long end of the curve. This improvement is mostly due to the fact that the yield curve has been segregated between money and capital markets and hence latent factors for longer maturities are different, as they carry information which is more relevant to yields on the long end, whereas latent factors influencing the short end have less predictive ability on long end yields.

Figure 8-2 Fitted Versus Observed Money Market Yields.



It should be mentioned that most of the research has been focused on yields up to 10 year maturities. Models fitting yields in the short end up to 10 years always perform better than for those trying to fit longer maturities such as 20 and 30 years. If the state space for the capital markets is run by dropping from the model the 5 years and leaving only the maturities comprising 10, 15, 20 and 30 years the fitted values become even closer to the observed long end yields. This effect is mostly attributed to the fact that the front end of the curve appears to have less information influencing yields on the longer maturities.

Figure 8-3 Fitted Versus Observed Capital Market Yields.



Tables 8-3 and 8-4 show the OLS (robust) results as discussed in (12). Recalling that the Level u_t is the most important factor governing the yields, the macroeconomic data influencing this factor is analysed in this section. Table 8-3 and Table 8-4 show that coefficients for the macroeconomic factors influencing the Level, Slope and Seasonal components is smaller for capital market latent factors compared to money market latent factors. In fact, the signs of the coefficients only seem to be in agreement for the case of the Level. For the Slope factor, only the consumer confidence coefficients are similar in size and exhibit the same sign. For the Seasonal component the coefficients for unemployment and consumer confidence exhibit same signs however, they differ in size significantly.

Table 8-3 OLS (robust) results of Money Market Latent factors versus Macroeconomic data.

	Level u_t	Slope v_t	Season γ_{1t}	$\hat{\mu}$
Ln Unemployment	-9.655936 (.5953772)	-1.348114 (.1271292)	.4216055 (.301257)	2.1329
Ln PPI	18.67764 (2.140462)	3.700983 (.7038401)	1.658626 (1.0074)	4.6036
Ln ECB M3	-8.055411 (.7723205)	-1.299466 (.2410876)	-.1551028 (.3770993)	8.8127
Ln Con. Conf. Index	-1.070341 (.556391)	-.9127945 (.1554375)	-.1715393 (.1920829)	4.4767
Intercept	13.47607 (5.312529)	1.323285 (1.304486)	-6.404112 (2.249984)	
R-squared	0.9267	0.7702	0.2578	

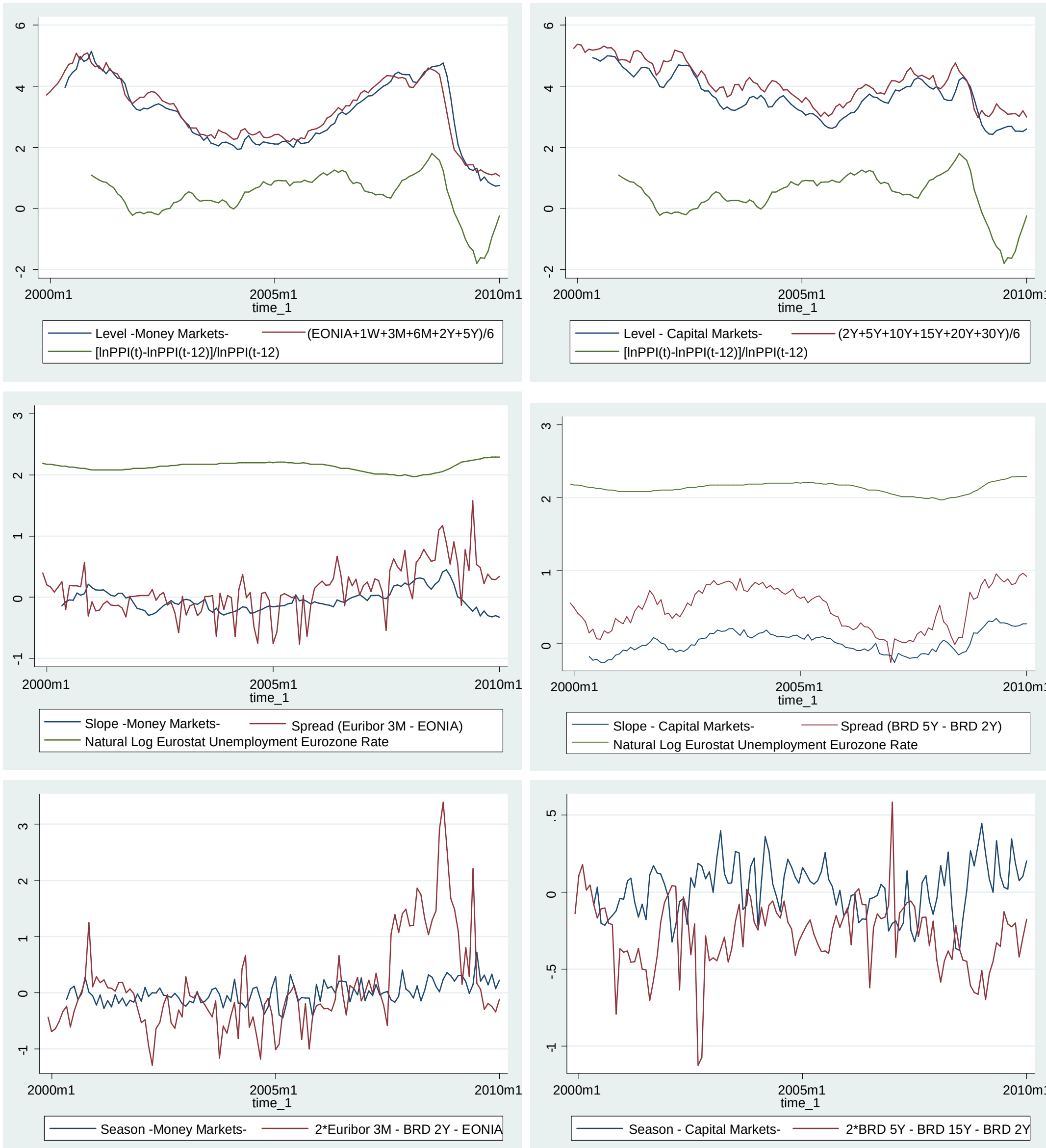
Table 8-4 OLS results of Capital Market Latent factors versus Macroeconomic data.

	Level u_t	Slope v_t	Season γ_{1t}	$\hat{\mu}$
Ln Unemployment	-4.60098 (.5729704)	.4937988 (.1106354)	.083843 (.2393815)	2.1329
Ln PPI	2.715887 (2.305689)	-2.10111 (.3090523)	-1.455954 (1.062349)	4.6036
Ln ECB M3	-2.710198 (.7925468)	.7113546 (.1122904)	.3672107 (.378095)	8.8127
Ln Con. Conf. Index	.9895847 (.4375565)	-.8902952 (.084333)	-.9289367 (.2541476)	4.4767
Intercept	20.46274 (4.951114)	6.344468 (.8901571)	7.462491 (2.146008)	
R-squared	0.8047	0.8633	0.3356	

The theoretical interpretation of the effects of the macroeconomic data on the Slope and Seasonal component factors are left for the reader to go through the exercise. However, the interpretation of the level is less challenging, as it appears to be pretty much in line with economic theory. For example table 8-3 and 8-4 show that increases in unemployment rate results in a fall in yields. This makes sense as the yield curve studied is the risk free curve and hence if unemployment increases, expected aggregate consumption growth is expected to be lower with the subsequent fall in risk-free asset yields. On the other hand, a fall in unemployment is expected to decrease inflationary pressures so that central banks have no reason for keeping policy rates high and hence are likely to introduce rate cuts. In same fashion, if the price level PPI increases this is expected to result in an increase in the short rate as a consequence of central bank policy, but also an increase in the price level is expected to rise the overall level of interest rates, mostly, in order to compensate investors for the loss in value on real money balances. An increase in money supply M3 is expected to result in a fall in interest rates, as seen in the classical IS-LM models. The sign of the consumer confidence index is not as expected by the theory. In addition, the coefficient does not appear to be very significant and its contribution to the overall variance is negligible.

Figure 8-4 shows the latent factors, which are the level, the slope and seasonal component and their empirical counterparts. Here, the empirical counterparts differ to those of Diebold, Rudebusch and Aruoba (2006) as in this research the yield curve is segregated into money markets and capital markets. Notice that by doing so it is possible to account for different behaviours of the level in the front end and the level in the long end of the curve. For both cases the level fits very well the observed empirical counterparts. Correlations between Money and Capital Market levels u_t with their empirical counterparts (EONIA+…+5Y)/6 and (2Y+…+30Y)/6 is of 0.93 and 0.88 respectively. The correlation between the money and capital market levels with current inflation $\left((\ln PPI_t - \ln PPI_{t-12})/\ln PPI_{t-12}\right)$is of 0.34 and 0.06 respectively. Similar to Diebold, Rudebusch and Aruoba (2006) we link yield levels with inflation, as suggested by the Fischer equation, as seen also in prominent macro-finance literature such as Hördahl, Tristani and Vestin (2003) and Rudebusch and Wu (2003), amongst others.

Figure 8-4 Level, slope, seasonal components and their empirical counterparts.



For the case of the slope, it can be shown that in both cases it is possible to replicate very well the trend however Money Market slope is less volatile than its empirical counterpart. Interestingly, the slope for the Capital Market follows a similar pattern to that of its empirical counterpart. Correlations between Money and Capital Market slopes v_t with their empirical counterparts (Euribor3M-EONIA) and (5Y-2Y) is of 0.14 and 0.78 respectively. The correlation between the money and capital market levels with current unemployment is of 0.57 and 0.53 respectively. So these results appear to be in support u_t representing interest rates levels for money and capital markets and v_t being the slope which seems to be more relevant for the Capital Market and less relevant for the Money market curve. The seasonal component is rather inconclusive, as for the Money Market the empirical counterpart is much more volatile, particularly during the Lehman collapse. Somehow a better picture is observed for the Capital Markets seasonal component however still, they do not seem to match as nicely as it did for the level or the slope factors.

So far, this paper has concentrated in applying a local level model with a stochastic slope and a stochastic seasonal component with no feedback. Thus, innovations in the latent variables do not feed back to the macroeconomy. This assumption can be tested via a basic VAR model, orthogonal impulse response functions as well as the forecast error variance decompositions and the classical Granger Causality test, all of these will be taken care of in this section.

Results for lag selection

For the lag order selection criteria for a series vector autoregressions of order 1, a prediction error (FPE), Akaike's information criterion (AIC), Schwarz's Bayesian information criterion (SBIC), and the Hannan and Quinn information criterion (HQIC) are used. According to our results, the FPE and AIC selected 2 lags, and SBIC and HQIC selected 1 lag for the Money Markets curve. So for simplicity's sake and following the theoretical advantages of using SBIC and HQIC over FPE and AIC, as discussed in Lütkepohl (2005, 148-152) 1 lag is selected for the VAR. For the Capital Markets curve the SBIC and HQIC selected 4 and 1 lags accordingly. Here, in order to keep consistency, the lags selected have also been of 1 order.

Tables 8-5 and 8-6 show VAR results for the coefficients and standard errors for the Money Market and Capital Market latent factors with the respective macroeconomic variables. The macroeconomic variables used exhibit a significant autocorrelation to their one-period lags and appear not to be a function of the other macroeconomic or latent factors. On the contrary, the latent factors do seem to be influenced by some of its own lags, as well as by lagged macroeconomic data. For the reader's convenience, coefficients which are significantly different from zero have been bold highlighted. According to the VAR results, there appears to be very little feedback from latent factors to the macroeconomy and a rather significant feedback from the macroeconomy to the latent factors. This does not mean that there is no feedback at all from latent factors to the macroeconomy, but rather that this feedback is weaker. This is a striking result, as it would appear that interest rate levels provide little feedback to the macroeconomy according to the period and variables analysed.

Figures 8-5 and 8-6 show the orthogonalised impulse response functions and, they seem to confirm this view. These results are in line to those seen in Jakas (2011) and in line with the VAR results, which suggests that little or no feedback is observed between latent factors and the macroeconomy, however there is a clear statistical relationship from lagged macro variables to the latent factors particularly to the level.

Table 8-5 VAR results of Money Market Latent factors versus Macroeconomic data.

	$\ln U_{t-1}$	$\ln PPI_{t-1}$	$\ln M3_{t-1}$	$\ln CC_{t-1}$	Level u_{t-1}	Slope v_{t-1}	Season $\gamma_{1,t-1}$	Intercept
LnU_t	**1.007** **(.0274)**	-.1177 (.0650)	.0385 (.0256)	-.1051 (.0143)	.0074 (.0028)	-.0092 (.0102)	.0038 (.0044)	.6337 (.1336)
$LnPPI_t$	-.0270 (.0157)	**.9863** **(.0372)**	.0049 (.0146)	.0462 (.0082)	-.0047 (.0016)	.0083 (.0058)	.0003 (.0025)	-.1130 (.0764)
$LnM3_t$	-.0531 (.0187)	.1021 (.0444)	**.9559** **(.0174)**	.0012 (.0097)	-.0050 (.0019)	.0094 (.0069)	-.0053 (.0030)	.0470 (.0911)
$LnCC_t$	-.0738 (.0595)	.0794 (.1410)	-.0366 (.0553)	**1.015** **(.0311)**	-.0131 (.0061)	-.0156 (.0222)	.0100 (.0096)	.0830 (.2895)
Level u_t	**-1.2932** **(.4520)**	**5.7258** **(1.0709)**	**-1.828** **(.4207)**	**.9297** **(.2356)**	**.8646** **(.0465)**	**-.6903** **(.1688)**	-.1798 (.0730)	**-11.2878** **(2.1988)**
Slope v_t	.0765 (.1868)	**1.292** **(.4426)**	-.3223 (.1737)	-.0614 (.0974)	.04016 (.0192)	**.6247** **(.0698)**	-.0715 (.0301)	**-3.140** **(.9087)**
Season γ_{1t}	**1.8691** **(.5679)**	-.3229 (1.3452)	.8502 (.5284)	-.3829 (.2961)	.2130 (.0585)	**-.4671** **(.2120)**	.0437 (.0917)	**-8.9613** **(2.7621)**

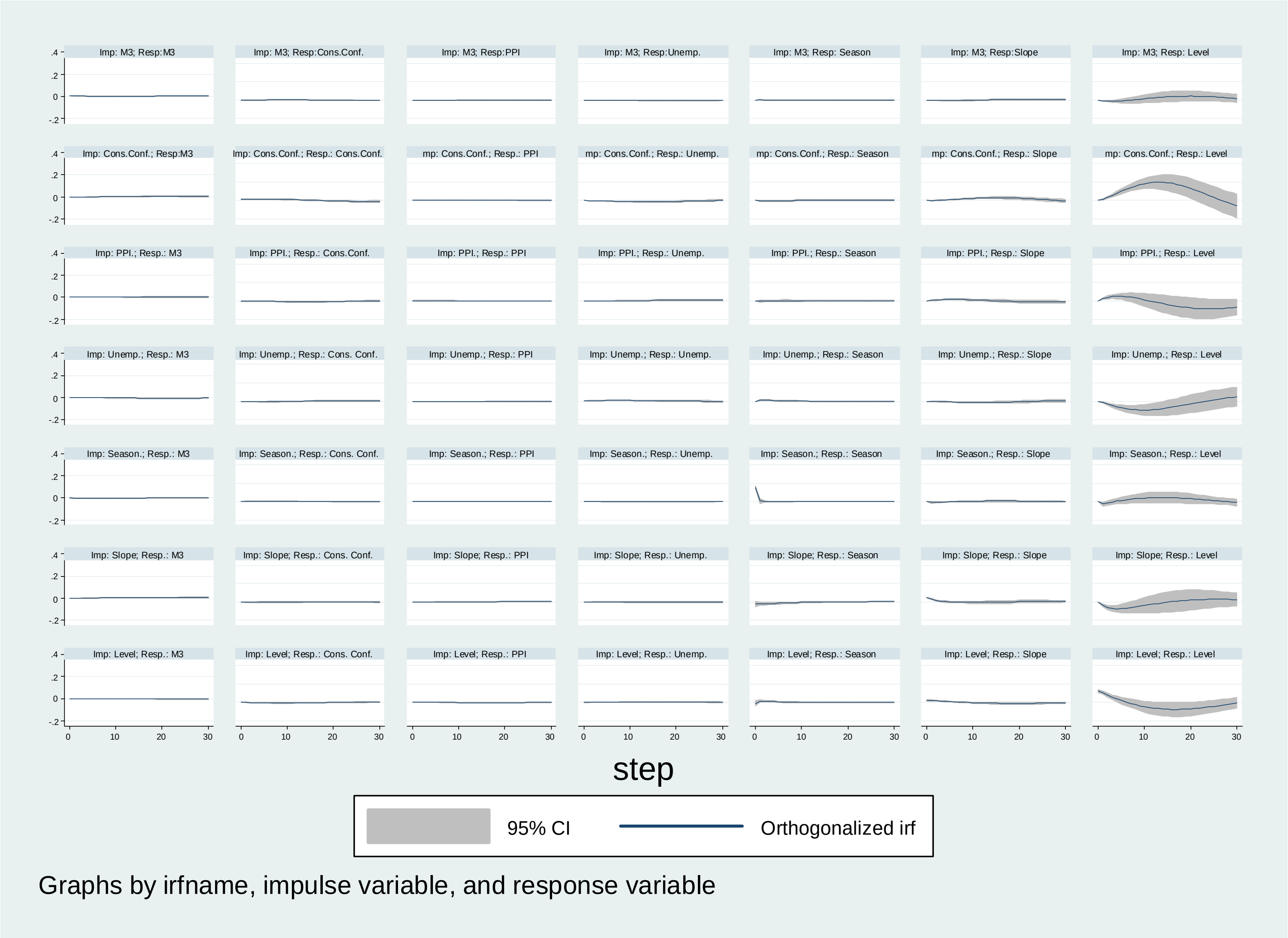
Note: Sample: 2000m6 - 2010m1; No. of obs = 116; all equations significant with P>chi2 at P-values of 0.0000; All equations with R-sq > 0.91 except for season at 0.36 , and LnU_t = Euro-Zone Unemployment rate; $LnPPI_t$ = Euro-Zone Production Price Index; $LnM3_t$ = ECB M3 Money Aggregate and, $LnCC_t$ = Euro-Zone Consumer Confidence Index.

Table 8-6 VAR results of Capital Market Latent factors versus Macroeconomic data.

	$\ln U_{t-1}$	$\ln PPI_{t-1}$	$\ln M3_{t-1}$	$\ln CC_{t-1}$	Level u_{t-1}	Slope v_{t-1}	Season $\gamma_{1,t-1}$	Intercept
LnU_t	**.9861** **(.0174)**	.0261 (.0531)	-.0018 (.0190)	-.0865 (.0161)	.0098 (.0024)	.0222 (.0139)	.0076 (.0052)	.2763 (.1667)
$LnPPI_t$	.1667 (.0102)	**.9456** **(.0310)**	.0184 (.0111)	.0477 (.0094)	-.0041 (.0013)	.0057 (.0081)	-.0051 (.0030)	-.0784 (.0974)
$LnM3_t$	-.0371 (.0123)	.0257 (.0376)	**.9793** **(.0134)**	-.0033 (.0114)	-.0052 (.0016)	-.0163 (.0099)	.0079 (.0036)	.1833 (.1181)
$LnCC_t$	.0395 (.0395)	-.2772 (.1202)	.0874 (.0430)	**1.003** **(.0365)**	-.0115 (.0053)	-.0351 (.0316)	-.0178 (.0117)	.4482 (.3774)
Level u_t	**-.5870** **(.2156)**	-.8975 (.6563)	.0902 (.2352)	**.4419** **(.1993)**	**.8358** **(.0291)**	.2504 (.1727)	**-.6970** **(.0640)**	3.2088 (2.0599)
Slope v_t	**.1872** **(.0951)**	-.2362 (.2896)	.1164 (.1038)	**-.3608** **(.0879)**	.0319 (.0126)	**.7204** **(.0762)**	**.0794** **(.0282)**	1.1639 (.9091)
Season γ_{1t}	.3303 (.3049)	.6967 (.9279)	-.0833 (.3325)	-.1485 (.2818)	**.1340** **(.0412)**	**.7537** **(.2443)**	**.2891** **(.0905)**	-3.0001 (2.9124)

Note: Sample: 2000m6 - 2010m1; No. of obs = 116; all equations significant with P>chi2 at P-values of 0.0000; All equations with R-sq > 0.92 except for season at 0.46, and LnU_t = Euro-Zone Unemployment rate; $LnPPI_t$ = Euro-Zone Production Price Index; $LnM3_t$ = ECB M3 Money Aggregate and, $LnCC_t$ = Euro-Zone Consumer Confidence Index.

Figure 8-5 Impulse Response Functions for Money Market Latent and Macroeconomic variables.



Note: "M3" for ECB M3 Money Aggregate; "Cons.Conf." as Euro-Zone Consumer Confidence Index; "PPI" for Euro-Zone Production Price Index and "Unemp." for Euro-Zone Unemployment Rate.

Figure 8-6 Impulse Response Functions for Capital Market Latent and Macroeconomic variables.

Note: "M3" for ECB M3 Money Aggregate; "Cons.Conf." as Euro-Zone Consumer Confidence Index; "PPI" for Euro-Zone Production Price Index and "Unemp." for Euro-Zone Unemployment Rate.

