Deutsches
Forschungszentrum
für Künstliche
Intelligenz GmbH

Research
Report
RR-98-03

Developing a Matrix Characterization for $\mathcal{MELL}$

Heiko Mantel

September 1998

Deutsches Forschungszentrum für Künstliche Intelligenz GmbH

Postfach 20 80
67608 Kaiserslautern, FRG
Tel.: + 49 (631) 205-3211
Fax: + 49 (631) 205-3210
E-Mail: info@dfki.uni-kl.de

Stuhlsatzenhausweg 3
66123 Saarbrücken, FRG
Tel.: + 49 (681) 302-5252
Fax: + 49 (681) 302-5341
E-Mail: info@dfki.de

WWW: http://www.dfki.de

Deutsches Forschungszentrum für Künstliche Intelligenz

DFKI GmbH

German Research Center for Artificial Intelligence

Founded in 1988, DFKI today is one of the largest nonprofit contract research institutes in the field of innovative software technology based on Artificial Intelligence (AI) methods. DFKI is focusing on the complete cycle of innovation — from world-class basic research and technology development through leading-edge demonstrators and prototypes to product functions and commercialization.

Based in Kaiserslautern and Saarbrücken, the German Research Center for Artificial Intelligence ranks among the important "Centers of Excellence" worldwide.

An important element of DFKI's mission is to move innovations as quickly as possible from the lab into the marketplace. Only by maintaining research projects at the forefront of science can DFKI have the strength to meet its technology transfer goals.

DFKI has about 115 full-time employees, including 95 research scientists with advanced degrees. There are also around 120 part-time research assistants.

Revenues for DFKI were about 24 million DM in 1997, half from government contract work and half from commercial clients. The annual increase in contracts from commercial clients was greater than 37% during the last three years.

At DFKI, all work is organized in the form of clearly focused research or development projects with planned deliverables, various milestones, and a duration from several months up to three years.

DFKI benefits from interaction with the faculty of the Universities of Saarbrücken and Kaiserslautern and in turn provides opportunities for research and Ph.D. thesis supervision to students from these universities, which have an outstanding reputation in Computer Science.

The key directors of DFKI are Prof. Wolfgang Wahlster (CEO) and Dr. Walter Olthoff (CFO).

DFKI's six research departments are directed by internationally recognized research scientists:

- ❏ Information Management and Document Analysis (Director: Prof. A. Dengel)
- ❏ Intelligent Visualization and Simulation Systems (Director: Prof. H. Hagen)
- ❏ Deduction and Multiagent Systems (Director: Prof. J. Siekmann)
- ❏ Programming Systems (Director: Prof. G. Smolka)
- ❏ Language Technology (Director: Prof. H. Uszkoreit)
- ❏ Intelligent User Interfaces (Director: Prof. W. Wahlster)

In this series, DFKI publishes research reports, technical memos, documents (eg. workshop proceedings), and final project reports. The aim is to make new results, ideas, and software available as quickly as possible.

Prof. Wolfgang Wahlster
Director

Developing a Matrix Characterization for $\mathcal{MELL}$

Heiko Mantel

DFKI-RR-98-03

My special thanks go to Christoph Kreitz. He motivated me to develop a matrix characterization for linear logic when he was my advisor. He guided me through my diploma thesis which was the predecessor of the work presented in this report and contributed many valuable hints and comments. Afterwards he continuously encouraged me to keep working on this interesting and challenging topic.

Many other people commented on this work especially in the early stages. In particular I would like to thank: Serge Autexier, Wolfgang Bibel, Daniel Korn, Jens Otten, Frank Pfenning, Enno Sandner, and Stephan Schmitt.

ISSN 0946-008X

Developing a Matrix Characterization for $\mathcal{MELL}$

Heiko Mantel
German Research Center for Artificial Intelligence – DFKI
Stuhlsatzenhausweg 3, 66123 Saarbrücken, Germany,
mantel@dfki.de

Contents

1 Introduction

Linear logic [Gir87] has become known as a very expressive formalism for reasoning about action and change. During its rather rapid development linear logic has found applications in logic programming [HM94, Mil96], modeling concurrent computation [GG91], planning [MTV91], and many others. The expressiveness of linear logic, however, results in a high complexity. Propositional linear logic is undecidable and even the multiplicative fragment ($\mathcal{MLL}$) is already $\mathcal{NP}$-complete [LW94]. The complexity of the multiplicative exponential fragment ($\mathcal{MELL}$) is still unknown.

Proof search in linear logic is difficult to automate. Girard's sequent calculus [Gir87], although covering all of linear logic, contains too many redundancies to be useful for efficient automatic proof search. Attempts to remove permutabilities from sequent proofs [And92, GP94] and to add proof strategies [Tam94] have provided significant improvements but some redundancies remain because of the use of sequent calculi. Proof nets [DR89], on the other hand, can handle only a fragment of the logic.

Matrix characterizations of logical validity, originally developed as foundation of the *connection method* for classical logic [And81, Bib81, Bib87], avoid many kinds of redundancies contained in sequent calculi and yield a very compact representation of the search space. They have successfully been extended to intuitionistic and modal logics [Wal90] and serve as a basis for a uniform proof method for classical and non-classical logics [OK96] as well as for a uniform method for translating these matrix proofs into sequent proofs [SK96]. Resource management similar to multiplicative linear logic is already addressed by the *linear connection method* [Bib86]. Fronhöfer [Fro96] gives a matrix characterization of $\mathcal{MLL}$ which can also capture some aspects of weakening and contraction but does not appear to generalize any further. In [KMOS97] we have developed a matrix characterization for $\mathcal{MLL}$ and have shown how to extend the uniform proof search and translation procedures accordingly.

In this report we present a matrix characterization for the full multiplicative exponential fragment including the constants $\mathbf{1}$ and $\bot$. This characterization uses Andreoli's focusing principle [And92] as one of its major design steps and does not appear to share the limitations of the previous approaches.

Our approach includes a methodology for step-wisely developing such a characterization. By introducing a series of intermediate calculi the development of a matrix characterization for $\mathcal{MELL}$ becomes manageable. Each newly introduced calculus adds one more condensing principle to the previous one and can be proven correct and complete with respect to it. We expect that this methodology will generalize to further fragments of linear logic as well as to other logics.

In section 2 we give a brief introduction to $\mathcal{MELL}$. We define the syntax of $\mathcal{MELL}$ and give an intuitive explanation of the connectives. By adapting Smullyan's uniform notation to $\mathcal{MELL}$ we create a compact representation of a linear logic sequent calculus. Based on the notion of multiplicities, i.e. an eager handling of contraction and a lazy handling of weakening, we present a dyadic and a triadic calculus which condense the search space in section 3. These are auxiliary calculi in the development of the matrix characterization. In section 4 we develop a calculus Σ_{pos} which operates on positions in a formula tree instead of on subformulas. In order to express the peculiarities of some connectives we insert special positions into the formula tree. The matrix characterization is presented in section 5. We define a notion of complementarity for $\mathcal{MELL}$ and prove it to be sufficient and necessary for validity in $\mathcal{MELL}$. The matrix characterization allows us to reason about the existence of a sequent proof rather than concentrating on the construction of a specific proof, thus condensing the search space. We conclude with some remarks on related work in section 6.

2 Multiplicative Exponential Linear Logic

Linear logic [Gir87] treats formulas like resources which disappear after their use unless they are explicitly marked as reusable. From a proof theoretical point of view, linear logic can be seen as the outcome of removing the structural rules for contraction and weakening from the classical sequent calculus and re-introducing them in a controlled manner. Linear negation $^\perp$ is involutive like classical negation. The two different traditions for writing the sequent rule for conjunction result in two different conjunctions $\otimes$ and $\&$ and two different disjunctions $⅋$ and $\oplus$. The constant `true` splits up into $\mathbf{1}$ and $\top$ and `false` into $\perp$ and $\mathbf{0}$. The unary connectives ? and ! mark formulas for a controlled application of weakening and contraction. Quantifiers $\forall$ and $\exists$ can be added as usual.

Linear logic can be divided into the multiplicative, additive, and exponential fragment. While in the multiplicative fragment resources are used exactly once, resource sharing is enforced in the additive fragment. Exponentials mark formulas as reusable. All fragments exist on their own right and can be combined freely. The full power of linear logic comes from combining all of them.

In this section we give a brief introduction to linear logic. Throughout this report we will focus on multiplicative exponential linear logic ($\mathcal{MELL}$), the combination of the multiplicative and exponential fragments, leaving the additive fragment and the quantifiers out of consideration. $^\perp$, $\otimes$, $⅋$, $\multimap$, $\mathbf{1}$, $\perp$, !, and ? are the connectives of $\mathcal{MELL}$.

In addition to Girard's article [Gir87] general introductions to the syntax, semantics, and proof theory of linear logic can be found in [Tro92, Gal91, Sce90, Lin92, Ale94].

In subsection 2.1, we define the syntax of $\mathcal{MELL}$ and explain the meaning of the logical connectives in subsection 2.2 on an intuitive level. A sequent calculus for $\mathcal{MELL}$ is presented in subsection 2.3. The concept of multiplicities which is fundamental to matrix characterizations is introduced in subsection 2.4. In order to simplify meta-reasoning about calculi we adapt Smullyan's uniform notation to $\mathcal{MELL}$ and present a sequent calculus based on uniform notation in subsection 2.5.

2.1 The Syntax of $\mathcal{MELL}$

Definition 1 *Formulas* are defined recursively from a set $\mathcal{P}^0$ of propositions.

1. Each $A \in \mathcal{P}^0$ is a formula.

2. $\mathbf{1}$ and $\bot$ are formulas.

3. If F is a formula then $F^\bot$ is a formula.

4. If F_1 and F_2 are formulas then $F_1 \otimes F_2$ and $F_1 ⅋ F_2$ are formulas.

5. If F is a formula then $?F$ and $!F$ are formulas.

The formulas $P \in \mathcal{P}^0$ are called *atomic*. $\mathbf{1}$ and $\bot$ are called *constants*.
The set of all $\mathcal{MELL}$-formulas is denoted by *wff*.

$^\bot$ is the linear logic negation. A formula $F^\bot$ is pronounced *nil* F. $\mathbf{1}$ and $\bot$ are the multiplicative versions of *true* and *false*. They are pronounced *one* and *bottom*. $\otimes$ and ⅋ are the multiplicative variants of *conjunction* and *disjunction*. A formula $F_1 \otimes F_2$ is pronounced F_1 *tensor* F_2 and a formula $F_1 ⅋ F_2$ is pronounced F_1 *par* F_2. ? and ! are the exponentials *why-not* and *off-course*.
As convention we use F, G, and H as meta-variables for formulas. A, B, and C are used as meta-variables for atomic formulas. Γ and Δ are used as meta-variables for multi-sets of formulas. All meta-variables are used with indices as well.

Definition 2 We define the *set of subformulas* of a given formula F recursively.

1. F is a subformula of F.

2. If $G^\bot$, $?G$, or $!G$ is a subformula of F, so is G.

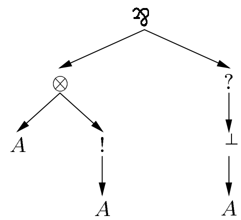
3. If $G_1 \otimes G_2$ or $G_1 ⅋ G_2$ is a subformula of F, so are G_1 and G_2.

If two formulas are syntactical identical then they are regarded as the same formula. However, when considering subformulas we talk about occurrences of formulas. Thus, two (occurrences of) subformulas are equal if they occur at the same place in a formula (and are therefore also syntactical identical).

Definition 3 The *major subformula* of a formula $F^\bot$, $?F$, or $!F$ is F. The *major subformulas* of a formula $F_1 \otimes F_2$ or $F_1 ⅋ F_2$ are F_1 and F_2.

We define $succ_1$ and $succ_2$ as functions which yield the major subformulas of a given formula, e.g. $succ_1(F_1 \otimes F_2) = F_1$ and $succ_2(F_1 \otimes F_2) = F_2$. $succ_2$ is undefined for formulas $F^\bot$, $?F$, or $!F$. Both functions are undefined for atomic formulas and for constants.
$succ_1$ and $succ_2$ induce an ordering $\prec$ on the subformulas of a formula. $F \prec G$ shall hold if G is a major subformula of F, i.e. $G = succ_1(F)$ or $G = succ_2(F)$. We define $\ll$ as the transitive closure of $\prec$.
A formula F can be represented as a graph. We define the concept of a *formula tree*. For each occurrence of a subformula of F there is a corresponding node in the tree. The edges connect subformulas with their major subformulas. Two occurrences of a subformula are equal if they correspond to the same node in the formula tree.

Example 4 $F = (A \otimes !A) \parr ?(A^{\perp})$ is a formula. The major subformulas of it are $succ_1(F) = A \otimes !A$ and $succ_2(F) = ?(A^{\perp})$. Note that there are three different occurrences of A within F. The formula tree for F is depicted below.



2.2 On the Meaning of the $\mathcal{MELL}$-Connectives

Linear logic formulas should be understood as resources which can be produced while others are consumed. A multi-set of formulas specifies a state in which exactly those resources exist which are specified by the formulas in the multi-set. Transitions from one state to another state are again specified by linear logic formulas.

A formula $F_1 \multimap F_2$ specifies a transition during which F_1 is consumed while F_2 is produced. A resource $\mathbf{1}$ can always be consumed without producing any other formula. A resource $\perp$ can always be constructed without consuming any other formula. Thus, $\mathbf{1} \multimap F$ means that F can be constructed without consuming anything else, and, $F \multimap \perp$ means that F can be consumed without producing anything. Linear negation $^{\perp}$ expresses the difference between resources which are consumed and resources which are produced, i.e. $F^{\perp} \multimap \perp$ is equivalent to $\mathbf{1} \multimap F$ and $\mathbf{1} \multimap F^{\perp}$ is equivalent to $F \multimap \perp$. The production or consumption of a resource $F_1 \otimes F_2$ means that both resources F_1 and F_2 are produced or consumed, respectively. A resource $F_1 \parr F_2$ is equivalent to ${F_1}^{\perp} \multimap F_2$ as well as to ${F_2}^{\perp} \multimap F_1$. A resource $!F$ can be consumed as often as needed. It can be understood as a machine which produces copies of F. To produce a resource $!F$ one must be able to produce F any number of times. Clearly, this can be achieved only by the use of machines. An intuitive explanation of $?F$ is difficult. ? is the dual to !.

2.3 A Sequent Calculus for $\mathcal{MELL}$

Definition 5 A *sequent* is a pair $\langle \Gamma, \Delta \rangle$ of multi-sets of formulas. Γ is the *antecedent* and Δ is the succedent of the sequent. We usually denote a sequent by $\Gamma \longrightarrow \Delta$.

Informally, a sequent $\Gamma \longrightarrow \Delta$ is valid if the formulas in Δ can be constructed while all formulas in Γ are used up during the construction.

A *sequent calculus* is composed of a set of *sequent rules*. Each sequent rule consists of one *conclusion* and possibly multiple *premises*. Each premise as well as the conclusion is a sequent. Formulas which occur in a premise but not in the conclusion are called *side formulas*. A formula which occurs in the conclusion but not in any premise and has the side formulas as subformulas is called *principal formula*. All other formulas are the *context* of the rule. A rule which requires that the context formulas in the conclusion are of a special kind is called *context sensitive*.

A sequent calculus for $\mathcal{MELL}$ is depicted in table 1. The rules are divided into four different groups, i.e. identity, negation, multiplicative fragment, and exponential fragment. The principal formula of each rule in the left hand side column lies in the antecedent. Therefore, these rules are called *left rules*. Likewise the rules in the right hand side column are called *right rules*. The d, c, w, and p in the rules for exponentials abbreviate *dereliction*, *contraction*, *weakening*, and *promotion*. Note that the rules *axiom*, $\bot_{ax}$, $\mathbf{1}_{ax}$, and the promotion rules are context sensitive. $!\Gamma$ and $?\Delta$ in the promotion rules enforce that all context formulas in the antecedent (succedent) have ! (?) as main connective.
The sequent calculus has a nice symmetry. Negation shifts a formula from the antecedent into the succedent and vice versa (rules ${}^{\bot}l$ and ${}^{\bot}r$). There are similarities between the rules $\otimes l$, $⅋\, r$, and $\multimap r$, as well as between $\otimes r$, $⅋\, l$, and $\multimap l$. Further, there are similarities between the rules for $\mathbf{1}$ and $\bot$ as well as between ? and !. These similarities will be exploited in subsection 2.5.
A *derivation* $\mathcal{D}$ is a tree where all edges are labeled with sequents and all inner nodes are labeled with inference rule names. The sequents adjacent to such a node must match the corresponding sequent rule. The root node must have only one adjacent edge. The label of this edge is called the *conclusion* of $\mathcal{D}$. The sequents which are adjacent to a leaf node which is not labeled are the *open goals* of $\mathcal{D}$. A derivation without any open goals is a *proof* of the conclusion.
An inference rule can be applied forwards as well as backwards. If one has derived all premises of a rule then one can conclude the conclusion by a *forward inference*. If one wants to derive a sequent which matches the conclusion of an inference rule then one can reduce that sequent into subgoals by a *backward inference*. Searching for a proof of a sequent by forward inferences is called *synthetic proof search*. In this approach one starts with axioms and tries to derive the sequent which shall be proven. *Analytic proof search* is the orthogonal approach where search is performed by backward inferences. Here, one starts with the sequent which shall be proven and reduces it into subgoals until an axiom rule can be applied. In this report we usually take the analytic point of view. When a sequent $\Gamma \longrightarrow \Delta$ is provable we sometimes write $\Gamma \vdash \Delta$ or being more precise $\vdash \Gamma \longrightarrow \Delta$.
The calculus in table 1 incorporates no cut-rule. Cut-elimination for linear logic has been proven in [Gir87]. The lack of a cut-rule ensures that all formulas which occur in a derivation of a sequent S are sub-formulas of some formula in S.

Example 6 We want to prove the sequent $\longrightarrow (A\otimes !A)⅋\, ?(A^{\bot})$. The only applicable rule is $⅋\, r$.

$$\frac{\longrightarrow (A\otimes !A), ?(A^{\bot})}{\longrightarrow (A\otimes !A)⅋\, ?(A^{\bot})}\ ⅋\, r$$

Now we have a choice between $\otimes r$, $d-?$, $w-?$, and $c-?$. Though it will turn out to be the wrong choice we decide to try $\otimes r$ first.

$$\frac{\longrightarrow A, ?(A^{\bot}) \quad \longrightarrow !A}{\longrightarrow (A\otimes !A), ?(A^{\bot})}\ \otimes r$$

When applying the rule $\otimes r$ one must decide on a partition of the context onto the different branches. We decide to put the only context formula into the left

identity

$$\frac{}{A \longrightarrow A}\ \mathit{axiom}$$

negation

$$\frac{\Gamma \longrightarrow \Delta, F}{\Gamma, F^{\perp} \longrightarrow \Delta}\ {}^{\perp}l \qquad \frac{\Gamma, F \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, F^{\perp}}\ {}^{\perp}r$$

multiplicative fragment

$$\frac{}{\perp \longrightarrow \cdot}\ \perp_{ax} \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \perp}\ \perp r$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \mathbf{1} \longrightarrow \Delta}\ \mathbf{1}l \qquad \frac{}{\cdot \longrightarrow \mathbf{1}}\ \mathbf{1}_{ax}$$

$$\frac{\Gamma, F_1, F_2 \longrightarrow \Delta}{\Gamma, F_1 \otimes F_2 \longrightarrow \Delta}\ \otimes l \qquad \frac{\Gamma_1 \longrightarrow \Delta_1, F_1 \quad \Gamma_2 \longrightarrow \Delta_2, F_2}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2, F_1 \otimes F_2}\ \otimes r$$

$$\frac{\Gamma_1, F_1 \longrightarrow \Delta_1 \quad \Gamma_2, F_2 \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, F_1 \parr F_2 \longrightarrow \Delta_1, \Delta_2}\ \parr\, l \qquad \frac{\Gamma \longrightarrow \Delta, F_1, F_2}{\Gamma \longrightarrow \Delta, F_1 \parr F_2}\ \parr\, r$$

$$\frac{\Gamma_1 \longrightarrow \Delta_1, F_1 \quad \Gamma_2, F_2 \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, F_1 \multimap F_2 \longrightarrow \Delta_1, \Delta_2}\ \multimap l \qquad \frac{\Gamma, F_1 \longrightarrow \Delta, F_2}{\Gamma \longrightarrow \Delta, F_1 \multimap F_2}\ \multimap r$$

exponential fragment

$$\frac{\Gamma, F \longrightarrow \Delta}{\Gamma, !F \longrightarrow \Delta}\ d-! \qquad \frac{\Gamma \longrightarrow \Delta, F}{\Gamma \longrightarrow \Delta, ?F}\ d-?$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, !F \longrightarrow \Delta}\ w-! \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, ?F}\ w-?$$

$$\frac{\Gamma, !F, !F \longrightarrow \Delta}{\Gamma, !F \longrightarrow \Delta}\ c-! \qquad \frac{\Gamma \longrightarrow \Delta, ?F, ?F}{\Gamma \longrightarrow \Delta, ?F}\ c-?$$

$$\frac{!\Gamma, F \longrightarrow ?\Delta}{!\Gamma, ?F \longrightarrow ?\Delta}\ p-? \qquad \frac{!\Gamma \longrightarrow ?\Delta, F}{!\Gamma \longrightarrow ?\Delta, !F}\ p-!$$

Table 1: A sequent calculus for first order $\mathcal{MELL}$

hand branch. From the structure of the right hand branch we can already see that this will never lead to an axiom rule. A different partition of the context would not help. For the only other partition the left hand branch would not be provable. Thus, $\otimes r$ was not a good choice. The formula $?(A^{\perp})$ should be copied first. Thus, we decide to backtrack and try $c - ?$ instead. Now we are able to complete the proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A}\,axiom}{\longrightarrow A, A^{\perp}}\,{}^{\perp}r}{\longrightarrow A, ?(A^{\perp})}\,d-? \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A}\,axiom}{\longrightarrow A, A^{\perp}}\,{}^{\perp}r}{\longrightarrow A, ?(A^{\perp})}\,d-?}{\longrightarrow !A, ?(A^{\perp})}\,p-!}{\longrightarrow (A\otimes !A), ?(A^{\perp}), ?(A^{\perp})}\,\otimes r}{\longrightarrow (A\otimes !A), ?(A^{\perp})}\,c-?$$

Remark 7 Compared to a sequent calculus for classical logic the calculus in table 1 lacks rules for contraction and weakening for arbitrary formulas. The rules $w - !$, $w - ?$, $c - !$, and $c - ?$ allow weakening and contraction only for formulas which are marked with the appropriate exponential.
The constants incorporate already part of the power of weakening. The rules $\mathbf{1}l$ and $\perp r$ allow weakening for constants. This results in a similar behavior of formulas $?F$ and $\perp$ in the succedent if weakening is applied. The same relation holds between formulas $!F$ and $\mathbf{1}$ in the antecedent.

Weakening and contraction are essential, i.e. there exist sequents (with exponentials) which are valid but cannot be proven without using these rules. For example, any proof of the sequent $\longrightarrow (A\otimes !A) \parr ?(A^{\perp})$ requires contraction. The sequent $\longrightarrow \mathbf{1} \parr ?A$ can only be proven if weakening is used.
If one looks at a sequent proof in which the contraction rule has been applied our previously defined notion of an occurrence of a subformula does not seem to be appropriate any more. Subformulas which allow the application of contraction can be copied in a proof. After being duplicated the copies may be treated in different ways, e.g. they may go into different branches of the derivation like the copies of $?(A^{\perp})$ in example 6. It seems appropriate to consider the copies as different occurrences of a subformula in the derivation. Thus, now we have identified two different notions of occurrence of a formula – one for formula trees and one for derivations. This difference becomes apparent in other sub-structural logics like relevance logics as well. In the sequel we usually consider the later notion and will explicitly mention if we use the first one.

2.4 Multiplicities

In subsection 2.3 it has been argued that weakening and contraction are essential. However, in (analytic) proof search it seems rather difficult to decide when one of these rules should be applied. Not to copy a formula might make any further proof attempt fail as we have seen in example 6. However, there are no bounds for the number of copies required. Thus, it is difficult to say how many copies will suffice.

