Unstable minimal surfaces of annulus type in manifolds

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Zusammenfassung

Es sei (N, h) eine vollständige zusammenhängende Riemannsche Mannigfaltigkeit und Γ, Γ_1, Γ_2 Jordansche Kurven in N. Weiter sei M eine Kreisscheibe bzw. ein Kreisring in $\mathbb{R}^2$ und $\widetilde{\Gamma}(\subset N)$ bezeichne die Kurve Γ bzw. $\Gamma_1 \cup \Gamma_2$. Bei dem Plateauschen Problem, bezeichnet mit $\mathcal{P}(\widetilde{\Gamma})$, untersucht man Minimalflächen $X : M \to N$, die von $\tilde{\Gamma}$ berandet sind, als Lösungen des folgenden Systems:

(1) $\tau_h(X) = 0$ auf M,

(2) $|X_u|_h^2 - |X_v|_h^2 = \langle X_u, X_v \rangle_h = 0$ auf M,

(3) $X|_{\partial M}$ bildet ∂M schwach monoton auf $\tilde{\Gamma}$ ab,

wobei τ_h der Gleichung für harmonische Funktionen in (N, h) entspricht.
Die Lösungen des obigen Systems sind genau die stationären Punkte des Dirichletschen Integrals E,

$$E(X) = \frac{1}{2} \int_M |\nabla X|^2 dM, \qquad \text{in}$$

$$\mathcal{C}(\widetilde{\Gamma}) = \{f \in H^{1,2} \cap C^0(\overline{M}, N) \mid f(\partial M) = \widetilde{\Gamma}, \text{ schwach monoton}\}.$$

Unter instabilen Minimalflächen versteht man die stationären Punkte von E, die keine lokalen Minimierer in $\mathcal{C}(\tilde{\Gamma})$ sind.
Eine Methode zur Untersuchung instabiler stationären Punkte von Variationsproblemen ist die sogenannte Ljusternik-Schnirelmann Theorie zusammen mit dem Minimax-Prinzip. Eine Anwendung dieser Theorie auf instabile Minimalflächen vom Typ der Kreisscheibe in $\mathbb{R}^n$ wurde erst im Jahr 1984 präsentiert in [St3] (siehe auch [St1]).
Später wurden auch instabile Minimalflächen vom Kreisring-Typ vom selben Autor in [St4] untersucht, und im Hauptsatz wurde die Existenz von Minimalflächen vom Typ des Kreisrings in $\mathbb{R}^n$ bewiesen unter gewissen Bedingungen an die Lösungen von $\mathcal{P}_{\mathbb{R}^n}(\Gamma_i), i = 1, 2$ und $\mathcal{P}_{\mathbb{R}^n}(\Gamma_1, \Gamma_2)$.

In der vorliegenden Arbeit wird ein solcher Existenzsatz in Riemannschen Mannigfaltigkeiten bewiesen. Hier betrachten wir Jordansche Kurven $\Gamma_1, \Gamma_2 \subset N$ mit $\mathrm{dist}(\Gamma_1, \Gamma_2) > 0$, die von der Klasse C^3 und diffeomorph zu S^1 sind. Weiterhin erfülle $(N, h)(\supset \Gamma_1, \Gamma_2)$ eine der folgenden Eigenschaften:

(C1) Es gibt $p \in N$ mit $\Gamma_1, \Gamma_2 \subset B(p, r)$, wobei $B(p, r)$ die Normalumgebung aller ihrer Punkte ist. Dabei ist $r < \pi/(2\sqrt{\kappa})$, und κ bezeichne eine obere Schranke der Schnittkrümmung von N.

(C2) N ist kompakt mit nicht positiver Schnittkrümmung.

Unter diesen Bedingungen ist die Existenz und die Eindeutigkeit der harmonischen Fortsetzung für eine gegebene Parametriesierung der Randkurven bewiesen ([HKW3], [JK], [ES], [Le],[Hm]).

Hier werden zunächst sowohl im Fall (C1) als auch im Fall (C2) geeignete Rahmenbedingungen vorgestellt. Z.B. wird ein Funktionenraum $\overline{\mathcal{M}} \ni (x^1, x^2, \rho)$ wird eingeführt, wobei x^1 bzw. x^2 eine Randparametrisierung für Γ_1 bzw. Γ_2 und $\rho \in [0, 1)$ die Größe des Kreisrings ($A_\rho \in \mathbb{R}^2$) oder Einheitkreise (falls $\rho = 0$) beschreibt, sowie eine Art von Tangentialraum für $x \in \overline{\mathcal{M}}$ und das Funktional $\mathcal{E} : \overline{\mathcal{M}} \to \mathbb{R}$,

$$\mathcal{E}(x) := \frac{1}{2} \int |\nabla \mathcal{F}(x)|^2 d\omega.$$

Hier bezeichne $\mathcal{F}$ die harmonische Fortsetzung in N.
Als nächstes wird bewiesen, dass das Funktional $\mathcal{E}$ genügend Regularität besitzt, um die Ljusternik-Schnirelmann Theorie anzuwenden. Anschließend werden die kritischen Punkte so definiert, dass die harmonische Fortsetzungen davon genau die Lösungen vom $\mathcal{P}(\Gamma_i)$, $i = 1, 2$, oder $\mathcal{P}(\Gamma_1, \Gamma_2)$ werden. Dazu wird die $H^{2,2}$-Regularität der harmonischen Fortsetzung kritischer Punkte von $\mathcal{E}$ bewiesen.
Die Gültigkeit einer sogenannten Palais-Smale Bedingung (die für obige Methode notwendig ist) wird ebenfalls bewiesen. Hier wird wegen der konformen Invarianz des Dirichletschen Integrals, eine Drei-Punkten-Bedingung gefordert allerdings nur für die Flächen vom Typ der Kreisscheibe. Da dies für den Typ des Kreisrings nicht möglich ist, wird der obige Ansatz passend (in einer Mannigfaltigkeit) normalisiert.
Weil wir instabile Flächen vom Kreisring-Typ suchen, wird wie im Euklidischen Fall folgende Eigenschaft nützlich: Die Energie der Flächen vom Typ des Kreisrings ($\rho \leq \rho_0$) ist größer als die Energie der (zwei) Kreisscheiben-Typ harmonischen Flächen mit einer gleimäßigen positiven Konstante (abhängig von ρ) auf einer gewissen Menge von x^i.
Im Kapitel 5 wird diese Eigenschaft auf den Fall der Riemannschen Mannigfaltigkeiten verallgemeinert, allerdings unter etwas starkeren Bedingungen als im Euklidischen Fall. Diese reichen aber für den Beweis des Hauptsatzes aus.
Schließlich wird im Hauptsatz bewiesen: Es existiere ein Kreisring, dessen Energie ein striktes lokales Minimum ist (in einer gewisser Klasse, je nach (C1), (C2)). Weiterhin seien alle Lösungen $\mathcal{F}^i$ der $\mathcal{P}(\Gamma_i), i = 1, 2$ absolute Energie-Minimierer mit $\text{dist}(\mathcal{F}^1, \mathcal{F}^2) > 0$. Dann existiert mindestens eine instabile Minimalfläche vom Typ des Kreisrings.
Als Korollare werden konkrete Bedingungen für die dreidimensionale Kugel S^3 bzw. den dreidimensionalen hyperbolischen Raum H^3 vorgestellt, welche konstante Krümmung 1 bzw. -1 besitzen. Haben Γ_1, Γ_2 Totalkrümmung $\leq 4\pi$, so impliziert beim hyperbolischen Raum H^3 die Existenz strikt Energie-Minimierer vom Kreisring-Typ die Existenz einer instabilen Keisring-Minimalfläche (wegen der Eindeutigkeit der Kreisscheibe-Minimalfläche berandet durch $\Gamma_i, i = 1, 2$).

Chapter 1

Introduction

1.1 Minimal surfaces

The study of minimal surfaces begins with the problem to find a surface with the smallest area for a given closed curve Γ in the three-dimensional Euclidean space, $\mathbb{R}^3$. In 1762, this problem was discussed by J.L. Lagrange. He derived an equation which the solutions of the above area-minimizing problem have to satisfy, assuming that the solution surface can be described by the equation $z = z(x, y)$ (named a nonparametric minimal surface). This equation is called the Euler-Lagrange equation for a nonparametric minimal surface. Let us be more precise: let C be the projection of the given curve Γ onto the (x, y)-plane and D its interior, and $z = z(x, y) \in C^2(D) \cap C^1(\overline{D})$ be a surface of smallest area.
We consider a surface $z_\varepsilon(x, y) = z(x, y) + \varepsilon\xi(x, y)$ nearby z and with the same boundary, i.e. $\xi(x, y)$ is an arbitrary function with suitable regularity, vanishing on C and $|\varepsilon|$ is a small real number. Then for all small $|\varepsilon| > 0$, z_ε should not have surface area smaller than that of z. Hence the area integral, considered as a function of ε,

$$I(\varepsilon) = \int\int_D \sqrt{1 + z_{\varepsilon,x}^2 + z_{\varepsilon,y}^2}\,dxdy$$

has minimum at $\varepsilon = 0$, hence $I'(0) = 0$ and from this we obtain what is today called the minimal surface equation (nonparametric),

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 0. \tag{1.1}$$

Moreover, from (1.1), we observe that the mean curvature of the surface $z = z(x, y)$ is identically zero.

Now we consider the case of parametrized surfaces $X(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$. The area of X is

$$A(X) := \int_\Omega \sqrt{|X_u|^2|X_v|^2 - |X_u \cdot X_v|^2}\,d\omega.$$

Assume that $X : \overline{\Omega} \to \mathbb{R}^3$ is a regular (gradient not 0) surface of class C^2 with the normal vector $N : \overline{\Omega} \to \mathbb{R}^3$ defined by $N = \frac{1}{|X_u \times X_v|} X_u \times X_v$. Then the first variation of the area functional $A_\Omega(X)$ on Ω at X in the direction of a vector field $Y \in C_c^\infty(\Omega, \mathbb{R}^3)$ can be computed as follows:

$$\delta A_\Omega(X, Y) = -2 \int_\Omega \langle Y, N \rangle H d\omega,$$

where H is the mean curvature of the surface X. Hence the stationay points, including the absolute minimizer, of the area functional are exactly the surfaces of zero mean curvature and a regular C^2 surface is called minimal surface if the mean curvature of it is identically zero.

For this reason the term 'minimal surface' is customarily used for any surface of 'vanishing mean curvature' (not necessarily regular).

As we can see in (1.1), a plane is clearly a minimal surface. In 1769, Euler had shown that when the catenary is rotated about an external horizontal axis it generates a surface of smallest area, called 'catenoid' (first in Figure 1.1).
After that Lagrange had found an equation for nonparametric minimal surfaces as above, in 1776 J.M.B.C. Meusnier discoverded that the right helicoid (second in Figure 1.1) and the catenoid satisfy the equation (1.1).

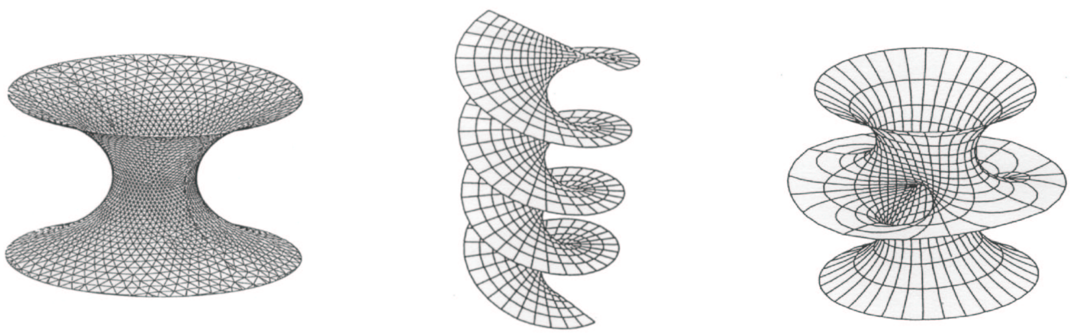

Figure 1.1: Catenoid, Helicoid and Costa minimal surface (source: [DHKW1])

The plane, the catenoid and the helicoid were the only known complete minimal surfaces in $\mathbb{R}^3$ with no self intersections until another complete minimal surface was discovered by Costa 1984 (third in Figure 1.1).

1.2 Plateau's Problem

(A)(Classical) Plateau's Problem of disc type and Dirichlet-Integral

A-i) In the 'Classical Plateau Problem' one asks for the existence of disc type surfaces of least area bounded by a given Jordan curve $\Gamma \subset \mathbb{R}^3$.
This problem was named in honor of the Belgian physicist J.A.F. Plateau. In 1873, Plateau described a multitude of experiments connected with the phenomenon of capillarity. In particular he noted that every contour consisting of single closed wire, whatever its geometric form is, bounds one soap film.

Recall that regular surfaces of least area have vanishing mean curvature, thus they are minimal surfaces which are defined as surfaces with mean curvature 0 as we have seen in the previous section. From this, we now formulate a more general version of Plateau's problem as follows:

$\mathcal{P}(\Gamma)$: Given a rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$, find a (disc type) minimal surface bounded by Γ.

This somewhat generalized problem is concerned with the stationary points of the area functional, whereas the 'Classical Plateau Problem' deals with the minimizers of area.

As formulated above, in (Classical) Plateau's Problem, we prescribe a special topology, i.e. 'disc', for minimal surfaces for a given rectifiable Jordan curve Γ. In other words, we consider only surfaces $X \in C^0(\overline{B}, \mathbb{R}^3)$.

A-ii) Lichtenstein's theorem says that every regular surface $X : B \to \mathbb{R}^3$ can be reparametrized by a regular change $\tau : B \to B$ of parameters such that $Y := X \circ \tau$ and with the conformality,

$$|Y_u|^2 = |Y_v|^2, \qquad \langle Y_u, Y_v \rangle = 0,$$

which is called isothermal or conformal parametrization.

Introducing this isothermal coordinate, the surfaces of mean curvature H (called H-surface) in $\mathbb{R}^n$ are written as follows:

$$\Delta X = 2H X_u \times X_v.$$

A surface $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ (not necessarily regular) with vanishing mean curvature bounded by Γ, namely a solution of $\mathcal{P}(\Gamma)$ satisfies the following conditions:

(1) $\Delta X = 0$, i.e. harmonic in $\mathbb{R}^3$,

(2) $|X_u|^2 - |X_v|^2 = \langle X_u, X_v \rangle = 0$,

(3) The restriction $X|_{\partial B}$ is weakly monotone onto Γ.

The solutions to this system may have branch points (gradient 0), self-intersections, they may be physically unstable and not to observe in the soap film experiment.

Introducing the Dirichlet's integral,

$$D(X) = \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2) d\omega,$$

and the space of admissible functions, of course including all the minimal surfaces,

$$\mathcal{C}(\Gamma) := \{X \in H^{1,2}(B, \mathbb{R}^3) | X|_{\partial B} \in C^0(\partial B; \Gamma) \quad \text{is a weakly monotone parametrization}\},$$

where $H^{1,2}(B, \mathbb{R}^3)$ be the Sobolev space of L^2-functions with square integrable distributional derivatives, we have:
$X \in \mathcal{C}(\Gamma)$ solves (1)-(2) if and only if X is critical point of D in the sense that

- $\frac{d}{d\varepsilon} D(X + \varepsilon\phi)|_{\varepsilon=0} = 0$, for all $\phi \in H_0^{1,2}(B, \mathbb{R}^3)$
- $\frac{d}{d\varepsilon} D(X \circ g_\epsilon^{-1}; B_\epsilon)_\varepsilon = 0$ for any diffeomorphism $g_\epsilon : \overline{B} \to \overline{B}_\varepsilon$, depending differentiably on $|\varepsilon| < \varepsilon_0$, with $g_0 = Id$.

Hence from the above Lichtenstein's theorem, the regular minimal surfaces, i.e. the stationary surfaces of area functional are exactly the stationary (or critical) points of D.

A-iii) The 'Classical Plateau Problem', asking for the existence of area minimizing minimal surface of disc type bounded by a closed curve, was solved during the 19th century for many special contours Γ. However a sufficiently general result, namely the existence of area minimizing minimal surface bounded by an Jordan curve, was obtained first in 1930/31 by J. Douglas and by T. Radó. They proposed a variational principle using the Dirichlet-integral $D(\cdot)$ instead of the area integral $A(\cdot)$. A considerable simplification of their methods was made by R. Courant and L. Tonelli.

In fact, for any diffeomorphism g of $\overline{B}$, $A(X \circ g) = A(X)$, whereas $D(\cdot)$ is invariant only under conformal transformations. And any attempt to solve the classical Plateau's problem by minimizing area functional $A(\cdot)$ fails due to lack of compactness.
Meanwhile we have the following relationship between $A(\cdot)$ and $D(\cdot)$:

- For $X \in H^{1,2}(B, \mathbb{R}^3)$,
$$A(X) \leq D(X)$$
with equality if and only if X is conformal.

- For given $\varepsilon > 0$, there exists a diffeomorphism $g : B \to B$ such that $X' = X \circ g$ satisfies:
$$D(X') \leq (1+\varepsilon)A(X') = (1+\varepsilon)A(X).$$

Thus we have that
$$\inf_{X \in \mathcal{C}(\Gamma)} A(X) = \inf_{X \in \mathcal{C}(\Gamma)} D(X),$$
and a minimizer of A is indeed a critial point of D.

Define $\mathcal{C}(\Gamma)^* \subset \mathcal{C}(\Gamma)$ as follows:
$$C(\Gamma)^* := \{X \in \mathcal{C}(\Gamma) | X(P_j) = Q_j, \;\; j = 1, 2, 3\}$$
with fixed oriented triple Q_j and $P_j := \mathrm{e}^{\frac{2\pi i j}{3}}$, $j = 1, 2, 3$. Difference of $\mathcal{C}(\Gamma)^*$ from $\mathcal{C}(\Gamma)$ is only conformal transformatioms under which D is invariant.

Then $\mathcal{C}(\Gamma)^*$ is closed with respect to the weak topology in $H^{1,2}(B, \mathbb{R}^n)$. Together with coerciveness of D in $\mathcal{C}(\Gamma)$ and weak lower semi-continuity on $H^{1,2}$ one obtains the existence of a D-minimizing solution of $\mathcal{P}(\Gamma)$, also proving that $\mathcal{C}(\Gamma) \neq \emptyset$. Finally we can also prove that the solution is in fact in the class of $C^2(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$.

(B) Generalized Plateau's Problem, specially of annulus type

B-i) The classical Plateau's problem reads as: find a disc-type minimal surface bounded by a given closed Jordan curve Γ. However it is by no means clear what the topological type of the surface of least area in a given configuration Γ will be, for example see Figure 1.2.

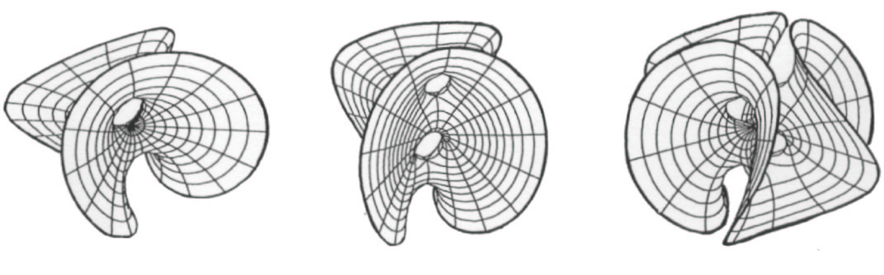

Figure 1.2: Minimal surfaces of different topolpgy types bounded by a Jordan curve (source: [DHKW1])

J.Douglas firstly stated Plateau's problem in the following general form:

General Problem of Douglas *Given in* $\mathbb{R}^3$ *a configuration* $\Gamma = \langle \Gamma_1, \Gamma_2, \cdots, \Gamma_k \rangle$ *consisting of* k *closed and mutually disjoint Jordan curves* Γ_j*, find a minimal surface of prescribed Euler characteristic, orientable or not, that span the contour* Γ.

For this he formulated the following result:

Theorem of Douglas *Let* α *denote the infimum of the Dirichlet integrals of all oriented connected surfaces of genus* g *spanning the given curves* $\Gamma_1, \cdots, \Gamma_k$ *and let* α^* *be the corresponding infimum over all oriented connected surfaces of genus less than* g *and all oriented, disconnected surfaces of total genus* g *consisting of two or more components spanning proper, non-empty,disjoint subsets of* $\Gamma_1 \cup \cdots \cup \Gamma_k$ *whose union equals* $\Gamma_1 \cup \cdots \cup \Gamma_k$*. If* $\alpha < \alpha^*$ *then there exists an oriented minimal surface of genus and having* $\Gamma_1 \cup \cdots \cup \Gamma_k$ *as boundary.*

Another approach to the general approach is due to Courant and Shiffman, they work with a class of surfaces of fixed topological type.

B-ii) To determine a minimal surface of annulus typ bounded by two given Jordan curves is a special case of the above 'General Problem of Douglas'.
More exactly, letting $A_\rho = \{w \in B \mid \rho < |w| < 1\} \subset \mathbb{R}^2$ with boundary $C_1 := \partial B, C_2 = \partial B_\rho$ and Γ_1, Γ_2 in $\mathbb{R}^3$ be the prescribed two closed Jordan curves with $\operatorname{dist}(\Gamma_1, \Gamma_2) > 0$, we can formulate Plateau's Problem $\mathcal{P}(\Gamma_1, \Gamma_2)$ as follows:
for some $\rho \in (0,1)$, a map $X \in C^2(A_\rho, \mathbb{R}^3) \cap C^0(\overline{A_\rho}, \mathbb{R}^3)$ is a solution of Plateau's Problem $\mathcal{P}(\Gamma_1, \Gamma_2)$ (i.e. an annulus type minimal surface bounded by $\Gamma_1 \cup \Gamma_2$) if

(1) $\Delta X = 0$,

(2) $|X_u|^2 - |X_v|^2 = \langle X_u, X_v \rangle = 0$,

(3) $X|_{C_i}$ is a weakly monotone map onto $\Gamma_i, i = 1, 2$.

In particular, if Γ_1 and Γ_2 are linked, then the sufficient condition of Douglas (see [Do2] or [Ni]) is satisfied and consequently there is an annulus-type minimal surface bounded by Γ_1 and Γ_2 whose area is less than the sum of area of two area minimizing surfaces of disc type which are bounded by Γ_1 and Γ_2 (Figure 1.3).

There does not always exist a minimal surface of annulus type for two given Jordan curves, although they are smooth, for example, the following result also by Nitsche:

Let Γ_1 *and* Γ_2 *be two Jordan curves in parallel planes. If there exists a plane which is orthogonal to the planes of these curves and which separates them, then* Γ_1 *and* Γ_2 *cannot bound a minimal surface of the type of the circular annulus.*

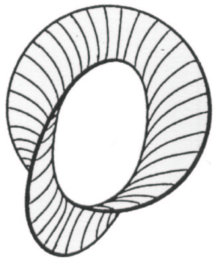

Figure 1.3: Minimal surface bounded by two interlocking Jordan curves (source: [DHKW2])

Even in a coaxial situation, we have conditions for the nonexistence of annulus-type minimal surface. Let

$$\begin{aligned}\Gamma_1 &= \{(\cos\theta, \sin\theta, -\frac{h}{2}) | 0 \le \theta \le 2\pi\} \subset \text{plane}, \ \ z = -\frac{h}{2} \\ \Gamma_2 &= \{(\cos\theta, \sin\theta, \frac{h}{2}) | 0 \le \theta \le 2\pi\} \subset \text{plane}, \ \ z = \frac{h}{2}.\end{aligned}$$

For small $h > 0$, there exist exactly two different annulus-type minimal surfaces, namely S_h^1 (smaller area) and S_h^2. The area of S_h^1 which is actually a catenoid, is smaller than 2π which is the sum of area of disc minimal surfaces bounded by Γ_1, Γ_2. And the area of S_h^2 may not be a relative minimum (see Figure 1.6). However if $h > 0$ is too large, there is no annulus minimal surface bounded by Γ_1, Γ_2.
In other words, if the distance of coaxial curves Γ_1, Γ_2 is not zero and small enough then there exist exactly two different annulus-type minimal surfaces, but if it is too large, there is no annulus minimal surface bounded by Γ_1, Γ_2.
Actually, Plateau remarked that as a soap film realized catenoid broke at that moment when the distance of two wires exceeded.
Moreover, Nitsche proved that the above nonexistence property also holds for general curves.

1.3 Unstable minimal surfaces

(A) Unstable minimal surfaces

As mentioned above, the area of a regular minimal surface is not always the minimum of the area functional A, although it is a stationary point A (or D). A regular minimal surface is called unstable if its surface is not a minimum among the neighboring surfaces with the same boundary. To determine the stability (or unstability) we may need to investigate the second variation of area functional.

Since nearby a regular minimal surface there are other surfaces with the same boundary but with larger area (see [Ni] §104), the area of unstable minimal surface is a stationary point of A, but neither a maximum nor a minimum.
These minimal surfaces cannot (easily) be produced experimentally, in contrast to stable minimal surfaces which are minimal in their neighborhood and so fulfil a basic property of 'real life' minimal surface as remarked by Plateau.

The following Figure 1.4 illustrates an example of unstable minimal surfaces bounded by a Jordan curve (a). In the case of (b) and (c), the surfaces (locally) minimize the area functional while the minimal surface (d) is unstable.

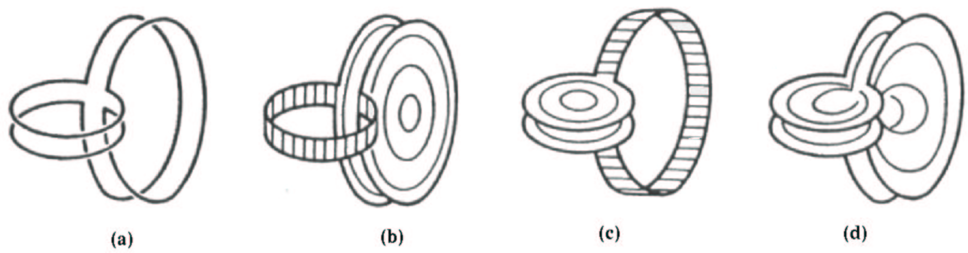

Figure 1.4: Minimal surfaces bounded by a Jordan curve (source: [Ni])

(B) Mountain pass lemma (Minimax-principle)

For a real valued function of n variables, we can sometimes guarantee the existence of the third stationary point (all of the first derivatives vanish) under the assumption of two relative minima. That is: for a domain $\Omega \subset \mathbb{R}^n$, let $f : \Omega \to \mathbb{R}$ be continuously differentiable in Ω. Assume that $f(\cdot)$ tends to infinity nearby any boundary point of Ω. If $f(\cdot)$ has strict relative minima at two distinct points p_1 and p_2 in Ω, then there exists an additional point p_0 such that $f(\cdot)$ is stationary at p_0, with

$$f(p_0) = \inf_{l \in L} \sup_{p \in l[0,1]} f(p), \tag{1.2}$$

where

$$L = \{l \in C([0,1];\Omega) | l(0) = p_1, \;\; l(1) = p_2\}.$$

The choice of the point p_0 is illustrated as in the Figure 1.5.

Here the lines indicate level sets of f. The function f has two relative strict minima at the points p_1 and p_2 - two valleys of the mountain - and is stationary at p_0. The

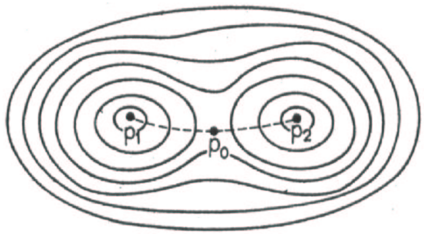

Figure 1.5: Mountain pass (source: [Ni])

mountain pass designates the lowest possible elevation among the passes from p_1 to p_2. From (1.2) we see that f (roughly speaking) takes its supremum at p_0 with respect to a curve from p_1 to p_2, on the other hand, takes its infimum in another direction, vertical to curves from p_1 to p_2. Thus, p_0 is neither maximizer nor minimizer of f.
In a certain situation this mountain pass Lemma is very useful to prove the existence of an unstable critical point (saddle point).

(C) Mountain Pass Lemma for Plateau's Problem

As we have seen, minimal surfaces are critical points of Dirichlet's integral. This fact suggests that we may relate Plateau's Problem to the theory of critical points in the global calculus of variations, due to M. Morse and L.A. Ljusternik and L. Schnirelmann. The unstable minimal surfaces and the Morse theory for the minimal surfaces had also been studied by J. Douglas, T. Rado, M. Shiffman and Morse-Tompkin. In 1939, M. Morse and M. Shiffman and Morse-Tompkin have suceeded in applying their theory to Plateau's Problem.

The so called Morse theory serves as a method for the information about the number of critical points. This Morse theory connects the topological structure of the level set and the number of the critical points of the functional in order to obtain the information.

As an example of an application of the above Mountain Pass Lemma to Plateau's Problem, a result (for nonparametric minimal surface) from that time (by the above authors in 1939) briefly reads: considering a space of admissible functions $\mathcal{A}$ (consisting of bounded harmonic mappings) let x_1, x_2 be two vectors in $\mathcal{A}$ and $\mathfrak{L}$ be a closed connected subset of $\mathcal{A}$ containing these two vectors. Let $d[\mathfrak{L}; x_1, x_2]$ denote the supremum of $D[x]$ for $x \in \mathfrak{L}$ and $d[x_1, x_2]$ denote the infimum of $d[\mathfrak{L}; x_1, x_2]$ over all subsets $\mathfrak{L}$. We then say, x_1 and x_2 are separated by a wall of elevation $d[x_1, x_2] - \max\{D[x_1], D[x_2]\}$ which is in fact nonnegative. If the two minmal surfaces x_1 and x_2 are separated by a wall of positive elevation, then there exists another vector in $\mathcal{A}$ which defines a minimal surface.

On the other hand, as a modern view of unstable critical points, the Ljusterink-Schnirelmann Theory (gradient line deformation) with the minimax-principle or Morse theory have been studied, for instance, by J. Milnor, R.S. Palais, S. Smale and J. Tromba. Here the functional has to be in the class of C^1.

(D) Ljusternik-Schnirelmann Theory with the minimax-principle for minimal surfaces

In 1983 ([St1], see also [St4] [St3]), M. Struwe gave another approach to unstable minimal surfaces of disc or annulus type for a given boundary in $\mathbb{R}^n$, extending the Ljusternik-Schnirelmann Theory on convex sets in Banach Spaces. With this he developed the Morse theory for minimal surfaces, inspired by the paper [BT], where the global structure of minimal surfaces is discussed. For higher topological structure in $\mathbb{R}^n$, it was studied in [JS].
In these papers, the space of boundary functions are taken with the following functional:

$$\mathcal{E}(x) := \frac{1}{2}\int_\Sigma H(x)d\omega,$$

where $H(x)$ is the harmonic extension of a boundary $x : \partial\Sigma \to \mathbb{R}^n$.

1.4 Generalization to Riemannian manifolds and results

(A) In a Riemannian manifold (N, h) with metric $(h_{\alpha\beta})$ of dimension 3, a uniform parametrized surface $X = (X^\alpha)$ satisfies of mean curvature H, in local coordinates,

$$\Delta X^\alpha - \Gamma^\alpha_{\beta\gamma}\nabla X^\beta \nabla X^\gamma = 2H\sqrt{h}h^{\alpha\gamma}(X_u \times X_v)_\alpha, \;\; \alpha = 1, \cdots, n,$$

where $\Gamma^\alpha_{\beta\gamma}$ is the christoffel symbol of metric h, $h = \det(h_{\alpha\beta})$ and $(h^{\alpha\gamma}) = (h_{\alpha\gamma})^{-1}$.
We now consider the generalized Plateau Problem, where $\mathbb{R}^n$, $n \geq 3$ is replaced by Riemannian manifold (N, h). Given curve $\Gamma \subset N$, the minimal surfaces satisfy:

(1) $\tau_h(X) = 0$,

(2) $|X_u|^2_h - |X_v|^2_h = \langle X_u, X_v\rangle_h = 0$,

(3) $X|_{\partial\Sigma}$ is weakly monotone onto Γ,

where $\tau_h := \Delta X^\alpha - \Gamma^\alpha_{\beta\gamma}\nabla X^\beta X^\gamma$ is the harmonic equation in (N, h) as the Euler-Lagrange eqution of energy functional.

This problem (also in the case of non-vanishing mean curvature H) has been studied, for instance, in [Mo1],[Gu],[He],[HH],[HKW1],[HKW2],[HKW3],[Grü1],[Qi].
Unstable minimal surfaces in a manifold of nonpositive curvature are discussed by Strömer in [Str] in 1980 where the argument of wall was used and the case of polygonal Jordan curve Γ is also investigated.
Recently in [Ho], unstable minimal surfaces of higher topological structure with one boundary in a nonpositively curved Riemannian manifold was studied by applying the method in [St3]. In particular, in the first part of this paper, the Jacobi field extension operater as the derivative of the harmonic extension was studied.

(B) Results

In this paper, we study unstable minimal surfaces of annulus type in manifolds. The Euclidean case was studied earlier in [St4] and the result is as follows: let us consider two given Jordan curves in $\mathbb{R}^n$ such that each of them bounds only energy (or area) minimizing minimal surfaces. Then the existence of an annulus-type minimal surface whose energy is a strict relative minimum in $\mathcal{C}(\Gamma_1, \Gamma_2)$ ensures the existence of an unstable minimal surface of annulus type. See Figure 1.6 below where (c) resp. (b) are stable minimal surfaces of disc type resp. annulus type (strict relative minimizer) and then (d) is an unstable minimal surface of annulus type.

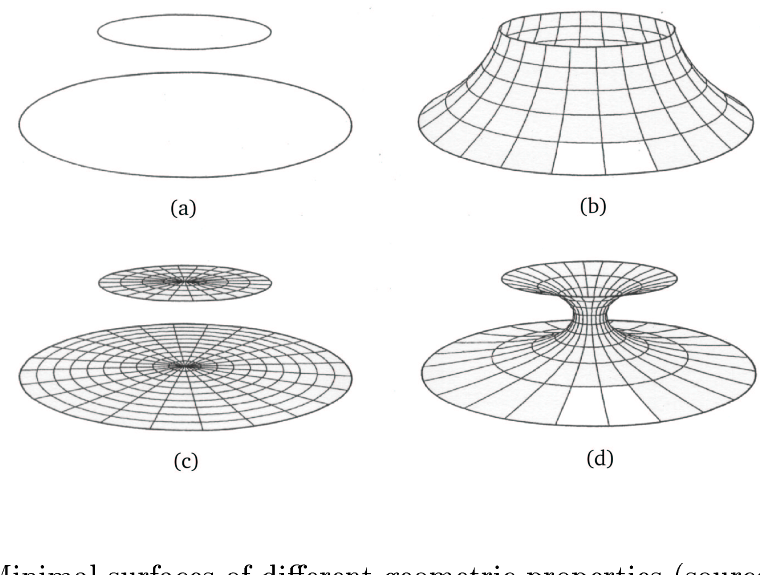

Figure 1.6: Minimal surfaces of different geometric properties (source: [DHKW1])

We want to generalize this result to manifolds satisfying some appropriate conditions, namely we will consider two boundary curves Γ_1, Γ_2 in a Riemannian manifold (N, h) such that one of the following holds.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p,r)$, where $B(p,r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

(C2) N is compact with nonpositive sectional curvature.

These conditions are related to the existence and the uniqueness of the harmonic extension for a given boundary parametrization. The compactness (stronger than the homogeneously regular conditon) of N in (C2) will be used for a technical computation.

We may further assume that N is isometrically embedded into $\mathbb{R}^k$ (see [Gro]).

First, in Chapter 3 we construct a suitable space of functions, $\overline{\mathcal{M}} = M^1 \times M^2 \times [0,1)$, where we have to distinguish the cases of (C1) and (C2). Here M^i is a set of parametrizations for $\Gamma_i, i = 1,2$ while the third variable $\rho \in [0,1)$ denotes the size of A_ρ or two discs for $\rho = 0$. