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Approaches for modeling and solving the multi-period Loss Offset Problem in German income tax system

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This thesis is dedicated to

my parents

for their continual support throughout the years

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Summary

The major objective of this thesis is to study optimization techniques applied in financial planning. As financial optimization is a diverse field, we restrict our work onto tax planning. Our effort is directed towards studying the Loss Offset Problem which arises in German income tax system. The Loss Offset Problem deals with a situation where individuals or companies confront a loss in some financial years and profits in the years before and after the “loss years”. When such a situation occurs, it is allowed to divide a loss amount into two parts: the loss carry-backward and loss carry-forward. This will reduce the taxable income in other years, therefore reduces tax payments. The problem is of significant importance for a number of reasons. First, potentials for optimization procedures exist as there is a trade-off between the amount of loss to be carried back and forward. Second, from international perspective over the last several years, German loss offset regulations are still rather generous as many other countries do not allow a tax loss carry-backward at all. Besides, we consider two possible choices of taxation options in each period. The focus of this study is the multi-period scenario. As we will see, this hides many interesting dynamics in the interactive behavior of decisions. We formulate the mathematical model so as to optimize an objective function subject to appropriate constraints. The objective function itself is a discontinuous, non-linear, non-convex function with recursive characteristics, which makes the problem difficult to solve. In order to achieve this goal, we first study the complexity of the problem in two cases: a 3-period-model and a multiple-period-model and then apply optimization algorithms from Operations Research to search for solutions. We discuss several algorithms and their corresponding commonly mentioned application. We differentiate between exact and heuristic algorithms. An exact algorithm attempts to obtain the global optimal solution, no matter how long it takes for computational time. However, such approaches do not always work. For many practical problems in business, it is unlikely to acquire a global optimal solution in an acceptable amount of time. To the contrary, a heuristic algorithm may discover a very good feasible solution in a given number of iterations, but not necessarily the optimal solution for the specific problem being considered. To refine our analysis, both types of algorithms need to be adapted, applied, and analyzed under different scenarios of data setting.

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1 Introduction

"Count what is countable, measure what is measurable, and what is not measurable, make measurable."

Galileo Galilei (1564 - 1642)

1.1 Problem Statement

This beginning section of the thesis gives statements about the relevance and originality of the problem to be studied.

We begin with the question: What is Financial Planning? Intuitively, Financial Planning is the process of making multiple target-oriented decisions concerning the cash inflows and outflows of individuals and organizations. In brief, it can be described as a generic term of several different subsets of research areas such as Investment Planning, Portfolio Optimization, Tax Planning or Cash Flow Management, and several more disciplines. For solving such problems, researchers can construct models that mostly deal with a multi-period planning scenario. In a work of Dinh and Schmidt [DS11], for example, a general framework for building linear optimization model for Cash Flow Management was formulated which is very closely related to Asset-Liability-Management. In case of an individual or household, this is a task of Personal Financial Planning [DS10]. Very often, the authors concentrate on the risky aspects of investment activities and the stochastic behaviour of model parameters [EDDMS14]. The effects of taxes, unfortunately, are neglected. This leads to suboptimal solutions in many decision situations, as we will discuss later. Especially in case of corporate finance, lack of transparent information on tax often causes problematic in the management level and a thorough consideration of tax is not taken during an investment process.

Tax planning, however, is one of the indispensable components of every investment and financial planning process. This is simple to explain because in nearly every country, the government charges taxes on economic activities of its citizens and enterprises, their incomes and their properties. When creating strategies for a best possible operating profit in investment projects, the investment planner cannot afford to ignore tax opportunities that may bring significant improvements to the final business result[1]. In this thesis we consider a nonlinear combinatorial optimization problem formulated according to the German income tax law[2], provide the mathematical formulation for modeling real-world constraints and propose several quantitative methods for optimally dealing with it. The consideration of tax effects in financial planning in Germany is all the more important due to the fact that German tax system does not offer the so called "postulate of neutrality"[3]. That means,

[1] In later chapters of this thesis we refer to this financial result as the final cumulative wealth.
[2] In German: Deutsches Steuerrecht.
[3] Neutralitätspostulat.

the advantageousness of decision alternatives before and after tax is not always fully differed [Sch08, p.23].

Solving a tax planning problem has the meaning of not giving any unnecessary cash amount from total income to the government in taxes. For that purpose, decisions have to be made, consequently or simultaneously, to take advantage of the complex prescriptions of tax law. In general, the first step in such a process is to analyze the principles of income taxation and understanding how they create negative effects on the total income. After that, a mathematical model can be built, where a specific objective function is optimized subject to some restrictions that might be formulated as equations and inequalities. In this case, we speak of an optimization problem. The complexity of our optimization problems, which might vary from case to case, will be discussed in coming parts of the thesis.

Nowadays, tax consultants operate in an increasingly complicated context than originally expected. Tasks range from the compilation of consolidated financial statements and the documentation of results to the offering of reliable solutions regarding tax declaration for different groups of clients; from income tax declarations to special solutions for inheritance taxes. Furthermore, consultants are not only challenged with commercial business tasks, but also with assignments regarding the personal aspects of their clients: real estate in foreign countries and retirement residences at the tax shelters are just some of many examples[4]. Understandably, there is a bunch of software products that support tax consultants with comprehensive solutions for multiple tasks such as accounting, balancing, tax declaration and consulting, as well as searching for optimal operational decisions.

On the other side, countries and governments all over the world have traditionally been considering tax planning as a legitimate practice. The profession as a tax planner or tax consultant has therefore been long acknowledged and, especially in Germany, appreciated. Tax consultants have been formulating and solving problems in taxation planning for their clients all the time. As mentioned above, simple problems start from calculating tax payments, declaring yearly income statements or gathering receipts for tax return purposes. More complicated problems concern the finding of an optimal solution among several ones. These kinds of problems always require, besides knowledge of tax law and experience of practical approaches, at least the fundamental of optimization techniques. Knowledge of operations research is therefore essential when it is about searching for possibilities in order to get the optimal result in a complex and interactive business environment.

Over time, the tax planning structures have become ever-more sophisticated. Modern tax planning consists in taking advantage of the technicalities of a tax system or of mismatches between two or more tax systems for the purpose of reducing tax liability. Such tax planning strategies can take a multitude of forms. For example, a very popular problem to decide is on depreciation methods for investment projects of individuals and firms. Another well-known type of decision is on how to balance the losses and deductions vertically and horizontally[5]. A good strategy might seek to combine several regulations of a tax system to create the best options and/or handle with multi periodical models. From a multinational point of view, tax planning also include double deductions (e.g., the same loss is deducted both in the state of source and residence) and double non-taxation (e.g., income which is not taxed in the source state is exempt in the state of residence) [Wac79,

[4] For more information on planning problems with tax paradises and tax shelters, see for instance [Wac79].
[5] We will explain loss and deduction in Chapter 3.

p.91]. Other strategies develop across various jurisdictions and effectively shift taxable profits towards states with beneficial tax regimes. A key characteristic of such practices is that they reduce tax liability through strictly legal arrangements which however contradict the intent of the law.

When discussing tax planning, we can break the general area of activity into two subdomains:

1. Accounting, revising, as well as tax declaration. Such solutions are typically offered to clients by tax consultants that hold Business Administration background and most importantly, comprehensive taxation knowledge of a certain section (i.e., for either business or individuals) or region (i.e., for which state or country).

2. Finding out optimal decisions and choices among several given possible alternatives according to tax laws and regulations. Those activities are mainly carried out by decision makers who possess a strong background on Operations Research and Quantitative Methods.

Ignoring the former completely, it is the latter domain that this thesis considers. Since there are numerous algorithms that can be used to obtain an optimal solution for various practical problems, we would like to review selected algorithms and apply the best ones for solving our problem. The measure of "best" we will use here is the number of steps used in an explicit algorithm in comparison to the result achieved.

In our problem which will be proposed in Chapter 3, there are two kinds of decisions to be made. The first decision relates to the choice between two given taxation options. In each period we have to decide which taxation option should be chosen, not in order to achieve a best result in that period, but to optimize the overall result, i.e., maximize the final wealth? The second decision arises when there are one or more loss periods, i.e., periods in which the taxable income is negative. In such periods where a loss occurs, the question is how to optimally transfer (or offset) this loss amount back and forward to other periods in order to reduce tax payments and therefore maximize final wealth?

We start with a scenario that is arguably one of the simplest possible, with three periods and one loss period, as formulated in [SSDK12]. Step by step, we indicate the factors affecting the structure of our model and the objective function. Then a dynamic, structural model of tax planning that simultaneously considers the loss offset regulations and tax options will be introduced. By ignoring sources of uncertainty, the model is a deterministic one. We point out that putting a loss in one year into a multi-period perspective will result into the question of how to utilize the regulation on loss offset in order to get maximal benefits. This offers a strategic advantage for the decision maker. The focus of the optimization model is therefore on the choice of

1. taxation options in each period, and
2. loss offset strategies in periods where there exists a loss

The goal is to find decisions on taxation options and loss carry-backward so that the final wealth is maximized, taking into account the underlying constraints. From an analytical perspective, the strategy of offsetting loss in one period to other periods leads to a reallocation of the taxable incomes in all planning periods such that the values of taxable income and marginal tax rates are distributed as homogenously as possible. We show that for increasing number of periods of the

planning horizon and the more the restrictions being considered, the optimization problem becomes quite unmanageable because of the interrelationships between several model parameters. Logically, the dimension of the objective function will also become larger when there are more than one loss period to be considered.

The difficulty of solving this problem stems from the non-linearity, non-convexity and discontinuity of constraints and the objective function. Consider the following simple cases:

1. If for example someone wishes to

   maximize $a \cdot x + b \cdot y$

   subject to $c \cdot x + d \cdot y < e$

   $f \cdot x + g \cdot y < h$

   then we have a linear optimization problem. The optimal values for x and y can be exactly determined using the standard methods such as the Simplex algorithm. For intuitive purposes, the graphical solving method can also be carried out because there are only two variables. Notice that the strict inequality $<$ is interpreted to mean the loose inequality $\leq$ that ensures continuity. The difficulty will rise upwards when there are more decision variables to be optimized. Furthermore, in many cases either the objective function or the constraints may not be linear. In such situations, we have much higher complexity to deal with.

2. If for example someone wishes to

   minimize $f_0(x)$

   subject to $f_i(x) \leq b_i, \ i = 1, \dots, n$

   whereas the objective function and **all** constraints satisfy

   $$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

   for any $\alpha + \beta = 1 \ and \ \alpha \geq 0, \beta \geq 0$

   then the problem is defined to be convex and there are also very efficient algorithms for determining the optimal values of our variables.

Unfortunately, the optimization problem we are going to consider in this work is not so straightforward to be solved by standard algorithms of linear and convex programming. Formulating the interrelationships between all the given tax laws and regulations, we have a combinatorial optimization problem with mixed-integer variables.

As a common technique in Combinatorial Optimization, it is helpful to divide the problem in sub-problems which are easier to solve. Basically, we have to split the whole search space into smaller ones. In each of these parts of the search space we then apply appropriate techniques to solve the problems. Two kinds of techniques are employed to establish the algorithm. The first one is *Brute-*

force-Search which is applied to divide the search space into uni-modal[6] parts in which the lower and upper bound of loss carry-backward can be precisely calculated. The other technique consists of some exact algorithms such as *Binary Search* or *Golden Ratio Search* which can be used to find the optimal amount of loss carry-backward. In case of the basic model of three periods we can identify quadratic parts of the "big" problem that can be solved conveniently. Then, the result of each part is compared with the others. In this way, we can guarantee global optimality. We point out that the generalized model for multiple periods becomes much more difficult to solve. We suggest the application of meta-heuristic methods and test the proposed algorithms on representative examples of a six-period-problem.

1.2 Research Objectives

The research objectives of this work can be stated as:

1. Modeling the Loss Offset Problem as a mixed integer nonlinear optimization problem. A basic formulation of the problem is already given in [SSDK12]**.** We verify and modify the new models with a differentiation between the 3-period and multi-period planning horizons. It is important to understand the fundamentals of the German income tax system regarding our problem. Then, the complete mathematical formulation of models is necessary. On one hand, approximations and simplifying assumptions must be made in order for the model to be tractable, i.e., capable of being solved. On the other hand, we analyze the constructed model, verify its correctness and discuss its complexity. (Chapter 3)

2. Developing algorithms to solve the problem. Because we search for optimality, exact search methods are on the first priority to be examined. We show that in the case we want to extend and generalize our model, the complexity will rise and exact search methods become inappropriate for generating solution. It is therefore necessary to apply heuristic methods, especially iterative and stochastic algorithms which belong to the family of global search. We also study and give proof of the characteristics that make the problem hard to be solved by analytical approaches. (Chapter 4)

3. Improving the understanding of the decision model for loss offset and the efficiency of proposed algorithms for specific cases of parameter sets. To do this, we study different scenarios of input data and present several propositions by making simple but useful observations. As a result, it can become understandable in which scenario either one taxation option or the other will always be the best choice in a certain period, regardless of other decision variables. This will help to reduce computation time of the solution process. (Chapter 5)

Mathematical modeling and optimization algorithms are the key components to problem-solving for our research of tax planning. A step-by-step analysis of the problem will be provided, and it is shown

[6] Uni-modality is explained in Chapter 2.

how the difficulties arise and can be surmounted. In every optimization task, modeling the problem is the first and most important step. Addressing the importance of the model design process, we focus on the correct formulation of models that allows the design of solution methods. Figure 1-1 shows the cyclical nature of mathematical modeling process.

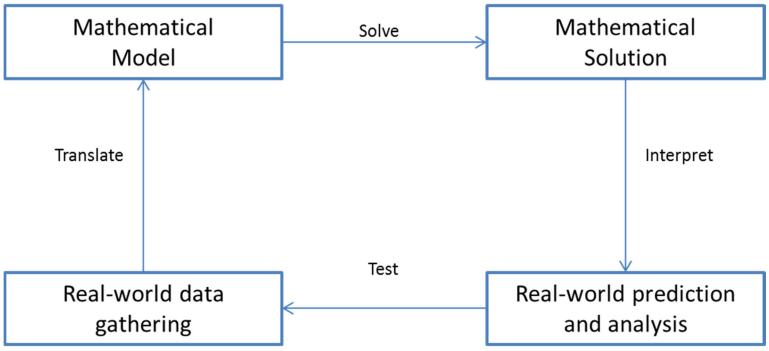


Figure 1-1 The process of modeling real-world problems [Rad10, p.4]

Once we have gone through the modeling process, the next step is to search for a combination of best decisions by using some algorithms. This is what we refer to as optimization. When we speak of "optimality", we may do so from several different points of view.

We will in general be interested in whether the optimum found by the algorithms is a global optimum or simply a local optimum. In order to guarantee a global optimum, we must apply exact methods such as *Exhaustive Search* or *Branch and Bound*. However, for many problem structures it is not possible to conduct this exact search process. We have to apply other techniques that may not deliver a global optimum but an approximation, i.e., near-optimal solution. We will discuss the complexity of the Loss Offset Problem in Chapter 4. For a first illustration of the whole process, Figure 1-2 shows a **simplified** sequence of making the decisions within the Loss Offset Problem that we will study and solve.

$$\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_T \\ \\ y_1 \\ y_2 \\ \dots \\ y_T \end{pmatrix} \longrightarrow \begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_T \end{pmatrix} \longrightarrow \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_T \end{pmatrix} \longrightarrow \begin{pmatrix} W_1 \\ W_2 \\ \dots \\ W_T \end{pmatrix}$$

Figure 1-2 The process of decision making of the Loss Offset Problem

The figure simply reveals the fact that if we change some values of variables x_t for loss carry-backwards and y_t for taxation options, the values of taxable income I_t will be affected. These, in

turn, will make change to the calculation of tax payments S_t. These, again, will affect the values of wealth W_t in each period and also final wealth W_T.

The problem belongs to a class of non-convex programs, having mixture of binary and continuous variables. Since the decision variables do not appear explicitly in the objective function, as we will see, we cannot perceive the real complexity created by them. We can try to substitute the variables themselves in the objective function; however, this can only be done for simplified cases. Such an attempt can be found in Appendix D. Furthermore, the optimization process is fully dynamic, with decisions and results in one period affecting the decisions in the next one and vice versa. In the most case, we cannot make any a-priori knowledge about the search space by analytical methods unless we try to plot the objective function with mathematical tools such as Matlab. Such plots with no standard behavior will be shown in Chapter 4.

As seen in Figure 1-3, we may have a search problem with multiple local optima and one global optimum which should be found. For large combinatorial problems such as the *Traveling Salesman Problem*, this is a hard task even with the support of advanced mathematical and information techniques.

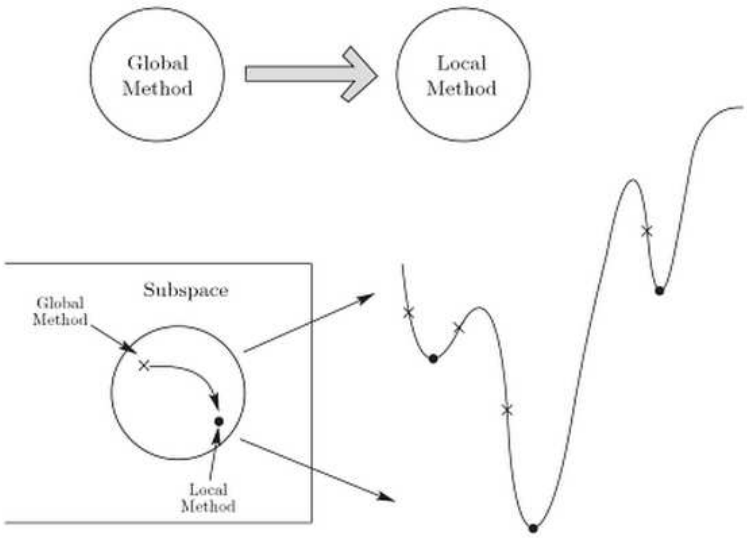


Figure 1-3 The problem of Global Optimization according to [PC09]

It is a well-known fact in optimization theory that meta-heuristic techniques can be used to approximate solutions to extremely difficult or impossible numeric maximization and minimization problems. The major advantage is that a heuristic approach does not require much information about the problem structure. The disadvantage of a heuristic algorithm is that it may be able to find the global optimum, but also one of the multiple local optima.

Generally, there are two main reasons for the application of heuristic methods [HL04, p.16]:

1. An optimal solution for the simplified mathematical model may be far from ideal for the real problem. For this reason, seeking for the ultimately optimal solution might be a scientific

incentive for operations researchers, but not for managers who make decisions in real business environment.

2. When pursuing the “science of the ultimate”, the researcher also has to consider the expense for the whole process of algorithm construction and operation. This is especially the case when the time or cost required to find the global optimal solution for a model of only small size would become very large. For large-scale models with increasing complexity the expenses would be tremendous and thus no more lucrative.

As any *Operations Research* analyst may agree with us, a great variety of optimization techniques compete for being the most efficient problem-solving approach. The question is anything but easy to answer. An interesting and widely quoted theorem of Wolpert and Macready [WM95] which is called *“No free lunch theorem“* addresses the efficiency of algorithms. The *“No free lunch theorem”* states that "any two algorithms are equivalent when their performance is averaged across all possible problems”. This implies that for any optimization algorithm, gaining additional performance over one class of problems is exactly paid for in performance over another class. Finding an universal algorithm that can efficiently solve multiple types of problems is therefore not possible. Keeping this in mind when solving our problem, we neither attempt to seek for the best optimization method, nor try to prove the absolute superiority of any selected algorithm.

1.3 Thesis Organization

With the research objectives and their requirements in mind, we organize the remaining chapters of this thesis as follows:

Chapter 2 provides a compact introduction to Operations Research and Optimization Theory. An algorithmic background is needed to obtain a basic understanding of modeling and solving a wide range of optimization problems. The first purpose is to illustrate the advantages of applying mathematical methods in business and engineering. Mathematical modeling has the advantage over model description in verbal form because it makes quantitative analysis possible. Second, we classify optimization problems in classes with linear, non-linear and integer features in order to obtain an insight into the complexity of each class. Besides, we illustrate the problems with higher complexity that belong to the category of Combinatorial Optimization.

After that, the fundamentals of search algorithms and optimization techniques are explained. In fact there are a considerable number of literatures describing different classes of algorithms and their applications. Each algorithm has its advantages and disadvantages, and the choice of which one to use, depends either on the nature of the problem or the requirement of the decision maker. Our goal is not to give a summary of all this knowledge, but rather to give a motivation enough to pursue the further study of our optimization problem in tax planning which will be discussed in the later chapters. As a general time-saving rule, we concentrate on those methods that can be applied in Chapter 4 to solve the optimization problem proposed in Chapter 3. We begin with four exact search methods: *Brute-force-Search*, *Branch and Bound*, *Binary Search* and *Golden Ratio Search*. After that we turn to the discussion on heuristics. First, we present the Greedy Algorithm and Hill Climbing as

two fundamental heuristic approaches. Then, the concept for the optimization of nonlinear functions using *Particle Swarm Optimization* and *Simulated Annealing* methodology is introduced. For the sake of completeness we also survey simulation methods in search and optimization problems. Because a thorough description of the algorithms is beyond the scope of this thesis, we purely provide the basic concepts and examples and refer to relevant research literature for more advanced details.

Chapter 3 has the main purpose of presenting the general mathematical formulation of the Loss Offset Problem. A careful modeling phase always precedes the optimization procedure. At the beginning we give an overview of literature on the topics of tax planning. There, we differ between the literature in German and non-German speaking communities. Moreover, we point out to which part of the literature we contribute specifically. Generally speaking, our main contributions to the existing literature on optimization in tax planning concern the three research objectives which have been mentioned in the last section.

Prerequisite for implementing any tax strategy is that the laws must be understood thoroughly. Therefore we describe the central issues and concepts of taxation in a nutshell. The goal is to provide an overview of the basic aspects of income taxation in Germany in order to help understanding what the background to the tax planning problem of this thesis is. We deal with the features of the real-world restrictions and its mathematical characteristics.

Assumptions must be made. For instance, there are several types of incomes that are treated differently under German income tax law. As will become evident in the sections for model building, the financial plans may get rather large and impropriate complicated if we allow for different income types. So, we will be simplifying the situation going forward by assuming there are only two types of income that will be considered in our mathematical models. With this assumption, we offer some models for explaining why it is necessary to consider the effects of taxes in investment and financial planning. Relevant concepts such as depreciation and time value of money are presented. After that we formulate the Loss Offset Problem that is to be solved. A section is dedicated to introducing the notations of the tax models that will be constructed.

We construct financial plans which can be used to analyze the effects of decision variables on the tax payments of each period and the value of final wealth. As a result, we create a basis for the ascending chapter in which optimization methods are applied to solve the problem. Furthermore, we show how the combination of several decision possibilities present challenges for the optimization models with even a limited number of periods and variables.

Chapter 4 deals with the construction of algorithms for solving our optimization problem. After having assembled a “toolkit” of optimization techniques and modeled the Loss Offset Problem in the previous chapters, we discuss the difficulty of solving the postulated problem, developing algorithms as well as the quality of the solutions that can be found. The application of appropriate algorithms is analyzed. Equally important, some numerical examples are presented to demonstrate the effectiveness and feasibility of the proposed algorithm.

In our problem, we have a system of equations and inequalities that define the search domain of variables and the search space of the objective function. Our task is then to maximize the final wealth subject to some constraints, i.e., to solve a maximization problem. As explained later, an optimization problem can usually also be modeled as a search problem, since we are searching for the optimum solution from among the solution space. Two cases are analyzed and solved separately:

the 3-period-problem with one loss, and the multiple-period-problem with more than one loss. We use MS-Excel as the spreadsheet software for constructing financial plans and analyzing decisions as well as interpreting optimization results. In the spreadsheet itself, we can also define advanced mathematical functions or commands as parts of an algorithm. Furthermore, it is convenient to write and run the desired algorithms with a macro programming language called VBA for Excel. The results delivered by the algorithms can be observed and interpreted easily.

In Chapter 5 we deliver some propositions from an analytical point of view. Our tasks in the previous chapters are to build the models and propose algorithms to solve these models. Now, the purpose is to classify scenarios in which some decisions can be proven to be always optimal. We present some logical findings that can improve the search process and reduce computing time of the proposed algorithms and methods. The marginal tax rate is the focus of our analysis.

Chapter 6 concludes the thesis by stating again the most important remarks. Furthermore, we present further ideas for future research. The results of this thesis can be summarized as: studying and modeling the real-world problem of loss offset in German income tax system as an optimization problem; designing the necessary algorithms to solve the problem; and using mathematical theory to justify possible algorithmic improvements for the solution process.

We have said what this thesis is about. It is also worthwhile to state what it is not about. It is not a comprehensive guide to approaches to solve all existing problems in German tax law. For the study of other problems, we refer to a valuable collection of literature in German language on laws and regulations for income tax in Germany. For the sake of completeness, in parts of this thesis we point out the connections and sometimes the relevance of other laws or prescriptions to our optimization model. It is mentioned once again, however, that a look for comprehensive treatments of all aspects in income taxation will not be found in this work.

2 Operations Research and Optimization Theory

The main purpose of this chapter is to explain optimization as the process of obtaining the best result under given conditions. We also introduce some appropriate algorithms used *in Operations Research (OR)* and *Optimization Theory* with the intention of applying them for solving the problems in later chapters. In Chapter 4, the methods we present here will be applied to solve the problems proposed in Chapter 3. We begin this chapter by giving a general discussion.

2.1 Introduction to Operations Research

Let us start with the following example, formulated in simple verbal form [Hei09, p. 287]:

Suppose we receive a task to use a pan balance to find one heavier coin among eight coins under the assumption that they all look alike and the other seven all have the same weight. So, the problem is that we have eight coins, and one of them is a heavy coin that weighs more than the others. With only a pan balance, what would the most efficient way to find the heavy coin? This is a simple problem for which we have to "construct" an algorithm.

The simplest solution that one may think of is to just weigh the coins in pairs. Take a pair of coins and put one coin on each side of the balance. If one is heavier, we have found the heavy coin and the process stops. If they weigh the same, we continue with the next pair. In the worst case, we have to make four trials. We may be interested in the question if this method is optimal. The answer is no, as James Hein has pointed out.

Now we examine a better method for finding the solution. We can put four of the coins on one side of the balance, and the other four coins on the other side of the balance. One side must be heavier than the other, because one side must contain the heavier coin. If the left side is heavier, then we discard the four coins on the right side. We then divide the remaining four coins into two groups of 2, and repeat the procedure by weighing one group against the other. The heavier of the two pairs then gets split and weighed. The coin that is heavier is our solution. This is a sure way to find the bad coin in three steps. OR analysts will quickly recognize this algorithm as a *Binary Search*, which can be represented by the following decision tree from page 287 of Hein's book:

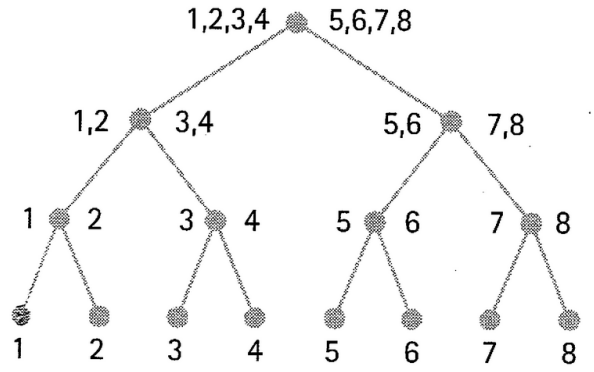


Figure 2-1 A binary decision tree

There is, however, another more “intelligent” way to solve the problem [Hei09, p.288]. We can apply another strategy of splitting the coins at each decision nodes. The revised, optimal decision tree is shown in the next figure.

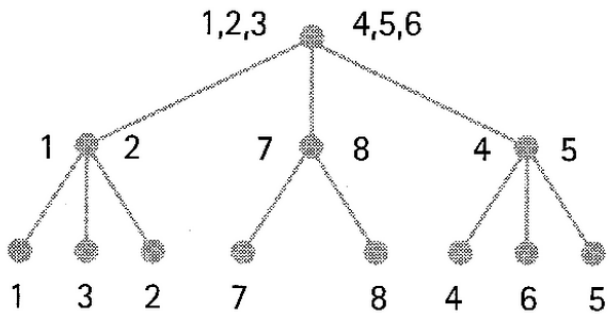


Figure 2-2 A modified decision tree

For this revised procedure, we begin by weighing two groups of coins: 1 through 3 and 4 through 6. If those groups have equal weight, then we go down the middle branch and see that the heavy coin must be 7 or 8. We follow a similar procedure if, after the first weighing, 1 through 3 is heavier. We go down the left branch and test 1 and 2. If those weigh the same, then the heavy coin must be coin number 3.

We have examined the problem of how many steps it takes to sort through n coins and have created three different ways to solve the problem. In mathematical analysis of algorithms, researchers are also concerned with the complexity of problems and the runtime of solution approaches. If we are seeking for the best way to fulfil a task, we are talking about optimization. Optimization is the mechanism by which one seeks for the most favorable value of a function or a process. By this definition, an optimization problem can usually also be modeled as a search problem, since we are

searching for the optimum solution from among the solution space. Without any loss of generality, we can assume that our optimization problems are of the maximization category[7].

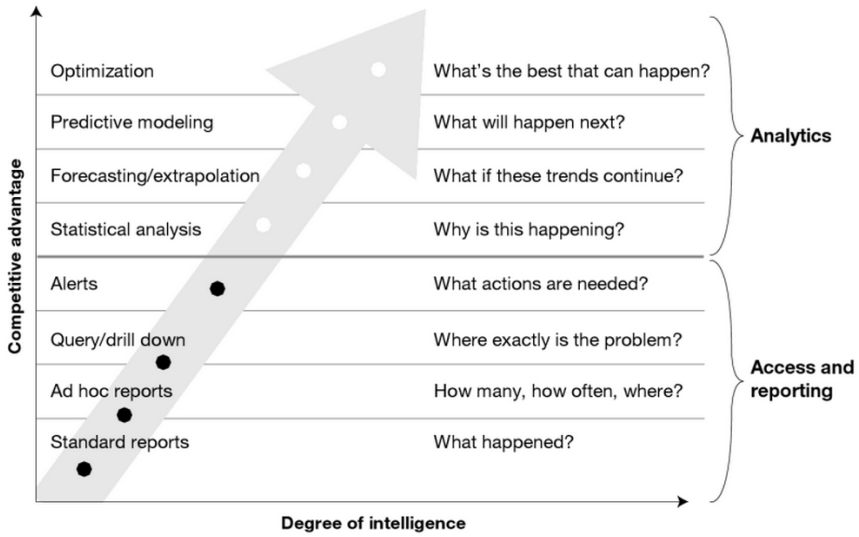


Figure 2-3 Business intelligence and Competitive advantage [DH07]

As can be seen in Figure 2-3, the more analytical work that is pursued by management issues, the higher the competitive advantage can be created against other competitors in the market [DH07]. Clearly, there are some industries and branches that are more amenable to quantitative methods than others. However, in general, we can obtain a concrete financial benefit when applying methods to make better decisions that can be quantified. Optimization always requires the more complicated tasks to be undertaken, in return offers the best possibility for planning and decision making on management level.

Dated back to the late 1960s, the use of computer systems began popular among researchers and practitioners. For purposes of modeling and solving analytical and often repetitive activities of business processes, applications are developed which are called *decision support systems (DSS)*. Often, some authors may argue that the original ideas of such applications were closely connected to the planning and optimization of military operations during World War II, although the computational strength of information systems at that time was unmatchable to today's computers [DH07, pg. 12]. However, because of this root, the term "*Operations Research*" has become more and more popular and also mentioned as a separate discipline in research literature.

Application areas of interest to *OR* analysts include finance, energy, production, logistics, health and several others. Specific areas of financial research include for example portfolio optimization, risk management, derivative pricing and hedging and taxation planning. Nowadays, strategic decision models that consider taxation effects become even more and more important as they avoids the possibility of decision makers on management level being blinded by the sophistication of the tax laws. In tax planning the aim is to construct a model which represents a situation in practice where gains and losses after tax are to be calculated depending on restrictions of tax laws. Then, we must

[7] In case of minimization problems, we simply need to change the sign of the objective function.

offer a rational approach to decision making in order to optimize some predefined goals. This requires model-building skills of *OR* practitioners for the financial process to be fully effective.

In a nutshell, *OR* is the science of applying analytical and quantitative methods to help making better decisions. Due to its advantage of helping to increase effectiveness and competitiveness, *OR* is becoming more and more important to governments and businesses in today's world. There are a great variety of books and other research publications on *OR* that explain how to build and test quantitative models as well as how to do basic analysis and develop optimization algorithms. Also, a great variety of areas of application such as in production, scheduling, supply chain management, finance and so on can be found in the literature. For getting an insight into the ideas and applications of *OR*, we refer to a Hillier and Lieberman's book, *"Introduction to Operations Research"* [HL04] which has become one of the classic texts on *OR*. The hallmark features of this book include solid coverage of fundamentals, an extensive set of interesting problems and cases, and state-of-the-practice operations research software used in conjunction with examples from the text. The book also features some topics in *OR* that are relevant for our work such as meta-heuristics, simulation, and spreadsheet modeling.

There are a huge number of academic publications on the application of *OR* to various fields such as economics, engineering, manufacturing and so on. For our purpose, we especially concentrate on problem types as well as solving techniques that are applied in finance. In fact, the application of *OR* techniques to Finance and Financial Markets is an extremely wide topic that many researchers have tried to describe in short-form. For many decades, mathematical models for financial modeling have been increasingly in use [Mer95]. As an example, researchers at Wells Fargo Bank recognized the role of mathematics in formulating new financial concepts and introduced the first indexed investment fund in 1971 [Ber92]. Since then, investment banks have developed the tradition of recruiting *OR* analysts, those that are sometimes referred to as quantitative researchers, for analyzing and solving financial problems. Other contributions are made by Ashford et al. [ABD88] or Board et al. [BSZ03]. Some authors assume that the decisions are made in a world of uncertainty [BMSM83]. Other authors made the assumption of certainty in their model building and optimization [Akd91].

As mentioned in [BSZ03], there is a close relationship between finance and *OR* that can be described as complementary. On one side *OR* techniques are needed to solve finance problems. On the other side, challenges in finance have motivated the development of new *OR* solutions as well as the modification of established ones. It should be noted that in many applications of *OR* in finance, the aim must not always be the optimization of some objective functions, but to generate feasible solutions to highly constrained problems [BSZ03]. For a better view of the techniques widely used in *OR*, in the following we list some of the most prominent ones which are already mentioned in the publication of Board et al. [BSZ03]:

- Monte Carlo Simulation
- Linear Programming
- Dynamic Programming
- Quadratic Programming (where Convex Optimization is a special field)
- Integer Programming

- Nonlinear Programming
- Goal Programming
- Game Theory
- Network Model
- Simulated Annealing

Among the above methods, *Monte Carlo Simulation* is a special technique that allows first intuition into a problem that cannot be easily solved by analytical procedures[8]. Also, an approximation of the possible solution can be made. For example, the use of *Monte Carlo Simulation* as an *OR* technique for the valuation of options was proposed in 1977 by Boyle [Boy77]. Later, financial analysts apply this technique to generate possible outcomes of the price of a financial instrument until a predefined maturity. In other financial modeling problems, quasi-*Monte Carlo Simulation* can contribute to speeding up the simulation process [JBT96]. We will come back to a discussion of this technique later. What follows is a short list of some examples that belong to the discipline of *OR*, widely separated in styles:

Portfolio optimization

- variables: amounts invested in different assets
- constraints: budget, minimum return, maximum risk
- objective: overall risk return variance

Production scheduling

- variables: operation of machines
- constraints: time budget, sequences of products, maximum operating cost
- objective: minimum manufacturing time

Data fitting

- variables: model parameters
- constraints: prior information
- objective: prediction error

2.2 Classification of Optimization Problems

In mathematics, optimization is the discipline that deals with the finding of maxima or minima of functions in a predefined search region. There is nearly no business which does not have to tackle with solving optimization problems. We will briefly discuss typical features of optimization problems in this section. The definition of what is an optimization problem is a fundamental thing and therefore should not require an additional treatment.

[8] For difficult problems, different calculus cannot be applied in searching for optima.

According to Clarke, any optimization problem may firstly concerns with three central themes [Cla90, p.18]:

- Existence - is there a solution to the problem?
- Necessary conditions - what clues are there to help identify solutions?
- Sufficient conditions - how can a candidate be confirmed to be a solution?

Practically, all real world problems have some physical bounds that are called constraints. Solutions that lie outside of these bounds are infeasible. Because constraints model real-world restrictions of a problem, they are represented by equations and inequalities. In other words, a constraint defines a component of solution feasibility. There is usually a bunch of constraints to be satisfied. If a solution violates one or more of its constraints then the solution is considered infeasible [WL01]. Figure 2-4 shows the constraints and feasible solution region of a linear optimization problem in 2D.

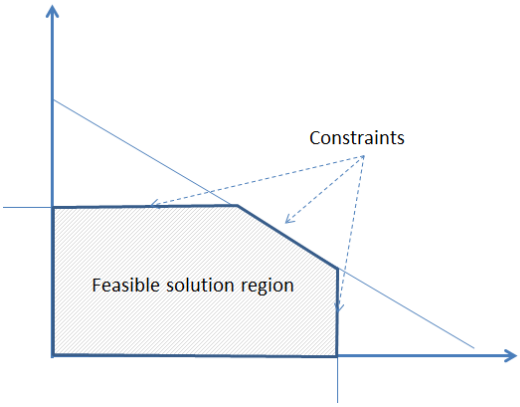


Figure 2-4 Feasible solution region in 2D

In a multi-dimensional problem, the constraints create the boundaries of a feasible solution space. With no constraints, the values of a mathematical model may lie at any point in a solution space. However, the question of how to satisfy one or more constraints is not the problematic issue in our case. As we will see in Chapter 3, having formulated a mathematical model for the Loss Offset Problem, we can always get a feasible solution. We would like to achieve a more challenging goal of finding the best solution, as depicted on the rightmost side of Figure 2-5.

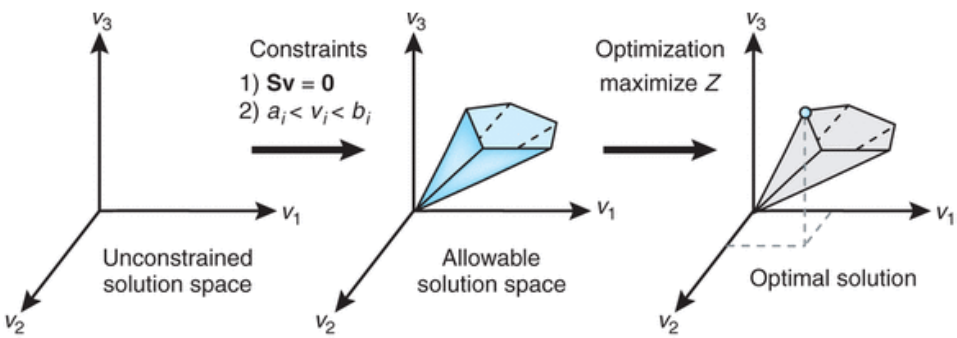


Figure 2-5 Feasible solution region and optimal solution in 3D [OTP10]

In literature on *OR* and mathematical programming, there are several concepts of dividing optimization problems into different categories according to their nature or complexity. A clear classification can rarely be made. In a natural way, we can consider the most popular concepts of categorizing optimization problems as follows:

- uni-variate vs. multi-variate problems
- continuous vs. discrete problems
- linear vs. non-linear problems
- deterministic vs. stochastic problems
- local vs. global problems
- dynamic vs. static problems

In this work, we are concerned with a combinatorial optimization problem under a deterministic assumption. The problems itself contains continuous as well as discrete decision variables, Constraints and objective function are piece-wise functions with linear and non-linear characteristics. Our task is to search for the global maximum of the objective function among several local maxima. Furthermore, the decisions should be made in a dynamic, simultaneous context. Overall, these combined features make the problem a complex one to model, understand and solve. In the following we present some crucial types of problem that have become intensive treatment in research as well as practice.

2.2.1 Linear Optimization

Linear Programming (LP) is a generalization of Linear Algebra and plays a unique role in optimization theory. It is capable of handling a variety of problems, ranging from optimizing schedules for airlines or lectures in a university to distributing oil from refineries to markets. The reason for this great versatility is the ease at which constraints can be incorporated into the model. As the name implies, Linear Programming is about linear constraints and minimizing (or maximizing) linear objective functions[9]. Usually, a linear problem can be formulated as:

Maximize

$$\sum_{j=1}^{n} c_j \, x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} \, x_j \;\leq\; b_i \; for \; i = 1,2,\dots,m$$

$$x_j \;\geq 0 \; for \; j = 1,2,\dots,n$$

The problems can be solved in polynomial time using interior point methods. The probably most popular algorithm used for solving linear problems is the simplex algorithm proposed by George

[9] Yet, a problem with some quadratic (or higher order) constraints can be handled as well by using a process called linearization.

Bernard Dantzig [DOW55]. Another famous algorithm was formulated by Nelder and Mead some years later [NM65].

To describe these techniques would take us too far afield. For the optimization problem that will be proposed in Chapter 4 of this thesis, not only the objective function but also several constraints have non-linear characteristics. Therefore, *Linear Programming* is in fact irrelevant for our work. For more details on this field of optimization we refer to [Mur83].

2.2.2 Non-linear Optimization

In the past many researchers as well as practitioners have come to the realization that a variety of problems in business and industry are in fact much more complicated to be solved by linear programming techniques. Frequently these problems have non-linear objective functions to be optimized rather than linear ones. In additional, the constraints may be quadratic, cubic or has interdependency on each other.

While linear programming is a technique that optimizes a given linear function subject to linear constraints, non-linear programming deals with a much broader variety of functions and inequalities. Furthermore, in nonlinear optimization it is well known that convex functions are easier to optimize than their non-convex counterparts. This is not only an empirical fact, but has been proven theoretically; see [BV04]. This is due to the fact that for convex optimization problems, local optimality always implies global optimality. A convex[10] problem may have the form:

Minimize $f_0(x)$

Subject to $f_i(x) \leq b_i, \ i = 1, \dots, n$

Whereas the objective function and ***all*** constraints satisfy

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

For any $\alpha + \beta = 1 \ and \ \alpha \geq 0, \beta \geq 0$

Several authors have been interested in analyzing and solving convex problems. For more details on convexity see [Lan13, Chapter 6]. Furthermore, in computational theory, if a problem is convex it permits the design of very efficient algorithms for exactly finding the global optimum of the objective function as well as giving a proof of global optimality. It is also important to mention that any linear optimization problem is often a special case of convex optimization, because a set of linear constraints will form a convex polytope. For an example of such a 3-dimensional polytope see [PS98, p.35]. In the following we offer two examples for understanding the behavior of convex and non-convex functions in 3-dimensional space. In Figure 2-6, consider the 3D surface plot of the function

$$f(x,y) = x^4 - 6 \cdot x^2 + 4 \cdot x + 12 + 2 \cdot (1 - y) + 3 \cdot y^2$$

[10] We also use the term "convexity" to refer to concavity.

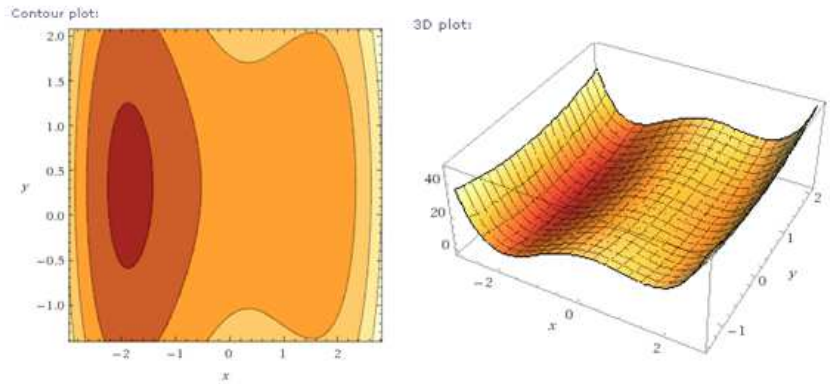


Figure 2-6 3D surface plot of a non-convex function

We have chosen this function with intention to conveniently demonstrate the two characteristics: non-linearity and non-convexity. The function is non-convex because it has more than one optimum. Searching for the global optimality of such a function is considerably more difficult than, for instance, the function

$$f(x,y) = -x^2 - y^2$$

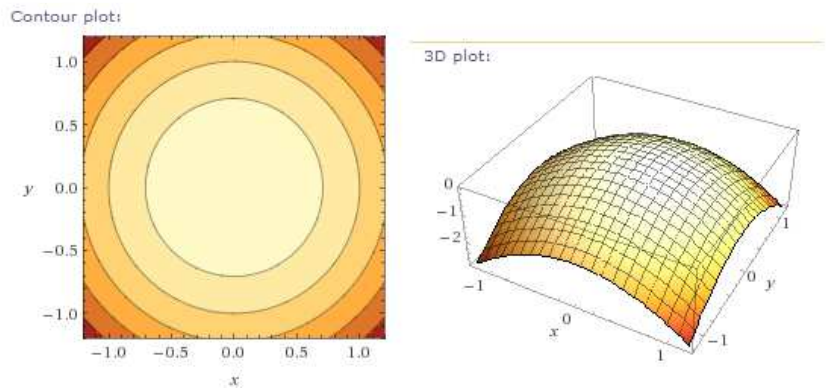


Figure 2-7 3D surface plot of a convex function

Figure 2-7 exposes the plot of the function with its unique maximum which can be found by standard techniques of convex optimization. The plots are created on wolframalpha.com.

A more complicated topic in non-linear optimization is solving multi-objective problems[11]. A general overview of nonlinear multi-objective optimization methods is given in [Mie01]. The basic features of several methods are introduced so that an appropriate method could be found for different purposes. The methods are classified according to the role of a decision maker in the solution process. The main emphasis is devoted to interactive methods where the decision maker progressively provides preference information so that the most satisfactory solution can be found. In most cases, such problems can be solved with evolutionary algorithms, see [FF98] or [Sch95].

[11] Notice that multi-objective problems can also arise in linear optimization.

2.2.3 Integer Optimization

We would also describe the next type of optimization problem: *Integer Optimization*. In integer problems, variables can only take integer values instead of real values. A special case of *Integer Programming* is *Binary Integer Programming*. For these problems, we have the additional restriction that each variable $x_i \in \{0,1\}$. There are many situations where we would like to use binary variables. For example, we might let a binary variable represent the decision in scheduling problems, which is 1 if at time t a machine operates, and 0 otherwise. *Binary Integer Programming* is, moreover, a specific case of a more general problem, namely *Integer Linear Programming*.

Much of the difficulty of the subject stems from the fact that a problem may have optimal real solutions and optimal integer solutions, but the optimal integer solutions need not be close to the optimal real-valued solutions. To understand this, we can consider the knapsack problem. We follow the presentation in [Fra80, p.127–128]. Knapsack problem is formulated verbally as: Given a knapsack with maximum capacity and a set of items where each item has some weight and profit value, which of the items should be packed into the knapsack to achieve the maximum profit subject to the capacity constraint? There is an interrelated trade-off between the choices. The problem is called a "yes" or "no" problem because for each selection an item must be entirely accepted or rejected, that is we cannot subdivide an item. These decisions can be mathematically represented by $x_i \in \{0,1\}$.

Similarly, later in our tax problem, we must choose to apply a taxation option or not, we cannot choose another number "between" 0 or 1 as a feasible solution because it has no meaning at all.

The application of *Integer Programming* can be found in various flow and matching problems, whereas the probably most prominent example is the *Traveling Salesman Problem (TSP)*. The *TSP* is considered to be unsolvable by polynomial algorithms and therefore described to be *NP-hard*. For a thorough discussion on complexity of the *TSP* see for instance [LMSK63] or [JM97]. In Section 2.3.3 we give a small example with solving the *TSP* with a *Greedy Algorithm*.

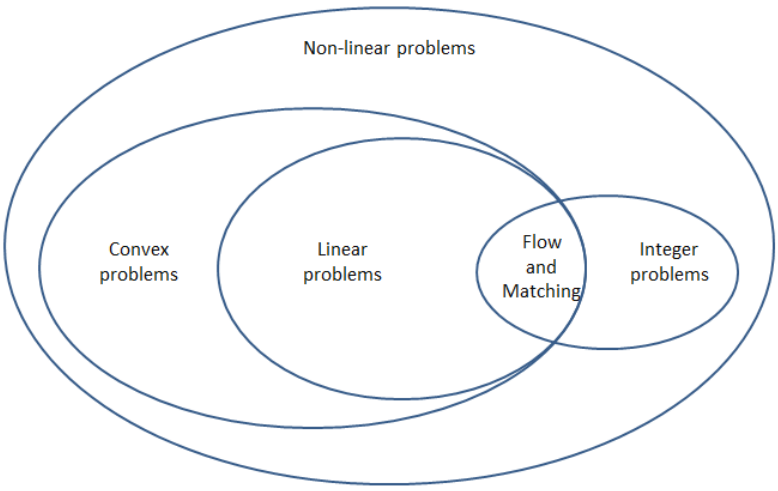


Figure 2-8 Classes of problems [PS98, p.2]

The above figure indicates a nesting of the problems we have mentioned so far. The illustration makes it quite easy to recognize the complexity issues. As we can see, the class of non-linear, non-convex problems contains all other classes of optimization problems. As a result, it implicates all the difficulties that may appear in solving problems of the subclasses. This is exactly the case of our Loss Offset Problem to be proposed and solved.

2.2.4 Combinatorial and Global Optimization

In many cases, optimization problems are very difficult to solve and an analytical search method is not feasible. Other solution methods may involve some compromises, either very long computation time or even not always finding the optimal solution. In the worst case, one has to make use of *Exhaustive Search* for finding the optimal solution[12]. There are also exceptions that certain problem classes can be solved reliably. Some of those have been discussed the previous sections such as:

- linear problems
- convex problems
- integer problems

We have surveyed these categories of optimization, for which established problem-solving approaches are available. Furthermore, when the above features can be proven to be fulfilled by a problem, then most of the existing approaches are analytical and deterministic, so that an optimization problem of these types can be dealt with very efficiently.

Combinatorial optimization is a subset of optimization that consists of finding an optimal item from a finite, discrete set of items [PS98]. In most cases, it is not problematic to find a feasible, good solution. The search for the global optimum, however, is extremely difficult. Once found, global optimality can only be guaranteed if an *Exhaustive Search* is conducted. Research literature on this topic is closely intertwined with those of global optimization and discrete optimization in which the most common involving problems are the traveling salesman problem and the minimum spanning tree problem. It is useful to mention that discrete optimization deals with problems in which all variables are discrete; whereas combinatorial optimization deals with problems in which all variables are discrete and finite. Thus, combinatorial optimization can be seen as a specific field of discrete optimization. Both of these terminologies are related to the global optimization, due to the fact that for a problem instance there may be several local optima, among which only one global optimum is desired.

[12] However, an exhaustive approach is not possible for large problems with thousands of variables and constraints.

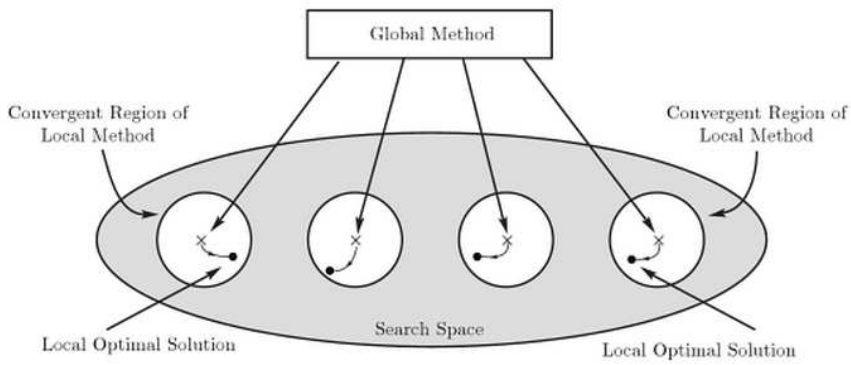


Figure 2-9 Illustration of a global-type algorithm [PC09, p.65]

The general problem in global optimization is to escape from a local solution and approach another, as shown in Figure 2-9 [PC09, p.65]. Thus, any global-type algorithm may identify several solutions and compare those with each other to find out which of them is the global optimum. Usually, there is a trade-off between computation time and quality of the solution. Because there may be an extremely large number of promising local regions, the strength of a global search strategy can be seen as the capability of finding the region in which the global optimum is contained. Due to the necessity of solving mathematical problems for global optimality in diverse areas, there has been ground-breaking progress in the algorithmic development of global optimization during the last several decades. A discussion of some important issues of global optimization can be found in [HT96]. Some important heuristic methods will be covered in the next section.

2.3 Methods for solving Optimization Problems

In this section we are concerned with some techniques to deal with optimization problems. The goal is to find a set of value assignments to certain variables that will satisfy specific mathematical equations and inequalities. They are also used when the goal is to find a variable assignment that will maximize or minimize a certain function of those variables. As mentioned in an earlier section, it is the task of a quantitative analyst to determine which problem-solving approach is most suitable for a given type of problem.

Some of the methods we discuss may be classified as *weak search methods*. Such approaches say that the search strategy should arise out of the interaction between the structure of the search agent and the proposed task. The search strategy chosen is assumed to be *weak* if the search agent has little knowledge about the task environment. The advantage of such an approach is that it avoids program synthesis, i.e., the behavior results from the interaction of agent and task rather than being explicitly programmed. While it is far from clear what the optimal method might be, it is our intention that the reasoning and examples in Chapter 4 will help to give a comparison for the methods proposed in this section.

2.3.1 Monte Carlo Simulation

The first technique for search problem considered here is the *Monte Carlo Simulation (MCS)* method, which is considered a valuable tool in predicting an unknown outcome. As the name of the method already indicates, we attempt to solve a problem by simulating the relationship between variables and objective function in a system or mathematical model. Starting from an initial random trial we can estimate how good a solution can be. In other word, we can draw an estimate and make certain assumption of an unknown value, here the objective function, by this method. In a complicated search problem, with *MCS* we may be able to generate the maximum and the minimum value of the search domain. The interval between them indicates a possible range of value in which the optimum is contained. Nowadays, a simulation task can usually be fulfilled by using available software tools with advanced built-in functions.

There are four major classes of software used for modeling and simulation as described in [HL04]:

- Spreadsheet software
- General-purpose programming language
- General-purpose simulation language
- Applications-oriented simulators

The reliability is a crucial issue in simulation. Mainly used as an approach in forecasting model in finance and especially option theory, the *MCS* will only be able to provide estimation that represents probabilities and not a deterministic value. Thus, what a simulation process can tell us is just how likely the resulting outcomes can be. However, the use of MCS is not only limited to finance, but can also be found in other disciplines of natural sciences [Bin78] and management sciences [KY13]. For instance, in [Čer85], the author proposes a Monte Carlo algorithm to find approximate solutions to the *TSP* by generating permutation of the cities of the tour randomly. They tested the algorithm on several examples and came to the surprising conclusion that it is possible to get very close to the optimal solution. However, the reliability could not be proven.

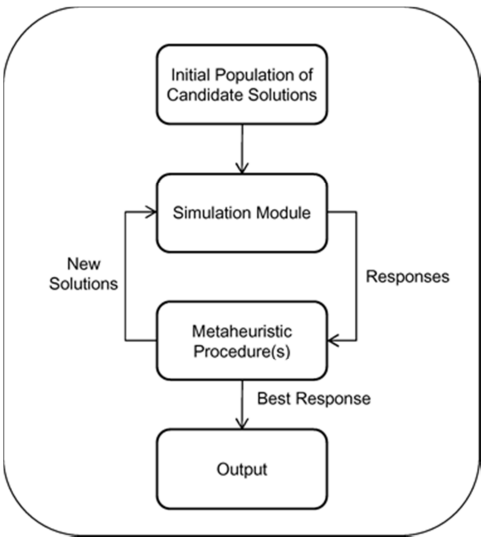


Figure 2-10 Simulation optimization with metaheuristic procedures [KY13]

Randomness is also the essence of simulation. In our application of *Monte Carlo Simulation* in Chapter 4, we make the assumption of a uniform distribution and not the standard normal distribution while generating random feasible variable values for calculating the final wealth. Also, there is one way we can improve the simulation technique. We can use it to best effect by providing the simulator some type of "knowledge" about possible solutions of the problem. The objective function is calculated based on a random value selected for each of the decision variables. The result is recorded and the process is repeated in hundreds or even thousands time. When finishing the process we have a significant number of different results obtained from the model. Based on the information from the simulation, we may make know which combination of variables may be the better one that leads to a higher value of the objective function. The more information we have, the higher probability of correctness we can have when making a statement about optimal decisions.

Because it is not the focus of this thesis to describe and apply simulation techniques, but just to apply the basic principles of simulation to model and make approximation of good feasible solutions for the Loss Offset Problem, there is no need going into further detail of this technique.

2.3.2 Exact Search Algorithms

2.3.2.1 Brute-force Search

The *Brute-force Search* algorithm, or *Exhaustive Search*, is very straightforward and can thus be defined in a few simple steps. When we have a search space, the simplest and the most obvious solution is to perform a *Brute-force Search* through the whole space to find the item we want. This means nothing more than purely list all of the possibilities of choices and analyze each choice. Because of its simplicity, the algorithm is quite widely used and actually performs pretty well in most cases where the problem structure is complex but the search domain is not so large. It is also very easy to describe and code so that we do not need an intensive discussion here. The exhaustive methods also include various algorithms based on decision tree such as depth-first search and breadth-first search. A decision tree considers the elements as vertices of a tree, and we need to traverse that tree in some special order by using tree pruning methods such as *Branch and Bound*.

The following example shows the *Brute-force Search* algorithm in action, looking for best solution in a simple *TSP* which is a combinatorial optimization problem.

Example 2-1

Starting from city A, the salesman has to travel through all other cities and come back to A. Which is the route with a minimal total cost? The graphical illustration of the problem is given in Figure 2-11 [MF04].

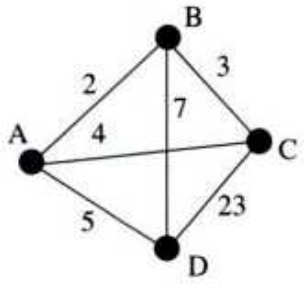


Figure 2-11 An example of the TSP

Figure 2-12 shows the decision tree constructed for this problem instance. The *Exhaustive Search* will calculate all values of total cost resulted by all possible paths. After analyzing all six possibilities, we can compare the values of the routes and find out the optimal route with minimal cost that is

$$A - C - B - D - A$$

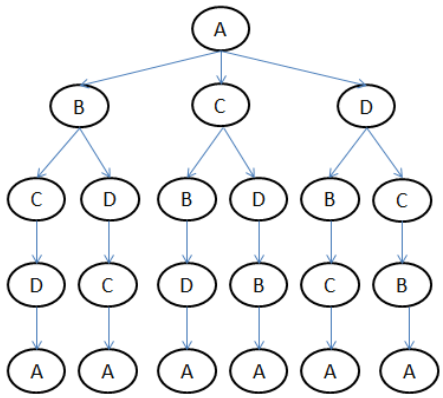


Figure 2-12 A decision tree of *Brute-force-Search* for the *TSP*

In some cases when we have a specific type of problem, the *Exhaustive Search* may be more efficient than other deterministic methods. For example, consider the search for a specific name in a phone book. We can apply a linear search that returns exact match to the lookup value by searching from A to Z. While sometimes slower than a *Binary Search*, this strategy may be better if the named to be found is located extremely near to the beginning of the list. Another easy way to understand *Exhaustive Search* is to consider the process of choosing item in a knapsack problem, which we present in the following section.

2.3.2.2 Branch and Bound

The *Branch and Bound* method is a well-known and established tree-search methodology that has been repeatedly applied in optimization problems in economics and other disciplines, especially whenever integer variables appear. Unlike general meta-heuristics, which at best work only in a probabilistic sense, many of the tree-search methods are guaranteed to find the exact or optimal solution, given a correct "branching and bounding" strategy and, of course, enough computation time. A detailed presentation of the method can be found in a work of Lawler and Wood [LW66] or Papadimitriou and Steiglitz [PS98, Chapter 18]. The general principle can be described as follows:

A complex problem is divided into several sub-problems that have lower level of complexity; a partitioning process which is called branching. After solving these sub-problems, we can calculate bounds for the solutions, compare them, and decide which sub-problem can be ignored in the next stage of branching. This process is referred to as bounding, i.e., reducing the solution space. By repeating this branching and bounding procedure, we try to construct a proof of optimality for the final solution. While it is hopeless to apply *Exhaustive Search* on large combinatorial optimization problems, the *Branch and Bound* method is much more promising. It is considered to be "the idea of intelligently enumerating all feasible solutions" [PS98, p.433]. Dakin [Dak65] recommends using depth-first-search as a branching strategy. An illustration of such a decision tree is shown below.

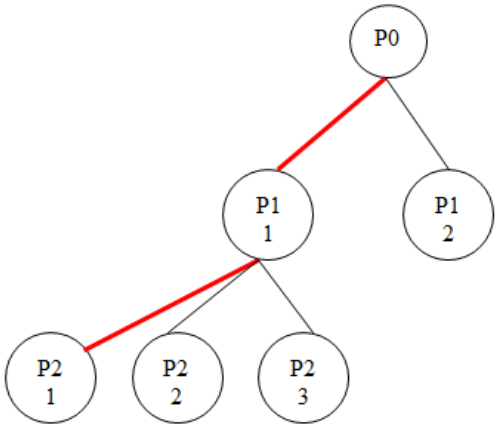


Figure 2-13 A decision tree for *Branch and Bound*

For a better understanding, let us briefly consider the following numerical example.

Example 2-2

Maximize $f(x,y) = x + y$

s.t. $4 \cdot x + y \leq 8$ (1)

$x + 2 \cdot y \leq 3$ (2)

$x, y \geq 0$ (3)

$x, y \in \mathbb{Z}$ (4)

Solution: Let us call the problem above P.

1. Relaxation: Define problem $P_0 = P$ without constraint (4). Solve the problem P_0 with the Simplex method [DD07, p.21]. We get the solution $UB_0 = 2{,}43$ with $x = 1{,}86$ and $y = 0{,}57$. This is the upper bound which represents the maximal value achievable with the relaxation of integer variables. The lower bound is $LB_0 = 0$ with $x = 0$ and $y = 0$.

2. Branching part: Divide problem P_0 into two sub-problems.

   $P_{11} = P_0$ with the additional constraint $x \geq 2$.

   Solution $UB_{11} = 2$ with $x = 2$ and $y = 0$.

   The lower bound is $LB_{11} = 2$ with $x = 2$ and $y = 0$.

   $P_{12} = P_0$ with the additional constraint $x \leq 1$.

   Solution $UB_{12} = 2$ with $x = 1$ and $y = 1$.

   The lower bound is $LB_{12} = 0$ with $x = 0$ and $y = 0$.

3. Analyzing problem P_{11}. Because $LB_{11} = UB_{11}$, the search process can terminate. The optimal solution is found with $x = 2$ and $y = 0$. The node P_{12} can be *killed* (the term *fathomed* is also used).

Of course, this was a very simple example that contains a quite intuitive structure. In solving practical problems, there are many important questions to be answered, for instance:

- What is the most suitable way to divide the search region into smaller regions?
- How many sub-regions will be enough?
- In what order should the sub-problems be solved?
- What kind of relaxation is allowed to be made?
- Should a weak bound or a strong bound be chosen?
- etc.

As insisted by numerous studies, there is no single answer for such questions. Each time when there is a new problem instance to be solved, a new strategy must be set up to deal with all the questions above. More complicated examples of the application of *Branch and Bound* algorithm for solving the *TSP* can be found in [LMSK63] and [HK70].

2.3.2.3 Binary Search

In the beginning of this section we have briefly explained how a binary decision tree can be constructed to solve a search problem. In fact, it can also be applied to solve optimization problems, provided that the problems fulfil certain conditions.

Binary Search, also known as a half-interval search, is one of the fundamental algorithms in computer science. In order to explore it, we first build up a theoretical backbone, then use that to implement our own solution algorithm properly and avoid errors in later chapters. In its simplest form, *Binary Search* is used to find a value in a sorted sequence. That is, for example, a list of last names in ascending or descending orders. This is called the search space that contains the entire sequence. For clarity, let us call the name to be found the target value. *Binary Search* maintains a contiguous subsequence of the starting sequence. At each step, the algorithm compares the midpoint value in the search space to the target value. Based on the comparison and because the sequence is sorted, it can then eliminate half of the search space. By repeating this loop, we will gradually get to with a search space consisting of a single element. That will be our target value.

The greatest advantage of *Binary Search* can be seen in its complexity. Since for each step *Binary Search* halves the search space, we can easily prove that *Binary Search* will never use more than $O(log_2 N)$ comparison steps to find the target value. The logarithm is an extremely slowly growing function. For a better understanding of just how efficient binary search is, consider the next figure that contains the plot of the logarithmic function. This may not seem very useful at the moment where we have simple problems, but we will see the efficiency of *Binary Search* when the search space contains hundreds of thousands of values.

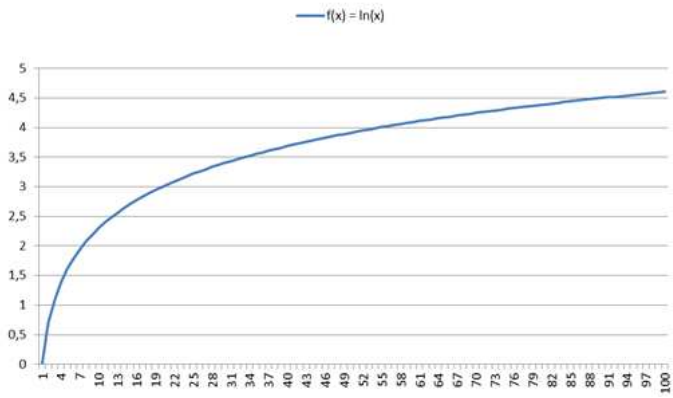


Figure 2-14 The logarithmic function

In order to make the concept of using *Binary Search* in optimization clear, we provide the following simple example considering a quadratic function.

Example 2-3

Find the maximum of the function $f(x) = -x^2 + x + 1$ in the interval $x \in [-2; 2]$ using binary search. It is obvious that this task can easily be solved by computing the first derivative of the function. The maximum is achieved with $x = 0{,}5$ as shown in the next figure. Next, explain the algorithm of *Binary Search*.

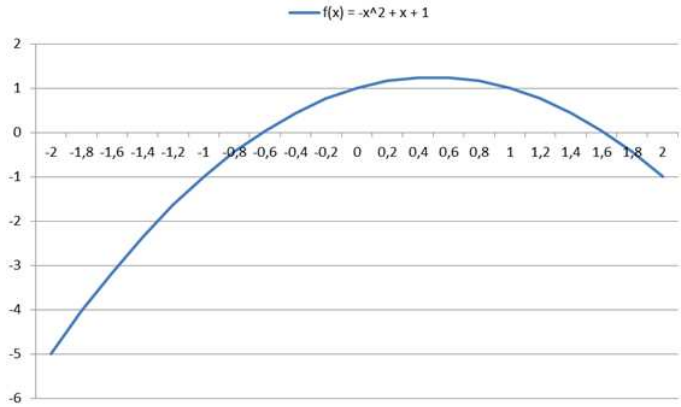


Figure 2-15 Optimization using *Binary Search*

Solution:

Step 1: Define the left and right bounds xL and xR of the search domain. We have $xL = -2$ and $xR = 2$. Denote ε as the tolerance to terminate the search algorithm. Let $\varepsilon = 10^{-10}$

Step 2: Calculate the function slope of the midpoint $x = 0$. We have $\frac{df(x)}{d(x)} = 1 > 0$. Thus, set $xL = 0$. The right bound remains the same $xR = 2$. Calculate $xR - xL = 4$ that is greater than ε. Continue the loop.

Step 3: Calculate the function slope of the midpoint $x = 1$. We have $\frac{df(x)}{d(x)} = -1 < 0$. Thus, set $xR = 1$. The left bound remains the same $xL = 0$. Calculate $xR - xL = 2$ that is greater than ε. Continue the loop.

Step 4: Calculate the function slope of the midpoint $x = 0{,}5$. We have $\frac{df(x)}{d(x)} = 0$. Thus, set $xR = xL = 0{,}5$. The optimal value of x is found. Calculate $xR - xL = 0$ that is smaller than ε. Stop the algorithm.

A disadvantage of *Binary Search* is the requirement about the search space. It should be sure that we have a sequence of rising or falling values so that the method can show its strength. Trying to use *Binary Search* on unsorted lists makes little sense. In such cases, it is better to use a plain linear search or blind search instead. An application of *Binary Search* trees can be found in [HT71].

In case of optimization, some complications can arise in the solution approach with *Binary Search*: sometimes the search process terminates at an undesired point, delivering no optimum. A proof for this is given in the following figure which plots the function $f(x) = 2 \cdot x \cdot sinx^2$ in the interval $x \in [-3; 3]$.

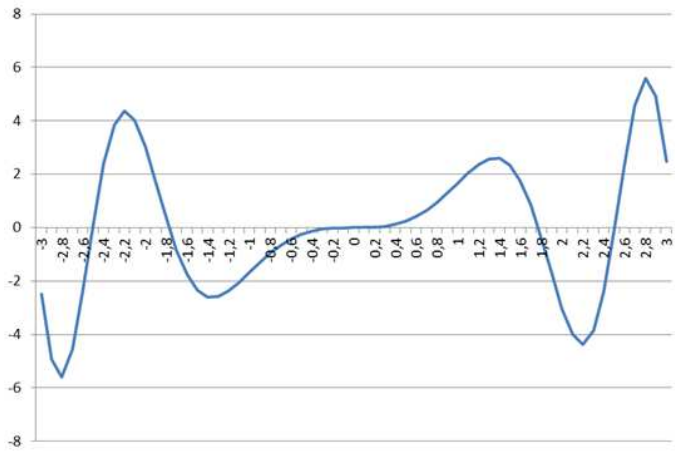


Figure 2-16 Function with multiple maxima and minima

Suppose we search for the global maximum of this function in the given interval. The first derivative of this function is $f'(x) = 2 \cdot sinx^2 + 4 \cdot x^2 \cdot cosx^2$. Solve $f'(x) = 0$ we get $x = 0$ so that no slope can be found at this point. *Binary Search* will therefore stops, having $x = 0$ as the solution.

The next limitation is related to the shape of the function. If the function is multimodal[13], i.e., there are several optima, *Binary Search* may deliver a suboptimal solution, i.e., a local optimum. There, we may be able to discard the half which does not contain the optimum, but in the worst case, the opposite will happen. This can be seen very clearly in the graph above. Another case *Binary Search* will also not work efficiently is when the function is discontinuous or when we cannot define the left and right bounds exactly. In Chapter 4 we will discuss the application of *Binary Search* for solving our problem.

There are many variations of the *Binary Search* method such as midpoint search, noisy search, and several other modified techniques. Thus, the implementation of this search algorithm must be adjusted for each separate problem setting. This might not be an easy undertaking. Knuth has mentioned in his book in Section 6.2.1: "Searching an Ordered Table. Sorting and Searching" that: "Although the basic idea of *Binary Search* is comparatively straightforward, the details can be surprisingly tricky…" [Knu97, pp. 409–426]. This fact can be confirmed by Bentley when he assigned a problem in a course for professional programmers and found that ninety percent of the programmers failed to code a *Binary Search* algorithm correctly after several hours of working on it [Ben00, p. 341].

[13] The term "unimodal" and "multimodal" will be discussed in brief.

2.3.2.4 Golden Ratio Search

In this section we discuss the *Golden Ratio Search*[14] *(GRS)*. In 1953, a method called Fibonacci Search was presented by Kiefer [Kie53], which is a method to find the maximum of a uni-modal function and has many applications in computer science. Thompson and Peart [TP68] then recommended the *Golden Ratio Search*, which is another version of the Fibonacci Search, for one-dimensional, uni-modal search problems. We shall briefly discuss what "uni-modal" means.

It is at first to mention that the term "modal" originally applies to data sets and probability distribution, and not in general to functions. However, in some literature the definition of "uni-modal" was extended to functions of real numbers as well. A common definition is given by Ushakov [Ush01] as follows:

A function $f(x)$ is a uni-modal function, if for some value m, it is monotonically increasing for $x \leq m$ and monotonically decreasing for $x \geq m$. In that case, the maximum value of $f(x)$is $f(m)$ and there are no other local maxima. As in the case of convexity, proving uni-modality often proofs to be hard. One popular way consists in using the definition of that property, but it turns out to be suitable for simple functions only. Another general method based on derivatives exists, but it does not succeed for every function despite its simplicity [Tou84]. Examples of uni-modal functions include quadratic polynomial functions with a negative quadratic coefficient.

Now, what is the *Golden Ratio*? And how does the search strategy work?

First, for our work the *Golden Ratio* is defined as

$$\phi = \frac{1+\sqrt{5}}{2} - 1$$

Thus, $\phi = 0{,}61833 \ldots$

Assume that we are given a uni-variate function f that is uni-modal on the interval $[l, u]$. Our task is to find the optimum of f on that interval. Denote by I_0 the length of this interval. In order to find exactly where the optimum is, we can iteratively narrow down this length by calculating new lower and upper bounds. This means:

Calculate $l' = l + \phi \cdot I_0$ and $u' = u - \phi \cdot I_0$.

If $f(l') \geq f(u')$, then set u' as the new lower bound, i.e., the new search interval is $[u', u]$.

Else, set l' as the new upper bound, i.e., the new search interval is $[l, l']$.

Repeat the procedure until the interval is small enough.

A graphical illustration is given below. The flowchart of the algorithm is given in Appendix B.

[14] Some literature may refer to this as the *Golden Section Search*.

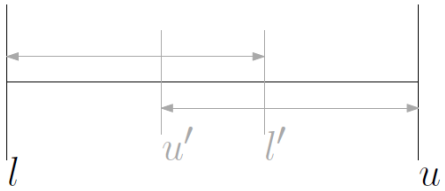


Figure 2-17 Reducing the search space with *Golden Ratio Search*

Alike *Binary Search*, *Golden Ratio Search* belongs to the class of deterministic search methods. One shortcoming of the *Golden Ratio Search* is, if it is carried out on multimodal functions, there is a high possibility that the search will return an incorrect value and there is no guarantee for the result found. Also, the implementation of the algorithms differs from case to case. On the other hand, despite its ubiquity and seeming simplicity, the method is apparently quite difficult to implement for multi-variate functions.

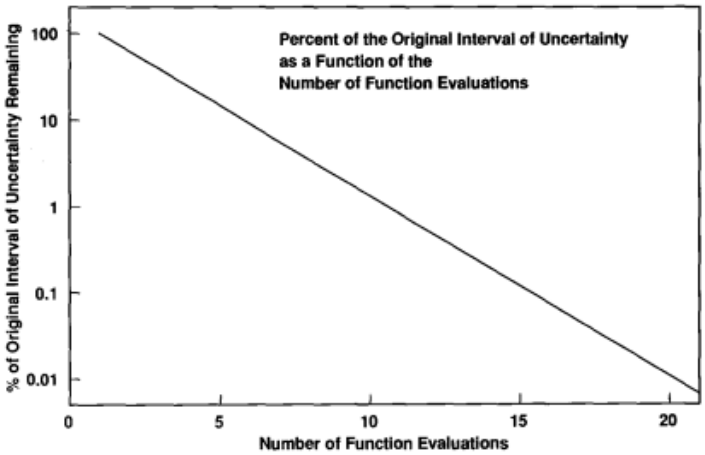


Figure 2-18 Percentage of the original search region remaining after N function evaluations by *Golden Ratio Search*

The paper of Jones and Grisso [JG92], although describing a different application domain, outlines the *Golden Ratio Search* for determining optimal values of functions described within spreadsheets. This is a very similar purpose as in our work that will be presented in chapter 4. Due to the *Golden Ratio*, this search technique reduces the search space by approximately 38,2% per iteration step and is therefore one of the fastest interval search techniques. Figure 2-18 illustrates the percentage of the original search region after each calculation of the function value [JG92]. Often, we define a percentage of the full range to stop. The full range is the range of boundaries confined from upper bound and lower bound, which are called right and left bounds in our later implementation.

2.3.3 Stochastic and Heuristic Algorithms

In the following we discuss some representative heuristic algorithms. Exact search algorithms are also referred to as deterministic algorithms, because they always show an identical procedure and therefore finding the same result when starting from the same start point. This, unfortunately, cannot be guaranteed by heuristic algorithms. Despite this lack of exactness, heuristics remains a vibrant topic of research with many exciting approaches under investigation such as swarm intelligence, *Ant Colony Optimization*, *Simulated Annealing*, or genetic and evolutionary algorithms[15]. Interest in such solution techniques is intense because only few optimization problems in real-world environment can be solved exactly in a reasonable amount of time. In German language, Nissen [Nis97] gave a good overview of the major types of evolutionary algorithms. For this type of algorithms, artificial intelligence is an important application field, for example see [FOW66]. Other authors have concentrated their studies on the application of behaviors of termites and ants; see [LB04] or [BD04]. One other important application field of research in which heuristic methods are widely in use is artificial neutral network [SADJ98].

2.3.3.1 Greedy Algorithm

A *Greedy Algorithm* deals with a problem by constructing a series of steps according to a very simple idea: assign the values for all of the decision variables one by one and at every step make the best available decision. The name greedy describes exactly how the mechanism is, namely the approach assumes a heuristic for decision making that provides the best possible move at each step, the best "profit". However, we can easily identify the problem with a greedy approach. Because the complete structure of a problem is not necessarily considered, taking the optimal decision at each separate step cannot not always guarantee the optimal solution overall. Therefore, this approach is also described as shortsighted [MF04, p.87].

An intuitive example for demonstrating the weakness of *Greedy Algorithm* is, again, the *TSP*. With a greedy approach, the salesman start from a city and proceed to the nearest unvisited city among several alternatives. The process is repeated at each coming city until every city has been visited. Consider the following simple example from [MF04].

[15] EA is also offered by the MS Excel Solver.

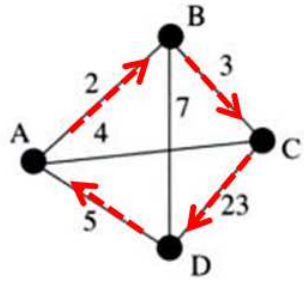


Figure 2-19 A greedy algorithm for the TSP

In Figure 2-19, the salesman starts from city A and wishes to visit three other cities B, C and D. Applying the basic version of the greedy algorithm, we can construct a tour with the sequence $A - B - C - D - A$ with a total cost of 33. The optimal tour with minimal cost is, however, $A - C - B - D - A$. The cost is then 19, which illustrates a significant better result.

Despite its simplicity, the application of greedy approaches is very broad and many variations have been constructed by researchers as well as practitioners. A research area in which greedy methods are widely popular is graphs, networks and matroids[16]. In some types of problem, the *Greedy Algorithm* actually leads to optimal solution [Jun12, Chapter 5]. Another important practical application of the algorithm is in metric space searching and message distribution in networks, where it also tends to deliver good experimental result [SZ04].

In complex optimization problems, *Greedy Algorithm* may be used to obtain a first insight into the problem structure. Furthermore, as for most non-deterministic algorithms, each time we have a new problem of a specific class, there is the need to adapt the *Greedy Algorithm* so that it can be appropriately applied. In Chapter 4 we will discuss this matter in the context of our optimization problem in tax loss planning.

2.3.3.2 Hill Climbing

One of the most prominent search methods that have been demonstrated and applied very often in *OR* as well as computer sciences is *Hill Climbing (HC)*. The strategy of *Hill Climbing* can be applied when the plot of an objective function shows no standard behavior. Although more advanced algorithms may give better results, in some cases hill climbing works just as well. An extension of this method is *Stochastic Hill Climbing*. The combination of those two search methods results into the optimization method called *Simulated Annealing*.

Basically, the *HC* technique for searching an optimum may consist of three steps.

1. Start with an initial solution, also called the starting point. Set current point as the starting point.
2. Make a move to a next solution, called the move operation.

[16] Matroids belong to a class of structures that plays an important role in combinatorial optimization.

3. If the move is a good move, then set the new point as the current point and repeat step (2). If the move is a bad move, terminate. The last current solution is the possible optimum solution.

Move operation: The move operation is problem dependent. In a discrete optimization problem, such as the Travelling Salesman Problem, a move operation would probably shuffle a couple of positions in the original solution. In problems where the search space is continuous, techniques such as those used in steepest ascent/descent methods are used to obtain the next solution.

Good/Bad Move: A move is said to be good, if a point obtained by the move operation improves the quality of the solution, as compared to the previous solution. A bad move is defined similarly.

Quality of solution: In all optimization problems, there is always at least on quantity (in single-objective optimization problems) that we want to improve. A quality of solution is judged on the basis of this quantity. In a function maximization problem, the function value itself is a measure of the quality.

Example 2-4

Let us now consider a simple one-peaked function $f(x) = -x^2 + 25$ with $-5 \leq x \leq 5$. We seek the maximum of this function. Notice that we have deliberately chosen an unrealistically simple problem to keep the ideas of *Hill Climbing* clear. It is obvious that the solution to this problem is $x = 0$, which yields a minimum function value of $f(x) = 25$, so using *HC* is not really necessary in this case. In Figure 2-20 is the plot of the function:

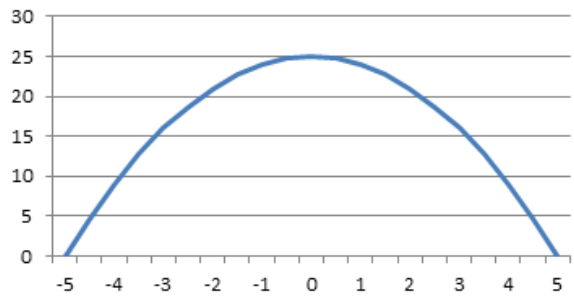


Figure 2-20 Example for *Hill Climbing*

We carry out the steps to find the optimum value of the function:

1. We start with an initial solution. Here for the starting point, we choose $x = -4$ and receive the value of $(x) = 9$. Set current point as the starting point.
2. Let the move operation take us to point $x = -3$. Consider the quality of the two solutions. The new solution is definitely improved, since the function value at the new point is $f(x) = 16$ and therefore greater than before, we keep moving, setting $x = -3$ as the current point.
3. Because the move was a good move, we repeat step (2). The move operation takes us to point $x = 0$ and we receive the function value $f(x) = 25$. Set $x = 0$ as the current point.
4. For the next move we set $x = 1$, the function value $f(x) = 24$. Because the function value becomes worse, we see that this was a bad move. We terminate our search, and set the latest current point $x = 0$ as our optimal solution.

As seen in the above example, simplicity is a hallmark of the *Hill Climbing* search. However, its simplicity comes at a price. In a multimodal landscape, *Hill Climbing* procedure will terminate at the first peak it finds. It may be the global maximum, but it could also be one of the several local maxima. This is illustrated in Figure 2-16 in the last section.

When there is multiple optima, the starting point for a *Hill Climbing* strategy is often a matter of arbitrary choice. Let us consider the simple function of $f(x) = x^3 - x^2 - 2$. We would like to find the maximum of this function on the interval $x \in [-0{,}5; 1{,}5]$. As seen in the plot in Figure 2-21, the global maximum of the function in the defined search domain is achieved at $x = 1{,}5$. However, if we set the starting point at $x = -0{,}5$ for a hill climbing strategy, the search process will terminate at $x = 0$ where we get a local maximum only.

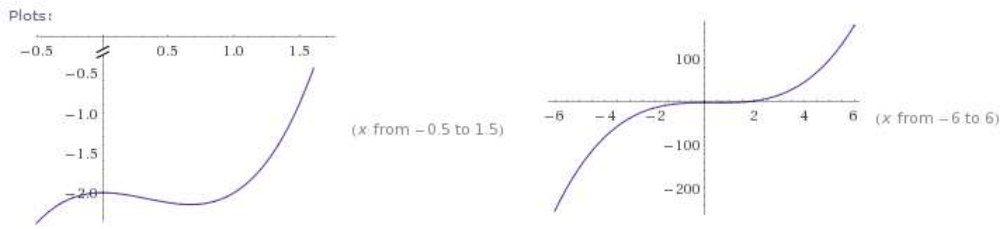


Figure 2-21 Example for problematic cases with *Hill Climbing*

In general, if the value of objective function is globally optimal, then it should not get worse if we simply change the starting point, i.e., replace the value x_i with a new value $x_i + s$, where s is a shift in the starting point.

2.3.3.3 Particle Swarm Optimization

In the previous section we have seen that a severe problem with *Hill Climbing* is that it will find only local optima. Unless the whole search space is convex, it may not be able to converge to a global maximum/minimum. Thus, other search algorithms such as *Stochastic Hill Climbing*, *swarm algorithms* and *Simulated Annealing* try to overcome this problem by combining arbitrary heuristics in specific ways. In most cases, a global algorithm may scan the search space by moving randomly in the search space according to some predefined rules instead of proceeding from item to item along the edges.

Particle Swarm Optimization (PSO) belongs to an important subclass of search methods called "Swarm Optimization" or "Swarm Intelligence" that views the elements of the search space globally. In the literature, this algorithm is related to genetic and evolutionary algorithms mostly because of its origin. The concept of *PSO* is illustrated by the natural behaviors of a flock of birds or a school of fishes. Originally, it is a population-based stochastic searching technique developed by James Kennedy, who was a social psychologist, and Russell Eberhart, an electrical engineer [KE95]. Following the idea of searching in 3-D space, *PSO* is well-suited to search problems with 2 or more

decision variables. In most situations, *PSO* must have some constraints on the range of possible values for these variables. Figure 2-22 illustrates the basic idea of the *PSO* algorithm. As we can see, there are a number of search agents, which are called particles. Effectively, we have a swarm of particles moving around in a N-dimensional search space, where each particle is considered as a point. Each particle adjusts its movement according to some rules that consider its own position and movement as well as the position and movement of other particles in the swarm.

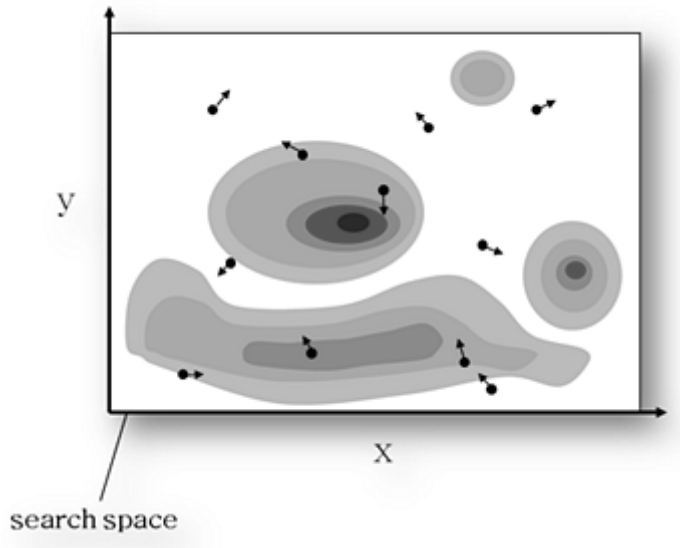


Figure 2-22 Illustration of the concept of *PSO* Algorithm

Thus, an intuitive explanation of the search process can be as follows:

1. Initialize the number of particles to be sent into the search space.
2. Set random positions, velocities and accelerations for all particles.
3. Determine which particle is "the best" by evaluating the function value.
4. Adjust all accelerations toward that best particle.
5. Update particles positions based on velocity, update velocity based on acceleration.
6. If the termination criterion is not met, go to step 3 again.

The steps 4 and 5 are the most "analytical" steps of the algorithms.

Among various evolutionary optimization techniques, *Genetic Algorithm (GA)* and *PSO* have attracted considerable attention [ES98]. The *PSO* is described as a robust stochastic evolutionary computation technique based on the movement and intelligence of swarms looking for the most fertile feeding location. Unlike the drawback of expensive computational cost of *GA*, *PSO* has better convergence speed. Similar to the *GA*, *PSO* is initialized with a population of random solutions. It starts with the random initialization of a population of individuals (particles) in the search space and works on the social behavior of the particles in the swarm. However, unlike the *GA*, each particle is assigned a random velocity, which determines the direction that the particle will fly through the search space.

For an optimization problem, each particle represents a potential solution. At each step of iteration, *PSO* tries to find the global best solution by adjusting the trajectory of each individual particle towards its own best location and towards the best particle of the swarm. The trajectory of each individual in the search space is adjusted by dynamically altering the velocity of each particle,

according to its own "flying experience and the flying experience of the other particles in the search space. Since the introduction of *PSO* algorithm, a number of different *PSO* varieties have been applied by researchers for solving complex problems in several application fields. Figure 2-23 depicts the flow chart for the general *PSO* algorithm [JC09].

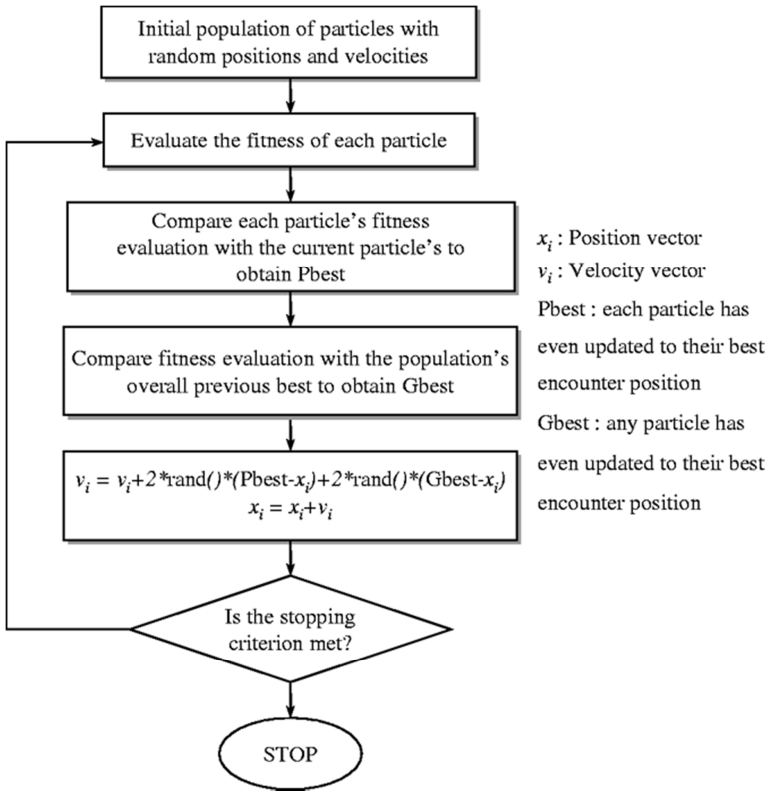


Figure 2-23 Flowchart of the *Particle Swarm Optimization* algorithm

The original *PSO* algorithm according to [PKB07] can be explained as:

1. Initialize a population array of particles with random positions and velocities on *D* dimensions in the search space.
2. loop.
3. For each particle, evaluate the desired objective function in *D* variables.
4. Compare particle's fitness evaluation with its *pbest*. If current value is better than *pbest*, then set *pbest* equal to the current value, and *p* equal to the current location *x* in *D*-dimensional space.
5. Identify the particle in the neighborhood with the best success so far, and assign its index to the variable *g* that represents the best neighbor.
6. Change the velocity and position of the particle according to the predefined equation as in [PKB07].
7. If a criterion is met (usually a sufficiently good fitness or a maximum number of iterations), exit loop.
8. end loop

The theoretical idea and detailed explanation of the *PSO* algorithm can be found in [KE95], whereas a discussion on the advantages and disadvantages of *PSO* in comparison with genetic algorithms is

given by [ES98]. One of the most difficult issues of applying *PSO* is to understand the effects and judicious selection of the various parameters for operating the algorithm. We will discuss this matter in Chapter 4 when solving numerical example problems.

2.3.3.4 Simulated Annealing

Simulated Annealing (SA) is a global optimization algorithm that was proposed by Kirkpatrick et al. [KGV83] in 1983 and has been recognized as an increasingly important method in optimization theory. Its application is popular in the area of combinatorial optimization where the objective function is defined in a discrete and finite domain. More insight into the algorithm is provided in a basic flowchart in Figure 2-24, which considers a minimization problem [CMMR87]. A much more complicated flowchart for solving our Loss Offset Problem is given in Appendix B.

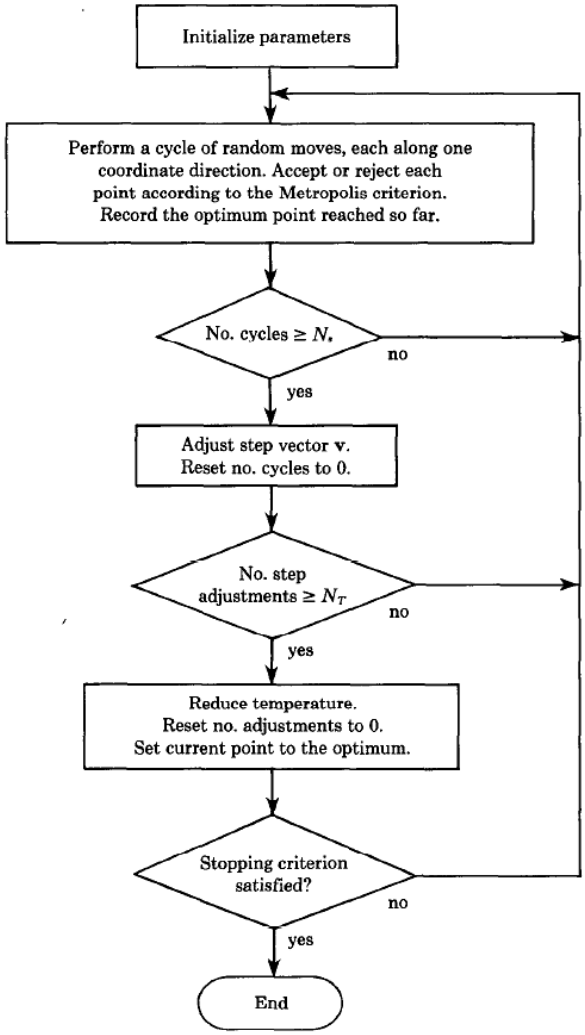


Figure 2-24 Flowchart for *Simulated Annealing* algorithm

As the name already implies, the main idea of *Simulated Annealing* is based on the analogy to statistical mechanics of annealing of solids. In an annealing process, a solid is heated up at a very high

temperature and then cooled down step by step until it reaches its minimum energy state. Thus, just as *Branch and Bound* can be seen as a general concept for exact search, *SA* is rather a strategy for solving combinatorial optimization problems in a heuristic way. As can be seen in the flowchart, the optimization with *SA* mainly goes through two major processes [Hea08]:

- First, for each temperature, the *Simulated Annealing* algorithm runs through a number of cycles. The number of cycles is predetermined by the decision maker. As a cycle runs, the inputs are randomized.
- Once the specified number of training cycles has been completed, a loop is finished and the temperature can be lowered. Once the temperature is lowered, it is determined whether or not the temperature has reached the lowest temperature allowed. If the temperature is not lower than the lowest temperature allowed, then the temperature is lowered and another cycle of randomizations will take place. If the temperature is lower than the lowest temperature allowed, the algorithm stops.

Literature on the *SA* algorithm, its application and variants is plentiful. Ingber examined the practicability of the algorithm [Ing93]. Martin and Otto [MO96] presented a method of combining *SA* with local search heuristics to solve combinatorial optimization problems. Vanderbilt and Louie combined *Monte Carlo Simulation* with the *SA* method [VL84]. Another contribution on the application of *SA* was made by Aarts et al. [AHKMS06] or Aarst and Korst [AK89]. With experimental examples, the authors show how it can be used to train artificial neural networks. The experiments indicate that good results can be obtained with little or no parameter tuning. For more interesting issues about optimization multi-model functions see for example [CMMR87], [GFR94], [Kir84], [BJS86], and [SDJ99].

So far, we have described the fundamentals of *OR* and its applications in various fields of problem. For the introductory purpose, most of our examples were chosen to be straightforward to understand, and the solutions arbitrarily easy to calculate. Later in this thesis, we discuss more realistic problems in tax planning that can be solved using the above discussed methods. As the complexity of problems increase, the algorithms will be able to show their effectiveness more clearly. As stated by Kirkpatrick et al. [KGV83], there is no guarantee that a heuristic algorithm that can find a near-optimal solution for one combinatorial problem will also works effectively for another. One way to test the heuristic algorithms is to make use of the Rosenbrock's function, which is a popular test function for optimization algorithms introduced by Howard H. Rosenbrock in 1960 [Ros60].

$$f(x,y) = (1-x)^2 + 100 \cdot (y - x^2)^2$$

With the presentation in this section we have merely scratched the surface of the rich and very complicated topic of Operations Research and Optimization Algorithms. One of the most common books is the one of Cormen et al named "Introduction to Algorithms". Its fame has led to the appellation of the abbreviation "CLRS". We encourage the interested reader to peruse other fundamental books such as [CLRS09] and the references therein for more details.

3 Tax planning and the Loss Offset Problem

The very first reason for planning activities from an OR point of view is the financial benefit that the decision maker can obtain. Because huge sums of money can be involved in modeling and solving a finance problem, applying OR methods can result in making or saving a lot of money, at least until competitors in the market begin building the same models or getting solutions using the same techniques [BSZ03].

The objective of this chapter is to study the Loss Offset Problem in tax planning. Due to its complexity, tax planning has no unified definition [Spi72]. In general, it can be seen as a subdomain of financial planning that contains various decision and optimization problems. We address the problem by initially presenting an overview of some essential matters that exist in research literature of tax planning. After that a section is dedicated to introducing the basics of German income tax law. Another section provides some models that explain the effects of taxation on business decisions. We then formulate the Loss Offset Problem as a mathematical model. This chapter does not intend to give profound discussion on tax regulations in Germany, the macroeconomic effects of before- and the after-tax income distribution, or analyze the effects of tax rates on individuals and households. It describes two effects of tax policy on tax payments. The first involves the impact of taxes for a given taxation option on total taxable income. The second involves loss offset strategies that are induced by taxpayers' information on gains and losses of multiple periods.

One of the most fundamental problems of investment planning is the following: suppose we are given an investment opportunity. At the beginning of the investment process, we make an investment which is often called the initial capital, e.g., for buying a machine, purchasing a building, or constructing a factory for manufacturing our products. In return, when selling products on the markets we receive a stream of cash inflows in future points in time. The problem is formulated as the question: given the initial capital I_0 and expected cash inflows CF_t, should the investment be made or not? This is a typical decision problem for which the solution is a yes-or-no answer. There is no variable which value to be determined. Also, there is no objective function which value is to be maximized or minimized. The answer for this problem can be either 'yes' or 'no', and depends upon the input data[17]. We will show that the answer can be different with and without the effect of taxation.

Mathematical modeling can be done according to the principles of described in [Sch99], [Wil01] or [SM06]. In fact, the multi-period cash flows then create a network flow structure [AMO93].We use MS Excel as a tool for modeling financial plans and constructing search and optimization algorithms. The advantage of creating a financial plan for taxation planning with such spreadsheet software is that it is easy to organize data and identify the arithmetic operations within several parameters. In form of a table, all values are given in cells, which are arranged in numbered rows and letter-named

[17] Nevertheless, we can certainly formulate this decision problem as an optimization problem by integrating a binary variable into the model. The "yes/no" decision will then become value 1/0 of the binary variable.

columns. By clicking on any cell, the financial planner can see which function is defined in it and how it is related to other variables and parameters.

3.1 Overview of tax planning models

Tax planning is important in a lot of economic sectors: in financial organizations as well as industrial and service-orientated enterprises. It is a breeding ground for modeling and solving problems and in fact, practitioners have been always seeking as well as utilizing optimization potentials that may appear. For individuals and households, income tax is an elementary but somehow confused topic. A good example is the choice whether the spouses of a household should declare their taxes together or not. The essential basis for any tax planning activity is the existence of decision alternatives that are permitted by law [Rie78, p.27]. In this case, tax planning is both legal from a juridical point of view as well as necessary from an economic perspective.

In the academic literature, researchers have examined potentials for optimization in the field of tax planning since decades. For example, one of the topics that have been intensively studied is the analysis of tax depreciation strategies in order to minimize the present value of expected tax payments levied on future cash flows. Several publications have been published in many fields like finance, accounting and operations research confirming the important role of tax planning. In the following we give a review of some significant publications on the topic of taxation. References for the tax law contain material to the associated problems, or at least significant parts thereof.

3.1.1 German literature

The fundamental idea of income taxation is quite simple: For every monetary unit an individual or a company earns, it pays a percentage to a taxing agency. Briefly speaking, every person shall be assessable and chargeable in respect of any income arising. Germany, like most other countries in modern time, employs a progressive income tax system in which higher income earners pay a higher tax rate compared to their lower earning counterparts. Mathematically, this represents a non-linear feature of every model which has the tax function in it. Thus, it is principally more difficult to solve problems involving the tax function because efficient algorithms for linear programming cannot be applied. We will discuss the complexity of such models in later parts.

Back into the history, researchers in the German-speaking community have been formulating and solving multiple problems related to taxation. However, literature on topics of quantitative tax planning still remains quite manageable. The model constructed and analyzed in this thesis builds on the German income tax law[18]. A short overview of existing literature on quantitative tax planning can make the planning ideas more comprehensive. With the term “German literature”, we mean both publications that are written in German language, as well as publications that are composed in English but deal with specific problems that arise in the German tax system.

[18] Deutsches Einkommenssteuergesetz.

One of the most prominent study groups that can be identified is probably the "*Quantitative Tax Research*". Many publications and discussion papers in *Quantitative Tax Research* offer a thorough understand of types of problems that can arise in real-world business activities. Niemann [Nie06], Knirsch and Schanz [KS08] investigate optimal repatriation amounts by applying a business tax planning model. The results are based on heuristic approaches such as tabu search or scatter search.

The advantageousness of applying different tax options in various financial planning models is a subject of research for authors in the past. Decades ago, researchers already recognized the potential of utilizing possible options of tax law in a legal way [Rie78, p.27]. For example, Schanz examines the evaluation of decisions under consideration of taxation [Sch08]. Wacker pointed out that in tax-relevant decision systems of corporates, most decisions are made to minimize tax payments in a single period [Wac79, p.35]. There are models in which aspects of tax such as depreciation, exemptions or tax rates are analyzed. Recently, Schanz and Schanz investigate several business decisions, e.g., the choice of location, buying or leasing decisions, or the proper mix of debt and equity in the company's capital structure [SS10]. These studies suggest that quantitative tax planning remains an interesting research field to be explored. However, researchers also point out that due to the complex nature of the topic, identifying and quantifying possible tax effects may represent great challenges.

Another line of research construct model for tax planning from a multinational perspective. International taxation is very complex [Lüh09]. Therefore, the term "tax planning" in international context has many interpretations and extremely various application forms [Lüh09, p.3]. Schanz's model, for example, gives a general explanation and anatomy of the situation where the profits from foreign investments are carried back to the home country of a taxpayer. For more information on principles of multinational taxation see [DB06].

Our work contributes to a growing theoretical literature on the relationship between tax planning and investment decisions, and its implications for tax strategies on a business context. In our opinion, this is one of the pioneer models of its kind with a forward-looking perspective. It gives possibility to the tax payers and tax planners to get views on how heavily the tax payments affect the financial planning process. As an important feature, the decision maker can amalgamate different types of choice; here we consider the choices of tax options with choices of loss offset, in a systematic way. We will later highlight the details, advantages as well as drawbacks of such models and demonstrate how optimization approaches can be integrated in the framework we propose.

3.1.2 Non-German literature

Looking from a tax perspective, there are many strategies for companies and individuals to benefit from multinational economic activities [FMPR07]. For example, an interesting discussion on the taxation of taxation income in different economies is given by Gordon. [Gor92]. International tax planning matters are discussed by Sandmo [San76]. Pethig and Wagener analyzed income tax competition and equilibrium tax rates on an international basis [PW07]. In the work of Scholes et al. [SWEMS01], one can understand how taxation influences asset prices, equilibrium returns, and the form and content of contractual agreements. One of the oldest books about tax planning is maybe the work of Casey in 1966 called "Tax planning ideas" [Cas66]. In the field of corporate tax planning,

there are many sophisticated topics. Merger and acquisition, carry-over of loss of a company, taxation of dividends and profits are discussed in [Cho74]. Dividend distortions and repatriation strategies are discussed in [ARN95], [DFH01], or [AG03]. One other important research direction is concerned with the capital structure under taxation such as described in the works of DeAngelo and Masulis [DM80], Knoeber and Flath [KF80] or the publication of Modigliani and Miller [MM58] in an earlier stage.

Welsch and Anthony confirmed the fact that "tax-planning considerations frequently govern the decision" [WA74, p.12]. Therefore, big companies "can afford a full-rounded tax team" [CH78, p.20]. However, Chown and Humble have proved in their study that the tax strategies for many large and middle-size corporations have not been developed appropriately [CH78]. That may lead to unnecessary disadvantages and operative risks in a competitive global market. The taxation of investments and savings from an international view is studied by Gordon [Gor86]. In Europe, Janeba and Peters examined the interest taxation with cross-border strategies [JP99]. Mintz and Smart [MS04] pointed out that international tax planning may have positive effects on real investment that can offset the negative consequences of loss. Bucovetsky and Haufler [BH08] also considered a model of income shifting and investment by multinationals. On the special topic of tax havens see for example [HR94].

However, since the objective of our tax optimization model is not to find a universal economic generating mechanism, but to choose an optimal combination of decisions, the level of detail needed for this task is more granular and it is at level of business administration and below. Thus "details" become important when an investor is faced with the situation how to make a decision particularly in regard to which parameters could be affected and its co-dependence with other decision possibilities in the past and future. Due to the tremendous dimension of tax systems, there are logically a vast number of further aspects that needed to be considered in constructing relevant models. In general, it is very hard to construct a model that mirror exactly the reality.

Most of existing literature with important similarity to our model has a focus on tax depreciation. Optimal tax depreciation was discussed by Wakeman [Wak80]. In [WWK02], the authors examine the choice between the two most commonly used methods in practice, i.e., the straight line depreciation method and an accelerated depreciation method, such as the double declining balance method and the sum of the years–digits method. Berg, Waegenaere and Wielhouwer analyze optimal tax depreciation with uncertain future cash-flows [BWW01]. They focus on an obvious fact that taxable income consists of cash-flows reduced by depreciation charges. Because the choice of the depreciation method affects taxable income in future periods, they try to minimize the present value of future tax payments by choosing a particular depreciation method among those that are accepted by the tax authorities.

In most of such models, no loss is considered; there is no decision about possible loss carry forwards and backwards that can reduce taxable income of multiple periods. Only until recently, Kulp and Hartmann examine the straight-line and accelerated depreciation methods under a situation where net-operating losses may be carried forward and backward [KH11]. The question to be answered is again which depreciation method is better in which specific cases. Their model is of stochastic nature, since the cash flows in future periods are assumed with some certain probabilities.

The most significant difference of the models for tax depreciation from our model is that in our model we assume a linear depreciation method. So, the depreciation is always a fixed amount that can be precisely calculated and subtracted from the net operating income of every single period. Because the amounts are fixed and always stay constant, they do not affect the model structure and thus can be ignored. In a lot of publications in general financial planning, a common assumption is that all of the incomes and expenses, including deductions and depreciation are deterministic values. We also use this assumption in our coming quantitative discussion.

3.2 German income tax system

In the last section we have reviewed the most important research directions in tax planning. As a next step we turn our attention to a new topic. In this thesis we are concerned with decision possibilities that lead to saving on income taxes. Therefore, the current section is designed to cover the basic structure of income tax system in the Federal Republic of Germany that is essential for studying the Loss Offset Problem. We first present some basic terminologies and concepts at a glance.

3.2.1 Fundamental issues

German Tax System

Just like the various social security schemes in Germany, the national tax system is rather well-structured but complicated. It is a well-known fact that the tax system in Germany is very strictly regulated. The German economy, as a social market one, needs a well-developed taxation framework in order to function properly and to benefit society on a sustained basis. The Federal Ministry of Finance[19] is responsible for financial sector legislation. Their work includes creating the legal framework to ensure that the financial and labor markets can fulfill their function in the economy as a whole. The revenue gained from tax collection in Germany is used for government spending and can be broken down into the areas of welfare (almost 50 percent), existing debts (approx. 15 percent), defense (approx. 9 percent), transport (approx. 9 percent), federal employees (3 percent) and sciences (3 percent) [Source: Statistical yearbooks[20] for various years].

With so many different options for individual employees as well as complexities for corporations and businesses, the use of a tax advisor when filling out a tax return is often considered. This leads to a fact that tax consulting has become one of the most acknowledged professions in Germany. There are many types of taxes such as Vehicle Tax, Property Tax, Value-Added Tax, Dog Tax, Church Tax, and of course, Income Tax and Sale Tax. The two most important types of Income Tax are the Wage Tax (applied for employees) and the Corporate Tax (applied for companies).

[19] Bundesministerium der Finanzen.
[20] Statistische Jahrbücher.

Income Tax, besides Sale Tax[21], is probably the most important area of any tax system. In Germany, income tax makes the biggest share of the total tax revenue, which is the income that is gained by the government through taxation. In the following we see an illustration of tax revenue of different types of tax in the Quarters from 2008 to 2012. The source of is from annual reports of the Federal Statistical Office[22]. Detailed information about macroeconomic statistics can be found on the website www.destatis.de, *Wiesbaden: Finanzen und Steuern. Steuerhaushalt. 2012. Erschienen am 24. April 2013.*

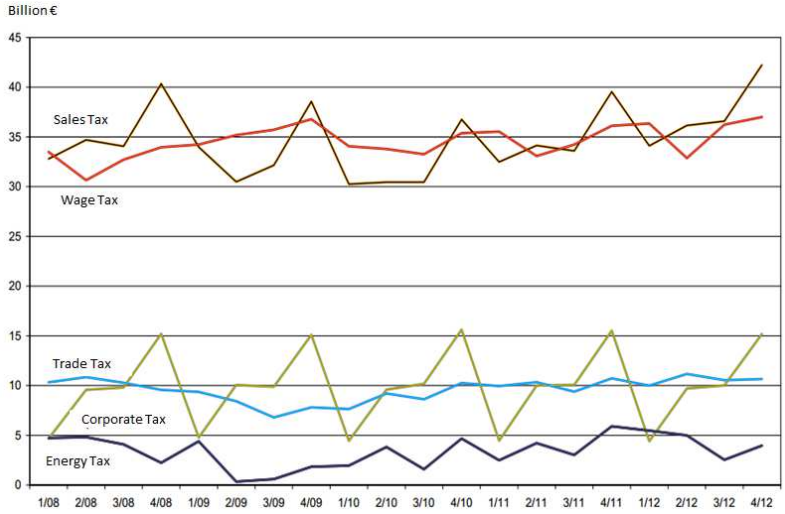


Figure 3-1 Tax Revenue and components in Germany from 2008 to 2012

In Figure 3-1 we recognize of the most important component of the income tax, namely the Wage Tax[23] that is levied on employees. Trade Tax[24] and Corporate Tax[25] make up a relatively small percentage of the total tax revenue.

German Income Tax System

Because we want to develop a model that deals with income tax, we will concentrate on this type of tax in following parts. Income tax is the payment which can be calculated based on the taxable income, whereas the taxable income is derived from the total income. Certainly at this point, some basic question should be raised: What is the taxable income and how is tax payment defined and quantified? We will explain those terminologies shortly after introducing the basic knowledge of the tax system.

The personal income tax is the centerpiece of any tax system and a key component of the welfare state. The German income tax law is detailed and complex. The fundamental principal of the German income tax law is based on individual performance capability. This principal means that the tax

[21] Umsatzsteuer.
[22] Statistisches Bundesamt.
[23] Lohnsteuer.
[24] Gewerbesteuer.
[25] Körperschaftsteuer.

burden is proportional to the economic situation of each individual [TK08, Chapter 4]. The paragraphs of Income tax law are defined in the EStG[26].

In total, the German income tax law alone already consists of 99 paragraphs. It is therefore impossible for us to present a thorough study but only to gain a short insight into income tax. Personal income tax is imposed on the income of individuals who are resident in Germany or have their normal place of abode there. For a German resident, all the income earned at home and abroad is subject to German tax. A foreign resident, who is employed in Germany, pays tax only on income earned in Germany. We now take a closer look at some definitions regarding income and income tax. The definition for incomes is very detailed and regulated in §§ 2 – 24b of the Income tax law.

Unrestricted tax liability is defined for individuals in §1 EStG as follows:

(1) 1 Natural individuals having their domicile or habitual residence in the country are liable to income tax. 2 For domestic purposes of this Act and of the Federal Republic of Germany's share of a part of the continental shelf, to the extent there exploring natural resources of the seabed and beneath the sea bed or be exploited or this energy is using renewable energies.

Whereas Section 8 of The Fiscal Code[27] of Germany gives a more precise regulation:

Section 8
Residence

Individuals shall be resident at the place at which they maintain a dwelling under circumstances from which it may be inferred that they will maintain and use such dwelling.

And Section 33 provides a definition of the "Taxpayer":

Section 33
Taxpayer

(1) Taxpayer shall mean any person who owes a tax, is liable for a tax, who is obliged to withhold and give to revenue authorities a tax which is due for account of a third party, to file a tax return, to provide collateral, to keep accounts and records or to discharge other obligations imposed by the tax laws.

(2) Taxpayer shall not mean a person who is obliged with regard to tax matters of a third person to provide information, to produce documents, to submit an expert opinion or to authorize entry to properties, business premises and offices.

3.2.2 Definition of income

Sources of income[28] (§2 and §13 to §22 EStG)

The basic of charge for the income tax is called the taxable income[29]. There are seven sources of income subject to income tax which are classified according to §2 and §13 to §22 of the German income tax law:

[26] Einkommenssteuergesetz.
[27] Abgabenordnung.
[28] Einkunftsarten.

• income from agriculture and forestry (1)

• income from trade and business operations (2)

• income from self-employed work (3)

• income from employed work (4)

• income from capital assets (5)

• income from rents and property (6)

• other income in the name of law §22 (7)

The tax on "income from employed work" (4) is identical with the Wage Tax we have just explained. Categories (2) and (3) (income from business and self-employment), on the other hand, are relevant for free-lancers who run a small business or trade[30] or who are self-employed artists, accountants, consultants, etc. Apart from this, if one has additional income from any other category, like interest payments on savings (5) or rental yields (6), it is also subject to taxes in Germany.

As mentioned above, an employee's Wage Tax may merely be a part of their overall income tax. One and the same taxpayer can have several sources of income such as income from business operations, from employed work and capital income. Such taxes are usually immediately transferred to the tax authorities when the earnings are paid out by employers or banks, so that similar taxes are never charged more than once on the same income for the same period. If one only has income from employed work in Germany, this is the easiest case. Usually, the employer simply withholds a certain percentage of the employee's gross salary and passes it on directly to the tax office. Under national German regulation, a liability for tax arises at the point at which the capital income accrues to the creditor. Because a tax will be deducted and paid to the tax office at the time of payment to the beneficiary, it is call a withholding tax. Under certain circumstances, no tax is withheld at source for residents (e.g., by granting an exemption or deduction order or submitting a non-assessment note).

Besides, there are textbooks that provide advance rulings on the treatment, for tax purposes, of precisely defined matters that have not been implemented yet. For information regarding tax deduction and exemption from German Income Tax, we refer to a publication of Kußmaul [Kuß10] which is a book that conveys the complexities of tax concepts and individual tax codes very clearly.

Because of the rising complexity when considering multiple sources of income, it is of great importance for us to make a restriction. In this thesis we do not examine all seven types of income, but differentiate between two major sources of income:

1. The first source of income is called ordinary income. This is the income that can be any type of income except for capital gains. It is taxed according to the German tax function[31] in §32a of the EStG.

[29] zu versteuerndes Einkommen.
[30] Gewerbebetrieb.
[31] Einkommensteuertarif.

2. The second source of income is called capital gains. Capital gains are taxed at a rate of 25%. The uniform tax rate is 25%. In our model we ignore the solidarity surcharge. We will explain the taxation of these two income types in the coming parts.

Tax brackets[32] (§ 38b EStG)

There are six different forms of tax brackets for the taxation of individuals in Germany according to §38b EStG):

- Single or separated workers with no children and no recent divorce/separation/widowing
- Single or separated workers with a child (additional child allowance may also be received)
- Married workers where only one partner is working (or widowed within the first year of wife's/husband's death)
- Married employees, both in jobs
- A mixture of the above two (such as married professional couple, one of whom has lost a spouse within the last year)
- Double income employees (with income received in more than one tax bracket)

For the sake of simplicity and clarity, we assume a single individual within the first tax bracket.

Tax due date[33]

In the reality, the tax declaration must be provided until 31st May of the year following the emergence of the tax. For our problem we assume that all transactions and payments happen at the end of each year, i.e., all the calculations are done in arrears.

Tax exempt income[34]

There are also several types of incomes that are exempt from personal income tax in Germany, for instance: annual total income of up to 8.354 EUR for a resident, capital income of up to 801 EUR, receipts from accident and health insurance, contributions by the employer to state pension plans.

Tax deductions and allowances[35] (§35a EStG)

The personal tax can be in German deducted. All forms of tax relief refer to the income of an individual. This applies in particular to costs immediately related to earnings, i.e., certain expenses of the taxpayer in excess of operating expenses and work-related expenses. For instance:

- donations
- child allowance
- church tax, which is fully deductible
- travel expenses to work and from work
- allowance for capital income
- profits on sales

[32] Steuerklassen.
[33] Abgabetermin für die Steuererklärung.
[34] Freibeträge.
[35] Steuerermäßigung.

In addition, there are so-called special expenses[36] which include

- certain types of insurance premiums
- expenses for additional retirement savings

Also, extraordinary financial burdens[37] can be deducted. In our models, we make the assumption that all the above mentioned deductions have been considered in our cash flow of ordinary income. The only type of deduction has affect the planning result is the amount of loss offset in each period.

The Dual Income Tax[38]

As in many other countries, Germany has also replaced the synthetic income tax system with a dual one by introducing the *Abgeltungssteuer* in 2009. The *Abgeltungssteuer* is a tax on savings interest and other capital gains that are made within the Federal Republic of Germany.

The dual income tax initially found its application in the Danish tax system. According to Sørensen [Sør94], Niels Christian Nielsen, an economist and member of a Danish committee on tax reform, was the first to advance the proposal of a dual income tax system [Nie78]. The paper considers the motives for the introduction of this new income tax system, ranging from abstract theoretical arguments to very pragmatic considerations. The dual income tax system levies a proportional tax rate on all net income (capital, wage and pension income less deductions) combined with progressive tax rates on gross labor and pension income. This implies that labor income is taxed at higher rates than capital income, and that the value of the tax allowances is independent of the income level. The dual income tax system deliberately moves away from the comprehensive income tax (global income tax) system which taxes all or most (cash) income less deductions (net income) according to the same rate schedule. The dual income tax was first implemented in the four Nordic countries (Denmark, Finland, Norway and Sweden) through a number of tax reforms from 1987 to 1993. It is therefore also known as the Nordic tax system or the Nordic Dual Income Tax. For more details on the system see [EG05] and [Boa04]. A contribution on its application for Germany was made by Spengel and Wiegard [EW04].

The result of the Dual Income Taxation is what we later refer to as the capital gains tax at the rate of 25%. When news about the *Abgeltungssteuer* first came to public attention, there was much scaremongering from banks or other institutions in their advertising that this *Abgeltungssteuer* is some kind of the next horrible regulation that the German authorities have invented to exploit income from the population. However, it actually brought German treatment of investment income into alignment with the way it has been treated in a lot of other places.

From one point of view, this tax results in an after-tax interest rate of only 75% of the offered interest rates offered to investors by banks, financial organizations or investment funds. From another point of view, such a tax may destroy the compounding effect over a long period of time, thus reducing the end value of one's savings by a huge amount.

[36] Sonderausgaben.
[37] Außergewöhnliche Belastungen.
[38] Duale Einkommensteuer.

In fact, it is an improvement if a person is in a higher personal income tax bracket than 25%. The reason is the following. Before the law for the *Abgeltungssteuer* is presented and became valid, any profit in investments or from savings got added to the personal income and taxed accordingly depending on how high this personal income is (max. income tax is 45% according to the tax function). Now applying this law, capital gains from savings and investments are taxed instantaneously at the source[39]at 25%. After that, those amounts of capital gains will not be added to other incomes to form the total amount of taxable income. This results in a lower taxable income and thus makes the tax payer suffer a lower tax rate as well as tax payment. Therefore, people whose total income is in high income tax brackets must now pay only 25% Abgeltungssteuer + *Solidarity Surcharge*[40] (+ *Church Tax*[41], if registered for this). It will then be at max. 28.5% in total all up and means a great amount of tax relief on investment profits in comparison with over 45% tax payment according to the tax function.

3.2.3 Income tax payment

3.2.3.1 Taxation options

Calculation of income tax payment

Income tax liability, of an entity for an income year, is the amount assessed as being the amount of income tax that the entity has to pay to the government. In general, businesses and individuals must submit an income tax return every year to declare whether they owe any taxes or are eligible for a tax refund.

In the reality there can be a transitional period, i.e., a company with a tax loss in the 2012-13 tax year that had a tax liability in the 2011-12 income year, can choose to claim the offset in the 2013 tax return. For simplicity, we assume that the tax payer operates on a calendar-year basis. A period is always a tax assessment period. We also use the word “tax payer” collectively for all relevant economic subjects such as individuals, households and firms.

The net income of each of the seven income sources is calculated separately (income after deduction of expenses) and then added together. From this total, tax allowances are deducted to arrive at the taxable income. Profits that do not fall within the seven income sources are tax free (e.g., lottery winnings, sale of privately owned capital assets except for the capital gains derived from transactions realized within a defined time frame). A basic personal allowance of the taxable income is not subject to taxation. Income tax rates vary from a minimum marginal tax rate of 14 % for income exceeding EUR 8.354, rising progressively to 42 % for a taxable income of EUR 52.882, to 45 % for income above EUR 250.731 for single individuals. To finance the reunification of Germany a surcharge called *Solidarity Surcharge* is levied from all taxpayers. The *solidarity surcharge* is currently 5.5 % of the relevant assessment basis, whereas the assessment basis is the income tax or corporation tax of all

[39] Quellensteuer.
[40] Solidaritätszuschlag.
[41] Kirchensteuer.

tax payers. For example, if an income tax 5000 € is calculated for a tax payer, he has to pay a solidarity surcharge of 275 € (5,5% von 5000 €).

Income from agriculture and forestry, trade and business and self-employment are calculated by deducting the operating expenditures from the operating revenues. Operating expenditures are expenditures arising from the company or self-employment. Income from employment, rent and leasing and other income are determined via the revenue from the respective type of income less all expenditure intended to acquire, secure and maintain the income (so-called work-related expenses). Income from capital assets is subject to special taxation. Note that dividends of corporations are also subject to withholding tax.

In what follows our attention will be restricted to two types of income in each assessment period: the ordinary income and the capital gains. Ordinary income can be characterized as income other than capital gain. By definition, ordinary income stands in contrast to capital gain which is defined as gain from the sale or exchange of a capital asset. A typical case where an income is not taxed as ordinary income is the case of interest payment. Ordinary income is taxed at the escalating marginal tax rates as defined in the basic income tax function. Capital gain is taxed at the constant marginal tax rates of 25%.

Basic income tax function[42] (§32a EStG)

In Germany is personal income tax progressively, i.e., higher income earners pay a higher tax rate compared to their lower earning counterparts. Such a progressive income tax promotes equal opportunities by reducing the inequality of after-tax income. From social point of view, this substantially contributes to social justice. In 2014 is tax rate from 0 % to 45 % (in 2007 was tax rate from 0 % to 42 %). The so-called solidarity surcharge at a rate of 5.5 % of income tax has to be paid on top of this. Tax base is different for single and married. Married couples may choose to be assessed either jointly or separately. However the case of married couples is not the subject of our study.

The progressive tax rates as defined in §32a EStG is presented as an excerpt in German language below.

(1) [1]Die tarifliche Einkommensteuer in den Veranlagungszeiträumen ab 2014 bemisst sich nach dem zu versteuernden Einkommen. [2]Sie beträgt vorbehaltlich der §§ 32b, 32d, 34, 34a, 34b und 34c jeweils in Euro für zu versteuernde Einkommen

1.

bis 8 354 Euro (Grundfreibetrag):

0;

2.

von 8 355 Euro bis 13 469 Euro:

(974,58 • y + 1 400) • y;

3.

von 13 470 Euro bis 52 881 Euro:

(228,74 • z + 2 397) • z + 971;

[42] Steuertarif.

4.	von 52 882 Euro bis 250 730 Euro: 0,42 • x – 8 239;
5.	von 250 731 Euro an: 0,45 • x – 15 761.

[3]„y“ ist ein Zehntausendstel des den Grundfreibetrag übersteigenden Teils des auf einen vollen Euro-Betrag abgerundeten zu versteuernden Einkommens. [4]„z“ ist ein Zehntausendstel des 13 469 Euro übersteigenden Teils des auf einen vollen Euro-Betrag abgerundeten zu versteuernden Einkommens. [5]„x“ ist das auf einen vollen Euro-Betrag abgerundete zu versteuernde Einkommen. [6]Der sich ergebende Steuerbetrag ist auf den nächsten vollen Euro-Betrag abzurunden.

These five segments result in a piecewise linear and continuous function which can be used to represent the tax payment as a function of taxable income. From now on we will refer to this as the *basic income tax function*. In the function, the value of 8.354 Euro is called the tax exempt income[43]. For taxable income lower than this value, no tax is levied. Figure 3-2 illustrates tax payment and the marginal tax rate as functions of the taxable income. The mathematical function is given in a later section of this chapter.

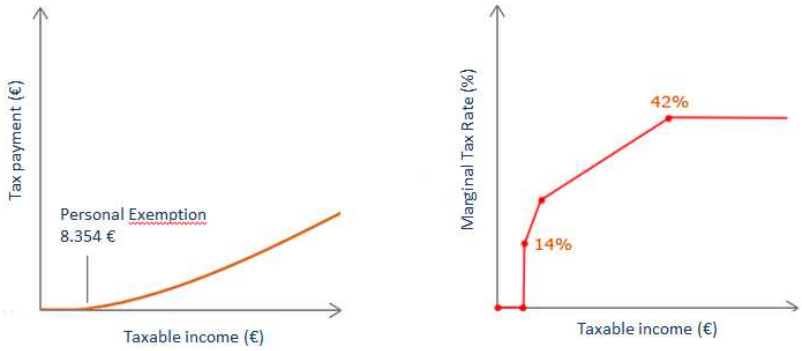


Figure 3-2 Illustration of the basic tax function in §32a EStG

Tax on capital gains[44] (§32d EStG)

The §32d EStG defines „Gesonderter Steuertarif für Einkünfte aus Kapitalvermögen“. As a result, capital gains are subject to a flat rate tax for income from capital assets invested after 1st Jan, 2009. According to the Federal Central Tax Office[45], capital gains tax is levied on capital income, which includes income from e.g., the following sources:

- all interest earned
- dividends
- capital gains
- investment yields

[43] Grundfreibetrag.
[44] Abgeltungssteuer.
[45] Bundeszentralamt für Steuern.

In many literatures about finance, capital gains and losses are realized from the sale or exchange of capital assets. Capital assets include properties, securities, bonds or cash account that an individual owns. In our work, we restrict to only capital gains that are generated from a cash account according to a given yearly interest rate. There is no default risk or volatility at all. This is intended to be representative of fairly common situations of individual taxpayers. For more details on assets and capital gains, see literatures on the topic of *"Portfolio Selection"* [Mar52] and other publication on ideas pioneered by Harry Markowitz. In our model we suppose that interest income is our only type of capital gains will be taxed at source. For a regular bank account, interest income continues to compound along as usual.

The tax rate for capital gains as defined in §32d EStG is presented as an excerpt in German language below. Due to the many specific terminologies of tax laws, we do not attempt to translate all the texts. The most important aspects of the laws that are relevant to our mathematical models will be emphasized in later parts in this chapter.

(1) [1]Die Einkommensteuer für Einkünfte aus Kapitalvermögen, die nicht unter § 20 Absatz 8 fallen, beträgt 25 Prozent. [2]Die Steuer nach Satz 1 vermindert sich um die nach Maßgabe des Absatzes 5 anrechenbaren ausländischen Steuern. [3]Im Fall der Kirchensteuerpflicht ermäßigt sich die Steuer nach den Sätzen 1 und 2 um 25 Prozent der auf die Kapitalerträge entfallenden Kirchensteuer. [4]Die Einkommensteuer beträgt damit

$$\frac{e - 4q}{4 + k}.$$

[5]Dabei sind „e" die nach den Vorschriften des § 20 ermittelten Einkünfte, „q" die nach Maßgabe des Absatzes 5 anrechenbare ausländische Steuer und „k" der für die Kirchensteuer erhebende Religionsgesellschaft (Religionsgemeinschaft) geltende Kirchensteuersatz.

Depreciation[46]

The depreciation is allowed in the form of a deduction against the taxpayer's income in the computation to arrive at the taxable income amount against which the tax rates are applied to compute the income tax under § 32a of the EStG. According to Germany's complex tax legislation, there are various sorts of tax breaks and tax deductions that apply to individuals and companies as well. For our model, we do not focus on depreciation methods. We assume that the straight-line is applied and determine our ordinary income before tax. The decisions to be made are therefore fully independent on depreciation strategies.

3.2.3.2 Treatment of gains and losses

Loss Offset[47] **(§10d EStG)**

Losses can occur for a number of reasons beyond the control of businesses and individuals. Since the success in earning income results in tax liability, it is logical to expect the government to give us a payment in return, a so called tax refund, when we have a loss resulted from business activities. As in many other countries, the German tax regulation provides planning opportunities for tax payers in

[46] Abschreibung für Abnutzung (AfA).
[47] Verlustabzüge.

case of losses, for instance when capital investments lose value. The net operating loss or tax loss can generally be used to recover past tax payments or reduce future tax payments. In this work, we do not differ between tax loss and operating loss. Below, we highlight the key issues that are necessary to know about the loss offset strategy, and discuss the interplay between the existing loss carry-forward rules and the loss carry-back ward rules.

Loss offsetting is a legal means of rearranging income of the tax payer. Loss carry-forward and -backward can be any kind of loss that leads to a negative total income in a year. The law § 10d EStG define the basic principle that in any period, the taxable income is calculated depending not only on negative and positive income of that period but also on the loss amounts of other periods. If there is no special prescription for a negative amount of a certain type of income, then this negative amount can be fully offset against the positive income in the same period. There are two types of balancing losses:

- Horizontal balancing of losses means offsetting losses and gains within the same type of income in different periods. For example, if we make a loss due to a bad investment in securities this year, we can only offset this loss next year if we make a profit from the same activity, i.e., with stocks, bonds or other derivatives.
- Vertical balancing of losses means offsetting losses and gains within different type of incomes in the same period. For example, if we make a loss due to a bad investment in securities this year, we can already offset this loss in the same year if we have positive income from other activities such as salary or income from rent and lease.

Obviously, there are a lot of further restrictions on the possibilities of balancing and offsetting losses. For example, a loss made by investment in a foreign country may be balanced against income from that same country only, regulated in § 2a EStG. A loss from trading with derivatives such as futures and options is also allowed to offset against profits from the same activities regulated in § 15 Section 4 EStG. Profits and losses realized by the sale of capital assets are regulated in § 23 Abs. 3 EStG. Furthermore, there are other restrictive conditions in the German Income Tax Law that are especially formulated in § 15a and § 15b EStG.

Loss carry-backward[48]:

Not every tax system allows for carrying back a loss to previous periods. However, industrial countries with an advanced financial system like Canada, France, the United Kingdom and the United States of America all currently have a loss carry backward mechanism. Especially in Australia, it is allowed to carry back for up to two years, that is, the regime allows a company to carry back losses to be offset against income tax liabilities in the two preceding income years by claiming a loss carry-backward in the income tax return.

In Germany, it is possible to claim up to EUR 1 Million as a refundable tax offset by carrying back losses to the preceding year. Instead of carrying forward the loss to deduct in later years, the taxpayer will get a refund. Logically, the taxpayer must have had taxable income and an income tax liability in the instantly preceding income year to be eligible to claim the loss carry-backward. Thus,

[48] Verlustrücktrag.

start-up companies or companies with ongoing periods of loss-making activity will not be eligible to claim the loss carry-backward as they will have no eligible prior year income tax liabilities. The mechanism to claim the loss carry backward is regulated in the German Tax Code. Originally thought of as a helping tool for small businesses in start-up phases, the loss carry backward is now, however, not limited to small business. Companies of all sizes can potentially access the loss carry back-ward offset. Remaining losses can be carried forward without limitations of time.

Loss carry-forward[49]:

A loss carry-forward is nothing more than a deduction of income that can be taken in a future year. In other words, income tax liabilities in future periods can be reduced by the amount of the loss in the current year. The logic behind this is to reduce tax liability of a year where the income is high if losses were experienced previously. According to current German income tax laws, losses can be accumulated if there are multiple years of loss consecutively, and can be carried forward for up to an indefinite number of years.

However, utilizing the losses of previous years by way of carry-forward is restricted by a concept called minimum taxation rule. Let us assume a sufficient loss carry-forward, i.e., the amount of loss carry-forward from $t-1$to t is greater than the total income in t. Then the rule can be interpreted as: Profits up to EUR 1m can be offset without restriction; above that amount only 60% of the exceeding profits can be offset. This means that in effect, 40 % of the profits exceeding EUR 1m will always be taxed regardless how high the existing loss carry-forward is. Losses not utilized are again continued to be carried forward to the next periods.

According to German Income Tax law, there are restrictions on the maximum amount of loss carry-forward and –backward to be recognized. The purpose of these restrictions is to ensure the minimal taxation principle[50]. In the following text of tax law §10d of the EStG, loss carry-backward is regulated in the first section, and loss carry-forward in the second section.

(1) [1]Negative Einkünfte, die bei der Ermittlung des Gesamtbetrags der Einkünfte nicht ausgeglichen werden, sind bis zu einem Betrag von 1 000 000 Euro, bei Ehegatten, die nach den §§ 26, 26b zusammenveranlagt werden, bis zu einem Betrag von 2 000 000 Euro vom Gesamtbetrag der Einkünfte des unmittelbar vorangegangenen Veranlagungszeitraums vorrangig vor Sonderausgaben, außergewöhnlichen Belastungen und sonstigen Abzugsbeträgen abzuziehen (Verlustrücktrag). [2]Dabei wird der Gesamtbetrag der Einkünfte des unmittelbar vorangegangenen Veranlagungszeitraums um die Begünstigungsbeträge nach § 34a Absatz 3 Satz 1 gemindert. [3]Ist für den unmittelbar vorangegangenen Veranlagungszeitraum bereits ein Steuerbescheid erlassen worden, so ist er insoweit zu ändern, als der Verlustrücktrag zu gewähren oder zu berichtigen ist. [4]Das gilt auch dann, wenn der Steuerbescheid unanfechtbar geworden ist; die Festsetzungsfrist endet insoweit nicht, bevor die Festsetzungsfrist für den Veranlagungszeitraum abgelaufen ist, in dem die negativen Einkünfte nicht ausgeglichen werden. [5]Auf Antrag des Steuerpflichtigen ist ganz oder teilweise von der Anwendung des Satzes 1 abzusehen. [6]Im Antrag ist die Höhe des Verlustrücktrags anzugeben.

(2) [1]Nicht ausgeglichene negative Einkünfte, die nicht nach Absatz 1 abgezogen worden sind, sind in den folgenden Veranlagungszeiträumen bis zu einem Gesamtbetrag der Einkünfte von 1 Million Euro unbeschränkt, darüber hinaus bis zu 60 Prozent des 1 Million Euro übersteigenden Gesamtbetrags der Einkünfte vorrangig vor Sonderausgaben, außergewöhnlichen Belastungen und sonstigen Abzugsbeträgen abzuziehen (Verlustvortrag). [2]Bei Ehegatten, die nach den §§ 26, 26b zusammenveranlagt werden, tritt an die Stelle des Betrags von 1 Million Euro ein Betrag von 2 Millionen Euro. [3]Der Abzug ist nur insoweit zulässig, als die Verluste nicht nach Absatz 1 abgezogen worden sind und in den vorangegangenen Veranlagungszeiträumen nicht nach Satz 1 und 2 abgezogen werden konnten.

[49] Verlustvortrag.
[50] Prinzip der Mindestbesteuerung.

(3) (weggefallen)

(4) [1]Der am Schluss eines Veranlagungszeitraums verbleibende Verlustvortrag ist gesondert festzustellen. [2]Verbleibender Verlustvortrag sind die bei der Ermittlung des Gesamtbetrags der Einkünfte nicht ausgeglichenen negativen Einkünfte, vermindert um die nach Absatz 1 abgezogenen und die nach Absatz 2 abziehbaren Beträge und vermehrt um den auf den Schluss des vorangegangenen Veranlagungszeitraums festgestellten verbleibenden Verlustvortrag. [3]Zuständig für die Feststellung ist das für die Besteuerung zuständige Finanzamt. [4]Bei der Feststellung des verbleibenden Verlustvortrags sind die Besteuerungsgrundlagen so zu berücksichtigen, wie sie den Steuerfestsetzungen des Veranlagungszeitraums, auf dessen Schluss der verbleibende Verlustvortrag festgestellt wird, und des Veranlagungszeitraums, in dem ein Verlustrücktrag vorgenommen werden kann, zu Grunde gelegt worden sind; § 171 Absatz 10, § 175 Absatz 1 Satz 1 Nummer 1 und § 351 Absatz 2 der Abgabenordnung sowie § 42 der Finanzgerichtsordnung gelten entsprechend. [5]Die Besteuerungsgrundlagen dürfen bei der Feststellung nur insoweit abweichend von Satz 4 berücksichtigt werden, wie die Aufhebung, Änderung oder Berichtigung der Steuerbescheide ausschließlich mangels Auswirkung auf die Höhe der festzusetzenden Steuer unterbleibt. [6]Die Feststellungsfrist endet nicht, bevor die Festsetzungsfrist für den Veranlagungszeitraum abgelaufen ist, auf dessen Schluss der verbleibende Verlustvortrag gesondert festzustellen ist; § 181 Absatz 5 der Abgabenordnung ist nur anzuwenden, wenn die zuständige Finanzbehörde die Feststellung des Verlustvortrags pflichtwidrig unterlassen hat.

Appeal procedure[51] (§10d EStG)

The practical approach for choosing the "joint" or "separate" taxation option is called *Günstigerprüfung*. We will highlight later the drawbacks and how this approach can be integrated in the framework we propose.

Usually, tax on income from employment is collected directly at source by the employer. When employment begins, the employee has to hand over to the employer a wage tax card[52] . This wage tax card includes the wage tax classification and personal allowances. Based on the taxable income of the employee, from each payment of remuneration the wage tax will be deducted.

If a taxpayer does not agree with an assessment or other specific decisions made by the tax authorities, he can appeal within a period of one month after receipt of the assessment or of the decision. The tax office that has made the original assessment or the first decision will deal with the first appeal. If no agreement between the taxpayer and tax office can be made, the appeal can be taken to a higher level, which is the Finance Court[53]. The decision of this court can again be appealed within one month at the Federal Finance Court[54].

A key simplifying assumption of our model is that the tax authority accepts the choice of tax options and loss offset without any complexity. In reality, one has to apply for a test and get approval from the tax office in order to choose which tax option to be applied on income. This is therefore an imperfect optimization environment. Thus, our planning techniques are described in a neutral and concise manner, without taking into account a specific jurisdiction, but based on the principles that underlie them.

[51] Günstigerprüfung
[52] Lohnsteuerkarte
[53] Finanzgericht
[54] Bundesfinanzhof

Making simplifying assumptions based on the issues discussed above, in our model we have two decision problems to be considered:

Decision problem Nr. 1: Which taxation option to choose?
As a result of these two sources of income, we have different tax rates while calculating tax payments on total income. In Germany, current tax law allows a choice between two taxation options. The possibility of selecting a taxation option is defined in **§ 32d EStG** subparagraph **6.**

Decision problem Nr. 2: To carry back, or carry forward?

Our presentation of the basic income tax structure here is not meant to be exhaustive. We only emphasize the most important aspects that are needed in our optimization models in later chapters. For those who are interested in the economic interpretations of tax functions and its related formulas, there is a great amount of excellent publications on income taxation in Germany that can be consulted for more detailed information, especially the many rules on deductions, exemptions, and so on.

3.3 The impact of tax on investment and financial planning

In this section we will go through the technical aspects of arriving at the mathematical model for the Loss Offset Problem. At the beginning we present some simple models with an intention of demonstrating the relevance of taxes in financial modeling and decision making in investment. We will see how the financial result can be improved by selecting given choices. Similar concepts can be found in [SS10], and our presentation is influenced and inspired by the authors also.

In order to understand the models in this section, there are no prerequisites apart from a willingness to adopt a pro-active stance towards the theory of taxation. Comprehension is helped by the use of a common notation throughout and detailed explanation of the formulas. We offer transparency by providing the availability of complete solutions in Excel and VBA forms.

Any tax planning task is in fact about implementing some strategies that can increase long-term, after-tax returns and incomes. Because this is a vast field of problems, a model that is concerned with tax planning can only represents some of the laws and regulations. It is nearly impossible to solve an optimization problem with constraints that fully mirror the real world. On the other side, problems can be formulated for different time frames. Some people concentrate on finding decisions that optimize the single-year-result, whereas others may be interested in long-term planning problems. In this work we emphasize the importance and necessity of making decisions in a multi-period context.

Our basic idea for presenting the next three models is the assessment of an investment proposal. We consider two cases: investment with and without the effects of taxation. For a given pre-tax net present value of an investment project or firm the impact of taxation is measured by the effect on the post-tax net present value. Obviously, there are different impact on the tax burden for different

scenarios regarding the assets invested in[55], as well as for different alternatives regarding the economic background such as profitability, real interest rate and inflation. This is a very complicated topic that is not intended to be handled in our work. Therefore, we make the simplifying assumption that we have only two types of income: the ordinary income, which we label as cash flow, and the capital gains. Tax payment will be imposed on these two kinds of income. Similarly, in our discussion of tax laws and in all calculation steps we assume that an individual entrepreneur is being considered.

Formulating the Loss Offset Problem in tax planning is a sophisticated task. Non-experts will have to confront with many comprehension questions, from fundamental to challenging ones. We mentioned in the introductory chapter that we have to handle a combinatorial optimization problem in which two arts of decision variables are integrated: (1) the tax options and (2) the loss carry-backward[56]. Thus, following questions might arise: Which are the tax options? How is a loss defined? And what kind of rules is applied for offsetting, i.e., carrying back and forward a loss?

Due to the necessity of clarity and definiteness, in the coming parts we carefully approach the problem in a systematic and detailed way. We try to avoid overly complicated explanations whilst providing a theoretical background on financial planning with taxation that leads to the formulation of the Loss Offset Problem. Based on the input data given, we will address the following problems:

- Calculate the present values of the project under taxation and without taxation.
- Make necessary adjustments for the decision if the project should be carried on.
- Derive conclusions that may be helpful for other financial decisions under taxation.

Let us have a preliminary remark on cash forecasting:

As mentioned by Ashford et al. [ABD88], a substantial portion of the time spent by financial managers was identified as being spent on planning and budgeting. These processes cover a variety of activities whereas the most important ones are the monitoring of cash balance and prediction of future cash flows. Furthermore, these activities are described as capital management that mostly belongs to the tasks of accountants rather than the decision makers with knowledge on Operations Research. For such reasons, in our work we assume to get information on cash inflows as deterministic values and do not have to consider any probabilistic feature in building models.

[55] For instance intangibles, industrial buildings, machinery, financial assets and inventories
[56] The loss carry-forward is not seen as a variable on our models.

3.3.1 Planning models with no decision variables

3.3.1.1 Assessment on a Present Value basis

Model 3-1 Assessment of an Investment Project without the effects of taxes

In the discussions to follow, we assume a fixed interest rate. Also, it is not our aim to construct models for randomness.

Given an interest rate of a risk-free asset, e.g., interest rate for money put in a bank account, that can be used as the discount rate i, we can assess if an investment should be made or not. Assume that i stays constant during our planning horizon. Consider an investment project over a time horizon T with initial investment cost I_0 that is expected to generate income represented by cash flows CF_t for $t = 1,2, \dots T$ in the future. Assume that the cash flows are deterministic and known in advance. For simplicity, also assume that all CF_t's are positive, i.e., there is no loss[57].

In order to get the correct answer for this decision problem, we need to mention a basic but important concept in financial calculation: the "time value of money" concept.

Time value of money:

The Time Value of Money concept has applications in many areas of finance including capital budgeting, bond valuation, and stock valuation. It states that a monetary unit we receive today is worth more than the same unit to be received in the future. This is due to the fact that any amount of money today can be invested at a certain interest rate to yield more than that money in the future[58]. For example, assuming an 8% interest rate, 100 EUR invested today will be worth 108 EUR in one year (100 multiplied by 1.08). Conversely, 100 EUR received one year from now is only worth 92.59 EUR today (100 divided by 1.08), assuming an 8% interest rate. Because of this universal fact, we would prefer to receive money today rather than the same amount in the future. The assumption is that the investments considered are all of equal risk, i.e., the discount rate is the same.

In order to assess an investment project, we must quantify the value of all cash flows that project creates through time. This depends upon the interest rate which can be earned on the investment. The value of the project today, thus, depends upon what future cash flows are worth in today's cash amount. According to The Time Value of Money concept we have two terminologies: Future Value and Present Value. Future Value describes the process of finding what an investment today will grow to in the future. Present Value describes the process of determining what a cash flow to be received in the future is worth in today's cash. The Present Value of an investment project is equal to the sum of the Present Values of the individual cash flows generated by that investment. For a thorough, actual discussion on the computation of the present value of future cash flows see for example [FSW13].

Back to our model, the Present Value of the project is the difference between the initial cost and the sum of all discounted cash flows and is defined by the following equation

[57] The treatment of losses is one of the two features the Loss Offset Problem considers and will be discussed in later models.

[58] It is in most cases obvious that money deposited in a savings account will earn interest.

$$PV = -I_0 + \sum_{t=1}^{T} \frac{CF_t}{(1+i)^t} \tag{3-1}$$

If the calculated value of PV is positive, the project is worth invested in. Conversely, if PV is zero or negative, there is no reason to carry on the investment project because the investor can simply put the money in a bank account for receiving a profit instead of invest in the project to finally receive a loss. We will illustrate this concept in Example 3-1.

Example 3-1

Suppose we apply following data for the above model:

$T = 3, I_0 = 30.000, CF_1 = CF_2 = CF_3 = 12.050, i = 10\%$

We expect to receive annual cash inflow of 12.050 in the next three years. The total amount of those cash flows is 36.150 and is, at the first sight, higher than the initial investment cost of 30.000. However, that is not the basis of our decision-making process.

In order to make a decision if the project should be carried out, we calculate the Present Value according to Equation (3-1)

$$PV = -30.000 + \sum_{t=1}^{3} \frac{12.050}{(1+0{,}1)^t}$$

$$PV = -33$$

We see that because the calculated Present Value is -33 which is smaller than zero, the investment should not be made.

The financial plan is presented in Figure 3-3.

1						
2	Discount Rate	10%				
3						
4	t		**0**	**1**	**2**	**3**
5	**CF_t** (cash flow before depreciation)		**-30.000**	**12.050**	**12.050**	**12.050**
6	Discounted Cash Flows			10.955	9.959	9.053
7	Sum of Discounted Cash Flows		29.967			
8	Present Value of the Project		-33			

Figure 3-3 Financial plan of Example 3-1

Our simple example above is called a decidable decision problem because it can be solved by an algorithm. Here, the mathematical formulation of this problem concerns no optimization. Although very simple, we formulate the decision procedure as follows.

Algorithm 3-1

Step 1: Calculate the Present Value of each future cash flow using the discount rate.

Step 2: Calculate the Present Value of the project by adding up all Present Values of the future cash flows.

Step 3: Calculate the difference between the value obtained in Step 2 and the given I_0. If the difference is greater than zero, then the answer produced is “yes”, otherwise it is “no”.

Now we switch to the modified model that takes the effects of taxation into account.

For the purpose of our coming models, the following definitions apply:

(a) "Tax" means income tax, business tax, corporation tax and, where applicable, capital gains tax, as well as withholding tax of a nature equivalent to any of these taxes;

(b) "Cash flows" mean all sorts of income which are defined as such under the German income tax law.

Model 3-2 Assessment of an Investment Project under consideration of taxes

Again, given an interest rate of a risk-free asset that can be used as the discount rate i, we now furthermore assume an average tax rate s that is imposed on all profits made by the investor. Assume that i and s stay constant during our whole planning horizon. We will assess if the investment should be made or not using i and s.

Consider an investment project over a time horizon T with initial investment cost I_0 that is expected to generate cash flows CF_t for $t = 1,2, \dots T$ in the future. Assume that the cash flows are deterministic and known in advance.

What kind of impact does the consideration of taxes generate on this model?

First, in each period there is now a tax payment S_t that is calculated as

$$S_t = s \cdot (CF_t - D_t) \tag{3-2}$$

where s is the average tax rate. In addition, D_t is the depreciation amount of period t. Depreciation has been discussed and explained in previous sections of this chapter. If we apply the linear depreciation rule for the calculation, then

$$D_t = \frac{I_0}{t} \tag{3-3}$$

Note that, in this case the parenthesized term $(\mathrm{CF_t} - \mathrm{D_t})$ in (3-2) can theoretically be referred to as the taxable income that is relevant for our later models.

Second, for the calculation of values we now have to apply the discount rate (or risk-free interest rate) ***after tax*** that is

$$i_s = i \cdot (1 - s) \tag{3-4}$$

The Present Value ***after tax*** of the project is now the difference between the initial cost and the sum of all discounted cash flows ***after tax***. It is defined by the following equation:

$$PV_{tax} = -I_0 + \sum_{t=1}^{T} \frac{CF_t - S_t}{(1 + i_s)^t} \tag{3-5}$$

Analog to Model 3-1, we now assess the project proposal based on the **PV** after tax. Again, if the calculated value of **PV** is positive, the project is worth invested in. Conversely, if **PV** is zero or negative, the investment will not be made. We will illustrate this calculation after tax in Example 3-2.

In order to make comparison to Model 3-1 possible, we re-use the same input data from Example 3-1.

Example 3-2

We apply previous data:

$T = 3, I_0 = 30.000, CF_1 = CF_2 = CF_3 = 12.050, i = 10\%$

In addition, let $s = 50\%$. The discount rate ***after tax*** is then

$$i_s = 5\%$$

Let us calculate the PV_{tax} of the investment by first defining D_t according to (3-3).

Logically, for all t we have

$$D_t = 10.000$$

The tax payments in each period are calculated by substituting (3-3) into (3-2)

$$S_1 = S_2 = S_3 = 0{,}5 \cdot (12.050 - 10.000) = 1.025$$

With the above values, we calculate the Present Value ***after tax*** according to Equation (3-5)

$$PV_{tax} = -30.000 + \sum_{t=1}^{3} \frac{12.050 - 1.025}{(1 + 0{,}05)^t}$$

$$PV_{tax} = 24$$

Taking the effects of taxes into account, we now obtain the Present Value after tax of the project $PV_{tax} = 24$ that is a positive value. This result suggests that the investment is lucrative and should be carried on. The financial plan for this case is shown in Figure 3-4.

	A	B	C	D	E	F
1						
2	Tax Rate	50%				
3	Discount Rate	10%				
4	Discount Rate after Tax	5%				
5						
6	t		0	1	2	3
7	**CF_t** (cash flow before depreciation)		-30.000	12.050	12.050	12.050
8	**D_t** (Depreciation)			10.000	10.000	10.000
9	Tax Payment			1.025	1.025	1.025
10	Discounted Cash Flows after Tax			10.500	10.000	9.524
11	Sum of Discounted Cash Flows after Tax		30.024			
12	Present Value of the Project		24			

Figure 3-4 Financial plan of Example 3-2

Though using the same data for the two examples, the results are contradictory: the Present Value of the first example is negative and implies that the investment project is a bad undertaking, whereas the result of second example shows exactly the opposite. This fact is called the tax paradox that has been mentioned and studied by some authors in the past, see [Göt06, pg.134] or [LES05, pg.371].

Comparing the result of Example 3-2 to that of Example 3-1, we recognize that tax effects might distort decision on business planning. A detailed discussion on how taxes distort business decisions can be found in [LES05, pg.386] or [BL95, pg.120]. It is obvious that taxes reduce incomes and profits. However, we have also recognized that complicated rules in the tax laws, for instance the depreciation assessment of taxation basis, offer great opportunities for decision makers to improve business result.

For our optimization models concerning the loss offset and tax options in later parts we consider not the present value of the cash flows, but the future value of final wealth. Therefore, in a next step we transform Model 3-2 into a model that calculates the accumulated amounts of periodical wealth. How the present value of project should be interpreted is discussed in [NBGHS09, pg.80], whereas the meaning of Present Value and Future Value are described in [Sei01, pg.81] .

Note that the model purely represents the concept of calculating incomes and wealth without any decision possibility. Thus, it is neither a decision problem nor an optimization problem.

3.3.1.2 Assessment on a Future Value basis

Model 3-3 Calculating Final Wealth of an Investment Project under consideration of taxes

In this deterministic taxation planning model, taxpayers have ordinary income and capital gains each year. Denote by A_t the ordinary income of the taxpayer in each year. It is assumed that the taxpayer knows A_t ex ante, i.e., at the beginning of the first period. In addition, we define B_t as the capital gains for each year. The amount of capital gains in a period t is calculated based on the wealth cumulated in the previous year W_{t-1} and a given risk-free interest rate i. We calculate the final wealth.

Given:

T: time horizon of the project in years.

i: the annual interest rate at which money can be reinvested.

s: average tax rate that is imposed on all arts of incomes.

CF_t: cash flow before depreciation received in t for $t = 1,2, \dots T$.

D_t: depreciation amount in t for $t = 1,2, \dots T$.

W_0: initial wealth in $t = 0$

Variable:

There is no decision variable.

Question:

Calculate the final wealth W_T of the project?

Solution:

For $t = 1,2, \dots T$, define

A_t as the cash flow after depreciation. This quantity is the first part of the assessment basis for calculating the tax payment. We simply have

$$A_t = CF_t - D_t \tag{3-6}$$

B_t as the capital gains. This quantity is the second part of the assessment basis for calculating the tax payment. The value of capital gains in one period is calculated depending on the wealth of the previous period.

$$B_t = i \cdot W_{t-1} \tag{3-7}$$

where W_t is the accumulated wealth in t according to the following equation

$$W_t = W_{t-1} + CF_t + B_t - S_t \tag{3-8}$$

That is, the wealth in a current period is the sum of:

- wealth in the previous period and
- cash flow before depreciation in current period and
- capital gains of current period (which itself depends on wealth in the previous period)

subtracted by tax payment S_t of the current period with

$$S_t = f(A_t + B_t) \tag{3-9}$$

Note that, in this case the parenthesized term $(A_t + B_t)$ in (3-9) can theoretically be referred to as the taxable income.

For simplicity in this model, we have assumed a common average tax rate s for both A_t and B_t. Therefore, equation (3-9) can be rewritten as

$$S_t = s \cdot (A_t + B_t) \tag{3-10}$$

Note in this case the similarity of equation (3-10) to equation (3-2). The only difference is that in this model we principally distinguish cash flow from capital gains. However, here we still apply a common tax rate for both types of income.

Now define I_t as the taxable income which in this case is the total amount of cash flow and capital gains:

$$I_t = A_t + B_t \tag{3-11}$$

Substituting equation (3-11) into (3-10) we have

$$S_t = s \cdot I_t \tag{3-12}$$

We have constructed the mathematical framework that takes taxation into account. The following example illustrates how the above equations can be used to calculate and assess final wealth of an investment project. We also provide the financial plan that helps improving comprehensibility.

Example 3-3

Using exactly the same data from previous Example 3-2:

$T = 3, I_0 = 30.000, CF_1 = CF_2 = CF_3 = 12.050, i = 10\%$

Assume an linear depreciation rule and let average tax rate $s = 50\%$ as well as initial wealth $W_0 = 0$.

We now calculate step by step the final wealth as follows:

In period 1:

Cash Flow: $A_1 = CF_1 - D_1 = 2.050$

Capital gains: $B_1 = i \cdot W_0 = 0$

Tax payment: $S_1 = s \cdot (A_1 + B_1) = 50\% \cdot 2.050 = 1.025$

Wealth: $W_1 = W_0 + CF_1 + B_1 - S_1 = 11.025$

For Period 2 and Period 3, we repeat the same calculation steps and get

$A_2 = 2.050, B_2 = 1.103, S_2 = 1.576, W_2 = 22.601$

$A_3 = 2.050, B_3 = 2.260, S_2 = 2.155$, and our final wealth is

$W_3 = W_2 + CF_3 + B_3 - S_3 = 34.756$

It may be interesting for us to compare this value to that of the situation in which the initial investment cost of 30.000 Euro is not invested into the project but invested, into let say some money market accounts, at the risk-free interest rate after tax $i_s = 5\%$. In the latter case, final wealth in Period 3 is simply computed according to the compound interest rule:

$$W_3' = 30.000 \cdot (1 + i_s)^3 = 34.729$$

Comparing the two values, we clearly see that W_3' is smaller than W_3. Thus, even under taxation the investment project is profitable, i.e., creates a higher final wealth than the risk-free investment. This result is consistent with the result generated by Model 3-2. Figure 3-5 illustrates the calculation steps and final result in spreadsheet form.

	A	B	C	D	E	F
1						
2	Average Tax Rate	50%				
3	Interest Rate	10%				
4	Interest Rate after Tax	5%				
5						
6	t		0	1	2	3
7	**CF_t** (cash flow before depreciation)		-30.000	12.050	12.050	12.050
8	**D_t** (Depreciation)			10.000	10.000	10.000
9	**A_t** (Cash Flow after Depreciation)			2.050	2.050	2.050
10	**B_t** (Capital gains)			0	1.103	2.260
11	**S_t** (Tax Payment)			1.025	1.576	2.155
12	**W_t** (Wealth of Period t)		0	11.025	22.601	34.756
13						
14	Wealth generated by risk-free investment					34.729

Figure 3-5 Financial plan of Example 3-3

In the case of the taxable economic activities, we have seen that it is a generally well-known fact that taxes can heavily affect economic and financial decision processes as confirmed in [WD80] or [Wag05]. We presented some examples to illustrate this fact. As a result, we have shown that the

consideration of just very basic tax law elements in an investment process distorts the decision if the investment should be undertaken or not. Our goal was to understand why sometimes it is better to invest under taxation.

In what follows we extend the model to deal with two possible taxation options. In order for any comparison of models to be meaningful, the objectives of the models must first be stipulated. In our work, we consider a single objective of maximizing final wealth.

3.3.2 Planning models with decision variables

3.3.2.1 Taxation options

Model 3-4 Planning with decision on taxation options

We recall a point that has been discussed in the last chapter. Prior to the validity of the law of a flat rate tax on capital gains, in Germany interest income was taxed at the personal marginal income tax rate which could become extremely high. Thus, the 25% tax rate of the *Abgeltungsteuer* is actually an improvement. For individuals who are below the personal income tax bracket of 25% there might be an advantage to choose the joint taxation. However, the final decision must be made by carefully quantifying the two choices. In some case, investors are taxed on the investment profits in the same way as how high their tax rate for ordinary income is. It is therefore tempting to utilize the advantage from this flat rate tax.

As mentioned before, the tax optimization model proposed in this work differs from other models in the combination of two primary choices allowed by German tax law. From quantitative point of view, we would like to answer the question how to justify a decision on joint or separate taxation. In other words, we would like to know which tax option to apply in which cases for a better final result (final wealth). For each period t, we let y_t represent the choice of which tax option to apply, see (3-15).

Given:

T: Time horizon of the project in years.

i : Interest rate for the calculation of capital gains on an annual basis

A_t : Ordinary income in period t

W_0: Initial wealth in $t = 0$

Variable:

y_t : Taxation option which is applied for period t (see below)

Question:

Maximize the final wealth

$$W_T = W_0 + \sum_{t=1}^{T}(A_t + B_t) - \sum_{t=1}^{T} S_T \tag{3-13}$$

Solution:

In order to calculate the final wealth which is to be maximized, we need to determine the values of a bunch of variables and parameters. Those are presented in the following.

B_t : Capital gains in period t as defined in (3-7)

$$B_t = i \cdot W_{t-1} \tag{3-14}$$

y_t : Taxation option which is applied for period t

$$y_t = \begin{cases} 0 & \text{if ordinary income and capital gains are added and taxed together} \\ 1 & \text{if ordinary income and capital gains are taxed seperatly} \end{cases} \tag{3-15}$$

In each period the taxable income has to be calculated depending ordinary income A_t and capital gains B_t. Basically, this is the issue of whether to calculate the taxable income as sum of A_t and B_t or not.

I_t: Taxable income in period t

$$I_t = \begin{cases} A_t + B_t & if\ y_t = 0 \\ A_t & if\ y_t = 1 \end{cases} \tag{3-16}$$

$f(.)$: Tax function formulated in the German tax law (EStG §32)

$$f(I) = \begin{cases} 0 & \text{if } I \leq 8.354 \\ (974{,}58 \cdot a + 1.400) \cdot a & \text{if } 8.355 \leq I \leq 13.469 \\ (228{,}74 \cdot b + 2.397) \cdot b + 971 & \text{if } 13.470 \leq I \leq 52.881 \\ 0{,}42 \cdot I - 8.239 & \text{if } 52.882 \leq I \leq 250.730 \\ 0{,}45 \cdot I - 15.761 & \text{if } 250.731 \leq I \end{cases} \tag{3-17}$$

with

$$a = \frac{(I - 8.354)}{10.000} \tag{3-18}$$

and

$$b = \frac{(I - 13.469)}{10.000} \tag{3-19}$$

a and b are auxiliary parameters for the second and third domains of the tax function.

$g(.)$: Tax function for capital gains using 25% flat rate. Note that this function could be only applied for capital gains and in no circumstances for ordinary income. Therefore we have

$$g(B_t) = 0{,}25 \cdot B_t \tag{3-20}$$

Now, we denote the total tax payment in period t by S_t. The total tax payment in a period depends on:

- the amount of A_t
- the amount of B_t
- possibly the amount of deduction from losses in other periods, if there is any
- the taxation option to be chosen for period t

In a reduced-form, we formulate equation for calculating the total tax payment as

$$S_t = \begin{cases} f(I_t) & if\ y_t = 0 \\ f(I_t) + g(B_t) & if\ y_t = 1 \end{cases} \tag{3-21}$$

Substituting (3-16) and (3-20) into (3-21) we obtain the equation that is more intuitive

$$S_t = \begin{cases} f(A_t + B_t) & if\ y_t = 0 \\ f(A_t) + 0{,}25 \cdot B_t & if\ y_t = 1 \end{cases} \tag{3-22}$$

W_t: Wealth in period t

$$W_t = W_{t-1} + A_t + B_t - S_t \tag{3-23}$$

Finally, the objective function is wealth in T that is formulated in (3-13) or also

$$W_T = W_{T-1} + A_T + B_T - S_T \tag{3-24}$$

For the sake of clarity in future mathematical formulation, let us call AB_t the total income defined by

$$AB_t = A_t + B_t \tag{3-25}$$

Note that in this model we do not have any loss, i.e., all the cash flows of ordinary income and capital gains are positive. In this context, this model do not represents an optimization potential. Decision on taxation options can be made by simple comparison of the corresponding values. We illustrate this with the next numerical example.

Example 3-4

In this example we explore the advantageousness of the choice between the two available taxation options described above using the following data.

Given:

Number of time periods $T = 3$,

Initial wealth $W_0 = 30.000$,

Amounts of ordinary income $A_1 = 20.000$, $A_2 = 15.000$, $A_3 = 10.000$

Annual interest rate $i = 10\%$

Variables:

Taxation options in three periods y_1, y_2, y_3.

Question:

Which taxation option should be chosen in each period in order to maximize final wealth?

Solution:

Let us begin with Period 1. Recall that the total tax payment now depends on the choice of taxation option. First we calculate the capital gains

$B_1 = i \cdot W_0 = 10\% \cdot 30.000 = 3.000$, see (3-14)

Then we determine the remaining parameters according to the variable y_1.

Case 1: Choose joint taxation, i.e., the taxable income contains both ordinary income and capital gains. Thus:

$y_1 = 0$, see (3-15)

$I_1 = 20.000 + 3.000 = 23.000$, see (3-16)

$S_1 = f(I_1) = f(23.000) = 3.463$, see (3-17) and (3-21)

$W_1 = W_0 + A_1 + B_1 - S_1 = 30.000 + 20.000 + 3.000 - 3.463 = 49.537$, see (3-23)

Case 2: Choose separate taxation, thus i.e., the taxable income contains only ordinary income. Thus:

$y_1 = 1$, see (3-15)

$I_1 = 20.000$, see (3-16)

$S_1 = f(I_1) + g(B_1) = f(20.000) + g(3.000) = 2.634 + 750 = 3.384$, see (3-17) and (3-21)

$W_1 = W_0 + A_1 + B_1 - S_1 = 30.000 + 20.000 + 3.000 - 3.384 = 49.616$, see (3-23)

Making a brief comparison between the tax payments of two cases, we choose $y_1 = 1$ because in this case the tax payment is lower, and thus results into a higher wealth of the period. Applying the same procedure to period 2 and 3, we can find the optimal values of y_t ´s in all three period such that the final wealth is maximized. Note that the "*optimality*" is simply acquired by comparing the two possibilities in each period and there are three comparisons in total.

For a better graphical illustration, we create two financial plans in spread sheet form. The first one in displayed in Figure 3-6 and shows a supobtimal solution. In this case, we choose $y_1 = 1, y_2 = 0$, $y_3 = 1$. This combination of taxation options leads to a value W_t of 81.719.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rate)	0,1				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**20.000**	**15.000**	**10.000**
7	**B_t** (capital yield)		0	3.000	4.962	6.696
8	**AB_t** (total income)		0	23.000	19.962	16.696
9	**y_t** (taxation variant: 0 or 1)			***1***	***0***	***1***
10	**I_t** (total taxable income)		0	20.000	19.962	10.000
11	**S_t (total tax payment)**		0	3.384	2.624	1.931
12	**W_t** (end of period wealth)		**30.000**	49.616	66.954	81.719

Figure 3-6 Choices of taxation options – Suboptimal Solution

In the figure, the decision variables can be seen on line 9. The optimal solution, i.e., the set of choices of variables y_t that maximizes final wealth, is shown in Figure 3-7. For $y_1 = 1, y_2 = 1, y_3 = 0$, the maximum value W_t of 81.925 is obtained.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rate)	0,1				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**20.000**	**15.000**	**10.000**
7	**B_t** (capital yield)		0	3.000	4.962	6.700
8	**AB_t** (total income)		0	23.000	19.962	16.700
9	**y_t** (taxation variant: 0 or 1)			***1***	***1***	***0***
10	**I_t** (total taxable income)		0	20.000	15.000	16.700
11	**S_t (total tax payment)**		0	3.384	2.584	1.769
12	**W_t** (end of period wealth)		**30.000**	49.616	66.994	81.925

Figure 3-7 Choices of taxation options – Optimal Solution

The interpretation of these results is quite straightforward if we take a look at the tax rate that is applied on the taxable income in each case. We explain this in the following:

In the first period, the marginal tax rate which is applied on $A_t = 20.000$ alone is 26,96 %, whereas the marginal tax rate applied on B_t is 25 % no matter how high B_t is. This means that if we add B_t to A_t, the marginal tax rate according to §32a that is applied on $A_t + B_t = 23.000$ will be 28,33 %, definitely higher than 26,96 % and of course also higher than 25 %. This leads to a worse result. Therefore we choose $y_t = 1$.

In the second period, the marginal tax rate which is applied on $A_t = 15.000$ alone is 24,67 %, whereas the marginal tax rate applied on B_t is 25 % no matter how high B_t is. This means that if we add B_t to A_t, the marginal tax rate according to §32a that is applied on $A_t + B_t = 19.962$ will be 26,94 % that is definitely higher than 24,67 % and of course also higher than 25 %. This again leads to a worse result. Therefore we choose $y_t = 1$.

In the third period, the marginal tax rate which is applied on $A_t = 10.000$ alone is 17,21 %, whereas the marginal tax rate applied on B_t is 25 % no matter how high B_t is. This means that if we add B_t to A_t, the marginal tax rate according to §32a that is applied on $A_t + B_t = 16.700$ will be 25,45 %. Here we see the difference of this result to those of Period 1 and Period 2. The marginal tax rate now is 25,45 % that is still higher than 17,21 % and also higher than the flat rate tax of 25 %. This joint taxation option, however, leads to a better result, i.e., a lower total tax payment, than the separate taxation option. Therefore we choose $y_t = 0$.

The explanation of how we came to these values of marginal tax rate and how we choose which taxation option will be presented in section 3.x "Discussion of the complexity of models". Then we will briefly come back to this model and analyze it in greater detail.

Table 3-1 shows all possibility of taxation options. Because we have 3 periods, each with 2 possible choices on taxation options, there are $2^3 = 8$ combinations of these options.

y_1	y_2	y_3	W_3
0	***0***	***0***	81.790
0	***0***	***1***	81.627
0	***1***	***0***	81.833
0	***1***	***1***	81.670
1	***0***	***0***	81.882
1	***0***	***1***	81.719
1	***1***	***0***	81.925
1	***1***	***1***	81.762

Table 3-1 Enumeration of all possible combinations of taxation options for Example 3-4

With the above models and examples, we see how the implementation of tax into the planning model affects decisions of business entities, i.e., enterprises or household. Analogically, taking into account tax laws and regulations in a basic level leads to contrary decisions compared to the case where the laws, especially the rights for different choices, are analyzed in detail.

Up to this point, we have discussed several scenarios under the assumption that all of the income $\mathrm{A_t}$ and capital gains $\mathrm{B_t}$ are positive for $\mathrm{t} = 1$ to T. Let us now turn to a more challenging situation that results into the Loss Offset Problem.

At various times during recessions or depressions of the economy, income can fall dramatically. In such cases, planning can be done to postpone or even totally avoid capital gains taxation. This is related to the choices of taxation options that we have introduced in the last section. Another case of decision making is when a loss occurs/ several losses occur. In such cases, regulations for loss offset afford individuals and firms an opportunity to make use of the loss in one period to reduce tax payments in other periods. Unused losses can be carried forward.

3.3.2.2 Loss carry-backward and loss carry-forward

To understand the implications of the later models, it is useful to consider the next simplified case. First, observe that, if there were no loss period, then the optimal decision on taxation option in each periods would simply be the one that results into less tax payment in that period. It becomes more difficult to decide when there is a loss. With the loss offset strategy, the decision maker seeks to redistribute income from the high-income-periods to low-income-periods. To understand the redistributive motive in a simple way, let us suppose that we have no interest rate, i.e., capital gains in all periods are equal to zero. Then, the decision variable of the model is the loss carry-backward alone.

At a first sight, this might seems simple: if a tax payer makes a loss of EUR -50.000, he should be able to declare a loss carry-backward of EUR -50.000. However, in reality there are more restrictions to the maximal amount of loss carry-backward that can be declared and accepted by the Tax Office. For example, according to current law loss carry-backward can be EUR 1 Million at maximum. Moreover, the amount in reality also depends on the gains in the prior period. The

question is how much of loss to carry back to get the maximal final wealth. These issues will be surveyed in our coming model.

Model 3-5 Offsetting a loss

We continue with Model 3-4.

For the sake of simplicity, assume the interest rate is zero so that B_t for all periods is zero. Thus all income of the tax payer will be made out of A_t alone. We do not have to consider the taxation options as in the previous model. As usual, we assume that for each period the amount of income is known in advance. In a certain period the taxpayer incurs expenses that there would be a loss in total. In this case A_t is a negative value, representing a loss.

The basic issue of this optimization model is merely the German tax function. The interpretation of the mathematical function is as follows. No income tax is charged on the basic allowance, which is €8,354[59]. Beyond this threshold, the marginal tax rate increases linearly from 14% to 24% for a taxable income of €13,469. In the subsequent interval up to a taxable income of €52,881, the marginal tax rate increases linearly from 24% to 42%. The last change of rates occurs at a taxable income of €250,730 when the marginal tax rate jumps from 42% to 45%. The course of the marginal tax rate and the resulting average tax rate result into 5 functions and therefore made the objective function a discrete, nonlinear one.

Let us assume a planning horizon of three years and that the tax payer makes a loss in the second period. We have the fact that A_2 is now a negative number. As we have explained, the loss A_2 can now be divided into two parts. We denote:

- x_2 as the loss carry-backward from period 2 to period 1,
- xf_2 as the loss carry-forward from period 2 to period 3.

The fraction of loss that is not used as loss carry-backward is defined as the loss carry-forward. Logically, we have

$$x_2 + xf_2 = A_2 \tag{3-26}$$

The loss carry-backward x_2 will reduce the taxable income of period 1, thus lead to a recalculation of I_1 and tax payment S_1. Because we do not have B_1 (due to interest rate $i = 0$), according to formula (3-16) we have $I_1 = A_1$ and $S_1 = f(I_1)$, with $f(.)$ is the tax function in (3-17).

For the recalculation, we denote IR_1 as the taxable income of period 1 after the renewed assessment, i.e., after the loss carry-backward x_2 is made

$$IR_1 = A_1 + x_2 \tag{3-27}$$

Furthermore, denote SR_1 as the tax payment in period 1 after recalculation, SR_1 is calculated based on IR_1, i.e., $SR_1 = f(IR_1)$.

[59] We consider only values for unmarried individuals.

Now, because after the recalculation process the tax payment SR_1 is always smaller than or equal to S_1, in period 2 the tax payer will receive a tax refund which can be denoted as SRF_2.

$$SRF_2 = S_1 - SR_1 \tag{3-28}$$

This tax refund will be added to wealth in period 2. In general, we can rewrite formula (3-23) as

$$W_t = W_{t-1} + A_t + B_t - S_t + SRF_t \tag{3-29}$$

In this simplified model, we have $B_2 = 0$ and because A_2 is negative, no tax payment must be made, i.e., $S_2 = 0$. Thus, (3-29) becomes

$$W_2 = W_1 + A_2 + SRF_2 \tag{3-30}$$

Continuing our calculation, the loss carry-forward xf_2 will reduce taxable income of period 3, thus reduce the tax payment S_3. Again, because $\mathrm{B}_3 = 0$ (due to interest rate = 0), according to formula (3-16) we have $I_3 = A_3$ but now of course reduced by xf_2

$$I_3 = A_3 + xf_2 \tag{3-31}$$

and $S_3 = f(I_3)$.

The last step in our procedure is to calculate the wealth in period 3, which is the objective function to be maximized

$$W_3 = W_2 + A_3 - S_3 \tag{3-32}$$

It is important to mention that in this model and the following ones, we only consider the loss carry-backward x_t as a decision variable. The loss carry-forward will be automatically calculated as the unused amount of loss in a period.

Now let us turn back to the equation (3-26). We have define that sum of loss carry-backward x_2 and loss carry-forward xf_2 must equals to A_2. However, there are other restrictions. We have to formulate the formula that defines the upper bound of the loss carry-backward. In other words, we now define the maximum loss carry-backward that can be allowed. Note that because the loss carry-backward is a negative number, “upper bound” means “lower bound” and “maximum” means “minimum” in mathematical sense.

First, according to (3-26) we have the intuitive inequality

$$x_2 \geq A_2 \tag{3-33}$$
$$x_2 \leq 0 \tag{3-34}$$

According to §10d EStG, the maximum amount of loss carry-backward allowed is EUR 1m

$$x_2 \geq -1.000.000 \tag{3-35}$$

Furthermore, the loss carry-backward from period 2 to period 1 is intended to reduce the taxable income in period 1. Therefore it cannot be higher than the taxable income in period 1. Mathematically, this means

$$x_2 \geq -A_1 \tag{3-36}$$

The maximum possible loss carry-backward can now be written as

$$x_{2max} = \max(-1.000.000;\ A_2;\ -A_1) \tag{3-37}$$

The minimum possible loss carry-backward is zero in all cases

$$x_{2min} = 0 \tag{3-38}$$

In the following we present two numerical examples that illustrate our Model 3-5.

Example 3-5

In this example we explore the advantageousness of the choice of loss carry-backward and –forward described in Model 3-5 using the following data.

Given:

Number of time periods $T = 3$,

Initial wealth $W_0 = 30.000$,

Amounts of ordinary income $A_1 = 30.000$, $A_2 = -30.000$, $A_3 = 30.000$,

Annual interest rate $i = 0\%$

Variable:

The decision variable is the amount of loss carry-backward in second period x_2.

Question:

How high should the loss carry-backward in period 2 be in order to maximize final wealth?

Solution:

Let us begin with Period 1. Tax payment depends only on $A_1 = 30.000$.

$I_1 = A_1 = 30.000$,

$S_1 = f(I_1) = f(30.000) = 5.559,$

$W_1 = W_0 + A_1 - S_1 = 30.000 + 30.000 - 5.559 = 54.441.$

In period 2, we can choose a value for loss carry-backward x_2 between 0 and x_{2max} .

According to (3-37), $x_{2max} = -30.000$. Thus

$$-30.000 \leq x_2 \leq 0$$

The remaining parameters which we need to identify are the loss carry-backward and loss carry-forward. For our purpose, let assume that we decide to carry EUR 10.000 of loss back, i.e., $x_2 = -10.000$. Then:

$xf_2 = -20.000$, see (3-26)

$IR_1 = A_1 + x_2 = 20.000$, see (3-27)

$SR_1 = f(IR_1) = f(20.000) = 2.634,$

The tax refund that we receive in period 2 in case of $x_2 = -10.000$ is

$SRF_2 = S_1 - SR_1 = 5.559 - 2.634 = 2.925$, see (3-28)

Wealth in period 2 is

$W_2 = W_1 + A_2 + SRF_2 = 27.336$, see (3-30)

In period 3, the taxable income $I_3 = A_3 = 30.000$ will now be reduced by $xf_2 = -20.000$, therefore

$I_3 = A_3 + xf_2 = 30.000 - 20.000 = 10.000$, see (3-31)

$S_3 = f(I_3) = f(10.000) = 257$

$W_3 = W_2 + A_3 - S_3 = 57.109$

The graphical illustration of all above calculation steps can be viewed in Figure 3-8

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rate)	0				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**30.000**	**-30.000**	**30.000**
7	**B_t** (capital yield)		0	0	0	0
8	**AB_t** (total income)		0	30.000	-30.000	30.000
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***0***
10	**x_t max** (maximal loss carry backward)		0	0	-30.000	0
11	**I_t** (total taxable income)		0	30.000	0	10.000
12	**S_t (total tax payment)**		0	5.559	0	257
13	**IR_t (Recalculation of Taxable Income After L**		0	20.000	0	10.000
14	**SR_t (Recalculation of Tax Payment)**		0	2.634	0	257
15	**SRF_t (tax refund)**		0	0	2.925	0
16	**W_t** (end of period wealth)		**30000**	54.441	27.366	57.109
17	**x_t** (loss carry-backward)		0	***0***	***-10.000***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-20.000***	***0***

Figure 3-8 Choice of loss carry-backward –Suboptimal Solution

In the end we have obtained a value of $W_3 = 57.109$ that is in fact not the maximum. The optimal result, i.e., maximal wealth that can be achieved, is $W_3 = 57.313$ by choosing $x_2 = -15.000$. The financial plan for the optimal result is shown in Figure 3-9. At this junction we purely want to give an intuition of how the loss offset strategy works. It is obvious that when interest rate equals zero, the marginal tax rate in Period 1 and 3 should be equal in order to obtain the maximum final wealth. The search methods that we apply to calculate this optimal result will be presented and discussed in detail in Chapter 4.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rate)	0				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**30.000**	**-30.000**	**30.000**
7	**B_t** (capital yield)		0	0	0	0
8	**AB_t** (total income)		0	30.000	-30.000	30.000
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***0***
10	**x_t max** (maximal loss carry backward)		0	0	-30.000	0
11	**I_t** (total taxable income)		0	30.000	0	15.000
12	**S_t (total tax payment)**		0	5.559	0	1.343
13	**IR_t (Recalculation of Taxable Income After L**		0	15.000	0	15.000
14	**SR_t (Recalculation of Tax Payment)**		0	1.343	0	1.343
15	**SRF_t (tax refund)**		0	0	4.215	0
16	**W_t** (end of period wealth)		**30000**	54.441	28.657	57.313
17	**x_t** (loss carry-backward)		0	***0***	***-15.000***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-15.000***	***0***

Figure 3-9 Choice of loss carry-backward –Optimal Solution

With the last two models we have seen how the final wealth depends on two different types of choices:

- Taxation options: select if ordinary income and capital gains should be tax together or separately. We assess the selection in a sequential way.
- Loss carry-backward: the basic idea of the decision is to carry some amount of loss in period 2 back to period 1. We have to compute the maximum amount of loss in Period 2, in which the loss occurs, that is allowed to be carried back.

We have aimed to keep this introductory section on tax planning as basic as possible, whilst giving a fairly comprehensive overview of the main ideas that will be discussed and extended in following sections of this chapter. As a consequence of this aim, most of the data used for building a scenario of tax planning are fictional.

We are now in the position to describe a model that considers loss offset and taxation options simultaneously. The necessity to construct such a model can be argued by considering the situation when we change the interest rate from $0\ \%$ to $10\ \%$. Then, all B_t will be positive and thus the effect of the selection of taxation options will have to be carefully examined.

3.4 Mathematical Formulation of the Loss Offset Problem

The structure of model and mathematical formulation remain intact through the various modeling processes that we present in the current sections. We differentiate between three cases:

1. 3-Period-Problem with 1 loss period
2. Multiple-Period-Problem with 1 loss period
3. Multiple-Period-Problem with multiple loss periods

In order to provide a better way of keeping track on the topic, we attempt to make use of the examples in previous section; that is, we assume the same structure and change only the values of basic input data for our coming examples.

3.4.1 3-Period-Problem with 1 Loss Period

We consider a basic case of the Loss Offset Problem first, and then discuss the generalization in later part of this section. In this section, assume for simplicity that we have a planning horizon of three periods and only one loss period. For our demonstrative purpose we assume that it is the second period where loss occurs. Thus, loss amount in the second period can be divided into two components:

1. loss carry-backward that reduces the taxable income of the first period
2. loss carry-forward that reduces the taxable income of the third period

Notice that the amount of loss in 2nd period that can be offset as well as the amounts of taxable income in 1st and 3rd period depend on the tax option to be chosen in each period (see explanations in previous sections). The model in this section is partially based on [SSDK12].

The difference is that in [SSDK12], we made the assumption that the loss carry-backward directly reduces the taxable income and tax payment in the previous period, what is not possible in practice. Thus, in this model, we change the formulation such that the loss carry-backward leads to a recalculation of taxable income in the prior period, resulting into a tax refund. Furthermore, because tax rates and several tax laws may vary from year to year due to changes in the tax code or in the level of nominal incomes, we conduct our current analysis with all data from the year 2014.

Our task when building a model, which attempts to represent a real problem as close as possible, is to develop "general" formulas that cover all imaginable scenarios. This is not a simple task at all. For instance, the upper bound for a loss carry-backward in any year could be different. It depends on the input data and the decisions on tax options made in all earlier years. The value could be zero, could be the maximal amount allowed by law, or some numbers there-between. As we will see, the determination of the taxable income is subject to a complex set of specific rules as a consequence of the decision variables. In a first step, the taxable income is assessed according to the tax option. In a second step, the income depends on the strategy of offsetting losses in other periods.

Model 3-6 The Loss Offset Problem with 3 periods and 1 loss period

Based on the assumptions of Model 3-4 and Model 3-5, we again have a deterministic stream of cash flow that we label as the ordinary income A_t.

We have defined the capital gains B_t as the wealth in previous period W_{t-1} multiplies with interest rate. Let us assume that the wealth in every period[60] is always a positive number. Thus, in our models B_t will never be negative.

In a model without any loss period, the optimal tax option in a period t is the one that leads to less tax payment in ***that same*** period. This is not true anymore when there is a loss to be offset. Suppose t is a loss period, i.e., the ordinary income A_t is negative. The amount of loss in period t that can be used as carry-backward to $t-1$ is denoted by x_t. By definition x_t is always smaller or equal to zero.

The most representative case is when the loss occurs in the 2nd period. This is due to the fact that we only consider three planning periods such that:

- If the loss occurs in the 1st period, then set loss carry-backward equal to zero.
- If the loss occurs in the 3rd period, then set all of the loss as loss carry-backward.

In the following we suppose the loss is in the 2nd period so that there is a trade-off between loss carry-backward and -forward. The model can be formulated as

[60] For simplicity assume that the wealth is represented by the cash balance in a bank account.

Given:

$T = 3$: Time horizon of the project in years.

i : Interest rate for the calculation of capital gains on an annual basis

A_t : Ordinary income in period $t, for\ t = 1, 2, 3$

W_0: Initial wealth in $t = 0$

Variable:

y_t : Taxation option which is applied for period t

x_2 : Amount of loss carry-backward from period 2 to period 1

Question:

Maximize the final wealth

$$W_3 = W_0 + \sum_{t=1}^{3}(A_t + B_t) - \sum_{t=1}^{3} S_T \tag{3-39}$$

Constraints:

In period 1:

$$B_1 = i \cdot W_0 \tag{3-40}$$

We continue to formulate the choice of tax options as binary variables.

$$y_1 \in \{0; 1\} \tag{3-41}$$

$$I_1 = \begin{cases} A_1 + B_1 & if\ y_1 = 0 \\ A_1 & if\ y_1 = 1 \end{cases} \tag{3-42}$$

$$S_1 = \begin{cases} f(I_1) & if\ y_t = 0 \\ f(I_1) + g(B_1) & if\ y_t = 1 \end{cases} \tag{3-43}$$

$$W_1 = W_0 + A_1 + B_1 - S_1 \tag{3-44}$$

In period 2 (loss period):

$$B_2 = i \cdot W_1 \tag{3-45}$$

$$y_2 \in \{0; 1\} \tag{3-46}$$

$$I_2 = \begin{cases} max\ (A_2 + B_2; 0) & if\ y_2 = 0 \\ max(A_2; 0) & if\ y_2 = 1 \end{cases} \tag{3-47}$$

$$S_2 = \begin{cases} f(I_2) & if\ y_2 = 0 \\ f(I_2) + g(B_2) & if\ y_2 = 1 \end{cases} \tag{3-48}$$

Now we can claim a loss carry-backward $\boldsymbol{x_2}$ that leads to a recalculation of taxable income in period 1. We call this $\boldsymbol{IR_1}$.

$$IR_1 = \begin{cases} A_1 + B_1 + \boldsymbol{x_2} & if\ y_1 = 0 \\ A_1 + \boldsymbol{x_2} & if\ y_1 = 1 \end{cases} \tag{3-49}$$

This, in turn, leads to a recalculation of tax payment in period 1. We call this $\boldsymbol{SR_1}$.

$$SR_1 = \begin{cases} f(IR_1) & if\ y_1 = 0 \\ f(IR_1) + g(B_1) & if\ y_1 = 1 \end{cases} \tag{3-50}$$

And the tax refund is calculated as

$$SRF_2 = S_1 - SR_1 \tag{3-51}$$

Such that wealth in period 2 is

$$W_2 = W_1 + A_2 + B_2 - S_2 + SRF_2 \tag{3-52}$$

The restrictions on loss carry-backward can be formulated as

$$x_{2max} \leq x_2 \leq 0 \tag{3-53}$$

whereas

$$x_{2max} = max(-1.000.000; I_2; -I_1) \tag{3-54}$$

<u>In period 3:</u>

$$B_3 = i \cdot W_2 \tag{3-55}$$

$$y_3 \in \{0; 1\} \tag{3-56}$$

Now the loss carry-forward from period 2 leads to a reduction of taxable income in period 3. This is given as

$$xf_2 = \begin{cases} A_2 + B_2 - \boldsymbol{x_2} & if\ y_2 = 0 \\ A_2 - \boldsymbol{x_2} & if\ y_2 = 1 \end{cases} \tag{3-57}$$

$$I_3 = \begin{cases} A_3 + B_3 - \boldsymbol{xf_2} & if\ y_3 = 0 \\ A_3 - \boldsymbol{xf_2} & if\ y_3 = 1 \end{cases} \tag{3-58}$$

$$S_3 = \begin{cases} f(I_3) & if\ y_3 = 0 \\ f(I_3) + g(B_3) & if\ y_3 = 1 \end{cases} \tag{3-59}$$

And the final wealth is

$$W_3 = W_2 + A_3 + B_3 - S_3 \tag{3-60}$$

Considering this model, two kinds of variables appear in the mathematical model:

- Binary variables y_t represent yes-no-type decisions on taxation options. These are restricted to the values 0 or 1.
- A real-valued variable x_t that represents the amount of loss carry-backward. It is restricted to the values between 0 and an upper bound $x_{t\,max}$ that depends on the input data and other decision variables.

It is important to note that this formulation allows the decision maker to specify the upper bound of the loss carry-backward, i.e., the maximum amount possible allowed. This upper bound itself will change depending on different values of the binary variables. The discussion thus far has assumed that the loss period is the second one. By making such simplistic assumptions, intuitive solutions can be derived, leading to an immediate understanding and easiness of calculation. Continuing the example above, we now combine both types of decisions in one financial plan.

Example 3-6

Although the structure remains the same, we consciously choose different data in order to have a wide variety of scenarios. We will present the optimal solution that can be found with the algorithms to be presented in chapter 4.

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 1.000.000$$

$$A_1 = 50.000$$

$$A_2 = -90.000$$

$$A_3 = 50.000$$

Figure 3-10 shows a suboptimal solution, and the optimal solution is given in Figure 3-11.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**50.000**	**-90.000**	**50.000**
7	**B_t** (capital yield)		0	50.000	53.736	51.252
8	**AB_t** (total income)		0	100.000	-36.264	101.252
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-50.000	0
11	**I_t** (total taxable income)		0	50.000	0	0
12	**S_t (total tax payment)**		0	25.280	13.434	12.813
13	**IR_t (Recalculation of Taxable Income**		0	50.000	0	0
14	**SR_t (Recalculation of Tax Payment)**		0	25.280	13.434	12.813
15	**SRF_t (tax refund)**		0	0	0	0
16	**W_t** (end of period wealth)		**1.000.000**	1.074.720	1.025.022	1.113.461
17	**x_t** (loss carry-backward)		0	***0***	***0***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-90.000***	***0***

Figure 3-10 A 3-period-problem – Suboptimal Solution

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**50.000**	**-90.000**	**50.000**
7	**B_t** (capital yield)		0	50.000	53.736	52.276
8	**AB_t** (total income)		0	100.000	-36.264	102.276
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***0***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-36.264	0
11	**I_t** (total taxable income)		0	50.000	0	33.236
12	**S_t (total tax payment)**		0	25.280	0	19.672
13	**IR_t (Recalculation of Taxable Income**		0	30.500	0	33.236
14	**SR_t (Recalculation of Tax Payment)**		0	18.217	0	19.672
15	**SRF_t (tax refund)**		0	0	7.063	0
16	**W_t** (end of period wealth)		**1.000.000**	1.074.720	1.045.519	1.128.123
17	**x_t** (loss carry-backward)		0	***0***	***-19.500***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-16.764***	***0***

Figure 3-11 A 3-period-problem – Optimal Solution

3.4.2 Multiple-Period-Problem with 1 Loss Period

In the last part, we have dealt with a very simple model to explain the idea and mechanism of offsetting loss in a period. We assume that the loss in a year can only affect incomes and tax payments in the two years: the one before and the one after it. In fact, a planning horizon may not necessarily be limited to three years, but can consist of four, five or more years. Therefore, a loss in one single year can affect more than only two neighboring years. Taking this fact into consideration is the objective of this section. We must make one more adjustment to our model in order to deal with the unused loss amount. With the above model in mind, let us now consider the problem in a slightly more complicated manner. Suppose we have a planning horizon of more than 3 periods. In this

situation, the loss in period 2 can not only be carried forward to period 3, but also to several forthcoming periods when its amount is big enough. Clearly there seems to be something wrong with the formulation in the previous model.

In the case of Multiple-Period-Problem with 1-Loss-Period, the meaning of loss carry-backward remains the same, i.e., it reduces the taxable income in only ***one*** previous period[61]. However, the loss carry-forward may be able to reduce the taxable incomes in ***multiple*** future periods. Here, we recognize the necessity of re-modeling the problem. We must modify the constraints of our mathematical program such that a deduction from loss carry-forward in several future periods is possible.

Model 3-7 The Loss Offset Problem with Multiple Periods and 1 Loss Period

For definiteness, we shall consider a concrete situation. Let us say there are four years of planning horizon and loss occurs in 2nd year. This problem is similar to the 3-Period-Problem with 1-Loss-Period where there was only one loss carry-backward. We now allow the loss carry-forward to be able to affect not only the 3rd year but also the 4th year.

This model differs very little from the previous model. In fact, we only need to modify the loss carry-forward amount to be taken to multiple periods. We avoid redundancy by not formulating a new model, and refer to the next section. There, the formulation of the general Loss Offset Problem will be given in details. In order to understand this model more clearly, we next consider a numerical example.

Example 3-7

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 1.000.000$$

$$A_1 = 50.000$$

$$A_2 = -90.000$$

$$A_3 = 50.000$$

We now take one more period into consideration

$$A_4 = 150.000$$

For this scenario, we suppose that in the 4th period a high positive value of ordinary income can be predicted. As seen in Figure 3-12, an intuitive decision of carrying back a high amount of loss is not reasonable. Also, if all or none of the loss amount is claimed as loss carry-backward, the result will not be optimal.

The optimal solution for the problem is given in Figure 3-13.

[61] Again mentioned, the loss carry-backward in *t* reduces taxable income in *t-1* ,thus results in a tax refund in *t* that is added to the wealth in *t*

	A	B	C	D	E	F	G
1	**Notation**						
2							
3	i (interest rat	0,05					
4							
5	t		0	**1**	**2**	**3**	**4**
6	**A_t** (cash flow after depreciation)		0	**50.000**	**-90.000**	**50.000**	**150.000**
7	**B_t** (capital yield)		0	50.000	53.736	51.891	56.324
8	**AB_t** (total income)		0	100.000	-36.264	101.891	206.324
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-50.000	0	0
11	**I_t** (total taxable income)		0	50.000	0	10.000	150.000
12	**S_t (total tax payment)**		0	25.280	13.434	13.230	68.842
13	**IR_t (Recalculation of Taxable Income**		0	0	0	10.000	150.000
14	**SR_t (Recalculation of Tax Payment)**		0	12.500	13.434	13.230	68.842
15	**SRF_t (tax refund)**		0	0	12.780	0	0
16	**W_t** (end of period wealth)		**1.000.000**	1.074.720	1.037.802	1.126.463	1.263.945
17	**x_t** (loss carry-backward)		0	***0***	***-50.000***	***0***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-40.000***	***0***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-40.000***	***0***	***0***

Figure 3-12 A 4-period-problem – Suboptimal Solution

	A	B	C	D	E	F	G
1	**Notation**						
2							
3	i (interest rat	0,05					
4							
5	t		0	**1**	**2**	**3**	**4**
6	**A_t** (cash flow after depreciation)		0	**50.000**	**-90.000**	**50.000**	**150.000**
7	**B_t** (capital yield)		0	50.000	53.736	51.312	55.736
8	**AB_t** (total income)		0	100.000	-36.264	101.312	205.736
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-50.000	0	0
11	**I_t** (total taxable income)		0	50.000	0	0	113.000
12	**S_t (total tax payment)**		0	25.280	13.434	12.828	53.155
13	**IR_t (Recalculation of Taxable Income**		0	47.000	0	0	113.000
14	**SR_t (Recalculation of Tax Payment)**		0	24.080	13.434	12.828	53.155
15	**SRF_t (tax refund)**		0	0	1.200	0	0
16	**W_t** (end of period wealth)		**1.000.000**	1.074.720	1.026.222	1.114.706	1.267.287
17	**x_t** (loss carry-backward)		0	***0***	***-3.000***	***0***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-87.000***	***0***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-87.000***	***-37.000***	***0***

Figure 3-13 A 4-period-problem – Optimal Solution

3.4.3 Multiple-Period-Problem with Multiple Loss Periods

What discussed in the two previous sections applies to a simple case of three or four periods where only one loss occurs. What we need, however, in most of the cases is a solution for 2 or more losses and for values T of planning horizon which are not necessarily small. The multi-periodic case involving more than three planning periods is more complicated and the mathematical program does not have the straightforward form. The solution is no longer intuitive.

This section now considers how all the theory described above comes together in a general mathematical framework to deal with all possible scenarios. We present the interaction between decision variables by considering the case when there are more than three periods and also more

than one loss period. These assumptions correspond to a real-world problem. For example, how to model the constraints if two loss periods occur consecutively? Our next step is to construct a model for such a case also. A planning horizon of T periods is given. Within this horizon, any arbitrary number of loss periods is allowed. Also the arrangement of loss periods can be random. As was mentioned in the previous section, a loss carry-forward can be used as deduction in several future periods. These interdependencies require a modeling approach in order to determine the value of final wealth.

The ***Loss Offset Problem*** can be formulated as follows:

Model 3-8 The Loss Offset Problem

Given:

A_t: Ordinary income

W_0: Initial wealth

i: Interest rate

Variables:

y_t: Taxation options in every period t

x_t: Amount of loss carry-backward in every period t, no matter if it is a loss period or not

Question:

Maximize final wealth

$$W_T = W_{T-1} + AB_T - S_T + SRF_T \tag{3-61}$$

Subject to:

For all $t = 1\ to\ T$

Capital gains

$$B_t = i \cdot W_{t-1} \tag{3-62}$$

Taxation options

$$y_t \in \{0; 1\} \tag{3-63}$$

Total income

$$AB_t = A_t + B_t \tag{3-64}$$

The upper bound of loss carry-backward in any period t is determined by the expression:

$$x_{t\,max} = max(-1.000.000; min(0; (1-y_t)\cdot AB_t + y_t \cdot A_t); -\max(0; I_{t-1}) \quad \textbf{(3-65)}$$

We can make an observation, that the loss carry-backward can take any value in an interval between the max loss carry-backward and zero. The width of this interval is narrower with the choice of

- joint taxation in the loss period
- and/or separate taxation in the period prior to the loss period

Suppose we have prior knowledge indicating that two loss periods exist consecutively. The variable for loss backward in the second period in this series can be promptly set to zero. This has been included in our mathematical formulation of $x_{t\,max}$.

The following equation provides us with an efficient formulation for dealing with the restrictions of §10d of the EStG. The taxable income, which is the assessment basis for calculating tax payment, is

$$I_t = (1-y_t)\cdot \max(0; AB_t + \max(xff_{t-1}; \min(-1.000.000 - 0{,}6\cdot(AB_t - 1.000.000); 0)))$$
$$+ y_t \cdot \max(0; A_t + \max(xff_{t-1}; \min(-1.000.000 - 0{,}6\cdot(A_t - 1.000.000); 0))) \quad \textbf{(3-66)}$$

Tax payment

$$S_t = (1-y_t)\cdot f(I_t) + y_t \cdot (f(I_t) + 0{,}25\cdot B_t) \quad \textbf{(3-67)}$$

Recalculation of taxable income

$$IR_t = \max(I_t + x_{t+1}; 0) \quad \textbf{(3-68)}$$

Recalculation of tax payment

$$SR_t = (1-y_t)\cdot f(IR_t) + y_t \cdot (f(IR_t) + 0{,}25\cdot B_t) \quad \textbf{(3-69)}$$

Tax refund

$$SRF_t = S_{t-1} - SR_{t-1} \quad \textbf{(3-70)}$$

The amount of loss carry-backward that can be claimed in a period is restricted to

$$x_{t\,max} \le x_t \le 0 \quad \textbf{(3-71)}$$

The amount of resulted loss carry-forward in a period is given by

$$xf_t = \min(y_t \cdot (A_t - x_t) + (1-y_t)\cdot(AB_t - x_t); 0) \quad \textbf{(3-72)}$$

We need an auxiliary parameter. Denote by $\boldsymbol{xff_t}$ the unused amount of loss carry-forward in a period t. Therefore

$$xff_t = \min(min(xff_{t-1};0) + y_t \cdot max(A_t;0) + (1-y_t)\cdot max(AB_t;0)) + xf_t - I_t;0) \tag{3-73}$$

At the end of a period, the cumulative wealth is

$$W_t = W_{t-1} + AB_t - S_t + SRF_t \tag{3-74}$$

Let us consider the next example. For demonstrative purpose, we fix the time horizon to six years. To keep matters simple, let us suppose that there are two loss periods: in the second and the fifth period. The concept is identical to that of the last two examples.

Example 3-8

	A	B	C	D	E	F	G	H	I
1	**Notation**								
2									
3	i (interest rat	0,05							
4									
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**
6	**A_t** (cash flow after depreciation)		0	**2.000.000**	**-2.000.000**	**2.000.000**	**1.000.000**	**-2.000.000**	**3.000.000**
7	**B_t** (capital yield)		0	50.000	107.164	12.522	104.823	142.414	49.534
8	**AB_t** (total income)		0	2.050.000	-1.892.836	2.012.522	1.104.823	-1.857.586	3.049.534
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***0***	***0***	***0***	***0***
10	**x_t max** (maximal loss carry backward)		0	0	-1.000.000	0	0	-819.500	0
11	**I_t** (total taxable income)		0	2.050.000	0	405.009	819.500	0	1.191.948
12	**S_t (total tax payment)**		0	906.739	0	166.493	353.014	0	520.616
13	**IR_t (Recalculation of Taxable Income**		0	2.050.000	0	405.009	819.500	0	1.191.948
14	**SR_t (Recalculation of Tax Payment)**		0	906.739	0	166.493	353.014	0	520.616
15	**SRF_t (tax refund)**		0	0	0	0	0	0	0
16	**W_t** (end of period wealth)		**1.000.000**	2.143.261	250.425	2.096.454	2.848.263	990.677	3.519.595
17	**x_t** (loss carry-backward)		0	***0***	***0***	***0***	***0***	***0***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-1.892.836***	***0***	***0***	***-1.857.586***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-1.892.836***	***-285.323***	***0***	***-1.857.586***	***0***

Figure 3-14 A 6-period-problem – Suboptimal Solution

	A	B	C	D	E	F	G	H	I
1	**Notation**								
2									
3	i (interest rat	0,05							
4									
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**
6	**A_t** (cash flow after depreciation)		0	**2.000.000**	**-2.000.000**	**2.000.000**	**1.000.000**	**-2.000.000**	**3.000.000**
7	**B_t** (capital yield)		0	50.000	107.664	34.201	113.772	146.326	68.672
8	**AB_t** (total income)		0	2.050.000	-1.892.336	2.034.201	1.113.772	-1.853.674	3.068.672
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-1.000.000	0	0	-1.000.000	0
11	**I_t** (total taxable income)		0	2.000.000	0	1.000.000	1.000.000	0	1.749.300
12	**S_t (total tax payment)**		0	896.739	26.916	442.789	462.682	36.582	788.592
13	**IR_t (Recalculation of Taxable Income**		0	1.000.000	0	1.000.000	250.700	0	1.749.300
14	**SR_t (Recalculation of Tax Payment)**		0	446.739	26.916	442.789	125.498	36.582	788.592
15	**SRF_t (tax refund)**		0	0	450.000	0	0	337.184	0
16	**W_t** (end of period wealth)		**1.000.000**	2.153.261	684.009	2.275.421	2.926.511	1.373.439	3.653.519
17	**x_t** (loss carry-backward)		0	***0***	***-1.000.000***	***0***	***0***	***-749.300***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-1.000.000***	***0***	***0***	***-1.250.700***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-1.000.000***	***0***	***0***	***-1.250.700***	***0***

Figure 3-15 A 6-period-problem – Optimal Solution

We have now covered the main insights of the chapter 3. Recall that in chapter 2 we have presented the optimization methods and throughout this current chapter we explained the problem of loss offset as well as discussed the cases of 3-period- and multi-period-models. In the next Chapter, we apply the search approaches in chapter 2 to solve the problem of chapter 3.

Because it offers a generalization of all possible scenarios and therefore also covers the simpler cases, from now on we will only consider the Multiple-Period-Problem with Multiple Loss Periods that has been the focus of this section. Before continuing with the optimization approaches, we would like to verify the correctness of this model.

3.4.4 Verification of the mathematical model

One of the major reasons for verifying a model is the concern that it may be incorrect. Building a model is like computing a large software program, it must be debugged. Here, bugs in a model come in a variety of forms. A variable name may be assigned incorrectly, the sign of a parameter may be wrong, a coefficient may be in a wrong constraint, missing data are in the constraints, or maybe a decimal point may be misplaced.

Verifying a model becomes difficult when the model is large. For our purpose a model is large if neither its formulation nor its solution report and interpretation fit on one printed page. We may be interest in only parts of the model or parts of the solution report. The difficulty will be apparent if these parts are scattered over several pages. This is exactly the case for our model that has been established in the preceding section. A more precise description of the reality makes the model more complicated not only in the constraints but also in the objective function. An important distinguishing feature of the proposed problem is that it can be well defined. That is, the objective function is to maximize the final wealth, and the relevant variables and parameters in the model can be exactly quantified in monetary terms.

Our approach of verification of the model is straightforward. We generate different scenarios by altering the input data, i.e., ordinary income A_t, initial wealth W_0, and the interest rate i. We call each new scenario a test case. For each test case, we change the variable values and recalculate the resulting values of other parameters as can be seen in Figure 3-16. In total, we created ten test cases for checking the different changes in the values of final wealth. The correctness of models can then be easily observed and confirmed in our examples throughout this chapter and in the next chapter.

G11 =AUFRUNDEN((1-G9)*MAX(0;G8+MAX(F19;MIN(-1000000 - 0,6*(G8-1000000);0))) + G9*MAX(0;G6+MAX(F19;MIN(-1000000 - 0,6*(G6-1000000);0)));0)

	A	B	C	D	E	F	G	H	I	J	K
1	**Notation**										
2											
3	i (interest rat	0,05									
4											
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**		
6	**A_t** (cash flow after depreciation)		0	**60.000**	**-120.000**	**90.000**	**60.000**	**-120.000**	**90.000**		
7	**B_t** (capital yield)		0	50.000	54.027	50.841	57.174	61.470	58.570		
8	**AB_t** (total income)		0	110.000	-65.973	140.841	117.174	-58.530	148.570		
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***	***1***	***1***		
10	**x_t max** (maximal loss carry backward)		0	0	-60.000	0	0	-60.000	0		
11	**I_t** (total taxable income)		0	60.000	0	15.583	60.000	0	16.206		
12	**S_t (total tax payment)**		0	29.461	13.507	14.198	31.255	15.368	16.287		
13	**IR_t (Recalculation of Taxable Income**		0	14.417	0	15.583	13.794	0	16.206		
14	**SR_t (Recalculation of Tax Payment)**		0	13.700	13.507	14.198	15.343	15.368	16.287		
15	**SRF_t (tax refund)**		0	0	15.761	0	0	15.912	0		
16	**W_t** (end of period wealth)		**1.000.000**	1.080.539	1.016.820	1.143.463	1.229.382	1.171.397	1.303.680		
17	**x_t** (loss carry-backward)		0	***0***	***-45.583***	***0***	***0***	***-46.206***	***0***		
18	**xf_t** (loss carry-forward)		0	***0***	***-74.417***	***0***	***0***	***-73.794***	***0***		
19	**xff_t** (Not-used Loss Carry-Forward)		0	***0***	***-74.417***	***0***	***0***	***-73.794***	***0***		
20											
21				ok	ok	ok	ok	ok	ok		

Figure 3-16 A test case for verification

We have conducted the problem modelling with MS-Excel. Although sometimes biased towards the user who is concerned with developing and solving complex mathematical formulations of real problems, Excel has featured for the mathematician who is interested in displays of the tableau and visualization with charts or diagrams. This offers a great help in verifying the model. For the description of all necessary techniques on spreadsheet and macros VBA for optimization and simulation as well as Excel's basic and extended functionalities, we refer to a book of Şeref et al [ŞAW07]. In this book, the Solver as well as several methods for modeling, simulation, and working with large datasets is explained in details.

4 Construction of Algorithms for Solving the Loss Offset Problem

4.1 Discussion on complexity of the problem

The quantitative analysis of the proposed problem is subject of this section. In fact, the problem is much more complex than may appear at first sight. Among the many complexities, the focus here is primarily on two issues:

- The nonlinear, discontinuous characteristics of the constraints and objective function
- The mixed-integer characteristics of variables

Most of the global optimization problems proposed and discussed in fundamental books as well as recent research publications have made assumptions about the structure of models and problems. This section gives proof of some main characteristics of the variables, constraints and objective function that make the optimization worthwhile to study as well as tough so solve. Those characteristics are:

- Non-linearity
- Discontinuity
- Non-convexity

We will organize our discussion around the three features given above while keeping our tax related issues in mind.

If we cannot be sure about the behavior of a particular function, we can try graphing it over the expected range of the variables, i.e., the search space. This is what we do in the following and it will usually reveal whether the function is non-linear, non-convex, discontinuous or non-smooth. For a presentation and treatment of these features, we refer to the books of Clarke [Cla90] or Bounkhel [Bou11]. A much advanced treatment of non-smooth techniques in respect of vector functions can be seen in [JD07].

4.1.1 Non-linearity

The non-linearity of the objective function is very simple to prove because the tax function $f(.)$ formulated in the German tax law (EStG §32a) is a piece-wise nonlinear function as shown below.

$$f(I) = \begin{cases} 0 & \text{if } I \leq 8.354 \\ (974{,}58 \cdot a + 1.400) \cdot a & \text{if } 8.355 \leq I \leq 13.469 \\ (228{,}74 \cdot b + 2.397) \cdot b + 971 & \text{if } 13.470 \leq I \leq 52.881 \\ 0{,}42 \cdot I - 8.239 & \text{if } 52.882 \leq I \leq 250.730 \\ 0{,}45 \cdot I - 15.761 & \text{if } 250.731 \leq I \end{cases} \tag{4-1}$$

with

$$a = \frac{(I - 8.354)}{10.000} \tag{4-2}$$

and

$$b = \frac{(I - 13.469)}{10.000} \tag{4-3}$$

where a and b are auxiliary parameters for the second and third domains of the tax function. I represents the taxable income and can be seen as the input variable. Substituting a and b into $f(.)$, we get a function with linear as well as quadratic domains dependent on I. Let us consider a numerical example based on the $T = 3$ problem as an illustration.

Example 4-1

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 100.000$$

$$A_1 = 200.000$$

$$A_2 = -200.000$$

$$A_3 = 200.000$$

The following figure illustrate our financial plan for the case of joint taxation in all periods and no loss carry-backward in the loss period $t = 2$.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**200.000**	**-200.000**	**200.000**
7	**B_t** (capital yield)		0	5.000	11.357	1.925
8	**AB_t** (total income)		0	205.000	-188.643	201.925
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***0***
10	**x_t max** (maximal loss carry backward)		0	0	-188.643	0
11	**I_t** (total taxable income)		0	205.000	0	13.282
12	**S_t (total tax payment)**		0	77.861	0	927
13	**IR_t (Recalculation of Taxable Income**		0	205.000	0	13.282
14	**SR_t (Recalculation of Tax Payment)**		0	77.861	0	927
15	**SRF_t (tax refund)**		0	0	0	0
16	**W_t** (end of period wealth)		**100.000**	227.139	38.496	239.494
17	**x_t** (loss carry-backward)		0	***0***	***0***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-188.643***	***0***

Figure 4-1 Financial plan of example

In order to get a first intuition how the objective function behaves, we plot the graph by creating 1.000 values of loss carry-backward with an increment of approximately 200. Because we wish to understand the impact of x_t only, we fix the taxation options y_t. Note that in total we have $2^3 = 8$ combinations of y_t. Here, for representative purpose we consider merely two cases.

For all $y_t = 0$ we obtain the following plot of W_T.

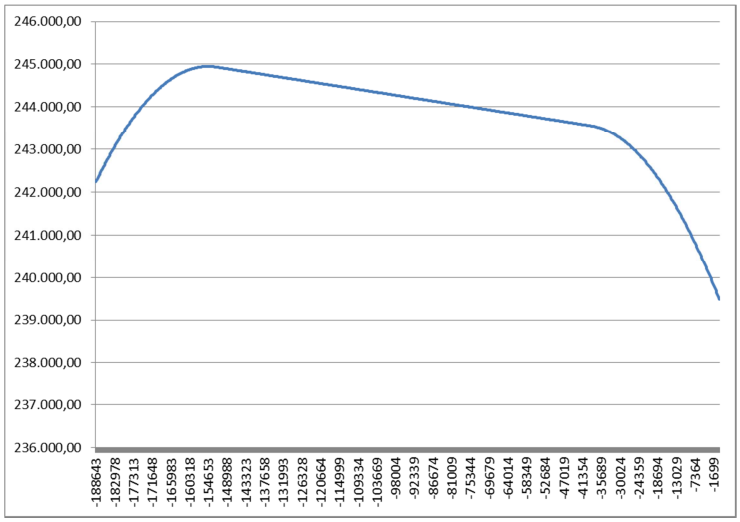


Figure 4-2 Proof of nonlinearity– Case 1

For all $y_t = 1$ we obtain the following plot of W_T.

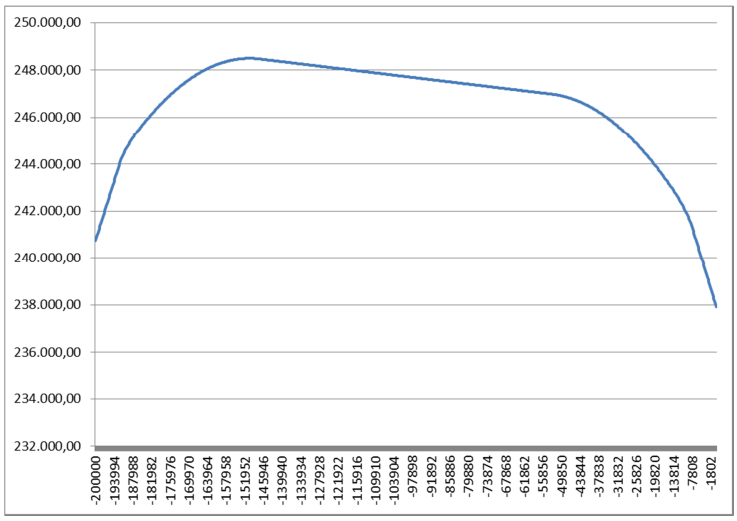


Figure 4-3 Proof of nonlinearity– Case 2

Observing the plots, we know that the maximal final wealth cannot be achieved by setting the loss carry-backward to zero ($x_2 = 0$) or to its bound ($x_2 = x_{2\,max}$). Comparing the two figures, we also see how the choice of taxation options affects the value of final wealth. In the first case, the maximum value that can be achieved is just below 245.000, whereas in the second case we can get at maximum a value above 248.000.

We can make the following observations:

- Parts of the objective function are nonlinear and other parts are linear. We can recognize that in the middle of the plots, the function contains a linear piece.
- At a first glance, the objective function is still a quasi-concave one. However if we take a closer look at the peak of our function, there is a saw-tooth pattern as shown in the next figure.

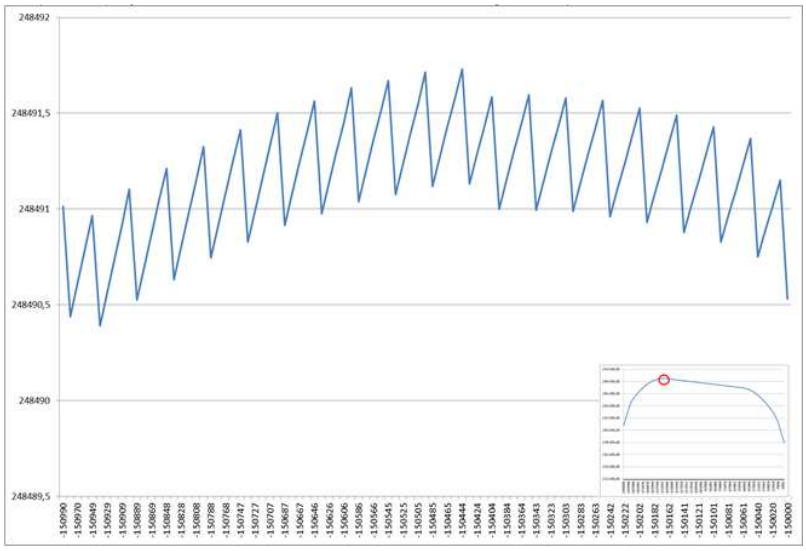


Figure 4-4 Proof of nonlinearity and non-convexity

In this figure, we plot the function in detail for a much smaller interval $x_t \in [-151.000; -150.000]$. By calculating values of the objective function with an increment EUR 10, we can see how it behaves. There are multiple local maxima of the function with extreme density over just a very small domain. Because of this fact it is difficult to find the ultimate global maximum of the objective function. This is also a proof for non-convexity.

4.1.2 Discontinuity

Next, we discuss briefly about the continuity of a mathematical function. A function that is continuous is a function whose graph has no breaks in it; i.e., we can draw a continuous curve. Generally speaking, a function is continuous if one can draw its graph without picking up the pen. In 2D, if the graph of a function is an unbroken curve with no gaps then it is said to be continuous. For example, on the graph of *f(x) = sin(x)*, the function is completely connected at all points. Many functions, however, will have isolated points where they are not connected.

From mathematical point of view, the progressive tax function defined in **§ 32d EStG** is called a piecewise function, since it is defined piece-by-piece. There are five pieces of the function in total,

each for a certain interval of taxable income. Examining this function for continuity, we see that the function adheres to our definition of being continuous.

Now let us consider not the tax function but the function for calculating total tax payment in period. Recall that S_t is a shorthand notation for the tax function. This tax function contains the binary variable as well as the loss carry-backward.

In a reduced-form, we formulate equation for calculating the total tax payment as

$$S_t = \begin{cases} f(I_t) & if\ y_t = 0 \\ f(I_t) + g(B_t) & if\ y_t = 1 \end{cases} \tag{4-4}$$

Similarly, SR_t denotes the tax payment in period 1 after recalculation and is calculated based on IR_{t}

$$SR_t = \begin{cases} f(IR_t) & if\ y_t = 0 \\ f(IR_t) + g(B_t) & if\ y_t = 1 \end{cases} \tag{4-5}$$

whereas

$$IR_{\mathrm{t}} = A_{\mathrm{t}} + x_{\mathrm{t+1}} \tag{4-6}$$

Because the discrete variable y_t can only take value of 0 or 1, the functions for calculating tax payment as well as tax refund will have "jumps" in them, resulting into the objective function for calculating final wealth is discontinuous. As evidence, consider Example 4-1. Let us fix the variables $y_1 = 1$, $y_2 = 1$ and let y_3 and x_2 move inside their definition domains. The plot of the objective function is shown below.