An application of a weakening rule deletes a formula from a sequent. While it may be nice to deal with smaller objects in a proof the benefit does not seem to be essential for subsequent rule applications. The context sensitive rules *axiom*, $\bot_{ax}$, $\mathbf{1}_{ax}$, as well as the promotion rules $p-!$ and $p-?$ may require a prior application of weakening. No other rule can become applicable by weakening. The rules *axiom*, $\bot_{ax}$, and $\mathbf{1}_{ax}$ require all kinds of weakening, i.e. $w-!$, $w-?$, $\mathbf{1}l$, and $\bot r$, while the promotion rules require only weakening of constants. Thus, as we will show later one may restrict the application of weakening rules to these cases. Such rules should only be applied if an application of an axiom or promotion rule follows immediately afterwards.
An application of a contraction rule duplicates a formula in a sequent. While it may be nice to have more resources there is no restriction on the number of such rule applications. Thus, one might apply contraction over and over again without getting any closer to an axiom. However, if one does not apply contraction before applying dereliction then the respective formula cannot be copied any more. Similarly, if one does not apply contraction before a rule where the proof branches then the respective formula moves into one branch but is not available in the other one.
Other authors [And92, CHP96] enforce contraction in these critical cases which guarantees that copies can be generated in both branches. Such a lazy handling of copies is appropriate for sequent calculus proof search but is not required in our approach where sequent calculi are developed only as intermediate steps and are not intended for actual proof search. As it is common in matrix characterizations we apply an eager approach. For each formula F on which weakening and contraction may be applied we assume an upper bound n on the number of copies required. Under this assumption it suffices to apply contraction n times as soon as F occurs as an active formula in the derivation. From then on no further applications of contraction for this formula need to be considered. For this purpose we define multiplicities.
Since, in general, no upper bound for the number of copies can be calculated[1] one can only try for a good guess. However, in proof search this is not a disadvantage compared to the lazy approach. A multiplicity must in general not be guessed in advance but can be modified during matrix based proof search.

Definition 8 A *multiplicity function* μ returns for each occurrence of a signed formula ν a natural number n. n is then called the multiplicity of ν.

This notion of multiplicity differs from the one defined in [Wal90] for modal logics and intuitionistic logic. While Wallen's notion of multiplicity is based on occurrences in a formula tree our notion is based on occurrences in a proof. This finer notion of multiplicity is required for a resource sensitive logic like linear logic.

2.5 Uniform Notation for $\mathcal{MELL}$

We want to prove properties of calculi. Most of the proofs about calculi will be done by induction over the structure of proofs. A case distinction over all rules

[1] Already propositional linear logic is undecidable.[LMSS92]

α	$\langle F_1 \otimes F_2, -\rangle$	$\langle F_1 \parr F_2, +\rangle$	$\langle F_1 \multimap F_2, +\rangle$
$succ_1(\alpha)$	$\langle F_1, -\rangle$	$\langle F_1, +\rangle$	$\langle F_1, -\rangle$
$succ_2(\alpha)$	$\langle F_2, -\rangle$	$\langle F_2, +\rangle$	$\langle F_2, +\rangle$

β	$\langle F_1 \otimes F_2, +\rangle$	$\langle F_1 \parr F_2, -\rangle$	$\langle F_1 \multimap F_2, -\rangle$
$succ_1(\beta)$	$\langle F_1, +\rangle$	$\langle F_1, -\rangle$	$\langle F_1, +\rangle$
$succ_2(\beta)$	$\langle F_2, +\rangle$	$\langle F_2, -\rangle$	$\langle F_2, -\rangle$

ν	$\langle !F, -\rangle$	$\langle ?F, +\rangle$
$succ_1(\nu)$	$\langle F, -\rangle$	$\langle F, +\rangle$

π	$\langle ?F, -\rangle$	$\langle !F, +\rangle$
$succ_1(\pi)$	$\langle F, -\rangle$	$\langle F, +\rangle$

o	$\langle F^{\perp}, -\rangle$	$\langle F^{\perp}, +\rangle$
$succ_1(o)$	$\langle F, +\rangle$	$\langle F, -\rangle$

τ	$\langle \perp, -\rangle$	$\langle \mathbf{1}, +\rangle$

ω	$\langle \mathbf{1}, -\rangle$	$\langle \perp, +\rangle$

lit	$\langle A, -\rangle$	$\langle A, +\rangle$

Table 2: Uniform notation for signed $\mathcal{MELL}$ formulas

of the calculus is needed in such proofs. Thus, we would benefit from a reduction of the number of calculus rules. Smullyan's uniform notation provides such a reduction. It exploits the symmetries in a calculus and represents symmetrical rules by one rule.

Definition 9 A signed formula $\langle F, k\rangle$ relates a formula F to a *polarity* $k \in \{+, -\}$. F is also called the *label* of $\langle F, k\rangle$.

We adapt Smullyan's uniform notation to $\mathcal{MELL}$. In table 2 types from α, β, ν, π, o, τ, ω, and *lit* are assigned to signed formulas depending on their label and their polarity. Atomic formulas $A \in \mathcal{P}^0$ have type *lit*. The association of types corresponds to symmetries in the sequent calculus which were pointed out in subsection 2.3. Two signed formulas are assigned the same type if their corresponding sequent calculus rules are symmetrical. The definition of $succ_1$ and $succ_2$ is extended to signed formulas in the table. Note that the polarity switches during the decomposition of formulas only when $^{\perp}$ or $\multimap$ is applied.
We use φ and ψ as meta-variables for signed formulas. α, β, ν, π, o, τ, and ω are used as meta-variables for signed formulas of the respective type. a is used as meta-variable for atomic signed formulas. Θ and Υ are used used as meta-variables for multi-sets of signed formulas. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$, Π, $\boldsymbol{o}$, $\boldsymbol{\tau}$, and Ω are used as meta-variables for multi-sets of signed formulas of the respective type. Ξ is used as meta-variable for sequences of signed formulas.
For a signed formula $\varphi = \langle F, k\rangle$ the functions *lab* and *pol* respectively return the label and the polarity of φ, i.e. $lab(\varphi) = F$ and $pol(\varphi) = k$. The functions *con* and *type* respectively return the main connective and the type of a signed formula.

Remark 10 Wallen [Wal90] distinguishes principal and secondary types. This distinction is not required for the theory developed in this report. Our types correspond to his primary types.

identity

$$\frac{}{\langle A,+\rangle, \langle A,-\rangle}\ axiom$$

negation

$$\frac{\Upsilon, succ_1(o)}{\Upsilon, o}\ o$$

multiplicative fragment

$$\overline{\tau}\ \tau \qquad\qquad \frac{\Upsilon}{\Upsilon, \omega}\ \omega$$

$$\frac{\Upsilon, succ_1(\alpha), succ_2(\alpha)}{\Upsilon, \alpha}\ \alpha \qquad\qquad \frac{\Upsilon_1, succ_1(\beta) \quad \Upsilon_2, succ_2(\beta)}{\Upsilon_1, \Upsilon_2, \beta}\ \beta$$

exponential fragment

$$\frac{\Upsilon, succ_1(\nu)}{\Upsilon, \nu}\ \nu \qquad\qquad \frac{\boldsymbol{\nu}, succ_1(\pi)}{\boldsymbol{\nu}, \pi}\ \pi$$

$$\frac{\Upsilon}{\Upsilon, \nu}\ w \qquad\qquad \frac{\Upsilon, \nu, \nu}{\Upsilon, \nu}\ c$$

Table 3: A unary sequent calculus Σ_1' for $\mathcal{MELL}$ in uniform notation

Definition 11 A *unary sequent* is a multi-set of signed formulas. We usually denote a unary sequent by Υ.

The calculus in table 1 is reformulated in table 3 using unary sequents. Rules which are similar in the original calculus melt into a single rule. Thus, the resulting calculus becomes more compact which makes reasoning about the system a lot easier since in inductive proofs less cases need to be considered. In Σ_1' the rules *axiom*, τ, and π are context sensitive.

Example 12 We prove the signed formula $\varphi = \langle (A \otimes !A) ⅋ ?(A^{\perp}), +\rangle$ in Σ_1'. We abbreviate the subformulas of F as shown in the table.

$lab(\varphi)$	φ
$(A \otimes !A) ⅋ ?(A^{\perp})$	α_0
$A \otimes !A$	β_{00}
A	a_{000}
$!A$	π_{001}
A	a_{0010}
$?(A^{\perp})$	ν_{01}
$A^{\perp}$	o_{010}
A	a_{0100}

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a_{000}, a_{0100}}\ axiom}{a_{000}, o_{010}}\ o}{a_{000}, \nu_{01}}\ \nu \quad \dfrac{\dfrac{\dfrac{\dfrac{}{a_{0010}, a'_{0100}}\ axiom}{a_{0010}, o'_{010}}\ o}{a_{0010}, \nu'_{01}}\ \nu}{\pi_{001}, \nu'_{01}}\ \pi}{\beta_{00}, \nu_{01}, \nu'_{01}}\ \beta}{\beta_{00}, \nu_{01}}\ c}{\alpha_0}\ \alpha$$

Theorem 13 (Correctness/Completeness) There exists a proof for a formula F in the sequent calculus in table 1 if and only if there exists a proof for the signed formula $\langle F, +\rangle$ in the unary calculus in table 3.

Proof. The proof is simple and can be done by induction over the structure of unary calculus proofs and proofs in Σ_1'.

Definition 14 A signed formula ν is *contracted* in a proof $\mathcal{P}$ if the contraction rule c is applied on ν in $\mathcal{P}$.

Definition 15 A proof $\mathcal{P}$ for a sequent S is *contraction normal* if one of the following conditions holds.

- $\mathcal{P}$ consists of an application of *axiom* or τ.
- $\mathcal{P}$ results from a contraction normal proof $\mathcal{P}'$ by application of one of the rules o, ω, α, ν, or w.
- $\mathcal{P}$ results from contraction normal proofs $\mathcal{P}'$ and $\mathcal{P}''$ for sequents S' and S'' by application of the rule β with S' and S'' as premises.
- $\mathcal{P}$ results from a contraction normal proof $\mathcal{P}'$ for a sequent S' by subsequent applications of the rule c on the same occurrence of a signed formula ν in S. Neither ν nor any copy of ν must be contracted in $\mathcal{P}'$.
- $S = \boldsymbol{\nu}, \pi$ and $\mathcal{P}$ results from a contraction normal proof $\mathcal{P}'$ by application of the rule π and no formula $\nu \in \boldsymbol{\nu}$ is contracted in $\mathcal{P}'$.

Lemma 16 If there exists a proof $\mathcal{P}$ for a sequent S in Σ_1' then there exists a contraction normal proof $\mathcal{P}'$ for S.

Proof. This can be followed from the results in [GP94]. Applications of the contraction rule can be moved towards the root in a proof.

3 Auxiliary Sequent Calculi

In this section we define the sequent calculi Σ_2' and Σ_3'. These calculi are intermediate steps on the way from the sequent calculus Σ_1' to a position calculus which will be presented in section 4. Σ_2' and Σ_3' are closely related to Andreoli's dyadic calculus Σ_2 and triadic calculus Σ_3 [And92]. Σ_3' exploits Andreolis focusing principle in order to condense the search space. However, they differ in the way rules for the exponentials are handled. While Andreolis calculi use a lazy strategy for contraction of generic formulas the calculi presented here use an eager strategy. Lazy contraction seems to be a good choice when proof search is done in a sequent calculus. Eager contraction supports the introduction of multiplicities into sequent calculus proof search. Multiplicities are a fundamental concept in matrix characterizations. Nevertheless, in proof search based on matrix characterizations lazy handling of multiplicities is still possible.
A dyadic calculus Σ_2' and a triadic calculus Σ_3' are presented in section 3.1 and 3.2. The calculi are proven to be sound and complete with respect to the sequent calculus Σ_1'. In subsection 3.3 they are related to the work of Andreoli. Note that the calculi presented in this section are introduced in order to prove facts about them rather than to do proof search with these calculi. Thus, efficiency of proof search in the calculi is not a major aspect.

identity

$$\frac{}{\Theta : \langle A, + \rangle, \langle A, - \rangle}\ axiom$$

negation

$$\frac{\Theta : \Upsilon, succ_1(o)}{\Theta : \Upsilon, o}\ o$$

multiplicative fragment

$$\frac{}{\Theta : \tau}\ \tau \qquad \frac{\Theta : \Upsilon}{\Theta : \Upsilon, \omega}\ \omega$$

$$\frac{\Theta : \Upsilon, succ_1(\alpha), succ_2(\alpha)}{\Theta : \Upsilon, \alpha}\ \alpha$$

$$\frac{\Theta_1 : \Upsilon_1, succ_1(\beta) \quad \Theta_2 : \Upsilon_2, succ_2(\beta)}{\Theta_1, \Theta_2 : \Upsilon_1, \Upsilon_2, \beta}\ \beta$$

exponential fragment

$$\frac{\Theta, succ_1(\nu)^{\mu(\nu)} : \Upsilon}{\Theta : \Upsilon, \nu}\ \nu \qquad \frac{\Theta : succ_1(\pi)}{\Theta : \pi}\ \pi$$

focusing

$$\frac{\Theta : \Upsilon, \varphi}{\Theta, \varphi : \Upsilon}\ focus$$

Table 4: A dyadic sequent calculus Σ_2' for $\mathcal{MELL}$ in uniform notation

3.1 A Dyadic Calculus

The dyadic calculus Σ_2' incorporates a specific handling of formulas of type ν. Sequents are split into two zones where formulas in one of the zones originate from ν-type formulas only. Compared to the dyadic calculus in [And92], Σ_2' additionally introduces the notion of multiplicities, i.e. an eager handling of generic formulas. Multiplicities are an essential ingredient of matrix characterizations.

Definition 17 A *dyadic sequent S* is a pair $\langle \Theta, \Upsilon \rangle$ of multi-sets of signed formulas. Usually, we write $\Theta : \Upsilon$ instead of $\langle \Theta, \Upsilon \rangle$. Θ is called the *unbounded zone* and Υ the *bounded zone* of S.

Definition 18 A *multiplicity function μ for a dyadic sequent* $\Theta : \Upsilon$ is a function which returns a natural number $\mu(\varphi')$ for each signed subformula φ' of any signed formula φ in Θ or Υ.

A sequent calculus for dyadic sequents is depicted in table 4. Derivations of a dyadic sequent S are defined with respect to a fixed multiplicity function μ for S. The multiplicity function is important whenever the ν-rule is applied. A signed formula φ is *derivable* in Σ_2' if $\cdot : \varphi$ is derivable for some multiplicity μ.

Example 19 We prove the signed formula $\varphi = \langle (A \otimes !A) \parr ?(A^{\perp}), + \rangle$ in Σ_2'. We abbreviate subformulas of φ as shown in the table.

$lab(\varphi)$	φ
$(A \otimes !A) \parr ?(A^{\perp})$	α_0
$A \otimes !A$	β_{00}
A	a_{000}
$!A$	π_{001}
A	a_{0010}
$?(A^{\perp})$	ν_{01}, ν'_{01}
$A^{\perp}$	o_{010}, o'_{010}
A	a_{0100}, a'_{0100}

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot : a_{000}, a_{0100}}\,\mathit{axiom}}{\cdot : a_{000}, o_{010}}\,o}{o_{010} : a_{000}}\,\mathit{focus} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdot : a_{0010}, a'_{0100}}\,\mathit{axiom}}{\cdot : a_{0010}, o'_{010}}\,o}{o'_{010} : a_{0010}}\,\mathit{focus}}{o'_{010} : \pi_{001}}\,\pi}{\boldsymbol{o}_{010}, o'_{010} : \beta_{00}}\,\beta}{\cdot : \beta_{00}, \nu_{01}}\,\nu}{\cdot : \alpha_0}\,\alpha$$

Except for the shift from unary to dyadic sequents there are a couple of differences between Σ_1' and Σ_2'.

- The rules *axiom* and τ of Σ_2' can be applied even when the context is not empty. Though, these formulas must be in the unbounded zone. Rules for explicit weakening of ν-type formulas are omitted in Σ_2'.
- The ν rule of Σ_2' uses the notion of multiplicity. Thus, the rule for explicit contraction is omitted in Σ_2'.
- While the π rule in Σ_1' requires all formulas in the context to be of type ν the π rule in Σ_2' requires all formulas of the context to be in the unbounded zone. All formulas in the unbounded zone are copied to the premise.
- Σ_2' has an additional rule *focus* for moving a formula from the unbounded zone into the bounded zone.

Remark 20 Note that formulas of type ω and ν are handled differently in Σ_2' with respect to weakening. While ω-type formulas are weakened by a separate rule ν-type formulas are weakened as members of the unbounded zone at the axioms. Alternatives to this approach would be possible. An explicit weakening rule could be introduced for the unbounded zone. Alternatively, ω-type formulas could be handled lazily in the rules π and *axiom*. The choice made will have an impact on the resulting matrix characterization.

Lemma 21 If there exists a proof $\mathcal{P}$ for the dyadic sequent $\Theta : \Upsilon$ then there exists a proof $\widetilde{\mathcal{P}}$ for the dyadic sequent $\Theta, \Theta' : \Upsilon$ for any multi-set Θ' of signed formulas.

Proof. The proof is by induction over the structure of $\mathcal{P}$ and is quite straightforward. The rules *axiom* and τ allow an arbitrary set Θ'. In all other rules each element of Θ' is propagated from the conclusion to one premise and the induction hypothesis can be applied.

Theorem 22 (Completeness) If there exists a proof $\mathcal{P}_1$ in Σ'_1 for a unary sequent $S_1 = \Upsilon, \boldsymbol{\nu}_{\overline{c}}$ such that any signed formula $\nu \in \Upsilon$ is contracted in $\mathcal{P}_1$ and that $\boldsymbol{\nu}_{\overline{c}}$ contains only signed formulas of type ν which are not contracted in $\mathcal{P}_1$ then there exists a multiplicity μ_2 for which the dyadic sequent $S_2 = succ_1(\boldsymbol{\nu}_{\overline{c}}) : \Upsilon$ can be derived in Σ'_2.[2]

Proof. Lemma 16 shows that we can restrict ourselves to the case where $\mathcal{P}_1$ is a contraction normal proof. We construct a multiplicity μ_2 and a proof $\mathcal{P}_2$ for S_2 by induction over the structure of $\mathcal{P}_1$.
Base case: $\mathcal{P}_1$ consists only of a single rule application. Two cases need to be distinguished.

- If $\mathcal{P}_1 = \left\{ \dfrac{}{\langle A,+\rangle,\langle A,-\rangle}\ axiom \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{}{\cdot:\langle A,+\rangle,\langle A,-\rangle}\ axiom \right.$.
- If $\mathcal{P}_1 = \left\{ \dfrac{}{\tau}\ \tau \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{}{\cdot:\tau}\ \tau \right.$.

Induction hypothesis: The theorem holds for any subproof $\mathcal{P}'_1$ of $\mathcal{P}_1$.
Induction step: We make a case distinction depending on the last rule application in $\mathcal{P}_1$.

- If $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}'_1}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},succ_1(o)}}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},o}\ o \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}'_2}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\Upsilon,succ_1(o)}}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\Upsilon,o}\ o \right.$.
- The cases where one of the rules α or β is applied as the last rule in $\mathcal{P}_1$ can be shown similarly.
- If $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}'_1}{\Upsilon,\boldsymbol{\nu}_{\overline{c}}}}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},\omega}\ \omega \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}'_2}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\Upsilon}}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\Upsilon,\omega}\ \omega \right.$.
- The case where w is applied can be shown with the help of lemma 21.
- If $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}'_1}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},succ_1(\nu)}}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},\nu}\ \nu \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}'_2}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\Upsilon,succ_1(\nu)}}{succ_1(\boldsymbol{\nu}_{\overline{c}}),succ_1(\nu):\Upsilon}\ focus \right.$.
- If $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}'_1}{\boldsymbol{\nu}_{\overline{c}},succ_1(\pi)}}{\boldsymbol{\nu}_{\overline{c}},\pi}\ \pi \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}'_2}{succ_1(\boldsymbol{\nu}_{\overline{c}}):succ_1(\pi)}}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\pi}\ \pi \right.$.
- If $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}'_1}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},\nu^{n+1}}}{\Upsilon,\boldsymbol{\nu}_{\overline{c}},\nu}\ c^n \right\}$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}'_2}{succ_1(\boldsymbol{\nu}_{\overline{c}}),succ_1(\nu)^{\mu_2(\nu)}:\Upsilon}}{succ_1(\boldsymbol{\nu}_{\overline{c}}):\Upsilon,\nu}\ \nu \right.$
  with $\mu_2(\nu) = n + 1$ and $\mu_2(\nu') = \mu'_2(\nu')$ for any $\nu' \neq \nu$.
  c^n abbreviates n subsequent applications of c on the same formula.

In the above case distinction we have considered only the cases where the side formulas of the last rule application in $\mathcal{P}_1$ are not of type ν or otherwise are contracted in $\mathcal{P}_1$. In the case where side formulas of type ν are not contracted in $\mathcal{P}_1$ an additional application of ν in $\mathcal{P}_2$ is needed in order to move them into the

[2] Throughout this report for a multi-set of formulas Θ and a function f (like $succ_1$) which is defined on elements of Θ we abbreviate the multi-set $\{f(e) \mid e \in \Theta\}$ by $f(\Theta)$.

unbounded zone. Besides this little difference the proof is identical to the one of the case which has been shown above.

Theorem 23 (Correctness) Let Θ_1^+ and Θ_2^- be multi-sets of signed formulas which contain only positive and negative signed formulas respectively. If there exists a proof $\mathcal{P}_2$ in Σ_2' for a dyadic sequent $S_2 = \Theta_1^+, \Theta_2^- : \Upsilon$ with multiplicity μ_2 then there exists a proof $\mathcal{P}_1$ in Σ_1' for the unary sequent $S_1 = ?\Theta_1^+, !\Theta_2^-, \Upsilon$.[3]

Proof. We construct a proof $\mathcal{P}_1$ for S_1 by induction over the structure of the proof $\mathcal{P}_2$. We show how the rule applied last in $\mathcal{P}_2$ can be translated into a set of Σ_1'-rules. For applications of the rules w and c we abbreviate a sequence of n rule applications e.g. by w^n. For brevity we sometimes leave n unspecified and write e.g. w^*.
Base case: $\mathcal{P}_2$ consists only of a single rule application.

- If $\mathcal{P}_2 = \left\{ \dfrac{}{\Theta_1^+, \Theta_2^- : \langle A, +\rangle, \langle A, -\rangle}\ axiom \right.$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{}{\langle A, +\rangle, \langle A, -\rangle}\ axiom}{?\Theta_1^+, !\Theta_2^-, \langle A, +\rangle, \langle A, -\rangle}\ w^* \right.$.

- If $\mathcal{P}_2 = \left\{ \dfrac{}{\Theta_1^+, \Theta_2^- : \tau}\ \tau \right.$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{}{\tau}\ \mathcal{T}}{?\Theta_1^+, !\Theta_2^-, \tau}\ w^* \right.$.

Induction hypothesis: The theorem holds for any subproof $\mathcal{P}_2'$ of $\mathcal{P}_2$.
Induction step: We make a case distinction depending on the last rule application in $\mathcal{P}_2$.

- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta_1^+, \Theta_2^- : \Upsilon, succ_1(o)}}{\Theta_1^+, \Theta_2^- : \Upsilon, o}\ o \right.$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}_1'}{?\Theta_1^+, !\Theta_2^-, \Upsilon, succ_1(o)}}{?\Theta_1^+, !\Theta_2^-, \Upsilon, o}\ o \right.$.

- The cases where one of the rules α or β is applied as the last rule in $\mathcal{P}_2$ can be shown similarly.

- if $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta_1^+, \Theta_2^-, succ_1(\nu)^{\mu_2(\nu)} : \Upsilon}}{\Theta_1^+, \Theta_2^- : \Upsilon, \nu}\ \nu \right.$

  then if $\mu_2(\nu) > 0$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}_1'}{?\Theta_1^+, !\Theta_2^-, \Upsilon, \nu^{\mu_2(\nu)}}}{?\Theta_1^+, !\Theta_2^-, \Upsilon, \nu}\ c^{\mu_2(\nu)-1} \right.$

  and if $\mu_2(\nu) = 0$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}_1'}{?\Theta_1^+, !\Theta_2^-, \Upsilon}}{?\Theta_1^+, !\Theta_2^-, \Upsilon, \nu}\ w \right.$.

- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta_1^+, \Theta_2^- : succ_1(\pi)}}{\Theta_1^+, \Theta_2^- : \pi}\ \pi \right.$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}_1'}{?\Theta_1^+, !\Theta_2^-, succ_1(\pi)}}{?\Theta_1^+, !\Theta_2^-, \pi}\ \pi \right.$.

- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta_1^+, \Theta_2^- : \Upsilon, \langle F, {}^{+}_{-}\rangle}}{\Theta_1^+, \Theta_2^-, \langle F, {}^{+}_{-}\rangle : \Upsilon}\ focus \right.$ then $\mathcal{P}_1 = \left\{ \dfrac{\dfrac{\mathcal{P}_1'}{?\Theta_1^+, !\Theta_2^-, \Upsilon, \langle F, {}^{+}_{-}\rangle}}{?\Theta_1^+, !\Theta_2^-, \Upsilon, \langle {}^{?}_{!} F, {}^{+}_{-}\rangle}\ d \right.$.

