Gomory-Chvátal Cutting Planes and the Elementary Closure of Polyhedra

Dissertation

zur Erlangung des Grades
Doktor der Ingenieurwissenschaften (Dr.-Ing.)
der Naturwissenschaftlich–Technischen Fakultät I
der Universität des Saarlandes
von

Friedrich Eisenbrand

Saarbrücken
2000

Tag des Kolloquiums:	6. Juli 2000
Dekan:	Prof. Dr. Rainer Schulze-Pillot-Ziemen
Berichterstatter:	Prof. Dr. Alexander Bockmayr
	Prof. Dr. Kurt Mehlhorn

Abstract

The elementary closure P' of a polyhedron P is the intersection of P with all its Gomory-Chvátal cutting planes. P' is a rational polyhedron provided that P is rational. The Chvátal-Gomory procedure is the iterative application of the elementary closure operation to P. The Chvátal rank is the minimal number of iterations needed to obtain P_I. It is always finite, but already in $\mathbb{R}^2$ one can construct polytopes of arbitrary large Chvátal rank. We show that the Chvátal rank of polytopes contained in the n-dimensional $0/1$ cube is $\mathrm{O}(n^2 \log n)$ and prove the lower bound $(1+\epsilon)\,n$, for some $\epsilon > 0$.

We show that the separation problem for the elementary closure of a rational polyhedron is NP-hard. This solves a problem posed by Schrijver.

Last we consider the elementary closure in fixed dimension. The known bounds for the number of inequalities defining P' are exponential, even in fixed dimension. We show that the number of inequalities needed to describe the elementary closure of a rational polyhedron is polynomially bounded in fixed dimension. Finally, we present a polynomial algorithm in varying dimension, which computes cutting planes for a simplicial cone from this polynomial description in fixed dimension with a maximal degree of violation in a natural sense.

Kurzzusammenfassung

Die elementare Hülle P' eines Polyeders P ist der Durchschnitt von P mit all seinen Gomory-Chvátal Schnittebenen. P' ist ein rationales Polyeder, falls P rational ist. Die Chvátal-Gomory Prozedur ist das wiederholte Bilden der elementaren Hülle, beginnend mit P. Die minimale Anzahl der Iterationen, die bis zum Erhalt der ganzzahligen Hülle P_I von P nötig sind, heißt der Chvátal-Rang von P. Der Chvátal-Rang eines rationalen Polyeders ist endlich. Jedoch lassen sich bereits im $\mathbb{R}^2$ Beispiele mit beliebig hohem Chvátal-Rang konstruieren. Wir zeigen, daß der Chvátal-Rang eines Polytops im n-dimensionalen 0/1 Würfel durch $\mathrm{O}(n^2 \log n)$ beschränkt ist, und beweisen die untere Schranke $(1+\epsilon)\, n$, für ein $\epsilon > 0$.

Wir zeigen, daß das Separationsproblem für die elementare Hülle eines rationalen Polyeders NP-hart ist. Dies löst ein von Schrijver formuliertes Problem.

Schließlich wenden wir uns der elementaren Hülle rationaler Polyeder in fester Dimension zu. Die bislang bekannten Schranken für die Anzahl der Ungleichungen, die zur Darstellung von P' benötigt werden, sind exponentiell, selbst in fester Dimension. Wir zeigen, daß in fester Dimension P' durch polynomiell viele Ungleichungen beschrieben werden kann. Wir entwerfen außerdem einen, in beliebiger Dimension polynomiellen, Algorithmus, der zu einem spitzen Kegel P eine Schnittebene aus der polynomiellen Darstellung von P' berechnet, die zudem einen maximalen Grad der Verletzung in einem natürlichen Sinne aufweist.

Acknowledgments

I thank Prof. Dr. Harald Ganzinger for giving me the opportunity to work in his excellent programming logics group at the Max-Planck-Institut für Informatik, and I wish to thank the Deutsche Forschungsgemeinschaft for granting me a scholarship as a member of the Graduiertenkolleg “Effizienz und Komplexität von Algorithmen und Rechenanlagen”.

I was most fortunate to have Prof. Dr. Alexander Bockmayr as my advisor. He introduced me to (integer) linear programming and to the theory of Gomory-Chvátal cuts. Most of the results presented here are joint work with him. I want to thank him for this very fruitful collaboration during the last three years and for his invaluable help and encouragement during the time I was writing this thesis.

Many thanks go to Prof. Dr. Günter M. Ziegler for inviting me to spend six weeks with his Discrete Geometry Group at the Technical University Berlin. Chapter 5 of this thesis contains the results that I found there. I benefited very much from many discussions we had and from his support.

I also want to thank my coauthors Mark Hartmann and Andreas S. Schulz.

I am grateful to Prof. Dr. Kurt Mehlhorn for his instant commitment to become a referee of this thesis.

Thanks to Volker Priebe for reading parts of the manuscript and his comments. Special thanks go to Anne, Christoph, Florent, René and Solofo for good vibrations.

Contents

1

Introduction

> Gab es Einwände, die man vergessen hatte? Gewiß gab es solche. Die Logik ist zwar unerschütterlich, aber einem Menschen, der leben will, widersteht sie nicht.
>
> (Franz Kafka, Der Prozeß)

1.1 Motivation

Integer programming is concerned with the optimization of a linear function over the integer points in a polyhedron P. Among the most successful methods for solving integer programming problems is the cutting plane method in combination with branch-and-bound. A Gomory-Chvátal cutting plane for P is an inequality $c^T x \leq \lfloor \delta \rfloor$, where c is an integral vector and $c^T x \leq \delta$ is valid for P, i.e., the halfspace defined by $c^T x \leq \delta$ contains P. The cutting plane $c^T x \leq \lfloor \delta \rfloor$ is valid for all integral points in P and thus for the convex hull of integral vectors in P, the integer hull P_I. The addition of a cutting plane to the system of inequalities defining P results in a better approximation of the integer hull. The intersection of a polyhedron with all its Gomory-Chvátal cutting planes is called the elementary closure P' of P. If P is rational, then P' is a rational polyhedron again. The successive application of the elementary closure operation to a rational polyhedron yields the integer hull of the polyhedron after a finite number of steps (Chvátal 1973*a*, Schrijver 1980). This successive application of the elementary closure operation is referred to as the Chvátal-Gomory procedure. The minimal number of rounds until P_I is obtained is the Chvátal rank of P.

Even in two dimensions, one can construct polytopes of arbitrary large Chvátal rank. Integer programming formulations of combinatorial optimization problems are most often polytopes in the 0/1 cube. This motivates the following question.

Question 1. *Can the Chvátal rank of polytopes in the 0/1 cube be polynomially bounded in terms of the dimension?*

In combinatorial optimization, cutting planes are often derived from the structure of the problem. But even then they most likely fit in the Gomory-Chvátal cutting plane framework. A polynomial separation routine for the elementary closure of a rational polyhedron would thus be a very powerful tool. The next question was posed as an open problem in (Schrijver 1986, p. 351).

Question 2. *Does there exist a polynomial separation algorithm for the elementary closure P' of a rational polyhedron P?*

Not much was known about the polyhedral structure of the elementary closure in general. In essence one has the following result (see, e.g. (Cook, Cunningham, Pulleyblank & Schrijver 1998)): If P is defined as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, then P' is the intersection of P with all Gomory-Chvátal cutting planes $c^T x \leq \lfloor \delta \rfloor$, $c \in \mathbb{Z}^n$, where $c^T = \lambda^T A$ with some $\lambda \in [0,1)^m$ and $\delta = \max\{c^T x \mid x \in P\}$. The infinity norm $\|c\|_\infty$ of any such vector $c = A^T \lambda$ from above can be estimated as follows: $\|c\|_\infty = \|A^T \lambda\|_\infty \leq \|A^T\|_\infty$. From this, only an exponential (in the input encoding of P) upper bound $\|A^T\|_\infty^n$ on the number of inequalities needed to describe P' can be derived. This is also exponential in fixed dimension n. Integer programming in fixed dimension is solvable in polynomial time (Lenstra 1983). It would be undesirable if the upper bound described above was tight. A deeper knowledge of the structure of the elementary closure is also important in the context of choosing effective cutting planes.

Question 3. *What is the structure of the elementary closure of a polyhedron? Can its complexity be polynomially bounded in fixed dimension?*

1.2 Outline

This thesis is concerned with the questions above.

After reviewing some preliminaries in chapter 2, we introduce the cutting plane method and the cutting plane proof system in chapter 3 in greater detail. We show how Gomory's (Gomory 1958) original algorithmic result implies the finiteness of the Chvátal-Gomory procedure. Apparently this has not been observed before for general polyhedra. A similar observation was made by Schrijver for polyhedra in the positive orthant.

In chapter 4 we are concerned with Question 1. We first study rational polytopes in the n-dimensional 0/1 cube that do not contain integral points. It turns out that their Chvátal rank can essentially be bounded by their dimension. Our main result in this chapter is an $\mathrm{O}(n^2 \log n)$ upper bound on the Chvátal rank of arbitrary polytopes in the 0/1 cube. We also present a family of polytopes in the n-dimensional 0/1-cube whose Chvátal rank is at least $(1+\epsilon)\,n$, for some $\epsilon > 0$. This improves the known lower bound n.

In chapter 5 we give a negative answer to Question 2 by showing that the separation problem for the elementary closure of a polyhedron is NP-hard.

Chapter 6 is concerned with Question 3. We prove that the elementary closure can be described with a polynomial number of inequalities in fixed dimension and we provide a polynomial algorithm (in varying dimension) for finding cutting planes from this description. First we inspect the elementary closure of rational simplicial cones. We show that it can be described with polynomially many inequalities in fixed dimension. Via a triangulation argument, we prove a similar statement for arbitrary rational polyhedra. Then we show that the elementary closure of a rational polyhedron can be constructed in polynomial time in fixed dimension. This yields a polynomial algorithm that constructs a cutting plane proof of $\mathbf{0}^T x \leq -1$ for rational polyhedra P with empty integer hull. Based on these results, we then develop a polynomial algorithm in varying dimension for computing Gomory-Chvátal cutting planes of pointed simplicial cones. These cutting planes are not only among those of maximal possible violation in a natural sense, but also belong to the polynomial description of P' in fixed dimension.

Each of the chapters 4–6 begins with a more detailed motivation and with a summary of the contributions that are presented there.

1.3 Sources

The material in chapter 4 is from the papers (Bockmayr & Eisenbrand 1997, Bockmayr, Eisenbrand, Hartmann & Schulz 1999, Eisenbrand & Schulz 1999). Chapter 5 is built on the paper (Eisenbrand 1999), and the results in chapter 6 are from the paper (Bockmayr & Eisenbrand 1999).

2

Preliminaries

We assume that the reader is familiar with basic set theory, linear algebra, and linear programming. Excellent references are the books of Lang (1971) and Schrijver (1986).

2.1 Basics and notation

If a set U is contained in a set V, we write $U \subseteq V$. If U is strictly contained in V, we write $U \subset V$. The symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ denote the set of real, rational, integer and natural numbers respectively.

If α is a real number, then $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α and $\lceil \alpha \rceil$ denotes the smallest integer larger than or equal to α. We define

$$\lfloor \alpha \rceil = \begin{cases} \lfloor \alpha \rfloor & \text{if } x \geq 0, \\ \lceil \alpha \rceil & \text{if } x < 0. \end{cases}$$

The *size* of an integer z is the number

$$\operatorname{size}(z) = \begin{cases} 1 & \text{if } z = 0 \\ 1 + \lfloor \log_2(|z|) \rfloor & \text{if } z \neq 0 \end{cases}$$

The size of a rational $r = p/q \in \mathbb{Q}$ is defined as $\operatorname{size}(p) + \operatorname{size}(q)$, where p and q are relatively prime integers.

Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ be functions from the natural numbers to the reals. The function f is in $\mathrm{O}(g)$ if there exists constants c and N such that $f(n) \leq c\, g(n)$ for all $n \in \mathbb{N}$ with $n \geq N$. We write $f = \mathrm{O}(g)$.

2.2 Basic number theory

We recall some basic number theory see e.g. (Niven, Zuckerman & Montgomery 1991). An integer a divides an integer b, $a \mid b$, if there exists some integer c with $ac = b$. A common divisor of integers $a_1, \ldots, a_n$ is an integer d dividing all a_i for $i \in \{1, \ldots, n\}$. The *greatest common divisor* of n integers $a_1, \ldots, a_n$, not all equal to 0, is the largest common divisor of $a_1, \ldots, a_n$. It is denoted by $\gcd(a_1, \ldots, a_n)$ and can be computed with the euclidean algorithm see e.g. (Knuth 1969). If $\gcd(a_1, \ldots, a_n) = 1$, then $a_1, \ldots, a_n$ are called *relatively prime*. $\mathbb{Z}_d$ denotes the *ring of residues* modulo d, i.e., the set $\{0, \ldots, d-1\}$ with addition and multiplication modulo d. We will often identify an element of $\mathbb{Z}_d$ with the natural number in $\{0, \ldots, d-1\}$ to which it corresponds. $\mathbb{Z}_d$ is a commutative ring but not a field if d is not a prime. However $\mathbb{Z}_d$ is a *principal ideal ring*, i.e., each ideal is of the form $\langle g \rangle = \{gx \mid x \in \mathbb{Z}_d\} \trianglelefteq \mathbb{Z}_d$. This follows since $\mathbb{Z}$ is a principal ideal domain. The ideal $\langle g \rangle \trianglelefteq \mathbb{Z}_d$ is equal to the ideal $\langle \gcd(d, g) \rangle \trianglelefteq \mathbb{Z}_d$. Therefore we can assume that g divides d, $g \mid d$. Thus each ideal of $\mathbb{Z}_d$ has a unique generator dividing d, call it the *standard generator*. The standard generator g of an ideal $\langle a_1, \ldots, a_k \rangle \trianglelefteq \mathbb{Z}_d$ is easily computed with the euclidian algorithm.

2.3 Linear algebra

If R is a commutative ring then R^n denotes the R-module of n-tupels of elements of R. In our applications R stands for $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}_d$. An element of R^n is interpreted as a column vector. The vector of all zeroes (ones) is denoted by $\mathbf{0}$ ($\mathbf{1}$) and the i-th unit vector (the vector of zeroes everywhere except in the i-th component, which is 1) is denoted by e_i, for $i = \{1, \ldots, n\}$. If U and V are nonempty subsets of R^n, then $U + V = \{u + v \mid u \in U, v \in V\}$. We write $U + v$ instead of $U + \{v\}$ for a singleton $v \in \mathbb{R}^n$.

The l_∞-*norm* $\|c\|_\infty$ of the vector $c \in \mathbb{R}^n$ is the largest absolute value of its entries: $\|c\|_\infty = \max\{|c_i| \mid i = 1, \ldots, n\}$. If $A \in \mathbb{R}^{m \times n}$, then $\|A\|_\infty$ denotes the row-sum norm, i.e., the number $\max\{\sum_{j=1}^n |a_{i,j}| \mid i = 1, \ldots, m\}$. The l_1-*norm* $\|c\|_1$ of c is the sum $\|c\|_1 = \sum_{i=1}^n |c_i|$. The *euclidean norm* $\|c\|_2$ of c is the sum $\|c\|_2 = \sqrt{\sum_{i=1}^n c_i^2}$. The euclidean norm of c is also denoted by $\|c\|$. For $w \in \mathbb{R}^n$, let $\lfloor w \rfloor$, $\lceil w \rceil$, $\lfloor w \rceil \in \mathbb{Z}^n$ be the vectors obtained by component-wise application of $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ and $\lfloor \cdot \rceil$.

If a matrix $A \in R^{m \times n}$ is given, then $A^{(j)}$, for $j \in \{1, \ldots, n\}$, denotes the j-th column of A and $A_{(i)}$ for $i \in \{1, \ldots, m\}$ denotes the i-th row of A.

If $A \in \mathbb{R}^{n \times n}$ then the inequality

$$|\det(A)| \leq \|A^{(1)}\| \cdots \|A^{(n)}\| \tag{2.1}$$

is known as the Hadamard inequality. The *size of a matrix* $A \in \mathbb{Q}^{m \times n}$, size$(A)$, is the number of bits needed to encode A, i.e., $\mathrm{size}(A) = mn + \sum_{i,j} \mathrm{size}(a_{i,j})$, see (Schrijver 1986, p. 29). The Hadamard inequality, together with Cramer's rule implies that $\mathrm{size}(A^{-1})$ is polynomially bounded by size(A) for a nonsingular matrix $A \in \mathbb{Q}^{n \times n}$.

Let S be a subset of $\mathbb{R}^n$,

- the *linear hull* of S, $\mathrm{lin}(S)$ is the subspace of $\mathbb{R}^n$ generated by S.
- the *affine hull* of S is the set $\mathrm{aff}(S) = \mathrm{lin}(S - s_0) + s_0$ for an arbitrary element $s_0 \in S$.
- the *convex hull* of S is the set

$$\mathrm{conv}(S) = \{\sum_{i=1}^{t} \lambda_i s_i \mid t \geq 1, \sum_{i=1}^{t} \lambda_i = 1, \\ \lambda_1, \ldots, \lambda_n \geq 0, s_1, \ldots, s_n \in S\}.$$

- the *conical hull* of S is the set

$$\mathrm{cone}(S) = \{\sum_{i=1}^{t} \lambda_i s_i \mid t \geq 1, \\ \lambda_1, \ldots, \lambda_n \geq 0, s_1, \ldots, s_n \in S\}.$$

The *(affine)-dimension* of a set of vectors $S \subseteq \mathbb{R}^n$ is the dimension of the subspace $\mathrm{aff}(S) - s_0$ of $\mathbb{R}^n$ for some $s_0 \in S$.

The following proposition is known as Carathéodory's theorem.

Theorem 2.1. *If $X \subseteq \mathbb{R}^n$ and $x \in \mathrm{cone}(X)$ then $x \in \mathrm{cone}(\{x_1, \ldots, x_d\})$ for some d linearly independent vectors $x_1, \ldots, x_d \in X$.*

If $X \subseteq \mathbb{R}^n$ and $x \in \mathrm{conv}(X)$, then $x \in \mathrm{conv}(\{x_0, \ldots, x_d\})$ for some $d+1$ affinely independent vectors $x_0, \ldots, x_d \in X$.

Let $S \subseteq \mathbb{R}^n$, $n > 1$ and let $i \in \{1, \ldots, n\}$. The *projection* $\pi_i(S) \subseteq \mathbb{R}^{n-1}$ is the set

$$\pi_i(S) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^T \mid \exists y \in \mathbb{R},\ (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)^T \in S\}. \tag{2.2}$$

2.4 Polyhedra and linear programming

In this section we give definitions and fundamental facts about polyhedra and linear programming. Excellent references for this topic are the books by Schrijver (1986), Nemhauser & Wolsey (1988) and Ziegler (1998).

A *polyhedron* P is a set of vectors of the form $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, for some matrix $A \in \mathbb{R}^{m \times n}$ and some vector $b \in \mathbb{R}^m$. We write $P(A, b)$. The polyhedron is *rational* if both A and b can be chosen to be rational. If P is bounded, then P is called a *polytope*. If P is given as $P(A, b)$, then the *size* of P is defined as $\mathrm{size}(P) = \mathrm{size}(A) + \mathrm{size}(b)$. Notice that the size of a polyhedron depends on its inequality representation.

An inequality $a^T x \leq \beta$ from $Ax \leq b$ is called an *implicit equality* if $a^T x = \beta$ for all $x \in \mathbb{R}^n$ satisfying $Ax \leq b$. The system $A^= x \leq b^=$ denotes the subsystem of implicit equalities in $Ax \leq b$ and $A^+ x \leq b^+$ denotes the subsystem of all other inequalities in $Ax \leq b$. If $P(A,b) \subseteq \mathbb{R}^n$, then $\dim(P(A,b)) = n - \mathrm{rank}(A^=)$.

Polyhedra can be described by a set of inequalities or equivalently as the Minkowski sum of a polytope with a cone (see Figure 2.1).

Theorem 2.2 (Decomposition theorem for polyhedra). *A set $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if $P = \mathrm{conv}(Q) + \mathrm{cone}(C)$ for some finite subsets Q, $C \subseteq \mathbb{R}^n$.*

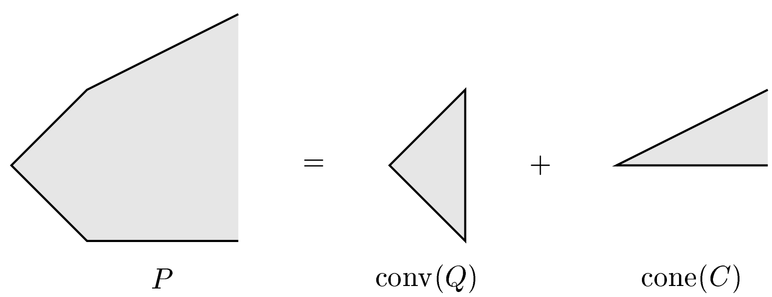

Figure 2.1: A polyhedron and its decomposition into conv(Q) and cone(C)

We say a polyhedron $P \subseteq \mathbb{R}^n$ is *full-dimensional* if $\dim(P) = n$. A *rational half space* is a set of the form $H = \{x \in \mathbb{R}^n \mid c^T x \leq \delta\}$, for some non-zero vector $c \in \mathbb{Q}^n$ and some $\delta \in \mathbb{Q}$. The half space H is then denoted by $(c^T x \leq \delta)$. The corresponding *hyperplane*, denoted by $(c^T x = \delta)$, is the set $\{x \in \mathbb{R}^n \mid c^T x = \delta\}$. A rational half space always has a representation in which the components of c are relatively prime integers. That is, we can chose $c \in \mathbb{Z}^n$ with $\gcd(c) = 1$.

An inequality $c^T x \leq \delta$ is called *valid* for a polyhedron P, if $(c^T x \leq \delta) \supseteq P$. A *face* of P is a set of the form $F = (c^T x = \delta) \cap P$, where $c^T x \leq \delta$ is valid for P. The inequality $c^T x \leq \delta$ is a *face-defining* inequality for F. Clearly F is a polyhedron. If $P \supset F \supset \emptyset$, then F is called *proper*. A maximal (inclusion wise) proper face of P is called a *facet* of P. If the face-defining inequality $c^T x \leq \delta$ defines a facet of P, then $c^T x \leq \delta$ is a *facet-defining* inequality. A proper face of P of dimension 0 is called a *vertex* of P. A vertex v of $P(A,b)$ is uniquely determined by a subsystem $A^v x \leq b^v$ of $Ax \leq b$, where A is nonsingular and $v = (A^v)^{-1} b$. A polytope P can be described as the convex hull of its vertices. A d-simplex is a polytope, which is the convex hull of $d + 1$ affinely independent points.

Proposition 2.3. *A full-dimensional polyhedron P has a unique (up to scalar multiplication) minimal representation by a finite set of linear inequalities. Those are the facet-defining inequalities.*

Proposition 2.4. *If P is given by the inequalities $Ax \leq b$, and if F is a face of P, then F is of the form $F = \{x \in P \mid A'x = b'\}$, for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. The *facet complexity* of P is the smallest number φ such that $\varphi \geq n$ and there exists a system $Ax \leq b$ of rational linear inequalities defining P such, that each inequality in $Ax \leq b$ has size at most φ. The *vertex complexity* of P is the smallest number ν, such that there exist rational vectors $q_1, \ldots, q_k, c_1, \ldots, c_t$, each of size at most ν, with

$$P = \operatorname{conv}(\{q_1, \ldots, q_k\}) + \operatorname{cone}(\{c_1, \ldots, c_t\}).$$

Theorem 2.5. *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron of facet complexity φ and vertex complexity ν. Then*

$$\nu \leq 4n^2\varphi \text{ and } \varphi \leq 4n^2\nu.$$

Linear programming concerns the maximization of a linear function $c^T x$, where x ranges over the elements in a polyhedron. The *linear programming problem* is:

> Given a rational matrix A and rational vectors b and c, determine $\max\{c^T x \mid x \in P(A, b)\}$.

Khachiyan's method (Khachiyan 1979), an extension of the ellipsoid method to linear programming, results in a polynomial algorithm for linear programming.

Proofs to the following facts can be found in the book of Schrijver (1986).

Theorem 2.6 (Farkas' Lemma). *The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is empty if and only if there exists a $\lambda \in \mathbb{R}^m_{\geq 0}$ with*

$$\lambda^T (A \mid b) = (0, \ldots, 0, -1).$$

Theorem 2.7 (Linear programming duality). *Let A be a matrix and b and c be vectors. Then*

$$\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid y \geq 0,\, y^T A = c^T\}$$

provided that both sets are not empty.

Proposition 2.8 (Complementary slackness). *Let A be a matrix and b and c be vectors. Suppose that the sets $\{x \mid Ax \leq b\}$ and $\{y \mid y \geq 0,\, y^T A = c^T\}$ are nonempty. Let $\hat{x}$ and $\hat{y}$ be feasible solutions to*

$$\max\{c^T x \mid Ax \leq b\} \text{ and } \min\{b^T y \mid y \geq 0,\, y^T A = c^T\} \tag{2.3}$$

respectively. Then the following are equivalent:

i. *$\hat{x}$ and $\hat{y}$ are optimal solutions of* (2.3)*;*

ii. $c^T\hat{x} = \hat{y}^T b$*;*

iii. *if a component of $\hat{y}$ is positive, then the corresponding inequality in $Ax \leq b$ is tight at $\hat{x}$, i.e., $\hat{y}^T(b - A\hat{x}) = 0$.*

Carathéodory's theorem and complementary slackness yield the following corollary.

Corollary 2.9. *Let A be a matrix and b and c be vectors. If the optimum of the LP-problems*

$$\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid y \geq 0,\, y^T A = c^T\}$$

is finite, then the optimum is attained at a vector $\hat{y}$ whose positive components correspond to linear independent rows of A.

A consequence of the discussed results is that for a given polyhedron $P = P(A, b)$, all valid inequalities $c^T x \leq \delta$ can be derived as a nonnegative linear combination and right-hand-side weakening from $Ax \leq b$:

$$c = \lambda^T A \text{ and } \delta \geq \lambda^T b \text{ for some } \lambda \geq 0. \tag{2.4}$$

2.5 The equivalence of separation and optimization

It is not necessary to have an explicit representation of a polyhedron P in terms of linear inequalities in order to optimize a linear function over P. It is enough to be able to solve the *separation problem*, which is: Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational vector $\hat{x} \in \mathbb{Q}^n$, decide whether $\hat{x}$ is in P and if not, compute a rational *separating inequality* $c^T x \leq \delta$ which is valid for P but not valid for $\hat{x}$.

The equivalence of separation and optimization, a result of Grötschel, Lovász & Schrijver (1988), decouples optimization from an explicit representation of a polyhedron P by linear inequalities.

More formally: Let for each $i \in \mathbb{N}$, $P_i \subseteq \mathbb{R}^{n_i}$ be a rational polyhedron such that, given $i \in \mathbb{N}$, one can compute the number n_i and an upper bound of the facet complexity φ_i of $P_i \subseteq \mathbb{R}^{n_i}$ in polynomial time (polynomial in size i). Then, the *separation problem* for the *class of polyhedra* $\mathscr{F} = (P_i \mid i \in \mathbb{N})$ is:

> Given $i \in \mathbb{N}$ and $\hat{x} \in \mathbb{Q}^{n_i}$, decide whether $\hat{x} \in P_i$ and if $\hat{x} \notin P_i$ compute a hyperplane $c^T x \leq \delta$ that separates $\hat{x}$ from P_i.

The *optimization problem* for the *class of polyhedra* $\mathscr{F} = (P_i \mid i \in \mathbb{N})$ is:

Given $i \in \mathbb{N}$ and $c \in \mathbb{Z}^{n_i}$, decide whether P_i is empty, $\max\{c^T x \mid x \in P_i\}$ is unbounded or compute an optimal solution $\hat{x} \in P_i$ of $\max\{c^T x \mid x \in P_i\}$.

Theorem 2.10 (Grötschel, Lovász & Schrijver (1988)). *For any class of polyhedra $\mathscr{F} = (P_i \mid i \in \mathbb{N})$, the separation problem is polynomially solvable if and only if the optimization problem is polynomially solvable.* [1]

2.6 Integer programming

The *integer linear programming problem* is:

Given a rational matrix A and rational vectors b and c, determine

$$\max\{c^T x \mid x \in P(A, b),\ x \text{ integral }\}.$$

Integer linear programming is NP-complete.

The polyhedron $P(A, b)$ from above is called the *linear programming relaxation*. The reason for the rationality assumption is that if P is a rational polyhedron, then the *integer hull* $P_I = \mathrm{conv}(P \cap \mathbb{Z}^n)$ of P is a rational polyhedron again.

Theorem 2.11. *If P is a rational polyhedron, then $P_I = \mathrm{conv}(\{x \mid x \in P \cap \mathbb{Z}^n\})$ is a rational polyhedron.*

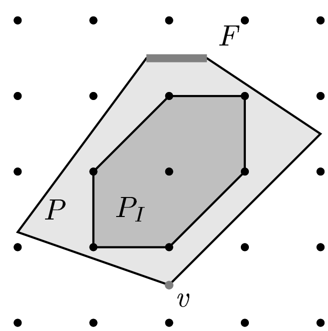

Figure 2.2: This picture illustrates a polyhedron P, one of its vertices v, one of its facets F and its integer hull P_I.

The integer linear programming problem can be reduced to the linear programming problem

$$\max\{c^T x \mid x \in P(A, b)_I\}.$$

[1] polynomial in size(i), size($\hat{x}$) and size(c)

However, an inequality description of P_I can be exponential. The integer hull of a non rational polyhedron is in general not a polyhedron.

For the decomposition of P_I one has the following estimates.

Proposition 2.12. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, then*

$$P_I = \operatorname{conv}(\{x_1, \dots , x_t\}) + \operatorname{cone}(\{y_1, \dots , y_s\}),$$

where $x_1, \dots , x_s, y_1, \dots , y_t$ are integral vectors of infinity norm at most $(n + 1)\Delta$, where Δ is the maximal absolute value of the subdeterminants of the matrix $(A \mid b)$.

Theorem 2.13. *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron of facet complexity φ. Then P_I has facet complexity at most $24n^5\varphi$.*

The polyhedron P is called *integral* if P is equal to its integer hull P_I. If P and Q are polyhedra with $Q \supseteq P$, then Q is called *weakening* of P, if $Q_I = P_I$.

2.7 Integer linear algebra

A *(rational) lattice* $\mathscr{L} = \mathscr{L}(A)$ is a subset of $\mathbb{R}^m$ of the form $\mathscr{L} = \{Ax \mid x \in \mathbb{Z}^n\}$, where $A \in \mathbb{Q}^{m\times n}$ is a rational matrix of full row rank. If A is in addition of full culumn rank, then A is called *basis* of $\mathscr{L}$. We refer to the books of Cassels (1997) and Lovász (1986) for basics about lattices.

A matrix $U \in \mathbb{Z}^{n\times n}$ is called *unimodular* if it is invertible and $U^{-1} \in \mathbb{Z}^{n\times n}$. One has the following fact.

Proposition 2.14. *A matrix $U \in \mathbb{Z}^{n\times n}$ is unimodular if and only if $\det(U) = \pm 1$.*

If $U \in \mathbb{Z}^{n\times n}$ is unimodular, then $\mathscr{L}(A) = \mathscr{L}(AU)$. The *Hermite normal form, HNF* of an integral matrix $A \in \mathbb{Z}^{m\times n}$ with full row rank is a nonnegative, nonsingular lower triangular matrix H, where each row has a unique maximal entry, located at the diagonal $h_{i,i}$ with $\mathscr{L}(A) = \mathscr{L}(H)$. The Hermite normal form exists for each integral matrix of full row rank. Conceptually, it can be traced back to the study of quadratic forms by Gauß (1801). See (Kannan & Bachem 1979), (Domich, Kannan & Trotter 1987), (Hafner & McCurley 1991) and (Storjohann & Labahn 1996) for polynomial algorithms concerning the computation of the Hermite normal form. It follows from this that every lattice has a basis.

Let $A \in \mathbb{Q}^{n\times n}$ be a basis of $\mathscr{L}$ and let B be another basis of $\mathscr{L}$. Then $B = AV_1$ and $A = BV_2$ with some integral matrices V_1 and V_2 in $\mathbb{Z}^{n\times n}$. By substitution one obtains $A = AV_1V_2$ and thus that $V_1V_2 = I$. This implies that V_1 and V_2 are unimodular.

Therefore the absolute value $|\det(A)|$ of the determinants of bases of $\mathscr{L}$ is an invariant of $\mathscr{L}$. This number is called the *lattice determinant* of $\mathscr{L}$ and is denoted by $\det(\mathscr{L})$.

The *dual lattice* $\mathscr{L}^*$ of a lattice $\mathscr{L} \subseteq \mathbb{R}^n$ is the set $\mathscr{L}^* = \{x \in \mathbb{R}^n \mid \forall y \in \mathscr{L}(A) : x^T y \in \mathbb{Z}\} \subseteq \mathbb{R}^n$.

Lemma 2.15. *Let $A \in \mathbb{Q}^{n \times n}$ have rank n. The dual lattice $\mathscr{L}^*(A)$ is the lattice $\mathscr{L}({A^{-1}}^T)$.*

Proof. Let a be the i-th row of A^{-1}. Then $a^T A = e_i^T$. Thus $a^T A x$ is an integer for each $x \in \mathbb{Z}^n$. Thus $\mathscr{L}({A^{-1}}^T) \subseteq \mathscr{L}^*(A)$.

Suppose that v is not in $\mathscr{L}({A^{-1}}^T)$. Then v^T can be written as $v^T = u^T A^{-1}$, where u is not integral. Then $v^T A = u^T A^{-1} A = u^T$ is not an integral vector. Thus $v^T \notin \mathscr{L}^*(A)$. □

Corollary 2.16. *If v is an element of the dual lattice of $\mathscr{L}(A)$, where A is integral, then v can be written as $v = u / \det(\mathscr{L}(A))$ with an integral vector u.*

2.8 Complexity

In Chapter 5 we prove computational complexity results for problems related to cutting planes. For this it is necessary to review some definitions and notations. The reader is refered to (Garey & Johnson 1979) and (Papadimitriou 1994) for further reference.

An *alphabet* is a finite nonempty set Σ, and a *language* is a subset of the *Kleene closure* Σ^* of Σ. The class NP is a class of languages for which membership has a short proof. In other words: a language $L \subseteq \Sigma^*$ is in NP, if there exists a language $L_1 \subseteq \Sigma^* \times \Sigma^*$ that is decidable in deterministic polynomial time, and a polynomial $p(X)$ with the property that for each $w \in \Sigma^*$ one has:

$$w \in L \iff \exists y \in \Sigma^*, |y| < p(|w|), (w, y) \in L_1.$$

If $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ are languages, then a *polynomial reduction* from L_1 to L_2 is a function $\tau : \Sigma_1^* \to \Sigma_2^*$, computable in polynomial time, such that for each $w \in \Sigma_1^*$ one has:

$$w \in L_1 \iff \tau(w) \in L_2.$$

In this case one says that L_1 can be *reduced* to L_2. A language $L \in \text{NP}$ is NP-complete, if each language in NP can be polynomially reduced to it.

3

The cutting plane method

3.1 Cutting planes

A cutting plane of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is an inequality that is valid for the integer hull P_I of P but not necessarily valid for P. In this chapter we assume that polytopes and polyhedra are always rational unless explicitly stated otherwise.

The simplest polyhedra are the rational half spaces. Their integer hull can be written down with little effort. If one has a rational half space $(c^T x \leq \delta)$ then it can be represented with $c \in \mathbb{Z}^n$ where the greatest common divisor of the components $\gcd(c_1, \ldots, c_n)$ is 1. The integer hull of this half space is the half space $(c^T x \leq \lfloor \delta \rfloor)$. This can for example be seen as follows: The subspace of $\mathbb{R}^n$ which is defined by the system $c^T x = 0$ is integral. The greatest common divisor $\gcd(c_1, \ldots, c_n) = 1$ has a representation $c^T y = 1$ with an integral vector $y \in \mathbb{Z}^n$. Each hyperplane $(c^T x = k)$, with $k \in \mathbb{Z}$ is the translation of $(c^T x = 0)$ with the vector $k\, y$ and is thus integral. Any point in $(c^T x \leq \lfloor \delta \rfloor)$ is in the convex hull of two consecutive hyperplanes $(c^T x = d)$ and $(c^T x = (d-1))$ for some $d \leq \delta$, $d \in \mathbb{Z}$ and thus is in the convex hull of integral vectors in $(c^T x \leq \lfloor \delta \rfloor)$. Therefore $(c^T x \leq \lfloor \delta \rfloor)$ is integral. Let us from now on assume that a half space $(c^T x \leq \delta)$ is always rational and that c in the representation above is always integral.

The case of two half spaces $(c_1^T x \leq \delta_1)$ and $(c_2^T x \leq \delta_2)$ is already more complicated. Assume that c_1 and c_2 are integral vectors with greatest common divisor $\gcd(c_{i,1}, \ldots, c_{i,n}) = 1$, $i = 1, 2$ and that $\delta_i \in \mathbb{Z}$. The half spaces represent the polyhedron $P \subseteq \mathbb{R}^n$ defined by the system

$$\begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \end{pmatrix} x \leq \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \tag{3.1}$$

There is a unimodular mapping U that transforms the matrix in (3.1) into a matrix of the form $\left(\begin{smallmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & a_3 & 0 & \cdots & 0 \end{smallmatrix}\right)$. Notice that the variables $x_3, \ldots, x_n$ are unconstrained and that the constraints of the integer hull of $\left(\begin{smallmatrix} 1 & 0 \\ a_2 & a_3 \end{smallmatrix}\right)(x_1, x_2)^T \leq (\delta_1, \delta_2)^T$ yield the integer hull of (3.1). Harvey (1999) presented an elementary algorithm which computes the integer hull of a rational polyhedron in $\mathbb{R}^2$ in polynomial time. The algorithm relies on diophantine approximations of rational numbers and is considerably more complicated than the one constraint case.

There does not seem to exist an elementary method to construct the linear description of the integer hull formed by three or more half spaces in polynomial time. It is possible though with an application of Lenstra's method (Lenstra 1983) as proposed by Cook, Hartmann, Kannan & McDiarmid (1992).

Rather than computing the integer hull P_I of P, the objective pursued by the *cutting plane method* is a better approximation of P_I. Here the idea is to intersect P with the integer hull of half spaces containing P. These will still include P_I but not necessarily P.

In the following we will study the theoretical framework of Gomory's cutting plane method (Gomory 1958) as given by Chvátal (1973*a*) and Schrijver (1980).

If the half space $(c^T x \leq \delta)$, $c \in \mathbb{Z}^n$, $\gcd(c_1, \ldots, c_n) = 1$ contains the polyhedron P, i.e. if $c^T x \leq \delta$ is valid for P, then $c^T x \leq \lfloor \delta \rfloor$ is valid for the integer hull P_I of P. The inequality $c^T x \leq \lfloor \delta \rfloor$ is called a *cutting plane* or *Gomory-Chvátal cut* of P. The geometric interpretation behind this process is that $(c^T x \leq \delta)$ is "shifted inwards" until an integer point of the lattice is in the boundary of the half space.

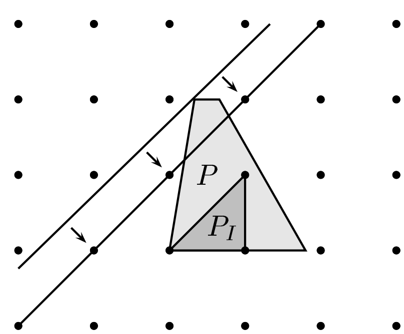

Figure 3.1: The half space $(-x_1 + x_2 \leq \delta)$ containing P is replaced by its integer hull $(-x_1 + x_2 \leq \lfloor \delta \rfloor)$. The darker region is the integer hull P_I of P.

The idea pioneered by Gomory (1958) is to apply these cutting planes to the integer programming problem. Cutting planes tighten the linear relaxation of an integer program and Gomory showed how to apply cutting planes successively until the resulting relaxation has an integral optimal solution.

3.2 The elementary closure

Cutting planes $c^T x \leq \lfloor \delta \rfloor$ of $P(A,b)$, $A \in \mathbb{R}^{m\times n}$ obey a simple *inference rule.* Clearly $\max\{c^T x \mid Ax \leq b\} \leq \delta$ and it follows from Corollary 2.9 that there exists a weight vector $\lambda \in \mathbb{Q}^m_{\geq 0}$ with at most n positive entries such that $\lambda^T A = c^T$ and $\lambda^T b \leq \delta$. Thus $c^T x \leq \lfloor \delta \rfloor$ follows from the following inequalities by weakening the right-hand-side if necessary:

$$\lambda^T A x \leq \lfloor \lambda^T b \rfloor,\ \lambda \in \mathbb{Q}^m_{\geq 0},\ \lambda^T A \in \mathbb{Z}^n. \tag{3.2}$$

Instead of applying cutting planes successively, one can apply all possible cutting planes at once. P intersected with all Gomory-Chvátal cutting planes

$$P' = \bigcap_{\substack{(c^T x \leq \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (c^T x \leq \lfloor \delta \rfloor) \tag{3.3}$$

is called the *elementary closure* of P.

The set of inequalities in (3.2), which describe P' is infinite. However, as observed by Schrijver (1980), a finite number of inequalities in (3.2) imply the rest.

Lemma 3.1. *Let P be the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. The elementary closure P' is the polyhedron defined by $Ax \leq b$ and the set of all inequalities $\lambda^T A x \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in [0,1)^m$ and $\lambda^T A \in \mathbb{Z}^n$.*

Proof. An inequality $\lambda^T A x \leq \lfloor \lambda^T b \rfloor$ with $\lambda \in \mathbb{Q}^m_{\geq 0}$ and $\lambda^T A \in \mathbb{Z}^n$ is implied by $Ax \leq b$ and $(\lambda - \lfloor \lambda \rfloor)^T A x \leq \lfloor (\lambda - \lfloor \lambda \rfloor)^T b \rfloor$, since

$$\lambda^T A x = (\lambda - \lfloor \lambda \rfloor)^T A x + \lfloor \lambda \rfloor^T A x \leq \lfloor (\lambda - \lfloor \lambda \rfloor)^T b \rfloor + \lfloor \lambda \rfloor^T b = \lfloor \lambda^T b \rfloor. \tag{3.4}$$

□

Corollary 3.2 (Schrijver (1980)). *If P is a rational polyhedron, then P' is a rational polyhedron.*

Proof. P can be described as $P(A,b)$ with integral A and b. There is only a finite number of vectors $\lambda^T A \in \mathbb{Z}^n$ with $\lambda \in [0,1)^m$. □

Remark 3.3. *This yields an exponential upper bound on the number of facets of the elementary closure of a polyhedron. The infinity norm $\|c\|_\infty$ of a possible candidate $c^T x \leq \lfloor \delta \rfloor$ is bounded by $\|A^T\|_\infty$, where the matrix norm $\|\cdot\|_\infty$ is the row sum norm. Therefore we have an upper bound of $\mathrm{O}(\|A^T\|_\infty^n)$ for the number of facets of the elementary closure of a polyhedron. In Chapter 6 we will prove a polynomial upper bound of the size of P' in fixed dimension.*

The following lemma is often useful. It states that if the i-th component of all elements of a polyhedron $P \subseteq \mathbb{R}^n$ is fixed to an integer, then the elementary closure P' of P is obtained by the elementary closure of the projection $\pi_i(P) \subseteq \mathbb{R}^{n-1}$. A proof is trivial.

Lemma 3.4. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron with $P \subseteq (x_i = z)$ for some $i \in \{1, \dots, n\}$ and some integer $z \in \mathbb{Z}$, then*

$$P' = \{(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n)^T \mid (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)^T \in \pi_i(P)'\}.$$

3.3 The Chvátal-Gomory procedure

The elementary closure operation can be iterated, so that successively tighter relaxations of the integer hull P_I of P are obtained. We define $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geq 0$. This iteration of the elementary closure operation is called the *Chvátal-Gomory procedure*. The *Chvátal rank* of a polyhedron P is the smallest $t \in \mathbb{N}_0$ such that $P^{(t)} = P_I$. In analogy, the *depth* of an inequality $c^T x \leq \delta$ which is valid for P_I is the smallest $t \in \mathbb{N}_0$ such that $(c^T x \leq \delta) \supseteq P^{(t)}$.

Chvátal (1973*a*) showed that every bounded polyhedron $P \subseteq \mathbb{R}^n$ has finite rank. Schrijver (1980) extended this result to rational polyhedra. The main ingredient to his result is the following observation, see also (Cook, Cunningham, Pulleyblank & Schrijver 1998, Lemma 6.33).

Lemma 3.5. *Let F be a face of a rational polyhedron P. If $c_F^T x \leq \lfloor \delta_F \rfloor$ is a cutting plane for F, then there exists a cutting plane $c_P^T x \leq \lfloor \delta_P \rfloor$ for P with*

$$F \cap (c_P^T x \leq \lfloor \delta_P \rfloor) = F \cap (c_F^T x \leq \lfloor \delta_F \rfloor).$$

Intuitively, this result means that that a cutting plane of a face F of a polyhedron P can be "rotated" so that it becomes a cutting plane of P and has the same effect on F.

Proof. Assume that $\delta_F = \max\{c_F^T x \mid x \in F\}$. Let F be defined by the half space $(c^T x \leq \delta) \supseteq P$, i.e., $F = P \cap (c^T x = \delta)$, where c and δ are integral and let $P = P(A, b)$. It follows from linear programming duality (Theorem 2.7) that there exists a nonnegative weight vector λ and some rational number μ with $c_F^T = \lambda^T A + \mu c^T$ and $\delta_F = \lambda^T b + \mu\delta$. Define $c_P^T = \lambda^T A + (\mu - \lfloor \mu \rfloor) c^T$ and observe that

$$c_P^T x \leq \lfloor \lambda^T b + (\mu - \lfloor \mu \rfloor)\delta \rfloor = \lfloor \delta_F \rfloor - \lfloor \mu \rfloor \delta$$

is a cutting plane for P. Notice further that

$$(c^T x = \delta) \cap (c_P^T x \leq \lfloor \delta_F \rfloor - \lfloor \mu \rfloor \delta) = (c^T x = \delta) \cap (c_F^T x \leq \lfloor \delta_F \rfloor).$$

Thus with $\delta_P = \lfloor \delta_F \rfloor - \lfloor \mu \rfloor \delta$ we see that

$$F \cap (c_P^T x \leq \lfloor \delta_P \rfloor) = F \cap (c_F^T x \leq \lfloor \delta_F \rfloor).$$

$\square$

This implies that a face F of P behaves under its closure F' as it behaves under the closure P' of P.

Corollary 3.6. *Let F be a face of a rational polyhedron P. Then*

$$F' = P' \cap F.$$

From this, one can derive that the Chvátal rank of rational polyhedra is finite.

Theorem 3.7 (Schrijver (1980)). *If P is a rational polyhedron, then there exists some $t \in \mathbb{N}$ with $P^{(t)} = P_I$.*

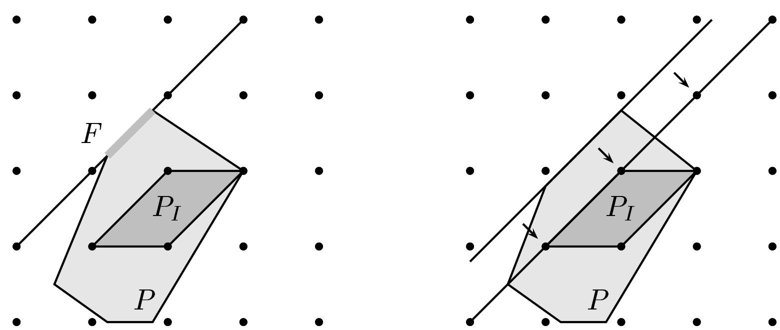

Figure 3.2: After a finite number of iterations F is empty. Then the half space defining F can be pushed further down. This is basically the argument that every inequality, valid for P_I eventually becomes valid for the outcome of the successive application of the elementary closure operation.

Proof. The argument proceeds by induction on the dimension of P.

One can assume P to be full-dimensional. Since otherwise, there exists a hyperplane $(c^T x = \delta)$ with integral c and $\gcd(c) = 1$ which contains P. If δ is not integral, one has immediately that $P' = \emptyset$. If δ is integral, we can apply a unimodular transformation, such that $(c^T x = \delta)$ becomes $(x_1 = \delta)$. Since the elementary closure operation and unimodular transformations commute (see Section 3.6) one has reduced to a case with one variable less (see Lemma 3.4).

If $\dim(P) = 0$, then clearly $P' = P_I$. Let $P_I = \emptyset$ and $\dim(P) > 0$. By Theorem 2.2 P is of the form $P = Q + \text{cone}(C)$ with some polytope Q and some finite set $C \subseteq \mathbb{Q}^n$.

Now cone(C) cannot be full dimensional. Otherwise there would be an integral point in P. Thus there exists a $c \in \mathbb{Z}^n$ which is perpendicular to the cone (see (Lang 1971)), i.e., for each $\mu \in \text{cone}(C)$ one has $c^T\mu = 0$. Since Q is bounded, there exist some $\delta_1, \delta_2 \in \mathbb{Z}$ with $\max\{c^Tx \mid x \in P\} \leq \delta_1$ and $\min\{c^Tx \mid x \in P\} \geq \delta_2$. Thus the minimal t such that $c^Tx \leq (\delta_2 - 1)$ is valid for $P^{(t)}$ is the Chvátal rank of P. Since the face F of P defined by $F = P \cap (c^Tx = \delta_1)$ is of lower dimension than P, one has that $F^{(t)} = \emptyset$ for some t. Thus, with Corollary 3.6, $(c^Tx \leq \delta_1 - \epsilon)$ is valid for $P^{(t)}$ for some $\epsilon > 0$ and thus $(c^Tx \leq \delta_1 - 1)$ is valid for $P^{(t+1)}$. By induction on $\delta_1 - \delta_2$ one can see that $c^Tx \leq (\delta_2 - 1)$ eventually becomes valid.

If $P_I \neq \emptyset$, let $c^Tx \leq \delta$ be valid for P_I. Clearly for each rational element μ of cone(C) one has $c^T\mu \leq 0$. Therefore $\max\{c^Tx \mid x \in P\}$ is bounded. An argument as given above shows that $c^Tx \leq \delta$ eventually becomes valid.

This is the termination argument of the Chvátal-Gomory procedure. □

Already in dimension 2, there exist rational polyhedra of arbitrarily large Chvátal rank (Chvátal 1973*a*). To see this, consider the polytopes

$$P_k = \text{conv}\{(0,0), (0,1)(k, \tfrac{1}{2})\},\ k \in \mathbb{N}. \tag{3.5}$$

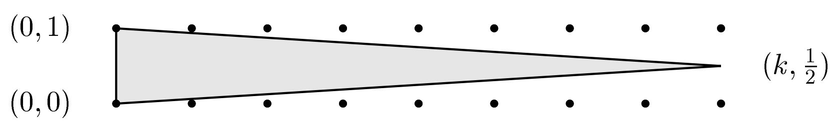

Figure 3.3:

One can show that $P_{(k-1)} \subseteq P'_k$. For this, let $c^Tx \leq \delta$ be valid for P_k with $\delta = \max\{c^Tx \mid x \in P_k\}$. If $c_1 \leq 0$, then the point $(0,0)$ or $(0,1)$ maximizes c^Tx, thus $(c^Tx = \delta)$ contains integral points. If $c_1 > 0$, then $c^T(k, \frac{1}{2}) \geq c^T(k-1, \frac{1}{2}) + 1$. Therefore the point $(k-1, \frac{1}{2})$ is in the half space $(c^Tx \leq \delta - 1) \subseteq (c^Tx \leq \lfloor\delta\rfloor)$. Unfortunately, this lower bound on the Chvátal rank of P_k is exponential in the encoding length of P_k which is $\mathrm{O}(\log(k))$.

Remark 3.8. *In Chapter 4 we will analyze the convergence of the method in the 0/1 cube in a more sophisticated way, yielding a polynomial upper bound on the Chvátal rank of polytopes in the 0/1 cube.*

3.4 Cutting plane proofs

An important property of polyhedra is the following rule to derive valid inequalities which is a consequence of linear programming duality (Theorem 2.7). If P is defined by the inequalities $Ax \leq b$, then the inequality $c^T x \leq \delta$ is valid for P if and only if there exists some $\lambda \in \mathbb{R}^m_{\geq 0}$ with

$$c = \lambda^T A \text{ and } \delta \geq \lambda^T b. \tag{3.6}$$

This implies that linear programming (in its decision version) belongs to the class NP $\cap$ co $-$ NP, because $\max\{c^T x \mid Ax \leq b\} \leq \delta$ if and only if $c^T x \leq \delta$ is valid for $P(A, b)$. A "No" certificate would be some vertex of P which violates $c^T x \leq \delta$. Interestingly, quite an amount of time went by until linear programming was found to be in P by Khachiyan (1979).

In integer programming there is an analogy to this rule. A sequence of inequalities

$$c_1^T x \leq \delta_1,\, c_2^T x \leq \delta_2, \ldots, c_m^T x \leq \delta_m \tag{3.7}$$

is called a *cutting-plane proof* of $c^T x \leq \delta$ from a given system of linear inequalities $Ax \leq b$, if $c_1, \ldots, c_m$ are integral, $c_m = c$, $\delta_m = \delta$, and if $c_i^T x \leq \delta_i'$ is a nonnegative linear combination of $Ax \leq b, c_1^T x \leq \delta_1, \ldots, c_{i-1}^T x \leq \delta_{i-1}$ for some δ_i' with $\lfloor \delta_i' \rfloor \leq \delta_i$. In other words, if $c_i^T x \leq \delta_i$ can be obtained from $Ax \leq b$ and the previous inequalities as a Gomory-Chvátal cut, by weakening the right-hand-side if necessary. Obviously, if there is a cutting-plane proof of $c^T x \leq \delta$ from $Ax \leq b$ then every integer solution to $Ax \leq b$ must satisfy $c^T x \leq \delta$. The number m here, is the *length* of the cutting plane proof.

The following proposition shows a relation between the length of cutting plane proofs and the depth of inequalities (see also (Chvátal, Cook & Hartmann 1989)). It comes in two flavors, one for the case $P_I \neq \emptyset$ and one for $P_I = \emptyset$. The latter can then be viewed as an analogy to Farkas' lemma.

Proposition 3.9. *Let $P(A, b) \subseteq \mathbb{R}^n$, $n \geq 2$ be a rational polyhedron.*

i. If $P_I \neq \emptyset$ and $c^T x \leq \delta$ with integral c has depth t, then $c^T x \leq \delta$ has a cutting plane proof of length at most $(n^{t+1} - 1)/(n - 1)$.

ii. If $P_I = \emptyset$ and $\operatorname{rank}(P) = t$, then there exists a cutting plane proof of $\mathbf{0}^T x \leq -1$ of length at most $(n + 1)(n^t - 1)/(n - 1) + 1$.

Proof. Let us first prove the following. If $P^{(t)} \neq \emptyset$ and $c^T x \leq \delta$ is valid for $P^{(t)}$ for some $c \in \mathbb{Z}^n$, then $c^T x \leq \lfloor \delta \rfloor$ has a cutting plane proof of length at most $(n^{t+1} - 1)/(n - 1)$. If $t = 0$, then the claim follows from Corollary 2.9. If $t > 0$, then $c^T x \leq \delta$ can be derived from n inequalities $c_i^T x \leq \lfloor \delta_i \rfloor$, $c_i \in \mathbb{Z}^n$, $i = 1, \ldots, n$, where each $c_i^T x \leq \delta_i$ is valid for $P^{(t-1)}$. By induction, each of the inequalities $c_i^T x \leq \lfloor \delta_i \rfloor$ has a cutting plane proof of length $(n^t - 1)/(n - 1)$. We obtain a cutting plane proof of $c^T x \leq \lfloor \delta \rfloor$ by concatenating those for the inequalities $c_i^T x \leq \lfloor \delta_i \rfloor$ with $c^T x \leq \lfloor \delta \rfloor$. The length of this proof is at most $n\,(n^t - 1)/(n-1) + 1 = (n^{t+1} - 1)/(n-1)$. (i) follows directly from this.

Let $P_I = \emptyset$. If $\operatorname{rank}(P) = 0$, then (ii) is simply Farkas' lemma and Carathéodory's theorem. Therefore let $\operatorname{rank}(P) = t \geq 1$. There exist $n+1$ inequalities $c_i^T x \leq \delta_i$, $i = 1, \ldots, n+1$ which are valid for $P^{(t-1)}$, such that a nonnegative linear combination of $c_i^T x \leq \lfloor \delta_i \rfloor$, $i = 1, \ldots, n+1$ yields $\mathbf{0}^T x \leq -1$. The cutting plane proofs of $c_i^T x \leq \lfloor \delta_i \rfloor$, $i = 1, \ldots, n+1$ and the inequality $\mathbf{0}^T x \leq -1$ form a cutting plane proof of $\mathbf{0}^T x \leq -1$. Its length is at most $(n+1)(n^t - 1)/(n-1) + 1$. $\square$

Due to this relation the Chvátal rank has a precise complexity theoretic meaning in the context of the question co − NP = NP (see e.g. (Nemhauser & Wolsey 1988, p. 227) and (Schrijver 1986, p. 352)). Suppose $\mathscr{F} = (P_i \mid i \in \mathbb{N})$ is a class of polyhedra (see § 2.5) for which linear programming is solvable in polynomial time:

Given $i \in \mathbb{N}$ and $c \in \mathbb{Q}^{n_i}$, compute $\max\{c^T x \mid x \in P_i\}$, where $P_i \subseteq \mathbb{R}^{n_i}$.

Consider then the integer programming problem for this class of polyhedra:

Given $i \in \mathbb{N}$ and $c \in \mathbb{Q}^{n_i}$, compute $\max\{c^T x \mid x \in \mathbb{Z}^{n_i} \cap P_i\}$, where $P_i \subseteq \mathbb{R}^{n_i}$.

If there exists a constant K such that for all $P_i \in \mathscr{F}$, $\operatorname{rank}(P_i) < K$ holds, then the integer programming problem for the class $\mathscr{F}$ in its decision version cannot be NP-complete, unless NP = co − NP. The fractional matching polytopes Q_G (see Example 4.3) are such a class of polyhedra, whose Chvátal rank is at most one as it was observed by Edmonds (1965).

Cutting plane proofs have been studied in the context of the fascinating field of propositional proof systems. After Haken (1985) showed that resolution was an exponential proof system for the unsatisfiability of propositional formulas, Cook, Coullard & Turán (1987) observed that cutting planes, when applied to polytopes resulting from propositional formulas, are a stronger proof system than resolution. They observed that the pigeon hole principle, which cannot be proved by resolution with a polynomial proof, could be proved by cutting planes with a polynomial proof. Eventually Pudlák (1997) was able to derive an exponential lower bound on the length of cutting plane proofs for propositional unsatisfiability. The question of whether each proof system for propositional logic is exponential or not is equivalent to the question whether co − NP = NP. See (Urquhart 1995, Pudlák 1999) for a survey on propositional proof systems.

3.5 The classical Gomory cut

Gomory (1958) derived cutting planes out of a simplex tableau of the current linear relaxation of the corresponding integer program. The classical Gomory cut therefore is defined for polyhedra in *standard form*, i.e.,

$$P = \{x \in \mathbb{R}^m \mid Ax = b,\, x \geq 0\}, \tag{3.8}$$

where $A \in \mathbb{R}^{m \times n}$ has rank m. The cut is derived from such a representation as follows. Let $a_{i,1}x_1 + \cdots + a_{i,n}x_n = b_i$ be the i-th equality of $Ax = b$. Notice that any integral $\hat{x} \in \mathbb{Z}^n$ satisfies

$$\{a_{i,1}\}\hat{x}_1 + \cdots + \{a_{i,n}\}\hat{x}_n \equiv \{b_i\} \pmod 1, \tag{3.9}$$

where $a \equiv b \pmod 1$ means that $a - b$ is an integer and $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$.

Since P is in the positive orthant we see that the inequality

$$\{a_{i,1}\}x_1 + \cdots + \{a_{i,n}\}x_n \geq \{b_i\} \tag{3.10}$$

is valid for all integral vectors in P. This is the classical Gomory cut. Note that it is derived from a row of the description $Ax = b$, $x \geq 0$. It is easy to see that this cutting plane can be obtained as a Gomory-Chvátal cutting plane. For this, add to the equality $a_{i,1}x_1 + \cdots + a_{i,n}x_n = b_i$ inequalities $-\{a_{i,j}\}x_j \leq 0$ for $j = 1, \dots, n$ to obtain

$$\lfloor a_{i,1} \rfloor x_1 + \cdots + \lfloor a_{i,n} \rfloor x_n \leq b_i. \tag{3.11}$$

Then we can round down the right-hand-side to obtain

$$\lfloor a_{i,1} \rfloor x_1 + \cdots + \lfloor a_{i,n} \rfloor x_n \leq \lfloor b_i \rfloor. \tag{3.12}$$

The Gomory-Chvátal cutting plane in (3.12) and $a_{i,1}x_1 + \cdots + a_{i,n}x_n = b_i$ yield the classical Gomory cut (3.10). More precisely, the polyhedron P intersected with the halfspace defined by (3.11) is the same polyhedron, as P intersected with the halfspace defined by (3.10).

In this sense, on the other hand, each Gomory-Chvátal cutting plane for P can be obtained by a classical Gomory cut derived from a suitable standard form representation of P. For this let $c^T x \leq \lfloor \delta \rfloor$ be an undominated Gomory-Chvátal cutting plane for P, with integral c and $\delta = \max\{c^T x \mid x \in P\}$. Undominated means that this cutting plane cannot be obtained from other valid inequalities for P' by a nonnegative linear combination and right-hand-side weakening. It follows from Lemma 3.1 that $c^T = \lfloor \lambda^T A \rfloor$ and $\delta = \lambda^T b$ for some $\lambda \in [-1, 1]^m$. Here λ can also be negative, since $Ax = b$ has the inequality description $Ax \leq b$, $-Ax \leq -b$ and the representation of c as $c^T = \lfloor \lambda^T A \rfloor$ comes from the fact that the nonnegativity constraints $-x \leq 0$ can only have multiplicative weights in $[0, 1)$ while applying Lemma 3.1 in this case. We now describe a suitable standard form representation of P whose first-row classical Gomory cut yields $c^T x \leq \lfloor \delta \rfloor$. Assume without loss of generality that the first component of λ is nonzero. The inhomogeneous system $Ax = b$ represents then the same set of vectors as the system $Cx = d$, where the first row of $(C \mid d)$ is is the row vector $(\lambda^T A \mid \lambda^T b)$ and where the other $m-1$ rows are the last $m-1$ rows of $(A \mid b)$. Observe that the classical Gomory cut derived from this first row is equivalent to $c^T x \leq \lfloor \delta \rfloor$, as the previous discussion has shown.

Gomory (1958) considered integer linear programs of the form

$$\max\{c^T x \mid Ax = b,\, x \geq 0,\, x \in \mathbb{Z}^n\}. \tag{3.13}$$

He added cuts derived as in (3.10) to the problem, with an additional slack variable to obtain a standard form representation again

$$\{a_{i,1}\}x_1 + \cdots + \{a_{i,n}\}x_n - y = \{b_i\}. \tag{3.14}$$

Since (3.9) holds this slack variable can be required to be integral. Therefore it remains to solve the problem

$$\begin{aligned} \max\{c^T x \mid Ax = b,\, &\sum_{j=1}^{n} \{a_{i,j}\}x_j - y = \{b_i\}, \\ &x \geq 0,\, x \in \mathbb{Z}^n,\, y \geq 0,\, y \in \mathbb{Z}\}. \end{aligned} \tag{3.15}$$

Gomory showed how to iteratively add cutting planes until an integral optimal solution is obtained, which then translates back to an integral optimal solution to the original problem. Notice that instead of (3.15) we can equivalently write

$$\begin{aligned} \max\{c^T x \mid Ax = b,\, &\sum_{j=1}^{n} \lfloor a_{i,j} \rfloor x_j + y = \lfloor b_i \rfloor, \\ &x \geq 0,\, x \in \mathbb{Z}^n,\, y \geq 0,\, y \in \mathbb{Z}\}. \end{aligned} \tag{3.16}$$

The next lemma clarifies how a Gomory-Chvátal cut of a polyhedron resulting from another one by the addition of slack variables, can be translated into a Gomory-Chvátal cut of the original polyhedron having the same effect. A proof is trivial.

Lemma 3.10. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with integral A and b and let $\widetilde{P} = \{(x, y) \in \mathbb{R}^{m+n} \mid Ax + y = b,\, y \geq 0\}$. If $(c_1^T, c_2^T)(x, y) \leq \lfloor \delta \rfloor$ is a Gomory-Chvátal cut of $\widetilde{P}$, then $(c_1^T - c_2^T A)x \leq \lfloor \delta - c_2^T b \rfloor$ is a Gomory-Chvátal cut of P and*

$$P \cap ((c_1^T - c_2^T A)x \leq \lfloor \delta - c_2^T b \rfloor) = \pi_y(\widetilde{P} \cap ((c_1^T, c_2^T)(x, y) \leq \lfloor \delta \rfloor)),$$

where $\pi_y(x, y) = x$.

Lemma 3.10 and the observation from (3.16) imply now that if we start with a polyhedron $P(A,b)$ with integral A and b in the positive orthant, then all cutting planes derived in the course of Gomory's original algorithm translate to iterated Gomory-Chvátal cuts of $P(A,b)$.

Theorem 3.11 (Gomory (1958)). *Let the integral inequality system $Ax \leq b$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ define a polyhedron $P(A,b)$ in the positive orthant and let $c^T x \leq \delta$, $c \in \mathbb{Z}^n$, $\delta \in \mathbb{Q}$ be valid for P. There exists an algorithm that computes a cutting plane proof for of $c^T x \leq \delta$ from the system $Ax \leq b$ on input A, b, c and δ.*

If $c^T x \leq \delta$ is from an inequality description of P_I, then Gomory's result is an algorithm, which adds cutting planes until $c^T x \leq \delta$ becomes valid. This yields the termination of the Chvátal-Gomory procedure for polyhedra in the positive orthant as observed by Schrijver (Schrijver 1986, p. 359).

Corollary 3.12. *If P is a rational polyhedron in the positive orthant, then there exists some $t \in \mathbb{N}$ with $P^{(t)} = P_I$.*

However we will show that Gomory's algorithm implies the convergence of the Chvátal-Gomory procedure for general rational polyhedra together with the simple observations concerning unimodular transformations in the following Section.

3.6 Unimodular transformations

Unimodular transformations have already been mentioned and used in this chapter. In this section we formalize the simple observation that unimodular transformations and the Chvátal-Gomory operation commute. Unimodular transformations also play a crucial role to relate the Chvátal rank of arbitrary polytopes in the 0/1-cube to the Chvátal rank of monotone polytopes, appearing in Section 4.6.

A *unimodular transformation* is a mapping

$$\begin{array}{rccc} u: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & Ux + v, \end{array}$$

where $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix, i.e., $\det(U) = \pm 1$, and $v \in \mathbb{Z}^n$.

Note that u is a bijection of $\mathbb{Z}^n$. Its inverse is the unimodular transformation $u^{-1}(x) = U^{-1}x - U^{-1}v$.

Consider the rational halfspace $(c^T x \leq \delta)$, $c \in \mathbb{Z}^n, \delta \in \mathbb{Q}$. The set $u(c^T x \leq \delta)$ is the rational halfspace

$$\begin{aligned} \{x \in \mathbb{R}^n \mid c^T u^{-1}(x) \leq \delta\} &= \{x \in \mathbb{R}^n \mid c^T U^{-1} x \leq \delta + c^T U^{-1} v\} \\ &= (c^T U^{-1} x \leq \delta + c^T U^{-1} v). \end{aligned}$$

Notice that the vector $c^T U^{-1}$ is also integral. Let S be some subset of $\mathbb{R}^n$. It follows that $(c^T x \le \delta) \supseteq S$ if and only if $(c^T U^{-1} x \le \delta + c^T U^{-1} v) \supseteq u(S)$.

Consider now the first elementary closure P' of some polyhedron P,

$$P' = \bigcap_{\substack{(c^T x \le \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (c^T x \le \lfloor \delta \rfloor).$$

It follows that

$$u(P') = \bigcap_{\substack{(c^T x \le \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (c^T U^{-1} x \le \lfloor \delta \rfloor + c^T U^{-1} v).$$

From this one can derive the next lemma.

Lemma 3.13. *Let P be a polyhedron and u be a unimodular transformation. Then*

$$u(P') = (u(P))'.$$

Corollary 3.14. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $c^T x \le \delta$ be a valid inequality for P_I. Let u be a unimodular transformation. The inequality $c^T x \le \delta$ is valid for $P^{(k)}$ if and only if $u(c^T x \le \delta)$ is valid for $(u(P))^{(k)}$.*

As an application of the previous discussion we will show that Gomory's algorithm implies the convergence of the Chvátal-Gomory procedure for general rational polyhedra. A similar observation was made by Schrijver (Schrijver 1986, p. 358) for polyhedra in the positive orthant. For this notice that we can assume that a rational polyhedron $P(A, b)$ is given with $A \in \mathbb{Z}^{m \times n}$ having full column rank, since otherwise we can transform A from the right with a unimodular matrix U into a matrix $(C \mid 0)$ where C has full column rank and 0 is a matrix with $k = n - \text{rank}(A)$ zero-columns. For this simply identify $\text{rank}(A)$ many linearly independent rows, and compute a unimodular matrix U, which transforms those rows into their Hermite normal form. Notice that $P(C, b)'$ yields $P((C \mid 0), b)'$ by adding k zero-columns to the linear description of $P(C, b)'$. But a polyhedron $P(A, b)$, with $A \in \mathbb{Z}^{m \times n}$ having full column rank can be transformed with a unimodular transformation into a polyhedron that lies in the positive orthant.

Lemma 3.15. *For each rational polyhedron $P(A, b) \subseteq \mathbb{R}^n$ with integral $A \in \mathbb{Z}^{m \times n}$ having full column rank and $b \in \mathbb{Z}^m$, there exists a unimodular transformation $u(x) = Ux + v$ such that $u(P)$ lies in the positive orthant $\mathbb{R}^n_{\ge 0}$.*

Proof. Let $A'x \le b'$ be a choice of inequalities of $Ax \le b$ with A' having full row rank and $\text{rank}(A) = \text{rank}(A')$. Let U be the unimodular matrix transforming A' from the right into its Hermite normal form H. Multiplying each column of H with -1 is a unimodular transformation. Thus assume that each entry on the diagonal of H is strictly negative. Then each member of the i-th row $h_{i,j}$ with $j < i$ can be replaced by the least positive remainder $h_{i,j} \pmod{h_{i,i}}$. This involves the addition of a column to a second one, a unimodular transformation. This can be iteratively done, starting at the first row. Therefore we can assume A in the description of P has a sub-matrix H of the form $h_{i,i} < 0$, $h_{i,j} \geq 0$ and $h_{i,j'} = 0$ for each $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, i-1\}$ and $j' \in \{i+1, \ldots, n\}$. In the so transformed polyhedron, lower bounds for each variable $-x_i \leq l_i$ can be derived. By eventually weakening the right-hand-sides, we can assume that l_i is integral. The translation of P with the integer vector $-(l_1, \ldots, l_n)$ lies in the positive orthant. □

This yields Theorem 3.7 as a corollary from Gomory's (Gomory 1958) original algorithmic result.

Corollary 3.16. *If P is a rational polyhedron, then there exists a natural number t with $P^{(t)} = P_I$.*

Proof. As we observed, we can assume that $P = P(A, b)$ where A is an integral matrix with full column rank. If P is not in the positive orthant, we can apply a unimodular transformation u to P with $u(P) \subseteq \mathbb{R}^n_{\geq 0}$. The result then follows from Lemma 3.13 and Corollary 3.12. □

Remark 3.17. *The "altered" Hermite normal form H with $h_{i,i} < 0$, $h_{i,j} \geq 0$ and $h_{i,j'} = 0$ for each $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, i-1\}$ and $j' \in \{i+1, \ldots, n\}$ from above has been used by Hung & Rom (1990) to compute cutting planes for simplicial cones P, which isolate a vertex of P_I.*

4

The Chvátal-Gomory procedure in the 0/1 cube

4.1 Motivation

Combinatorial optimization problems can often be modeled as an integer program. This typically involves the use of *decision variables.* Such a variable x can take the value 0 or 1, depending on the occurrence of a particular event.

Example 4.1. *A* stable set *of a graph $G = (V, E)$ is a subset $U \subseteq V$ with the property that $|\{v, w\} \cap U| \leq 1$ holds for each edge $\{v, w\} \in E$ of G. In other words not both nodes of an edge can be in the set U. The* maximum stable set problem *is: Given a graph $G = (V, E)$, find a maximal stable set. This can be modeled as an integer program using decision variables $x_v \in \{0, 1\}$ for all $v \in V$. Here $x_v = 1$ means that v belongs to the stable set and $x_v = 0$ means that v does not belong to the stable set U. The constraints are*

$$\begin{array}{rcll} x_v & \geq & 0 & \text{for all } v \in V, \\ x_v & \leq & 1 & \text{for all } v \in V, \\ x_u + x_v & \leq & 1 & \text{for all } \{u, v\} \in E. \end{array} \tag{4.1}$$

Call the polytope defined by (4.1) S_G. *Any integral solution to (4.1) corresponds to a stable set of G and the maximum stable set problem can be formulated as* $\max\{\mathbf{1}^T x \mid x \in S_G,\ x \text{ integral}\}$.

There are many more examples of combinatorial optimization problems which have a 0/1 formulation such as *maximum matching* or the famous *travelling salesman problem.*

Such combinatorial optimization problems can often successfully be attacked with cutting planes and branch-and-bound. Cutting planes which can be derived from the combinatorial structure of the problem are often most useful.

Example 4.2 (Continuation of Example 4.1). *Let $C = \{v_1, \ldots, v_{2k+1}\}$, $k \in \mathbb{N}$, be an* odd cycle *of G, i.e., an odd subset of nodes of G with $\{v_i, v_{i+1}\} \in E$, $i = \{1, \ldots, 2k\}$. If more than k nodes of C are selected, then at least two of them must be adjacent in the* *cycle and thus in G. Therefore the following inequalities are valid for $(S_G)_I$:*

$$\sum_{v \in C} x_v \leq \frac{|C|-1}{2}, \text{ for each odd cycle } C. \tag{4.2}$$

These inequalities are called odd cycle inequalities.

It is easy to see that the odd cycle inequalities are Gomory-Chvátal cutting planes of Q_G. They can be derived from (4.1) by adding the inequalities $x_u + x_v \leq 1$ for each edge $\{u, v\}$ of the cycle, dividing the resulting inequality by 2, and rounding the right-hand-side.

It is the case for most known combinatorially derived cutting planes that they are in fact Gomory-Chvátal cutting planes.

Example 4.3 (Matching). *A* matching *$M \subseteq E$ of a graph $G = (V, E)$ is a set of edges of G, where all edges are pairwise non adjacent. The 0/1 programming formulation is given by the constraints*

$$\begin{array}{rcll} x_e & \geq & 0 & \text{for all } e \in E, \\ \sum_{e \in \delta(v)} x_e & \leq & 1 & \text{for all } v \in V. \end{array} \tag{4.3}$$

Here $\delta(v)$ is the set of edges incident to the node v. Call the described polytope Q_G. It is clear that a 0/1 solution to (4.3) corresponds to a matching of G. If $U \subseteq V$ is an odd set of vertices, then the number of edges of a matching having both endpoints in U is at most $(|U| - 1)/2$. If $\gamma(U) \subseteq E$ is the set $\{\{u, v\} \in E \mid \{u, v\} \subseteq U\}$, then it is easy to see that the following inequalities are valid for the integer hull of $(Q_G)_I$:

$$\sum_{e \in \gamma(U)} x_e \leq (|U| - 1)/2, \text{ for all odd subsets } U \subseteq V. \tag{4.4}$$

Edmonds (1965) showed that $(Q_G)_I$ is described by the inequalities (4.3) and (4.4). The inequalities (4.4) are also Gomory-Chvátal cutting planes. For a given set U, sum the inequalities $\sum_{e \in \delta(v)} x_e \leq 1$ for each $v \in U$ and if an edge has only one endpoint in U add the inequality $-x_e \leq 0$. Then divide the outcome by 2 and round down.

There are many more examples of this kind and combinatorially derived cutting planes are very successful in practice. We have seen in (3.5) that the Chvátal rank of polytopes cannot be bounded in terms of the dimension. In fact there is an exponential lower bound of the Chvátal rank of polytopes in $\mathbb{R}^2$ in the length of the input encoding. Therefore we are motivated to study the convergence behavior of the elementary closure operation in the 0/1 cube. Our main result will be a polynomial upper bound in n on the Chvátal rank of polytopes in the n-dimensional 0/1 cube.

In polyhedral combinatorics, it has also been quite common to consider the depth of a class of inequalities as a measure of its complexity. Chvátal, Cook & Hartmann

(1989) (see also (Hartmann 1988)) answered questions and proved conjectures of Barahona, Grötschel & Mahjoub (1985), of Chvátal (1973*b*), and Grötschel & Pulleyblank (1986) on the behavior of the depth of certain inequalities relative to popular relaxations of the stable set polytope, the bipartite-subgraph polytope, the acyclic-subdigraph polytope, and the traveling salesperson polytope, respectively. The observed increase of the depth was never faster than a linear function of the dimension. We prove that this indeed has to be the case, as the depth of any inequality with coefficients bounded by a constant is $\mathrm{O}(n)$.

4.2 Outline

We first study the behavior of the Chvátal-Gomory procedure applied to polytopes $P \subseteq [0,1]^n$ with empty integer hull. It turns out that the Chvátal rank of a rational polytope is bounded by its dimension $\dim(P)$. We will further see that the case $\operatorname{rank}(P) = n$ and $P_I = \emptyset$ is rather pathological. Besides the $0 \leq x \leq 1$ constraints, one needs at least 2^n other constraints.

Then we study polytopes with nonempty integer hull. For this we have to consider the facet complexity of integral 0/1 polytopes. We will obtain a first upper bound on the Chvátal rank of polytopes in the n-dimensional 0/1 cube of $\mathrm{O}(n^3 \operatorname{size} n)$ by scaling the facet defining vectors of P_I. A more sophisticated application of scaling will eventually lead to an $\mathrm{O}(n^2 \operatorname{size} n)$ upper bound.

We then focus on monotone polyhedra. They reveal some nice features in the context of the Chvátal-Gomory procedure. Via a *monotonization* we will prove a $\|c\|_1 + n$ upper bound on the depth of an inequality $c^T x \leq \delta$, where $c \in \mathbb{Z}^n$. This is an explanation of the phenomenon described above, namely that the lower bounds on the depth of combinatorially derived valid inequalities were at most linear in the dimension. Combinatorially derived cutting planes usually have 0/1 components.

Finally, we construct a family of polytopes in the n-dimensional 0/1-cube whose Chvátal rank is at least $(1+\epsilon)\,n$, for some $\epsilon > 0$.

If $\operatorname{rank}(n)$ denotes the maximum Chvátal rank over all polytopes that are contained in $[0,1]^n$, then it is shown that

$$(1+\epsilon)\,n \quad \leq \quad \operatorname{rank}(n) \quad \leq \quad 3n^2 \operatorname{size}(n).$$

4.3 Polytopes in the 0/1 cube without integral points

Recall the termination argument of the Chvátal-Gomory procedure in § 3.3. Here one has used that the procedure terminates for those faces of P which do not include any integral points. In the following we will study the behavior of such faces of polytopes in the 0/1 cube. Such a face defines a polytope again. It turns out that the Chvátal rank of $P \subseteq [0,1]^n$ with $P_I = \emptyset$ is at most the dimension of P. Via a construction of Chvátal, Cook & Hartmann (1989) we will see that this bound is tight.

Lemma 4.4. *Let $P \subseteq [0,1]^n$ be a d-dimensional rational polytope in the 0/1 cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$; if $d > 0$, then $P^{(d)} = \emptyset$.*

Proof. The case $d = 0$ is obvious.

If $d = 1$, then P is the convex hull of two points $a, b \in [0,1]^n, a \neq b$. Since $P \cap \mathbb{Z}^n = \emptyset$, there exists an $i \in \{1, \dots, n\}$ such that $0 < a_i < 1$. If $a_i \leq b_i$ (resp. $a_i \geq b_i$), then $x_i \geq a_i$ (resp. $x_i \leq a_i$) is valid for P and $P' \subseteq (x_i = 1)$ (resp. $P' \subseteq (x_i = 0)$). Since $0 < a_i < 1$ and $\dim(P) = 1$, it follows $P' \subseteq \{b\}$. Likewise, we can show in the same way that $P' \subseteq \{a\}$. Together, we obtain $P' \subseteq \{a\} \cap \{b\} = \emptyset$.

The general case is proven by induction on d and n. If P is contained in $(x_n = 0)$ or $(x_n = 1)$, we are done by induction on n (see Lemma 3.4). Otherwise, the dimension of $P_0 = P \cap (x_n = 0)$ and $P_1 = P \cap (x_n = 1)$ is strictly smaller than d. By the induction hypothesis and Lemma 3.6 we get

$$P_0^{(d-1)} = P^{(d-1)} \cap (x_n = 0) = \emptyset$$

and

$$P_1^{(d-1)} = P^{(d-1)} \cap (x_n = 1) = \emptyset$$

It follows

$$0 < \min\{x_n \mid x \in P^{(d-1)}\} \leq \max\{x_n \mid x \in P^{(d-1)}\} < 1,$$

which implies $P^{(d)} = \emptyset$ (see Figure 4.1). □

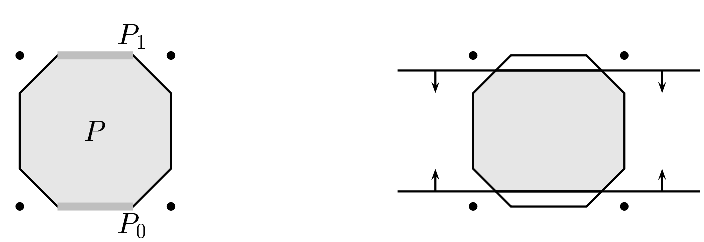

Figure 4.1: After P_0 and P_1 are empty, the Gomory-Chvátal cuts $x_n \geq \lceil \epsilon \rceil$ and $x_n \leq \lfloor 1-\epsilon \rfloor$ apply for some $\epsilon > 0$.

For each polytope $P \subseteq [0,1]^n$, there exists a rational polytope $P^* \supseteq P$ in the 0/1 cube with the same integer hull (see (Schrijver 1986), proof of Corollary 23.2a). Indeed, for each 0/1 point $y \notin P$, there exists a rational half space H_y containing P but not containing y. So

$$P^* = [0,1]^n \cap \bigcap_{\substack{y \in \{0,1\}^n \\ y \notin P}} H_y \tag{4.5}$$

has the desired properties. As $P^* \supseteq P$ implies $(P^*)^{(t)} \supseteq P^{(t)}$ we have proved the following corollary.

Corollary 4.5. *The Chvátal rank of polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ is at most n.*

The next lemma implies that the bound of Lemma 4.4 is tight. Its proof follows immediately from the proof of Lemma 7.2 in (Chvátal, Cook & Hartmann 1989).

Lemma 4.6. *Let F_j be the set of all vectors y in $\mathbb{R}^n$ such that j components of y are 1/2 and each of the remaining $n-j$ components are equal to 0 or 1. If a polyhedron P contains F_1, then $F_j \subseteq P^{(j-1)}$, for all $j = 1, \dots, n$.*

Proof. Let $(c^T x \leq \delta)$ contain F_{j-1}. We have to show that $(c^T x \leq \lfloor \delta \rfloor) \supseteq F_j$. Assume that $\delta = \max\{c^T x \mid x \in F_{j-1}\}$. Let $\hat{x} \in F_j$ and $I \subseteq \{1, \dots, n\}$ be the set of indices with $\hat{x}_i = 1/2$. If $c_i = 0$ for all $i \in I$, then $c^T \hat{x} \in \mathbb{Z}$, thus $c^T \hat{x} \leq \lfloor \delta \rfloor$.

If $c_i \neq 0$ for some $i \in I$, then $c^T(\hat{x} \pm 1/2 e_i) \leq \delta$, where e_i is the i-th unit vector. Therefore $c^T \hat{x} \leq \delta \pm 1/2 c_i$, which implies $c^T \hat{x} \leq \lfloor \delta \rfloor$. □

If we define P_n as the convex hull of F_1, then one has

$$P_n = \big\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq \frac{1}{2}, \text{ for all } J \subseteq \{1, \dots, n\},\ 0 \leq x \leq 1 \big\}, \tag{4.6}$$

$(P_n)_I = \emptyset$ and $F_n = \{(1/2, \dots, 1/2)\} \subseteq P_n^{(n-1)}$. Thus n is the smallest number such that $P_n^{(n)} = (P_n)_I = \emptyset$. We therefore have the following proposition.

Proposition 4.7. *There exist rational polytopes $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and Chvátal rank n.*

Notice that the number of inequalities describing P_n in (4.6) is 2^n, not counting the $0 \leq x \leq 1$ constraints. We will now show that this has to be the case.

Proposition 4.8. *Let $P \subseteq [0,1]^n$ be a rational polytope in the 0/1-cube with $P_I = \emptyset$ and rank$(P) = n$. Any inequality description of P has at least 2^n inequalities.*

Proof. For a polytope $P \subseteq \mathbb{R}^n$ and for some $i \in \{1, \dots, n\}$ and $\ell \in \{0,1\}$ let $P_i^\ell \subseteq \mathbb{R}^{n-1}$ be the polytope defined by

$$P_i^\ell = \{x \in [0,1]^{n-1} \mid (x_1, \dots, x_{i-1}, \ell, x_{i+1}, \dots, x_n)^T \in P\}.$$

Notice that, if P is contained in a facet $(x_i = \ell)$ of $[0,1]^n$ for some $\ell \in \{0,1\}$ and some $i \in \{1, \dots, n\}$, then the Chvátal rank of P is the Chvátal rank of P_i^ℓ (see Lemma 3.4).

We will prove now that any one-dimensional face F_1 of the cube satisfies $F_1 \cap P \neq \emptyset$. We proceed by induction on n.

If $n = 1$, this is definitely true since P is not empty and since F_1 is the cube itself. For $n > 1$, observe that any one-dimensional face F_1 of the cube lies in a facet $(x_i = \ell)$ of the cube, for some $\ell \in \{0,1\}$ and for some $i \in \{1, \dots, n\}$. Since P has Chvátal rank n it follows that $\tilde{P} = (x_i = \ell) \cap P$ has Chvátal rank $n - 1$. If the Chvátal rank of $\tilde{P}$ was less than that, P would vanish after $n - 1$ steps. It follows by induction that $(F_1)_i^\ell \cap \tilde{P}_i^\ell \neq \emptyset$, thus $F_1 \cap P \neq \emptyset$.

Now, each 0/1-point has to be cut off from P by some inequality, as $P_I = \emptyset$. If an inequality $c^T x \leq \delta$ cuts off two different 0/1-points simultaneously, then it must also cut off a 1-dimensional face of $[0,1]^n$. Because of our previous observation this is not possible, and hence there is at least one inequality for each 0/1-point which cuts off only this point. Since there are 2^n different 0/1-points in the cube, the claim follows. ☐

We conclude that in order to obtain a rational polytope in the n-dimensional 0/1 cube with empty integer hull and rank n, each 0/1 point has to be cut off by an individual inequality.

4.4 A first polynomial upper bound

To study the rank of polytopes with nonempty integer hull we first have to study the structure of facet defining inequalities of integral 0/1 polytopes. Hadamard's inequality can be used to show that an integral 0/1 polytope can be described by inequalities with integer normal vectors whose l_∞-norm is only exponential in n (see, e.g, (Padberg & Grötschel 1985, Theorem 2)).

Theorem 4.9. *An integral 0/1 polytope P can be described by a system of integral inequalities $Ax \leq b$ with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ such that each absolute value of an entry in A is bounded by $n^{n/2}$.*

Proof. We show the assertion for full dimensional integral 0/1 polytopes. Since any integral 0/1 polytope is a face of a full-dimensional 0/1 polytope, the assertion follows then easily. Let $v_1, \dots, v_n$ be n affinely independent 0/1 points lying in a facet of P. We will estimate the l_∞-norm of an integral vector c, which defines the hyperplane through these points. Any facet defining inequality of an integral 0/1 polytope is of this form. For symmetry reasons we can assume that $v_1 = 0$. Then c is the generator of the submodule of $\mathbb{Z}^n$ defined by the system

$$Vx = \mathbf{0}, \tag{4.7}$$

where $V \in \{0,1\}^{n-1\times n}$ is the matrix having $v_2,\dots,v_n$ as its rows. Assume without loss of generality that the first $n-1$ columns of V are linearly independent and call the corresponding matrix U. The solution $\hat{x}$ of the system $U\,x = -V^{(n)}$ yields a solution $(\hat{x}_1,\dots,\hat{x}_{n-1},1)^T$ of the system $V\,y = 0$. Cramer's rule (see (Lang 1971)) implies that $\hat{x}_i = \det(B_i)/\det(U)$ for $i = 1,\dots,n-1$, where B_i is obtained from U by replacing the i-th column by $-V^{(n)}$. Thus an integral solution to (4.7) is given by $(\det(B_1),\dots,\det(B_{n-1}),\det(U))^T$. The Hadamard bound (2.1) implies that each absolute value of these determinants is bounded by $n^{n/2}$. □

Alon & Vu (1997) (see also (Ziegler 1999)) showed that this upper bound, derived from the Hadamard bound is tight, i.e., there exist 0/1-polytopes with facets for which any inducing inequality $a^Tx \le \beta$, $a \in \mathbb{Z}^n$ satisfies $\|a\|_\infty \in \Omega(n^{n/2})$.

First we formulate and prove a lemma which is already in the termination argument of the Chvátal-Gomory procedure in Section 3.3, only specially shaped for the 0/1 cube, with the knowledge on polytopes in the 0/1 cube without integral points.

Lemma 4.10. *Let $P \subseteq [0,1]^n$ be a rational polytope with $P_I \neq \emptyset$. For $0 \neq c \in \mathbb{Z}^n$ let $\gamma = \max\{c^Tx \mid x \in P\}$ and $\delta = \max\{c^Tx \mid x \in P_I\}$. Then $c^Tx \le \delta$ is valid for $P^{(k)}$, for all $k \ge d\lceil\gamma - \delta\rceil$.*

Intuitively, the lemma says that any face-defining inequality $c^Tx \le \delta$ of P_I can be obtained from P by at most $d\lceil d_c\rceil$ iterations of the Chvátal-Gomory procedure, where $d_c = \gamma - \delta$ is the integrality gap of P with respect to c. A related result can be found in (Chvátal 1973*a*, Sect. 4), see also (Hartmann 1988, Lemma 2.2.7). This lemma yields an exponential upper bound on the Chvátal rank of polytopes in the 0/1 cube, since the integrality gap of a facet defining vector of P_I can be bounded by $\sum_{i=1}^n |c_i| \le n^{n/2+1}$, following Theorem 4.9.

Proof. If $d = 0$, then $P_I = P$ and the claim follows trivially. If $d = 1$ and $P \neq P_I$, then P is the convex hull of a 0/1 point a and some non-integral point $b \in [0,1]^n$. An argument similar to the one in Lemma 4.4 shows that $P' = \{a\} = P_I$, which implies the claim for $d = 1$, too.

So assume that $d \ge 2$. The proof is by induction on $\lceil\gamma - \delta\rceil$. The case $\lceil\gamma - \delta\rceil = 0$ is trivial, so suppose $\lceil\gamma - \delta\rceil > 0$.

If $\gamma \notin \mathbb{Z}$, then $c^T x \leq \lfloor \gamma \rfloor = \lceil \gamma \rceil - 1$ is valid for P'.

If $\gamma \in \mathbb{Z}$, then $F = (c^T x = \gamma) \cap P$ is a face of P without any integral points and $\dim(F) < d$. With Lemma 4.4 and since $d \geq 2$, we get $F^{(d-1)} = \emptyset$. Since $F^{(d-1)} = P^{(d-1)} \cap F$, we have $\max\{c^T x \mid x \in P^{(d-1)}\} < \gamma$, which implies that $c^T x \leq \gamma - 1$ is valid for $P^{(d)}$.

So in any case we see that $c^T x \leq \lceil \gamma \rceil - 1$ is valid for $P^{(d)}$. Let $\gamma' = \max\{c^T x \mid x \in P^{(d)}\}$. Then $\gamma' \leq \lceil \gamma \rceil - 1$ and since $\delta \in \mathbb{Z}$, it follows by induction that $c^T x \leq \delta$ is valid for $(P^{(d)})^{(k')}$, for all $k' \geq d(\lceil \gamma - \delta \rceil - 1) \geq d\lceil \gamma' - \delta \rceil$. This implies the claim. □

We now derive an $\mathrm{O}(d\,n^2 \log n)$ upper bound for the Chvátal rank of d-dimensional rational polytopes in the 0/1 cube. Here, the basic idea is to use scaling of the row vectors a^T of A, where $Ax \leq b$ is an integral inequality description if P_I. The sequence of integral vectors obtained from a^T by dividing it by decreasing powers of 2 followed by rounding gives a better and better approximation of a^T itself. One estimates the number of iterations of the Chvátal-Gomory rounding procedure needed until the face given by some vector in the sequence contains integer points, using the fact that the face given by the previous vector in the sequence also contains integer points. Although the size of the vector is doubled every time, the number of iterations of the Chvátal-Gomory rounding procedure in each step is at most quadratic.

The key is the following observation.

Lemma 4.11. *Let $P \subseteq [0,1]^n$ be a d-dimensional rational polytope with $P_I \neq \emptyset$. If $c \neq 0$ is an integral vector with $\mathrm{size}(\|c\|_\infty) \leq k$ and if $c^T x \leq \delta$ is valid for P_I, then $c^T x \leq \delta$ is valid for $P^{(k\,d\,n)}$.*

Proof. Assume that $\delta = \max\{c^T x \mid x \in P_I\}$. We proceed by induction on k.

For $k = 1$ note that $c \in \{-1, 0, 1\}^n$, so for $\gamma = \max\{c^T x \mid x \in P\}$ one has $\gamma - \delta \leq n$ and the claim follows with Lemma 4.10.

Now let $k > 1$ and write c as the sum $2c_1 + c_2$ with $c_1 = \lfloor c/2 \rceil$. Note that $\mathrm{size}(\|c_1\|_\infty) < \mathrm{size}(\|c\|_\infty)$ and that $c_2 \in \{-1, 0, 1\}^n$. Let $c_1^T x \leq \delta_1$ be a face-defining inequality for P_I. By the induction hypothesis it follows that $c_1^T x \leq \delta_1$ is valid for $P^{((k-1)dn)}$. Let $x_I \in P_I$ satisfy $c_1^T x_I = \delta_1$. Let $\gamma' = \max\{c^T x \mid x \in P^{((k-1)dn)}\}$. We will conclude that $\gamma' - \delta \leq n$ and the claim then follows again from Lemma 4.10. Let $\hat{x} \in P^{((k-1)dn)}$ satisfy $c^T \hat{x} = \gamma'$. Clearly $c^T(\hat{x} - x_I)$ is an upper bound on the integrality gap $\gamma' - \delta$. But

$$\begin{aligned} c^T(\hat{x} - x_I) &= 2c_1(\hat{x} - x_I) + c_2(\hat{x} - x_I) \\ &\leq c_2(\hat{x} - x_I) \\ &\leq n. \end{aligned}$$

This follows since x_I maximizes $\{c_1^T x \mid x \in P^{((k-1)dn)}\}$ and since c_2 and $\hat{x} - x_I$ are in $[-1, 1]^n$. $\square$

A polynomial upper bound on the Chvátal rank now follows easily.

Theorem 4.12. *Let $P \subseteq [0,1]^n$, $P_I \neq \emptyset$, be a d-dimensional rational polytope in the $0/1$ cube. The Chvátal rank of P is at most $(\lfloor n/2 \log_2 n \rfloor + 1)\, n\, d$.*

Proof. P_I is obtained by i iterations of the Chvátal-Gomory procedure if each inequality $c^T x \leq \delta$ out of the description delivered by Proposition 4.9 is valid for $P^{(i)}$. With Lemma 4.11 this is true for all $i \geq \operatorname{size}(n^{n/2})\, d\, n = (\lfloor n/2 \log_2 n \rfloor + 1)\, d\, n$ $\square$

We can now conclude with a polynomial upper bound on the Chvátal rank for polytopes in the $0/1$ cube.

Theorem 4.13. *The Chvátal rank of any polytope $P \subseteq [0,1]^n$ in the n-dimensional $0/1$ cube is at most $(\lfloor n/2 \log_2 n \rfloor + 1)\, n^2$.*

Proof. Let P^* be the construction from equation (4.5) in Sect. 4.3. The rank of P^* is an upper bound on the rank of P. Since P^* is rational either Lemma 4.4 or Theorem 4.12 applies to P^* and the result follows. $\square$

4.5 An $\mathrm{O}(n^2 \log n)$ upper bound

The weakness of the previous analysis is that the faces of the intermediate polytopes are taken to have worst case behavior d. In the following we will get rid of this nuisance. Observe the following. If a polytope $P \subseteq [0,1]^n$ does not intersect with two arbitrarily chosen facets of the cube, then $P' = \emptyset$. This implies the next lemma.

Lemma 4.14. *Let $P \subseteq [0,1]^n$ be a rational polytope and let $c^T x \leq \alpha$ be valid for P_I and $c^T x \leq \gamma$ be valid for P, where $\alpha \leq \gamma$, $\alpha, \gamma \in \mathbb{Z}$ and $c \in \mathbb{Z}^n$. If, for each $\beta \in \mathbb{R}$, $\beta > \alpha$, the polytope $F_\beta = P \cap (c^T x = \beta)$ does not intersect with two opposite facets of the $0/1$-cube, then the depth of $c^T x \leq \alpha$ is at most $2(\gamma - \alpha)$.*

Proof. Notice that $F'_\beta = \emptyset$ for each $\beta > \alpha$. The proof is by induction on $\gamma - \alpha$.

If $\alpha = \gamma$, there is nothing to prove. So let $\gamma - \alpha > 0$. Since $F'_\gamma = \emptyset$, Lemma 3.6 implies that $c^T x \leq \gamma - \epsilon$ is valid for P' for some $\epsilon > 0$ and thus the inequality $c^T x \leq \gamma - 1$ is valid for $P^{(2)}$. $\square$

To facilitate the argument we call a vector c *saturated* with respect to a polytope P, if $\max\{c^T x \mid x \in P\} = \max\{c^T x \mid x \in P_I\}$. If $Ax \leq b$ is an inequality description of P_I, then

$P = P_I$ if and only if each row vector of A is saturated with respect to P. In section 4.4, it is shown that an integral vector $c \in \mathbb{Z}^n$ is saturated after at most $n^2 \operatorname{size} \|c\|_\infty$ steps of the Chvátal-Gomory procedure. We now use Lemma 4.14 for a more sophisticated analysis of the convergence behavior of the Chvátal-Gomory procedure.

Proposition 4.15. *Let P be a rational polytope in the n-dimensional 0/1-cube. Any integral vector $c \in \mathbb{Z}^n$ is saturated with respect to $P^{(t)}$, for any $t \geq 2(n^2 + n \operatorname{size}(\|c\|_\infty))$.*

Proof. We can assume that $c \geq 0$ holds and that $P_I \neq \emptyset$. The proof is by induction on n and $\operatorname{size}(\|c\|_\infty)$. The claim holds for $n = 1, 2$ since the Chvátal rank of a polytope in the 1- or 2-dimensional 0/1-cube is at most 4.

So let $n > 2$. If $\operatorname{size}(\|c\|_\infty) = 1$, then the claim follows, e.g., from Theorem 4.20 below. So let $\operatorname{size}(\|c\|_\infty) > 1$. Write $c = 2c_1 + c_2$, where $c_1 = \lfloor c/2 \rfloor$ and $c_2 \in \{0,1\}^n$. By induction, it takes at most $2(n^2 + n \operatorname{size}(\|c_1\|_\infty)) = 2(n^2 + n \operatorname{size}(\|c\|_\infty)) - 2n$ iterations of the Gomory-Chvátal procedure until c_1 is saturated. Let $k = 2(n^2 + n \operatorname{size}(\|c\|_\infty)) - 2n$.

Let $\alpha = \max\{c^T x \mid x \in P_I\}$ and $\gamma = \max\{c^T x \mid x \in P^{(k)}\}$. The integrality gap $\gamma - \alpha$ is at most n. This can be seen as in the proof of Lemma 4.11: Choose $\hat{x} \in P^{(k)}$ with $c^T \hat{x} = \gamma$ and let $x_I \in P_I$ satisfy $c_1^T x_I = \max\{c_1^T x \mid x \in P^{(k)}\}$. One can choose x_I out of P_I since c_1 is saturated with respect to $P^{(k)}$. It follows that

$$\gamma - \alpha \leq c(\hat{x} - x_I) = 2c_1(\hat{x} - x_I) + c_2(\hat{x} - x_I) \leq n.$$

Consider now an arbitrary fixing of an arbitrary variable x_i to a specific value ℓ, $\ell \in \{0,1\}$. The result is the polytope

$$P_i^\ell = \{x \in [0,1]^{n-1} \mid (x_1, \dots, x_{i-1}, \ell, x_{i+1}, \dots, x_n)^T \in P\}$$

in the $(n-1)$-dimensional 0/1-cube for which, by the induction hypothesis, the vector $\widetilde{c}_i = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)$ is saturated after at most

$$2((n-1)^2 + (n-1) \operatorname{size}(\|\widetilde{c}_i\|_\infty)) \quad \leq \quad 2(n^2 + n \operatorname{size}(\|c\|_\infty)) - 2n$$

iterations.

It follows that

$$\alpha - \ell c_i \geq \max\{\widetilde{c}_i^T x \mid x \in (P_i^\ell)^{(k)}\} = \max\{\widetilde{c}_i^T x \mid x \in (P_i^\ell)_I\}.$$

If $\beta > \alpha$, then $(c^T x = \beta) \cap P^{(k)}$ cannot intersect with a facet of the cube, since a point in $(c^T x = \beta) \cap P^{(k)} \cap (x_i = \ell)$, $\ell \in \{0,1\}$, has to satisfy $c^T x \leq \alpha$.

With Lemma 4.14, after $2n$ more iterations of the Gomory-Chvátal procedure, c is saturated, which altogether happens after $2(n^2 + n \operatorname{size}(\|c\|_\infty))$ iterations. $\square$

We conclude this section with an $\mathrm{O}(n^2 \log n)$ upper bound on the Chvátal rank of polytopes in the 0/1 cube.

Theorem 4.16. *The Chvátal rank of a polytope in the n-dimensional 0/1 cube is bounded by a function in $\mathrm{O}(n^2 \log n)$.*

Proof. Each polytope Q in the 0/1-cube has a rational weakening P. Theorem 4.9 implies that the integral 0/1-polytope P_I can be described by a system of integral inequalities $P_I = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ such that each absolute value of an entry in A is bounded by $n^{n/2}$. We estimate the number of Chvátal-Gomory steps until all row-vectors of A are saturated. Proposition 4.15 implies that those row-vectors are saturated after at most $2(n^2 + n \operatorname{size} n^{n/2}) = \mathrm{O}(n^2 \log n)$ steps. □

4.6 Upper bounds through monotonization

As we have mentioned in § 4.1 for combinatorially derived inequalities, only a linear growth of their depth has been observed. We give an explanation to this phenomenon in this section. We show that any inequality $c^T x \leq \delta$ which is valid for the integer hull of a polytope P in the n-dimensional 0/1-cube, has depth at most $n + \|c\|_1$ with respect to P. This explains the linear growth of combinatorial inequalities that has been observed so far, since such inequalities rarely have components larger than 3. Compared with the bound of Proposition 4.15 and Lemma 4.11, then the bound shown here is superior for c with small entries.

We start by introducing the unimodular transformations of the cube, the *switching operations.*

4.6.1 The switching operations

The *i-th switching operation* is the unimodular transformation

$$\begin{array}{rlcl} \pi_i: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & (x_1, \dots, x_n) & \mapsto & (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_n), \end{array}$$

It has a representation

$$\begin{array}{rlcl} \pi_i: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & Ux + e_i, \end{array}$$

where U coincides with the identity matrix I_n except for $U_{(i,i)}$ which is -1. Note that the switching operation is a bijection of $[0,1]^n$. For the set $(c^T x \leq \delta)$ one has $\pi_i(c^T x \leq \delta) = \widetilde{c}^T x \leq \delta - c_i$. Here $\widetilde{c}$ coincides with c except for a change of sign in the i-th component.

4.6.2 Monotone polyhedra

A nonempty polyhedron $P \subseteq \mathbb{R}^n_{\geq 0}$ is called *monotone* if $x \in P$ and $0 \leq y \leq x$ imply $y \in P$. Hammer, Johnson & Peled (1975) observed that a polyhedron P is monotone if and only if P can be described by a system $x \geq 0,\ Ax \leq b$ with $A, b \geq 0$.

The next statements are proved in (Hartmann 1988) and (Chvátal, Cook & Hartmann 1989, p. 494). We include a proof of Lemma 4.18 for the sake of completeness.

Lemma 4.17. *If P is a monotone polyhedron, then P' is monotone as well.*

Lemma 4.18. *Let P be a monotone polytope in the 0/1-cube and let $w^Tx \leq \delta$, $w \in \mathbb{Z}^n$, be valid for P_I. Then $w^Tx \leq \delta$ has depth at most $\|w\|_1 - \delta$.*

Proof. The proof is by induction on $\|w\|_1$. If $\|w\|_1 = 0$, the claim follows trivially.

W.l.o.g., we can assume that $w \geq 0$ holds. Let $\gamma = \max\{w^Tx \mid x \in P\}$ and let $J = \{j \mid w_j > 0\}$. If $\max\{\sum_{j \in J} x_j \mid x \in P\} = |J|$, then, since P is monotone, $\hat{x}$ with

$$\hat{x}_i = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{otherwise} \end{cases}$$

is in P. Also $w^T\hat{x} = \gamma$ must hold. So $\gamma = \delta$ and the claim follows trivially. If $\max\{\sum_{j \in J} x_j \mid x \in P\} < |J|$, then $\sum_{j \in J} x_j \leq |J| - 1$ has depth at most 1. If $\|w\|_1 = 1$ this also implies the claim, so assume $\|w\|_1 \geq 2$. By induction the valid inequalities $w^Tx - x_j \leq \delta$, $j \in J$ have depth at most $\|w\|_1 - \delta - 1$. Adding up the inequalities $w^Tx - x_j \leq \delta$, $j \in J$ and $\sum_{j \in J} x_j \leq |J| - 1$ yields

$$w^Tx \leq \delta + (|J| - 1)/|J|.$$

Rounding down yields $w^Tx \leq \delta$ and the claim follows. □

4.6.3 The reduction to monotone weakenings

If one wants to examine the depth of a particular inequality with respect to a polytope $P \subseteq [0,1]^n$, one can apply a series of switching operations until all its coefficients become nonnegative. An inequality with nonnegative coefficients defines a (fractional) 0/1-knapsack polytope K. The depth of this inequality with respect to the convex hull of $P \cup K$ is then an upper bound on the depth with respect to P. We will show that $\mathrm{conv}(P \cup K)^{(n)}$ has a monotone rational weakening in the 0/1-cube.

Lemma 4.19. *Let $P \subseteq [0,1]^n$ be a polytope in the 0/1-cube, with $P_I = K_I$, where $K = \{x \mid c^Tx \leq \delta,\ 0 \leq x \leq 1\}$ and $c \geq 0$. Then, $P^{(n)}$ has a rational, monotone weakening Q in the 0/1-cube.*

Proof. We can assume that P is rational. Let $\hat{x}$ be a 0/1-point which is not contained in P, i.e., $c^T\hat{x} > \delta$. Let $I = \{i \mid \hat{x}_i = 1\}$. The inequality $\sum_{i\in I} x_i \leq |I|$ is valid for the cube and thus for P. Since $c \geq 0$, the corresponding face $F = \{x \mid \sum_{i\in I} x_i = |I|,\ x \in P\}$ of P does not contain any 0/1-points. Lemma 4.4 implies that $\sum_{i\in I} x_i \leq |I| - 1$ is valid for $P^{(n)}$.

Thus, for each 0/1-point $\hat{x}$ which is not in P, there exists a nonnegative rational inequality $a_{\hat{x}}^T x \leq \gamma_{\hat{x}}$ which is valid for $P^{(n)}$ and which cuts $\hat{x}$ off. Thus

$$\begin{aligned} &0 \leq x_i \leq 1, && i \in \{1, \dots, n\} \\ &a_{\hat{x}}^T x \leq \gamma_{\hat{x}}, && \hat{x} \in \{0,1\}^n,\ \hat{x} \notin P \end{aligned}$$

is the desired weakening. □

Theorem 4.20. *Let $P \subseteq [0,1]^n$, $P \neq \emptyset$ be a nonempty polytope in the 0/1-cube and let $c^T x \leq \delta$ be a valid inequality for P_I with $c \in \mathbb{Z}^n$. Then $c^T x \leq \delta$ has depth at most $n + \|c\|_1$ with respect to P.*

Proof. One can assume that c is nonnegative, since one can apply a series of switching operations. Notice that this can change the right hand side δ, but in the end δ has to be nonnegative since $P \neq \emptyset$. Let $K = \{x \in [0,1]^n \mid c^T x \leq \delta\}$ and consider the polytope $Q = \operatorname{conv}(P \cup K)$. The inequality $c^T x \leq \delta$ is valid for Q_I and the depth of $c^T x \leq \delta$ with respect to P is at most the depth of $c^T x \leq \delta$ with respect to Q. By Lemma 4.19, $Q^{(n)}$ has a monotone rational weakening S. The depth of $c^T x \leq \delta$ with respect to $Q^{(n)}$ is at most the depth of $c^T x \leq \delta$ with respect to S. But it follows from Lemma 4.18 that the depth of $c^T x \leq \delta$ with respect to S is at most $\|c\|_1 - \delta \leq \|c\|_1$. □

4.7 A lower bound

The Chvátal-Gomory procedure applies to general polyhedra. For the 0/1 cube other cutting plane approaches, relying on lift-and-project were invented by Balas, Ceria & Cornuéjols (1993), Sherali & Adams (1990) and Lovàsz & Schrijver (1991). These methods can also be defined via an operator like the Chvátal-Gomory operation this thesis is concerned with. In analogy, the rank defined by those operations is $\leq n$ for all polytopes in the 0/1 cube. We now give a lower bound that shows that the Chvátal rank of polytopes in the n-dimensional 0/1 cube exceeds n for infinitely many n.

We show that $\operatorname{rank}(n) > (1+\epsilon)\, n$, for infinitely many n, where $\epsilon > 0$. The construction relies on the lower bound result for the fractional stable-set polytope due to Chvátal, Cook & Hartmann (1989).

Let $G = (V, E)$ be a graph on n vertices. A *clique* of G is a nonempty set of vertices C where each two vertices in C are adjacent to each other. Let $\mathscr{C}$ be the family of all cliques of G and let $Q \subseteq \mathbb{R}^n$ be the fractional stable set polytope of G defined by the equations

$$\begin{aligned} \textstyle\sum_{v\in C} x_v \le 1 \quad & \text{for all } C \in \mathscr{C}, \\ x_v \ge 0 \quad & \text{for all } v \in V. \end{aligned} \tag{4.8}$$

The following lemma is proved in (Chvátal, Cook & Hartmann 1989, Proof of Lemma 3.1).

Lemma 4.21. *Let $k < s$ be positive integers and let G be a graph with n vertices such that every subgraph of G with s vertices is k-colorable. If P is a polyhedron that contains Q_I and the point $u = \frac{1}{k}\mathbf{1}$, then $P^{(j)}$ contains the point $x^j = (\frac{s}{s+k})^j u$.*

Let $\alpha(G)$ be the size of the largest independent subset of the nodes of G. It follows that $\mathbf{1}^T x \le \alpha(G)$ is valid for Q_I. One has

$$\mathbf{1}^T x^j = \frac{n}{k}(\frac{s}{s+k})^j \ge \frac{n}{k} e^{-jk/s} \quad ,$$

and thus x^j does not satisfy the inequality $\mathbf{1}^T x \le \alpha(G)$ for all $j < (s/k)\ln\frac{n}{k\alpha(G)}$.

Erdős (1962) proved that for every positive t there exist a positive integer c, a positive number δ and arbitrarily large graphs G with n vertices, cn edges, $\alpha(G) < tn$ such that every subgraph of G with at most δn vertices is 3 colorable. One wants that $\ln\frac{n}{k\alpha(G)} \ge 1$ and that s/k grows linearly, so by choosing some $t < 1/(3e)$, $k = 3$ and $s = \lfloor \delta n \rfloor$ one has that x^j does not satisfy the inequality $\mathbf{1}^T x \le \alpha(G)$ for all $j < (s/k)$.

We now give the construction. Let $P = \operatorname{conv}(P_n \cup Q)$ be the polytope that results from the convex hull of P_n defined in (4.6) and Q. $P_n \subseteq P$ contributes to the fact that $\frac{1}{2}\mathbf{1}$ is in $P^{(n-1)}$. Thus $x_0 = \frac{1}{3}\mathbf{1}$ is in $P^{(n-1)}$, since $\mathbf{0}$ also is in P. Since the convex hull of P is Q_I, it follows from the above discussion that the depth of $\mathbf{1}^T x \le \alpha(G)$ with respect to $P^{(n-1)}$ is $\Omega(n)$. Thus the depth of $\mathbf{1}^T x \le \alpha(G)$ is at least $(n-1) + \Omega(n) \ge (1+\epsilon)n$ for infinitely many n, where $\epsilon > 0$. We conclude.

Theorem 4.22. *There exists an $\epsilon > 0$ such that there exist, for infinitely many $n \in \mathbb{N}$, a polytope $P \subseteq \mathbb{R}^n$ with Chvátal rank at least $(1+\epsilon)\,n$.*

Remark 4.23. *The gap in between the lower bound $\Omega(n)$ and $\mathrm{O}(n^2 \log n)$ for the rank function $r(n)$ is still large. Lower bounds that are worse than linear are not known.*

5

Complexity of the elementary closure

5.1 Motivation

Gomory-Chvátal cuts exist since 1958 (Gomory 1958). They are a classic in integer programming. It is natural to ask for the complexity of the optimization problem over all cuts that can be derived from a polyhedron P. Of course there are a lot of Gomory-Chvátal cutting planes that can be derived from P. Indeed the matching polytope has an exponential number of facets, but this does not imply that optimization over P' is not possible in polynomial time. One can optimize over the matching polytope and the elementary closure analogon of other cutting plane approaches, based on lift-and-project (Lovàsz & Schrijver 1991, Balas, Ceria & Cornuéjols 1993, Sherali & Adams 1990) yield polyhedra with an exponential number of facets, over which one can optimize in polynomial time. The semidefinite operator of Lovàsz & Schrijver (1991) even yields convex sets that are not polyhedra. However, unlike the general Gomory-Chvátal cuts, these methods apply for the 0/1 cube only.

Also, as we observed in § 4.1, a lot of combinatorially derived cutting planes are in fact Gomory-Chvátal cutting planes. A polynomial separation routine for the Gomory-Chvátal cuts of a rational polyhedron P would be a powerful tool. This motivated Schrijver to pose the possibility of such an algorithm as an open problem in his book (Schrijver 1986).

5.2 Outline

We will prove that there exists no polynomial algorithm for the optimization problem over the elementary closure of a rational polyhedron unless P = NP. This solves the problem raised by Schrijver in (Schrijver 1986, p. 351). The proof also shows that minimizing the support of a nontrivial Chvátal-Gomory cut is NP-hard. At the heart of the proof is a result given by Caprara & Fischetti (1996) concerning the separation of so called $\{0, \frac{1}{2}\}$-cuts.

5.3 The NP-completeness of membership

We proceed by showing NP-completeness of the (non)-membership problem for the elementary closure. We consider the (non)-membership problem instead of the membership problem to avoid unnecessary technicalities involving the class co − NP.

Definition 5.1 (MEC)**.** The *membership problem for the elementary closure* is as follows:

> Given an integral matrix $A \in \mathbb{Z}^{m\times n}$, an integral vector $b \in \mathbb{Z}^m$ and a rational vector $\hat{x} \in \mathbb{Q}^n$, is $\hat{x} \notin P(A,b)'$?

The membership problem for the elementary closure is a subproblem of the separation problem for the elementary closure (see § 2.5) which is as follows: Given a polyhedron P and some $\hat{x} \in \mathbb{R}^n$, decide if $\hat{x} \in P$ and if not return an inequality $c^T x \leq \delta$, which is valid for P but not for $\hat{x}$.

First we have to show that MEC is in NP (see Section 2.8). For this let A, b and $\hat{x}$ be given with $\hat{x} \notin P(A,b)'$. We have to provide a short certificate for this. In fact, if $\hat{x}$ is not in the elementary closure $P(A,b)'$, then there exists a Gomory-Chvátal cut $c^T x \leq \lfloor\delta\rfloor$, which is not satisfied by $\hat{x}$ such that c can be written as $c^T = \lambda^T A$, where $\lambda \in [0,1]^m$. Notice then that $\|c\|_\infty \leq \|A^T\|_\infty$, where the matrix norm $\|\cdot\|_\infty$ is the row-sum-norm. Clearly $\hat{x}$ does not satisfy the inequality $c^T x \leq \lfloor\gamma\rfloor$, where $\gamma = \max\{c^T x \mid Ax \leq b\}$. Since linear programming is polynomial, this c serves as a polynomial certificate for the fact that $\hat{x}$ is not in $P(A,b)'$. Thus MEC is in NP.

To proceed, we have to show that each language $L \in$ NP can be polynomially reduced to MEC. We will reduce the so called $\{0,\frac{1}{2}\}$-*closure membership problem* to MEC. Caprara & Fischetti (1996) showed that the $\{0,\frac{1}{2}\}$-closure membership is NP-complete.

Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix, $b \in \mathbb{Z}^m$ be an integral vector, and let $P \subseteq \mathbb{R}^n$ be the polyhedron $P(A,b)$. A $\{0,\frac{1}{2}\}$-*cut* derived from A and b is a Gomory-Chvátal cut of P of the form $\lambda^T Ax \leq \lfloor\lambda^T b\rfloor$, where $\lambda^T A$ is integral and the components of λ are either 0 or $\frac{1}{2}$. The $\{0,\frac{1}{2}\}$-*closure* $P_{\frac{1}{2}}(A,b)$ derived from A and b is the intersection of P with all the $\{0,\frac{1}{2}\}$-cuts derived from A and b. Unlike the elementary closure, the $\{0,\frac{1}{2}\}$-closure of $P(A,b)$ depends on the description of P by A and b and thus is not a property of the polyhedron $P = P(A,b)$. Observe that $P(A,b) = P(2\cdot A, 2\cdot b)$, but no nontrivial $\{0,\frac{1}{2}\}$-cuts can be derived from the second description of the polyhedron, since there cannot be any rounding effect. Notice that the odd cycle inequalities (4.2) and the odd set constraints (4.4) are $\{0,\frac{1}{2}\}$-cuts.

Definition 5.2 (M0$\frac{1}{2}$)**.** The *membership problem for the* $\{0,\frac{1}{2}\}$-*closure* is as follows:

> Given an integral matrix $A \in \mathbb{Z}^{m\times n}$, an integral vector $b \in \mathbb{Z}^m$ and a rational vector $\hat{x} \in \mathbb{Q}^n$, is $\hat{x} \notin P_{\frac{1}{2}}(A,b)$?

Caprara & Fischetti (1996) show that M$0\frac{1}{2}$ is NP-complete. For the sake of completeness we state and prove their result below.

5.3.1 M$0\frac{1}{2}$ is NP-complete

This section follows closely (Caprara & Fischetti 1996, Sect. 3). Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ be integral and let $\hat{x} \in P(A,b)$. The vector $\hat{x}$ does not satisfy all $\{0,\frac{1}{2}\}$-cuts derived from A and b if and only if there exists some $\mu \in \{0,1\}^m$ with $\mu^T A \equiv 0 \pmod 2$ and $\mu^T b \equiv 1 \pmod 2$ such that the inequality $\mu^T(b - A\hat{x}) < 1$ is valid.

We will reduce M$0\frac{1}{2}$ to the problem of *decoding of linear codes* (Garey & Johnson 1979, p. 280). Here, one is given a matrix $Q \in \mathbb{Z}_2^{m\times n}$ and a vector $d \in \mathbb{Z}_2^m$, which together form a linear system $Qx = d$ over $\mathbb{Z}_2$. The problem is: Given Q, d and a natural number k, decide whether there exists a solution $\hat{x} \in \mathbb{Z}_2^n$ to the system $Qx = d$ with no more than k 1's. The NP-completeness of this decision problem immediately implies the NP-completeness of the following decision problem, by choosing $w = \mathbf{1}/(k+1)$.

Definition 5.3 (WCW). The *weighted codeword problem* is the following:

> Given a matrix $Q \in \{0,1\}^{r\times t}$, a vector $d \in \{0,1\}^r$ and a weight vector $w \in \mathbb{Q}^t_{\geq 0}$, decide whether there exists some $z \in \{0,1\}^t$ with
>
> $$Qz \equiv d \pmod 2 \text{ and } w^T z < 1.$$

We will see that one can reduce WCW to both M$0\frac{1}{2}$ and MEC, which implies that they are both NP-complete.

Theorem 5.4 (Caprara & Fischetti (1996)). M$0\frac{1}{2}$ *is* NP *complete.*

Proof. M$0\frac{1}{2}$ clearly is in NP. We show that WCW can be polynomially reduced to M$0\frac{1}{2}$.

For this let Q, d and w be an instance of WCW. Construct the following instance of M$0\frac{1}{2}$:

$$A = \left(\begin{array}{c|c} Q^T & \\ d^T & 2I_{t+1} \end{array} \right) \tag{5.1}$$

$$b = (2, \ldots, 2, 1)^T \tag{5.2}$$

$$\hat{x} = (\mathbf{0}^T, \mathbf{1}^T - \tfrac{1}{2}w^T, \tfrac{1}{2})^T, \tag{5.3}$$

where $\mathbf{0} = \{0\}^r$ and $\mathbf{1} = \{1\}^t$. Notice first that $\hat{x}$ is in $P(A,b)$ and observe that $b - A\hat{x} = (w_1, \ldots, w_t, 0)^T$. The point $\hat{x}$ does not satisfy all $\{0,\frac{1}{2}\}$-cuts derived from A and b if and only if there is a $\mu \in \{0,1\}^{t+1}$ with $\mu^T A \equiv 0 \pmod 2$, $\mu^T b \equiv 1 \pmod 2$ and $(w_1, \ldots, w_t, 0)\mu < 1$. In this case, the system forces the last entry of μ to be 1. Therefore the latter is satisfied if and only if there is a $z \in \{0,1\}^t$ with $Qz \equiv d \pmod 2$ and $w^T z < 1$, where z is to play the role $\mu^T = (z^T, 1)$. □

5.3.2 *MEC is NP-complete*

It will be shown that in the above reduction, the $\{0, \frac{1}{2}\}$-closure is in fact the elementary closure, so that the question, whether $\hat{x}$ is in the $\{0, \frac{1}{2}\}$-closure is the same as asking whether $\hat{x}$ is in the elementary closure. This establishes the NP-completeness of MEC via the same reduction of WCW to MEC.

The key is the following observation.

Lemma 5.5. *Let P be the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with A and b integral. If A is of the form $A = (C \mid 2I_m)$ for some integral matrix C, then $P' = P_{\frac{1}{2}}(A, b)$.*

Proof. Clearly $P_{\frac{1}{2}}(A, b) \supseteq P'$. For the reverse inclusion we simply show that each undominated Gomory-Chvátal cut of P is also a $\{0, \frac{1}{2}\}$-cut derived from the system (A, b). Recall from Lemma 3.1 that each undominated Gomory-Chvátal cut of P can be written as $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, where $\lambda^T A \in \mathbb{Z}^n$ and $\lambda \in [0, 1)^m$. However λ has to satisfy $\lambda^T 2I_m \in \mathbb{Z}^m$. Thus for $i = 1, \dots, m$ one has $2\lambda_i \in \mathbb{Z}$ and $0 \leq 2\lambda_i < 2$, i.e., $\lambda \in \{0, \frac{1}{2}\}^m$. □

Corollary 5.6. MEC *is* NP*-complete.*

Proof. We reduce WCW to MEC. Let Q, d and w be an instance of WCW. Construct an instance of MEC as given in the proof of Theorem 5.4. Since in this case $P_{\frac{1}{2}}(A, b) = P'$ the claim follows. □

Theorem 5.7. *If* P $\neq$ NP, *then optimizing over the elementary closure of a rational polyhedron cannot be done in polynomial time.*

Proof. If one could optimize over the elementary closure of a rational polyhedron in polynomial time, then one could also solve the separation problem for the elementary closure in polynomial time (see § 2.5), which is at least as hard as MEC. □

Hartmann, Queyranne & Wang (1999) give conditions under which an inequality has depth at most 1 and identify special cases for which they can test whether an inequality has rank at most 1. It follows from our results in this section that this cannot be done in general unless P = NP.

5.4 Minimizing the support of a cut

A Gomory-Chvátal cut $c^T x \leq \lfloor \delta \rfloor$ of P is *nontrivial*, if $\max\{c^T x \mid x \in P\} > \lfloor \delta \rfloor$. The *support* of a Gomory-Chvátal $c^T x \leq \lfloor \delta \rfloor$ is the minimal number of positive entries of a weight vector $\lambda \in \mathbb{R}^m_{\geq 0}$ with $\lambda^T A = c$ and $\lfloor \lambda^T b \rfloor = \lfloor \delta \rfloor$. It was recently suggested (Caprara, Fischetti & Letchford 2000, Letchford 1999) that nontrivial Gomory-Chvátal cuts with minimal support could be expected to be more effective. It is an application of the previous results that finding a Gomory-Chvátal cut with minimal support is NP-complete.

For this consider again an instance Q, d and k of the decoding of linear codes problem. The polyhedron $P(A, b)$ will be the same as in the proof of Theorem 5.4. Let $c^T x \leq \lfloor \delta \rfloor$ be a nontrivial Gomory-Chvátal cut, derived with the weight vector λ. Notice that λ can be recovered from c, since A has full row rank. Replacing λ by $\lambda - \lfloor \lambda \rfloor$ strengthens the cut and the number of positive entries does not increase. Therefore we can assume λ to be in $\{0, \frac{1}{2}\}^{t+1}$ as the proof of Lemma 5.5 suggests. We observe again, that the mapping $\pi_{t+1}(2\lambda)$ is 1-1 and onto into the solutions to the system $Qz \equiv d$.

Thus there exists a Gomory-Chvátal cut of support at most k if and only if there exists a solution z of the system $Qz \equiv d$ with at most k 1's. We summarize.

Proposition 5.8. *The following problem is* NP*-complete.*

> *Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Decide whether there exists a nontrivial Gomory-Chvátal cut of $P(A, b)$ of support at most k.*

6

The elementary closure in fixed dimension

6.1 Motivation

If the dimension n in the integer linear programming problem

$$\max\{c^T x \mid Ax \leq b,\, x \in \mathbb{Z}^n\}, \text{ where } A \in \mathbb{Z}^{m\times n} \text{ and } b \in \mathbb{Z}^m. \tag{6.1}$$

is fixed, then (6.1) becomes solvable in polynomial time (Lenstra 1983). Lenstra's algorithm decides whether a rational polyhedron $P(A, b)$ has empty integer hull or not. The integer programming problem can then be solved via binary search. In contrast to the case when P is centrally symmetric, i.e., $-x \in P$ whenever $x \in P$, where Minkowski's convex body theorem implies an upper bound on the volume of P if $P_I = \{\mathbf{0}\}$, P can have infinite volume and $P_I = \emptyset$. However a polyhedron $P \subseteq \mathbb{R}^n$ with empty integer hull has to be "flat" in some integral direction. More formally, let K be a convex body, i.e., a bounded, closed, full-dimensional and convex set and let $c \in \mathbb{R}^n$ be some vector. The *width of K along c* is the quantity

$$\max\{c^T x \mid x \in K\} - \min\{c^T x \mid x \in K\}$$

and the *width* of K is defined as the minimal width of K along any nonzero integral vector $c \in \mathbb{Z}^n$. The next theorem, called *flatness theorem*, is due to Khinchine (see (Kannan & Lovász 1988)).

Theorem 6.1. *There exists a function $f(n)$ depending only on the dimension n, such that each convex body $K \subseteq \mathbb{R}^n$ containing no integral vectors has width at most $f(n)$.*

This implies that the *integer feasibility problem*, which is: Given an integral system $Ax \leq b$, defining the rational polyhedron $P = P(A, b)$, decide whether $P_I = \emptyset$, is in $\mathrm{NP} \cap \mathrm{co} - \mathrm{NP}$ if n is fixed. This is because an integral vector in P must then lie in one of the constant number of lower dimensional polyhedra $P \cap (c^T x = \delta)$, where δ is an integer satisfying $\max\{c^T x \mid x \in P\} \geq \delta \geq \min\{c^T x \mid x \in P\}$ and where $0 \neq c \in \mathbb{Z}^n$ is a direction in which P is flat.

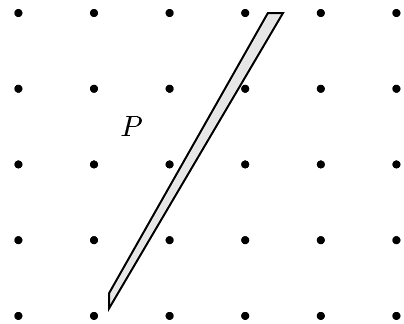

Figure 6.1: A polyhedron P with empty integer hull. P is flat in the direction $(-1, 1)$.

Lenstra's algorithm (Lenstra 1983) applies *lattice basis reduction*, and the *ellipsoid method* to find an integral point in P or a direction in which it is flat. Lovász & Scarf (1992) found a way to avoid the ellipsoid method. However, present algorithms for integer programming in fixed dimension are still far from being elementary.

Also there is a polynomiality result concerning the size of a defining system of the integer hull P_I of a rational polyhedron $P \subseteq \mathbb{R}^n$. Namely, the number of vertices of P_I is polynomially bounded in size(P), if the dimension n is fixed (Hayes & Larman 1983, Schrijver 1986, Cook, Hartmann, Kannan & McDiarmid 1992).

The Chvátal-Gomory procedure computes iteratively tighter approximations of the integer hull P_I of a polyhedron P, until P_I is finally obtained. We have seen in § 3.3 that the number of iterations t until $P^{(t)} = P_I$ is not polynomial in the size of the description of P, even in fixed dimension. Yet, if $P_I = \emptyset$ and $P \subseteq \mathbb{R}^n$, Cook, Coullard & Turán (1987) showed that there exists a number $t(n)$, such that $P^{(t(n))} = \emptyset$.

Theorem 6.2 (Cook, Coullard & Turán (1987)). *There exists a function $t(d)$, such that if $P \subseteq \mathbb{R}^n$ is a d-dimensional rational polyhedron with empty integer hull, then $P^{t(d)} = \emptyset$.*

Proof. If P is not full dimensional, then there exists a rational hyperplane $(c^T x = \delta)$ with $c \in \mathbb{Z}^n$ and $\gcd(c) = 1$ such that $P \subseteq (c^T x = \delta)$. If $\delta \notin \mathbb{Z}$, then $P' = \emptyset$. If $\delta \in \mathbb{Z}$, then there exists a unimodular matrix, transforming c into e_1. Thus P can be transformed via a unimodular transformation (see § 3.6) into a polyhedron where the first variable is fixed to an integer.

Thus we can assume that P is full-dimensional. The function $t(d)$ is inductively defined. Let $t(0) = 1$. For $d > 0$, let $c \in \mathbb{Z}^n$, $c \neq 0$ be a direction in which P is flat, i.e., $\max\{c^T x \mid x \in P\} - \min\{c^T x \mid x \in P\} \leq f(d)$. We "slice off" in this direction using Corollary 3.6. If $c^T x \leq \delta$, $\delta \in \mathbb{Z}$ is valid for P, then $c^T x \leq \delta - 1$ is valid for $P^{(t(d-1)+1)}$, since the face $F = P \cap (c^T x = \delta)$ has at most dimension $d-1$. Thus $c^T x \leq \delta - k$ is valid for $P^{(k\,(t(d-1)+1))}$. Since the integral vector c is chosen such that $\max\{c^T x \mid x \in P\} - \min\{c^T x \mid x \in P\} \leq f(d)$, $t(d) = (f(d)+2)(t(d-1)+1)$ satisfies our needs. □

Cook (1990) proved the existence of cutting plane proofs for integer infeasibility that can be carried out in polynomial space. These results raise the question whether it is possible to come up with a polynomial cutting plane algorithm for integer infeasibility in fixed dimension. Using binary search this would also yield a polynomial cutting plane algorithm for integer programming in fixed dimension.

In this context we are motivated to investigate the complexity of the elementary closure in fixed dimension. More precisely, we will study the question whether, in fixed dimension, the elementary closure P' of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, with A and b integer, can be defined by an inequality system whose size is polynomial in the size of A and b.

We have seen that P' can be described with an exponential number of inequalities in fixed dimension (see § 3.2 Remark 3.3). One can further restrict the cutting planes $c^T x \leq \lfloor \delta \rfloor$ to those corresponding to a *totally dual integral* system defining P (Edmonds & Giles 1977). A rational system $Ax \leq b$ is called *totally dual integral*, abbreviated TDI, if for each integral vector c, for which the minimum of the LP-duality equation

$$\max\{c^T x \mid Ax \leq b\} = \min\{y^T b \mid y \geq 0,\, y^T A = c\} \tag{6.2}$$

is finite, the minimum is attained at an integral optimal solution y. Giles & Pulleyblank (1979) showed that each rational polyhedron P can be represented by an integral TDI system. If P is given by an integral TDI system $Ax \leq b$, $A \in \mathbb{Z}^{m\times n}$, then P' is defined by $Ax \leq \lfloor b \rfloor$ (Schrijver 1980). This can be seen as follows. A Gomory-Chvátal cutting plane $c^T x \leq \lfloor \delta \rfloor$, with $\delta = \max\{c^T x | Ax \leq b\}$ can be derived as $(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor$with an integral $\lambda \geq 0$, since $Ax \leq b$ is a TDI system. But $\lfloor \lambda^T b \rfloor \geq \sum_{i=1}^m \lfloor \lambda_i b_i \rfloor \geq \sum_{i=1}^m \lambda_i \lfloor b_i \rfloor = \lambda^T \lfloor b \rfloor$. Thus each cut follows from the system $Ax \leq \lfloor b \rfloor$.

The number of inequalities of a minimal TDI-system defining a polyhedron P can still be exponential in the size of P, even in fixed dimension (Schrijver 1986, p. 317).

6.2 Outline

First we generalize a result of Hayes & Larman (1983) on the number of vertices of the integer hull of knapsack polyhedra so that it applies to general polyhedra. The possibility of such a generalization is mentioned in (Schrijver 1986, Cook, Hartmann, Kannan & McDiarmid 1992). By combining an observation concerning the number of simplices needed for a decomposition of P and the result of Cook, Hartmann, Kannan & McDiarmid (1992) we can prove an asymptotically better bound on the number of vertices of the integer hull of a rational polyhedron in fixed dimension than the one observed in (Cook, Hartmann, Kannan & McDiarmid 1992). Then we inspect the elementary closure of rational simplicial cones. We show that it can be described with polynomially many inequalities in fixed dimension. Via a triangulation argument, we prove a similar statement for arbitrary rational polyhedra. We show that the elementary closure of a rational polyhedron can be constructed in polynomial time in fixed dimension. This yields a polynomial algorithm that constructs a cutting plane proof of $\mathbf{0}^T x \leq -1$ for rational polyhedra P with empty integer hull. Based on these results, we then develop a polynomial algorithm in varying dimension for computing Gomory-Chvátal cutting planes of pointed simplicial cones. Our approach uses techniques from integer linear algebra like the Hermite and the Howell normal form of matrices. While the Hermite normal form has been applied to cut generation before (see e.g. (Hung & Rom 1990, Letchford 1999)), the cutting planes that we derive here are not only among those of maximal possible violation in a natural sense, but also belong to the polynomial description of P' in fixed dimension.

6.3 Vertices of the integer hull

If $P = P(A, b)$ is a rational polyhedron, then the number of extreme points of P_I can be polynomially bounded by size(P) in fixed dimension. This follows from a generalization of a result by Hayes & Larman (1983), see (Schrijver 1986, p. 256).

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, be a rational polyhedron where each inequality in $Ax \leq b$ has size at most φ. First, we can assume that P is full-dimensional since otherwise P is a face of a full-dimensional polyhedron of at most equal size. We want to estimate the number of integral vertices of P_I. Observe that we can assume that P is a polytope, since each vertex v of P_I satisfies $\|v\|_\infty \leq (n+1)\Delta$, where Δ is the maximal absolute value of the sub-determinants of the matrix $(A \mid b)$ (Proposition 2.12). We can impose this condition by adding $2n$-more inequalities

$$-(n+1)\Delta \leq x_i \leq (n+1)\Delta, \text{ for } i = 1, \dots, n. \tag{6.3}$$

Notice that the Hadamard bound (2.1) implies that the size of those inequalities is in $\mathrm{O}(\varphi)$ if n is fixed.

If we have a representation of the polytope P as the union of K n-simplices

$$P = \bigcup_{i \leq K} \Sigma_i, \tag{6.4}$$

then each vertex of P_I must be a vertex of the integer hull $(\Sigma_i)_I$ for some simplex Σ_i, $i \leq K$. The next lemma gives an upper bound on the minimal number K, such that P can be represented as the union of K simplices.

Lemma 6.3. *Let $P \subseteq \mathbb{R}^n$ be a d-dimensional polytope with m facets, where $d \geq 1$. Then P is the union of at most m^{d-1} d-simplices Σ. Each d-simplex Σ in this decomposition is spanned by vertices of P and barycenters $v = \frac{1}{k}\sum_{j=1}^{k} v_j$, $k \leq d+1$ of vertices $v_1, \ldots, v_k$ of P.*

Proof. The proof proceeds by induction on d. If $d = 1$, then P is a simplex itself. If $d > 1$, then P has $d+1$ affinely independent vertices $v_1, \ldots, v_{d+1}$. Consider the barycenter of these vertices $v = \frac{1}{d+1}\sum_{i=1}^{d+1} v_i$. Clearly v is in the relative interior of P and P is the union of the convex hulls of each facet F with v,

$$P = \bigcup_{F \text{ facet of } P} \operatorname{conv}(F \cup \{v\}). \tag{6.5}$$

A facet F of P is a $d-1$-dimensional polytope with at most $m-1$ facets. So, by induction, F is the union of at most $(m-1)^{d-2}$ simplices

$$F = \bigcup_{j \leq (m-1)^{d-2}} \Sigma_j^F. \tag{6.6}$$

Each simplex Σ_j^F in (6.6) is spanned by vertices of P and barycenters of at most d vertices of P, since each vertex of F is a vertex of P. Observe that

$$\operatorname{conv}(F \cup \{v\}) = \bigcup_{j \leq (m-1)^{d-2}} \operatorname{conv}(\Sigma_j^F \cup \{v\}). \tag{6.7}$$

The convex hull of the $d-1$-simplex Σ_j^F with v is a d-simplex. Therefore P is the union of at most $m\,(m-1)^{d-2} \leq m^{d-1}$ d-simplices which are spanned by vertices of P and barycenters of at most $d+1$-vertices of P. ☐

Summarizing the previous discussion, we have the following proposition.

Proposition 6.4. *If $P \subseteq \mathbb{R}^n$ is a rational d-dimensional polytope, where $d \geq 1$, defined by m inequalities, each of size at most φ, then P is the union of at most m^{d-1} simplices Σ_i, $i \leq m^{d-1}$, each of size $\mathrm{O}(\varphi)$, in fixed dimension n.*

Proof. Observe that the facet and vertex complexity are related via a multiplicative constant in Theorem 2.5 if the dimension n is fixed. In this case, the size of a barycenter $v = \frac{1}{k}\sum_{j=1}^{k} v_j$, of $k \leq n+1$ vertices $v_1, \ldots, v_k$ of P is also in $\mathrm{O}(\varphi)$. Thus the size of a d-simplex in the proof of Lemma 6.3 is in $\mathrm{O}(\varphi)$. ☐

Thus in order to show that the number of vertices of the integer hull of a rational polyhedron is polynomial in fixed dimension, we only need to derive such a bound where P is a full-dimensional rational simplex $\Sigma \subseteq \mathbb{R}^n$. We can further assume that $\mathbf{0}$ is a vertex of Σ. Otherwise we embed Σ into $\mathbb{R}^{n+1}$ as follows: Let $\Sigma = \operatorname{conv}(\{v_1, \dots, v_{n+1}\})$, then the embedding is defined as the simplex

$$\Sigma' = \operatorname{conv}\left(\left\{\mathbf{0}, \begin{pmatrix} 1 \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ v_{n+1} \end{pmatrix}\right\}\right). \tag{6.8}$$

So let $\Sigma \subseteq \mathbb{R}^n$ be a full-dimensional rational simplex with $\mathbf{0}$ being one of its vertices. A full dimensional simplex in $\mathbb{R}^n$ is defined by $n+1$ inequalities. Each choice of n inequalities in such a definition has linearly independent normal vectors, defining one of the vertices of Σ. Since $\mathbf{0}$ is one of the vertices, Σ is the set of all $x \in \mathbb{R}^n$ satisfying $Bx \geq 0$, $c^T x \leq \beta$, where $B \in \mathbb{Z}^{n\times n}$ is a nonsingular matrix, and $c^T x \leq \beta$ is an inequality. The inequality $c^T x \leq \beta$ can be rewritten as $a^T B x \leq \beta$, with $a^T = c^T B^{-1} \in \mathbb{Q}^n$. Let K be the knapsack polytope $K = \{x \in \mathbb{R}^n \mid x \geq 0,\, a^T x \leq \beta\}$. The vertices of Σ_I correspond exactly to the vertices of $\operatorname{conv}(K \cap \mathscr{L}(B))$.

Proposition 6.5. *Let $K \subseteq \mathbb{R}^n$ be a knapsack polytope given by the inequalities $x \geq 0$ and $a^T x \leq \beta$. Let $\mathscr{L}(B)$ be a lattice with integral and nonsingular $B \subseteq \mathbb{Z}^n$, then*

i. A vector $B\hat{x} \in \mathscr{L}(B)$ is a vertex of $\operatorname{conv}(K \cap \mathscr{L}(B))$ if and only if $\hat{x}$ is a vertex of the integer hull of the simplex Σ defined by $Bx \geq 0$ and $a^T B x \leq \beta$;

ii. if $v^{(1)}$ and $v^{(2)}$ are distinct vertices of $\operatorname{conv}(K \cap \mathscr{L}(B))$, then there exists an index $i \in \{1, \dots, n\}$ such that $\operatorname{size}(v_i^{(1)}) \neq \operatorname{size}(v_i^{(2)})$.

Proof. The convex hull of $K \cap \mathscr{L}(B)$ can be written as

$$\begin{aligned} \operatorname{conv}(K \cap \mathscr{L}(B)) &= \operatorname{conv}(\{x \mid x \geq 0,\, a^T x \leq \beta\,,\, x = By,\, y \in \mathbb{Z}^n) \\ &= \operatorname{conv}(\{By \mid By \geq 0,\, a^T By \leq \beta,\, y \in \mathbb{Z}^n\}). \end{aligned}$$

If one transforms this set with B^{-1}, one is faced with the integer hull of the described simplex Σ. Thus (i) follows.

For (ii) assume that $v^{(1)}$ and $v^{(2)}$ are vertices of $\operatorname{conv}(K \cap \mathscr{L}(B))$, with $\operatorname{size}(v_i^{(1)}) = \operatorname{size}(v_i^{(2)})$ for all $i \in \{1, \dots, n\}$. Then clearly $2v^{(1)} - v^{(2)} \geq 0$ and $2v^{(2)} - v^{(1)} \geq 0$. Also

$$a^T(2v^{(1)} - v^{(2)} + 2v^{(2)} - v^{(1)}) = a^T(v^{(1)} + v^{(2)}) \leq 2\beta,$$

therefore one of the two lattice points lies in K. Assume without loss of generality that $2v^{(1)} - v^{(2)} \in K \cap \mathscr{L}(B)$. Then $v^{(1)}$ cannot be a vertex since

$$v^{(1)} = 1/2\,(2v^{(1)} - v^{(2)}) + 1/2\, v^{(2)}.$$

□

If $K = \{x \in \mathbb{R}^n \mid x \geq 0, a^T x \leq \beta\}$ is the corresponding knapsack polytope to the simplex Σ, then any component $\hat{x}_i$, $i = 1, \dots, n$ of an arbitrary point $\hat{x}$ in K satisfies $0 \leq \hat{x}_i \leq \beta / a_i$. Thus the size of a vertex $\hat{x}$ of $\mathrm{conv}(K \cap \mathscr{L}(B))$ is in $\mathrm{O}(\mathrm{size}(K)) = \mathrm{O}(\mathrm{size}(\Sigma))$ in fixed dimension. This is because $\mathrm{size}(B^{-1}) = \mathrm{O}(\mathrm{size}(B))$ in fixed dimension. It follows from Proposition 6.5 that Σ_I can have at most $\mathrm{O}(\mathrm{size}(\Sigma)^n)$ vertices.

We can summarize.

Theorem 6.6. *If $P \subseteq \mathbb{R}^n$ is a rational polyhedron, then the number of vertices of P_I is polynomially bounded in* $\mathrm{size}(P)$ *when the dimension is fixed.*

The following upper bound on the number of vertices of P_I was proved by Cook, Hartmann, Kannan & McDiarmid (1992). Bárány, Howe & Lovász (1992) show that this bound is tight if P is a simplex.

Theorem 6.7. *If $P \subseteq \mathbb{R}^n$ is a rational polyhedron which is the solution set of a system of at most m linear inequalities whose size is at most φ, then the number of vertices of P_I is at most $2m^d(6n^2\varphi)^{d-1}$, where $d = \dim(P_I)$ is the dimension of the integer hull of P.*

This result yields an $\mathrm{O}(m^n \varphi^{n-1})$ upper bound on the number of vertices of P_I, where $P \subseteq \mathbb{R}^n$ is a rational polyhedron defined by at most m inequalities, each of size at most φ in fixed dimension. Interestingly, this bound is not tight.

Theorem 6.8. *If $P \subseteq \mathbb{R}^n$ is a rational polyhedron defined by m inequalities, each of size at most φ, then P_I has at most $\mathrm{O}(m^{n-1}\varphi^{n-1})$ vertices.*

Proof. Following the previous discussion we can again assume that P is a polytope. This involves the $2n$ additional equations (6.3) of size $\mathrm{O}(\varphi)$. P can then be described as the union of $\mathrm{O}(m^{n-1})$ simplices Σ, each of size $\mathrm{O}(\varphi)$. Theorem 6.7 implies that each simplex Σ in the decomposition of P has at most $\mathrm{O}(\varphi^{n-1})$ vertices. $\square$

6.4 The elementary closure of a rational simplicial cone

Consider a *rational simplicial cone*, i.e., a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^n$ and A has full row rank. If A is a square matrix, then P is called *pointed.*

Observe that P, P' and P_I are all full-dimensional. The elementary closure P' is given by the inequalities

$$(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor, \text{ where } \lambda \in [0,1]^n, \text{ and } \lambda^T A \in \mathbb{Z}^n. \tag{6.9}$$

Since P' is full-dimensional, there exists a unique (up to scalar multiplication) minimal subset of the inequalities in (6.9) that suffices to describe P'. These inequalities are the facets of P'. We will come up with a polynomial upper bound on their number in fixed dimension.

The vectors λ in (6.9) belong to the dual lattice $\mathscr{L}^*(A)$ of the lattice $\mathscr{L}(A)$. Recall that each element in $\mathscr{L}^*(A)$ is of the form μ/d, where $d = \det(\mathscr{L}(A))$ is the lattice determinant. It follows from the Hadamard inequality that size(d) is polynomial in size(A), even for varying n. Now (6.9) can be rewritten as

$$\frac{\mu^T A}{d}\, x \leq \left\lfloor \frac{\mu^T b}{d} \right\rfloor, \text{ where } \mu \in \{0, \dots, d\}^m, \text{ and } \mu^T A \in (d \cdot \mathbb{Z})^n. \tag{6.10}$$

Notice here that $\mu^T b/d$ is a rational number with denominator d. There are two cases: either $\mu^T b/d$ is an integer, or $\mu^T b/d$ misses the nearest integer by at least $1/d$. Therefore $\lfloor \mu^T b/d \rfloor$ is the only integer in the interval

$$\left[\frac{\mu^T b - d + 1}{d}, \frac{\mu^T b}{d}\right].$$

These observations enable us to construct a polytope Q, whose integral points will correspond to the inequalities (6.10). Let Q be the set of all (μ, y, z) in $\mathbb{R}^{2n+1}$ satisfying the inequalities

$$\begin{array}{rcl} \mu & \geq & 0 \\ \mu & \leq & d \\ \mu^T A & = & d\, y^T \\ (\mu^T b) - d + 1 & \leq & d\, z \\ (\mu^T b) & \geq & d\, z. \end{array} \tag{6.11}$$

If (μ, y, z) is integral, then $\mu \in \{0, \dots, d\}^n$, $y \in \mathbb{Z}^n$ enforces $\mu^T A \in (d \cdot \mathbb{Z})^n$ and z is the only integer in the interval $[(\mu^T b + 1 - d)/d,\ \mu^T b/d]$. It is not hard to see that Q is indeed a polytope. We call Q the *cutting plane polytope* of the simplicial cone $P(A, b)$

The correspondence between inequalities (their syntactic representation) in (6.10) and integral points in the cutting plane polytope Q is obvious. We now show that the facets of P' are among the vertices of Q_I.

Proposition 6.9. *Each facet of P' is represented by an integral vertex of Q_I.*

Proof. Consider a facet $c^T x \leq \delta$ of P'. If we remove this inequality (possibly several times, because of scalar multiples) from the set of inequalities in (6.10), then the polyhedron defined by the resulting set of inequalities differs from P', since P' is full-dimensional. Thus there exists a point $\hat{x} \in \mathbb{Q}^n$ that is violated by $c^T x \leq \delta$, but satisfies any other inequality in (6.10) (see Figure 6.2). Consider the following integer program:

$$\max\{(\mu^T A/d)\, \hat{x} - z \mid (\mu, y, z) \in Q_I\}. \tag{6.12}$$

Since $\hat{x} \notin P'$ there exists an inequality $(\mu^T A/d)x \leq \lfloor \mu^T b/d \rfloor$ in (6.10) with

$$(\mu^T A/d)\hat{x} - \lfloor \mu^T b/d \rfloor > 0.$$

Therefore, the optimal value will be strictly positive, and an integral optimal solution (μ, y, z) must correspond to the facet $c^T x \leq \delta$ of P'. Since the optimum of the integer linear program (6.12) is attained at a vertex of Q_I, the assertion follows. □

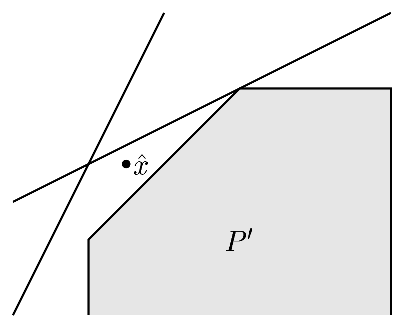

Figure 6.2: The point $\hat{x}$ lies "above" the facet $c^T x \leq \delta$ and "below" each other inequality in (6.10).

Remark 6.10. *Not each vertex of Q_I represents a facet of P'. In particular, if P is defined by nonnegative inequalities only, then $\mathbf{0}$ is a vertex of Q_I but not a facet of P'.*

Theorem 6.11. *The elementary closure of a rational simplicial cone $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where A and b are integral and A has full row rank, is polynomially bounded in* size(P) *when the dimension is fixed.*

Proof. Each facet of P' corresponds to a vertex of Q_I by Proposition 6.9. Recall from the Hadamard bound that $d \leq \|a_1\| \cdots \|a_n\|$, where a_i are the columns of A. Thus the number of bits needed to encode d is in $\mathrm{O}(n \text{ size}(P))$. Therefore the size of Q is in $\mathrm{O}(n \text{ size}(P))$. It follows from Theorem 6.7 that the number of vertices of Q_I is in $\mathrm{O}(\text{size}(P)^n)$ for fixed n, since the dimension of Q is $n + 1$. □

It is possible to explicitly construct in polynomial time a minimal inequality system defining P' when the dimension is fixed.

Observe first that the lattice determinant d in (6.11) can be computed with some polynomial Hermite normal form algorithm. If H is the HNF of A, then $\mathscr{L}(A) = \mathscr{L}(H)$ and the determinant of H is simply the product of its diagonal elements. Notice then that the system (6.11) can be written down. In particular its size is polynomial in the size of A and b, even in varying dimension, which follows from the Hadamard bound.

As noted in (Cook, Hartmann, Kannan & McDiarmid 1992), one can construct the vertices of Q_I in polynomial time. This works as follows. Suppose one has a list of vertices $v_1, \dots, v_k$ of Q_I. Let Q_k denote the convex hull of these vertices. Find an inequality description of Q_k, $Cx \leq d$. For each row-vector c_i of C, find with Lenstra's algorithm a vertex of Q_I maximizing $\{c^T x \mid x \in Q_I\}$. If new vertices are found, add them to the list and repeat the preceding steps, otherwise the list of vertices is complete. The list of vertices of Q_I yields a list of inequalities defining P'. With the ellipsoid method or your favorite linear programming algorithm in fixed dimension, one can decide for each individual inequality, whether it is necessary. If not, remove it. What remains are the facets of P'.

Proposition 6.12. *There exists an algorithm which, given a matrix $A \in \mathbb{Z}^{m \times n}$ of full row rank and a vector $b \in \mathbb{Z}^m$, constructs the elementary closure P' of $P(A, b)$ in polynomial time when the dimension n is fixed.*

6.5 The elementary closure of rational polyhedra

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, with integral A and b, be a rational polyhedron.

Any Gomory-Chvátal cut can be derived from a set of $\mathrm{rank}(A)$ inequalities out of $Ax \leq b$ where the corresponding rows of A are linear independent. Such a choice represents a simplicial cone C and it follows from Theorem 6.11 that the number of inequalities of C' is polynomially bounded by $\mathrm{size}(C) \leq \mathrm{size}(P)$.

Theorem 6.13. *The number of inequalities needed to describe the elementary closure of a rational polyhedron $P = P(A, b)$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, is polynomial in $\mathrm{size}(P)$ in fixed dimension.*

Proof. An upper bound on the number of inequalities that are necessary to describe P' follows from the sum of the upper bounds on the number of facets of C' where C is a simplicial cone, formed by $\mathrm{rank}(A)$ inequalities of $Ax \leq b$. There are at most $\binom{m}{\mathrm{rank}(A)} \leq m^n$ ways to choose $\mathrm{rank}(A)$ linear independent rows of A. Thus the number of necessary inequalities describing P' is $\mathrm{O}(m^n \, \mathrm{size}(P)^n)$ for fixed n. $\square$

Following the discussion at the end of Section 6.4 and using again Lenstra's algorithm, it is now easy to come up with a polynomial algorithm for constructing the elementary closure of a rational polyhedron $P(A, b)$ in fixed dimension. For each choice of $\mathrm{rank}(A)$ rows of A defining a simplicial cone C, compute the elementary closure C' and put the corresponding inequalities in the partial list of inequalities describing P'. At the end, redundant inequalities can be deleted.

Theorem 6.14. *There exists a polynomial algorithm that, given a matrix $A \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$, constructs an inequality description of the elementary closure of $P(A,b)$.*

6.6 Cutting plane proofs of $\mathbf{0}^T x \leq -1$

If the rational polyhedron P has empty integer hull, then Theorem 6.2 together with Proposition 3.9 implies the existence of a cutting plane proof of $\mathbf{0}^T x \leq -1$ which has constant length in fixed dimension. This was observed by Cook, Coullard & Turán (1987). Their result is only of existential nature. It follows from our results that one can construct a cutting plane proof of $\mathbf{0}^T x \leq -1$ whose length can be bounded according to (ii) in Proposition 3.9.

Theorem 6.15. *For fixed n, there exists a polynomial algorithm which computes a cutting plane proof of $\mathbf{0}^T x \leq -1$ of length bounded $(n+1)(n^t-1)/(n-1)+1$ if its input is a matrix $A \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$ defining a rational polyhedron $P = P(A,b)$ with empty integer hull and Chvátal rank t.*

Proof. Since t is a constant in fixed dimension, one can construct integral inequality descriptions $C_1 x \leq d_1, \ldots, C_t x \leq d_t$, of $P^{(1)}$, $P^{(2)}, \ldots, P^{(t)}$ with the algorithm proposed in Theorem 6.14. Each inequality in the system $C_i x \leq d_i$ was derived from at most n inequalities from the previous system $C_{i-1} x \leq d_{i-1}$ for $i = 2, \ldots, n$. As one constructs $C_i x \leq d_i$, one remembers the parents of each inequality. An inequality from the last system $C_t x \leq d_t$ thus has a cutting plane proof of length at most $1 + n + \ldots + n^{t-1} = (n^t-1)/(n-1)$ (recall that the original inequalities in $Ax \leq b$ do not contribute to the length of the proof) which can be computed by backtracking the parents. Using linear programming, one can find at most $n+1$ inequalities from the system $C_t x \leq d_t$, from which $\mathbf{0}^T x \leq -1$ can be derived. The concatenations of the cutting plane proofs of these inequalities and $\mathbf{0}^T x \leq -1$ is the desired proof. □

6.7 Finding cuts for simplicial cones

In § 6.4 we saw that the vertices of Q_I include the facets of the elementary closure P' of a simplicial cone $P(A,b)$. In practice the following situation often occurs. The matrix A is invertible and one wants to find a cutting plane that cuts of the extreme point of the pointed cone P, $\hat{x} = A^{-1}b$. It is easy to see that the scenario of Gomory's corner polyhedron (Gomory 1967) (see also (Schrijver 1986, p. 364)) is of this nature. We shortly describe it. As the method of choice for solving linear relaxations is most likely the simplex method, one is faced with an integer programming problem in standard form

$$\max\{c^T x \mid Ax = b,\ x \geq 0,\ x \text{ integral}\}, \tag{6.13}$$

where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Clearly one can assume that A has full row rank. An optimal solution $\hat{x}$ to the linear relaxation of (6.13) is characterized by a set $B \subseteq \{1, \dots, n\}$ corresponding to m linearly independent columns of A, called a *basis*. Without loss of generality assume that B corresponds to the first m columns of A. Let $N = \{m+1, \dots, n\}$ be the index set corresponding to the variables which do not belong to the basis B. We also use B and N to denote the matrices corresponding to the first m columns of A and the last $n - m$ columns of A respectively, i.e., $A = (B \mid N)$. Then $\hat{x}$ is of the form

$$\hat{x} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}. \tag{6.14}$$

The point $\hat{x}$ also is the optimum to the linear program

$$\max\{c^T x \mid Ax = b,\, x_N \geq 0\}. \tag{6.15}$$

Then consider the integer program resulting from (6.15).

$$\max\{c^T x \mid Ax = b,\, x_N \geq 0,\, x \text{ integral}\}. \tag{6.16}$$

Compared to (6.13) one has dropped thus the nonnegativity of the basis variables. The integer programming problem (6.16) is an upper bound to (6.13) which one can use in a branch-and-cut framework. The polyhedron described in (6.15) is a pointed simplicial cone in an affine subspace of $\mathbb{R}^n$. Via unimodular transformations, one can translate this integer programming problem (6.15) into an integer programming problem over a pointed simplicial cone.

In this section, we will show how to generate cutting planes for pointed simplicial cones. Following § 6.4, they will have the special property that they correspond to vertices of the integer hull of the cutting plane polytope Q and thus belong to a family of inequalities which grows only polynomially in fixed dimension. While the separation problem for the elementary closure is NP-hard (see § 5) in general, these cutting planes can be computed in polynomial time in varying dimension.

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a rational pointed simplicial cone, where $A \in \mathbb{Z}^{n \times n}$ and $b \in \mathbb{Z}^n$. Let $d = |\det(A)|$ denote the absolute value of the determinant of A. Let Q be the cutting plane polytope of P defined by the inequalities in (6.11). We will find a face-defining inequality of Q_I that represents the cutting planes with a maximal rounding effect. This relates to the study of maximally violated mod k-cuts by Caprara, Fischetti & Letchford (2000). A cutting plane

$$(\mu/d)^T Ax \leq \lfloor (\mu/d)^T b \rfloor$$

can be found by solving the following linear system over $\mathbb{Z}_d$,

$$\mu^T (A \mid b) = (0, \dots, 0, \nu), \tag{6.17}$$

where ν/d for $\nu \in \{0, \dots, d-1\}$ is the desired value for the rounding effect $(\mu^T b)/d - \lfloor(\mu^T b)/d\rfloor$. If P is a simplicial cone, then this rounding effect is the *amount of violation* of the cutting plane by the extreme point $\hat{x}$ of P. Caprara, Fischetti & Letchford (2000) fix ν in the system (6.17) to the maximal possible value $d-1$. However, there does not have to exist a solution to (6.17) when ν is set to $d-1$. We show here that the maximal ν, denote it by $\nu_{\max}$, for which a solution to (6.17) exists, can be computed efficiently.

For this we have to reach a little deeper into the linear algebra tool-box. In the following we will make extensive use of the Hermite and Howell normal form of an integer matrix. The Hermite normal form belongs to the standard tools in integer programming. Hung & Rom (1990) for example use a variant of the Hermite normal form to generate cutting planes of simplicial cones P, such that the outcome $\tilde{P}$ has in integral vertex. Letchford (1999) uses the Hermite normal form to cut off the minimal face of a simplicial cone $P(A, b)$. We use the Hermite normal form because it allows us to represent the image and kernel of matrices $A \in \mathbb{Z}_d^{m \times n}$ in a convenient way. Notice that $\mathbb{Z}_d$ is not a field if d is not a prime. Therefore, standard Gaussian elimination does not apply for these tasks in general.

6.7.1 The Howell and Hermite normal form

Let us study the column-span of a matrix $B \in \mathbb{Z}_d^{m \times n}$

$$\operatorname{span}(B) = \{x \in \mathbb{Z}_d^m \mid \exists y \in \mathbb{Z}_d^n,\, By = x\}.$$

The column-span of an integral matrix $B \in \mathbb{Z}^{m \times n}$ is defined accordingly. We write $\operatorname{span}_{\mathbb{Z}_d}(B)$ and $\operatorname{span}_{\mathbb{Z}}(B)$ to distinguish if necessary. The span of an empty set of vectors is the submodule $\{\mathbf{0}\}$ of $\mathbb{Z}_d^m$.

Consider the set of vectors $S(i) \subseteq \operatorname{span}(B)$, $i = 0, \dots, m$, whose first i components are 0. Clearly $S(i)$ is a $\mathbb{Z}_d$-submodule of $\operatorname{span}(B)$. We say that a nonzero matrix B is in *canonical form* if

i. B has no zero column, i.e., a column containing zeroes only,

ii. B is in *column-echelon form*, i.e., if the first occurrence of a nonzero entry in column j is in row i_j, then $i_j < i_{j'}$, whenever $j < j'$ (the columns form a staircase "downwards"),

iii. $S(i)$ is generated by the columns of B belonging to $S(i)$.

Notice that if d is a prime, then (iii) is automatically satisfied, since $\mathbb{Z}_d$ has no zero-divisors.

Example 6.16. *Consider the matrix $B = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ in $\mathbb{Z}_4$. Clearly B satisfies the conditions (i) and (ii). But B does not satisfy the condition (iii), since the vector $\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$ is in $\mathrm{span}_{\mathbb{Z}_4}(B)$ but not in the column-span of those column vectors of B that belong to $S(1)$, since there are none. A canonical form of this matrix would be the matrix $\widetilde{B} = \left(\begin{smallmatrix} 2 & 0 \\ 3 & 2 \end{smallmatrix}\right)$*

We now motivate this concept in the context of the decision problem, whether a vector belongs to the column-span of a matrix in canonical form or not. If $B \in \mathbb{Z}_d^{m \times n}$ is in canonical form and $y \in \mathbb{Z}_d^m$ is given, then it is easy to decide whether $y \in \mathrm{span}_{\mathbb{Z}_d}(B)$. For this, let i be the number of leading zeroes of y. Clearly $y \in \mathrm{span}_{\mathbb{Z}_d}(B)$ if and only if $y \in S(i)$. Conditions ii) and iii) imply that if $y \in S(i)$, then there exists a unique column b of B with exactly i leading zeroes and

$$b_{i+1} \cdot x = y_{i+1} \tag{6.18}$$

being a solvable equation in $\mathbb{Z}_d$. It is an elementary number theory task to decide, whether such an x exists and if so to find one (see e.g. (Niven, Zuckerman & Montgomery 1991, p. 62)). Now subtract $x\, b_{i+1}$ times column b from y. The result is in $S(i+1)$. One proceeds until the outcome is in $S(n)$, which implies that $y \in \mathrm{span}_{\mathbb{Z}_d}(B)$, or the conditions discussed above fail to hold, which implies that $y \notin \mathrm{span}_{\mathbb{Z}_d}(B)$.

Storjohann & Mulders (1998) show how to compute a canonical form of a matrix A with $\mathrm{O}(mn^{\omega-1})$ basic operations in $\mathbb{Z}_d$, where $\mathrm{O}(n^\omega)$ is the time required to multiply two $n \times n$ matrices. The number ω is less then or equal to 2.37 as found by Coppersmith & Winograd (1990). In the rest of this chapter, we use the O-notation to count basic operations in $\mathbb{Z}_d$ like addition, multiplication, or (extended)-gcd computation of numbers in $\{0, \dots, d-1\}$. The bit-complexity of a basic operation in $\mathbb{Z}_d$ is $\mathrm{O}(\mathrm{size}(d) \log \mathrm{size}(d) \log\log \mathrm{size}(d))$ as found by Schönhage & Strassen (1971) (see also (Aho, Hopcroft & Ullman 1974)). Recall that $\mathrm{size}(d) = \mathrm{O}(n\ \mathrm{size}(A))$.

Storjohann & Mulders (1998) give Howell (1986) credit for the first algorithm and the introduction of the canonical form and call it *Howell normal form.* However, there is a simple relation to the *Hermite normal form.*

Proposition 6.17. *Let $A \in \mathbb{Z}_d^{m \times n}$be a nonzero matrix and let H be the Hermite normal form of $(A \mid d \cdot I)$ where $(A \mid d \cdot I)$ is interpreted as an integer matrix. Then a canonical form of A is the matrix H' which is obtained from H by deleting the columns $h^{(i)}$ with $h_{i,i} = d$ (notice that $h_{i,i} \mid d$).*

Proof. Clearly, $\mathrm{span}_{\mathbb{Z}_d}(H') \subseteq \mathrm{span}_{\mathbb{Z}_d}(A)$ and H' is in column-echelon form. We need to verify iii). Let $u \in \mathrm{span}_{\mathbb{Z}_d}(A)$ with $u \in S(i)$, where i is maximal. Property iii) is guaranteed if $i = m$. If $i < m$, then $u_{i+1} \neq 0$. Interpreted over $\mathbb{Z}$, this means that $0 < u_{i+1} < d$. Clearly $u \in \mathrm{span}_{\mathbb{Z}}(H)$, and since $u_{i+1} \in h_{i+1,i+1} \cdot \mathbb{Z}$ (recall that H is a lower triangular matrix with nonzero diagonal elements and that u_{i+1} is the first nonzero entry of u), it follows that the column $h^{(i+1)}$ appears in H'. After subtracting $u_{i+1}/h_{i+1,i+1}$ times the column $h^{(i+1)}$ from u, the result will be in $S(i+1)$ and, by induction, the result will be in the span of the columns of H' belonging to $S(i+1)$. All together we see that u is in the span of the vectors of H' belonging to $S(i)$. □

It is now easy to see that the canonical forms of a matrix A have a unique representative B that, using the notation of ii), satisfies the following additional conditions that we will assume for the rest of the chapter:

iv. the elements of row i_j are reduced modulo $b_{i_j,j}$ (interpreted over the integers) and

v. the natural number $b_{i_j,j}$ divides d.

6.7.2 Determining the maximal amount of violation

We now apply the canonical form to determine the maximal amount of violation $\nu_{\max}/d$. Notice that $P \neq P_I$ if and only if there exists a $\nu \neq 0$ such that (6.17) has a solution. If $(A \mid b)^T$ consist in $\mathbb{Z}_d$ of zeroes only, then $P = P_I$. Otherwise let H be the canonical form of $(A \mid b)^T$, which can be found with $\mathrm{O}(n^\omega)$ basic operations in $\mathbb{Z}_d$ (Storjohann & Mulders 1998). Since $P \neq P_I$, the last column of H is of the form $(0, \ldots, 0, g)^T$, for some $g \neq 0$. The ideal $\langle g \rangle \trianglelefteq \mathbb{Z}_d$ generated by g is exactly the set of ν such that (6.17) is solvable for μ. Since $g \mid d$, the largest $\nu \in \{1, \ldots, d-1\} \cap \langle g \rangle$ is

$$\nu_{\max} = d - g.$$

Thus we can compute $\nu_{\max}$ in $\mathrm{O}(n^\omega)$ basic operations in $\mathbb{Z}_d$ and the inequality

$$(b^T/d, \mathbf{0}^T, -1)(\mu, y, z) = \mu^T b/d - z \leq \nu_{\max}/d \tag{6.19}$$

will be valid for Q_I, defining a nonempty face of Q_I,

$$F = (Q_I \cap (\mu^T b/d - z = \nu_{\max}/d)). \tag{6.20}$$

Theorem 6.18. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a rational simplicial cone, where $A \in \mathbb{Z}^{n\times n}$ is of full rank, $b \in \mathbb{Z}^n$ and $d = |\det(A)|$. Then one can compute in $\mathrm{O}(n^\omega)$ basic operations of $\mathbb{Z}_d$ the maximal possible amount of violation $\nu_{\max}/d$. Here, $\nu_{\max}$ is the maximum number $\nu \in \{0, \ldots, d-1\}$ for which there exists a cutting plane $(\mu/d)^T Ax \leq \lfloor(\mu^T b)/d\rfloor$ separating $A^{-1}b$ with $(\mu^T b)/d - \lfloor(\mu^T b)/d\rfloor = \nu/d$.*

6.7.3 Computing vertices of Q_I

We proceed by computing a vertex of F, which will also be a vertex of Q_I. First we find in $O(n^\omega)$ basic operations of $\mathbb{Z}_d$, a solution $\hat{\mu}$ to

$$\mu^T(A \mid b) = (0, \dots, 0, \nu_{\max}). \tag{6.21}$$

Let $K \in \mathbb{Z}_d^{n \times k}$ represent the kernel of $(A \mid b)^T$, i.e.,

$$\operatorname{span}_{\mathbb{Z}_d}(K) = \{x \in \mathbb{Z}_d^n \mid x^T(A \mid b) = (0, \dots, 0)\}.$$

The canonical form of K again can be computed in time $O(n^\omega)$ (Storjohann & Mulders 1998). The solution set of (6.21) is the set of vectors

$$\mathscr{S} = \{\hat{\mu} + \bar{\mu} \mid \bar{\mu} \in \operatorname{span}_{\mathbb{Z}_d}(K)\}. \tag{6.22}$$

Notice that $\mathscr{S}$ is the set of integral vectors in F. Vertices of Q_I will be obtained as minimal elements of $\mathscr{S}$ with respect to some ordering on $\mathscr{S}$. For $i = 1, \dots, n$ and a permutation σ of $\{1, \dots, n\}$, we define a quasi-ordering $\leq_\sigma^i$ on $\mathscr{S}$ by

$$\mu \leq_\sigma^i \tilde{\mu} \quad \text{iff} \quad (\mu_{\sigma(1)}, \dots, \mu_{\sigma(i)}) \leq_{\text{lex}} (\tilde{\mu}_{\sigma(1)}, \dots, \tilde{\mu}_{\sigma(i)}).$$

Here, $\leq_{lex}$ denotes the *lexicographic ordering* on $\{0, \dots, d-1\}^i$, i.e., $u \leq_{lex} v$ if $u = v$ or the leftmost nonzero entry in the vector difference $v - u$ is positive. The lexicographic ordering is a total order.

Proposition 6.19. *If $\mu \in \mathscr{S}$ is minimal with respect to $\leq_\sigma^n$, then (μ, y, z) is a vertex of Q_I, where y and z are determined by μ according to (6.11).*

Proof. Assume without loss of generality that $\sigma = \text{id}$. Let $\mu \in \mathscr{S}$ be minimal with respect to $\leq_\sigma^n$ and suppose that $\mu = \sum_{j=1,\dots,l} \alpha_j \mu^{(j)}$ is a convex combination of vertices of Q_I, where each $\mu^{(j)} \neq \mu$ and $\alpha_j > 0$. Clearly, each $\mu^{(j)}$ is in $\mathscr{S}$. Therefore, there exists an index $i \in \{1, \dots, n\}$ such that $\mu_i \leq \mu_i^{(j)}$, for all $j \in \{1, \dots, l\}$, and $\mu_i < \mu_i^{(j)}$, for some $j \in \{1, \dots, l\}$. Since $\alpha_j \geq 0$ and $\sum_{i=1,\dots,l} \alpha_j = 1$, we have $\sum_{j=1,\dots,l} \alpha_j \mu_i^{(j)} > \mu_i$, a contradiction. □

We now show how to compute a minimal element $\mu \in \mathscr{S}$ with respect to $\leq_\sigma^n$. For simplicity we assume that $\sigma = \text{id}$, but the algorithm works equally well for any other permutation. For $\mu \in \mathscr{S}$, we call $(\mu_1, \dots, \mu_i)$ the *i-prefix* of μ. We will construct a sequence $\mu^{(i)}, i = 0, \dots, n$, of elements of $\mathscr{S}$ with the property that the i-prefix of $\mu^{(i)}$ is minimal among all i-prefixes of elements in $\mathscr{S}$ with respect to the $\leq_{\text{lex}}$ order. Since $\leq_{\text{lex}}$ is a total order, the i-prefix of $\mu^{(i)}$ is unique and the i-prefix of $\mu^{(j)}$ is the i-prefix of $\mu^{(i)}$, for all $j \geq i$. In other words, the j-prefix of $\mu^{(j)}$ coincides with the i-prefix of $\mu^{(i)}$ except possibly in the last $(j-i)$ components.

Define $K(i) \subseteq \mathrm{span}_{\mathbb{Z}_d}(K)$ as the $\mathbb{Z}_d$-submodule of $\mathrm{span}_{\mathbb{Z}_d}(K)$ consisting of those elements having a zero in their first i components. For $j \geq i$, the vector $\mu^{(j)}$ is obtained from $\mu^{(i)}$ by adding an element of $K(i)$. Suppose that K is in canonical form and let $K^{(i)}$ be the submatrix of K consisting of those columns of K that lie in $K(i)$. Notice that $K^{(i)}$ is in canonical form, too, and that $\mathrm{span}_{\mathbb{Z}_d}(K^{(i)}) = K(i)$.

We initialize $\mu^{(0)}$ with an arbitrary element of $\mathscr{S}$. Suppose we have constructed $\mu^{(i)}$. By the preceding discussion, $\mu^{(i+1)}$ is of the form $\mu^{(i)} + \mu$, for some $\mu \in K(i)$. We have to take care of the $(i+1)$-st component. Let κ be the first column of $K^{(i)}$ and let g be the $(i+1)$-st component of κ. If $g = 0$, then $\mu^{(i)}$ is minimal with respect to $\leq^{i+1}$. Otherwise the smallest component that we can get in the $(i+1)$-st position is is the least positive remainder r of the division of $\mu^{(i)}_{i+1}$ by g (remember that $g \mid d$). We have $\mu^{(i)}_{i+1} = qg + r$ with an appropriate natural number q and some $r \in \{1, \ldots, g-1\}$. Thus, by subtracting $q\kappa$ from $\mu^{(i)}$, we obtain a vector $\mu^{(i+1)}$ that is minimal with respect to $\leq^{i+1}$. Notice that the computation of $\mu^{(i+1)}$ from $\mu^{(i)}$ involves $\mathrm{O}(n)$ elementary operations in $\mathbb{Z}_d$. Repeating this construction n times we get the following theorem.

Theorem 6.20. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a rational simplicial cone, where $A \in \mathbb{Z}^{n \times n}$ is of full rank, $b \in \mathbb{Z}^n$ and $d = |\det(A)|$. Then one can compute in $\mathrm{O}(n^\omega)$ basic operations of $\mathbb{Z}_d$ a vertex of Q_I corresponding to a cutting plane $(\mu/d)^T Ax \leq \lfloor (\mu/d)^T b \rfloor$ separating $A^{-1}b$ with maximal possible amount of violation $\nu_{\max}/d$.*

In practice one would want to generate several cutting planes for P. Here is a simple heuristic to move from one cutting plane corresponding to a vertex of Q_I to the next. If one has computed some $\mu \in \mathscr{S}$ then it can be easily checked, whether a component of μ can be individually decreased. This works as follows. Suppose we are interested in the i-th component μ_i. Compute the standard generator g of the ideal of the i-th components of $\mathrm{span}_{\mathbb{Z}_d}(K)$. Recall that $g \mid d$. Now μ_i can be individually decreased, if $g < \mu_i$. In this case we swap rows i and 1 of K and components i and 1 of μ and proceed as discussed in the previous paragraph. This "swapping" corresponds to another permutation. It results in a new order $\leq_\sigma$ and a new vertex of Q_I.

Summary

In this thesis we study a prominent approach to integer programming, the so-called cutting plane method. A Gomory-Chvátal cutting plane (Gomory 1958, Chvátal 1973a) for a polyhedron P is an inequality $c^T x \leq \lfloor \delta \rfloor$, where c is an integral vector and $c^T x \leq \delta$ is valid for P, i.e., the halfspace defined by $c^T x \leq \delta$ contains P. The cutting plane $c^T x \leq \lfloor \delta \rfloor$ is valid for all integral points in P and thus for the convex hull of integral vectors in P, the integer hull P_I. The addition of a cutting plane to the system of inequalities defining P results in a better approximation of the integer hull. The intersection of a polyhedron with all its Gomory-Chvátal cutting planes is called the elementary closure P' of P. If P is rational, then P' is a rational polyhedron again. Schrijver (1980) showed that the successive application of the elementary closure operation to a rational polyhedron yields the integer hull of the polyhedron after a finite number of steps. Chvátal (1973a) observed this for polytopes. This successive application of the elementary closure operation is referred to as the Chvátal-Gomory procedure. The minimal number of rounds until P_I is obtained is the Chvátal rank of P. We observe that the finiteness of the Chvátal rank of rational polyhedra can also be derived from Gomory's original algorithmic result (Gomory 1958). A similar observation was made by Schrijver (1986) for polyhedra in the positive orthant.

Even in two dimensions, one can construct polytopes of arbitrary large Chvátal rank. Integer programming formulations of combinatorial optimization problems are most often polytopes in the 0/1 cube. Therefore we study the Chvátal rank of polytopes that are contained in the 0/1 cube. First we investigate rational polytopes in the n-dimensional 0/1 cube that do not contain integral points. It turns out that their Chvátal rank can essentially be bounded by their dimension. Then we study polytopes with nonempty integer hull. For this we have to consider the facet complexity of integral 0/1 polytopes. We obtain a first upper bound on the Chvátal rank of polytopes in the n-dimensional 0/1 cube of $\mathrm{O}(n^3 \log n)$ by scaling the facet defining vectors of P_I. A more sophisticated application of scaling eventually leads to an $\mathrm{O}(n^2 \log n)$ upper bound. We then present a

family of polytopes in the n-dimensional 0/1-cube whose Chvátal rank is at least $(1+\epsilon)\,n$, for some $\epsilon > 0$. This improves the known lower bound n. So if $\mathrm{rank}(n)$ denotes the maximum Chvátal rank over all polytopes that are contained in $[0,1]^n$, then it is shown that $(1+\epsilon)\,n \leq \mathrm{rank}(n) \leq 3\,n^2\,\mathrm{size}(n)$.

In combinatorial optimization, cutting planes are often derived from the structure of the problem. But even then they most likely fit in the Gomory-Chvátal cutting plane framework. A polynomial separation routine for the elementary closure of a rational polyhedron would thus be a very powerful tool. Schrijver posed the existence of such an algorithm as an open problem in his book (Schrijver 1986). We give a negative answer to this question by showing that the separation problem for the elementary closure of a polyhedron is NP-hard.

Not much was known about the polyhedral structure of the elementary closure in general. In essence one has the following result (see, e.g. (Cook, Cunningham, Pulleyblank & Schrijver 1998)): If P is defined as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, then P' is the intersection of P with all Gomory-Chvátal cutting planes $c^T x \leq \lfloor \delta \rfloor$, $c \in \mathbb{Z}^n$, where $c^T = \lambda^T A$ with some $\lambda \in [0,1)^m$ and $\delta = \max\{c^T x \mid x \in P\}$. The infinity norm $\|c\|_\infty$ of any such vector $c = A^T\lambda$ from above can be estimated as follows: $\|c\|_\infty = \|A^T\lambda\|_\infty \leq \|A^T\|_\infty$. From this, only an exponential (in the input encoding of P) upper bound $\|A^T\|_\infty^n$ on the number of inequalities needed to describe P' can be derived. This is also exponential in fixed dimension n. Integer programming in fixed dimension is solvable in polynomial time (Lenstra 1983). There is also a polynomiality result concerning the size of a defining system of the integer hull P_I of a rational polyhedron $P \subseteq \mathbb{R}^n$. Namely, $\mathrm{size}(P_I)$ is polynomially bounded in $\mathrm{size}(P)$, if the dimension n is fixed (Hayes & Larman 1983, Schrijver 1986, Cook, Hartmann, Kannan & McDiarmid 1992). It would be undesirable if the upper bound described above was tight. A deeper knowledge of the structure of the elementary closure is also important in the context of choosing effective cutting planes. We prove that the elementary closure can be described with a polynomial number of inequalities in fixed dimension and we provide a polynomial algorithm (in varying dimension) for finding cutting planes from this description. First we inspect the elementary closure of rational simplicial cones. We show that it can be described with polynomially many inequalities in fixed dimension. Via a triangulation argument, we prove a similar statement for arbitrary rational polyhedra. Then we show that the elementary closure of a rational polyhedron can be constructed in polynomial time in fixed dimension. This yields a polynomial algorithm that constructs a cutting plane proof of $\mathbf{0}^T x \leq -1$ for rational polyhedra P with empty integer hull. Based on these results, we then develop a polynomial algorithm in varying dimension for computing Gomory-Chvátal cutting planes of pointed simplicial cones. These cutting planes are not only among those of maximal possible violation in a natural sense, but also belong to the polynomial description of P'

in fixed dimension.

Zusammenfassung

In dieser Arbeit untersuchen wir einen bedeutenden Ansatz zur Lösung ganzzahliger Programme, das sogenannte Schnittebenenverfahren. Eine Gomory-Chvátal Schnittebene (Gomory 1958, Chvátal 1973*a*) eines Polyeders P ist eine Ungleichung $c^T x \leq \lfloor \delta \rfloor$, wobei c ein ganzzahliger Vektor und die Ungleichung $c^T x \leq \delta$ für P gültig ist, das heißt, daß jeder Punkt, der in P liegt, auch die Ungleichung $c^T x \leq \delta$ erfüllt. Die Schnittebene $c^T x \leq \lfloor \delta \rfloor$ ist für jeden ganzzahligen Punkt in P gültig, also auch für die konvexe Hülle der ganzzahligen Punkte in P, die sogenannte ganzzahlige Hülle P_I von P. Da eine Schnittebene im allgemeinen nicht für das Polyeder P gültig ist, führt ihre Hinzunahme zu einer besseren Approximation der ganzzahligen Hülle P_I, als dies P selbst ist. Der Durchschnitt von P mit all seinen Gomory-Chvátal Schnittebenen ist die elementare Hülle P' von P. Falls P ein rationales Polyeder ist, dann ist auch die elementare Hülle von P ein rationales Polyeder. Schrijver (1980) zeigte, daß das wiederholte Bilden der elementaren Hülle eines rationalen Polyeders P nach endlich vielen Schritten zu der ganzzahligen Hülle von P führt. Chvátal (1973*a*) zeigte dies zuvor für den Fall, daß P ein Polytop ist. Dieses wiederholte Bilden der elementaren Hülle nennt man das Chvátal-Gomory Verfahren. Die minimale Anzahl an Iterationen, die nötig ist, um P_I zu erhalten, nennt man den Chvátal-Rang von P. Wir zeigen, daß die Endlichkeit des Chvátal-Ranges rationaler Polyeder (Chvátal 1973*a*, Schrijver 1980) bereits aus Gomorys algorithmischem Ergebnis (Gomory 1958) folgt. Für den Fall, daß das Polyeder im positiven Orthanten ist, wurde dies von Schrijver (1986) beobachtet.

Bereits im zweidimensionalen Raum läßt sich eine Familie von rationalen Polytopen konstruieren, für die sich keine obere Schranke des Chvátal-Ranges angeben läßt. Formulierungen kombinatorischer Optimierungsprobleme als ganzzahliges Programm sind für gewöhnlich Polytope im 0/1 Würfel. Daher interessieren wir uns für den Chvátal-Rang von Polytopen, die im 0/1 Würfel enthalten sind. Zunächst untersuchen wir rationale Polytope, deren ganzzahlige Hülle leer ist. Es stellt sich heraus, daß deren Chvátal-Rang im wesentlichen durch ihre Dimension beschränkt ist. Dann wenden wir uns den Polytopen im 0/1 Würfel zu, deren ganzzahlige Hülle nichtleer ist. Dazu müssen wir die Komplexität von Facetten ganzzahliger 0/1 Polytope betrachten. Durch Skalieren dieser Facetten leiten wir eine erste polynomielle Schranke $\mathrm{O}(n^3 \log n)$ des Chvátal-Ranges von Polytopen im n-dimensionalen 0/1 Würfel her. Eine geschicktere Anwendung der Skalierungsmethode führt schließlich zu einer $\mathrm{O}(n^2 \log n)$ oberen Schranke. Dann konstruieren wir eine Familie von Polytopen im n-dimensionalen 0/1 Würfel, deren Chvátal-Rang mindestens $(1+\epsilon)\,n$ ist, für ein $\epsilon > 0$. Dies verbessert die bisher bekannte untere Schranke n. Wenn die Funktion $\mathrm{rank}(n)$ den maximalen Chvátal-Rang von Polytopen im n-dimensionalen 0/1 Würfel bezeichnet, dann zeigen wir $(1+\epsilon)\,n \leq \mathrm{rank}(n) \leq 3\,n^2\,\mathrm{size}(n)$.

Zum Lösen kombinatorischer Optimierungsprobleme mit ganzzahliger Programmierung werden Schnittebenen oft aus der Kombinatorik des Problems abgeleitet. Aber auch dann sind sie meist Gomory-Chvátal Schnittebenen. Eine polynomielle Separationsroutine für die elementare Hülle wäre daher ein mächtiges Werkzeug. Dies motivierte Schrijver, die Frage nach der Existenz einer solchen Routine als offenes Problem in seinem Buch (Schrijver 1986) zu formulieren. Wir geben eine negative Antwort auf diese Frage, indem wir zeigen, daß das Separationsproblem für die elementare Hülle eines rationalen Polyeders NP-hart ist.

Es war nicht sehr viel über die Struktur der elementaren Hülle bekannt. Man weiß im wesentlichen das folgende (siehe (Cook, Cunningham, Pulleyblank & Schrijver 1998)): Wenn P definiert ist als $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ wobei $A \in \mathbb{Z}^{m \times n}$ und $b \in \mathbb{Z}^m$, dann ist P' der Durchschnitt von P mit allen Gomory-Chvátal Schnittebenen $c^T x \leq \lfloor \delta \rfloor$, $c \in \mathbb{Z}^n$, wobei sich c als $c^T = \lambda^T A$ mit $\lambda \in [0,1)^m$ schreiben läßt und δ das Maximum $\delta = \max\{c^T x \mid x \in P\}$ ist. Die Maximumnorm $\|c\|_\infty$ eines solchen $c = A^T \lambda$ kann wie folgt abgeschätzt werden: $\|c\|_\infty = \|A^T \lambda\|_\infty \leq \|A^T\|_\infty$. Daraus ergibt sich die exponentielle (in der binären Eingabelänge) obere Schranke $\|A^T\|_\infty^n$ für die Anzahl der Ungleichungen, die zur Darstellung von P' benötigt werden. Diese Schranke ist auch exponentiell, wenn man die Dimension n festhält. Ganzzahlige Programme in fester Dimension können jedoch in polynomieller Zeit gelöst werden (Lenstra 1983). Auch gibt es eine polynomielle obere Schranke für die Ungleichungsdarstellung der ganzzahligen Hülle P_I eines rationalen Polyeders P in fester Dimension (Hayes & Larman 1983, Schrijver 1986, Cook, Hartmann, Kannan & McDiarmid 1992). Es wäre nicht wünschenswert, stellte sich heraus, daß es eine solche polynomielle obere Schranke für die Darstellung von P' in fester Dimension nicht gibt. Genaueres Wissen von der Struktur der elementaren Hülle erscheint auch hilfreich im Kontext des Problems effektive Schnittebenen zu wählen. Wir beweisen, daß die elementare Hülle eine polynomielle Darstellung in fester Dimension besitzt und wir beschreiben einen in beliebiger Dimension polynomiellen Algorithmus, der uns Schnittebenen aus dieser Darstellung berechnet. Zuerst untersuchen wir die elementare Hülle von simplizialen Kegeln. Wir zeigen, daß sie eine polynomielle Darstellung hat und verallgemeinern dies auf beliebige rationale Polyeder durch Triangulierung. Dann beweisen wir, daß die elementare Hülle eines rationalen Polyeders in fester Dimension in polynomieller Zeit berechnet werden kann. Dies führt zu einem polynomiellen Algorithmus, der für rationale Polyeder mit leerer ganzzahliger Hülle in fester Dimension einen Schnittebenenbeweis für die Ungleichung $\mathbf{0}^T x \leq -1$ herleitet. Basierend auf diesen Erkenntnissen entwickeln wir schließlich einen Algorithmus, der Schnittebenen von spitzen simplizialen Kegeln berechnet. Dieser Algorithmus ist polynomiell in beliebiger Dimension. Die Besonderheit der berechneten Schnittebenen ist nicht nur die, daß sie einen maximalen Grad der Verletzung in einem natürlichen Sinne aufweisen, sondern auch, daß sie zu der zuvor beschriebenen polynomiellen Darstellung von P' in fester Dimension gehören.

Bibliography

Aho, A. V., Hopcroft, J. E. & Ullman, J. D. (1974), *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading.

Alon, N. & Vu, V. H. (1997), 'Anti-Hadamard matrices, coin weighing, threshold gates, and indecomposable hypergraphs', *Journal of Combinatorial Theory* **79**(Series A), 133–160.

Balas, E., Ceria, S. & Cornuéjols, G. (1993), 'A lift-and-project cutting plane algorithm for mixed 0-1 programs', *Mathematical Programming* **58**, 295 – 324.

Barahona, F., Grötschel, M. & Mahjoub, A. (1985), 'Facets of the bipartite subgraph polytope', *Mathematics of Operations Research* **10**, 340–358.

Bárány, I., Howe, R. & Lovász, L. (1992), 'On integer points in polyhedra: a lower bound', *Combinatorica* **12**(2), 135 – 142.

Bockmayr, A. & Eisenbrand, F. (1997), On the Chvátal rank of polytopes in the 0/1 cube, Research Report MPI-I-97-2-009, Max-Planck-Institut für Informatik.

Bockmayr, A. & Eisenbrand, F. (1999), Cutting planes and the elementary closure in fixed dimension, Technical Report MPI-I-99-2-008, Max-Planck-Institut für Informatik.

Bockmayr, A., Eisenbrand, F., Hartmann, M. & Schulz, A. (1999), 'On the Chvátal rank of polytopes in the 0/1 cube', *Discrete Applied Mathematics* **98**, 21–27.

Caprara, A. & Fischetti, M. (1996), '$\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts', *Mathematical Programming* **74**, 221–235.

Caprara, A., Fischetti, M. & Letchford, A. N. (2000), 'On the separation of maximally violated mod-k cuts', *Mathematical Programming* **87**(1), 37–56.

Cassels, J. W. S. (1997), *An introduction to the geometry of numbers*, Classics in Mathematics, Springer.

Chvátal, V. (1973*a*), 'Edmonds polytopes and a hierarchy of combinatorial problems', *Discrete Mathematics* **4**, 305 – 337.

Chvátal, V. (1973*b*), 'Flip-flops in hypohamiltonian graphs', *Canadian Mathematical Bulletin* **16**, 33–41.

Chvátal, V., Cook, W. & Hartmann, M. (1989), 'On cutting-plane proofs in combinatorial optimization', *Linear Algebra and its Applications* **114/115**, 455–499.

Cook, W. (1990), 'Cutting-plane proofs in polynomial space', *Mathematical Programming* **47**, 11–18.

Cook, W., Coullard, C. R. & Turán, G. (1987), 'On the complexity of cutting plane proofs', *Discrete Applied Mathematics* **18**, 25 – 38.

Cook, W., Cunningham, W. H., Pulleyblank, W. R. & Schrijver, A. (1998), *Combinatorial Optimization*, John Wiley.

Cook, W., Hartmann, M., Kannan, R. & McDiarmid, C. (1992), 'On integer points in polyhedra', *Combinatorica* **12**(1), 27 –37.

Coppersmith, D. & Winograd, S. (1990), 'Matrix multiplication via arithmetic progressions.', *J. Symb. Comput.* **9**(3), 251–280.

Domich, P., Kannan, R. & Trotter, L. (1987), 'Hermite normal form computation using modulo determinant arithmetic.', *Mathematics of Operations Research* **12**(1), 50–59.

Edmonds, J. (1965), 'Maximum matching and a polyhedron with 0,1-vertices', *Journal of Research of the National Bureau of Standards* **69**, 125–130.

Edmonds, J. & Giles, R. (1977), 'A min-max relation for submodular functions on graphs', *Ann. discrete Math.* **1**, 185–204.

Eisenbrand, F. (1999), 'On the membership problem for the elementary closure of a polyhedron', *Combinatorica* **19**(2), 297–300.

Eisenbrand, F. & Schulz, A. S. (1999), Bounds on the Chvátal rank of polytopes in the 0/1 cube, *in* G. Cornuéjols, R. Burkard & G. Woeginger, eds, 'Integer Programming and Combinatorial Optimization, IPCO 99', Springer, LNCS 1610, pp. 137–150.

Erdős, P. (1962), 'On circuits and subgraphs of chromatic graphs', *Mathematika* **9**, 170–175.

Garey, M. R. & Johnson, D. S. (1979), *Computers and Intractability. A Guide to the Theory of NP-Completeness*, Freemann.

Gauß, C. F. (1801), *Disquisitiones arithmeticae*, Gerh. Fleischer Iun.

Giles, F. R. & Pulleyblank, W. R. (1979), 'Total dual integrality and integer polyhedra', *Linear Algebra and its Applications* **25**, 191 – 196.

Gomory, R. E. (1958), 'Outline of an algorithm for integer solutions to linear programs', *Bull. AMS* **64**, 275 – 278.

Gomory, R. E. (1967), 'Faces of an integer polyhedron', *Proc. Nat. Acad. Sci.* **57**, 16 – 18.

Grötschel, M., Lovász, L. & Schrijver, A. (1988), *Geometric Algorithms and Combinatorial Optimization*, Vol. 2 of *Algorithms and Combinatorics*, Springer.

Grötschel, M. & Pulleyblank, W. (1986), 'Clique tree inequalities and the symmetric travelling salesman problem', *Mathematics of Operations Research* **11**, 537–569.

Hafner, J. L. & McCurley, K. S. (1991), 'Asymptotically fast triangularization of matrices over rings', *SIAM J. Comput.* **20**(6), 1068–1083.

Haken, A. (1985), 'The intractability of resolution', *Theoretical Computer Science* **39**, 297 – 308.

Hammer, P. L., Johnson, E. & Peled, U. N. (1975), 'Facets of regular 0-1 polytopes', *Mathematical Programming* **8**, 179 – 206.

Hartmann, M. (1988), Cutting planes and the complexity of the integer hull, Technical Report 819, School of Operations Research and Industrial Engineering, Cornell University.

Hartmann, M., Queyranne, M. & Wang, Y. (1999), On the Chvátal rank of certain inequalities, *in* G. Cornuéjols, R. Burkard & G. Woeginger, eds, 'Integer Programming and Combinatorial Optimization, IPCO 99', Springer, LNCS 1610, pp. 218–233.

Harvey, W. (1999), 'Computing two-dimensional integer hulls', *SIAM J. Comput.* **28**(6), 2285–2299.

Hayes, A. C. & Larman, D. G. (1983), 'The vertices of the knapsack polytope', *Discrete Applied Mathematics* **6**, 135 – 138.

Howell, J. A. (1986), 'Spans in the module $(Z_m)^s$', *Linear and Multilinear Algebra* **19**, 67–77.

Hung, M. S. & Rom, W. O. (1990), 'An application of the Hermite normal form in integer programming', *Linear Algebra and its Applications* **140**, 163–179.

Kannan, R. & Bachem, A. (1979), 'Polynomial algorithms for computing the Smith and Hermite normal forms of an integer matrix', *SIAM J. Comput.* **8**(4), 499–507.

Kannan, R. & Lovász, L. (1988), 'Covering minima and lattice-point-free convex bodies', *Annals of Mathematics* **128**, 577 – 602.

Khachiyan, L. (1979), 'A polynomial algorithm in linear programming', *Doklady Akademii Nauk SSSR* **244**, 1093–1097.

Knuth, D. (1969), *The art of computer programming*, Vol. 2, Addison-Wesley.

Lang, S. (1971), *Linear Algebra*, Addison-Wesley.

Lenstra, H. W. (1983), 'Integer programming with a fixed number of variables', *Mathematics of Operations Research* **8**(4), 538 – 548.

Letchford, A. N. (1999), Totally tight rank-1 Chvátal-Gomory cuts. Manuscript.

Lovász, L. (1986), *An algorithmic theory of numbers, graphs and convexity*, SIAM.

Lovász, L. & Scarf, H. E. (1992), 'The generalized basis reduction algorithm', *Mathematics of Operations Research* **17**(3), 751 – 764.

Lovàsz, L. & Schrijver, A. (1991), 'Cones of matrices and set-functions and 0-1 optimization', *SIAM J. Opt.* **1**(2), 166–190.

Nemhauser, G. L. & Wolsey, L. A. (1988), *Integer and Combinatorial Optimization*, John Wiley.

Niven, I., Zuckerman, H. S. & Montgomery, H. L. (1991), *An Introduction to the Theory of Numbers, fifth edition*, Wiley.

Padberg, M. W. & Grötschel, M. (1985), Polyhedral computations, *in* E. L. Lawler, J. K. Lenstra, A. Rinnoy Kan & D. B. Shmoys, eds, 'The Traveling Salesman Problem', John Wiley & Sons, pp. 307–360.

Papadimitriou, C. H. (1994), *Computational Complexity*, Addison Wesley.

Pudlák, P. (1997), 'Lower bounds for resolution and cutting plane proofs and monotone computations', *Journal of Symbolic Logic* **62**(3), 981–988.

Pudlák, P. (1999), On the complexity of the propositional calculus, *in* 'Logic Colloquium'97', Cambridge Univ. Press, pp. 197–218.

Schönhage, A. & Strassen, V. (1971), 'Schnelle Multiplikation grosser Zahlen. (Speedy multiplication of great numbers).', *Computing* **7**, 281–292.

Schrijver, A. (1980), 'On cutting planes', *Annals of Discrete Mathematics* **9**, 291 – 296.

Schrijver, A. (1986), *Theory of Linear and Integer Programming*, John Wiley.

Sherali, H. D. & Adams, W. P. (1990), 'A hierarchy of relxations between the continuous and convex hull representations for zero-one programming problems', *SIAM J. Discr. Math.* **3**(3), 411–430.

Storjohann, A. & Labahn, G. (1996), Asymptotically fast computation of Hermite normal forms of integer matrices, *in* Y. N. Lakshman, ed., 'ISSAC 96', ACM Press, pp. 259–266.

Storjohann, A. & Mulders, T. (1998), Fast algorithms for linear algebra modulo n, *in* 'ESA'98', number 1461 *in* 'LNCS', Springer, pp. 139–150.

Urquhart, A. (1995), 'The complexity of propositional proofs', *Bulletin of Symbolic Logic* **1**(4), 425–467.

Ziegler, G. M. (1998), *Lectures on Polytopes*, number 152 *in* 'Graduate Texts in Mathematics', revised edition edn, Springer.

Ziegler, G. M. (1999), Lectures on 0/1-polytopes, *in* G. Kalai & G. Ziegler, eds, 'Polytopes - Combinatorics and Computation', Birkhäuser. to appear.