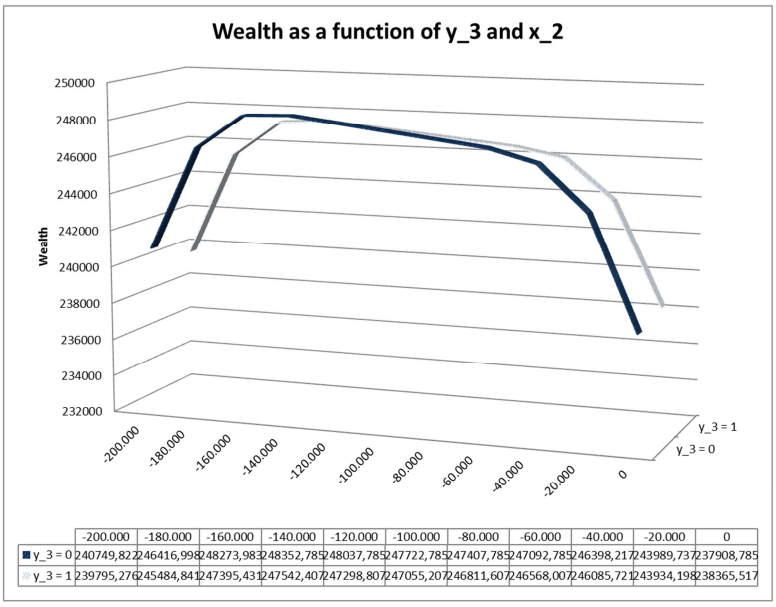


Figure 4-5 Proof of discontinuity

This is a 3D plot that shows the interrelationship between the two decision variables and the final wealth. Over the whole domain, the function pieces now have a different value at some y_3 and x_2, and we can see in the graph that our objective function seems to "jump" from one branch to the other. This jump makes the function discontinuous and we refer to this as a jump discontinuity.

4.1.3 Non-convexity

This part is dedicated to inspecting the convexity/concavity of the objective function. We will see that concavity is not given globally.

Let us consider a numerical example based on the $T = 6$ problem as an illustration.

Example 4-2

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 1.000.000$$

$$A_1 = 60.000;\ A_2 = -120.000;\ A_3 = 90.000;\ A_4 = 60.000;\ A_5 = -120.000;\ A_6 = 90.000$$

Here, we do not want to find the optimal solution to this problem but only to examine its convexity. As a first step, suppose that we choose the separate taxation option in all periods and set, in all loss periods, the total amount of the resulting losses as loss carry-backward. Mathematically, this means $y_t = 1\ for\ all\ t = 1, \dots, T$ and $x_t = x_{t\ max}\ for\ all\ t = 1, \dots, T$. The financial plan is shown below.

	A	B	C	D	E	F	G	H	I
1	**Notation**								
2									
3	i (interest rat	0,05							
4									
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**
6	**A_t** (cash flow after depreciation)		0	**60.000**	**-120.000**	**90.000**	**60.000**	**-120.000**	**90.000**
7	**B_t** (capital yield)		0	50.000	54.027	50.841	57.174	61.470	58.570
8	**AB_t** (total income)		0	110.000	-65.973	140.841	117.174	-58.530	148.570
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-60.000	0	0	-60.000	0
11	**I_t** (total taxable income)		0	60.000	0	15.583	60.000	0	16.206
12	**S_t (total tax payment)**		0	29.461	13.507	14.198	31.255	15.368	16.287
13	**IR_t (Recalculation of Taxable Income**		0	14.417	0	15.583	13.794	0	16.206
14	**SR_t (Recalculation of Tax Payment)**		0	13.700	13.507	14.198	15.343	15.368	16.287
15	**SRF_t (tax refund)**		0	0	15.761	0	0	15.912	0
16	**W_t** (end of period wealth)		**1.000.000**	1.080.539	1.016.820	1.143.463	1.229.382	1.171.397	1.303.680
17	**x_t** (loss carry-backward)		0	***0***	***-45.583***	***0***	***0***	***-46.206***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-74.417***	***0***	***0***	***-73.794***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-74.417***	***0***	***0***	***-73.794***	***0***

Figure 4-6 Financial plan of example

Now we let the two variables for loss carry-backward vary in their definition domains. We create the 3D surface plot of the objective function according to the visualization principles described by Thaller [Tha98]. The resulting 3D surface plot is given from two different perspectives below.

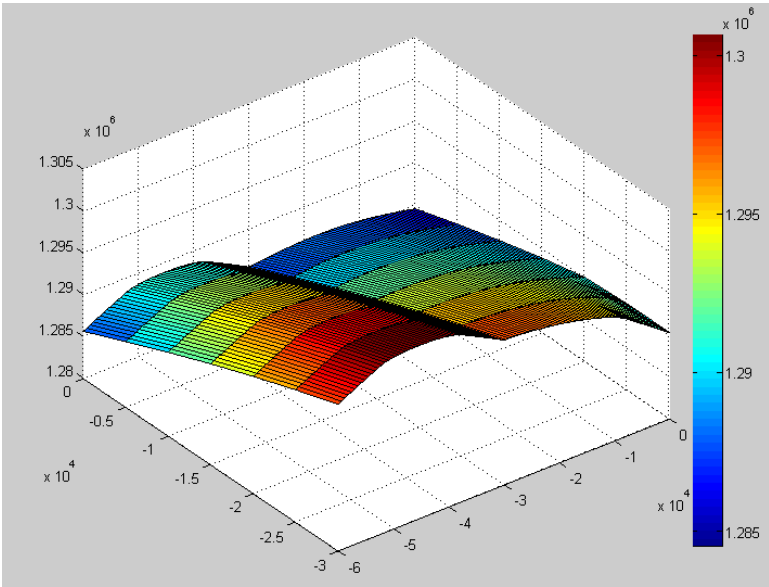


Figure 4-7 Proof of non-convexity (1)

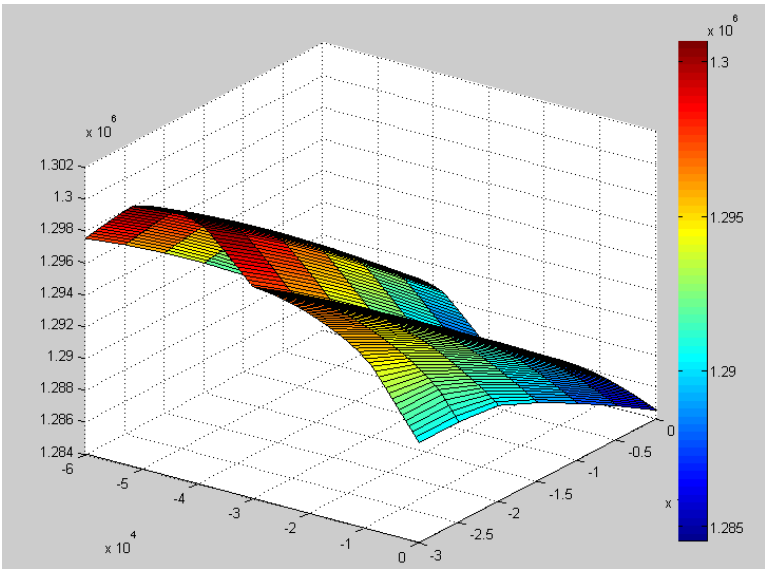


Figure 4-8 Proof of non-convexity (2)

We clearly see a proof of non-convexity. It is important to mention that this is only the case where we have fixed all binary variables. Furthermore, it is a single combination from a set of $2^6 = 64$ different combinations of $0/1$ variables. For each new combination, we expect a different plot for the objective function.

As a conclusion, we can prove that our problem is non-linear, discontinuous and non-convex. We have now discussed the difficulty of solving the postulated problem by analytical approaches. As a result, our main solution approaches are based on heuristic methods instead of analytical methods. Therefore, in this chapter of seeking for the solution, “optimization” is often used referred to our

attempt to find a global optimum. To avoid confusion, we make it clear that the word "optimization" is never used in the sense that a global optimum will be found and can be proven to be one.

4.2 Algorithms for solving the problem

We now describe algorithms for solving the optimization problem, i.e., to pass from a basic feasible solution to an optimal solution. Here we search for the global optimum. The value of the global optimum is unique, whereas there might be several solutions that deliver the same value of the objective function. As we shall recognize, the fact is that constraints to be considered are reducing the freedom of choices since the range space of the constraint derivative determines a certain upper and lower bound to be fixed. For the sake of separating the effects of choices on tax options and choices on loss offset, we first propose a consecutive method of decision. This technique allows us to differentiate between distinctive effects of different types of choices.

Our way of proceeding is as follows:

- First, we apply an exact algorithm for finding the optimal solution.
- Second, by using the solutions from the exact algorithm, we evaluate the performance of the heuristic algorithm for a set of problem instances.

For any specific situation, determining the optimal way to combine the two strategies will likely require careful consideration. Fixing the combination of tax options, we try to strike a balance between the amounts of loss carry-backward and –forward such that the discounted marginal tax rates in all periods are as low as possible. By using information from the financial plan, we gain additional insight into the manner in which variable values change with respect to each other when the input data changes.

4.2.1 Exact Methods

The formulation of the problem has been given. We have formulated and explained the objective function that represents a quantitative measure of the "goodness" of our complex tax planning model, i.e., the final cumulative wealth. We can try using deterministic approaches to solve the 3-period-problems.

Algorithm 4-1 Combine *Exhaustive Search* with *Binary Search*

1. List all possible combinations of binary variables.
2. For each combination of binary variables, proceed with binary search to calculate the optimal value of loss carry-backward.

   The *Binary Search* algorithm:

   - Divide the search interval in half.
   - Check the midpoint by calculating its slope $f(x+\Delta)-f(x)$.

- If the slope is positive, then set the midpoint as the new left-side bound (lower bound), i.e., consider the right half and discard the left half.
- Else, set the midpoint as the new right-side bound (upper bound), i.e., consider the left half and discard the right half.
- Loop, until the remaining search interval is smaller than ε.

3. Compare the final wealth of all cases. Choose the combination of binary variables and the value of loss carry-backward that deliver the highest final wealth.

Algorithm 4-2 Combine *Exhaustive Search* with *Golden Ratio Search*

1. List all possible combinations of binary variables.
2. For each combination of binary variables, proceed with *Golden Ratio Search* to calculate the optimal value of loss carry-backward.

   The *Golden Ratio Search* algorithm:

   - Determine two points within the search interval according to the *Golden Ratio*.
   - Calculate the function values at these two points.
   - Narrow the search interval according to the maximization rule as described in chapter 2.
   - Loop, until the remaining search interval is smaller than ε.

3. Compare the final wealth of all cases. Choose the combination of binary variables and the value of loss carry-backward that deliver the highest final wealth.

4.2.2 Stochastic and Heuristic Methods

Deterministic approaches are characterized by the exact reproducibility of the steps taken by the algorithm, in the same problem and initial conditions. In contrast, stochastic approaches produce samples of prospective solutions in the search space iteratively. Therefore, it is almost impossible to reproduce exactly the same sequence of samples in two distinct experiments, even in the same initial conditions. Thus, the results generated by the algorithms proposed in this section vary each time we apply them.

First, we attempt to solve the problem by a greedy algorithm.

Algorithm 4-3 Greedy Algorithm

The word *greedy* in *greedy algorithm* reflects the origin of these approaches in decision theory, whereby the algorithms only consider the best state to be chosen with no consideration of future states. In practice, the approach of *Günstigerprüfung* used by tax consultants and authorities can be described as a *greedy algorithm*. The *Günstigerprüfung* is a German terminology that describes a so-called *Appeal Procedure* that has been explained in chapter 3.

The approach can be described as follows:

1. Begin with period 1, choose the taxation option that results into less tax payment in the same period. Thus, the wealth in 1st period is "maximized".
2. In the second period, again, choose the taxation option that results into less tax payment in the same period. Suppose the second period is a loss period, then the loss carry-backward will then be claimed so as to receive the maximal amount of tax refund in the second period. This way, the wealth in the 2nd period is also "maximized".
3. Repeat the same procedure in each future period, until the final wealth is calculated. The decisions in the final period should also lead to a "maximized" final wealth.

Unfortunately, we can easily show that the performance of such a simple greed is quite poor, even for small and easy problem structures. A maximized value of cumulated wealth in some periods is not sufficient, and in most cases even irrelevant, to achieve a maximal final wealth. For this reason, we do not need to proceed with a further discussion on the greedy approach anymore. Instead, we concentrate on the other heuristic and stochastic algorithms below.

Algorithm 4-4 *Monte Carlo Simulation*

It is emphasized that a *Monte Carlo Simulation* process is **not** considered to be a solution approach to solve the Loss Offset Problem, neither for the simple 3-period case nor the complicated multi-period case. The intention is simply to test and verify the other algorithms by showing that an optimal solution delivered by MCS is never better than the solutions found by our exact and heuristic approaches.

1. Define $\varepsilon > 0$ with $\varepsilon \rightarrow 0$
2. Define N as number of random sampling
3. For $i = 1$ to N
4. For each period t, generate a random variable $y_t \in [0,1]$
   a. If $y_t \leq 0{,}5 - \varepsilon \quad \rightarrow \; y_t = 0$
   b. If $y_t \geq 0{,}5 + \varepsilon \quad \rightarrow \; y_t = 1$
   c. If $y_t = 0{,}5 \qquad \rightarrow$ Repeat Step 2

   Thus we receive a Laplace probability both possibility (0 und 1) such that

$$\mathrm{P}(0) = \mathrm{P}(1) = \frac{1}{2}$$

   d. Generate $x_t \in [x_{t\,max}; 0]$
5. Evaluate the objective function $W_{T\,max} := W_{T1}$
   a. If $W_{T\,i} \leq W_{T\,max}$ then discard $W_{T\,i}$
   b. If $W_{T\,i} > W_{T\,max}$ then set $W_{T\,max} := W_{Ti}$. Save the result of $W_{T\,max}, y_t, x_t$
6. Go to step 3

In the following we present the solution approaches with *PSO* and *SA*. Due to the fact that the original algorithms must be customized for an appropriate application to the Loss Offset Problem, our algorithms become quite large.

Algorithm 4-5 *Particle Swarm Optimization*

Step 1: Parameters setting.

p_n: number of particles

$itende$: number of iteration steps to be carried out

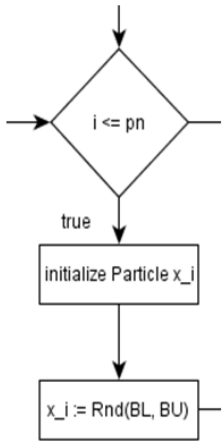
ω: factor of influence of current velocity

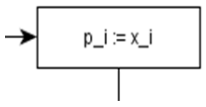
φ_p: factor of influence of best swarm position

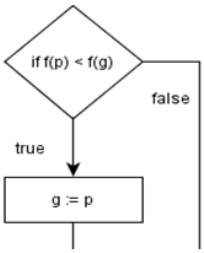
φ_g: factor of influence of best found position of particle

Step 2: Initializing process.

For each of the particles, assign a random value for each of the variables: taxation options, loss carry-backward, and velocity. These values must be set between the lower and upper bound.

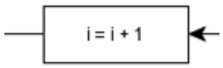






The first assignment is also the best assignment of each particle. Test, if this position is better than the swarm position. If this is true, set the swarm position to be this position.

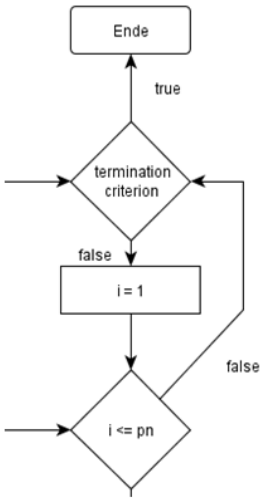
Repeat this step for all particles.



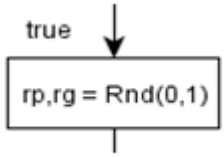
End of step 2.

Step 3: Iteration of the *PSO*.

As long as the termination criterion is not met, proceed with the iteration. In this iteration, each particle receives a new position.

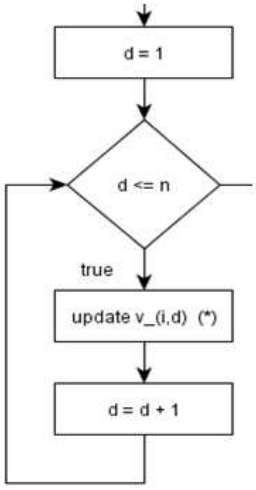


First, two random variables between 0 and 1 creates a greater variance in the movement of particles.



Then, all of the velocities are updated according to the given function. A new velocity is calculated as the sum of:

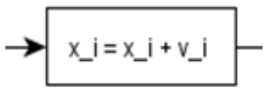
- The old velocity is multiplied with ω.
- The difference between current position of particle and best found position of particle is multiplied with φ_p and the random value r_p.
- The difference between current position of particle and swarm position is multiplied with φ_g and the random value r_g.
-



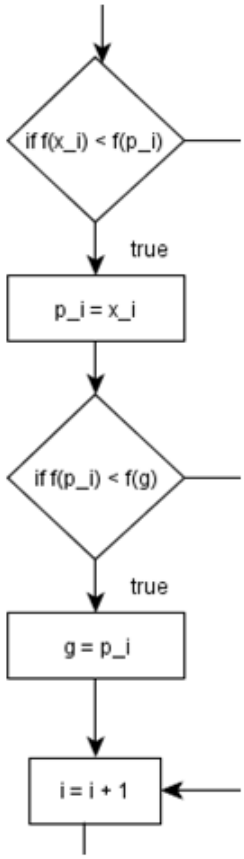
This applies for all variables for taxation options and loss carry-backwards.

$$\mathbf{v}_{i,d} \leftarrow \omega\, \mathbf{v}_{i,d} + \varphi_p\, r_p\, (\mathbf{p}_{i,d} - \mathbf{x}_{i,d}) + \varphi_g\, r_g\, (\mathbf{g}_d - \mathbf{x}_{i,d})$$

Now, the new values for all taxation options and loss carry-backwards can be calculated. If a value exceeds its bound, it will be set to equal the bound.



The final step is to test if the new position of each particle is better than its best found position. If true, then set the new position to be the best found position. Furthermore, if a new position is better than the swarm position, then update the swarm position also. This step is repeated on all particles.



The complete flowcharts can be found in Appendix B.

Algorithm 4-6 *Simulated Annealing*

Input:

tv: Taxation option variable

lc: Loss carry-backward variable

lcu: Loss carry-backward upper bound variable

ew: End wealth (final wealth)

Step 1: Initializing the specific parameters.

r_T: Temperature reduction coefficient

N_T: The number of times a "solution vector component" needs to be updated before a temperature reduction step

T: Initial temperature

ϵ: Max difference allowed between the end wealth of two solution vectors

N_ϵ: Number of wealth, each stored at a temperature decrement, to be compared during termination check.

$tvsize = UBOUND(tv)$: Highest index in tv array. May be -1 if array is empty

$lcsize = UBOUND(lc)$: Highest index in lc array. May be -1 if array is empty

$n = tvsize + lcsize + 1$: Size of the solution vector

$k = Neps$: Counter used during termination check

Step 2: Creating arrays.

The following solution vectors are divided into two components, the first component contains values of the consecutive taxation variables and the second part contains values of the consecutive loss carryback variables. In case either tv or lc vector is empty the corresponding component in the solution vector would not exist.

$x(n)$: Current solution vector with $n + 1$ components

$x_N(n)$: Neighboring solution vector with $n + 1$ components

$x_{opt}(n)$: Current best solution vector with $n + 1$ components

$f * (N_\epsilon - 1)$: Vector contains end wealth of current solution vector saved during consecutive temperature reductions. Initially its size is $N_\epsilon - 1$.

Step 3: Initializing array.

The current solution vector is first randomly initialized and its values are written into the loss allocation model. The taxation variables in this vector are assigned a random value between 0 and 1. However this value is rounded to the nearest integer when writing into the model. The loss carry-backward components of the solution vector are assigned a random value between their upper bound and 0. The algorithm initializes the loss carry-backward of the earliest period first and then the remaining loss carry-backwards of the later periods, if any exists. The algorithm makes an assumption that the loss carryback upper bound for a particular period is dependent only on the loss carry back values of the previous periods, if they exist.

$$\begin{array}{ll} x(i) = RND(0,1) & i = 0 \ldots tvsize \\ tv(i).value = CEIL(x(i)) & i = 0 \ldots tvsize \\ x(i) = RND(lco(i - tvsize - 1).value, 0) & i = tvsize + 1 \ldots n \\ lc(i - tvsize - 1).value = x(i) & i = tvsize + 1 \ldots n \end{array}$$

The end wealth belonging to the current solution vector is then stored into the variable $currentWealth$.

$$currentWealth = ew.value$$

Initially neighboring solution vector and current best solution vector are same as current solution vector and their wealth is the same as current end wealth.

$$
\begin{aligned}
x_{opt}(i) &= x(i) && i=0\ldots n \\
x_N(i) &= x(i) && i=0\ldots n \\
optimalWealth &= currentWealth \\
neighbourWealth &= currentWealth
\end{aligned}
$$

At the beginning, $f*$ vector is filled with the value of the variable $currentWealth$.

$$f*(0) = f*(1) = \cdots = f*(N_S - 1) = currentWealth$$

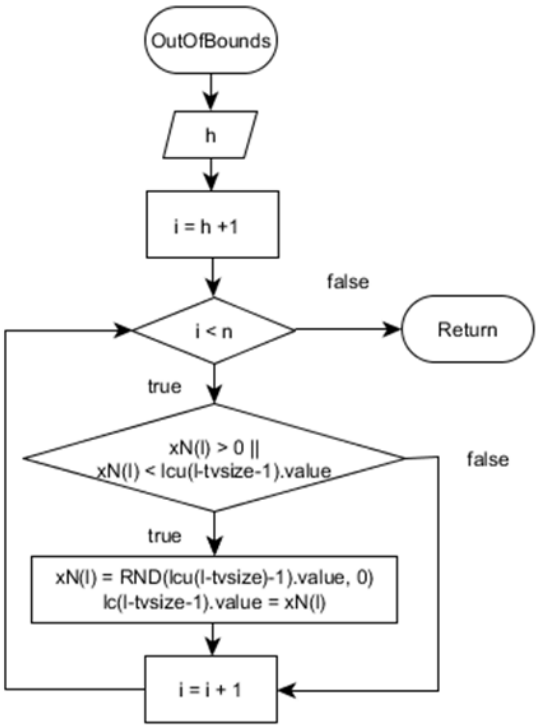
Step 4: Iteration.

The following steps are repeated N_T times.

For each index h in solution vector following steps are repeated:

1. A new neighbor is generated by changing the component h of the current solution vector. If index h represents a taxation component, i.e., $h \leq tvsize$ then the hth component is assigned a new random value between 0 and 1. In the other case if index h represents a loss carry-backward component, i.e., $h > tvsize$ the new neighbor has a different random loss carry-backward value. These changes to the neighbor are written to the loss allocation model. In order to ensure that the new neighbor is valid and within the feasible solution region a method "OutOfBounds" is invoked to a make an invalid neighboring solution vector valid. The "Out-of-bounds" method simply assigns the out of bound loss carryback components in the neighboring solution vector a random value within its limits. In case a taxation component is changed all the existing loss carryback values are verified. In case a loss carryback component is changed all the later period's loss carryback values are verified by OutOfBounds. If this method updates the neighboring vector, it writes these changes in the loss allocation model.

$$
\begin{aligned}
&x_N(h) = RND(0,1) && if\ h \leq tvsize \\
&x_N(h) = RND(lcu(h - tvsize - 1).value, 0) && otherwise \\
&tv(h).value = CINT(x_N(h)) && if\ h \leq tvsize \\
&lc(h - tvsize - 1).value = x_N(h) && otherwise \\
&OutOfBounds(tvsize, lcu, lc, tv) && if\ h \leq tvsize \\
&OutOfBounds(h, lcu, lc, tv) && otherwise
\end{aligned}
$$



The neighbor wealth is then read out from the loss allocation model and stored in $neighborWealth$ variable.

$$neighborWealth = ew.value$$

2. Next the probability of accepting the new neighbor is computed. If the neighbor solution vector has a greater end wealth compared to the current solution vector the neighbor is accepted with probability 1. In the other case the neighbor is accepted with probability computed using the Metropolis criterion.

$$\begin{aligned} p &= 1 && \text{if } neighbourWealth > currentWealth \\ p &= e^{((neighbourWealth - currentWealth)/T)} && otherwise \end{aligned}$$

3. If the neighbor is accepted i.e., $p > RND(0;1)$ the current solution vector is set as the neighbor solution vector. And its end wealth is also updated.

$$\begin{aligned} x(i) &= x_N(i) && i = 0 \ldots n \\ currentWealth &= neighbourWealth \end{aligned}$$

In case the neighbor has a higher end wealth compared to the current best solution vector, then the current best solution vector and its end wealth are set to those of the neighbor.

4. If the neighbor was rejected the neighbor solution vector is reset back to the current solution vector. The loss allocation model is also updated with the contents of the current solution vector.

$$\begin{aligned} x_N(i) &= x(i) && i = 0 \ldots n \\ tv(i).value &= CEIL(x(i)) && i = 0 \ldots tvsize \\ lc(i - tvsize - 1).value &= x(i) && i = tvsize + 1 \ldots n \end{aligned}$$

Step 5: Reducing temperature.

$$T = r_T \cdot T$$

Vector size of $f*$ is incremented by one. This additional space is assigned the end wealth of the current solution vector.

$$\begin{gathered} Redim\ f*(k) \\ f*(k) = currentWealth \end{gathered}$$

Step 6: Termination.

To check for termination the following condition is verified:

$$|f*(k) - f*(k-i)| \leq \epsilon \quad i = 1 \ldots N_\epsilon \ \wedge \ |f*(k) - optimalWealth| \leq \epsilon$$

This means the difference between the current end wealth, the end wealth stored at the previous $Neps$ temperature reduction steps and the end wealth belonging to the current best solution vector is checked to be within ϵ.

In case the condition is not satisfied the current solution vector and neighboring solution vector are set to current best solution vector. Their wealth is also updated accordingly and variable k is incremented by 1.

$$\begin{gathered} x(i) = x_{opt}(i) \qquad i = 0 \ldots n \\ x_N(i) = x_{opt}(i) \qquad i = 0 \ldots n \\ currentWealth = optimalWealth \\ neighbourWealth = optimalWealth \\ k = k + 1 \end{gathered}$$

In case the condition is satisfied the current best solution vector is written into the loss allocation model. We have the output.

$$\begin{gathered} tv(i).value = CEIL(xopt(i))\ if\ i \leq tvsize \\ lc(i - tvsize - 1).value = xopt(i)\ otherwise \end{gathered}$$

The complete flowcharts can be found in Appendix B.

As an aside: all of the proposed algorithms used for the complex model can naturally be applied to solve the simple one. Nevertheless, the opposite is not true due to the rising complexity when there are more than one loss period. Then, the problem structure becomes fundamentally different.

4.3 Numerical examples

4.3.1 Examples for the 3-period-problem

In this part we provide some numerical examples, first based on the $T = 3$ problem, to demonstrate the search process. Let us continue with Example 4-1. We have analyzed the complexity of this problem and can now solve it using the proposed algorithms.

For this example, we apply *Exhaustive Search* together with *Binary Search* and find the following results. We see in row 10 that the highest final wealth can be achieved with all $y_t = 1$ and a loss carry-backward $x_2 = -150.390$. The same applies for *Golden Ratio Search.*

As described in previous sections, the solution can be found with several search algorithms. Below we present the results.

As for the termination criterion of *Binary Search*, we choose $\varepsilon = 10$ and for calculating the slope $f(x + \Delta) - f(x)$, we set $\Delta = 10$.

Solution with *Binary Search*:

	A	B	C	D	E
1					
2	y_1	y_2	y_3	x_2	W_3
3	0	0	0	-154.746	244.945
4	0	0	1	-156.220	245.830
5	0	1	0	-154.688	246.793
6	0	1	1	-155.469	247.654
7	1	0	0	-149.554	245.790
8	1	0	1	-150.660	246.660
9	1	1	0	-150.000	247.645
10	1	1	1	-150.390	248.491

Figure 4-9 Solution with *BS* for Example 4-1

As for the termination criterion of *Golden Ratio Search*, we choose $\varepsilon = 10$. There is no need to calculate any slope because the algorithm compare to concrete values with each other.

Solution with *Golden Ratio Search*:

The best value can be found with $x_2 = -150.390$, resulting into a corresponding value of $W_3 = 248.491$. This result is identical to the one found by *Binary Search*.

	A	B	C	D	E
1					
2	y_1	y_2	y_3	x_2	W_3
3	0	0	0	-155.571	244.950
4	0	0	1	-155.795	245.831
5	0	1	0	-155.418	246.798
6	0	1	1	-156.039	247.655
7	1	0	0	-149.988	245.790
8	1	0	1	-151.160	246.660
9	1	1	0	-150.156	247.645
10	1	1	1	-151.161	248.491

Figure 4-10 Solution with *GRS* for Example 4-1

The approaches *SA* and *PSO* deliver the same optimal results with $W_3 = 248.491$.

For the *PSO* algorithm, we use the following configuration:

Number of particles $p_n = 15$

Number of iteration steps to be carried out $itende = 10$

Factor of influence of current velocity $\omega = 0{,}85$

Factor of influence of best swarm position $\varphi_p = 0{,}3$

Factor of influence of best found position of particle $\varphi_g = 0{,}5$

Solution with *PSO*:

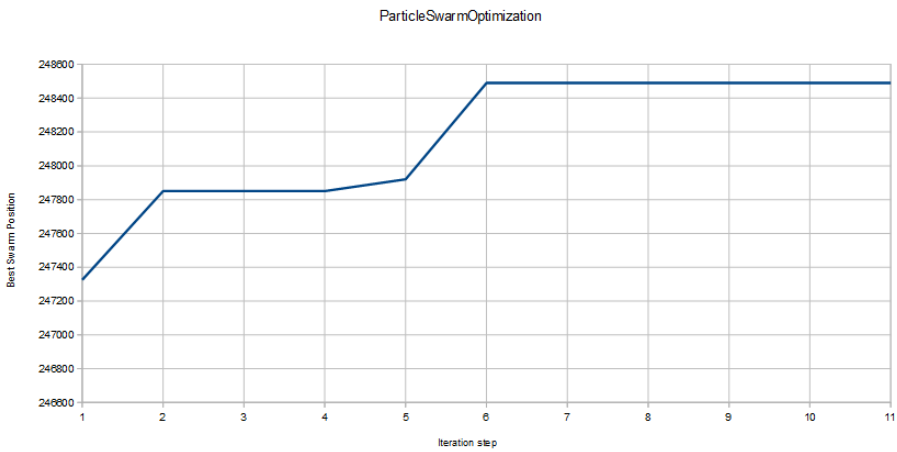


Figure 4-11 Solution process with *PSO* Example 4-1

For the *SA* algorithm, we use the following configuration:

Temperature reduction coefficient $r_T = 0{,}85$

The number of times a "solution vector component" needs to be updated before a temperature reduction step $N_T = 10$

Initial temperature $T = 100$

Max difference allowed between the final wealth of two solution vectors $\epsilon = 10$

Number of wealth, each stored at a temperature decrement, to be compared during termination check $N_\epsilon = 4$

Solution with *SA*:

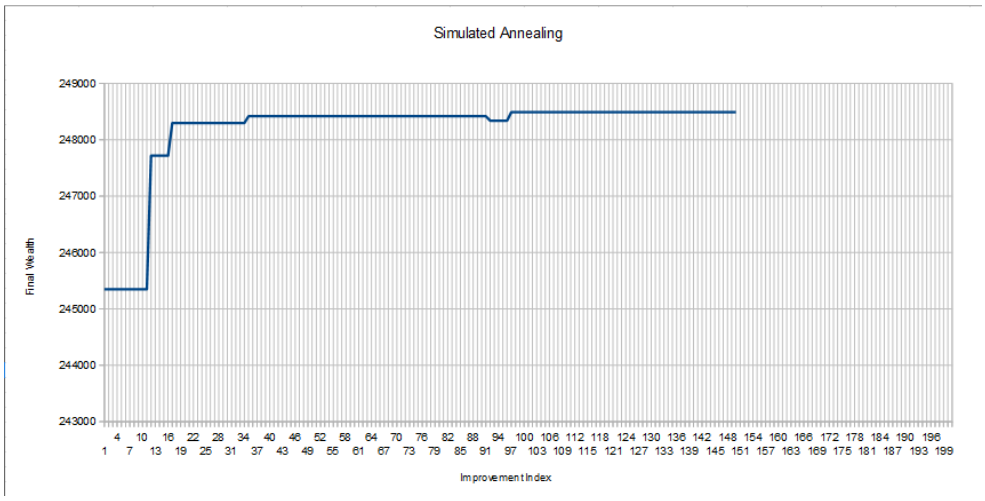


Figure 4-12 Solution process with *SA* Example 4-1

The financial plan with optimal taxation options, loss carry-backward and the resulting maximal final wealth is displayed in the next figure.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**200.000**	**-200.000**	**200.000**
7	**B_t** (capital yield)		0	5.000	11.400	4.984
8	**AB_t** (total income)		0	205.000	-188.600	204.984
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-200.000	0
11	**I_t** (total taxable income)		0	200.000	0	150.390
12	**S_t (total tax payment)**		0	77.011	2.850	56.171
13	**IR_t (Recalculation of Taxable Income**		0	49.610	0	150.390
14	**SR_t (Recalculation of Tax Payment)**		0	13.872	2.850	56.171
15	**SRF_t (tax refund)**		0	0	63.139	0
16	**W_t** (end of period wealth)		**100.000**	227.989	99.678	248.491
17	**x_t** (loss carry-backward)		0	***0***	***-150.390***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-49.610***	***0***

Figure 4-13 Optimal financial plan of Example 4-1

In the following is a simple example for demonstrating another logical aspect of the model. Using the same data as in Example 4-1, we construct a new scenario. Now suppose that the manager expects a high income in the third period, which is, logically, subject to a higher tax rate and tax payment.

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 100.000$$

$$A_1 = 200.000$$

$$A_2 = -200.000$$

$$A_3 = 600.000$$

The following financial plan shows the optimal decisions on taxation options and loss carry-backward found by our solution approaches.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**200.000**	**-200.000**	**600.000**
7	**B_t** (capital yield)		0	5.000	11.400	1.827
8	**AB_t** (total income)		0	205.000	-188.600	601.827
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-200.000	0
11	**I_t** (total taxable income)		0	200.000	0	400.000
12	**S_t (total tax payment)**		0	77.011	2.850	164.696
13	**IR_t (Recalculation of Taxable Income**		0	200.000	0	400.000
14	**SR_t (Recalculation of Tax Payment)**		0	77.011	2.850	164.696
15	**SRF_t (tax refund)**		0	0	0	0
16	**W_t** (end of period wealth)		**100.000**	227.989	36.539	473.670
17	**x_t** (loss carry-backward)		0	***0***	***0***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-200.000***	***0***

Figure 4-14 Optimal financial plan (2) of Example 4-1

Interpretation of the solution:

Explanation of the choice on taxation options is analog to the last example. Now, all of the resulting loss amount should be used as loss carry-forward in order to get the maximal final wealth. The reason is very straightforward. In the first period, the maximum marginal tax rate is 42 % according to the tax function in §32a EStG. In the first period, the maximum marginal tax rate is 45 %. Thus, for every monetary unit, let say EUR 1 of loss carried backward, we can save at maximum 42 %, i.e., EUR 0,42 (in form of tax refund in the second period). However, because the loss carry-forward will then be reduced by EUR 1, we must pay more EUR 0,45 of tax in the third period. Although the amount of EUR 0,42 in the second period can be reinvested at an after tax interest rate, it cannot compensate the additional tax payment of EUR 0,45 in the third period. This leads to a suboptimal final result. In other words, for every monetary unit of loss carried forward, we can reduce more tax payment in the third period than the amount of tax refund we can received in case of using it as a loss carry-

backward. This can be confirmed by the plot of the objective function. In the following figure, we fix the binary variables so that final wealth is only dependent on the amount of loss carry-backward.

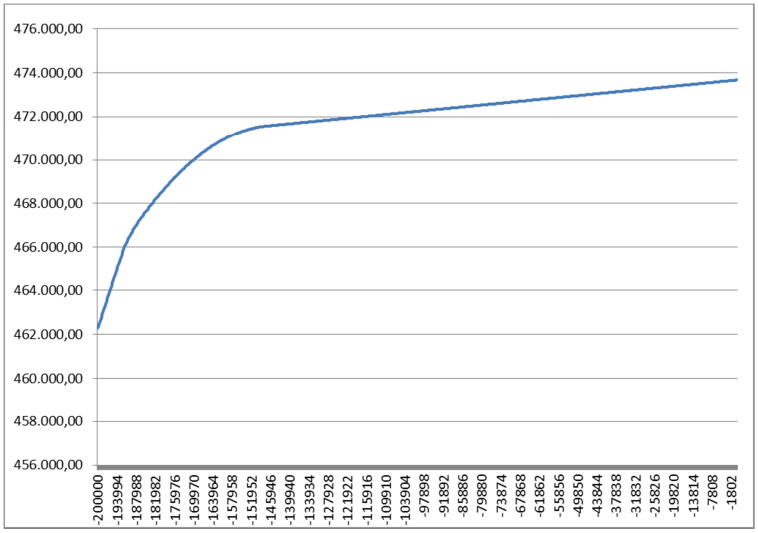


Figure 4-15 Plot of the objective function as a function of x_t for example

For a further illustration of our explanation, consider the next figure. We set loss carry-backward $x_2 = -1{,}000$ and received a worsened W_3. We spare a more detailed discussion of the numerical values in the financial plan.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**200.000**	**-200.000**	**600.000**
7	**B_t** (capital yield)		0	5.000	11.400	1.848
8	**AB_t** (total income)		0	205.000	-188.600	601.848
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-200.000	0
11	**I_t** (total taxable income)		0	200.000	0	401.000
12	**S_t (total tax payment)**		0	77.011	2.850	165.151
13	**IR_t (Recalculation of Taxable Income**		0	199.000	0	401.000
14	**SR_t (Recalculation of Tax Payment)**		0	76.591	2.850	165.151
15	**SRF_t (tax refund)**		0	0	420	0
16	**W_t** (end of period wealth)		**100.000**	227.989	36.959	473.656
17	**x_t** (loss carry-backward)		0	***0***	***-1.000***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-199.000***	***0***

Figure 4-16 Suboptimal financial plan of Example 4-1

Considering this example a, we see that a major risk in a loss carry-forward is that a high-earning year never comes in time to take advantage of the earlier loss. If claiming too much loss carry-forward, we may have a disadvantage of not utilizing the loss carry-backward, a possibility that is limited in time.

Example 4-3

Above, we have presented examples for demonstrating some simple logics of the 3-period-model. Now we should assume some arbitrary values in a new numerical example. The benefit of our optimization model in a more difficult decision making situation should become clear.

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 500.000$$

$$A_1 = 20.000$$

$$A_2 = -90.000$$

$$A_3 = 50.000$$

At this juncture we emphasize that the input data for cash flows may seem to be unrealistic. This is due to the fact the sum of all cash flows as given above is negative and represents an unprofitable investment no matter how good the planning strategy is. However, we have the following argumentation for making use such data:

- On one side, let us assume that for some reasons the tax planer would yet like to consider only three periods.
- On the other side, remember that the data above represents cash flows after depreciation. Suppose that the amount of depreciation is calculated to be EUR 100.000 per year. Cash flows before depreciation are thus as follows.

t	0	1	2	3
Cash flow before depreciation		120.000	10.000	150.000
Depreciation		100.000	100.000	100.000
A_t (cash flow after depreciation)		**20.000**	**-90.000**	**50.000**

Figure 4-17 Explanation of the depreciation effect

Therefore, we have seen that the scenario is justified. Solution to the problem can be found with approaches as presented in the previous example.

Solution with *Binary Search*:

	A	B	C	D	E
1					
2	y_1	y_2	y_3	x_2	W_3
3	0	0	0	-18.458	545.821
4	0	0	1	-30.235	546.608
5	0	1	0	-31.641	546.363
6	0	1	1	-39.375	542.846
7	1	0	0	-5.938	546.454
8	1	0	1	-15.000	543.389
9	1	1	0	-15.000	542.297
10	1	1	1	-15.000	536.436

Figure 4-18 Solution with *BS* for Example 4-3

Solution with *Golden Ratio Search*:

	A	B	C	D	E
1					
2	y_1	y_2	y_3	x_2	W_3
3	0	0	0	-18.738	545.821
4	0	0	1	-30.911	546.608
5	0	1	0	-45.000	546.265
6	0	1	1	-45.000	542.846
7	1	0	0	-6.525	546.454
8	1	0	1	-20.000	543.389
9	1	1	0	-20.000	542.297
10	1	1	1	-20.000	536.436

Figure 4-19 Solution with *GRS* for Example 4-3

Solution found by *PSO*:

y_1	y_2	y_3	x_2	W_3
0	0	1	-30.225,00	546.608,02

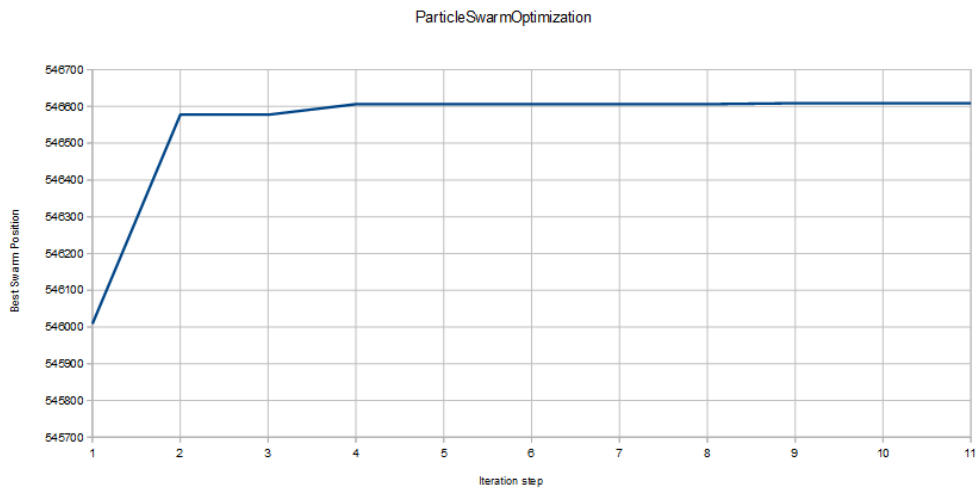


Figure 4-20 Solution with *PSO* for Example 4-3

Solution found by *SA*:

y_1	y_2	y_3	x_2	W_3
0	0	1	-30.361,64	546.608,34

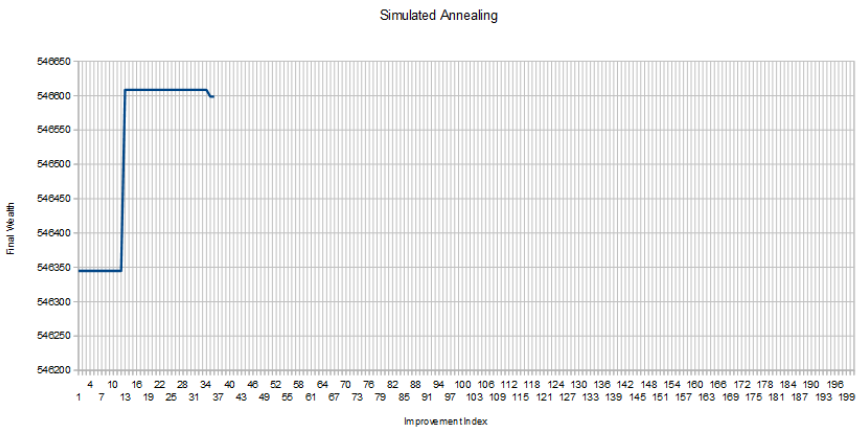


Figure 4-21 Solution with *SA* for Example 4-3

The financial plan with optimal solution is given below. Notice that the small deviation in cell F16 is purely caused by rounding up of values and therefore represents no error.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,05				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**20.000**	**-90.000**	**50.000**
7	**B_t** (capital yield)		0	25.000	26.710	24.022
8	**AB_t** (total income)		0	45.000	-63.290	74.022
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-45.000	0
11	**I_t** (total taxable income)		0	45.000	0	16.946
12	**S_t (total tax payment)**		0	10.803	0	7.838
13	**IR_t (Recalculation of Taxable Income**		0	14.764	0	16.946
14	**SR_t (Recalculation of Tax Payment)**		0	1.285	0	7.838
15	**SRF_t (tax refund)**		0	0	9.518	0
16	**W_t** (end of period wealth)		**500.000**	534.197	480.425	546.609
17	**x_t** (loss carry-backward)		0	***0***	***-30.236***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-33.054***	***0***

Figure 4-22 Optimal financial plan of Example 4-3

We consider the next scenario with a small change. Now let us assume an interest rate $i = 0{,}1$ instead of $i = 0{,}05$ as in the previous scenario. Changes in the taxation options can be observed.

Solution with *BS*:

	A	B	C	D	E
1					
2	y_1	y_2	y_3	x_2	W_3
3	0	0	0	-22.496	595.985
4	0	0	1	-30.658	604.632
5	0	1	0	-31.924	604.074
6	0	1	1	-56.875	607.480
7	1	0	0	-5	602.266
8	1	0	1	-4.375	608.003
9	1	1	0	-6.640	607.024
10	1	1	1	-15.000	597.299

Figure 4-23 Solution with *BS* for Example 4-3 (2)

Solution with *GRS*:

	A	B	C	D	E
1					
2	y_1	y_2	y_3	x_2	W_3
3	0	0	0	-22.912	595.985
4	0	0	1	-30.740	604.632
5	0	1	0	-33.049	604.074
6	0	1	1	-57.376	607.480
7	1	0	0	-62	602.262
8	1	0	1	-4.721	608.003
9	1	1	0	-12.361	607.008
10	1	1	1	-20.000	597.299

Figure 4-24 Solution with *GRS* for Example 4-3 (2)

The optimal solutions found with *PSO* and *SA* show a small difference.

Solution with *PSO*:

y_1	y_2	y_3	x_2	W_3
1	0	1	-4.028,27	608.003,34

Solution with *SA*:

y_1	y_2	y_3	x_2	W_3
1	0	1	-3.689,08	608.001,56

The optimal financial plan is given with the result found by *BS*.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,1				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**20.000**	**-90.000**	**50.000**
7	**B_t** (capital yield)		0	50.000	55.487	52.149
8	**AB_t** (total income)		0	70.000	-34.513	102.149
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***0***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-20.000	0
11	**I_t** (total taxable income)		0	20.000	0	19.862
12	**S_t (total tax payment)**		0	15.134	0	15.634
13	**IR_t (Recalculation of Taxable Income**		0	15.625	0	19.862
14	**SR_t (Recalculation of Tax Payment)**		0	13.998	0	15.634
15	**SRF_t (tax refund)**		0	0	1.136	0
16	**W_t** (end of period wealth)		**500.000**	554.866	521.489	608.003
17	**x_t** (loss carry-backward)		0	***0***	***-4.375***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-30.138***	***0***

Figure 4-25 Optimal financial plan of Example 4-3 (2)

The final scenario that we want to examine in the following is when the interest rate is assumed to be relatively low, $i = 0{,}03$ instead of $0{,}05$. In this case we see that the joint taxation is optimal.

Solution with *BS*:

	A	B	C	D	E
1					
2	**y_1**	**y_2**	**y_3**	**x_2**	**W_3**
3	0	0	0	-22.968	523.221
4	0	0	1	-30.625	521.203
5	0	1	0	-30.625	520.582
6	0	1	1	-30.625	517.155
7	1	0	0	-15.000	520.822
8	1	0	1	-15.000	517.393
9	1	1	0	-15.000	516.738
10	1	1	1	-15.000	513.340

Figure 4-26 Solution with *BS* for Example 4-3 (3)

Solution with *GRS*:

	A	B	C	D	E
1					
2	**y_1**	**y_2**	**y_3**	**x_2**	**W_3**
3	0	0	0	-22.946	523.222
4	0	0	1	-35.000	521.203
5	0	1	0	-35.000	520.582
6	0	1	1	-35.000	517.155
7	1	0	0	-20.000	520.822
8	1	0	1	-20.000	517.393
9	1	1	0	-20.000	516.738
10	1	1	1	-20.000	513.340

Figure 4-27 Solution with *GRS* for Example 4-3 (3)

Solution with *PSO*:

y_1	y_2	y_3	x_2	W_3
0	0	0	-22.743,31	523.222,64

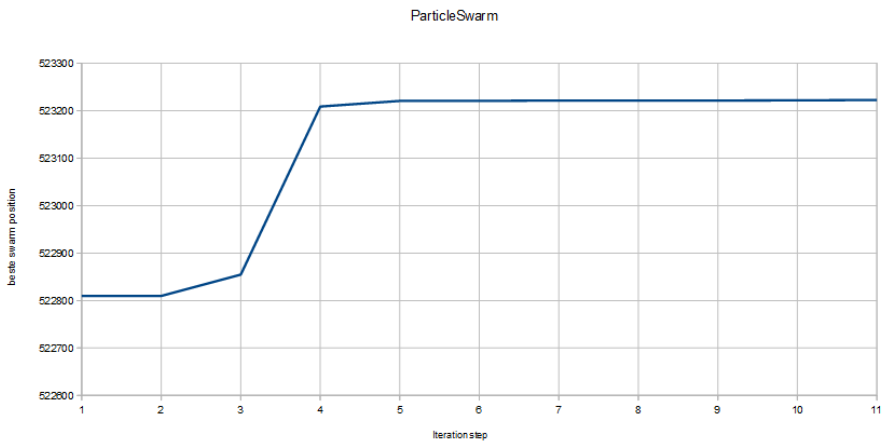


Figure 4-28 Solution with *PSO* for Example 4-3 (3)

Solution with *SA*:

y_1	y_2	y_3	x_2	W_3
0	0	0	-22.958,77	523.221,98

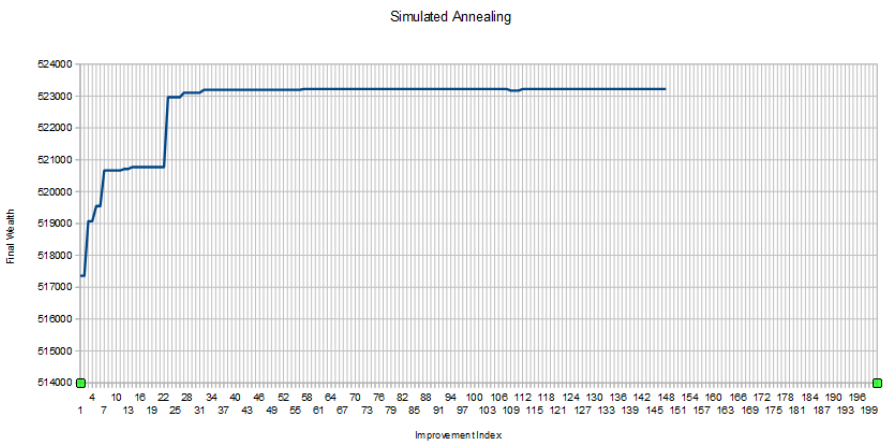


Figure 4-29 Solution with *SA* for Example 4-3 (3)

The optimal financial plan with the representative result found by *GRS* is shown below.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,03				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**20.000**	**-90.000**	**50.000**
7	**B_t** (capital yield)		0	15.000	15.835	13.806
8	**AB_t** (total income)		0	35.000	-74.165	63.806
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***0***
10	**x_t max** (maximal loss carry backward)		0	0	-35.000	0
11	**I_t** (total taxable income)		0	35.000	0	12.587
12	**S_t (total tax payment)**		0	7.192	0	767
13	**IR_t (Recalculation of Taxable Income**		0	12.054	0	12.587
14	**SR_t (Recalculation of Tax Payment)**		0	651	0	767
15	**SRF_t (tax refund)**		0	0	6.541	0
16	**W_t** (end of period wealth)		**500.000**	527.808	460.184	523.222
17	**x_t** (loss carry-backward)		0	***0***	***-22.946***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-51.219***	***0***

Figure 4-30 Optimal financial plan of Example 4-3 (3)

Plot of the objective function for all $y_t = 0$.

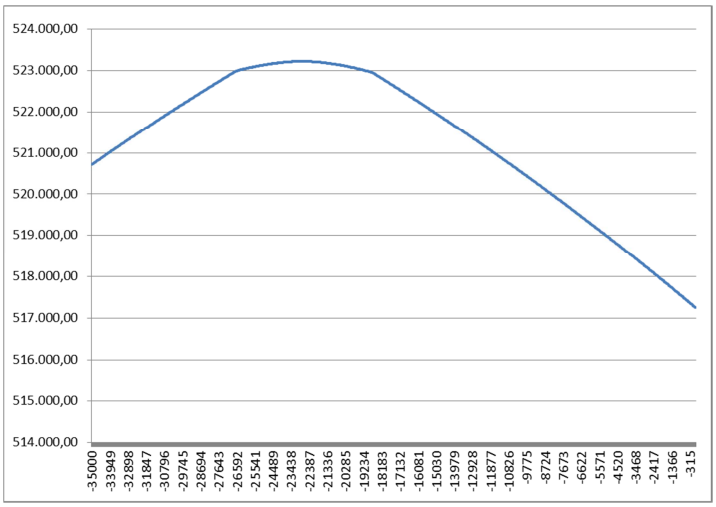


Figure 4-31 Plot of the objective function with $y_t = 0$

At this point it may be interesting for us to consider the case when all $y_t = 1$. The plot is shown in the figure below. We can see that for a large interval of loss carry-backward the value of final wealth remains indifferent.

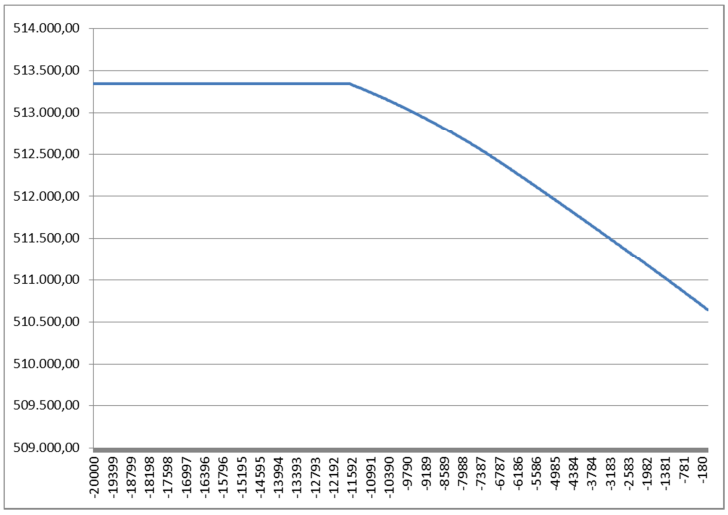


Figure 4-32 Plot of the objective function with $y_t = 1$

We can recognize a linear part of the function. This can be explained as follows. The maximal loss carry-backward possible is $x_{2\,max} = -20.000$. Nevertheless it is not necessary to carried such a high amount back, because the taxable income in the first period is only 20.000 (capital gains are taxed separately). In order to reduce this income to the exemption level of EUR 8.354, it is enough to carry EUR -11.646 of loss back. The illustration can be found in the figure below.

	A	B	C	D	E	F
1	**Notation**					
2						
3	i (interest rat	0,03				
4						
5	t		0	**1**	**2**	**3**
6	**A_t** (cash flow after depreciation)		0	**20.000**	**-90.000**	**50.000**
7	**B_t** (capital yield)		0	15.000	15.859	13.595
8	**AB_t** (total income)		0	35.000	-74.141	63.595
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-20.000	0
11	**I_t** (total taxable income)		0	20.000	0	0
12	**S_t (total tax payment)**		0	6.384	3.965	3.399
13	**IR_t (Recalculation of Taxable Income**		0	8.354	0	0
14	**SR_t (Recalculation of Tax Payment)**		0	3.750	3.965	3.399
15	**SRF_t (tax refund)**		0	0	2.634	0
16	**W_t** (end of period wealth)		**500.000**	528.616	453.144	513.341
17	**x_t** (loss carry-backward)		0	***0***	***-11.646***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-78.354***	***0***

Figure 4-33 Suboptimal financial plan of Example 4-3 (3)

4.3.2 Examples for the multi-period-problem

Solving the Multi-period problem is fundamentally much more difficult than in the case of having only 3 periods. Even after fixing the binary variables y_t, we still cannot always visualize the objective function graphically. With two variables x_t we have a three dimensional function which can be plotted as a 3-D surface, consequently with n x_t we have a polyhedron in $n + 1$ dimensional space.

Another difficulty stems from the fact that the upper bound of a loss carry-backward cannot be fixed during the optimization procedure, although it can be determined precisely. For a better understanding of this problem, consider the next example.

Example 4-4

Let the interest rate $i = 0{,}05$; the initial wealth W_0 and A_t be

$$W_0 = 1.000.000$$

$$A_1 = 100.000;\ A_2 = -100.000;\ A_3 = 100.000;\ A_4 = 100.000;\ A_5 = -100.000;\ A_6 = 100.000$$

As a first step, suppose that we choose the joint taxation option in all periods and set, in all loss periods, the total amount of the resulting losses as loss carry-backward. Mathematically, this means $y_t = 0\ for\ all\ t = 1, \dots, T$ and $x_t = x_{t\ max}\ for\ all\ t = 1, \dots, T$. The financial plan for this case is shown below.

	A	B	C	D	E	F	G	H	I
1	**Notation**								
2									
3	i (interest rat	0,05							
4									
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**
6	**A_t** (cash flow after depreciation)		0	**100.000**	**-100.000**	**100.000**	**100.000**	**-100.000**	**100.000**
7	**B_t** (capital yield)		0	50.000	54.762	53.451	58.313	63.316	62.252
8	**AB_t** (total income)		0	150.000	-45.238	153.451	158.313	-36.684	162.252
9	**y_t** (taxation variant: 0 or 1)		0	***0***	***0***	***0***	***0***	***0***	***0***
10	**x_t max** (maximal loss carry backward)		0	0	-45.238	0	0	-36.684	0
11	**I_t** (total taxable income)		0	150.000	0	153.451	158.313	0	162.252
12	**S_t (total tax payment)**		0	54.761	0	56.210	58.252	0	59.907
13	**IR_t (Recalculation of Taxable Income**		0	104.762	0	153.451	121.629	0	162.252
14	**SR_t (Recalculation of Tax Payment)**		0	35.761	0	56.210	42.845	0	59.907
15	**SRF_t (tax refund)**		0	0	19.000	0	0	15.407	0
16	**W_t** (end of period wealth)		**1.000.000**	1.095.239	1.069.001	1.166.242	1.266.302	1.245.025	1.347.371
17	**x_t** (loss carry-backward)		0	***0***	***-45.238***	***0***	***0***	***-36.684***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***0***	***0***	***0***	***0***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***0***	***0***	***0***	***0***	***0***

Figure 4-34 Suboptimal financial plan of Example 4-4, with $y_t = 0$

This is a suboptimal solution. We can find the optimal solution by the proposed heuristic approaches. However, before solving the problem we would like to discuss some of its features.

As mentioned before, when setting up the mathematical program to solve the problem we do not necessarily have a fixed value on the right-hand side of the inequalities. The maximal allowable loss carry-backward in any later loss period depends on two factors:

- the taxation options in **all prior periods**, and
- the amount of loss carry-backward that has been claimed in **all prior loss periods**.

This represents very complicated and intertwined relationships of the variables in the function for determining the maximum loss carry-backward. In order to make the situation easier to be analyzed, let us assume a fixed combination of taxation options and examine only the interaction between the loss carry-backwards themselves.

For the scenario above where we chose joint taxation for every period, we have two resulting losses in the second period and the fifth period. We now plot the function that represents maximum loss carry-backward in the fifth period depending on the value of loss carry-backward claimed in the

second period. We see that the maximal value of loss carry-backward in period 5 is $x_{5\,max} = -36.684$ when $x_2 = x_{2\,max}$ and will become $x_{5\,max} = -36.713$ when $x_2 = 0$.

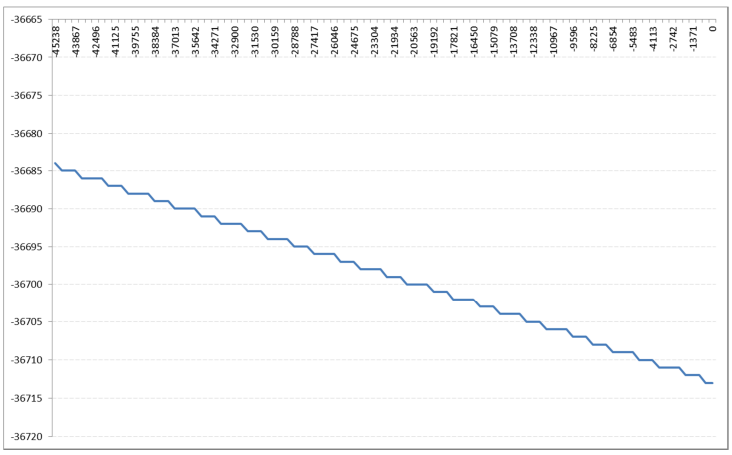


Figure 4-35 Plot of $x_{5\,max}$ as a function of x_2 for $x_{2\,max} \leq x_2 \leq 0$, with $y_t = 0$

In short, we do not have an $x - y$ plane with fixed intervals as shown in Figure 4-36 below.

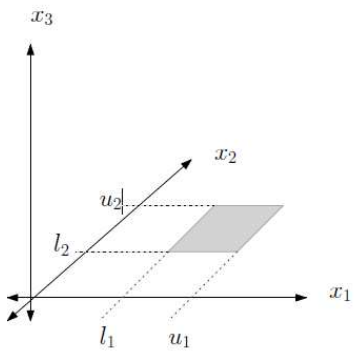


Figure 4-36 3D search space with fixed intervals

The function we have is a step-wise quasi-linear line and simply means that if we choose to claim none of the loss in period 2 as loss carry-backward, a higher loss carry-backward is possible in period 5, and vice versa. So, a decision should be made by the tax planner if it is favorable to set more or less loss carry-backward in the second period in order to claim the loss carry-backward in the fifth period, with the overall goal of maximizing final wealth. Moreover, we have to keep in mind that all taxation options have been fixed. Continuing with the above fixed combination of taxation options, we can plot the final wealth as a function of only the loss carry-backwards in period 2 and 5. We then have a three dimensional space that results into a 3D surface plot as in Example 4-2.

Now, we choose the separate taxation option in all periods, this means $y_t = 1\ for\ all\ t = 1, \dots, T$. In this case, $x_{5\,max} = -100.000$ is a constant and therefore does not depend on the chosen value of x_2 anymore. The plot is shown in Figure 4-37.

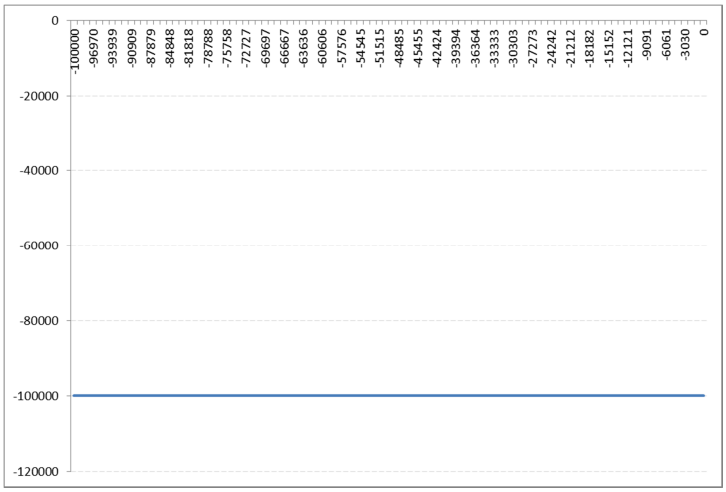


Figure 4-37 Plot of $x_{5\,max}$ as a function of x_2 for $x_{2\,max} \leq x_2 \leq 0$, with $y_t = 1$

<u>Optimal solution:</u>

In this part we apply the two heuristic algorithms to solve the problem. We again make use of the parameter configuration as presented previously.

Solution with *PSO*:

Figure 4-38 shows that after seven iterations the optimal result has been found. No improvement could be made.

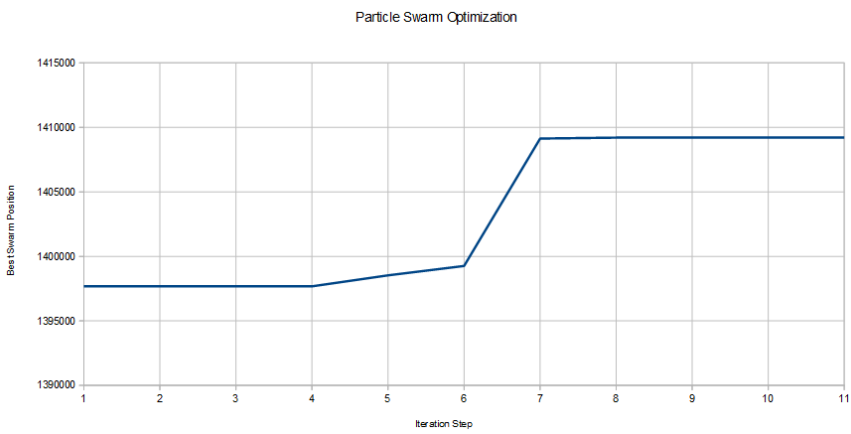


Figure 4-38 Solution process with *PSO* for Example 4-4

The optimal financial plan found by *PSO* is shown in Figure 4-39.

	A	B	C	D	E	F	G	H	I
1	**Notation**								
2									
3	i (interest rat	0,05							
4									
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**
6	**A_t** (cash flow after depreciation)		0	**100.000**	**-100.000**	**100.000**	**100.000**	**-100.000**	**100.000**
7	**B_t** (capital yield)		0	50.000	55.187	53.356	59.666	65.216	63.766
8	**AB_t** (total income)		0	150.000	-44.813	153.356	159.666	-34.784	163.766
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-100.000	0	0	-100.000	0
11	**I_t** (total taxable income)		0	100.000	0	52.498	100.000	0	52.760
12	**S_t (total tax payment)**		0	46.261	13.797	27.150	48.678	16.304	29.862
13	**IR_t (Recalculation of Taxable Income**		0	47.502	0	52.498	47.240	0	52.760
14	**SR_t (Recalculation of Tax Payment)**		0	24.278	13.797	27.150	26.591	16.304	29.862
15	**SRF_t (tax refund)**		0	0	21.983	0	0	22.086	0
16	**W_t** (end of period wealth)		**1.000.000**	1.103.739	1.067.112	1.193.319	1.304.307	1.275.305	1.409.210
17	**x_t** (loss carry-backward)		0	***0***	***-52.498***	***0***	***0***	***-52.760***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-47.502***	***0***	***0***	***-47.240***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-47.502***	***0***	***0***	***-47.240***	***0***

Figure 4-39 Optimal financial plan of Example 4-4 by *PSO*, with $y_t = 1$

Solution with *SA*

In Figure 4-40 we see that the near-optimal result has been found with approximately 15 first accepted neighbors.

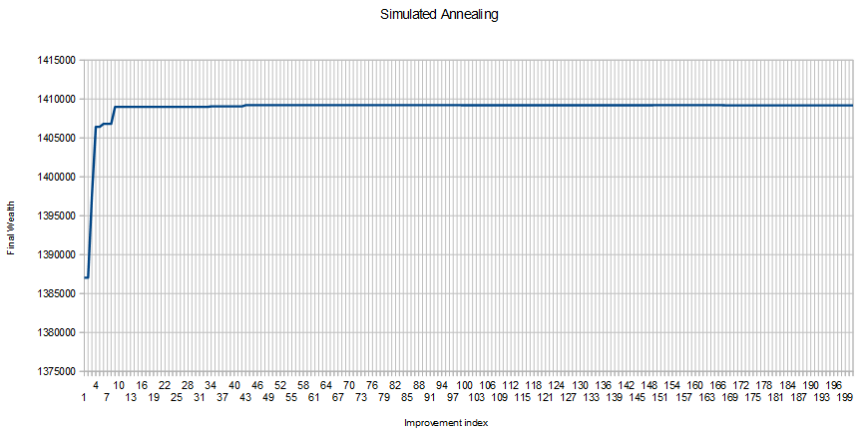


Figure 4-40 Solution process with *SA* for Example 4-4

The optimal financial plan found by *SA* is shown in Figure 4-41.

	A	B	C	D	E	F	G	H	I
1	**Notation**								
2									
3	i (interest rat	0,05							
4									
5	t		0	**1**	**2**	**3**	**4**	**5**	**6**
6	**A_t** (cash flow after depreciation)		0	**100.000**	**-100.000**	**100.000**	**100.000**	**-100.000**	**100.000**
7	**B_t** (capital yield)		0	50.000	55.187	53.312	59.666	65.216	63.735
8	**AB_t** (total income)		0	150.000	-44.813	153.312	159.666	-34.784	163.735
9	**y_t** (taxation variant: 0 or 1)		0	***1***	***1***	***1***	***1***	***1***	***1***
10	**x_t max** (maximal loss carry backward)		0	0	-100.000	0	0	-100.000	0
11	**I_t** (total taxable income)		0	100.000	0	50.309	100.000	0	51.236
12	**S_t (total tax payment)**		0	46.261	13.797	26.234	48.678	16.304	29.220
13	**IR_t (Recalculation of Taxable Income**		0	49.691	0	50.309	48.764	0	51.236
14	**SR_t (Recalculation of Tax Payment)**		0	25.155	13.797	26.234	27.197	16.304	29.220
15	**SRF_t (tax refund)**		0	0	21.106	0	0	21.480	0
16	**W_t** (end of period wealth)		**1.000.000**	1.103.739	1.066.236	1.193.314	1.304.302	1.274.695	1.409.209
17	**x_t** (loss carry-backward)		0	***0***	***-50.309***	***0***	***0***	***-51.236***	***0***
18	**xf_t** (loss carry-forward)		0	***0***	***-49.691***	***0***	***0***	***-48.764***	***0***
19	**xff_t (**Not-used Loss Carry-Forward)		0	***0***	***-49.691***	***0***	***0***	***-48.764***	***0***

Figure 4-41 Optimal financial plan of Example 4-4 by *SA*, with $y_t = 1$

For testing purpose we also apply the *Monte Carlo Simulation* as described in the algorithmic part. We conduct a simulation process consisted of 1.000 iterations. Figure 4-42 shows the first 10 best results. Compared with data the previous figures, it is clear that our results found by *PSO* and *SA* are both better.

	A	B	C	D	E	F
1		**x_2**			**x_5**	**W_6**
2		-49.011			-50.685	1.409.179
3		-52.262			-56.365	1.409.140
4		-47.642			-53.315	1.409.124
5		-56.827			-51.837	1.409.119
6		-52.528			-57.314	1.409.110
7		-49.069			-56.350	1.409.109
8		-56.565			-53.941	1.409.107
9		-53.279			-57.123	1.409.106
10		-46.320			-52.823	1.409.067

Figure 4-42 Result of Example 4-4 found by *Monte Carlo Simulation*

5 Some remarks on the models

Rather than just providing algorithms and methods for solving the problem, we also want to get some analytical insight in how to approach the solution in a more elegant way. This is, as already mentioned in the discussion on complexity of models, not an easy task. In the last chapter we have discussed the problem without making any consideration of the input data. Next, we make some observations about relationships between gains and losses that can help discover bounds for variables, therefore reduce computation time.

For the loss offset optimization problem, as described in chapter 4, input data plays a very important role when determining the complexity of a specific problem instance and thus the computation time of the algorithms to be applied. Out attempt is based on a simple fact. Sometimes, given the input data we can immediately recognize which taxation option should be chosen in a period. In other cases, we may be able to make a decision that all of the loss amount should be carried-back or carried-forward. The main reason for the existence of this chapter is therefore the presence of complex interrelationships of parameters in the model. In particular, we examine the amounts of ordinary income and capitals gains to get some useful hints.

5.1 Marginal tax rate vs. average tax rate

We differentiate between average income tax rate and marginal tax rate. The average income tax rate in for an individual taxpayer is the total tax payment divided by the taxable income. The marginal tax rate, differently, represents the ratio between the tax payment on every additional monetary unit of the total taxable income and that monetary unit itself. In Appendix C, we see that the average tax rate is always lower than the marginal tax rate. It is our purpose to reduce the marginal tax rate because it affects the last unit of additional income. In other words, we try to escape from the high tax brackets by reducing marginal tax rate.

We can make the following observations:

The income tax function is divided into five subdomains: a basic allowance and four income brackets with different mathematical characteristics. As we can see from the tax function, within each income brackets 2 and 3, the tax payment is calculated as a quadratic function of the taxable income. Therefore, two remarks are to be considered. First, the tax payment over-proportionally increases with rising taxable income. Second, the marginal tax rate linearly increases with taxable income.

For taxable income in the brackets 4 and 5, tax payment is a linear function of taxable income. There is no difficulty recognizing the two remarks again. First, the tax payment linearly increases with taxable income. Second, the marginal tax rate remains constant as taxable income continues to rise. The highest nominal marginal individual income tax rate is 45 %.

Recall that the tax function $f(.)$ formulated in the German tax law (EStG §32a) is a piece-wise nonlinear function.

$$f(I) = \begin{cases} 0 & \text{if } I \leq 8.354 \\ (974{,}58 \cdot a + 1.400) \cdot a & \text{if } 8.355 \leq I \leq 13.469 \\ (228{,}74 \cdot b + 2.397) \cdot b + 971 & \text{if } 13.470 \leq I \leq 52.881 \\ 0{,}42 \cdot I - 8.239 & \text{if } 52.882 \leq I \leq 250.730 \\ 0{,}45 \cdot I - 15.761 & \text{if } 250.731 \leq I \end{cases} \tag{5-1}$$

with

$$a = \frac{(I - 8.354)}{10.000} \tag{5-2}$$

and

$$b = \frac{(I - 13.469)}{10.000} \tag{5-3}$$

where a and b are auxiliary parameters for the second and third domains of the tax function. I represents the taxable income and can be seen as the input variable. We need to substitute a and b into $f(.)$ and compute the first derivative of the function $f(.)$ with respect to I.

A function need not have a derivative if it is not continuous. Here, our function is a piece-wise, discontinuous function. The derivative $\frac{\partial f}{\partial I}$ of function $f(.)$ at any chosen input value of I measures the rate of change of the function near that input value. In our economic context, this changing rate represents the marginal tax rate.

This is written according to the Leibniz's notation as:

$$\frac{\partial f}{\partial I} = \begin{cases} 0 & \text{if } I \leq 8.354 \\ 0{,}0000194916 \cdot I - 0{,}0228328264 & \text{if } 8.355 \leq I \leq 13.469 \\ 0{,}0000045748 \cdot I + 0{,}1780820188 & \text{if } 13.470 \leq I \leq 52.881 \\ 0{,}42 & \text{if } 52.882 \leq I \leq 250.730 \\ 0{,}45 & \text{if } 250.731 \leq I \end{cases} \tag{5-4}$$

Formula (5-4) permits us to calculate the marginal tax rate of any arbitrary amount of income and make comparison to the flat tax rate of 25 %. That is, the marginal tax rate of income may take on different values over different domains. For instance, if taxable income according to (3-16) is 15.000 EUR, we apply the third domain and get a marginal tax rate of 24,67 %.

At this juncture, we note that the value of I_t can vary depending on the choice y_t of taxation option that is 1 or 0. It can be either A_t only, or $A_t + B_t$. Furthermore, I_t also depends on the amount of loss offset that will be decided subject to some constraints. These intertwined relationships make it

extremely hard to compute the partial derivative of the objective function for given values of y_t and x_t. For this reason, we apply the finite difference method for approximating the derivative when using *Binary Search* to find the optimum of the objective function. For simplicity of calculation, we use the Lagrange's notation $f'(.)$ in later parts.

From the above function, an interesting issue emerges. We would like to know from which amount of taxable income, the joint taxation option is to be preferred over the separate one, and vice versa. Thus we need to solve the equation

$$f'(.) = g'(.)$$

For the second domain:

$$0{,}0000194916 \cdot I - 0{,}0228328264 = 0{,}25 \qquad \text{for } 8.355 \leq I \leq 13.469$$

We get $I = 13.997{,}47$ that is not in the defined interval and therefore not acceptable.

For the third domain:

$$0{,}0000045748 \cdot I + 0{,}1780820188 = 0{,}25 \qquad \text{for } 13.470 \leq I \leq 52.881$$

We get $I = 15.720{,}46$.

For simplicity we round the value up to EUR 15.720.

This represent a so called “break-even-point”, at which we can determine which taxation option is more profitable, i.e., leads to less tax payment in a period. Using this knowledge, we will formulate some propositions in the next section. Figure 5-1 illustrates the graphical solution of finding the break-even-point.

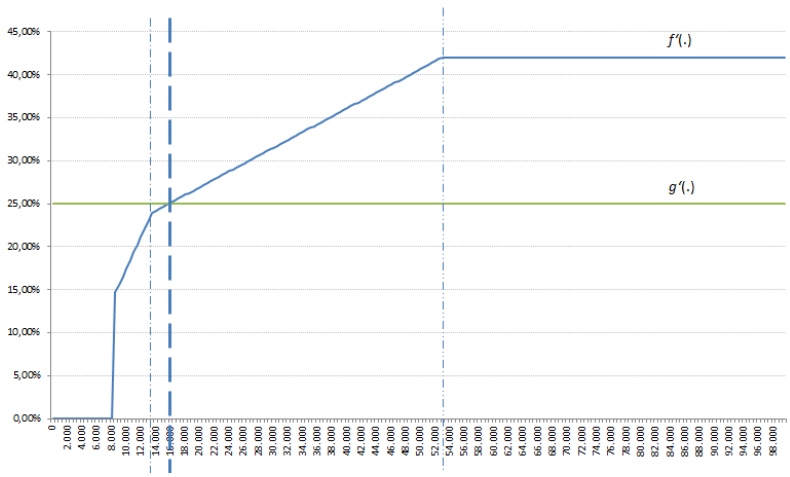


Figure 5-1 The break-even-point for calculating marginal tax rates

5.2 Some propositions

Proposition 5-1

Consider a period in which A_t and B_t are non-negative.

If $8.354 < A_t + B_t \leq 15.720$*, then always choose joint taxation, i.e.,* $y_t = 0$.

Proof:

In case of joint taxation, the sum of ordinary income and capital gains is subject to a common marginal tax rate that is lower than 25 %. In case of separate taxation, the amount of B_t is subject to the tax rate of 25 %, resulting in a higher total tax payment. Independent on the possible loss offset strategy, the choice $y_t = 0$ is always better.

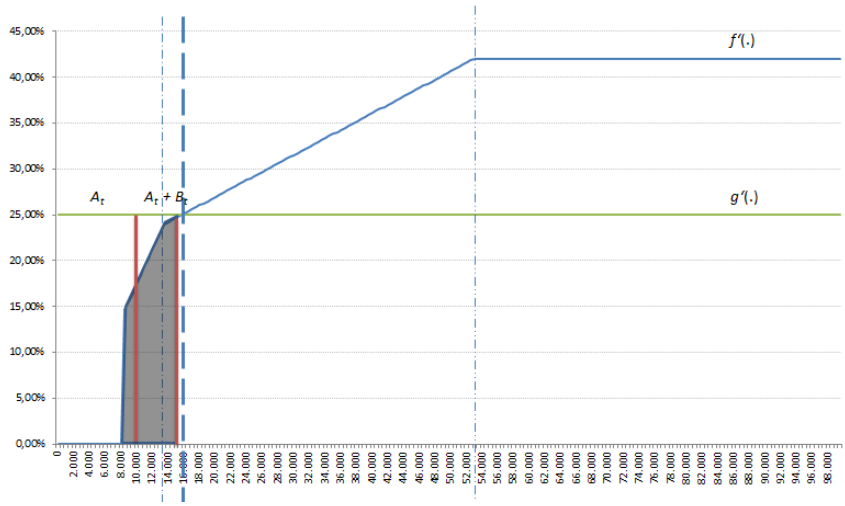


Figure 5-2 A case where joint taxation is favorable

In Figure 5-2, the tax payment is represented by the dark grey area. This is the case where A_t and B_t is added and taxed together. If B_t is taxed separately with 25 % tax rate, we have the tax payment as shown in Figure 5-3. The light grey area is the tax payment levied on A_t alone. We clearly see the the total tax payment is higher.

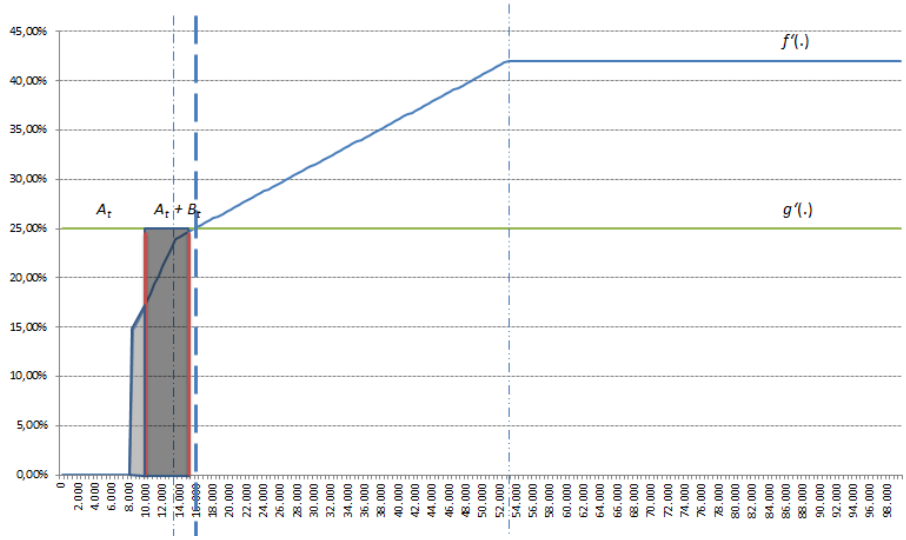


Figure 5-3 A case where separate taxation is detrimental

Proposition 5-2

The following proposition is an immediate consequence of the above proposition.

Consider a period in which A_t and B_t are non-negative.

If $A_t + B_t \leq 8.354$, then always choose joint taxation, i.e., $y_t = 0$.

Proof:

If the sum of ordinary income and capital gains is below the exempt level, then no tax payment must be made. Independent on the possible loss offset strategy in other periods, the choice $y_t = 0$ is always better.

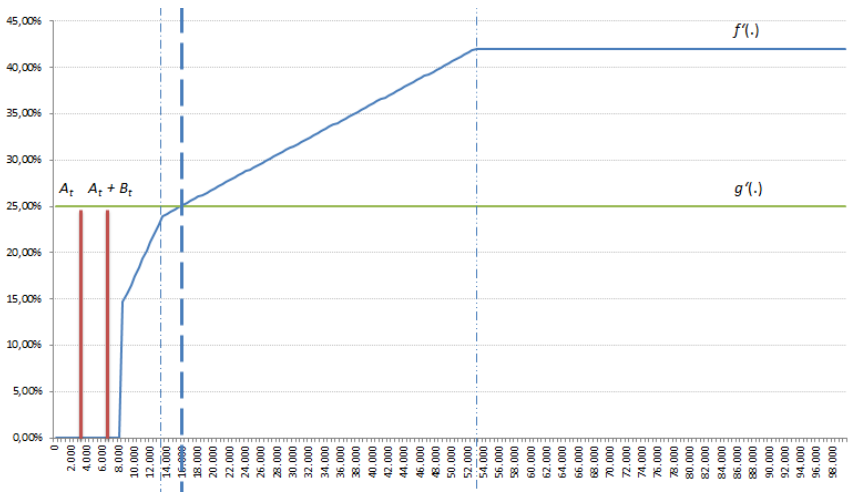


Figure 5-4 A case where joint taxation leads to no tax payment

Figure 5-4 shows that with joint taxation, $f(A_t + B_t) = 0$, there is no tax payment at all. In contrast, when B_t is taxed separately, the tax payment is represented as the grey square in Figure 5-5. Then, $f(A_t) = 0$ but $g(B_t) > 0$.

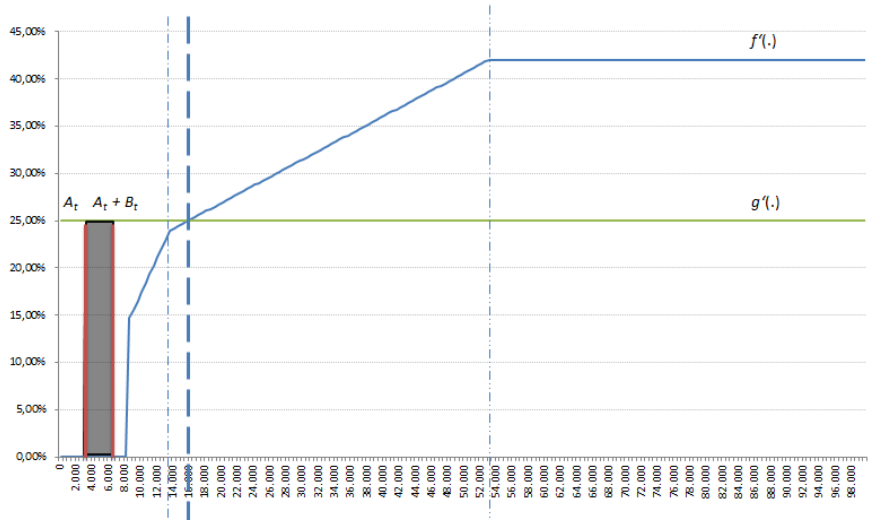


Figure 5-5 A case where separate taxation lead to unnecessary tax payment

Proposition 5-3

Consider a period in which A_t and B_t are non-negative.

If $A_t + B_t > 15.700$ and $A_t \leq 15.700$, then there is at first sight no clear clue if we should choose joint taxation, i.e., $y_t = 0$ or separate taxation, i.e., $y_t = 1$.

Proof:

In proof is given in graphical form in the next figures.

Figure 5-6 shows the tax payment (dark grey area) in case of joint taxation.

Figure 5-7 shows the tax payment (light grey area) in case of separate taxation.

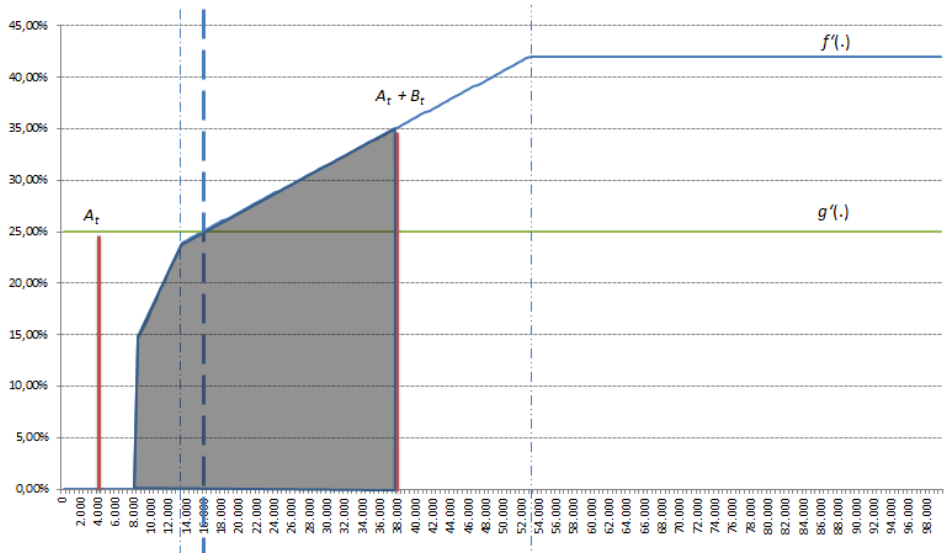


Figure 5-6 No clear indication of a better taxation option (1)

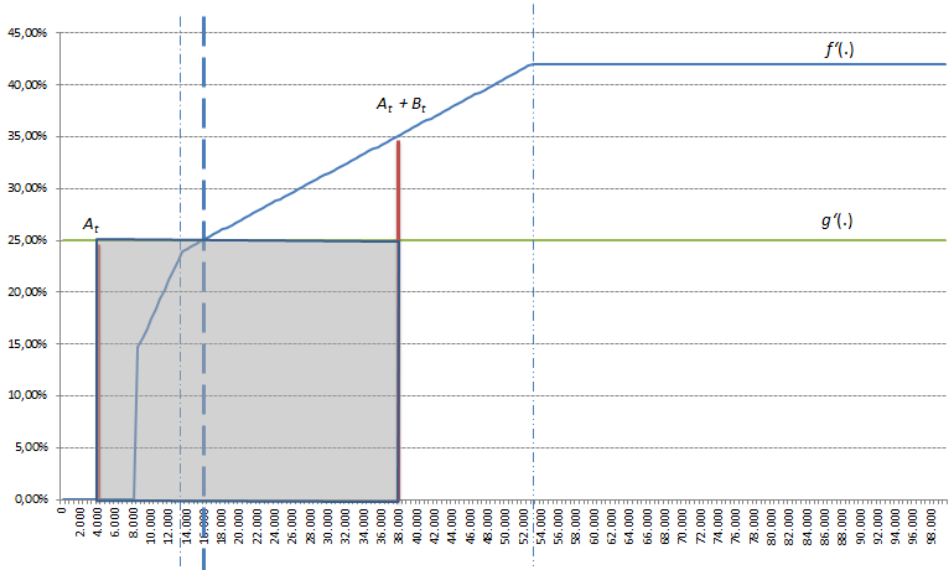


Figure 5-7 No clear indication of a better taxation option (2)

In order to decide which taxation option is better, we have to compare the two areas 0 and 1 as shown in Figure 5-8. If area 1 is larger than area 0, then choose the joint taxation. Else, the separate taxation is more profitable.

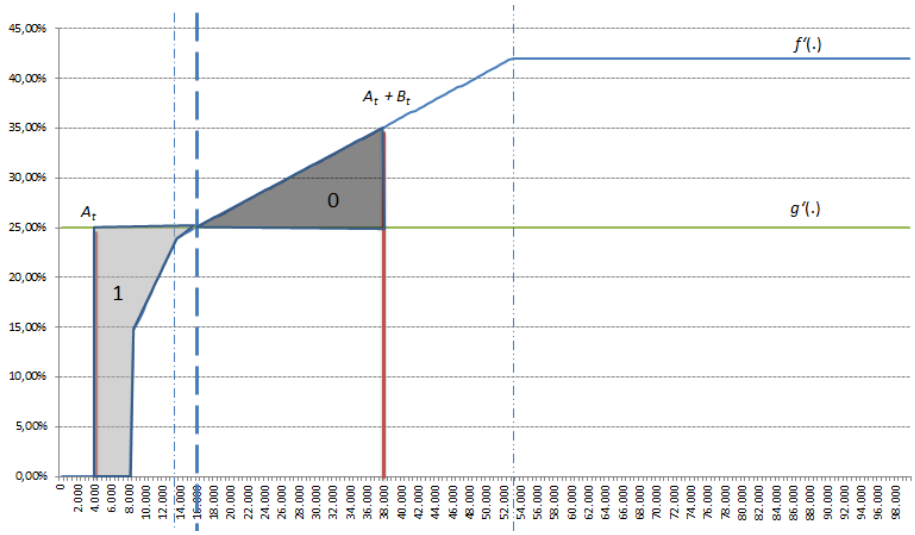


Figure 5-8 No clear indication of a better taxation option (3)

Proposition 5-4

Consider a period in which A_t and B_t are non-negative.

If $A_t + B_t > 15.700$ and $A_t \geq 15.700$, then temporarily choose joint taxation, i.e., $y_t = 1$. It will be more lucrative to switch to $y_t = 0$ if and only if there is a loss deduction that is high enough to offset against B_t.

Proof:

In case of joint taxation, the sum of ordinary income and capital gains is subject to a marginal tax rate that is always higher than 25 %. In case of separate taxation, the amount of B_t is subject to the tax rate of 25 %, resulting in a lower total tax payment. Without considering a possible loss offset strategy in other periods, the choice $y_t = 1$ is temporarily better.

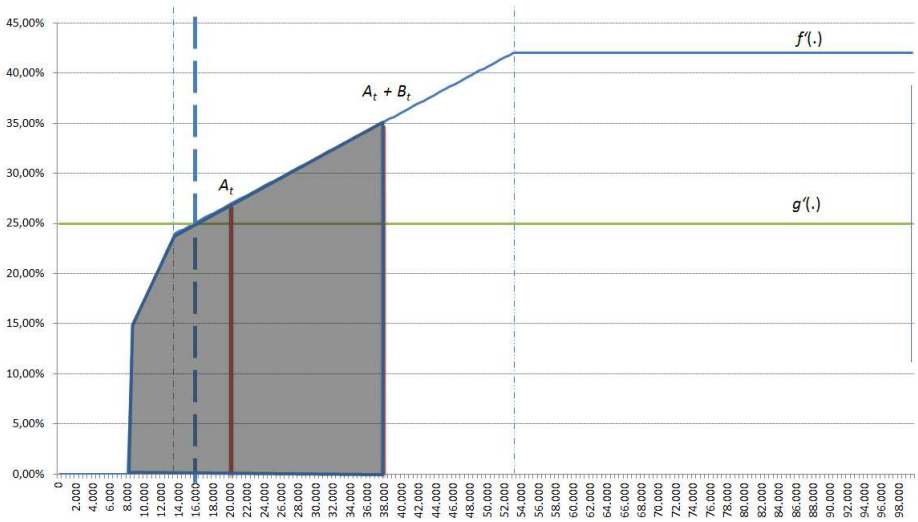


Figure 5-9 A case where joint taxation is detrimental

If there is a loss deduction that can fully „erase“ the trapezoid created by A_t, $A_t + B_t$, the $x\ axis$, and $f'(.)$, as shown in Figure 5-10, then the joint taxation should be preferred.

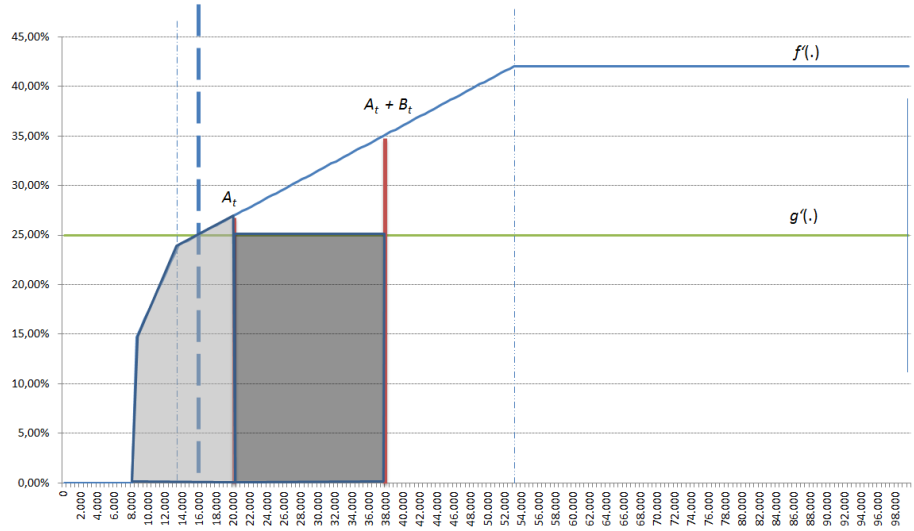


Figure 5-10 A case where separate taxation is favorable

As a conclusion, we can get some knowledge from examining the break-even-point. We calculate the point of intersection of $f'(.)$ and $g'(.)$ and find out that the value of taxable income at this point is EUR 15.720,46. We can make draws from the above propositions and examples to distill steps of solving the problem.

For every period t with $A_t > 0$ we have:

- If option $y_t = 0$ is already chosen, then it will always remain $y_t = 0$.
- If option $y_t = 1$ is already chosen, then it will be more lucrative only if there is a loss deduction that fully „erase“ the white trapezoid as seen in the figure above.

Thus far, we have seen that the decision for taxation options $y_t = 0\ or\ 1$ in each period is determined by comparing the tax payments. This was basically based on the marginal tax rates. A quantitative analysis of input information can provide some knowledge for making good decisions. However, a complete research on such relationships is impossible due to the great number of possible scenarios. In fact, this is the reason why we need to apply the proposed algorithms in finding the optimal combination of variables.

6 Conclusions and Future Works

6.1 Conclusions

This thesis is concerned with a general model for solving the Loss Offset Problem that arises in German income tax system. The problem deals with the situation where a tax payer suffers a loss generated by economic activities. According to income tax law, the tax payer can choose to offset the loss that is made in a financial year. Furthermore, two taxation options regarding the ordinary income and capital gains are available. However, the intuitive interpretation of this analysis framework becomes complex if more constraints are involved. This happens by "optimal" interpretations of tax rules. Therefore we adapt selected assumptions on the laws and construct mathematical models for the analysis. As a result, we derive a rigorous framework for a general model which allows us to integrate different constraints.

When the research topic was begun, it was hoped that a definitive answer could be found for the question: "What is the best algorithms for solving the Loss Offset Problem in German income taxation?". Instead of giving a definitive answer, we explore classes of algorithms that are capable of finding local as well as global optimum for our problem. The problem has been identified as a non-linear combinatorial optimization problem with mixed integer variables.

German taxation is often regarded as confusing and opaque. Indeed, the complex system seems rather complicated to anyone who want to do tax planning tasks. As a result, in tax planning there are a great variety of intriguing situations. The revised texts have been studied to ensure that we construct a model incorporating the latest tax laws. However, due to the volatile nature of the law affecting the planning of income tax, we did not attempt to concentrate on a detail treatment of tax laws but rather modeled the quantitative aspects of selected decision possibilities. By building the models with selected assumptions and presenting some appropriate quantitative problem-solving approaches, our purpose was to make a contribution that brings clarity to an area that was previously covered with black boxes.

Many tax planning models for income tax are concerned with a single-period context. Planning over many periods, however, has a distinct tax advantage over planning over one period because an integrated solution can be generated. The analysis presented in this thesis shows that there is a very substantial need for multi-period viewpoint in tax planning especially for the Loss Offset Problem. Taken together, existing tax regulations provide precious opportunity for individuals to reduce their tax payments by making use of loss carry-forward and -backward, thus reducing their taxable incomes in response to high marginal tax rates.

From an analytical point of view, we have developed a model that provides an overall and thorough concept of a multi-period optimization problem to gain the broadest possible advantage from planning losses and gains. A propagated aim in this thesis is the study of the problem structure and its complexity. Thus, we also provide an analysis of different scenarios that can be modeled over a wide variety of the input data. We also find the optimal strategy for choosing the taxation options in the absence of other regulations. We compare the results provided by *Monte Carlo Simulation* to

those of other optimization approaches and show that applying our proposed optimization processes can lead to significant tax advantage, i.e., minimize overall tax payments and therefore maximize wealth.

The methodologies for determining the optimal decision strategies have been presented. Given the complexity of the problem, standard mathematical methods fall short in providing a global optimal solution. The greedy approach, as applied in practice, delivers no satisfying result. The heuristic algorithms developed for this problem have been shown to perform well for the given instances. However, improvements in the heuristic methods can be done. Furthermore, we present numerical examples for verifying the model as well as testing the algorithms. We illustrate the differences that arise between the problem-solving methods. For a simple scenario with three periods, a combination of *Exhaustive Search* for the binary variables and *Binary Search (BS)* or *Golden Ratio Search (GRS)* for the real-valued variable can always delivers optimal result. For a more complicated problem with multiple periods, we need to apply heuristic search methods such as *Particle Swarm Optimization (PSO)* or *Simulated Annealing (SA)* to obtain near-optimal solutions. Although greatly different in nature and origin, the two methods have one common feature: they are able to deal with non-linear combinatorial problems, and have proven to be a good selection for our optimization problem. Different parameter settings of the input data show that *PSO* and *SA* provide good results for several scenarios of the problem and are applicable to any arbitrary number of planning periods. However, one limitation to the application of these methods is that it is not possible to test explicitly if the solution is a globally optimal one.

6.2 Future Works

The study of taxes represents a tremendous topic where optimization potentials exist in a numerous way. Due to the fact that this thesis deals with a fascinating new area in quantitative tax study, researchers can keep adding fresh material to the body of knowledge in the future. This final section present some open problems and alludes to possible extensions of this work.

No matter how good a model is constructed, there are its limitations. The models we constructed in the chapter 3, and their parameters, must always be somewhat uncertain in the reality. The model can be made to be more complete when the uncertainties have been rigorously quantified.

Thus, the choice of assumptions is still a problem which requires further attention. Generally speaking, it is nearly impossible to model all realistic features in a mathematical model. We have made the assumption that all input data are deterministic and known in advance. The challenging treatment of changing or unstable economic environment is mostly still neglected and must be considered in practice. Uncertainty in a model can arise due to either incomplete information or unpredictable changes in the future. In particular, the facts of more sophisticated cash flow structure as well as the volatile interest rates make the need to investigate those issues more urgent than it was the case that was proposed in our models. For dealing with such requirements, researchers can construct stochastic models which assume that the uncertainty of cash flows is known by statistical distribution. We also suggest the application of procedures such as scenario analysis and sensitivity analysis to tackle uncertainty in a more deterministic manner.

Another point is the depreciation method, which is assumed to be a straight-line method in our work. The cash flows that represent ordinary income, i.e., gain or loss will be particularly affected by the depreciation method. This is a realistic feature that could be integrated into the model. However, the mathematical formulation of the model is then expected to be much more complicated. Furthermore, solutions might be only generated with heuristic algorithms and no global optimum can be guaranteed.

On the other side, there is still plenty of room for further upgrades of the proposed algorithms to deal with the constructed problems and their more complicated variations. We are particularly interested in the analysis and improvement of the proposed heuristic search approaches. The reason is very simple. The Loss Offset Problem, due to the binary variables that represent taxation options, is a combinatorial optimization problem. The difficulty in solving this problem is similar to that of the *Traveling Salesman Problem*, for which no exact search methods are known except for the *Exhaustive Search*. For small problem instances with a reasonable number of periods, we can apply *Exhaustive Search* for the binary variables. However, this becomes impossible in larger problem settings. Thus, heuristics are indispensable in a general model with any arbitrary number of periods. In this sense, the choice of *PSO* and *SA* parameters can have a large impact on optimization performance.

Investment and financial planning may interact not only for taxation effects, but of course for other reasons. Therefore to get some deeper insight in the problem modeling and solving solution further research should consider more factors in the tax optimization models. Finally, it is not enough to model a real-world problem as a complex mathematical program; we need to be able to solve such models effectively. In other words, when developing a model, it is crucial to make assumptions so that the real-world problem is tractable. It is to be expected that the more new restrictions are implemented, the higher the probability that the problem can only be solved effectively by heuristic algorithms, e.g., those that are based on swarm and stochastic methodologies. We encourage other researchers to work further on the field of business as well as personal tax planning and develop more efficient algorithms and methods for solving the existing complex problems.

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Appendix

Appendix A: Relevant tax laws

In the following outlines of the relevant tax laws in German language are provided. All of the laws are of year 2014.

The basic tax function for ordinary income:

§ 32a Einkommensteuertarif

(1) [1]Die tarifliche Einkommensteuer im Veranlagungszeitraum 2013 bemisst sich nach dem zu versteuernden Einkommen. [2]Sie beträgt vorbehaltlich der §§ 32b, 32d, 34, 34a, 34b und 34c jeweils in Euro für zu versteuernde Einkommen

1. bis 8 130 Euro (Grundfreibetrag):

   0;

2. von 8 131 Euro bis 13 469 Euro:

   (933,70 • y + 1 400) • y;

3. von 13 470 Euro bis 52 881 Euro:

   (228,74 • z + 2 397) • z + 1 014;

4. von 52 882 Euro bis 250 730 Euro:

   0,42 • x – 8 196;

5. von 250 731 Euro an:

   0,45 • x – 15 718.

[3]„y“ ist ein Zehntausendstel des den Grundfreibetrag übersteigenden Teils des auf einen vollen Euro-Betrag abgerundeten zu versteuernden Einkommens. [4]„z“ ist ein Zehntausendstel des 13 469 Euro übersteigenden Teils des auf einen vollen Euro-Betrag abgerundeten zu versteuernden Einkommens. [5]„x“ ist das auf einen vollen Euro-Betrag abgerundete zu versteuernde Einkommen. [6]Der sich ergebende Steuerbetrag ist auf den nächsten vollen Euro-Betrag abzurunden.

Tax rate for capital gains:

§ 32d Gesonderter Steuertarif für Einkünfte aus Kapitalvermögen

(1) [1]Die Einkommensteuer für Einkünfte aus Kapitalvermögen, die nicht unter § 20 Absatz 8 fallen, beträgt 25 Prozent. [2]Die Steuer nach Satz 1 vermindert sich um die nach Maßgabe des Absatzes 5 anrechenbaren ausländischen Steuern. [3]Im Fall der Kirchensteuerpflicht ermäßigt sich die Steuer nach den Sätzen 1 und 2 um 25 Prozent der auf die Kapitalerträge entfallenden Kirchensteuer. [4]Die Einkommensteuer beträgt damit

$$\frac{e - 4q}{4 + k} \quad .$$

[5]Dabei sind „e“ die nach den Vorschriften des § 20 ermittelten Einkünfte, „q“ die nach Maßgabe des Absatzes 5 anrechenbare ausländische Steuer und „k“ der für die Kirchensteuer erhebende Religionsgesellschaft (Religionsgemeinschaft) geltende Kirchensteuersatz.

(6) [1]Auf Antrag des Steuerpflichtigen werden anstelle der Anwendung der Absätze 1, 3 und 4 die nach § 20 ermittelten Kapitaleinkünfte den Einkünften im Sinne des § 2 hinzugerechnet und der tariflichen Einkommensteuer unterworfen, wenn dies zu einer niedrigeren Einkommensteuer einschließlich Zuschlagsteuern führt (Günstigerprüfung). [2]Absatz 5 ist mit der Maßgabe anzuwenden, dass die nach dieser Vorschrift ermittelten ausländischen Steuern auf die zusätzliche tarifliche Einkommensteuer anzurechnen sind, die auf die hinzugerechneten Kapitaleinkünfte entfällt. [3]Der Antrag kann für den jeweiligen Veranlagungszeitraum nur einheitlich für sämtliche Kapitalerträge gestellt werden. [4]Bei zusammenveranlagten Ehegatten kann der Antrag nur für sämtliche Kapitalerträge beider Ehegatten gestellt werden.

Regulations for loss offset:

§ 10d Verlustabzug

(1) [1]Negative Einkünfte, die bei der Ermittlung des Gesamtbetrags der Einkünfte nicht ausgeglichen werden, sind bis zu einem Betrag von 1 000 000 Euro, bei Ehegatten, die nach den §§ 26, 26b zusammenveranlagt werden, bis zu einem Betrag von 2 000 000 Euro vom Gesamtbetrag der Einkünfte des unmittelbar vorangegangenen Veranlagungszeitraums vorrangig vor Sonderausgaben, außergewöhnlichen Belastungen und sonstigen Abzugsbeträgen abzuziehen (Verlustrücktrag). [2]Dabei wird der Gesamtbetrag der Einkünfte des unmittelbar vorangegangenen Veranlagungszeitraums um die Begünstigungsbeträge nach § 34a Absatz 3 Satz 1 gemindert. [3]Ist für den unmittelbar vorangegangenen Veranlagungszeitraum bereits ein Steuerbescheid erlassen worden, so ist er insoweit zu ändern, als der Verlustrücktrag zu gewähren oder zu berichtigen ist. [4]Das gilt auch dann, wenn der Steuerbescheid unanfechtbar geworden ist; die Festsetzungsfrist endet insoweit nicht, bevor die Festsetzungsfrist für den Veranlagungszeitraum abgelaufen ist, in dem die negativen Einkünfte nicht ausgeglichen werden. [5]Auf Antrag des Steuerpflichtigen ist ganz oder teilweise von der Anwendung des Satzes 1 abzusehen. [6]Im Antrag ist die Höhe des Verlustrücktrags anzugeben.

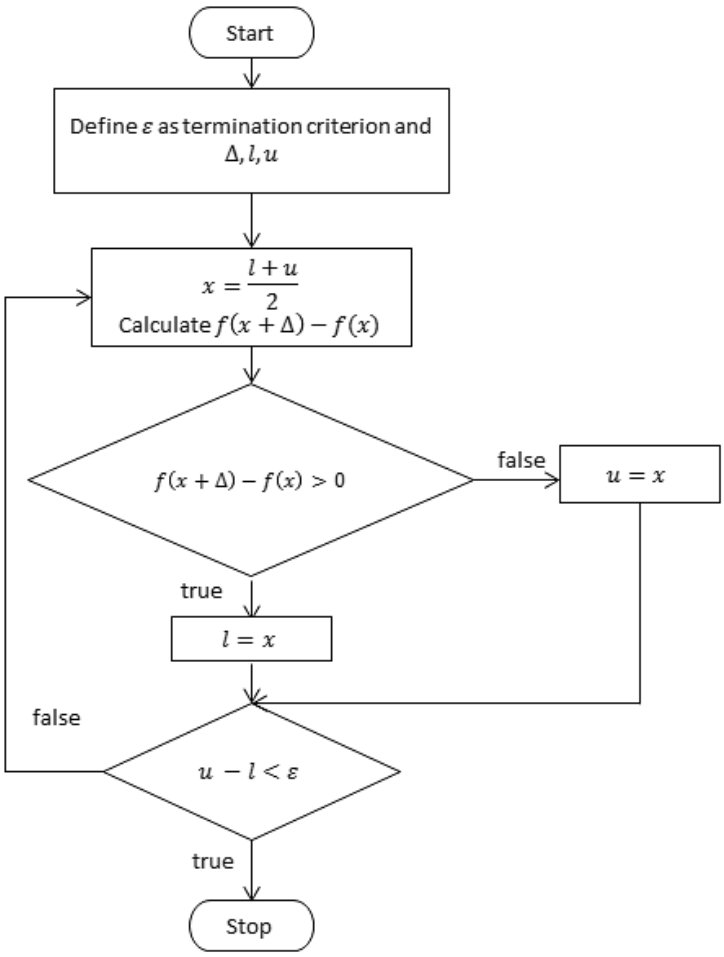
(2) [1]Nicht ausgeglichene negative Einkünfte, die nicht nach Absatz 1 abgezogen worden sind, sind in den folgenden Veranlagungszeiträumen bis zu einem Gesamtbetrag der Einkünfte von 1 Million Euro unbeschränkt, darüber hinaus bis zu 60 Prozent des 1 Million Euro übersteigenden Gesamtbetrags der Einkünfte vorrangig vor Sonderausgaben, außergewöhnlichen Belastungen und sonstigen Abzugsbeträgen abzuziehen (Verlustvortrag). [2]Bei Ehegatten, die nach den §§ 26, 26b zusammenveranlagt werden, tritt an die Stelle des Betrags von 1 Million Euro ein Betrag von 2 Millionen Euro. [3]Der Abzug ist nur insoweit zulässig, als die Verluste nicht nach Absatz 1 abgezogen worden sind und in den vorangegangenen Veranlagungszeiträumen nicht nach Satz 1 und 2 abgezogen werden konnten.

(3) (weggefallen)

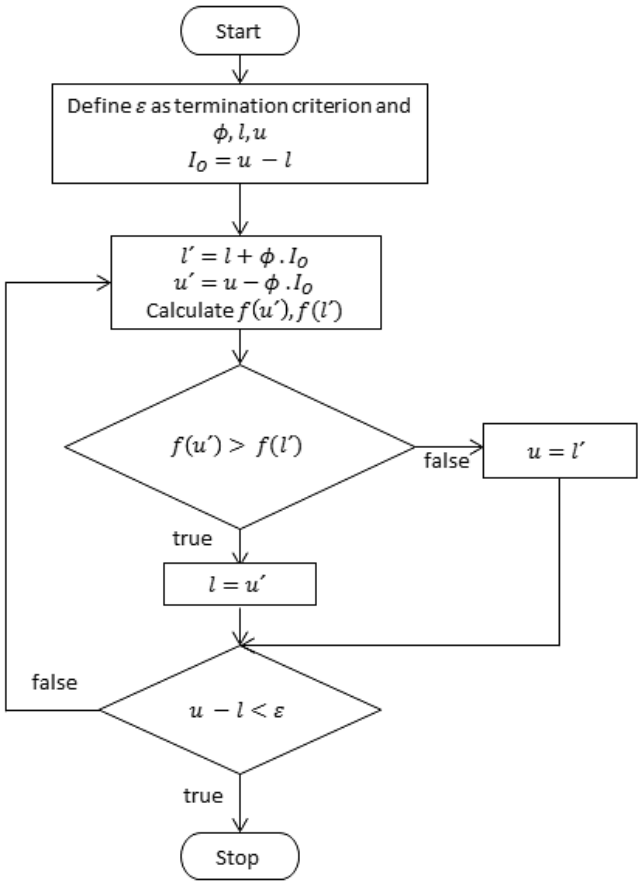
(4) [1]Der am Schluss eines Veranlagungszeitraums verbleibende Verlustvortrag ist gesondert festzustellen. [2]Verbleibender Verlustvortrag sind die bei der Ermittlung des Gesamtbetrags der Einkünfte nicht ausgeglichenen negativen Einkünfte, vermindert um die nach Absatz 1 abgezogenen und die nach Absatz 2 abziehbaren Beträge und vermehrt um den auf den Schluss des vorangegangenen Veranlagungszeitraums festgestellten verbleibenden Verlustvortrag. [3]Zuständig für die Feststellung ist das für die Besteuerung zuständige Finanzamt. [4]Bei der Feststellung des verbleibenden Verlustvortrags sind die Besteuerungsgrundlagen so zu berücksichtigen, wie sie den Steuerfestsetzungen des Veranlagungszeitraums, auf dessen Schluss der verbleibende Verlustvortrag festgestellt wird, und des Veranlagungszeitraums, in dem ein Verlustrücktrag vorgenommen werden kann, zu Grunde gelegt worden sind; § 171 Absatz 10, § 175 Absatz 1 Satz 1 Nummer 1 und § 351 Absatz 2 der Abgabenordnung sowie § 42 der Finanzgerichtsordnung gelten entsprechend. [5]Die Besteuerungsgrundlagen dürfen bei der Feststellung nur insoweit abweichend von Satz 4 berücksichtigt werden, wie die Aufhebung, Änderung oder Berichtigung der Steuerbescheide ausschließlich mangels Auswirkung auf die Höhe der festzusetzenden Steuer unterbleibt. [6]Die Feststellungsfrist endet nicht, bevor die Festsetzungsfrist für den Veranlagungszeitraum abgelaufen ist, auf dessen Schluss der verbleibende Verlustvortrag gesondert festzustellen ist; § 181 Absatz 5 der Abgabenordnung ist nur anzuwenden, wenn die zuständige Finanzbehörde die Feststellung des Verlustvortrags pflichtwidrig unterlassen hat.

Appendix B: Flowcharts

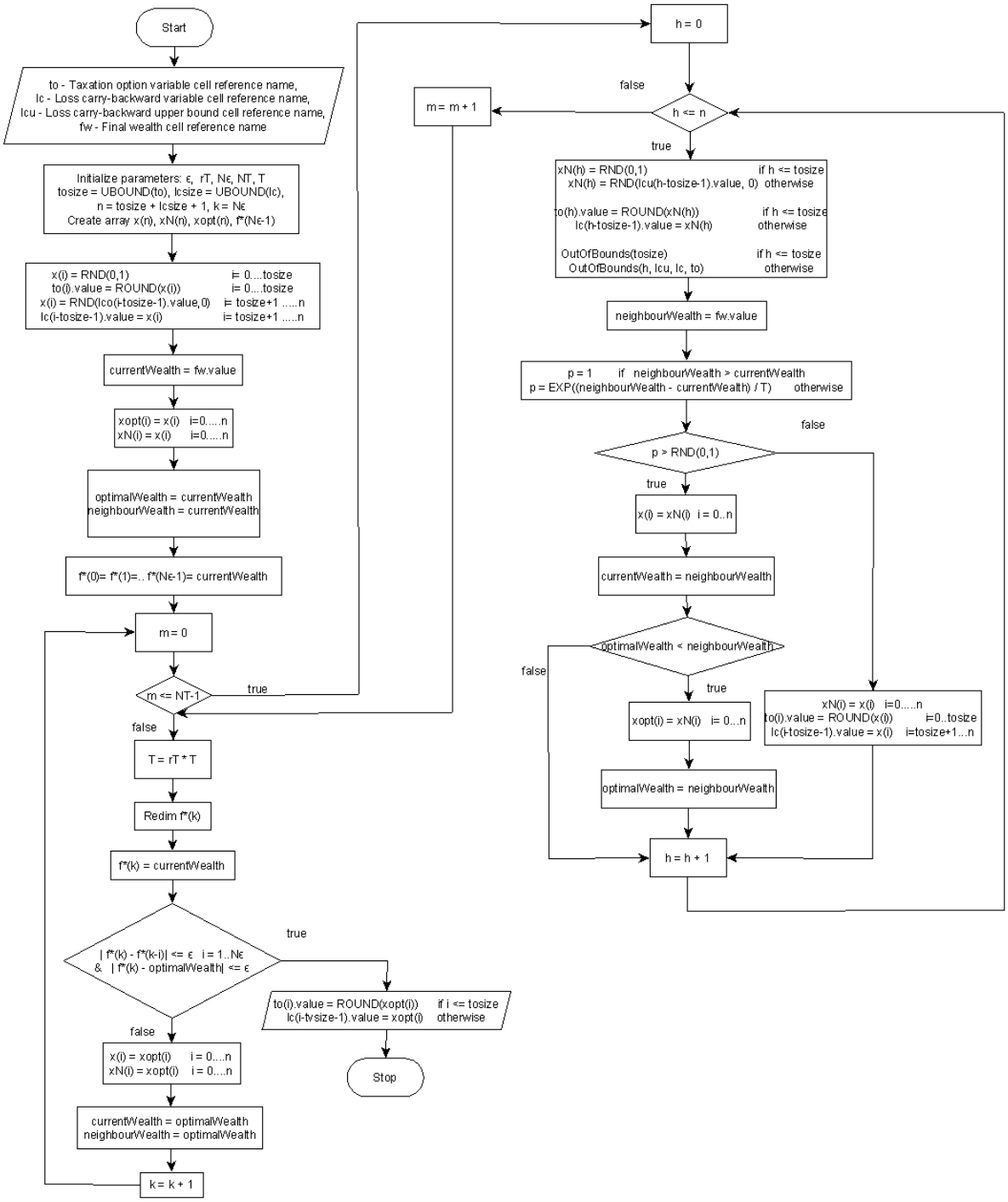
Flowchart of the *Binary Search* algorithm



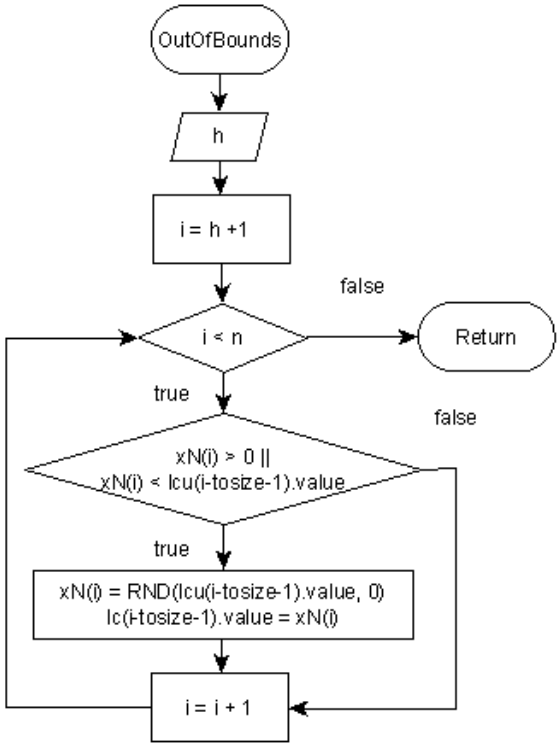
Flowchart of the *Golden Ratio Search* algorithm



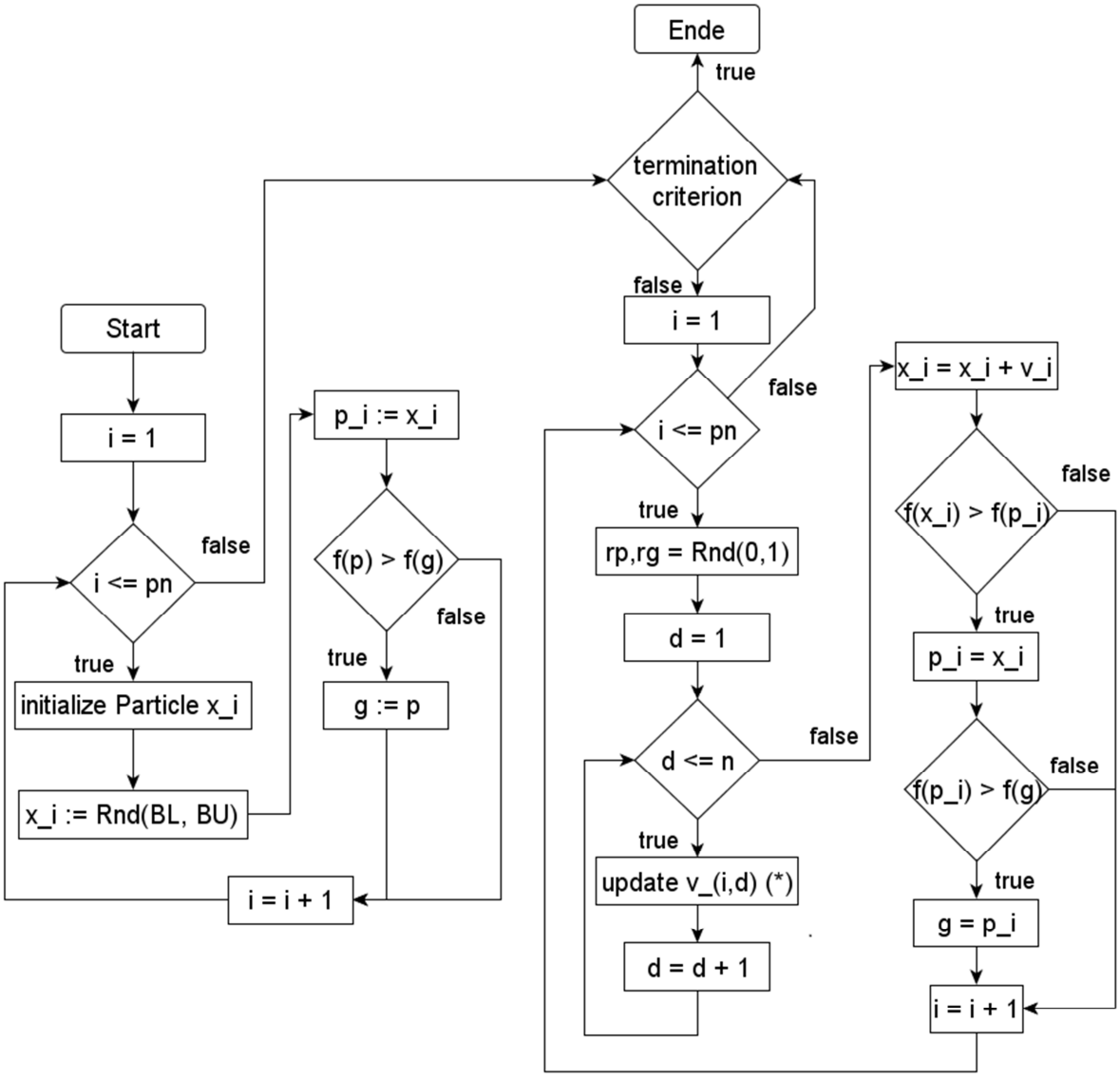
Flowchart of the *Simulated Annealing* algorithm



Flowchart of the *OutOfBounds* test function for *Simulated Annealing* algorithm



Flowchart of the *Particle Swarm Optimization* algorithm



(*) $\mathbf{v}_{i,d} \leftarrow \omega\, \mathbf{v}_{i,d} + \varphi_p\, r_p\, (\mathbf{p}_{i,d} - \mathbf{x}_{i,d}) + \varphi_g\, r_g\, (\mathbf{g}_d - \mathbf{x}_{i,d})$

Appendix C: Marginal tax rates

This appendix presents a comparison of the marginal and average tax rates for representative amounts of income. Source: BMF.

Examples of basic table of tax rates			
Taxable income	Marginal tax rate	Average tax rate	Tax to be assessed
5,000 €	0 %	0 %	0 €
10,000 €	17.49 %	2.94 %	294 €
20,000 €	26.95 %	13.39 %	2,677 €
30,000 €	31.53 %	18.67 %	5,601 €
40,000 €	36.10 %	22.46%	8,983 €
50,000 €	40.68 %	25.65 %	12,823 €
60,000 €	42.00 %	28.34 %	17,004 €
70,000 €	42.00 %	30.29 %	21,204 €
80,000 €	42.00 %	31.76 %	25,404 €
90,000 €	42.00 %	32.89 %	29,604 €
100,000 €	42.00 %	33.80 %	33,804 €
150,000 €	42.00 %	36.54 %	54,804 €
200,000 €	42.00 %	37.90 %	75,804 €
250,000 €	42.00 %	38.72 %	96,804 €
300,000 €	45.00 %	39.76 %	119,282 €

Appendix D: Structure of the objective function

In this appendix we consider a numerical example for the 3-period-case. In order to understand the structure of the objective function and its behavior, we assume a simplified tax function that is linear over its whole domain. We then use Maple to compute W_3 as a function of all binary and real-valued variables. The calculation steps were made for each period 1, 2, and 3.

$W0 := 100000$ $\quad A1 := 200000$ $\quad A2 := -200000$ $\quad A3 := 200000$

$i := 0.05$

$B1 := W0 \cdot i$

$$5000.00$$

$S1 := (1 - y1) \cdot (0.42 \cdot (A1 + B1)) + y1 \cdot (0.42 \cdot A1 + 0.25 \cdot B1)$

$$86100.0000 - 850.0000\, y1$$

$SR1 := (1 - y1) \cdot (0.42 \cdot (A1 + B1 + x2)) + y1 \cdot (0.42 \cdot (A1 + x2) + 0.25 \cdot B1)$

$$(1 - y1)\,(86100.0000 + 0.42\, x2) + y1\,(85250.0000 + 0.42\, x2)$$

$W1 := W0 + A1 + B1 - S1$

$$2.189000000\, 10^5 + 850.0000\, y1$$

$B2 := W1 \cdot i$

$$10945.00000 + 42.500000\, y1$$

$S2 := (1 - y2) \cdot 0 + y2 \cdot (0.25 \cdot B2)$

$$0.25\, y2\,(10945.00000 + 42.500000\, y1)$$

$SRF2 := S1 - SR1$

$$86100.0000 - 850.0000\, y1 - (1 - y1)\,(86100.0000 + 0.42\, x2) - y1\,(85250.0000 + 0.42\, x2)$$

$W2 := W1 + A2 + B2 - S2 + SRF2$

$$1.159450000\, 10^5 + 42.500000\, y1 - 0.25\, y2\,(10945.00000 + 42.500000\, y1) - (1 - y1)\,(86100.0000 + 0.42\, x2) - y1\,(85250.0000 + 0.42\, x2)$$

$simplify(W2)$

$$29845. + 892.5000000\, y1 - 2736.250000\, y2 - 10.62500000\, y2\, y1 - 0.4200000000\, x2$$

$B3 := W2 \cdot i$

$$5797.250000 + 2.12500000\, y1 - 0.0125\, y2\, (10945.00000 + 42.500000\, y1) - 0.05\, (1 - y1)\, (86100.0000 + 0.42\, x2) - 0.05\, y1\, (85250.0000 + 0.42\, x2)$$

$S3 := (1 - y3) \cdot (0.42 \cdot (A3 + B3 + (1 - y2) \cdot (A2 + B2 - x2) + y2 \cdot (A2 - x2))) + y3 \cdot (0.42 \cdot (A3 + (1 - y2) \cdot (A2 + B2 - x2) + y2 \cdot (A2 - x2)) + 0.25 \cdot B3)$

$$\begin{aligned}
&(1 - y3)\, \big(86434.84500 + 0.8925000000\, y1 \\
&\quad - 0.005250\, y2\, (10945.00000 + 42.500000\, y1) \\
&\quad - 0.0210\, (1 - y1)\, (86100.0000 + 0.42\, x2) \\
&\quad - 0.0210\, y1\, (85250.0000 + 0.42\, x2) + 0.42\, (1 - y2)\, \big(\\
&\quad -1.890550000\, 10^5 + 42.500000\, y1 - x2\big) + 0.42\, y2\, (\\
&\quad -200000 - x2)\big) + y3\, \big(85449.31250 + 0.42\, (1 - y2)\, \big(\\
&\quad -1.890550000\, 10^5 + 42.500000\, y1 - x2\big) + 0.42\, y2\, (\\
&\quad -200000 - x2) + 0.5312500000\, y1 \\
&\quad - 0.003125\, y2\, (10945.00000 + 42.500000\, y1) \\
&\quad - 0.0125\, (1 - y1)\, (86100.0000 + 0.42\, x2) \\
&\quad - 0.0125\, y1\, (85250.0000 + 0.42\, x2)\big)
\end{aligned}$$

$simplify(S3)$

$$\begin{aligned}
&5223.645000 + 36.59250000\, y1 - 4654.361250\, y2 \\
&\quad - 18.07312500\, y2\, y1 - 0.4288200000\, x2 \\
&\quad - 253.6825000\, y3 - 7.586250000\, y3\, y1 \\
&\quad + 23.25812500\, y3\, y2 + 0.09031250000\, y3\, y2\, y1 \\
&\quad + 0.003570000000\, y3\, x2
\end{aligned}$$

$W3 := W2 + A3 + B3 - S3$

$$3.217422500\ 10^5 + 44.62500000\ y1 - 0.2625\ y2\ (10945.00000 + 42.500000\ y1) - 1.05\ (1 - y1)\ (86100.0000 + 0.42\ x2) - 1.05\ y1\ (85250.0000 + 0.42\ x2) - (1 - y3)\ \big(86434.84500 + 0.8925000000\ y1 - 0.005250\ y2\ (10945.00000 + 42.500000\ y1) - 0.0210\ (1 - y1)\ (86100.0000 + 0.42\ x2) - 0.0210\ y1\ (85250.0000 + 0.42\ x2) + 0.42\ (1 - y2)\ \big(-1.890550000\ 10^5 + 42.500000\ y1 - x2\big) + 0.42\ y2\ (-200000 - x2)\big) - y3\ \big(85449.31250 + 0.42\ (1 - y2)\ \big(-1.890550000\ 10^5 + 42.500000\ y1 - x2\big) + 0.42\ y2\ (-200000 - x2) + 0.5312500000\ y1 - 0.003125\ y2\ (10945.00000 + 42.500000\ y1) - 0.0125\ (1 - y1)\ (86100.0000 + 0.42\ x2) - 0.0125\ y1\ (85250.0000 + 0.42\ x2)\big)$$

$simplify(W3)$

$$-0.09031250000\ y3\ y2\ y1 + 1781.298750\ y2 + 7.586250000\ y3\ y1 + 900.5325000\ y1 + 253.6825000\ y3 - 0.01218000000\ x2 + 6.916875000\ y2\ y1 - 23.25812500\ y3\ y2 - 0.003570000000\ y3\ x2 + 2.261136050\ 10^5$$

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Hilfsmittel verfasst habe. Die aus anderen Quellen übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet. Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form in einem Verfahren zur Erlangung eines akademischen Grades vorgelegt.

Saarbrücken, 31.03.2014

Hai-Dung Dinh