[3]For a multi-set of formulas Θ and a unary connective c (like ? or !) we abbreviate the multi-set $\{\langle cF, k\rangle \mid \langle F, k\rangle \in \Theta\}$ by $c\Theta$.

3.2 A Triadic Calculus

The triadic calculus Σ'_3 extends Σ'_2 by the focusing principle. Under the focusing principle a formula must be focused before it may be reduced. In order to focus a formula no other formula must be focused. Once a formula is focused a sequence of rule applications on that formula is enforced. Hence the focusing principle considerably reduces the non-determinism in proof search.

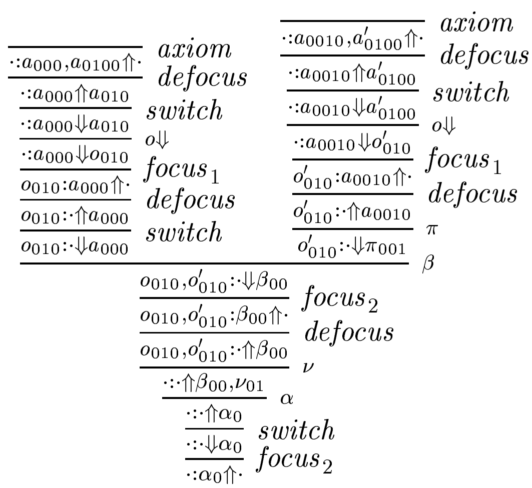
Definition 24 Let φ be a signed formula, Θ and Υ be multi-sets of signed formulas, and Ξ be a sequence of signed formulas. Then the triple $\Theta : \Upsilon \Uparrow \Xi$ as well as $\Theta : \Upsilon \Downarrow \varphi$ are *triadic sequents*. Θ is called the *unbounded zone*, Υ the *bounded zone*, and φ as well as Ξ the focused zone.

Definition 25 A *multiplicity function* μ *for a triadic sequent* $\Theta : \Upsilon \Uparrow \Xi$ *or* $\Theta : \Upsilon \Downarrow \varphi$ is a function which returns a natural number for signed subformulas of each signed formula in Θ, Υ, Ξ, and φ.

The triadic sequent calculus is depicted in table 3.2. Derivations of a triadic sequent S are defined with respect to a fixed multiplicity function μ for S. The multiplicity function is important whenever the rule ν is applied. A signed formula φ is derivable in Σ'_3 if the sequent $\cdot : \varphi \Uparrow \cdot$ is derivable for some multiplicity μ.

Example 26 We prove the signed formula $\varphi = \langle (A \otimes !A) \parr ?(A^\perp), + \rangle$ in Σ'_3. We abbreviate subformulas of φ as shown in the table.

$lab(\varphi)$	φ
$(A \otimes !A) \parr ?(A^\perp)$	α_0
$A \otimes !A$	β_{00}
A	a_{000}
$!A$	π_{001}
A	a_{0010}
$?(A^\perp)$	ν_{01}
$A^\perp$	o_{010}, o'_{010}
A	a_{0100}, a'_{0100}



After a formula has been moved into the focused zone by an application of $focus_1$ or $focus_2$ a couple of rule applications on that formula are enforced. After the formula has been focused the zone is in synchronous mode ($\Downarrow$). The synchronous mode enforces that exactly one formula is in the focused zone. This formula is reduced if it is of type o, β or π. While the mode remains synchronous after an application of $o\Downarrow$ or β it switches to asynchronous mode when π is applied. For all other types the only applicable rule is *switch* which enforces a switch into asynchronous mode. In asynchronous mode a (possibly empty) list of formulas is allowed in the focused zone. Formulas of type o, ω, α, and ν are reduced.

identity

$$\frac{}{\Theta : \langle A,+\rangle, \langle A,-\rangle \Uparrow \cdot}\ axiom$$

negation

$$\frac{\Theta : \Upsilon \Downarrow succ_1(o)}{\Theta : \Upsilon \Downarrow o}\ o\Downarrow \qquad \frac{\Theta : \Upsilon \Uparrow \Xi, succ_1(o)}{\Theta : \Upsilon \Uparrow \Xi, o}\ o\Uparrow$$

multiplicative fragment

$$\frac{}{\Theta : \tau \Uparrow \cdot}\ \tau \qquad \frac{\Theta : \Upsilon \Uparrow \Xi}{\Theta : \Upsilon \Uparrow \Xi, \omega}\ \omega$$

$$\frac{\Theta : \Upsilon \Uparrow \Xi, succ_1(\alpha), succ_2(\alpha)}{\Theta : \Upsilon \Uparrow \Xi, \alpha}\ \alpha$$

$$\frac{\Theta_1 : \Upsilon_1 \Downarrow succ_1(\beta) \quad \Theta_2 : \Upsilon_2 \Downarrow succ_2(\beta)}{\Theta_1, \Theta_2 : \Upsilon_1, \Upsilon_2 \Downarrow \beta}\ \beta$$

exponential fragment

$$\frac{\Theta, succ_1(\nu)^{\mu(\nu)} : \Upsilon \Uparrow \Xi}{\Theta : \Upsilon \Uparrow \Xi, \nu}\ \nu \qquad \frac{\Theta : \cdot \Uparrow succ_1(\pi)}{\Theta : \cdot \Downarrow \pi}\ \pi$$

focusing

$$\frac{\Theta : \Upsilon \Downarrow \varphi}{\Theta, \varphi : \Upsilon \Uparrow \cdot}\ focus_1 \qquad \frac{\Theta : \Upsilon \Downarrow \varphi}{\Theta : \Upsilon, \varphi \Uparrow \cdot}\ {focus_2}^{*}$$

$$\frac{\Theta : \Upsilon, \varphi \Uparrow \Xi}{\Theta : \Upsilon \Uparrow \Xi, \varphi}\ defocus^{**} \qquad \frac{\Theta : \Upsilon \Uparrow \varphi}{\Theta : \Upsilon \Downarrow \varphi}\ switch^{***}$$

$*$	In $focus_2$ φ must not be of type *lit* or τ.
$**$	In *defocus* φ must be of type *lit*, τ, β, or π.
$***$	In *switch* φ must be of type *lit*, τ, ω, α, or ν.

Table 5: A triadic sequent calculus Σ_3' for $\mathcal{MELL}$ in uniform notation

Only an application of ν moves the reduced formula into the unbounded zone. Formulas of other types must be moved from the focused zone into the bounded zone. When no formulas are left in the focused zone either one of the axiom rules *axiom*, τ or one of the focus rules may be applicable. That this principle of rule application is complete was discovered by Andreoli [And92]. It is called the *focusing principle*. Σ_3' differs from Andreolis triadic calculus mainly in the eager handling of resources by multiplicities.
Since the focusing principle enforces a sequence of rule applications after a formula has been focused less permutations of rule applications must be considered in proof search. Thus, the representation of the search space is more compact compared to a calculus like Σ_1'. The matrix characterization developed in this report exploits the focusing principle. However, it yields a representation with even less redundancies than a calculus like Σ_3' can.

Lemma 27 (Inversion) Let Θ, Υ be multi-sets of signed formulas and Ξ be a list of signed formulas.

1. If $\Theta_1 : \Upsilon_1 \Uparrow \mathit{succ}_1(\beta), \Xi_1$ and $\Theta_2 : \Upsilon_2 \Uparrow \mathit{succ}_2(\beta), \Xi_2$ can be derived in Σ_3' then so can $\Theta_1, \Theta_2 : \Upsilon_1, \Upsilon_2, \beta \Uparrow \Xi_1, \Xi_2$.
2. If $\Theta : \Upsilon \Uparrow \varphi, \Xi$ can be derived in Σ_3' then so can $\Theta, \varphi : \Upsilon \Uparrow \Xi$.
3. If $\Theta : \Upsilon \Uparrow \Xi$ can be derived in Σ_3' and $\Xi \equiv \Xi'$, i.e. the two sequences differ only in the order of elements, then $\Theta : \Upsilon \Uparrow \Xi'$ can be derived in Σ_3' as well.

Proof. The proofs of the inversion lemmas 1–3 are rather long. They can be carried out along the same lines as the proofs of the inversion lemmas in [And92].

Theorem 28 (Completeness) If there exists a proof $\mathcal{P}_2$ in Σ_2' for a dyadic sequent $S_2 = \Theta : \Upsilon$ for a multiplicity μ_2 then there exists a multiplicity μ_3 such that the triadic sequent $S_3 = \Theta : \cdot \Uparrow \Upsilon$ is derivable in Σ_3'.[4]

Proof. We construct a proof $\mathcal{P}_3$ for S_3 by induction over the structure of $\mathcal{P}_2$. For applications of the rule *defocus* we abbreviate a sequence of n rule applications by $\mathit{defocus}^n$. For brevity we sometimes leave n unspecified and write e.g. $\mathit{defocus}^*$.
Base case: $\mathcal{P}_3$ consists only of a single rule application.

- If $\mathcal{P}_2 = \left\{ \frac{}{\Theta:\langle A,+\rangle,\langle A,-\rangle}\ \mathit{axiom} \right.$ then $\mathcal{P}_3 = \left\{ \dfrac{\overline{\Theta:\langle A,+\rangle,\langle A,-\rangle \Uparrow \cdot}\ \mathit{axiom}}{\Theta:\cdot \Uparrow \langle A,+\rangle,\langle A,-\rangle}\ \mathit{defocus}^2 \right.$.
- if $\mathcal{P}_2 = \left\{ \frac{}{\Theta:\tau}\ \tau \right.$ then $\mathcal{P}_3 = \left\{ \dfrac{\overline{\Theta:\tau \Uparrow \cdot}\ \tau}{\Theta:\cdot \Uparrow \tau}\ \mathit{defocus} \right.$

Induction hypothesis: We assume that the theorem holds for any subproof $\mathcal{P}_2'$ of $\mathcal{P}_2$.
Induction step: We make a case distinction depending on the last rule application in $\mathcal{P}_2$. By lemma 27.3 we do not need to consider triadic sequents $\Theta : \Upsilon \Uparrow \Xi$ where the main formula φ of the last rule application in $\mathcal{P}_2$ does occur anywhere within Ξ. It is sufficient to consider the case where φ is at the end of Ξ.

[4] Note that the in the first place Υ is used as a multi-set of signed formulas but in the second as a sequence. Lemma 27.3 gives us the necessary freedom to do so.

- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta{:}\Upsilon, succ_1(o)}}{\Theta{:}\Upsilon, o}\, o \right.$ then $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta{:}\cdot\Uparrow\Upsilon, succ_1(o)}}{\Theta{:}\cdot\Uparrow\Upsilon, o}\, o\Uparrow \right.$.
- The case where one of the rules ω, α, or ν is applied as the last rule in $\mathcal{P}_2$ can be shown similarly. For the application of the ν-rule we set $\mu_3(\nu) = \mu_2(\nu)$.
- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta_1{:}\Upsilon_1, succ_1(\beta)} \qquad \dfrac{\mathcal{P}_2''}{\Theta_2{:}\Upsilon_2, succ_2(\beta)}}{\Theta_1,\Theta_2{:}\Upsilon_1,\Upsilon_2,\beta}\, \beta \right.$ then
  $\Theta_1 : \cdot \Uparrow \Upsilon_1, succ_1(\beta)$ and $\Theta_2 : \cdot \Uparrow \Upsilon_2, succ_2(\beta)$ can be derived by hypothesis. From lemma 27.1 we conclude that $\Theta_1, \Theta_2 : \beta \Uparrow \Upsilon_1, \Upsilon_2$ is derivable. By application of *defocus* we receive a proof for $\Theta_1, \Theta_2 : \cdot \Uparrow \Upsilon_1, \Upsilon_2, \beta$.
- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta{:}succ_1(\pi)}}{\Theta{:}\pi}\, \pi \right.$ then $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{P}_3'}{\Theta{:}\cdot\Uparrow succ_1(\pi)}}{\Theta{:}\cdot\Downarrow\pi}\,\pi}{\Theta{:}\pi\Uparrow\cdot}\, focus_2}{\Theta{:}\cdot\Uparrow\pi}\, defocus \right.$.
- If $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta{:}\Upsilon,\varphi}}{\Theta,\varphi{:}\Upsilon}\, focus \right.$ then
  $\Theta : \cdot \Uparrow \Upsilon, \varphi$ is derivable by hypothesis. By lemma 27.2 we can conclude, that $\Theta, \varphi : \cdot \Uparrow \Upsilon$ can be derived.

Theorem 29 (Correctness) If there exists a proof $\mathcal{P}_3$ in Σ_3' for a triadic sequent $S_3 = \Theta : \Upsilon \Uparrow \Xi$ (or $S_3 = \Theta : \Upsilon \Downarrow \varphi$) with multiplicity μ_3 then there exists a proof $\mathcal{P}_2$ in Σ_2' for the dyadic sequent $S_2 = \Theta : \Upsilon, \Xi$ (or $S_2 = \Theta : \Upsilon, \varphi$) with multiplicity μ_2.

Proof. We construct a proof $\mathcal{P}_2$ for S_2 by induction over the structure of $\mathcal{P}_3$. We show how the rule applied last in $\mathcal{P}_3$ can be translated into a set of Σ_2'-rules.
Base case: $\mathcal{P}_2$ consists only of a single rule application.

- if $\mathcal{P}_3 = \left\{ \dfrac{}{\Theta{:}\langle A,+\rangle,\langle A,-\rangle\Uparrow\cdot}\, axiom \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{}{\Theta{:}\langle A,+\rangle,\langle A,-\rangle}\, axiom \right.$
- if $\mathcal{P}_3 = \left\{ \dfrac{}{\Theta{:}\tau\Uparrow\cdot}\, \tau \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{}{\Theta{:}\tau}\, \tau \right.$

Induction hypothesis: We assume that the theorem holds for any subproof $\mathcal{P}_3'$ of $\mathcal{P}_3$.
Induction step: We make a case distinction depending on the last rule application in $\mathcal{P}_3$.

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta{:}\Upsilon\Downarrow succ_1(o)}}{\Theta{:}\Upsilon\Downarrow o}\, o\Downarrow \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta{:}\Upsilon, succ_1(o)}}{\Theta{:}\Upsilon, o}\, o \right.$.
- The cases where one of the rules $o\Uparrow$, ω, α, β, ν, or π is applied as the last rule in $\mathcal{P}_3$ can be shown similarly.
- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta{:}\Upsilon\Downarrow\varphi}}{\Theta,\varphi{:}\Upsilon\Uparrow\cdot}\, focus_1 \right.$ then $\mathcal{P}_2 = \left\{ \dfrac{\dfrac{\mathcal{P}_2'}{\Theta{:}\Upsilon,\varphi}}{\Theta,\varphi{:}\Upsilon}\, focus \right.$.
- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta_1{:}\Upsilon\Downarrow\varphi}}{\Theta_1{:}\Upsilon,\varphi\Uparrow\cdot}\, focus_2 \right.$ then $\mathcal{P}_2 = \mathcal{P}_2'$.

- The cases where one of the rules *defocus* or *switch* is applied as the last rule in $\mathcal{P}_3$ can be shown similarly.

3.3 Relation to Andreolis Work

Andreoli [And92] developed his triadic calculus as a foundation of the logic programming language *LinLog*. His calculus is intended for actual proof search. The dyadic calculus served as an intermediate step in the development of the triadic calculus from an ordinary sequent calculus like the one in table 1.
Our calculi Σ_2' and Σ_3' are quite similar to Andreolis calculi. However, they are not intended for proof search but both calculi serve as intermediate steps in the development of a matrix characterization. For this purpose Andreolis calculi were modified in some aspects. The most fundamental change is the introduction of the notion of multiplicities, i.e. the eager handling of contraction. Besides that the following adaptions have been made.

- In order to reduce the number of sequent rules an adaption of Smullyans uniform notation to linear logic is used rather than a restriction to one-sided sequents.
- The unbounded resources are distributed onto the premises when the β-rule is applied. In Andreolis calculi they are copied to both premises.
- The *axiom* and τ rule are modified.

One should note that Andreolis calculi cover all of first order linear logic while Σ_2' and Σ_3' cover only propositional $\mathcal{MELL}$. Using Andreolis results, an extension to full linear logic would be possible. Nevertheless, in order to serve as a useful intermediate step in the development of a matrix characterization for these fragments further modifications to Andreolis calculi will be necessary. What proper modifications would be is currently not known.

4 Position Calculus

The position calculus presented in this section is the last intermediate step in the development of the matrix characterization. In this calculus sequents of positions are the objects to be proven. Thus, positions play the role of signed formulas in the previous section. Although positions might appear subtle at first sight they simplify the proof of the characterization theorem in the subsequent section tremendously. The difference between an occurrence of a subformula in a formula and an occurrence in a derivation is reflected by the difference between the notion of basic positions and the notion of positions.
In subsection 4.1 and 4.2 the concepts of basic position trees and position trees are introduced. These trees are constructed from formula trees using a set of general rewrite rules and a specific multiplicity. Position forests are defined in subsection 4.3. They are the fundamental syntactic concept since position sequents (in subsection 4.4) and matrices (in section 5) are defined as position trees. A calculus for position sequents is presented in subsection 4.4 and is shown to be sound and complete.

4.1 Basic Position Trees

Definition 30 A *basic position bp* has a label $lab(bp) \in wff$, a polarity $pol(bp) \in \{+,-\}$, and a type $Ptype(bp) \in \{\alpha, \beta, \nu, \pi, o, \tau, \omega, lit, \phi^M, \psi^M, \phi^E, \psi^E\}$. A basic position with type *lit* is called atomic.

Note that a signed formula can have a type from $\{\alpha, \beta, \nu, \pi, o, \tau, \omega, lit\}$ only. The additional types ϕ^M, ψ^M, ϕ^E, and ψ^E may be associated with basic positions. Basic positions which have one of these types are called *special basic positions*. In the sequel we will identify (non-special) basic positions and the corresponding signed formulas.

We use *bp* as a meta-variable for basic position. α, β, ν, π, o, τ, ω, ϕ^M, ψ^M, ϕ^E, and ψ^E are used as meta-variables for basic positions of the respective type. a is used as meta-variable for basic positions of type *lit*. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$, Π, $\boldsymbol{o}$, $\boldsymbol{\tau}$, Ω, Φ^M, Ψ^M, Φ^E, and Ψ^E are used as meta-variables for multi-sets of basic positions of the respective type.

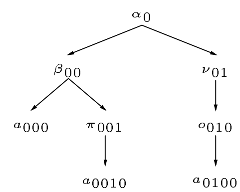
The function *con* returns for a basic position *bp* the main connective of $lab(bp)$. $sform(bp) = \langle lab(bp), pol(bp)\rangle$ is the signed formula corresponding to *bp*.

Let V be a set of basic positions and E be a set of pairs of basic positions. For a directed graph $\mathcal{G}_b = (V, E)$ where V is the set of nodes and E is the set of edges we define a couple of rewrite rules in figure 1. A rewrite rule R can be applied if a subgraph matches the left hand side of the rule. The dotted arrows match edges with the direction depicted or linear subgraphs which contain only nodes of type o where all edges in the subgraph have the direction depicted. A special case are the rewrite rules R_a and R_τ. They can only be applied if a node of type *lit* or τ has either no incoming adjacent edge or there is at most one incoming edge which is adjacent to a linear subgraph which contains only o-type basic positions and has no incoming edge. When a rewrite rule R is applied to a subgraph $\mathcal{G}'$ then $\mathcal{G}'$ may be rewritten to the right hand side of R. Note that the rewrite rules remove edges, insert special basic positions and edges but do not remove any nodes. The label and polarity of an inserted node equals that of the successor node, i.e. $lab(\phi^M) = lab(succ_1(\phi^M))$ and $pol(\phi^M) = pol(succ_1(\phi^M))$. The only exception are inserted nodes of type ψ^E for which they equal the corresponding value of the predecessor node. The rewrite rules are fundamental for the definition of basic position trees.

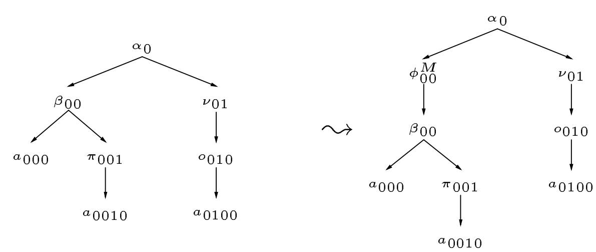
The *basic position tree* for a signed formula φ is a directed tree $\mathcal{T}_b = (V, E)$ where the set of nodes V is a set of basic positions. $\mathcal{T}_b$ originates from the formula tree of φ where each node in the formula tree is understood as a basic position. In order to construct the basic position tree $\mathcal{T}_b$ the rewrite rules in figure 1 must be applied as long as any of them are applicable. From now on we will consider basic position trees only rather than arbitrary graphs of basic positions.

The function $succ_1$ and $succ_2$ are redefined for basic position trees such that for a node $v \in V$, $succ_1(v)$ and $succ_2(v)$ equal the left and right successor of v, respectively. The orderings $\prec$ and $\ll$ are defined from $succ_1$ and $succ_2$ like for formulas, i.e. $bp_1 \prec bp_2$ iff $bp_2 = succ_1(bp_1)$ or $bp_2 = succ_2(bp_1)$. $\ll$ is the transitive closure of $\prec$.

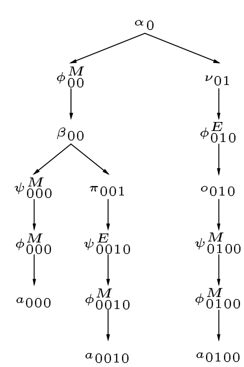
Example 31 We explain the application of a rewrite rule at the example of the rule R^{α}_{β}. The formula tree for $\langle (A \otimes !A) \parr ?(A^{\perp}), + \rangle$ is displayed below. Signed subformulas are abbreviated like in example 26.



β_{00} is a successor of α_0. There is no node in between the two nodes. Thus, the subtree consisting of α_0, β_{00}, and the edge which links these nodes matches the left hand side pattern of the rule R^{α}_{β}. The tree is rewritten to the tree depicted below.



This tree is neither a formula tree nor a position tree. The rewrite rules R^{β}_{a}, R^{π}_{a}, and R^{ν}_{a} are still applicable. After these rewrite rules have been applied the following basic position tree results.



The rewrite system in figure 1 separates layers of subformulas within a formula tree. A rewrite rule $R^{t_1}_{t_2}$ inserts special positions wherever a subformula of type t_1 has a subformula of type t_2. Thus, layers of type t_1 are separated from layers of type t_2. The rewrite system is confluent and Noetherian, i.e. the order of rule applications does not matter and the insertion of positions will eventually terminate.

Basic positions are a technique to identify different occurrences of a formula within a formula tree. Theoretically, it is not essential how basic positions look like and how they are assigned to the nodes of a tree as long as they have the above properties. However, for technical reasons we define a specific denotation. Basic positions are denoted by strings over $\{0, 1\}$ which point into a basic position tree. Let $\mathcal{T}_b = (V, E)$ be a basic position tree. The denotation of a basic position is assigned recursively to vertices from V. The root of $\mathcal{T}_b$ is denoted by '0'. If a vertex v with denotation 'p' has one successor vertex v_1 in the tree then v_1 is

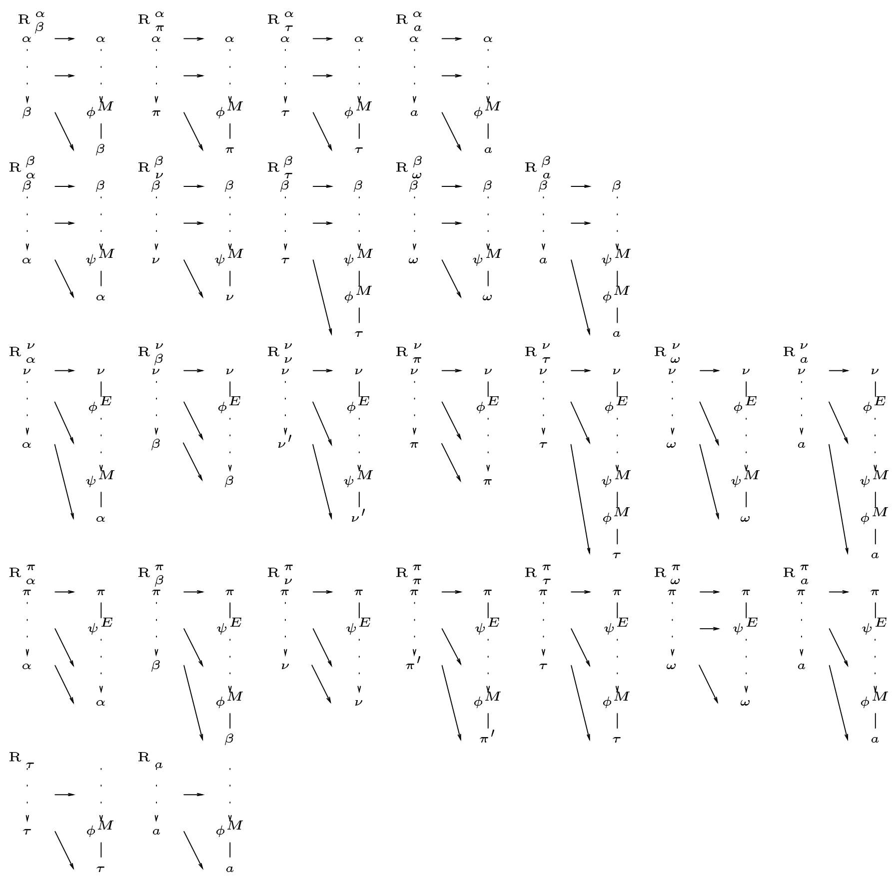
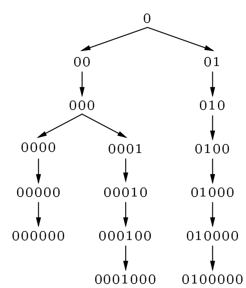


Figure 1: Insertion of special positions for basic position trees

denoted by '$p0$'. If v has two successors v_1 and v_2 then v_1 is denoted by '$p0$' and v_2 by '$p1$'. In the sequel we will sometimes refer to positions by their denotation. Note that we have used a similar numbering technique already in example 12. Appending 0 to the position of a node with two successors results in the position of the left successor and appending 1 in the position of the right successor. Thus, the denotation of a basic position indicates the path from the root of a tree to a specific node.