In light of the above results, a Granger Causality test is performed. Here it is possible to observe in tables 8-7 and 8-8 below that feedback from latent factors to the macro variables exist. The test though does not tell us the size of these feedbacks. For example, in the case of Money Market yields (see table 8-7) it is possible to see that Consumer Confidence and the Level granger-cause unemployment rate and PPI, amongst others (see bold font in columns $lnCC_{t-1}$ and $Level_{t-1}$). It can also be seen that monetary aggregate M3 is granger-caused by unemployment, PPI and the Level (with *p*-values: 0.005, 0.021 and 0.009 respectively). Consumer Confidence is only granger-caused by the Level (with a *p*-value: 0.032). The Level is granger-caused by all macro and latent factors. The Slope is granger-caused by PPI, the Level and by the Seasonal component (with *p*-values: 0.004, 0.037 and 0.018 respectively). Interestingly, the Seasonal component is granger-caused by the unemployment rate, the Level and the Slope (with *p*-values: 0.001, 0.000 and 0.028 respectively). The Granger Causality test using Capital Market latent factors have the following discrepancies with respect to the Money Market latent factors; 1) PPI does not granger-cause M3 (*p*-value: 0.495), 2) PPI, M3 and the Slope do not granger-cause the Level (*p*-values: 0.172, 0.701 and 0.147, respectively), 3) Unemployment and Consumer Confidence index granger-cause the Slope, 4) PPI does not granger-cause the slope and 4) Unemployment does not granger-causes the seasonal component.

Table 8-7 Money Markets Granger Causality test.
For Prob > chi2, so that if below p-values ≤ 0.05 the Ho "Excluded variable does not granger cause the equation of the endogenous variable" is rejected.

Excluded / Equation	lnU_{t-1}	$lnPPI_{t-1}$	$lnM3_{t-1}$	$lnCC_{t-1}$	Level u_{t-1}	Slope v_{t-1}	Season $\gamma_{1,t-1}$	All
LnU_t	**-**	0.070	0.132	**0.000**	**0.008**	0.365	0.388	0.000
$LnPPI_t$	0.085	-	0.735	**0.000**	**0.004**	0.152	0.883	0.000
$LnM3_t$	**0.005**	**0.021**	-	0.894	**0.009**	0.175	0.075	0.000
$LnCC_t$	0.214	0.573	0.508	-	**0.032**	0.481	0.297	0.000
Level u_t	**0.004**	**0.000**	**0.000**	**0.000**	**-**	**0.000**	**0.014**	0.000
Slope v_t	0.682	**0.004**	0.064	0.528	**0.037**	-	**0.018**	0.000
Season $\gamma_{1,t}$	**0.001**	0.810	0.108	0.196	**0.000**	**0.028**	-	0.000

Note: LnU_t = Euro-Zone Unemployment rate; $LnPPI_t$ = Euro-Zone Production Price Index; $LnM3_t$ = ECB M3 Money Aggregate and, $LnCC_t$ = Euro-Zone Consumer Confidence Index.

Table 8-8 Capital Markets Granger Causality test.

For Prob > chi2, so that if below p-values ≤ 0.05 the Ho "Excluded variable does not granger cause the equation of the endogenous variable" is rejected.

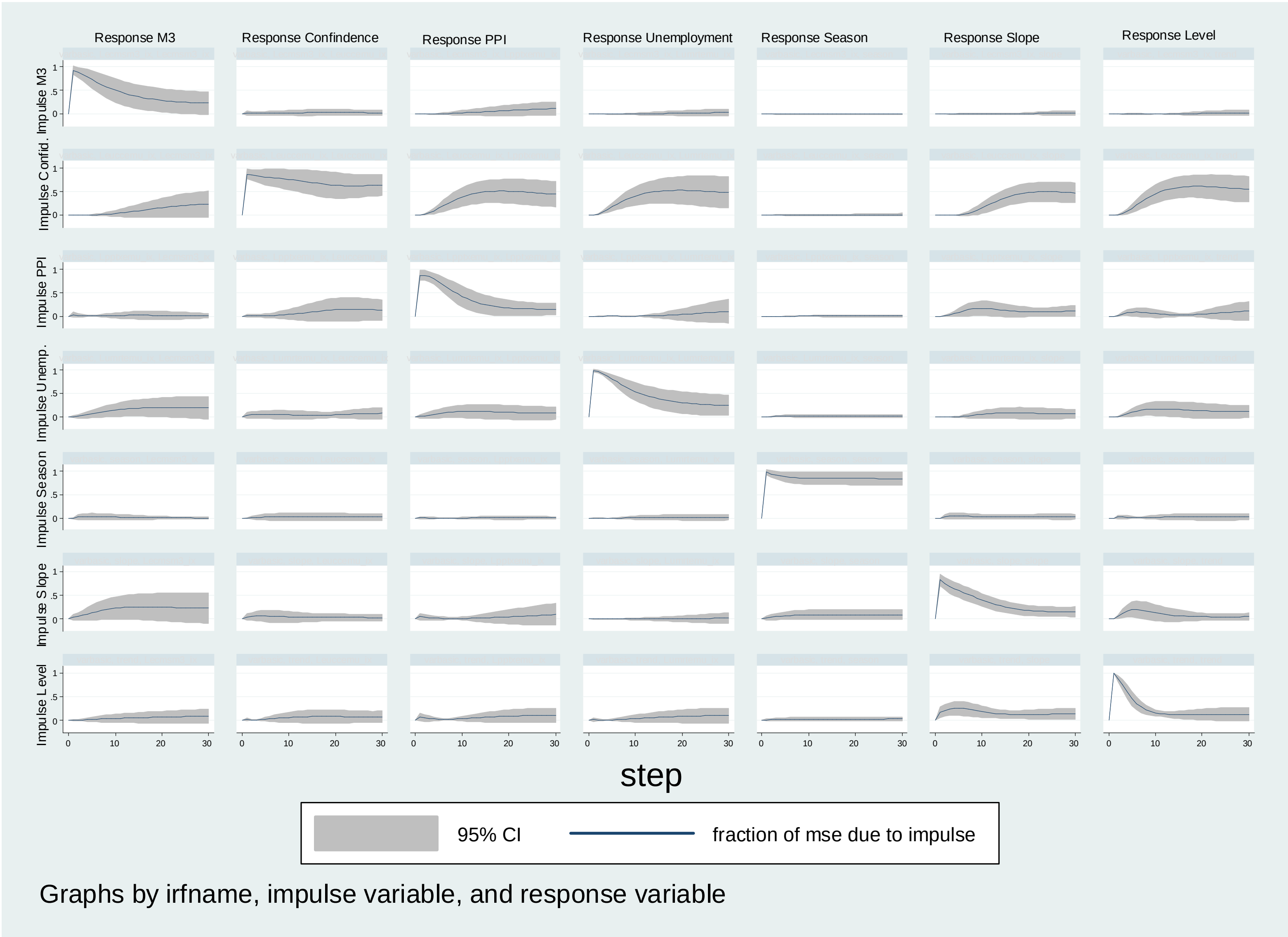
Excluded / Equation	lnU_{t-1}	$lnPPI_{t-1}$	$lnM3_{t-1}$	$lnCC_{t-1}$	Level u_{t-1}	Slope v_{t-1}	Season $\gamma_{1,t-1}$	All
LnU_t	-	0.624	0.927	**0.000**	**0.000**	0.112	0.138	0.000
$LnPPI_t$	0.155	-	0.097	**0.000**	**0.003**	0.481	0.090	0.000
$LnM3_t$	**0.003**	0.495	-	0.769	**0.002**	0.100	**0.030**	0.000
$LnCC_t$	0.318	**0.021**	**0.042**	-	**0.030**	0.268	0.129	0.000
Level u_t	**0.006**	0.172	0.701	**0.027**	-	0.147	**0.000**	0.000
Slope v_t	**0.049**	0.415	0.262	**0.000**	**0.013**	-	**0.005**	0.000
Season $\gamma_{1,t}$	0.279	0.453	0.802	0.598	**0.001**	**0.002**	-	0.000

Note: LnU_t = Euro-Zone Unemployment rate; $LnPPI_t$ = Euro-Zone Production Price Index; $LnM3_t$ = ECB M3 Money Aggregate and, $LnCC_t$ = Euro-Zone Consumer Confidence Index.

In summary, it could be said that there is a clear effect from the macroeconomy to the yield curve and from the yield curve to the macroeconomy however, from what we have learned from VAR tables 8-5 and 8-6 and impulse response figures 5 and 6, feedback from the yield curve to the macroeconomy seems to be weak.

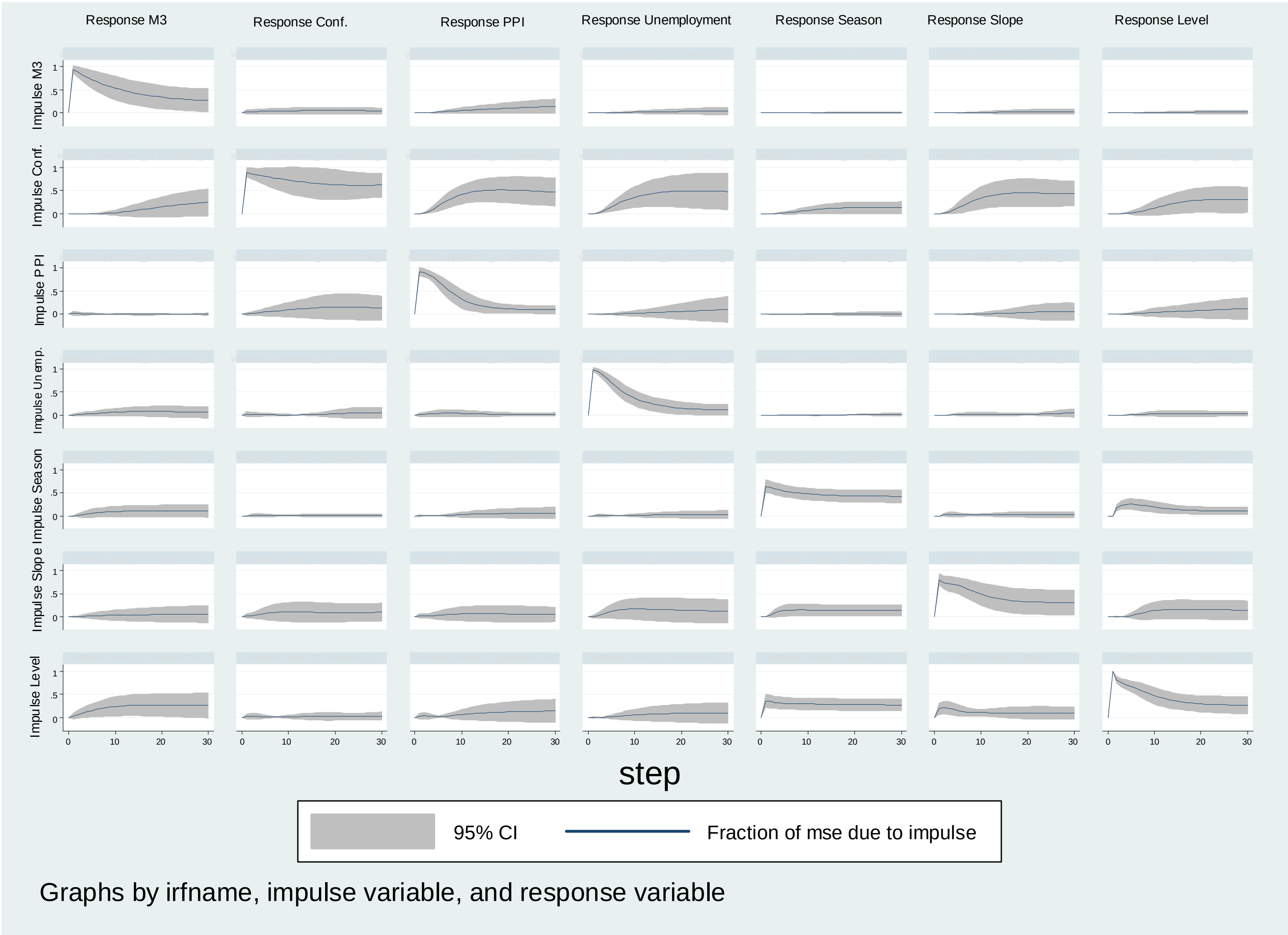
Figures 8-7 and 8-8 show the Cholesky forecast error variance decomposition (FEVD) to orthogonal shocks. Here, it is also possible to observe that orthogonal shocks cause permanent effects on some of the latent and macro factors. For example, shocks on consumer confidence index (second row) has permanent effects on PPI, unemployment, the slope and the level, thus these shocks do not die away, but in turn persist in time. Interestingly, own variable orthogonal shocks, thus auto-shocks, in some factors appear to have permanent effects too or exhibit persistence, as they appear to last several periods before they die away (see for example the diagonal charts in figures 8-7 and 8-8 below, for instance, see the consumer confidence index and the seasonal component diagonal charts). Another interesting outcome is that the level, the slope and the monetary aggregate M3 have weak impact on the price level (PPI). This can be seen –presumably- as a result of the ECB being successful in anchoring long term inflation expectations at low levels. The price level seems to be mostly influenced by orthogonal shocks on consumer confidence index which in turn seems to be independent of all other factors.

Figure 8-7 Forecast Variance Decomposition to Cholesky orthogonal shocks for Money Market latent and macroeconomic variables.



Note: "M3" for ECB M3 Money Aggregate; "Confid." or "Confidence" as Euro-Zone Consumer Confidence Index; "PPI" for Euro-Zone Production Price Index and "Unemp." or "Unemployment" for Euro-Zone Unemployment Rate.

Figure 8-8 Forecast Variance Decomposition to Cholesky orthogonal shocks for Capital Market latent and macroeconomic variables.



Note: "M3" for ECB M3 Money Aggregate; "Confid." or "Confidence" as Euro-Zone Consumer Confidence Index; "PPI" for Euro-Zone Production Price Index and "Unemp." or "Unemployment" for Euro-Zone Unemployment Rate.

Interestingly enough, a Johansen test for cointegration has been also performed for both the Money Markets as well as the Capital Markets latent factors applying a vector error correction model of two lags on the Level, Slope, Season, and unemployment rate, M3, PPI and Consumer Confidence index. The results show that Money Markets exhibit at least four cointegrating equations and that Capital Markets exhibit three cointegrating equations. This suggests that the latent factors and macroeconomic variables used are highly cointegrated.

Section IV. Calibrating an Affine Term Structure Model with Latent Factors

In this section an affine term structure model is calibrated with the money market latent factors discussed in previous sections and entered into equations (13) to (32). Thus a no-arbitrage model is fitted by calibrating the state vector with the latent variables: obtained from the local level model. Figure 9 below shows that the affine approach to yield curve modelling seems to fit quite well the observed yields, even for the 10 year maturities. However, this deteriorates as the maturity gets longer as seen also in most of the empirical research. Clearly, these results suggest that short term components in the yield curve (money market level, slope and season) which have high predictive power on the front end of the curve, exhibits a diminishing predictive power as maturities become larger. It could also be interpreted that the front and the long end of the curve are governed by different factors which appear not to have much in common, thus casting some doubt on the use of a no-arbitrage model for these maturities spectrum. The results shown in figure 8-9 seem to support the approach of breaking the yield curve in two types of markets: the money markets and the capital markets. For money markets being the yields governed by short term latent factors, hence comprising the maturities from overnight (EONIA) to 2-5 year German Govies, and for capital markets being the yields governed by long term latent factors, hence for maturities ranging from 2-5 years up to the 30 year German Govies. Figures 8-10 and 8-11 show the average yield curve fitted using the affine term structure model and the coefficients for equation (17) discussed in section 2. Here it can be seen that all coefficients are positive and decrease as maturities get closer to the 30 years (or 360 months), with the exception of the $A(N)/N$ which increases as maturity becomes larger.

Figure 8-9 Yield curve fitted with an affine term structure model using latent factors as state variables.

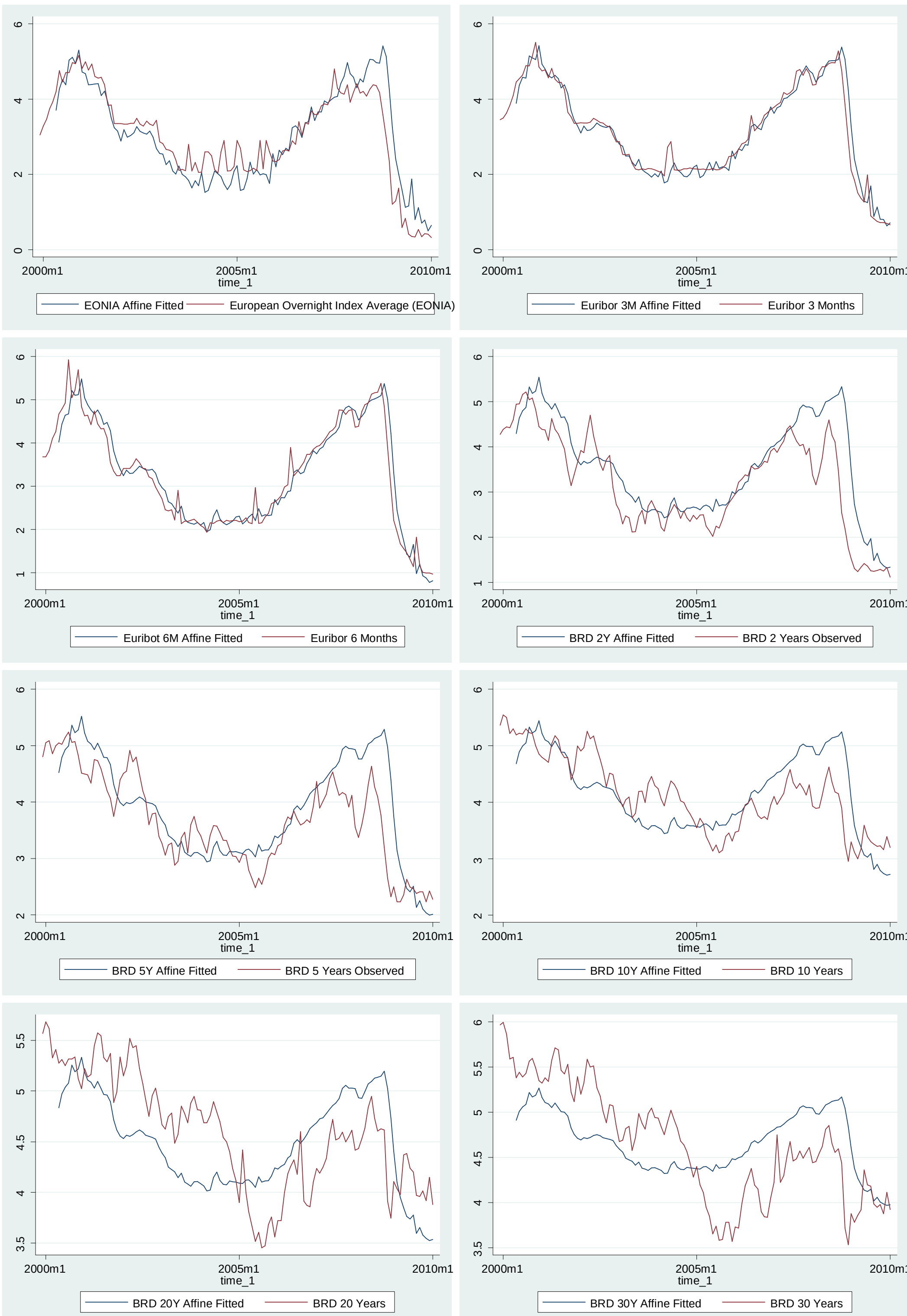


Figure 8-10 Yield curve fitted with an affine term structure model using latent factors as state variables.

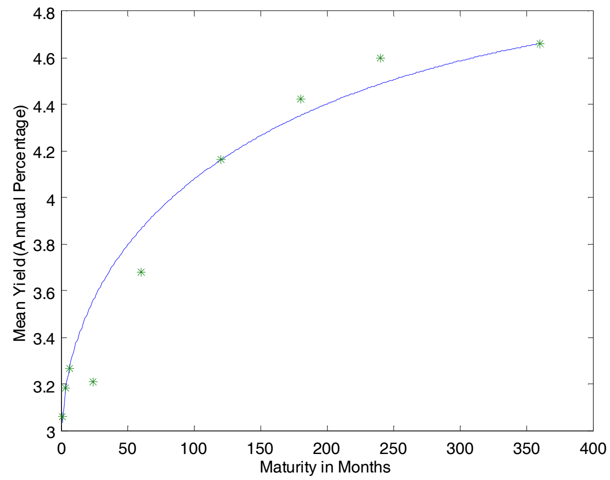
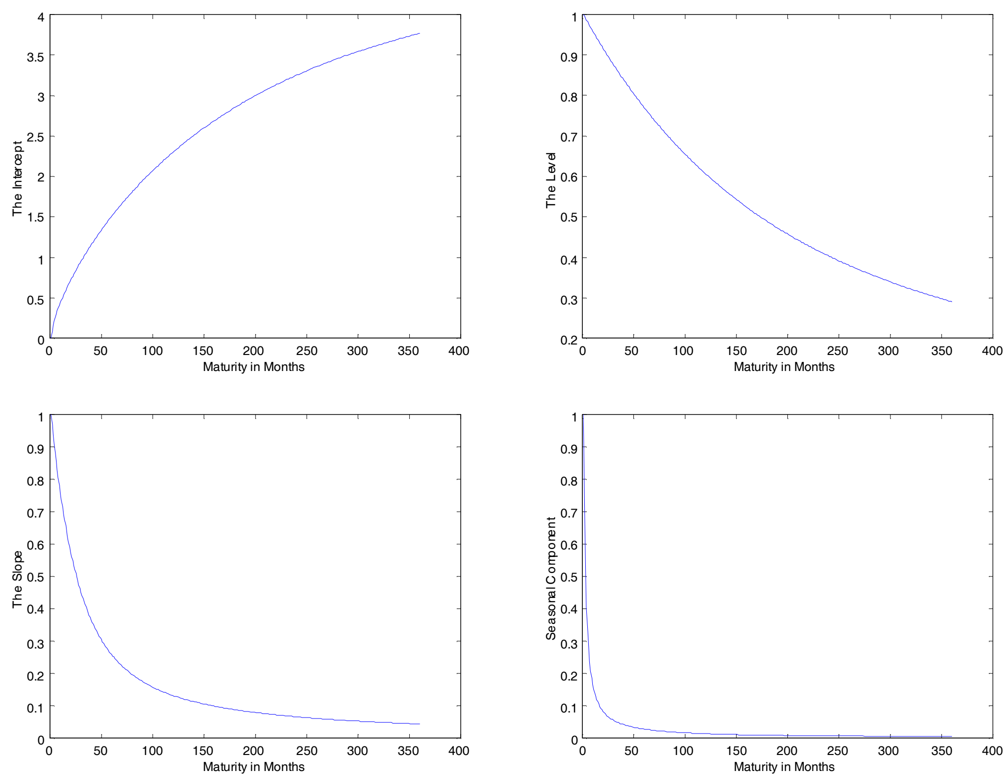


Figure 8-11 Fitted coefficients $A(N)_i / N$ and $B(N)_i / N$ as in equation (17).



Section V. Policy Implications

In light of these results, it appears that capital and money markets as well as the ECB seem to react to changes in macroeconomic variables which in turn result in movements in the level, the slope and the seasonal components of yields. However, the data give the impression that yields exhibit a rather limited or timid feedback on the wider economy, in support of the local level model with no feedback instead of a VAR model as seen in Diebold, Rudebusch and Aruoba (2006). Therefore the ECB, in terms of its Policy Rate and monetary aggregates, can only ensure interest rates are low in times when consumption growth is low in order to not make things worse, but according to the data and period analysed, there is no evidence that innovation in yields create a response from macro variables, similar to the results seen in Diebold, Rudebusch and Aruoba (2006). In addition, our results confirm that the level is the most important factor contributing to yield curve movements followed by the slope, and that this is the case for both the money as well as for the capital market yields. In times of high consumption growth, thus when consumer confidence is high and unemployment is low, the central bank is expected to increase interest rate levels in order to anchor long term inflation to low levels. However, the data does not support the existence of a significant feedback from yields' latent factors to PPI, but a rather one-way effect from PPI to yields' latent factors only. Therefore it appears that in times of low consumption growth, central banks can only limit their action to low interest rates in order to ensure that the economy does not deteriorate further, as there appears to be little evidence of interest rates influencing the wider economy. However, we remain cautious about our results, as the number of variables used for determining the state of the economy has been limited to four.

Section VI. Conclusions and Final Remarks

In this paper a local level model with a stochastic slope and a stochastic seasonality have been calibrated using European yields. The analysis involved the use of the state space methodology to a structural equation model which, in state space terminology was to estimate an unobserved state or latent factors being the level, the slope and the seasonality to an observation or measurement equation linking the observed yields to the unobserved latent factors. The results confirm the views of Diebold, Rudebusch and Aruoba (2006) and, provide strong evidence of macroeconomic effects on yields however and, weaker evidence of yield curve effects on the macroeconomy. This essay has also explored the possibility of breaking the yield curve in two: the money market and the capital market yield curves. It has been shown that by doing so results are more encouraging than those seen in the no-arbitrage experiences. In addition, a discrete time affine term structure model (hence no-arbitrage) has been calibrated with the level, the slope and the seasonal component and both, the local level model as well as the no-arbitrage term structure model performed quite well in explaining yield curve movements. However, similar to most of current literature, the explanatory power diminishes as maturities become larger.

9. Conclusion and Final Remarks

The aims and objectives of this thesis –outlined under chapter 1- are to link empirically term structure dynamics to economic theory. In the process of doing so, this set of essays has explored the use of term structure models in order to link macroeconomic data to European government bond yields. Results were compared to the outcomes seen in classical IS-LM models, consumption asset pricing theory and Taylor Rules. This work also shows how term structure models can be used for other purposes, not only for modelling yields but also for modelling innovations in governments' revenues as a result of shocks on –for instance– in unemployment rate. In the preceding paragraphs, a summary of our conclusions is presented.

Chapter two presented slight adjustments to the classical consumption based asset pricing model by introducing in the utility function unemployment data and survey data such as consumers' confidence index. The incorporation of a monetary aggregate has also been extensively discussed in this chapter and its inclusion in the model appears to us to be robust enough. This chapter has also tested empirically a term structure model which has been adapted to the above mentioned data departing somehow from the no-arbitrage approach. We have shown that using current theoretical developments and a few state space variables such as European unemployment data, the European Consumers' Confidence Index, European Production Price Index (PPI) and a monetary aggregate such as ECB M3 for Europe, it is possible to explain yield curve movements with strikingly very good results. Unemployment and consumer confidence index have exhibited a shift and a slope effect on the yield curve, for front-end yields moving faster than in the long end resulting in steepening or flattening effects. Production price index has a twist effect on the yield curve (flattening or steepening of the curve) which results in lower-end yields shifting in opposite directions to the long end. This empirical work shows that yields are negatively correlated to money supply, as expected in classical IS-LM models. And that money supply exhibits a slope effect, with the lower end of the curve shifting faster than the longer end. In the light of these results, we suggest that further analysis is needed to understand what other variables or mechanisms are governing the longer end of the curve, as for the lower end our results show that the variables used have very high predictive ability already. In addition, we also suggest further research on cross-sectional data across non-EU countries. In this essay we have used macroeconomic data to explain yield curve movements and we show that GIIPS yields are more sensitive to macroeconomic shocks compared to the German benchmark. In fact, we see that most of the GIIPS bond yields –in contrast to the German benchmark- exhibit positive coefficients for the unemployment rate, indicating that GIIPS-governments' cost of financing will be penalised when unemployment increases. This means that these governments cannot undertake counter-cyclical fiscal stimulus by issuing new debt in times of low consumption growth and high unemployment.

In Chapter 3 some algebra was presented and analysed, and shown step by step how to get to the no-arbitrage approach. This chapter is of interest to academics involved in teaching affine term structure models under the continuous time approach. The affine term structure models developed in this chapter start with the basic pricing equation, the pricing equation for asset returns and the holding period returns. The chapter also explains the relationship of bond prices and Ito's lemma and finally the fundamental pricing equation for fixed income securities is obtained, as this expression is later required for the definition of an affine term structure model. Discussions are linked to prominent research such as Piazzesi (2010) and results are compared to those of Cochrane (2005) and Duffie and Kan (1996). Here, we have learned that there are still some conceptual differences in the literature which requires further attention that goes beyond the scope of this thesis. Here, for instance, we point out the fundamental differential equation of bonds or fixed income securities. We have shown that depending on the assumptions, the closed solutions lead to different results for the calculations of the parameters $A(N)/N$ and $B_i(N)/N$.

Chapter 4 explores the use of macro-finance models in order to explain the links of macroeconomics and the state vector used for calibrating a term structure model. Some of the findings in Chapter 4 are applied to the models discussed in chapters 5, 6, 7 and 8. Here it is worth mentioning that in order to link macroeconomic policy to term structure dynamics the use of a macro-finance model would help the researcher select state variables that are supported by economic theory. This chapter also shows that when an asset becomes riskier, investors are looking into the solvency profile of the issuer rather than inflation and monetary policy data. For instance, the Greek government cannot finance its self by influencing the time path of inflation and central bank monetary aggregates. Therefore, in this chapter we have learned that if the bond is a risk-free asset, hence, solvency shocks on the issuer are small (as for the case of German government bonds) the state vector will comprise the natural logarithms of unemployment rate, natural logarithms of consumer confidence index (as in proposition 1); natural logarithms of monetary aggregate levels and natural logarithms of the price level (modified proposition 2). If the bond is risky (solvency shocks for issuer are large), the model will be calibrated with the natural logarithms of unemployment rate, natural logarithms of government surplus to GDP ratio and natural logarithms of total government debt to GDP ratio (proposition 3).

Chapter 5 some of the algebra and concepts seen in the continuous time affine term structure literature are plugged into the discrete time approach. Starting point for this paper has been the celebrated papers from Backus, Foresi, Telmer (1996-98). In addition, some of the developments seen on the continuous approach as documented in Piazzesi (2010), Singleton (2006), Le, Singleton and Dai (2010) and Duffie and Kan (1996) have been explored and adapted to the discrete time approach. This chapter focused mainly on the multifactor cases of affine term structure models, as the weaknesses seen on the one factor models under Vasicek (1977) and CIR (1985) have been very well documented already in Backus, Foresi and Telmer (1998). Novelty of this research is that the multifactor affine term structure models under the Vasicek (1977) and the CIR (1985) process were calibrated using observed Interbank and German sovereign yields and European macroeconomic data as well as Greek sovereign yields. For the European and German yield curve, we calibrate macroeconomic data such as Euro-Zone Unemployment rate, Euro-Zone Production Price Index, Euro-Zone monetary aggregates M3 and Euro-Zone Consumer Confidence Index similar to previous chapters. For the Greek yield curve we use Greece’s sovereign budget deficit-to-GDP ratio, Greek unemployment rate and Greek sovereign debt-to-GDP ratio. In this chapter we show that our results are in line with our analysis from chapter 4. The results are encouraging and the models fit the observed yields as well as give evidence of a reasonable predictive-ability. Main findings can be summarised as follows: In the case of the interbank rates and German sovereigns, an increase in unemployment results in a fall in yields on risk free assets and the curve is expected to steepen with front end yields falling faster than the long end. An increase in production prices are expected to result in yield curve flattening, with yields in the front end increasing at a faster pace than the long end. In contrast to the OLS model from chapter 2, that reproduces a twist with front-end yields moving in opposite direction to long-end. This is an effect that the affine models struggle to reproduce. An increase in monetary aggregate M3 is expected to result in yield curve steepening, with yields in the front end falling faster than the long end. Finally, an increase in the consumer confidence index is expected to result in yields flattening, with front end yields increasing faster than the long end. This means that when the economy is booming risk free asset yields are expected to flatten and when the economy is under recession risk free asset yields are expected to steepen. From a portfolio management perspective, a representative investor would have incentives to short risk free assets in times when yields are at their steepest levels and set a curve flattening strategy shorting the front end allocating greater weight than to the long end. A more conservative strategy would be to short 2 years versus long the 10 and 20 years onwards, as the profits from the front end are expected to outweigh the losses on the long end. For the case of Greek government bonds the model shows that if governments engage in counter cyclical fiscal policies when unemployment is high, this will only be possible if these policies do not result in a significant deterioration of the debt-to-GDP ratio. Governments that either exhibit a positive correlation of their yields to aggregate consumption growth or cannot monetise their deficits need to ensure lower deficits and debt burdens during booming periods so that they can still have capacity to issue new debt for the rainy days. Governments in this sense need to assess, if they are subject to large or small solvency shocks. From the Greek case this chapter has shown that a deterioration of government's deficit-to-GDP ratio results in a fall in yields. This is mainly because the increase in spending helps to boom the economy. However, this is more than offset by the deterioration of its debt-to-GDP ratio, thus a deterioration of the latter ratio will more than offset any positive effects stemming from any budget-deficit-induced counter-cyclical policies, in line with proposition 3 from chapter 4. We have learned that debt and deficit ratios can play a role in times of financial distress.

Chapter 6 shows how to apply a continuous time affine term structure model. We show that our model performs very well as long as the state space variables selected have significant explanatory power over the short rate, which throughout this thesis has been the European Overnight Index Average (EONIA). Specifically, this paper confirms that the state variables that can be used for this purpose are: (i) the EU unemployment, (ii) EU production price index, (iii) monetary aggregate ECB M3, and (iv) the EU consumer confidence index, in line with the findings discussed on previous chapters. These variables account for the EONIA rate over a period of time ranging from Dec. 1999 to Jan 2011 remarkably well. In addition to that, the proposed states variables not only are observed to work well during times of financial stability, but they also perform very well during periods of extended financial distress. We see that our calculations in the continuous time approach are consistent with our findings applying the discrete approach and that calculations perform better on the front end of the yield curve rather than on the long end. This seems to be the case because front end yields are more sensitive to the state variables, whereas this dissipates as maturities become larger. In summary, we have learned that yields are high in times when unemployment is low and when consumer confidence, M3 and the price levels are high. During times of boom the EU benchmark yield curve exhibits a flat shape, with front end yields almost as high as the 30 year bonds and during times of recessions, the yield curve shows a steeper shape with long term yields exhibiting greater spreads versus short maturity bond yields. Our findings are thus in line with the discrete approach from chapter 5 and with modern asset pricing theory, and they provide evidence of Taylor's (1993) central bank policy rules as well as of the classical outcomes seen on IS-LM models. In this chapter, the intention has been to show that current yield curve levels are indeed explained by current economic fundamentals and that its behaviour is in line with economic theory.

Chapter 7 discussed the use of affine term structure models in order to explain links between term structure and the theory of the price level. We have learned that it does not matter if the theory of the price level is at work or not. Mainly because either ways any deterioration of the cumulated surplus or deficit leads to higher financing costs either via inflation or via increases in the credit spreads to the benchmark curves. We have also learned that the front end is where the risk lies as seen for the Greek yield curve. Clearly, if for a given government its yields are not seen as a market benchmark and therefore exhibit increasing yields in times when aggregate marginal utility is high and consumption growth is low, or if the government does not control monetary policy, then the governments must avoid deficits and fund counter-cyclical policies with reserves for this purpose. Reserves must be financed via long end of the curve 20- to 30-year maturity bonds. Only those governments lucky enough to enjoy the so-called risk-free status are allowed to issue in the front end, as an increase in the financing costs due to increases in total debt outstanding are more than compensated by falls in the short term rate. This can only work when governments' solvency shocks are small. However, for those governments' exhibiting intermediate of large solvency shocks any deterioration on their fundamentals will result in a deterioration of their financing costs. This chapter also shows how useful affine term structure models can be for modelling the effects of macroeconomic variables on governments' revenues. For instance, as an example we show how the effect of innovations in the unemployment rate has on governments' fiscal imbalances. Governments which are more sensitive to macroeconomic risks affecting their fiscal imbalances are likely to exhibit higher spreads if they run deficits, thus limiting their ability to counter-cyclical policies in times when consumption growth is low and aggregate marginal utility is high.

In Chapter 8 a local level model with a stochastic slope and a stochastic seasonality has been calibrated using European yields. The analysis involved the use of the state space methodology to a structural equation model which, in state space terminology was to estimate an unobserved state or latent factors being the level, the slope and the seasonality to an observation or measurement equation linking the observed yields to the unobserved latent factors. The results confirm the views of Diebold, Rudebusch and Aruoba (2006) and, provide strong evidence of macroeconomic effects on yields however and, weaker evidence of yield curve effects on the macroeconomy. This essay has also explored the possibility of breaking the yield curve in two: the money market and the capital market yield curves. It has been shown that by doing so results improve significantly compared to those seen in the no-arbitrage experiences. In addition, a discrete time affine term structure model (hence, in absence of arbitrage) has been calibrated with the level, the slope and the seasonal component and both, the local level model as well as the no-arbitrage term structure model performed quite well in explaining yield curve movements. However, similar to most of current literature, the explanatory power diminishes under the no-arbitrage approach as maturities become larger. This can be improved by breaking the yields in two (money market yields and capital market yields), unfortunately at the expense of violating the no-arbitrage condition.

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