Example 32 The basic position tree for the formula $\langle (A \otimes !A) \parr ?(A^{\perp}), + \rangle$ is depicted below. Each node of the tree is marked with the basic position. The types and connectives for the basic positions are depicted in the table. Note, that position 00010 has connective ! and not A.

bp	$Ptype(bp)$	$con(bp)$	bp	$Ptype(bp)$	$con(bp)$
0	α	$\parr$	00	ϕ^M	$\otimes$
000	β	$\otimes$	0000	ψ^M	A
00000	ϕ^M	A	000000	lit	A
0001	π	!	00010	ψ^E	!
000100	ϕ^M	A	0001000	lit	A
01	ν	?	010	ϕ^E	$\perp$
0100	o	$\perp$	01000	ψ^M	A
010000	ϕ^M	A	0100000	lit	A



4.2 Position Trees

There is a one-to-one correspondence between non-special basic positions in a position tree and occurrences of subformulas in a formula tree. In this subsection we define the concept of *positions* as an analog to occurrences of subformulas in a derivation. A *position tree* originates from a basic position tree by copying subtrees with a root of type ν according to a multiplicity function.
A position p has a label $lab(p)$, a polarity $pol(p)$, and a type $Ptype(p)$ like a basic position. We denote positions by strings over $\{0, 1, 0^n\}_{n \in \mathbb{N}}$. Essentially, denotations of positions equal that of basic positions where copies of ν-positions are distinguished by different exponents.
We use p as meta-variable for positions. α, β, ν, π, o, τ, ω, ϕ^M, ψ^M, ϕ^E, and ψ^E are used as meta-variables for positions of the respective type. a is used as meta-variable for positions of type *lit*. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$, Π, $\boldsymbol{o}$, $\boldsymbol{\tau}$, Ω, Φ^M, Ψ^M, Φ^E, and Ψ^E are used as meta-variables for multi-sets of positions of the respective type. Ξ is used as meta-variable for lists of positions.

Definition 33 A *multiplicity function μ for positions* is a function from positions to natural numbers. For any position p, $\mu(p)$ is called the *multiplicity* of p.

Remark 34 The difference between basic positions and positions is required because of the resource sensitivity of linear logic. The positions in [Wal90] correspond to our basic positions. The logics investigated there are not resource sensitive. For a resource sensitive logic the notion of multiplicity must be based on positions rather than on basic positions.

Definition 35 Let $\mathcal{T}_b = (V_b, E_b)$ be a basic position tree and μ be a multiplicity function. We define the *set of positions Pos for* $\mathcal{T}_b$ *and* μ recursively. Each position p is associated with a corresponding basic position $bp(p)$.

- According to the denotation of basic positions the root of $\mathcal{T}_b$ is denoted by 0.
  - 0 is a position in Pos with $bp(0) = 0$.
- Let p be a position in Pos with corresponding basic position $bp(p)$.
  - If $Ptype(p) \in \{\alpha, \beta\}$ then $p0$ and $p1$ are positions in Pos with corresponding basic positions $bp(p0) = bp(p)0$ and $bp(p1) = bp(p)1$.
  - If $Ptype(p) \in \{o, \pi, \phi^M, \psi^M, \phi^E, \psi^E\}$ then $p0$ is a position in Pos with corresponding basic position $bp(p0) = bp(p)0$.
  - If $Ptype(p) = \nu$ then for every $m \leq \mu(p)$ $p0^m$ is a position in Pos with corresponding basic position $bp(p0^m) = bp(p)0$.

Definition 36 Let $\mathcal{T}_b$ be a basic position tree and μ a multiplicity function. Let V be the set of positions for $\mathcal{T}_b$ and μ. Let E be a set of edges such that E contains an edge from position p to all positions $p0$, $p1$, and $p0^i$ which are contained in V. Then $\mathcal{T} = (V, E)$ is a position tree for $\mathcal{T}_b$ and μ.

We define functions $succ_1$, $succ_2$, and $succ_1^i$ for $i \in \mathbb{N}$. If p has a left successor $p0$ in the position tree then $succ_1(p) = p0$ holds. If p has a right successor $p1$ in the position tree then $succ_2(p) = p1$ holds. If p has a successor $p0^i$ in the position tree then $succ_1^i(p) = p0^i$ holds. In all other cases the functions are undefined. The functions *lab*, *con*, *pol*, *type*, and *sform* are defined with respect to basic positions, i.e. $lab(p) = lab(bp(p))$, $con(p) = con(bp(p))$, $pol(p) = pol(bp(p))$, $Ptype(p) = Ptype(bp(p))$, and $sform(p) = sform(bp(p))$.
Position trees can be represented graphically like basic position trees.

Example 37 The position tree for the formula $\langle (A \otimes !A) \parr ?(A^\perp), + \rangle$ and the multiplicity μ with $\mu(01) = 2$ is depicted in figure 2.

4.3 Position Forests

As formulas can be represented by formulas trees, sequents can be represented by *sequent forests*, i.e. collections of formula trees. For each formula in a sequent there exists a tree in the corresponding forest. Different zones in a sequent are represented by different zones in the forest. For example, a sequent forest for a dyadic sequent has one zone with formula trees from the unbounded zone and one zone with formula trees from the bounded zone. Triadic sequents for signed formulas require sequent forests with three zones.

Example 38 The sequent forest for the sequent $o_{010}, o'_{010} : \cdot \Uparrow \beta_{00}$ from example 26 is depicted below.

bp	$Ptype(bp)$	$con(bp)$
0	α	⅋
00	ϕ^M	$\otimes$
000	β	$\otimes$
0000	ψ^M	A
00000	ϕ^M	A
000000	lit	A
0001	π	!
00010	ψ^E	!
000100	ϕ^M	A
0001000	lit	A
01	ν	?
010^1,010^2	ϕ^E	A
$010^1 0$,$010^2 0$	o	$\perp$
$010^1 00$,$010^2 00$	ψ^M	A
$010^1 000$,$010^2 000$	ϕ^M	A
$010^1 0000$,$010^2 0000$	lit	A

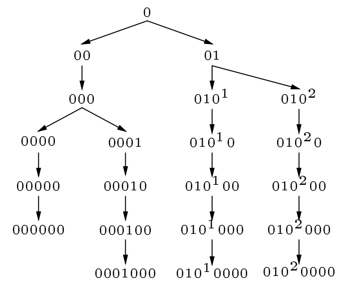
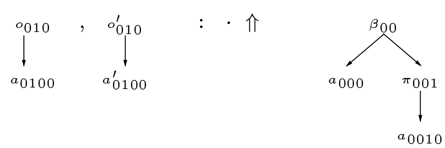
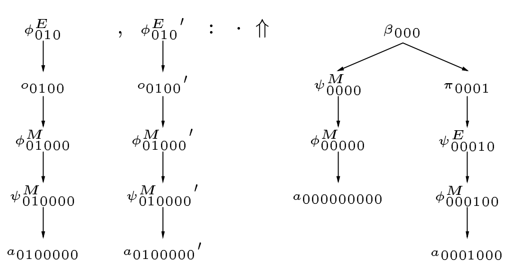


Figure 2: A Position Tree (Example 37)



For a triadic sequent $\Theta : \Upsilon \Uparrow \Xi$ or $\Theta : \Upsilon \Downarrow \varphi$ we define the corresponding *basic position forest* $\mathcal{F}_b$. Like a triadic sequent the sequent forest $\mathcal{F}_b$ has a mode from $\{\Uparrow, \Downarrow\}$. $\mathcal{F}_b$ has an unbounded, a bounded, and a focused zone. For each formula in Θ, Υ, or Ξ and for φ the corresponding basic position tree is added to the respective zone. Then the basic position trees are modified at their roots by the rewrite rules in figure 3. The exponent of a rewrite rule defines the zone in which the rule can be applied. The rules $\mathrm{R}^{\Theta}_{?}$, $\mathrm{R}^{\Upsilon}_{?}$, $\mathrm{R}^{\Xi}_{?}$, and $\mathrm{R}^{\varphi}_{?}$ can be applied on trees in the unbounded zone, bounded zone, focused zone (mode $\Uparrow$) and focused zone (mode $\Downarrow$), respectively. Note that the rewrite rules $\mathrm{R}^{\Upsilon}_{\tau}$ and $\mathrm{R}^{\Upsilon}_{a}$ require a node of type o at the root of the pattern.

Example 39 The basic position forest for the sequent $o_{010}, o'_{010} : \cdot \Uparrow \beta_{00}$ from example 26 is depicted below.



A *position forest* is constructed from a basic position forest $\mathcal{F}_b$ and a multiplicity μ. Each tree in $\mathcal{F}_b$ is modified to a position tree by the transformation described

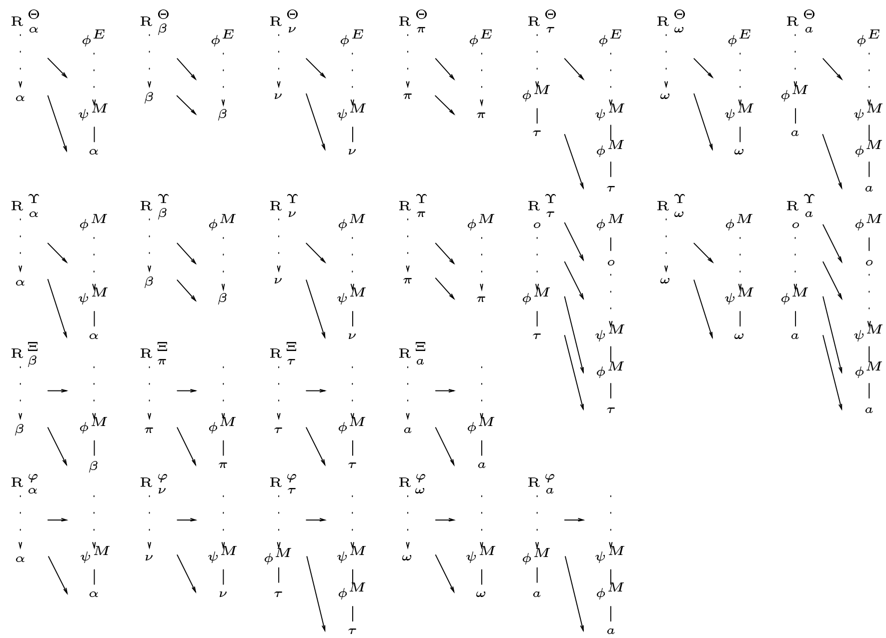


Figure 3: Insertion of special positions for basic position sequents

in definition 35 and 36. The modifications by rewrite rules from figure 3 to the roots of the trees are preserved.
Position forests will be the fundamental syntactic concept for the subsequent subsections and sections. In the position calculus introduced in subsection 4.4 position forests are understood as sequents. In the presentation of the matrix characterization position forests will be understood as matrices.

4.4 A Position Calculus

Definition 40 A *position sequent* is a position forest with three zones and a mode from $\{\Uparrow, \Downarrow\}$.

Usually, we denote a tree by its root if the rest of the tree is obvious from the context. Due to the definition of the rewrite rules, trees in the unbounded zone always have a root of type ϕ^E, trees in the bounded zone always have a root of type ϕ^M, trees in the focused zone always have a type from $\{\alpha, \nu, o, \omega, \phi^M\}$ when the sequent is in asynchronous mode and have a root of type β, π, o, ψ^M, or ψ^E when the sequent is in synchronous mode.
A sequent calculus Σ_{pos} for position sequents is depicted in table 4.4. Note that, derivations of a position sequent S are defined with respect to a fixed multiplicity function μ for S.
A close look at the position calculus reveals the motivation of the insertion of special positions. When a triadic sequent is transformed into a position sequent, the rewrite rules $\mathrm{R}^{\Theta}_{?}$ guarantee that each tree in the unbounded zone has a root of type ϕ^E and the rewrite rules $\mathrm{R}^{\Upsilon}_{?}$ guarantee that each tree in the bounded zone has a root of type ϕ^M.[5] These are hereditary properties, i.e. the rewrite

[5]The question mark represents any of α, β, ν, π, o, ω, τ, a for which a rewrite rule is defined.

rules in figure 1 are defined so that in a position calculus derivation all sequents have these properties. The insertion of a node of type ϕ^E by a rewrite rule $\mathrm{R}^{\nu}_{?}$ expresses the potential of a subtree to move into the unbounded zone during a derivation. Similarly, the insertion of a node of type ϕ^M by a rule expresses the potential of subtrees to move into the bounded zone during a derivation.
For most rules in the position calculus there is a one-to-one correspondence to a rule of the triadic calculus (depicted in table 3.2). However, there are subtle differences. The π rule of the position calculus does not switch the mode of the sequent. The mode is switched by the rules ψ^M and ψ^E only. The ψ^M rule corresponds to the *switch*-rule of Σ_3'. A rewrite rule $\mathrm{R}^{\pi}_{?}$ guarantees that after a node of type π always a node of type ψ^E is inserted. Thus, the π rule together with the ψ^E rule resemble the π rule of the triadic calculus. The reduction of the π type formula and the switching of the mode are split into separate rule applications. The ψ^E rule requires an empty bounded context.

Theorem 41 (Completeness) If there exists a proof $\mathcal{P}_3$ in Σ_3' for a triadic sequent $S_3 = \Theta : \Upsilon \Uparrow \Xi$ or $S_3 = \Theta : \Upsilon \Downarrow \varphi$ for a multiplicity μ_3 then there exists a multiplicity $\widetilde{\mu_3}$ such that the corresponding position sequent $\widetilde{S_3}$ is derivable in Σ_{pos}.

Proof. We denote the position tree corresponding to a signed formula φ by $\widetilde{\varphi}$. The list of position trees corresponding to a list of signed formulas Ξ is denoted by $\widetilde{\Xi}$. We denote the multi-set of position trees corresponding to multi-sets of signed formulas Θ and Υ by $\widetilde{\Theta}$ and $\widetilde{\Upsilon}$. The position sequent corresponding to a triadic sequent S_3' is denoted by $\widetilde{S_3'}$. According to the definition of position sequents $\widetilde{S_3} = \Phi^E : \Phi^M \Uparrow \widetilde{\Xi}$ or $\widetilde{S_3} = \Phi^E : \Phi^M \Downarrow \widetilde{\varphi}$ holds where $\widetilde{\Xi}$ contains only positions with type in $\{\alpha, \nu, o, \omega, \phi^M\}$ and $\widetilde{\varphi}$ has a type in $\{\beta, \pi, o, \psi^M, \psi^E\}$.
We construct a multiplicity $\widetilde{\mu_3}$ and a proof $\widetilde{\mathcal{P}_3}$ for $\widetilde{S_3}$ by induction over the structure of $\mathcal{P}_3$.
Base cases:

- If $\mathcal{P}_3 = \left\{ \frac{}{\Theta:\langle A,+\rangle,\langle A,-\rangle \Uparrow \cdot}\ axiom \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\langle A,+\rangle}, \widetilde{\langle A,-\rangle} \Uparrow \cdot$. ϕ_1^M and ϕ_2^M occur at the roots of $\widetilde{\langle A,+\rangle}$ and $\widetilde{\langle A,-\rangle}$ according to the rewrite rule R_{a}. No other special position is inserted into the two trees. $Ptype(\phi_1^M) = lit = Ptype(\phi_2^M)$, $lab(\phi_1^M) = lab(\phi_2^M)$, and $pol(\phi_1^M) \neq pol(\phi_2^M)$. The *axiom* rule is applicable on $\widetilde{S_3}$.
$\widetilde{\mathcal{P}_3} = \left\{ \frac{}{\widetilde{\Theta} : \phi_1^M, \phi_2^M \Uparrow \cdot}\ axiom \right.$
- The case where $\mathcal{P}_3$ consists only of one application of the τ rule can be shown similarly.

Induction hypothesis: The theorem holds for any subproof $\mathcal{P}_3'$ of $\mathcal{P}_3$.
Induction step: We make a case distinction depending on the last rule application in $\mathcal{P}_3$. Only the first case is shown in full detail.

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta:\Upsilon\Downarrow succ_1(o)}}{\Theta:\Upsilon\Downarrow o}\ o\Downarrow \right.$ then $\widetilde{S_3} = \Phi^E : \Phi^M \Downarrow \widetilde{o}$.
Let $S_3' = \Theta : \Upsilon \Downarrow succ_1(o)$. We distinguish the following cases depending on the type of $succ_1(o)$ and show that $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow succ_1(\widetilde{o})$.

$$\frac{}{\Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot}\ axiom^*$$

$$\frac{\Phi^E : \Phi^M \Downarrow succ_1(o)}{\Phi^E : \Phi^M \Downarrow o}\ o\Downarrow \qquad \frac{\Phi^E : \Phi^M \Uparrow \Xi, succ_1(o)}{\Phi^E : \Phi^M \Uparrow \Xi, o}\ o\Uparrow$$

$$\frac{}{\Phi^E : \phi_1^M \Uparrow \cdot}\ \tau^{**} \qquad \frac{\Phi^E : \Phi^M \Uparrow \Xi}{\Phi^E : \Phi^M \Uparrow \Xi, \omega}\ \omega$$

$$\frac{\Phi^E : \Phi^M \Uparrow \Xi, succ_1(\alpha), succ_2(\alpha)}{\Phi^E : \Phi^M \Uparrow \Xi, \alpha}\ \alpha$$

$$\frac{\Phi_1^E : \Phi_1^M \Downarrow succ_1(\beta) \quad \Phi_2^E : \Phi_2^M \Downarrow succ_2(\beta)}{\Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta}\ \beta$$

$$\frac{\Phi^E : \Phi^M \Downarrow succ_1(\pi)}{\Phi^E : \Phi^M \Downarrow \pi}\ \pi \qquad \frac{\Phi^E, succ_1^1(\nu), \ldots, succ_1^{\mu(\nu)}(\nu) : \Phi^M \Uparrow \Xi}{\Phi^E : \Phi^M \Uparrow \Xi, \nu}\ \nu$$

$$\frac{\Phi^E : \Phi^M \Downarrow succ_1(\phi_1^E)}{\Phi^E, \phi_1^E : \Phi^M \Uparrow \cdot}\ focus_1 \qquad \frac{\Phi^E : \Phi^M \Downarrow succ_1(\phi_1^M)}{\Phi^E : \Phi^M, \phi_1^M \Uparrow \cdot}\ {focus_2}^{***}$$

$$\frac{\Phi^E : \Phi^M, \phi_1^M \Uparrow \Xi}{\Phi^E : \Phi^M \Uparrow \Xi, \phi_1^M}\ \phi^M \qquad \frac{\Phi^E : \Phi^M \Uparrow succ_1(\psi_1^M)}{\Phi^E : \Phi^M \Downarrow \psi_1^M}\ \psi^M$$

$$\frac{\Phi^E : \cdot \Uparrow succ_1(\psi_1^E)}{\Phi^E : \cdot \Downarrow \psi_1^E}\ \psi^E$$

$*$	In *axiom* must hold $Ptype(succ_1(\phi_1^M)) = lit = Ptype(succ_1(\phi_2^M))$, $lab(\phi_1^M) = lab(\phi_2^M)$, and $pol(\phi_1^M) \neq pol(\phi_2^M)$.
$**$	In τ must hold $Ptype(succ_1(\phi_1^M)) = \tau$.
$***$	In $focus_2$ must hold $Ptype(succ_1(\phi_1^M)) \notin \{lit, \tau\}$.

Table 6: A position calculus Σ_{pos} for $\mathcal{MELL}$ in uniform notation

– If $succ_1(o) = \alpha$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ with $succ_1(\psi^M) = \alpha$ according to $\mathrm{R}^{\varphi}_{\alpha}$. The rewrite rule $\mathrm{R}^{\varphi}_{\alpha}$ ensures that in $\widetilde{S}_3$ the position ψ^M occurs between o and α.

– If $succ_1(o) = \beta$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\beta}$. In $\widetilde{S}_3$ no position occurs between o and β.

– If $succ_1(o) = \nu$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ with $succ_1(\psi^M) = \nu$ according to $\mathrm{R}^{\varphi}_{\nu}$. The rewrite rule $\mathrm{R}^{\varphi}_{\nu}$ ensures that in $\widetilde{S}_3$ the position ψ^M occurs between o and ν.

– If $succ_1(o) = \pi$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\pi}$. In $\widetilde{S}_3$ no position occurs between o and π.

– If $succ_1(o) = o'$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{o'}$. In $\widetilde{S}_3$ no position occurs between o and o'.

– If $succ_1(o) = \omega$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ with $succ_1(\psi^M) = \omega$ according to $\mathrm{R}^{\varphi}_{\omega}$. The rewrite rule $\mathrm{R}^{\varphi}_{\omega}$ ensures that in $\widetilde{S}_3$ the position ψ^M occurs between o and ω.

– If $succ_1(o) = \tau$ then $\widetilde{S}_3{}' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ with $succ_1(\psi^M) = \phi^M$ and $succ_1(\phi^M) = \tau$ according to $\mathrm{R}^{\varphi}_{\tau}$. This rewrite rule ensures that in $\widetilde{S}_3$ the positions ψ^M and ϕ^M occur between o and τ.

– If $succ_1(o) = a$ then $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\psi^M}$ with $succ_1(\psi^M) = \phi^M$ and $succ_1(\phi^M) = a$ according to R^{φ}_{a}. This rewrite rule ensures that in $\widetilde{S}_3$ the positions ψ^M and ϕ^M occur between o and a.

By induction hypothesis a proof $\widetilde{\mathcal{P}_3'}$ for $\widetilde{S}_3'$ exists. We construct a proof for $\widetilde{S}_3$ by an application of $o \Downarrow$.

$$\widetilde{\mathcal{P}_3} = \left\{ \frac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Downarrow succ_1(\widetilde{o})}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Downarrow\widetilde{o}}\, o \Downarrow \right.$$

- The case where $o \Uparrow$ is the last rule applied in $\mathcal{P}_3$ can be shown similarly.

- The case where ω is the last rule applied in $\mathcal{P}_3$ is trivial since no rewrite rule inserts special positions at the root of a tree consisting just of an ω-type formula which is in the focused zone in mode $\Uparrow$.

- If $\mathcal{P}_3 = \left\{ \frac{\dfrac{\mathcal{P}_3'}{\Theta:\Upsilon\Uparrow\Xi, succ_1(\alpha), succ_2(\alpha)}}{\Theta:\Upsilon\Uparrow\Xi,\alpha}\, \alpha \right.$ then $\widetilde{S}_3 = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}, \widetilde{\alpha}$.

  Let $S_3' = \Theta : \Upsilon \Uparrow \Xi, succ_1(\alpha), succ_2(\alpha)$. According to the rewrite rules $\mathrm{R}^{\Xi}_{?}$ and $\mathrm{R}^{\alpha}_{?}$ holds $\widetilde{S}_3' = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}, succ_1(\widetilde{\alpha}), succ_2(\widetilde{\alpha})$. Thus, after an application of α on $\widetilde{S}_3$ the induction hypothesis can be applied.

$$\widetilde{\mathcal{P}_3} = \left\{ \frac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Uparrow\widetilde{\Xi}, succ_1(\widetilde{\alpha}), succ_2(\widetilde{\alpha})}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Uparrow\widetilde{\Xi},\widetilde{\alpha}}\, \alpha \right.$$

- If $\mathcal{P}_3 = \left\{ \frac{\dfrac{\mathcal{P}_3'}{\Theta_1:\Upsilon_1\Downarrow succ_1(\beta)} \quad \dfrac{\mathcal{P}_3''}{\Theta_2:\Upsilon_2\Downarrow succ_2(\beta)}}{\Theta_1,\Theta_2:\Upsilon_1,\Upsilon_2\Downarrow\beta}\, \beta \right.$ then

  $\widetilde{S}_3 = \widetilde{\Theta_1}, \widetilde{\Theta_2} : \widetilde{\Upsilon_1}, \widetilde{\Upsilon_2} \Downarrow \widetilde{\beta}$. Let $S_3' = \Theta_1 : \Upsilon_1 \Downarrow succ_1(\beta)$ and $S_3'' =$

$\Theta_2 : \Upsilon_2 \Downarrow succ_2(\beta)$. According to the rewrite rules $\mathrm{R}^{\varphi}_{?}$ and $\mathrm{R}^{\beta}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta_1} : \widetilde{\Upsilon_1} \Downarrow succ_1(\widetilde{\beta})$ and $\widetilde{S_3''} = \widetilde{\Theta_2} : \widetilde{\Upsilon_2} \Downarrow succ_2(\widetilde{\beta})$. Therefore, after an application of β the induction hypothesis can be applied.

$$\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}'}}{\widetilde{\Theta_1}:\widetilde{\Upsilon_1}\Downarrow succ_1(\widetilde{\beta})} \qquad \dfrac{\widetilde{\mathcal{P}_3''}}{\widetilde{\Theta_2}:\widetilde{\Upsilon_2}\Downarrow succ_2(\widetilde{\beta})}}{\widetilde{\Theta_1},\widetilde{\Theta_2}:\widetilde{\Upsilon_1},\widetilde{\Upsilon_2}\Downarrow\widetilde{\beta}}\ \beta \right.$$

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta, succ_1(\nu)^{\mu(\nu)}:\Upsilon\Uparrow\Xi}}{\Theta:\Upsilon\Uparrow\Xi,\nu}\ \nu \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}, \widetilde{\nu}$.

  Let $S_3' = \Theta, succ_1(\nu)^{\mu(\nu)} : \Upsilon \Uparrow \Xi$. According to the rewrite rules $\mathrm{R}^{\Theta}_{?}$ and $\mathrm{R}^{\nu}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta}, succ_1(\widetilde{\nu})^{\mu(\nu)} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}$. Therefore , after an application of ν the induction hypothesis can be applied.

$$\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}, succ_1^1(\widetilde{\nu}),\ldots,succ_1^{\mu(\nu)}(\widetilde{\nu}):\widetilde{\Upsilon}\Uparrow\widetilde{\Xi}}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Uparrow\widetilde{\Xi},\widetilde{\nu}}\ \nu \right.$$

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta:\cdot\Uparrow succ_1(\pi)}}{\Theta:\cdot\Downarrow\pi}\ \pi \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \cdot \Downarrow \widetilde{\pi}$.

  Let $S_3' = \Theta : \cdot \Uparrow succ_1(\pi)$. According to the rewrite rules $\mathrm{R}^{\Xi}_{?}$ and $\mathrm{R}^{\pi}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow succ_1(succ_1(\widetilde{\pi}))$ where $succ_1(\widetilde{\pi}) = \psi^E$. Therefore, after an application of π and ψ^E the induction hypothesis can be applied.

$$\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\cdot\Uparrow succ_1(succ_1(\widetilde{\pi}))}}{\widetilde{\Theta}:\cdot\Downarrow succ_1(\widetilde{\pi})}\ \psi^E}{\widetilde{\Theta}:\cdot\Downarrow\widetilde{\pi}}\ \pi \right.$$

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta:\Upsilon\Downarrow\varphi}}{\Theta,\varphi:\Upsilon\Uparrow\cdot}\ focus_1 \right.$ then $\widetilde{S_3} = \widetilde{\Theta}, \widetilde{\varphi} : \widetilde{\Upsilon} \Uparrow \cdot$.

  Let $S_3' = \Theta : \Upsilon \Downarrow \varphi$. According to the rewrite rules $\mathrm{R}^{\Theta}_{?}$ and $\mathrm{R}^{\varphi}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\varphi}$. Therefore, after an application of $focus_1$ the induction hypothesis can be applied.

$$\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Downarrow succ_1(\widetilde{\varphi})}}{\widetilde{\Theta},\widetilde{\varphi}:\widetilde{\Upsilon}\Uparrow\cdot}\ focus_1 \right.$$

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta:\Upsilon\Downarrow\varphi}}{\Theta:\Upsilon,\varphi\Uparrow\cdot}\ focus_2 \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\Upsilon}, \widetilde{\varphi} \Uparrow \cdot$.

  The side condition of the $focus_2$ rule ensures that $Ptype(\varphi) \notin \{lit, \tau\}$. Let $S_3' = \Theta : \Upsilon \Downarrow \varphi$. According to the rewrite rules $\mathrm{R}^{\Upsilon}_{?}$ and $\mathrm{R}^{\varphi}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\varphi}$. Therefore, after an application of $focus_2$ the induction hypothesis can be applied.

$$\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Downarrow succ_1(\widetilde{\varphi})}}{\widetilde{\Theta}:\widetilde{\Upsilon},\widetilde{\varphi}\Uparrow\cdot}\ focus_1 \right.$$

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta:\Upsilon,\varphi\Uparrow\Xi}}{\Theta:\Upsilon\Uparrow\Xi,\varphi}\ \mathit{defocus} \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow \widetilde{\Xi}, \widetilde{\varphi}$.

  The side condition of the *defocus* rule ensures that $\mathit{Ptype}(\varphi) \notin \{\mathit{lit}, \tau, \beta, \pi\}$. Let $S_3' = \Theta : \Upsilon, \varphi \Uparrow \Xi$. According to the rewrite rules $\mathrm{R}^{\Xi}_{?}$ and $\mathrm{R}^{\Upsilon}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon}, \widetilde{\varphi} \Uparrow \widetilde{\Xi}$ and a position ϕ^M occurs at the root of $\widetilde{\varphi}$. Therefore, after an application of ϕ^M the induction hypothesis can be applied.

  $\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\widetilde{\Upsilon},\widetilde{\varphi}\Uparrow\widetilde{\Xi}}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Uparrow\widetilde{\Xi},\widetilde{\varphi}}\ \phi^M \right.$

- If $\mathcal{P}_3 = \left\{ \dfrac{\dfrac{\mathcal{P}_3'}{\Theta:\Upsilon\Uparrow\varphi}}{\Theta:\Upsilon\Downarrow\varphi}\ \mathit{switch} \right.$ then $\widetilde{S_3} = \widetilde{\Theta} : \widetilde{\Upsilon} \Downarrow \widetilde{\varphi}$.

  The side condition of *switch* ensures that $\mathit{Ptype}(\varphi) \in \{\mathit{lit}, \tau, \omega, \alpha, \nu\}$. Let $S_3' = \Theta : \Upsilon \Uparrow \varphi$. According to the rewrite rules $\mathrm{R}^{\Xi}_{?}$ and $\mathrm{R}^{\varphi}_{?}$ holds $\widetilde{S_3'} = \widetilde{\Theta} : \widetilde{\Upsilon} \Uparrow, \widetilde{\varphi}$ and a position ψ^M occurs at the root of $\widetilde{\varphi}$. Therefore, after an application of ψ^M the induction hypothesis can be applied.

  $\widetilde{\mathcal{P}_3} = \left\{ \dfrac{\dfrac{\widetilde{\mathcal{P}_3'}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Uparrow succ_1(\widetilde{\varphi})}}{\widetilde{\Theta}:\widetilde{\Upsilon}\Downarrow\widetilde{\varphi}}\ \psi^M \right.$

Theorem 42 (Correctness) If there exists a proof $\mathcal{P}$ in Σ_{pos} for a position sequent $S = \Phi^E : \Phi^M \Uparrow \Xi$ or $S = \Phi^E : \Phi^M \Downarrow \varphi$ for a multiplicity μ then there exists a multiplicity μ_3 such that the corresponding triadic sequent $S_3 = \mathit{sform}(\Phi^E) : \mathit{sform}(\Phi^M) \Uparrow \mathit{sform}(\Xi)$ or $S_3 = \mathit{sform}(\Phi^E) : \mathit{sform}(\Phi^M) \Downarrow \mathit{sform}(\varphi)$ is derivable in Σ_3'.

Proof. The proof can be carried out by induction on the structure of $\mathcal{P}$. An application of one of the rules *axiom*, $o\Downarrow$, $o\Uparrow$, τ, ω, α, β, ν, ψ^E, focus_1, focus_2, ϕ^M, or ψ^M in $\mathcal{P}$ results in an application of the rule *axiom*, $o\Downarrow$, $o\Uparrow$, τ, ω, α, β, ν, π, focus_1 , focus_2, *defocus*, or *switch*, respectively in the proof $\mathcal{P}_3$ of S_3. If the rule π is the last rule applied in $\mathcal{P}$ then no corresponding rule application is required in $\mathcal{P}_3$.

5 A Matrix Characterization for $\mathcal{MELL}$

A matrix characterization is a representation of the proof search space. Compared to a search space representation by a sequent calculus matrix characterizations can avoid many redundancies and are therefore better suited as a basis for automated proof search. Bibel distinguishes three kinds of redundancies which are typical for sequent calculi but which can be avoided by a good matrix characterization [Bib93]. These are *notational redundancies*, *redundancies due to permutabilities of rules*, and *redundancies due to irrelevant reductions of formulas.*

Notational Redundancies

The top-level objects in sequent calculus proof search are sequents. In a naive implementation of a sequent calculus based proof search, for every rule application

large parts of the conclusion are copied from the conclusion to the premises without change. This *notational redundancy* would cause a major overhead during proof search. One might argue that the application of techniques like e.g. structure sharing circumvent these redundancies. However, using such techniques one complicates the representation considerably. In matrix based proof search, notational redundancies need not be avoided explicitly since the formulation of the search space representation is already free from these redundancies.
During sequent calculus proof search formulas are decomposed by the application of rules in a stepwise manner until an axiom-rule becomes applicable. The reduction of a connective requires the application of a calculus rule. In matrix based proof search formulas need not be decomposed. Proof search is driven by connections. Each connection corresponds roughly to the application of an axiom-rule in the sequent calculus. In general, both, sequent based and matrix based proof search require some sort of bookkeeping in order to make backtracking possible. However, the bookkeeping in the sequent based approach must take the stepwise decomposition of formulas into account while in the matrix based approach only the axiom-rules need to be considered.

Permutabilities of Rules
If the order of two subsequently applied calculus rules is irrelevant for a proof they are called *permutable*. For a sequent calculus some rules might be permutable while others are not. Certain rules might be permutable in just one direction, i.e. the application of one rule can be permuted towards the root of a proof tree if the other rule has been applied before but not the other way round. The order of non-permutable rules can be understood as the essential part of a sequent proof. Permutabilities on the other hand cause an unnecessary non-determinism for proof search since all possible orders of rule applications are considered. Different techniques for removing such non-determinism from sequent calculus based proof search have been developed. Andreoli's focusing principle is one such technique. However, a good matrix characterization captures only the essential parts of a proof. This corresponds to the order of applications of non-permutable rules. No elaborated techniques need to be employed in order to remove the permutabilities from the search space because they do not occur in the representation.

Irrelevant Reductions
A very striking feature of matrix characterizations is the removal of irrelevant reductions, i.e. reductions of formulas which do not take part in an axiom. Since sequent calculus based proof search is connective-oriented such reductions must be considered. Matrix based proof search is connection oriented. Because connections correspond to axioms, in general, irrelevant reductions are not considered during matrix based proof search.

The matrix characterization for $\mathcal{MELL}$ presented in this section is a very compact search space representation. Compared to a sequent calculus search space representation (like by Σ_1') it avoids many redundancies. Since our matrix characterization is based on Andreoli's triadic calculus it is free from all permutabilities which are removed by his focusing principle.

Our matrix characterization for $\mathcal{MELL}$ is defined in a style which was first used by Wallen for intuitionistic logic and modal logics [Wal90]. It is based on the matrix characterization for multiplicative linear logic in [Man96] and [KMOS97] and extends that characterization by multiplicative constants and by exponentials.
After the presentation of some fundamental concepts in subsection 5.1, the complementarity of $\mathcal{MELL}$ matrices is defined in subsection 5.2. The correctness and the completeness of the matrix characterization is demonstrated in section 5.3 and 5.4, respectively. The fundamental characterization theorem is stated in section 5.5.

5.1 Fundamental Concepts

Definition 43 A *matrix* $\mathcal{M}$ is a position forest $\mathcal{F}$ with three zones and a mode in $\{\Downarrow, \Uparrow\}$. The multiplicity of $\mathcal{M}$ is defined as the multiplicity of $\mathcal{F}$.

We use $\mathcal{M}$ as meta-variable for matrices.
The set of positions in a matrix $\mathcal{M}$ is denoted by $Pos(\mathcal{M})$. Motivated by Σ_{pos}, we define some subsets of $Pos(\mathcal{M})$. The set of *axiom positions* $AxPos(\mathcal{M})$ contains all positions with principal type τ or *lit*. Axiom positions can take part in the rules *axiom* and τ. The set of *weakening positions* $WeakPos(\mathcal{M})$ contains all positions with principal type ω and all positions of type ν with $\mu(\nu) = 0$. Weakening positions can be weakened explicitly by the rules ω or ν (for $\mu(\nu) = 0$). The set of *leaf positions* is defined as $LeafPos(\mathcal{M}) = AxPos(\mathcal{M}) \cup WeakPos(\mathcal{M})$. The set $\boldsymbol{\beta}(\mathcal{M})$, $\Phi^M(\mathcal{M})$, $\Psi^M(\mathcal{M})$, $\Phi^E(\mathcal{M})$, and $\Psi^E(\mathcal{M})$ is the set of all positions in $Pos(\mathcal{M})$ of type β, ϕ^M, ψ^M, ϕ^E, and ψ^E, respectively. The set of *special positions* is defined as $SpecPos(\mathcal{M}) = \Phi^M(\mathcal{M}) \cup \Psi^M(\mathcal{M}) \cup \Phi^E(\mathcal{M}) \cup \Psi^E(\mathcal{M})$. The functions Pos, $AxPos$, $WeakPos$, $LeafPos$, $\boldsymbol{\beta}$, Φ^M, Ψ^M, Φ^E, Ψ^E, and $SpecPos$ are applied on matrices as well as on position trees.

Definition 44 A *path* is a set of positions. We define the set $Paths(\mathcal{T})$ of paths for a position tree $\mathcal{T}$ recursively.

1. The set $P = \{0\}$ which contains the root of $\mathcal{T}$ is a path through $\mathcal{T}$.
2. If $P \cup \{\alpha\}$ is a path through $\mathcal{T}$ then $P \cup \{succ_1(\alpha), succ_2(\alpha)\}$ is a path through $\mathcal{T}$.
3. If $P \cup \{\beta\}$ is a path through $\mathcal{T}$ then $P \cup \{succ_1(\beta)\}$ and $P \cup \{succ_2(\beta)\}$ are paths through $\mathcal{T}$.
4. If $P \cup \{p\}$ is a path through $\mathcal{T}$ and $Ptype(p) \in \{o, \pi, \phi^M, \psi^M, \phi^E, \psi^E\}$ then $P \cup \{succ_1(p)\}$ is a path through $\mathcal{T}$.
5. If $P \cup \{\nu\}$ is a path through $\mathcal{T}$ and $\mu(\nu) > 0$ then $P \cup \bigcup_{i \leq \mu(\nu)} \{succ_1^i(\nu)\}$ is a path through $\mathcal{T}$.

We extend the definition of $Paths$ to sets of position trees. The *set of paths through a set of position trees* $\mathcal{F}_s$ is defined recursively.

- If $\mathcal{F}_s = \emptyset$ then $Paths(\mathcal{F}_s) = \emptyset$.

- If $\mathcal{F}_s = \{\mathcal{T}\}$ then $Paths(\mathcal{F}_s) = Paths(\mathcal{T})$.
- If $\mathcal{F}_s = \{\mathcal{T}\} \cup \mathcal{F}'_s$ with $\mathcal{F}_s \neq \emptyset$ then $Paths(\mathcal{F}_s) = \{P_1 \cup P_2 | P_1 \in Paths(\mathcal{T}), P_2 \in Paths(\mathcal{F}'_s)\}$.

We extend the definition of *Paths* to matrices. The *set of paths through a matrix* is defined by

$$\begin{array}{rcll} Paths(\Phi^E : \Phi^M \Downarrow \varphi) & = & Paths(\Phi^E \cup \Phi^M \cup \{\varphi\}) & \text{and} \\ Paths(\Phi^E : \Phi^M \Uparrow \Xi) & = & Paths(\Phi^E \cup \Phi^M \cup \Xi) & . \end{array}$$

We use P as a meta-variable for paths.
A path through $\mathcal{M}$ which is a subset of $LeafPos(\mathcal{M})$ is called a *path of leafs*. The *set of paths of leafs* through $\mathcal{M}$ is the subset of $Paths(\mathcal{M})$ which contains all paths of leafs through $\mathcal{M}$. We denote the set of paths of leafs through $\mathcal{M}$ by $LPaths(\mathcal{M})$. Note that positions from $LeafPos(\mathcal{M})$ are not deconstructed in the recursive definition of paths. Thus, a path of leafs contains only irreducible positions.
A comparison of the reduction rules in the definition of paths through a matrix with Σ_{pos}-rules shows that there is a close relation between paths and sequents in a derivation. Let S be a position sequent (i.e. a matrix) then for any Σ_{pos}-derivation of S and any position sequent S' which occurs in that derivation there is a path P through S which is a superset of S', i.e. $S' \subseteq P$.

Definition 45 A *connection* in a matrix $\mathcal{M}$ is a subset of $LeafPos(\mathcal{M})$. A connection is either a two-element set $\{p_1, p_2\}$ where p_1 and p_2 are positions with $Ptype(p_1) = lit = Ptype(p_2)$, $lab(p_1) = lab(p_2)$, and $pol(p_1) \neq pol(p_2)$ or a one-element set $\{p_1\}$ with $Ptype(p_1) = \tau$.

We use C as a meta-variable for connections and $\mathcal{C}$ as a meta-variable for sets of connections.
A connection C is on a path P if $C \subseteq P$ holds, i.e. a connection is on a path if it is contained in it. There is a correspondence between a connection and the main positions in the application of an *axiom* or τ rules in Σ_{pos}. From the correspondence between paths and sequents it becomes apparent that the existence of a connection on a path P shows the potential of the sequents represented by P to result in an axiom.

Definition 46 Let $\mathcal{M}$ be a matrix. A *weakening map* for $\mathcal{M}$ is a subset of $\Phi^E(\mathcal{M}) \cup WeakPos(\mathcal{M})$.

We use $\mathcal{W}$ as a meta-variable for weakening maps.
Intuitively, in the corresponding Σ_{pos}-proof the elements of a weakening map are weakened explicitly by applications of ω or ν (positions of type ω or ν) or implicitly in an application of *axiom* (positions of type ϕ^E).

Definition 47 Let $\mathcal{M}$ be a matrix. A prefix is a string s from $(\Phi^M(\mathcal{M}) \cup \Psi^M(\mathcal{M}) \cup \Phi^E(\mathcal{M}) \cup \Psi^E(\mathcal{M}))^*$. For any position $p \in Pos(\mathcal{M})$ we define its *prefix* $pre_{\mathcal{M}}(p)$ by induction on the structure of p.

- Base case ($p = 0$):
  - If $Ptype(0) = \phi^M$, $Ptype(0) = \psi^M$, $Ptype(0) = \phi^E$, or $Ptype(0) = \psi^E$ then $pre_{\mathcal{M}}(0) = \phi_0^M$, $pre_{\mathcal{M}}(0) = \psi_0^M$, $pre_{\mathcal{M}}(0) = \phi_0^E$, or $pre_{\mathcal{M}}(0) = \psi_0^E$, respectively.
  - If $Ptype(0) \notin \{\phi^M, \psi^M, \phi^E, \psi^E\}$ then $pre_{\mathcal{M}}(0) = \varepsilon$.
- Step case ($p = p'i$ with $i \in \{0, 1\} \cup \bigcup_{m \le \mu(p')} \{0^m\}$):
  - If $Ptype(p) = \phi^M$, $Ptype(p) = \psi^M$, $Ptype(p) = \phi^E$, or $Ptype(p) = \psi^E$ then $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}}(p')\phi_p^M$, $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}}(p')\psi_p^M$, $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}}(p')\phi_p^E$, or $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}}(p')\psi_p^E$, respectively.
  - If $Ptype(p) \notin \{\phi^M, \psi^M, \phi^E, \psi^E\}$ then $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}}(p')$.

The prefix of a position p can be retrieved from the position tree by collecting all special positions on the path from the root of the tree to p. If $p_1 \ll p_2$ holds for two positions $p_1, p_2 \in Pos(\mathcal{M})$ then $pre_{\mathcal{M}}(p_1)$ is an initial substring of $pre_{\mathcal{M}}(p_2)$. We define a substring relation $\angle$ such that for two strings t_1 and t_2 holds $t_1 \angle t_2$ iff t_1 is an initial substring of t_2, e.g. $pre_{\mathcal{M}}(p_1) \angle pre_{\mathcal{M}}(p_2)$ holds.

Definition 48 A *multiplicative prefix substitution* is an idempotent mapping $\sigma_M : \Phi^M \to (\Phi^M \cup \Psi^M)^*$. An *exponential prefix substitution* is an idempotent mapping $\sigma_E : \Phi^E \to (\Phi^M \cup \Psi^M \cup \Phi^E \cup \Psi^E)^*$. A *multiplicative exponential prefix substitution* is an idempotent mapping $\sigma_{ME} : (\Phi^M \cup \Phi^E) \to (\Phi^M \cup \Psi^M \cup \Phi^E \cup \Psi^E)^*$ such that the restriction of σ_{ME} to Φ^M is a multiplicative prefix substitution and that the restriction to Φ^E is an exponential prefix substitution.
Assuming σ_{ME} as the identity on positions from Ψ^M and Ψ^E we denote the homomorphic extension of a prefix substitution σ_{ME} to strings from $(\Phi^M \cup \Psi^M \cup \Phi^E \cup \Psi^E)^*$ also by σ_{ME}.

In the sequel we refer to a multiplicative exponential prefix substitution σ_{ME} simply by σ. A prefix substitution substitutes elements from Φ^M and Φ^E by strings of special positions. Therefore, elements from Φ^M and Φ^E are called *variable special positions* while elements from Ψ^M and Ψ^E are called *constant special positions*.

5.2 Complementarity

Definition 49 A set of connections $\mathcal{C}$ *spans* a matrix $\mathcal{M}$ iff for every path $P \in LPaths(\mathcal{M})$ there is a connection $C \in \mathcal{C}$ which is on P, i.e. $C \subseteq P$.

Definition 50 Let $\mathcal{M}$ be a matrix. The pair $\langle \mathcal{C}, \mathcal{W} \rangle$ consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$ is *linear* for $\mathcal{M}$ iff the following conditions are satisfied:

- For any two connections $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ holds $C_1 \cap C_2 = \emptyset$.
- For any position p in any connection $C \in \mathcal{C}$ holds for any predecessor ϕ^E of p in $\mathcal{M}$ that $\phi^E \notin \mathcal{W}$.

Definition 51 A pair $\langle \mathcal{C}, \mathcal{W} \rangle$ consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$ has the *relevance* property for $\mathcal{P}$ if for any $p \in \mathit{LeafPos}(\mathcal{M})$ one of the following conditions holds.

- There is a connection $C \in \mathcal{C}$ with $p \in C$.
- $p \in \mathcal{W}$ (Thus, also $p \in \mathit{WeakPos}(\mathcal{M})$.).
- There is a $\phi^E \in \mathcal{W}$ which is a predecessor of p.

Definition 52 Let $\mathcal{M}$ be a matrix, $\mathcal{C}$ be a set of connections for $\mathcal{M}$, and $\mathcal{W}$ be a weakening map for $\mathcal{M}$. The *cardinality of the pair* $\langle \mathcal{C}, \mathcal{W} \rangle$ is defined as the sum of the connections and the number of β-type positions which are affected by weakening.

$$\mathit{card}(\langle \mathcal{C}, \mathcal{W} \rangle) = |\mathcal{C}| + \sum_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)|.$$

Definition 53 Let $\mathcal{M}$ be a matrix, $\mathcal{C}$ be a set of connections for $\mathcal{M}$, and $\mathcal{W}$ be a weakening map for $\mathcal{M}$. The pair $\langle \mathcal{C}, \mathcal{W} \rangle$ has the *cardinality property* for $\mathcal{M}$ if

$$\mathit{card}(\langle \mathcal{C}, \mathcal{W} \rangle) = \boldsymbol{\beta}(\mathcal{M}) + 1.$$

Definition 54 Let $\mathcal{M}$ be a matrix and Pos be the set of positions of $\mathcal{M}$. A prefix substitution σ is *admissible* for $\mathcal{M}$ if the following condition is fulfilled.

- σ results from unification, i.e. if $pu \angle \sigma(\mathit{pre}_{\mathcal{M}}(v))$ for some $v \in \mathit{SpecPos}(M)$ then $pu = \sigma(\mathit{pre}_{\mathcal{M}}(u))$.

Definition 55 Let $\mathcal{M}$ be a matrix. The pair $\langle \mathcal{C}, \mathcal{W} \rangle$ consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$ is *unifiable* if there exists an admissible prefix substitution $\sigma : (\Phi^M(\mathcal{M}) \cup \Phi^E(\mathcal{M})) \to (\Phi^M(\mathcal{M}) \cup \Psi^M(\mathcal{M}) \cup \Phi^E(\mathcal{M}) \cup \Psi^E(\mathcal{M}))$ such that the following conditions hold.

- For any connection $C \in \mathcal{C}$ holds $\forall p_1, p_2 \in C. \sigma(\mathit{pre}_{\mathcal{M}}(p_1)) = \sigma(\mathit{pre}_{\mathcal{M}}(p_2))$.
- For any $p' \in (\mathcal{W} \cap \mathit{WeakPos}(\mathcal{M}))$ there is a connection $C \in \mathcal{C}$ such that for $p \in C$ holds $\sigma(\mathit{pre}_{\mathcal{M}}(p')) \angle \sigma(\mathit{pre}_{\mathcal{M}}(p))$.
- For any $\phi^E \in (\mathcal{W} \cap \Phi^E)$ there is a connection $C \in \mathcal{C}$ such that for $p \in C$ holds $\sigma(\mathit{pre}_{\mathcal{M}}(\phi^E)) = \sigma(\mathit{pre}_{\mathcal{M}}(p))$.

If the conditions hold for an admissible prefix substitution σ then σ is called a *unifier* for $\langle \mathcal{C}, \mathcal{W} \rangle$.
Note that in this setting the third condition is equivalent to that for some σ holds $\sigma(\mathit{pre}_{\mathcal{M}}(\phi^E)) \angle \sigma(\mathit{pre}_{\mathcal{M}}(p))$. The equivalence can be shown along the lines mentioned in remark 20. We require the property as stated for technical reasons. The proof of lemma 66 is simpler with this formulation.

Definition 56 A matrix $\mathcal{M}$ is *complementary* iff there are a set of connections $\mathcal{C}$, a weakening map $\mathcal{W}$, and a string substitution σ such that the following conditions are satisfied.

- $\mathcal{C}$ spans $\mathcal{M}$.
- $\langle \mathcal{C}, \mathcal{W} \rangle$ is linear for $\mathcal{M}$.
- $\langle \mathcal{C}, \mathcal{W} \rangle$ has the relevance property for $\mathcal{M}$.
- $\langle \mathcal{C}, \mathcal{W} \rangle$ has the cardinality property for $\mathcal{M}$.
- σ is an unifier for $\langle \mathcal{C}, \mathcal{W} \rangle$.

If the conditions are satisfied for $\mathcal{M}$, $\mathcal{C}$, $\mathcal{W}$, and σ then $\mathcal{M}$ *is complementary for* $\mathcal{C}$, $\mathcal{W}$, and σ.

The complementarity of a matrix ensures the existence of a corresponding Σ_{pos}-proof. Each requirement captures an essential aspect of such a proof. Thus, there are relations between basic concepts in matrix proofs and Σ_{pos}-proofs. Paths are related to sequents. A connection on a path expresses the potential to close a Σ_{pos}-branch by an application of *axiom* or τ which involves the connected positions. A weakening map $\mathcal{W}$ contains all positions which are explicitly weakened by the rules ω and ν (for $\mu(\nu) = 0$) or implicitly weakened in *axiom* and τ. The unifiability of prefixes guarantees that connected positions can move into the same Σ_{pos}-branch and that positions in $\mathcal{W}$ can be weakened in some branch. Linearity and relevance resemble the lack of contraction and weakening for arbitrary formulas, while cardinality expresses the absence of the rule of mingle depicted below, i.e. a proof can only branch at the reduction of β-type positions.

$$\frac{\Gamma_1 \longrightarrow \Delta_1 \quad \Gamma_2 \longrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2} \; \mathit{mingle}$$

Since complementarity captures the essential aspects of Σ_{pos}-proofs but no unimportant technical details the search space is once more condensed. Problems like e.g. context splitting at the reduction of β-type positions simply do not occur.

5.3 Correctness

The unifiability property requires that each position in a weakening map $\mathcal{W}$ is related to some connection C in a set of connections $\mathcal{C}$. Possibly, a position p in $\mathcal{W}$ is related to more than one connection. For the remainder of this subsection we consider a relation where each $p \in \mathcal{W}$ is related to exactly one connection in $\mathcal{C}$. Let $\mathit{AssSet}(C)$ be the union of C and the set of all positions in $\mathcal{W}$ which are related to C. A matrix can be seen as a collection of trees, i.e. a forest. We add for each connection C additional edges to that graph which link all positions in $\mathit{AssSet}(C)$. We define a relation *AssRel* on trees of the matrix such that *AssRel* holds for trees which are connected by connections. The relation is extended to an equivalence relation. The equivalence classes of that relation correspond to maximally connected subgraphs of the matrix.

Definition 57 Let $\mathcal{M}$ be a matrix, $\langle \mathcal{C}, \mathcal{W} \rangle$ be a pair consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$, and σ be a prefix substitution. We define a relation *AssRel* on position trees. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be position trees in $\mathcal{M}$.

$$\mathit{AssRel}(\mathcal{T}_1, \mathcal{T}_2) \quad \mathit{iff} \quad \exists C \in \mathcal{C}. \exists p_1 \in \mathit{Pos}(\mathcal{T}_1), p_2 \in \mathit{Pos}(\mathcal{T}_2). p_1, p_2 \in \mathit{AssSet}(C)$$

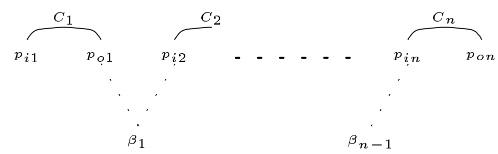


Figure 4: β-Chain

Let $\sim$ be the reflexive transitive closure of $AssRel$. $\sim$ is an equivalence relation. A *maximally connected component* in $\mathcal{M}$ is a non-empty set of position trees which is an equivalence class of $\sim$.

We define a function *FCons* which reduces a set of connections to the connections of which all elements are within a certain forest.

Definition 58 Let $\mathcal{F}$ be a forest of position trees and $\mathcal{C}$ be a set of connections. We define a function *FCons* by

$$FCons(\mathcal{F},\mathcal{C}) \;=\; \{C \in \mathcal{C} \mid \forall p \in C.p \in Pos(\mathcal{F})\} \;.$$

Remark 59 For the considerations which are carried out in the sequel, a definition of *FCons* by

$$FCons(\mathcal{F},\mathcal{C}) \;=\; \{C \in \mathcal{C} \mid \forall p \in AssSet(C).p \in Pos(\mathcal{F})\}$$

would be equivalent.

Definition 60 Let $\mathcal{F}$ be a forest of position trees and $\mathcal{W}$ be a weakening map. We define a function *FWeak* by

$$FWeak(\mathcal{F},\mathcal{W}) \;=\; \{p \in \mathcal{W} \mid p \in Pos(\mathcal{F})\} \;.$$

The following two definitions are visualized in figure 4 and 5.

Definition 61 Let $\mathcal{M}$ be a matrix, $\mathcal{C}$ be a set of connections, and $\mathcal{W}$ be a weakening map. A *β-chain* for $\mathcal{C}$ and $\mathcal{W}$ in $\mathcal{M}$ is a sequence $\langle(p_{i1},p_{o1}),\ldots,(p_{in},p_{on})\rangle$ such that for all p_{ij} and p_{oj} with $p_{ij} \neq p_{oj}$ there is a $C_j \in \mathcal{C}$ with $p_{ij}, p_{oj} \in AssSet(C_j)$ and such that for all p_{oj}, p_{ij+1} there is a position $\beta_j \in Pos(\mathcal{M})$ with $\beta_j \ll p_{oj}$ and $\beta_j \ll p_{ij+1}$ and there is no position $p \in Pos(\mathcal{M})$ with $p \ll p_{oj}$, $p \ll p_{ij+1}$, and $\beta_j \ll p$. Further, for all j_1, j_2 with $j_1 \neq j_2$ must hold $C_{j_1} \neq C_{j_2}$.

Definition 62 Let $\mathcal{M}$ be a matrix, $\mathcal{C}$ be a set of connections, and $\mathcal{W}$ be a weakening map. A β-circle for $\mathcal{C}$ and $\mathcal{W}$ in $\mathcal{M}$ is a β-chain $\langle(p_{i1},p_{o1}),\ldots,(p_{in},p_{on})\rangle$ for which there is a position $\beta \in Pos(\mathcal{M})$ such that $\beta \ll p_{on}$ and $\beta \ll p_{i1}$ and that there is no position $p \in Pos(\mathcal{M})$ with $p \ll p_{oj}$, $p \ll p_{ij+1}$, and $\beta_j \ll p$.

Note that the existence of a β-circle and the existence of a unifier for $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{W}$ are mutually exclusive. If a β-circle exists for $\mathcal{C}$ and $\mathcal{W}$ in $\mathcal{M}$ then there cannot be a unifier and vice versa.

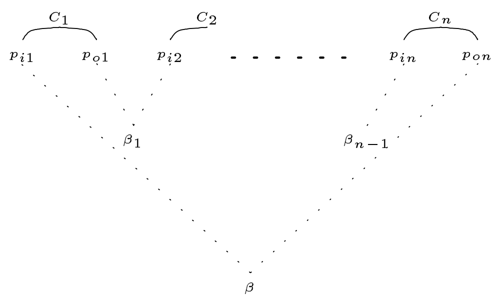


Figure 5: β-Circle

Lemma 63 Let $\mathcal{M}$ be a matrix, $\langle \mathcal{C}, \mathcal{W} \rangle$ be a pair consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$, and σ be a prefix substitution. We assume that

- $\langle \mathcal{C}, \mathcal{W} \rangle$ is linear for $\mathcal{M}$,
- $\langle \mathcal{C}, \mathcal{W} \rangle$ has the relevance property for $\mathcal{M}$, and
- σ is a unifier for $\langle \mathcal{C}, \mathcal{W} \rangle$.

For any maximally connected component $\mathcal{F}$ in $\mathcal{M}$ holds

$$|\boldsymbol{\beta}(\mathcal{F})| \quad < \quad |FCons(\mathcal{F}, \mathcal{C})| + \Sigma_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)|$$

Proof. We abbreviate $FCons(\mathcal{F}, \mathcal{C})$ by $\mathcal{C}_\mathcal{F}$ and $FWeak(\mathcal{F}, \mathcal{W})$ by $\mathcal{W}_\mathcal{F}$. According to definition 57, 58, and 60, $\langle \mathcal{C}_\mathcal{F}, \mathcal{W}_\mathcal{F} \rangle$ is linear and relevant for $\mathcal{F}$. σ is a unifier for $\langle \mathcal{C}_\mathcal{F}, \mathcal{W}_\mathcal{F} \rangle$ and $\mathcal{F}$.
The proof of the lemma is by induction over the size of $\boldsymbol{\beta}(\mathcal{F})$.
Base case: $|\boldsymbol{\beta}(\mathcal{F})| = 0$
The proposition holds because $\mathcal{F}$ is not empty and because the relevance property holds for $\mathcal{C}_\mathcal{F}$, $\mathcal{W}_\mathcal{F}$, and $\mathcal{M}$, i.e. $|\mathcal{C}_\mathcal{F}| > 0$ and $\Sigma_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)| = 0$.
Induction hypothesis: The lemma holds for $|\boldsymbol{\beta}(\mathcal{F})| < n + 1$.
Step case: Assume that all roots in $\mathcal{F}$ have type β or are atomic. Otherwise, if there is no root of type α or o the base case can be applied, and if there is a root of type α or o this root can be removed yielding a forest $\mathcal{F}'$ with fewer positions for which linearity, relevance, and unifiability are fulfilled.
Choose a root $\beta \in Pos(\mathcal{F})$ which separates $\mathcal{F}$ into two maximally connected components, i.e. the removal of β results in maximally connected components $\mathcal{F}_1$ and $\mathcal{F}_2$. Such a root exists because otherwise there would be a circle which contradicts the assumption that σ is a unifier. Let $\mathcal{C}_1 = FCons(\mathcal{F}_1, \mathcal{C})$, $\mathcal{C}_2 = FCons(\mathcal{F}_2, \mathcal{C})$, $\mathcal{W}_1 = FWeak(\mathcal{F}_1, \mathcal{W})$, and $\mathcal{W}_2 = FWeak(\mathcal{F}_2, \mathcal{W})$.
The following equations hold for $\mathcal{F}_1$ and $\mathcal{F}_2$.

$$|\boldsymbol{\beta}(\mathcal{F})| \quad = \quad |\boldsymbol{\beta}(\mathcal{F}_1)| + |\boldsymbol{\beta}(\mathcal{F}_2)| + 1 \qquad (1)$$

$$|\mathcal{C}_\mathcal{F}| \quad = \quad |\mathcal{C}_1| + |\mathcal{C}_2| \qquad (2)$$

Furthermore, one of the following two equations holds.

$$\Sigma_{\phi^E \in \mathcal{W}_\mathcal{F}} |\boldsymbol{\beta}(\phi^E)| \quad = \quad \Sigma_{\phi^E \in \mathcal{W}_1} |\boldsymbol{\beta}(\phi^E)| + \Sigma_{\phi^E \in \mathcal{W}_2} |\boldsymbol{\beta}(\phi^E)| \qquad (3)$$

$$\Sigma_{\phi^E \in \mathcal{W}_\mathcal{F}} |\boldsymbol{\beta}(\phi^E)| \quad = \quad \Sigma_{\phi^E \in \mathcal{W}_1} |\boldsymbol{\beta}(\phi^E)| + \Sigma_{\phi^E \in \mathcal{W}_2} |\boldsymbol{\beta}(\phi^E)| + 1 \qquad (4)$$

holds.
According to the induction hypothesis and the equations above the following relations hold.

$$\begin{array}{rcl}|\boldsymbol{\beta}(\mathcal{F})| & = & |\boldsymbol{\beta}(\mathcal{F}_1)| + |\boldsymbol{\beta}(\mathcal{F}_2)| + 1 \\ & \leq & |\mathcal{C}_1| + \Sigma_{\phi^E \in \mathcal{W}_1} |\boldsymbol{\beta}(\phi^E)| - 1 + \\ & & |\mathcal{C}_2| + \Sigma_{\phi^E \in \mathcal{W}_2} |\boldsymbol{\beta}(\phi^E)| - 1 + 1 \\ & = & |\mathcal{C}_1| + \Sigma_{\phi^E \in \mathcal{W}_1} |\boldsymbol{\beta}(\phi^E)| + \\ & & |\mathcal{C}_2| + \Sigma_{\phi^E \in \mathcal{W}_2} |\boldsymbol{\beta}(\phi^E)| - 1 \\ & \leq & |\mathcal{C}_{\mathcal{F}}| + \Sigma_{\phi^E \in \mathcal{W}_{\mathcal{F}}} |\boldsymbol{\beta}(\phi^E)| - 1 \\ & < & |\mathcal{C}_{\mathcal{F}}| + \Sigma_{\phi^E \in \mathcal{W}_{\mathcal{F}}} |\boldsymbol{\beta}(\phi^E)|\end{array}$$

Thus, the proposition holds for $\boldsymbol{\beta}(\mathcal{F}) = n$.

Corollary 64 Let $\mathcal{M}$ be a matrix, $\langle \mathcal{C}, \mathcal{W} \rangle$ be a pair consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$, and σ be a prefix substitution such that all preconditions of theorem 63 hold and that $\langle \mathcal{C}, \mathcal{W} \rangle$ has the cardinality property for $\mathcal{M}$. Then there is exactly one connected component in $\mathcal{M}$.

Definition 65 Let $\sigma : (\Phi^E \cup \Phi^M) \rightarrow (\Phi^E \cup \Psi^E \cup \Phi^M \cup \Psi^M)^*$ be a prefix substitution. Let $\sigma' : (\Phi^E \cup \Phi^M) \rightarrow (\Phi^E \cup \Psi^E \cup \Phi^M \cup \Psi^M)^*$ such that for all $\phi \in \Phi^E \cup \Phi^M$ holds $\sigma'(\phi) = \varepsilon$. We define the *grounded substitution* $\sigma^{\triangle}$ for σ by $\sigma^{\triangle} = \sigma' \circ \sigma$.

Note that $\sigma^{\triangle}$ equals σ except for that all variables are removed from the images. If σ is a unifier for $\langle \mathcal{C}, \mathcal{W} \rangle$ then so is $\sigma^{\triangle}$.

Lemma 66 Let $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \psi_1^E$ be a matrix, $\langle \mathcal{C}, \mathcal{W} \rangle$ be a pair consisting of a set of connections $\mathcal{C}$ and a weakening map $\mathcal{W}$, and σ be a grounded prefix substitution such that all preconditions of corollary 64 hold. For any $p \in \mathit{LeafPos}(\mathcal{M})$ there exists a string s such that $\sigma(\mathit{pre}_{\mathcal{M}}(p)) = \psi_1^E.s$ holds.

Proof. We prove the lemma by contradiction.
Assume that the set $\mathit{Aux} = \{p \in \mathit{LeafPos}(\mathcal{M}) | \exists s. \sigma(\mathit{pre}_{\mathcal{M}}(p)) \neq \psi_1^E.s\}$ is not empty.
Let $\mathcal{F}$ be the position forest of all trees in $\mathcal{M}$ with leafs in Aux. If one leaf of a tree is in Aux then all leafs of that tree are in Aux due to the way in which special positions occur in position trees. In a prefix, variable and constant positions occur mutually alternatingly. Let $\overline{\mathcal{F}}$ be the position forests which contains all trees from $\mathcal{M}$ which are not in $\mathcal{F}$.
By corollary 64 $\mathcal{M}$ has only one connected component. Thus, there is a tree $\mathcal{T}$ in $\mathcal{F}$ with a position $p \in \mathit{Pos}(\mathcal{T})$ such that there is a tree $\mathcal{T}'$ in $\overline{\mathcal{F}}$ with a position $p' \in \mathit{Pos}(\mathcal{T}')$ such that there is a connection $C \in \mathcal{C}$ with $p, p' \in \mathit{AssSet}(C)$. If possible we choose $\mathcal{T}$ and p such that $\mathit{Ptype}(p) \in \{\mathit{lit}, \tau\}$. Otherwise, we choose $\mathcal{T}$, p, $\mathcal{T}'$, and p' such that $\mathit{Ptype}(p') \in \{\mathit{lit}, \tau\}$. Due to the definition of connections at least one of the two choices is possible.
We distinguish the two cases.

1. $\mathit{Ptype}(p) \in \{\mathit{lit}, \tau\}$
   There is a string s such that $\psi_1^E.s = \sigma(\mathit{pre}_{\mathcal{M}}(p')) \underline{\angle} \sigma(\mathit{pre}_{\mathcal{M}}(p))$ because of the unifiability condition. This contradicts the choice of $\mathcal{T}$.

2. $Ptype(p') \in \{lit, \tau\}$
   Let ϕ be the root of $\mathcal{T}$. Because of the unifiability condition there is a string s such that $\sigma(\phi) = \sigma(pre_{\mathcal{M}}(p)) \angle \sigma(\mathcal{M})p' = \psi_1^E.s$ or there are strings s, s', and a position $\psi \in \Psi^E(\mathcal{M}) \cup \Psi^M(\mathcal{M})$ such that $\sigma(\phi).\psi.s = \sigma(pre_{\mathcal{M}}(p)) \angle \sigma(pre_{\mathcal{M}}(p')) = \psi_1^E.s'$. According to the way special positions are inserted before positions of type ω and ν, this contradicts the choice of $\mathcal{T}$ or the unifiability condition.

Due to the definition of prefix substitutions, a position of type ψ^E may not occur in the value of a position of type ϕ^M under σ. Therefore, Φ^M must be empty.

Definition 67 We define a function *wgt* which returns the weight of a matrix, i.e. a natural number.

$$wgt(\Phi^E : \Phi^M \Downarrow F) = \sum_{\phi \in (\Phi^E \cup \Phi^M)} (2 * |Pos(\phi)|) + 2 * |Pos(F)| + 1$$
$$wgt(\Phi^E : \Phi^M \Uparrow \Xi) = \sum_{\phi \in (\Phi^E \cup \Phi^M)} (2 * |Pos(\phi)|) + \sum_{p \in \Xi} (2 * |Pos(p)| + 1)$$

An ordering $<_{wgt}$ on matrices is defined by

$$\mathcal{M} <_{wgt} \mathcal{M}' \quad \textit{iff} \quad wgt(\mathcal{M}) < wgt(\mathcal{M}').$$

Theorem 68 (Correctness) Let $\mathcal{M}$ be a matrix, $\mathcal{C}$ be a set of connections, $\mathcal{W}$ be a weakening map, and σ a string substitution. If $\mathcal{M}$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ then a position calculus proof $\mathcal{P}$ exists for $\mathcal{M}$.

Proof. We prove the theorem by Noetherian induction over the weight of matrices. There is no base case. In the following considerations we assume σ to be a grounded substitution. As we pointed out earlier any unifier can be transformed into a grounded one.
Induction hypothesis: We assume that the theorem holds for any matrix $\mathcal{M}'$ with $\mathcal{M}' <_{wgt} \mathcal{M}$.
Step case: We make a case distinction over the structure of $\mathcal{M}$.
Recall from subsection 4.4 that for a matrix $\Phi^E : \Phi^M \Uparrow \Xi$ any position in Ξ has a type from $\{\omega, o, \alpha, \nu, \phi^M\}$ and that for a matrix $\Phi^E : \Phi^M \Downarrow p$ the position p has a type from $\{o, \beta, \pi, \psi^M, \psi^E\}$.

- $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \cdot$
  We distinguish two cases.

  1. For all $\phi \in \Phi^M \cup (\Phi^E \setminus \mathcal{W})$ holds $succ_1(\phi) \in LeafPos(\mathcal{M})$.
     Due to the way special positions are inserted holds $\Phi^E = \emptyset$ and for all $\phi^M \in \Phi^M$ holds $Ptype(succ_1(\phi^M)) \in \{lit, \tau\}$. Because of the relevance property there is $\phi_1^M \in \Phi^M$ such that $\{succ_1(\phi_1^M)\} \in \mathcal{C}$ or there are $\phi_1^M, \phi_2^M \in \Phi^M$ such that $\{succ_1(\phi_1^M), succ_1(\phi_2^M)\} \in \mathcal{C}$.
     If there is no $\phi_3^M \in \Phi^M$ with $\phi_3^M \notin C$ then the *axiom* rule is applicable in the first case and the τ rule in the second case.
     Let us assume that there is a $\phi_3^M \in \Phi^M$ with $\phi_3^M \notin C$. Because of the relevance property this leads to a contradiction with the cardinality condition:

$$\begin{array}{rcl} \beta(\mathcal{M})+1 & = & \sum_{\phi^E \in \mathcal{W}} \beta(\phi^E)+1 \\ & < & \sum_{\phi^E \in \mathcal{W}} \beta(\phi^E)+2 \ . \\ & \leq & card(\langle \mathcal{C}, \mathcal{W} \rangle) \end{array}$$

2. There exists $\phi \in \Phi^M \cup (\Phi^E \setminus \mathcal{W})$ with $succ_1(\phi) \notin LeafPos(\mathcal{M})$.
   We determine a position $\phi' \in \Phi^M \cup (\Phi^E \setminus \mathcal{W})$. If $\sigma(\phi) = \varepsilon$ then we take $\phi' = \phi$. Otherwise, holds $\sigma(\phi) = \psi.s$ for some $\psi \in \Psi^E \cup \Psi^M$. Due to the insertion of special positions and the admissibility of σ, there is exactly one position p of type ϕ^E or ϕ^M with $p \ll \psi$. We choose $\phi' = p$.
   With our choice of ϕ' holds $\sigma(\phi') = \varepsilon$.
   We distinguish two cases:
   - $\phi' \in \Phi^E \setminus \mathcal{W}$
     Let $\mathcal{M}' = \Phi^E \setminus \{\phi'\} : \Phi^M \Downarrow succ_1(\phi')$. Then $focus_1$ can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ with $p \notin Pos(\phi')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Since $\sigma(\phi') = \varepsilon$, σ is a unifier for $\mathcal{M}'$ and $\langle \mathcal{M}, \mathcal{W} \rangle$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ {focus_1} \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 1$.
   - $\phi' \in \Phi^M$
     Let $\mathcal{M}' = \Phi^E : \Phi^M \setminus \{\phi'\} \Downarrow succ_1(\phi')$. Then $focus_2$ can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ with $p \notin Pos(\phi')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Since $\sigma(\phi') = \varepsilon$, σ is a unifier for $\mathcal{M}'$ and $\langle \mathcal{M}, \mathcal{W} \rangle$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ {focus_2} \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 1$.

- $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, \omega$
  Let $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \Xi$. Then the ω rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}') \cup \{\omega\}$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. $\langle \mathcal{C}, \mathcal{W} \setminus \{\omega\} \rangle$ has the relevance property for $\mathcal{M}'$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W} \setminus \{\omega\}$, and σ. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ \omega \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 3$.

- $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, o$
  Let $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \Xi, succ_1(o)$. Then the $o \Uparrow$ rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ {o\Uparrow} \right.$ is a proof for $\mathcal{M}$ by induction hypothesis since $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 2$.

- $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, \alpha$
  Let $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \Xi, succ_1(\alpha), succ_2(\alpha)$. Then the α rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \alpha \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 1$.

- $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, \nu$
  Let $\mathcal{M}' = \Phi^E, succ_1^1(\nu), \ldots, succ_1^{\mu(\nu)}(\nu) : \Phi^M \Uparrow \Xi$. Then rule ν can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \nu \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 3$.

- $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, \phi_1^M$
  Let $\mathcal{M}' = \Phi^E : \Phi^M, \phi_1^M \Uparrow \Xi$. Then the ϕ^M rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \phi^M \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 1$.

- $\mathcal{M} = \Phi^E : \Phi^M \Downarrow o$
  Let $\mathcal{M}' = \Phi^E : \Phi^M \Downarrow succ_1(o)$. Then the $o \Downarrow$ rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} o\Downarrow \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 2$.

- $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \beta$
  According to corollary 64, $\mathcal{M}$ has exactly one connected component. The removal of β yields two connected components. $\mathcal{M}' = \Phi_1^E : \Phi_1^M \Downarrow succ_1(\beta)$ and $\mathcal{M}'' = \Phi_2^E : \Phi_2^M \Downarrow succ_2(\beta)$ can be constructed for for $succ_1(\beta) = p_1$ and $succ_2(\beta) = p_2$ such that each $\mathcal{M}'$ and $\mathcal{M}''$ has one connected component, $\Phi_1^E \cup \Phi_2^E = \Phi^E$, $\Phi_1^E \cap \Phi_2^E = \emptyset$, $\Phi_1^M \cup \Phi_2^M = \Phi^M$, and $\Phi_1^M \cap \Phi_2^M = \emptyset$. Let $\mathcal{C}_1 = FCons(\mathcal{M}', \mathcal{C})$, $\mathcal{W}_1 = FWeak(\mathcal{M}', \mathcal{W})$, $\mathcal{C}_2 = FCons(\mathcal{M}'', \mathcal{C})$, and $\mathcal{W}_2 = FWeak(\mathcal{M}'', \mathcal{W})$.

  The following equations hold.
  $\mathcal{C}' \cup \mathcal{C}'' = \mathcal{C}$, $\mathcal{C}' \cap \mathcal{C}'' = \emptyset$, $\mathcal{W}' \cup \mathcal{W}'' = \mathcal{W}$, and $\mathcal{W}' \cap \mathcal{W}'' = \emptyset$.

  The following properties hold.

  - $\mathcal{C}'$ ($\mathcal{C}''$) spans $\mathcal{M}'$ ($\mathcal{M}''$).
  - $\langle \mathcal{C}', \mathcal{W}' \rangle$ ($\langle \mathcal{C}'', \mathcal{W}'' \rangle$) is linear for $\mathcal{M}'$ ($\mathcal{M}''$).
  - $\langle \mathcal{C}', \mathcal{W}' \rangle$ ($\langle \mathcal{C}'', \mathcal{W}'' \rangle$) has the relevance property for $\mathcal{M}'$ ($\mathcal{M}''$).

– σ is a unifier for $\mathcal{M}'$ ($\mathcal{M}''$).

The following relations hold.

$$\begin{array}{rcl} |\boldsymbol{\beta}(\mathcal{M}')| + 1 + |\boldsymbol{\beta}(\mathcal{M}'')| + 1 & = & |\boldsymbol{\beta}(\mathcal{M})| + 1 \\ & = & |\mathcal{C}| + \Sigma_{\phi^E \in \mathcal{W}} |\boldsymbol{\beta}(\phi^E)| \\ & \geq & |\mathcal{C}'| + \Sigma_{\phi^E \in \mathcal{W}'} |\boldsymbol{\beta}(\phi^E)| + \\ & & |\mathcal{C}''| + \Sigma_{\phi^E \in \mathcal{W}''} |\boldsymbol{\beta}(\phi^E)| \end{array}$$

From lemma 63 we conclude with $|\mathcal{C}'| + \Sigma_{\phi^E \in \mathcal{W}'} |\boldsymbol{\beta}(\phi^E)| \geq |\boldsymbol{\beta}(\mathcal{M}')| + 1$ and $|\mathcal{C}''| + \Sigma_{\phi^E \in \mathcal{W}''} |\boldsymbol{\beta}(\phi^E)| \geq |\boldsymbol{\beta}(\mathcal{M}'')| + 1$ that the cardinality property holds for $\mathcal{C}'$, $\mathcal{W}'$, $\mathcal{M}'$ as well as for $\mathcal{C}''$, $\mathcal{W}''$, $\mathcal{M}''$.

Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}'$, $\mathcal{W}'$, and σ and $\mathcal{M}''$ is complementary for $\mathcal{C}''$, $\mathcal{W}''$, and σ. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'} \quad \dfrac{\mathcal{P}''}{\mathcal{M}''}}{\mathcal{M}} \, \beta \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') < wgt(\mathcal{M})$ and $wgt(\mathcal{M}'') < wgt(\mathcal{M})$.

- $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \pi$
  Let $\mathcal{M}' = \Phi^E : \Phi^M \Downarrow succ_1(\pi)$. Then the π rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$. Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \, \pi \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 2$.

- $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \psi_1^M$
  Let $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow succ_1(\psi_1^M)$. Then the ψ^M rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ with $p \notin Pos(\psi_1^M)$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$.
  We construct a substitution σ' from σ. σ' equals σ except for that ψ_1^M is removed from all values. The prefixes of positions in $\mathcal{M}$ equal the prefixes $\mathcal{M}$ except for that ψ_1^M is missing. Thus, σ' is a unifier for $\mathcal{M}'$ and $\langle \mathcal{C}, \mathcal{W} \rangle$.
  Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ'. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \, \psi^M \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 2$.

- $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \psi_1^E$
  Let $\mathcal{M}' = \Phi^E : \cdot \Uparrow succ_1(\psi_1^E)$. Then the ψ^E rule can be applied on $\mathcal{M}$ with premise $\mathcal{M}'$. Φ^M is empty according to lemma 66. Thus, rule ψ^E is applicable. $LPaths(\mathcal{M}) = LPaths(\mathcal{M}')$, $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}')$, $\boldsymbol{\beta}(\mathcal{M}) = \boldsymbol{\beta}(\mathcal{M}')$ and for all $p \in Pos(\mathcal{M}')$ with $p \notin Pos(\psi_1^E)$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$.
  We construct a substitution σ' from σ. σ' equals σ except for that ψ_1^E is removed from all values. The prefixes of positions in $\mathcal{M}$ equal the prefixes $\mathcal{M}$ except for that ψ_1^E is missing. Thus, σ' is a unifier for $\mathcal{M}'$ and $\langle \mathcal{C}, \mathcal{W} \rangle$.
  Therefore, $\mathcal{M}'$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ'. $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \, \psi^E \right.$ is a proof for $\mathcal{M}$ by induction hypothesis because $wgt(\mathcal{M}') = wgt(\mathcal{M}) - 2$.

5.4 Completeness

We define functions *ConSet*, *WeakMap*, and *red* which respectively return a set of connections, a weakening map, and a reduction ordering for a Σ_{pos}-proof of a matrix $\mathcal{M}$. From the reduction ordering a substitution can be constructed. We show that $\mathcal{M}$ is complementary for the constructed set of connections, the weakening map, and the substitution. Each requirement of complementarity is demonstrated separately.

Definition 69 Let $\mathcal{M}$ be a matrix and $\mathcal{P}$ be a Σ_{pos}-proof for $\mathcal{M}$. We define a function *ConSet* which calculates a set of connections for $\mathcal{M}$ from $\mathcal{P}$ by recursion over the structure of $\mathcal{P}$.

- $ConSet\left(\overline{\Phi^E{:}\phi_1^M,\phi_2^M \Uparrow \cdot}\ ^{axiom} \right) = \{\{succ_1(\phi_1^M), succ_1(\phi_2^M)\}\}$
- $ConSet\left(\overline{\Phi^E{:}\phi_1^M \Uparrow \cdot}\ ^{\tau} \right) = \{\{succ_1(\phi_1^M)\}\}$
- If the last rule applied in $\mathcal{P}$ is one of $o \Downarrow$, $o \Uparrow$, ω, α, ν, π, $focus_1$, $focus_2$, ϕ^M, ψ^M, ϕ^E, ψ^E then it has one premise. Let $\mathcal{P}'$ be the corresponding subproof.
  $ConSet(\ \mathcal{P}\ ) = ConSet(\ \mathcal{P}'\ )$
- If the last rule applied in $\mathcal{P}$ is β then it has two premises. Let $\mathcal{P}'$ and $\mathcal{P}''$ be the corresponding subproofs.
  $ConSet(\ \mathcal{P}\ ) = ConSet(\ \mathcal{P}'\ ) \cup ConSet(\ \mathcal{P}''\ )$

Definition 70 Let $\mathcal{M}$ be a matrix and $\mathcal{P}$ be a Σ_{pos}-calculus proof for $\mathcal{M}$. We define a function *WeakMap* which calculates a weakening map for $\mathcal{M}$ from $\mathcal{P}$ by recursion over the structure of $\mathcal{P}$.

- $WeakMap\left(\overline{\Phi^E{:}\phi_1^M,\phi_2^M \Uparrow \cdot}\ ^{axiom} \right) = \Phi^E$
- $WeakMap\left(\overline{\Phi^E{:}\phi_1^M \Uparrow \cdot}\ ^{\tau} \right) = \Phi^E$
- If the last rule applied in $\mathcal{P}$ is one of $o \Downarrow$, $o \Uparrow$, α, π, $focus_1$, $focus_2$, ϕ^M, ψ^M, ϕ^E, ψ^E then it has one premise. Let $\mathcal{P}'$ be the corresponding subproof.
  $WeakMap(\ \mathcal{P}\ ) = WeakMap(\ \mathcal{P}'\ )$
- If the last rule applied in $\mathcal{P}$ is ω then it has one premise. Let $\mathcal{P}'$ be the corresponding subproof.
  $$WeakMap\left(\dfrac{\dfrac{\mathcal{P}'}{\Phi^E{:}\Phi^M \Uparrow \Xi}}{\Phi^E{:}\Phi^M \Uparrow \Xi,\omega}\ \omega \right) = WeakMap\left(\dfrac{\mathcal{P}'}{\Phi^E{:}\Phi^M \Uparrow \Xi} \right) \cup \{\omega\}$$
- If the last rule applied in $\mathcal{P}$ is ν then it has one premise. Let $\mathcal{P}'$ be the corresponding subproof.
  - If $\mu(\nu) = 0$ then
    $$WeakMap\left(\dfrac{\dfrac{\mathcal{P}'}{\Phi^E{:}\Phi^M \Uparrow \Xi}}{\Phi^E{:}\Phi^M \Uparrow \Xi,\nu}\ \nu \right) = WeakMap\left(\dfrac{\mathcal{P}'}{\Phi^E{:}\Phi^M \Uparrow \Xi} \right) \cup \{\nu\}$$
  - If $\mu(\nu) > 0$ then $WeakMap(\ \mathcal{P}\ ) = WeakMap(\ \mathcal{P}'\ )$

- If the last rule applied in $\mathcal{P}$ has two premises with subproofs $\mathcal{P}'$ and $\mathcal{P}''$ then
$\mathit{WeakMap}\left(\ \mathcal{P}\ \right) = \mathit{WeakMap}\left(\ \mathcal{P}'\ \right) \cup \mathit{WeakMap}\left(\ \mathcal{P}''\ \right)$

Definition 71 Let $\mathcal{M}$ be a matrix and $\mathcal{P}$ be a Σ_{pos}-calculus proof for $\mathcal{M}$. We define a function *red* which returns the ordering $\mathit{red}(\mathcal{M}) \subseteq \mathit{SpecPos}(\mathcal{M})^2$ for $\mathcal{P}$ by recursion over the structure of $\mathcal{P}$.

- $\mathit{red}\left(\ \overline{\Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot}\ ^{axiom}\ \right) = \emptyset$
- $\mathit{red}\left(\ \overline{\Phi^E : \phi_1^M \Uparrow \cdot}\ ^{\tau}\ \right) = \emptyset$
- If the last rule applied in $\mathcal{P}$ is one of $o \Downarrow$, $o \Uparrow$, ω, α, ν, π, ϕ^M, or ϕ^E then it has one premise. Let $\mathcal{P}'$ be the corresponding subproof.
$\mathit{red}\left(\ \mathcal{P}\ \right) = \mathit{red}\left(\ \mathcal{P}'\ \right)$
- If $\mathcal{M} = \Phi^E, \phi_1^E : \Phi^M \Uparrow \cdot$ and $\mathcal{M}' = \Phi^E : \Phi^M \Downarrow \mathit{succ}_1(\phi_1^E)$ then
$\mathit{red}\left(\dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ \mathit{focus}_1 \right) = \mathit{red}\left(\dfrac{\mathcal{P}'}{\mathcal{M}'} \right) \cup \{(\phi_1^E, p) \mid p \in \mathit{SpecPos}(\mathcal{M}')\}.$
- If $\mathcal{M} = \Phi^E : \Phi^M, \phi_1^M \Uparrow \cdot$ and $\mathcal{M}' = \Phi^E : \Phi^M \Downarrow \mathit{succ}_1(\phi_1^M)$ then
$\mathit{red}\left(\dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ \mathit{focus}_2 \right) = \mathit{red}\left(\dfrac{\mathcal{P}'}{\mathcal{M}'} \right) \cup \{(\phi_1^M, p) \mid p \in \mathit{SpecPos}(\mathcal{M}')\}.$
- If $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \psi_1^M$ and $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \mathit{succ}_1(\psi_1^M)$ then
$\mathit{red}\left(\dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ \psi^M \right) = \mathit{red}\left(\dfrac{\mathcal{P}'}{\mathcal{M}'} \right) \cup \{(\psi_1^M, p) \mid p \in \mathit{SpecPos}(\mathcal{M}')\}.$
- If $\mathcal{M} = \Phi^E : \cdot \Downarrow \psi_1^M$ and $\mathcal{M}' = \Phi^E : \cdot \Uparrow \mathit{succ}_1(\psi_1^E)$ then
$\mathit{red}\left(\dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ \psi^E \right) = \mathit{red}\left(\dfrac{\mathcal{P}'}{\mathcal{M}'} \right) \cup \{(\psi_1^E, p) \mid p \in \mathit{SpecPos}(\mathcal{M}')\}.$
- If the last rule applied in $\mathcal{P}$ is β then it has two premises. Let $\mathcal{P}'$ and $\mathcal{P}''$ be the corresponding subproofs.
$\mathit{red}\left(\ \mathcal{P}\ \right) = \mathit{red}\left(\ \mathcal{P}'\ \right) \cup \mathit{red}\left(\ \mathcal{P}''\ \right)$

For any proof $\mathcal{P}$ the relation $\mathit{red}(\mathcal{P})$ is irreflexive, antisymmetric, and transitive. Thus, it is an ordering. Instead of $(p_1, p_2) \in \mathit{red}(\mathcal{P})$ we also write $p_1 \sqsubset_{\mathcal{P}} p_2$. $\sqsubset_{\mathcal{P}}$ expresses the order in which special positions are reduced in $\mathcal{P}$. Therefore, we call $\sqsubset_{\mathcal{P}}$ the *reduction ordering of* $\mathcal{P}$.

Lemma 72 Let $\mathcal{M}$ be a matrix, $\mathcal{P}$ be a position calculus proof for $\mathcal{M}$, and $p_1, p_2 \in (\Psi^M(\mathcal{M}) \cup \Psi^E(\mathcal{M}))$ be positions in $\mathcal{M}$ with $p_1 \neq p_2$. If there is a position $p \in \mathit{SpecPos}(\mathcal{M})$ with $p_1 \sqsubset_{\mathcal{P}} p$ and $p_2 \sqsubset_{\mathcal{P}} p$ then either $p_1 \sqsubset_{\mathcal{P}} p_2$ or $p_2 \sqsubset_{\mathcal{P}} p_1$ holds.

Proof. If there is a position p with the above properties then p_1 and p_2 are reduced in the same branch of $\mathcal{P}$.

That $\sigma_{\mathcal{P}}$ resembles a tree ordering is fundamental for the following definition as well.

Definition 73 We construct a mapping $\sigma_{\mathcal{P}} : (\Phi^M(\mathcal{M}) \cup \Phi^E(\mathcal{M})) \to (\Psi^M(\mathcal{M}) \cup \Psi^E(\mathcal{M}))^2$ from $\sqsubset_{\mathcal{P}}$. Let $\phi \in (\Phi^M(\mathcal{M}) \cup \Phi^E(\mathcal{M}))$ be a variable special position in $\mathcal{M}$. We define $\sigma_{\mathcal{P}}(\phi) = Z = \psi_1 \ldots \psi_n$ if $Z \in (\Psi^M(\mathcal{M}) \cup \Psi^E(\mathcal{M}))^*$ is a string with the following properties. Since $\sigma_{\mathcal{P}}$ resembles a tree ordering, Z always exists and is unique.

sortedness For all $i \in \{1, \ldots, n-1\}$ holds $\psi_i \sqsubset_{\mathcal{P}} \psi_{i+1}$.

prior reduction For all $i \in \{1, \ldots, n\}$ holds $\psi_i \sqsubset_{\mathcal{P}} \phi$.

exclusivity For all $p \in \mathit{SpecPos}(\mathcal{M})$ holds $p \ll \phi \Rightarrow p \sqsubset_{\mathcal{P}} \psi_1$.

maximality For any $\psi \in (\Psi^M(\mathcal{M}) \cup \Psi^E(\mathcal{M}))$ which does not occur in Z with $\psi \sqsubset_{\mathcal{P}} \phi$ there is a $p \in \mathit{SpecPos}(\mathcal{M})$ with $p \ll \phi$ and $\psi \sqsubset_{\mathcal{P}} p$.

$\sigma_{\mathcal{P}}$ is idempotent by construction because it is a grounded prefix substitution.

Lemma 74 $\sigma_{\mathcal{P}}$ is admissible for $\mathcal{M}$.

Proof. Let $v \in \mathit{SpecPos}(\mathcal{M})$ with $pu \angle \sigma_{\mathcal{P}}(\mathit{pre}_{\mathcal{M}}(v))$. The proof is by contradiction. Assume that $pu \neq \sigma_{\mathcal{P}}(\mathit{pre}_{\mathcal{M}}(u))$. If $pu \angle \sigma_{\mathcal{P}}(\mathit{pre}_{\mathcal{M}}(u))$ then u occurs in p. This violates sortedness or prior reduction. Thus, there is a position $w \in p$ ($p = p'wp''$) for which $p'w \angle \sigma_{\mathcal{P}}(\mathit{pre}_{\mathcal{M}}(w))$ does not hold. Let w be the first position in p with this property. The existence of w violates sortedness, prior reduction, exclusivity, or maximality. This contradiction implies that $pu = \sigma_{\mathcal{P}}(\mathit{pre}_{\mathcal{M}}(u))$ holds.

5.4.1 The Spanning Property

Lemma 75 If $\mathcal{P}$ is an arbitrary position calculus proof for a matrix $\mathcal{M}$ then the set of connections $\mathit{ConSet}(\mathcal{P})$ spans $\mathcal{M}$.

Proof. We prove the lemma by induction over the structure of $\mathcal{P}$.
Base case: $\mathcal{P}$ consists only of a single rule application.

- If $\mathcal{M} = \Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot$ and $\mathcal{P} = \left\{ \overline{\mathcal{M}}\ \mathit{axiom} \right.$ then for all $P \in \mathit{LPaths}(\mathcal{M})$ holds $\mathit{succ}_1(\phi_1^M), \mathit{succ}_1(\phi_2^M) \in P$ by definition 44. By definition 69 holds $\mathit{ConSet}(\mathcal{P}) = \{\{\mathit{succ}_1(\phi_1^M), \mathit{succ}_1(\phi_2^M)\}\}$. Thus, $\mathit{ConSet}(\mathcal{P})$ spans $\mathcal{M}$.

- The case where τ is the last rule applied in $\mathcal{P}$ can be shown similarly.

Induction hypothesis: We assume that the lemma holds for any subproof $\mathcal{P}'$ of $\mathcal{P}$.
Induction step:

- If $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, o$, $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \Xi, \mathit{succ}_1(o)$, and $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\ o\Uparrow \right.$ then the lemma holds by induction hypothesis since $\mathit{LPaths}(\mathcal{M}) = \mathit{LPaths}(\mathcal{M}')$ and $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$.

- The case where $o \Downarrow$, α, π, focus_1, focus_2, ϕ^M, ψ^M, or ψ^E is the last rule applied in $\mathcal{P}$ can be shown similarly.

- If $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, \omega$, $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \Xi$, and $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \omega \right.$ then for all paths $P \in \mathit{LPaths}(\mathcal{M})$ there is a path $P' \in \mathit{LPaths}(\mathcal{M}')$ with $P = P' \cup \{\omega\}$. Thus, the lemma follows from the induction hypothesis.

- The case where ν is the last rule applied can be shown like the case for the ω rule if $\mu(\nu) = 0$ and like the case for the $o \Uparrow$ rule if $\mu(\nu) > 0$.

- The case where β is the last rule applied is the most interesting one. If $\mathcal{M} = \Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta$, $\mathcal{M}' = \Phi_1^E : \Phi_1^M \Downarrow \mathit{succ}_1(\beta)$, $\mathcal{M}'' = \Phi_2^E : \Phi_2^M \Downarrow \mathit{succ}_2(\beta)$, and $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'} \quad \frac{\mathcal{P}''}{\mathcal{M}''}}{\mathcal{M}} \beta \right.$ then for all $P \in \mathit{LPaths}(\mathcal{M})$ holds by definition 69 either $(Pos(\beta) \cap P) \subseteq Pos(succ_1(\beta))$ or $(Pos(\beta) \cap P) \subseteq Pos(succ_2(\beta))$. Without loss of generality we assume the first case. Thus, there exists a path $P' \in \mathit{LPaths}(\mathcal{M}')$ such that $P' \subseteq P$. By induction hypothesis there is a connection $C \in \mathit{ConSet}(\mathcal{P}')$ with $C \subseteq P'$. Therefore, the lemma holds.

5.4.2 The Linearity Property

Lemma 76 If $\mathcal{P}$ is an arbitrary position calculus proof for a matrix $\mathcal{M}$ then $\langle \mathit{ConSet}(\mathcal{P}), \mathit{WeakMap}(\mathcal{P}) \rangle$ is linear for $\mathcal{M}$.

Proof. We prove the lemma by induction over the structure of $\mathcal{P}$.
Base case: If $\mathcal{P}$ consists only of an application of one of the rules *axiom* or τ then the lemma follows from the construction of $\mathit{ConSet}(\mathcal{P})$ and $\mathit{WeakMap}(\mathcal{P})$. There is only one connection in $\mathit{ConSet}(\mathcal{P})$, and $\mathit{WeakMap}(\mathcal{P})$ contains no predecessor of a position in this connection.
Induction hypothesis: We assume that the lemma holds for any subproof $\mathcal{P}'$ of $\mathcal{P}$.
Induction step:

- If one of $o \Downarrow$, $o \Uparrow$, α, π, focus_1, focus_2, ϕ^M, ψ^M, or ψ^E is the last rule applied in $\mathcal{P}$ then the lemma holds by induction hypothesis. Any of these rules has one premise. Let $\mathcal{P}'$ be the corresponding subproof. By definition 69 holds $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$ and by definition 70 holds $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}')$.

- If $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\Phi^E : \Phi^M \Uparrow \Xi}}{\Phi^E : \Phi^M \Uparrow \Xi, \omega} \omega \right.$ then the lemma holds by induction hypothesis since $\mathit{ConSet}(\mathcal{P}) = \mathit{ConSet}(\mathcal{P}')$ and $\mathit{WeakMap}(\mathcal{P}) = \mathit{WeakMap}(\mathcal{P}') \cup \{\omega\}$.

- The case where ν is the last rule applied can be shown like the case for the ω rule if $\mu(\nu) = 0$ and like the case for the $o \Downarrow$ rule if $\mu(\nu) > 0$.

- If $\mathcal{M} = \Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta$, $\mathcal{M}' = \Phi_1^E : \Phi_1^M \Downarrow \mathit{succ}_1(\beta)$, $\mathcal{M}'' = \Phi_2^E : \Phi_2^M \Downarrow \mathit{succ}_2(\beta)$, and $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'} \quad \frac{\mathcal{P}''}{\mathcal{M}''}}{\mathcal{M}} \beta \right.$ then $\mathit{WeakMap}(\mathcal{P}') \subseteq Pos(\mathcal{M}')$, $\mathit{WeakMap}(\mathcal{P}'') \subseteq Pos(\mathcal{M}'')$, and for any connection $C' \in \mathit{ConSet}(\mathcal{P}')$ and for any connection $C'' \in \mathit{ConSet}(\mathcal{P}'')$ holds $C' \subseteq Pos(\mathcal{M}')$ and $C'' \subseteq Pos(\mathcal{M}'')$. Since $Pos(\mathcal{M}') \cap Pos(\mathcal{M}'') = \emptyset$ the lemma follows from the induction hypothesis.

5.4.3 The Relevance Property

Lemma 77 If $\mathcal{P}$ is an arbitrary position calculus proof of a matrix $\mathcal{M}$ then $\langle ConSet(\mathcal{P}), WeakMap(\mathcal{P})\rangle$ has the relevance property for $\mathcal{M}$.

Proof. We prove the lemma by induction over the structure of $\mathcal{P}$.
Base case: If $\mathcal{P}$ consists only of an application of one of the rules *axiom* or τ then the lemma follows from the construction of $ConSet(\mathcal{P})$ and $WeakMap(\mathcal{P})$.
Induction hypothesis: We assume that the lemma holds for any subproof $\mathcal{P}'$ of $\mathcal{P}$.
Induction step:

- If one of $o\Downarrow$, $o\Uparrow$, α, π, $focus_1$, $focus_2$, ϕ^M, ψ^M, or ψ^E is the last rule applied in $\mathcal{P}$ then the lemma holds by induction hypothesis. Any of these rules has one premise. Let $\mathcal{M}'$ be the premise and $\mathcal{P}'$ be the corresponding subproof. By definition 69 and definition 70 holds $ConSet(\mathcal{P}) = ConSet(\mathcal{P}')$ and $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}')$.

- If $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\Phi^E:\Phi^M\Uparrow\Xi}}{\Phi^E:\Phi^M\Uparrow\Xi,\omega}\ \omega \right.$ then the lemma holds by induction hypothesis since $ConSet(\mathcal{P}) = ConSet(\mathcal{P}')$, $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}') \cup \{\omega\}$, and $LeafPos(\mathcal{M}) = LeafPos(\mathcal{M}') \cup \{\omega\}$.

- The case where ν is the last rule applied can be shown like the case for the ω rule if $\mu(\nu) = 0$ and like the case for the $o\Downarrow$ rule if $\mu(\nu) > 0$.

- If $\mathcal{M} = \Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta$, $\mathcal{M}' = \Phi_1^E : \Phi_1^M \Downarrow succ_1(\beta)$, $\mathcal{M}'' = \Phi_2^E : \Phi_2^M \Downarrow succ_2(\beta)$, and $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'} \quad \dfrac{\mathcal{P}''}{\mathcal{M}''}}{\mathcal{M}}\ \beta \right.$ then $ConSet(\mathcal{P}) = ConSet(\mathcal{P}') \cup ConSet(\mathcal{P}'')$ and $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}') \cup WeakMap(\mathcal{P}'')$. Thus, the lemma follows from the induction hypothesis.

5.4.4 The Cardinality Property

Lemma 78 Let $\mathcal{P}$ be an arbitrary position calculus proof for a matrix $\mathcal{M}$ then $\langle ConSet(\mathcal{P}), WeakMap(\mathcal{P})\rangle$ has the cardinality property for $\mathcal{M}$.

Proof. We prove the lemma by induction over the structure of $\mathcal{P}$.
Base case: $\mathcal{P}$ consists only of a single rule application.

- If $\mathcal{M} = \Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot$ and $\mathcal{P} = \left\{ \overline{\mathcal{M}}\ axiom \right.$ then

$$\begin{aligned} & card(\langle ConSet(\mathcal{P}), WeakMap(\mathcal{P})\rangle) \\ = \;& |ConSet(\mathcal{P})| + \textstyle\sum_{\phi^E \in WeakMap(\mathcal{P})} |\boldsymbol{\beta}(\phi^E)| \\ = \;& 1 + \textstyle\sum_{\phi^E \in \Phi^E} |\boldsymbol{\beta}(\phi^E)| \\ = \;& 1 + |\boldsymbol{\beta}(\mathcal{M})| \end{aligned}$$

- The case where τ is the last rule applied in $\mathcal{P}$ can be shown similarly.

Induction hypothesis: We assume that the lemma holds for any subproof $\mathcal{P}'$ of $\mathcal{P}$.
Induction step:

- If one of $o\Downarrow$, $o\Uparrow$, ω, α, ν, π, $focus_1$, $focus_2$, ϕ^M, ψ^M, or ψ^E is the last rule applied in $\mathcal{P}$ then the lemma holds by induction hypothesis. Any of these rules has one premise. Let $\mathcal{P}'$ be that premise and $\mathcal{P}'$ be the corresponding subproof. The following equations hold:

$$\begin{array}{lcl} |\boldsymbol{\beta}(\mathcal{M})| & = & |\boldsymbol{\beta}(\mathcal{M}')| \\ |ConSet(\mathcal{P})| & = & |ConSet(\mathcal{P}')| \\ \sum_{\phi^E \in WeakMap(\mathcal{P})} |\boldsymbol{\beta}(\phi^E)| & = & \sum_{\phi^E \in WeakMap(\mathcal{P}')} |\boldsymbol{\beta}(\phi^E)| \end{array}$$

- If $\mathcal{M} = \Phi_1^E, \Phi_2^E : \Phi_1^M, \Phi_2^M \Downarrow \beta$, $\mathcal{M}' = \Phi_1^E : \Phi_1^M \Downarrow succ_1(\beta)$, $\mathcal{M}'' = \Phi_2^E : \Phi_2^M \Downarrow succ_2(\beta)$, and $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'} \quad \dfrac{\mathcal{P}''}{\mathcal{M}''}}{\mathcal{M}}\, \beta \right.$ then the lemma follows from the induction hypothesis. The following equations hold:

$$\begin{array}{lcl} |\boldsymbol{\beta}(\mathcal{M})| & = & |\boldsymbol{\beta}(\mathcal{M}')| + |\boldsymbol{\beta}(\mathcal{M}'')| + 1 \\ |ConSet(\mathcal{P})| & = & |ConSet(\mathcal{P}') \cup ConSet(\mathcal{P}'')| \\ \sum_{\phi^E \in WeakMap(\mathcal{P})} |\boldsymbol{\beta}(\phi^E)| & = & \sum_{\phi^E \in WeakMap(\mathcal{P}')} |\boldsymbol{\beta}(\phi^E)| \\ & & + \sum_{\phi^E \in WeakMap(\mathcal{P}'')} |\boldsymbol{\beta}(\phi^E)| \end{array}$$

5.4.5 The Unifiability Property

Lemma 79 If $\mathcal{P}$ is an arbitrary Σ_{pos}-proof for a matrix $\mathcal{M}$ then $\sigma_{\mathcal{P}}$ is a unifier for the pair $\langle ConSet(\mathcal{P}), WeakMap(\mathcal{P})\rangle$.

Proof. The admissibility of $\sigma_{\mathcal{P}}$ follows from lemma 74. We prove the lemma by induction over the structure of $\mathcal{P}$.
Base Case: $\mathcal{P}$ consists only of a single rule application.

- Let $\mathcal{M} = \Phi^E : \phi_1^M, \phi_2^M \Uparrow \cdot$ and $\mathcal{P} = \left\{ \overline{\mathcal{M}}\ ^{axiom} \right.$.
Then $ConSet(\mathcal{P}) = \{\{succ_1(\phi_1^M), succ_1(\phi_2^M)\}\}$, $WeakMap(\mathcal{P}) = \Phi^E$, and $red(\mathcal{P}) = \emptyset$. Thus, $\sigma_{\mathcal{P}}(pre_{\mathcal{M}}(succ_1(\phi_1^M))) = \sigma_{\mathcal{P}}(pre_{\mathcal{M}}(succ_1(\phi_2^M))) = \varepsilon$ and for all $\phi^E \in \Phi^E$ holds $\sigma_{\mathcal{P}}(pre_{\mathcal{M}}(\phi^E)) = \varepsilon$.

- The case where τ is the last rule applied in $\mathcal{P}$ can be shown similarly.

Induction hypothesis: We assume that the lemma holds for any subproof $\mathcal{P}'$ of $\mathcal{P}$.
Induction step:

- Let $\mathcal{M} = \Phi^E : \Phi^M \Uparrow o$, $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow succ_1(o)$, and $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\, o\Uparrow \right.$.
Then $ConSet(\mathcal{P}) = ConSet(\mathcal{P}')$, $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}')$, $red(\mathcal{P}) = red(\mathcal{P}')$, and $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'}$. $pre_{\mathcal{M}}(succ_1(p)) = pre_{\mathcal{M}'}(succ_1(p))$ holds for all $p \in LeafPos(\mathcal{M}) \cup \Phi^E(\mathcal{M})$.

- The case where one of $o\Downarrow$, α, β, π, or ϕ^M is the last rule applied in $\mathcal{P}$ can be shown similarly.

- Let $\mathcal{M} = \Phi^E : \Phi^M \Uparrow \Xi, \omega$, $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow \Xi$, and $\mathcal{P} = \left\{ \dfrac{\dfrac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}}\, \omega \right.$. Then $ConSet(\mathcal{P}) = ConSet(\mathcal{P}')$, $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}') \cup \{\omega\}$, and $red(\mathcal{P}) = red(\mathcal{P}')$. Therefore, $\sigma_{\mathcal{P}} = \sigma_{\mathcal{P}'}$. For all $p \in LeafPos(\mathcal{M}) \cup \Phi^E(\mathcal{M})$ with $p \neq \omega$ holds $pre_{\mathcal{M}}(succ_1(p)) = pre_{\mathcal{M}'}(succ_1(p))$ and for $p = \omega$ holds $pre_{\mathcal{M}}(succ_1(p)) = \varepsilon$.

- The case where ν is the last rule applied in $\mathcal{P}$ can be shown like the case for ω for $\mu(\nu) = 0$ and like the case for $o \Uparrow$ for $\mu(\nu) > 0$.

- Let $\mathcal{M} = \Phi^E, \phi_1^E : \Phi^M \Uparrow \cdot$, $\mathcal{M}' = \Phi^E : \Phi^M \Downarrow succ_1(\phi_1^E)$, and $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \; focus_1 \right.$. Then $ConSet(\mathcal{P}) = ConSet(\mathcal{P}')$, $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}')$, and $red(\mathcal{P}) = red(\mathcal{P}') \cup \{(\phi_1^E, p) \mid p \in SpecPos(\mathcal{M}')\}$. For all $p \in LeafPos(\mathcal{M}) \cup \Phi^E(\mathcal{M})$ with $p \in Pos(\phi_1^E)$ holds $pre_{\mathcal{M}}(p) = \phi_1^E.pre_{\mathcal{M}'}(p)$ and by definition 73 (prior reduction) $\sigma_{\mathcal{P}}(\phi_1^E.pre_{\mathcal{M}'}(p)) = \sigma_{\mathcal{P}'}(pre_{\mathcal{M}'}(p))$. For all $p \in LeafPos(\mathcal{M}) \cup \Phi^E(\mathcal{M})$ with $p \notin Pos(\phi_1^E)$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$ and $\sigma_{\mathcal{P}}(pre_{\mathcal{M}}(p)) = \sigma_{\mathcal{P}'}(pre_{\mathcal{M}'}(p))$.

- The case where $focus_2$ is the last rule applied in $\mathcal{P}$ can be shown similarly.

- Let $\mathcal{M} = \Phi^E : \Phi^M \Downarrow \psi_1^M$, $\mathcal{M}' = \Phi^E : \Phi^M \Uparrow succ_1(\psi_1^M)$, and $\mathcal{P} = \left\{ \frac{\frac{\mathcal{P}'}{\mathcal{M}'}}{\mathcal{M}} \; \psi^M \right.$. Then $ConSet(\mathcal{P}) = ConSet(\mathcal{P}')$, $WeakMap(\mathcal{P}) = WeakMap(\mathcal{P}')$, and $red(\mathcal{P}) = red(\mathcal{P}') \cup \{(\psi_1^M, p) \mid p \in SpecPos(\mathcal{M}')\}$. For all $p \in LeafPos(\mathcal{M}) \cup \Phi^E(\mathcal{M})$ with $p \in Pos(\psi_1^M)$ holds $pre_{\mathcal{M}}(p) = \psi_1^M.pre_{\mathcal{M}'}(p)$ and $\sigma_{\mathcal{P}}(\psi_1^M.pre_{\mathcal{M}'}(p)) = \psi_1^M.\sigma_{\mathcal{P}'}(pre_{\mathcal{M}'}(p))$. For all $p \in LeafPos(\mathcal{M}) \cup \Phi^E(\mathcal{M})$ with $p \notin Pos(\phi_1^E)$ holds $pre_{\mathcal{M}}(p) = pre_{\mathcal{M}'}(p)$ and $\sigma_{\mathcal{P}}(pre_{\mathcal{M}}(p)) = \psi_1^M.\sigma_{\mathcal{P}'}(pre_{\mathcal{M}'}(p))$ by definition 73 (sortedness and maximality).

- The case where ψ^E is the last rule applied in $\mathcal{P}$ can be shown similarly.

5.4.6 Completeness Theorem

Theorem 80 (Completeness) Let $\mathcal{M}$ be a matrix. If there exists a Σ_{pos}-proof $\mathcal{P}$ for $\mathcal{M}$ then $\mathcal{M}$ is complementary for $ConSet(\mathcal{P})$, $WeakMap(\mathcal{P})$, and $\sigma_{\mathcal{P}}$.

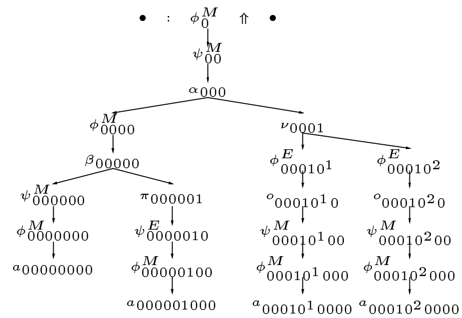
Proof. The theorem holds because of lemma 75, 76, 77, 78, and 79.

5.5 Characterization Theorem

The characterization theorem presented in this subsection implies that the validity of a formula and the complementarity of the corresponding matrix for some multiplicity are equivalent. It can serve as a foundation for matrix based proof search methods for $\mathcal{MELL}$. The matrix characterization yields a condensed representation of the search space which can be exploited by efficient proof search methods in the same way as for other logics [OK96]. A general proof method has been extended uniformly to multiplicative linear logic, as shown in [KMOS97]. Along the same lines an extension to $\mathcal{MELL}$ is possible.

Theorem 81 (Characterization Theorem) A formula φ is valid in $\mathcal{MELL}$ if and only if for some multiplicity the corresponding matrix is complementary.

Proof. The correctness follows from theorems 68, 42, 29, 23, and the correctness of Σ_1'. Completeness follows from theorems 80, 41, 28, 22, and the completeness of Σ_1'.



p	$pre_{\mathcal{M}}(p)$
$a_{00000000}$	$\phi^M_0 \psi^M_{00} \phi^M_{0000} \psi^M_{000000} \phi^M_{0000000}$
$a_{000001000}$	$\phi^M_0 \psi^M_{00} \phi^M_{0000} \psi^E_{0000010} \phi^M_{00000100}$
$a_{00010^1 000}$	$\phi^M_0 \psi^M_{00} \phi^E_{00010^1} \psi^M_{00010^1 00} \phi^M_{00010^1 000}$
$a_{00010^2 000}$	$\phi^M_0 \psi^M_{00} \phi^E_{00010^2} \psi^M_{00010^2 00} \phi^M_{00010^2 000}$

Figure 6: Example Matrix and Prefixes of Leaf Positions

We illustrate the theorem by a matrix proof for our running example.

Example 82 Let $\mathcal{M}$ be the matrix for $\varphi = \langle (A \otimes !A) \parr ?(A^\perp), + \rangle$ from figure 6. We choose $\mathcal{C} = \{\{a_{00000000}, a_{00010^2 0000}\}, \{a_{000001000}, a_{00010^1 0000}\}\}$, $\mathcal{W} = \emptyset$, and
$$\sigma = \{\phi^M_{0000} \backslash \varepsilon,\ \phi^M_{0000000} \backslash \psi^M_{00010^2 00},\ \phi^M_{00000100} \backslash \psi^M_{00010^1 00},\ \phi^E_{00010^1} \backslash \psi^E_{0000010},\ \phi^M_{00010^1 000} \backslash \varepsilon,\ \phi^E_{00010^2} \backslash \psi^M_{000000},\ \phi^M_{00010^2 000} \backslash \varepsilon\} .$$
Then $\mathcal{M}$ is complementary for $\mathcal{C}$, $\mathcal{W}$, and σ. Consequently φ is valid in $\mathcal{MELL}$.

6 Conclusion

A matrix characterization of logical validity has been presented for the full multiplicative exponential fragment of linear logic ($\mathcal{MELL}$). It extends our characterization for $\mathcal{MLL}$ [KMOS97] by the exponentials ? and ! and the multiplicative constants **1** and $\perp$. Our extension, as pointed out in [Fro96], is by no means trivial and goes beyond all existing matrix characterizations for fragments of linear logic.
In the process a methodology has been outlined for developing matrix characterizations from sequent calculi and for proving them correct and complete. It introduces a series of intermediate calculi, which step-wisely remove redundancies from sequent proofs while capturing their essential parts, and arrives at a matrix characterization as the most compact representation for proof search.
If applied to modal or intuitionistic logics, this methodology would essentially lead to Wallen's matrix characterization [Wal90]. In order to capture the resource sensitivity of linear logic, however, several refinements have been introduced. The notion of multiplicities is based on positions instead of basic positions. Different types of special positions are used. The novel concept of weakening maps makes us able to deal with the aspects of resource management. In linear logic, weakening can only be applied on certain formulas. A matrix proof must ensure

that it is possible to weaken all positions which take not part in an axiom in the corresponding sequent proofs. This is ensured by weakening maps together with a modified unifiability requirement.
Fronhöfer has developed matrix characterizations for various variations of the multiplicative fragment of linear logic [Fro96]. Compared to his work for linear logic our characterization captures additionally the multiplicative constants and the controlled application of weakening and contraction. In fact, we are confident that our methodology will extend to further fragments of linear logic as well as to other resource sensitive logics, such as affine or relevant logics.
In the future we plan to extend our characterization to quantifiers, which again is a non-trivial problem although much is known about them in other logics. Furthermore, the development of *matrix systems* [MS97] as a general theory of matrix characterizations has become possible. These systems include a uniform framework for defining notions of complementarity and a methodology for supporting the proof of characterization theorems.
The matrix characterization presented is a condensed representation of the search space. In general, matrix characterizations are known as a foundation for efficient proof search procedures for classical, modal and intuitionistic logics [OK96] and $\mathcal{MLL}$ [KMOS97]. We expect that these proof procedures can now be extended to $\mathcal{MELL}$ and a wide spectrum of other logics, as soon as our methodology has led us to a matrix characterization for them.

References

[Ale94] Vladimir Alexiev. Applications of linear logic to computation: An overview. *Bull. of the IGPL*, 2(1):77–107, 1994.

[And81] Peter B. Andrews. Theorem-Proving via General Matings. *Journal of the Association for Computing Machinery*, 28(2):193–214, 1981.

[And92] Jean-Marc Andreoli. Logic Programming with Focussing Proofs in Linear Logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[Bib81] Wolfgang Bibel. On Matrices with Connections. *Journal of the ACM*, 28:633–645, 1981.

[Bib86] Wolfgang Bibel. A Deductive Solution for Plan Generation. *New Generation Computing*, 4(2):115–132, 1986.

[Bib87] Wolfgang Bibel. *Automated Theorem Proving*. Vieweg, 2nd edition, 1987.

[Bib93] Wolfgang Bibel. *Deduction*. Academic Press Inc, San Diego, CA, 1993.

[CHP96] Iliano Cervesato, Joshua S. Hodas, and Frank Pfenning. Efficient Resource Management for Linear Logic Proof Search. In R. Dyckhoff, H. Herre, and P. Schroeder-Heister, editors, *Proceedings of the Fifth International Workshop on Extensions of Logic Programming -*

ELP'96, number 1050 in LNAI, pages 67–81. Springer Verlag, March 1996.

[DR89] V. Danos and L. Regnier. The Structure of the Multiplicatives. *Archive for Mathematical Logic*, 28:181–203, 1989.

[Fro96] Bertram Fronhöfer. *The Action-as-Implication Paradigm.* CS Press, 1996.

[Gal91] Jean Gallier. Constructive Logics. Part II: Linear Logic and Proof Nets. Technical Report 9, Digital Equipment Corporation, 1991.

[GG91] V. Gehlot and C. Gunter. Normal Process Representatives. In *Proc. 5-th Annual IEEE Symposium on Logic in Computer Science*, pages 200–207, Los Alamitos, California, 1991. IEEE Computer Society Press.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[GP94] Didier Galmiche and Guy Perrier. On proof normalization in linear logic. *Theoretical Computer Science*, 135:67–110, 1994.

[HM94] J.S. Hodas and D. Miller. Logic Programming in a Fragment of Linear Logic. *Journal of Information and Computation*, 110(2):327–365, 1994.

[KMOS97] Christoph Kreitz, Heiko Mantel, Jens Otten, and Stephan Schmitt. Connection-Based Proof Construction in Linar Logic. In *CADE-14*, LNAI 1249, pages 207–221. Springer Verlag, 1997.

[Lin92] Patrick Lincoln. Linear Logic. *ACM SIGACT Notices*, 23(2):29–37, 1992.

[LMSS92] Patrick Lincoln, John Mitchell, Andre Scedrov, and Natarajan Shankar. Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56:239–311, 1992.

[LW94] Patrick Lincoln and T. Winkler. Constant multiplicative linear logic is NP-complete. *Theoretical Computer Science*, 135:155–169, 1994.

[Man96] Heiko Mantel. Eine Matrixcharakterisierung für ein Fragment der Linearen Logik. Diplomarbeit, TH-Darmstadt, Germany, December 1996.

[Mil96] D. Miller. A multiple-conclusion specification logic. *Theoretical Computer Science*, 165(1):201–232, 1996.

[MS97] Heiko Mantel and Enno Sandner. A Framework for Connection Calculi. Technical Report WV 97-09, Knowledge Representation and Reasoning Group, Artificial Intelligence Institute, Computer Science Departement, Dresden University of Technology, Germany, 1997.

[MTV91] M. Masseron, C. Tollu, and J. Vauzeilles. Generating Plans in Linear Logic. In *Foundations of Software Technology and Theoretical Computer Science*, LNCS. Springer Verlag, 1991.

[OK96] Jens Otten and Christoph Kreitz. A Uniform Proof Procedure for Classical and Non-Classical Logics. In *KI-96: Advances in Artificial Intelligence*, LNAI 1137, pages 307–319. Springer Verlag, 1996.

[Sce90] A. Scedrov. A Brief Guide to Linear Logic. *Bulletin of the European Association for Theoretical Computer Science*, 41:154–165, June 1990.

[SK96] Stephan Schmitt and Christoph Kreitz. Converting Non-Classical Matrix Proofs into Sequent-Style Systems. In *CADE-13*, LNAI 1104, pages 418–432. Springer Verlag, 1996.

[Tam94] Tanel Tammet. Proof Strategies in Linear Logic. *Journal of Automated Reasoning*, 12:273–304, 1994.

[Tro92] Anne Troelsta. *Lectures on Linar Logic, CSLI Lecture Notes, no 29.* Center for the Study of Language and Information, Stanford University, 1992.

[Wal90] Lincoln Wallen. *Automated Deduction in Nonclassical Logic.* MIT Press, 1990.

RR-98-03
Research Repor

Developing a Matrix Characterization for $\mathcal{MELL}$

Heiko Mantel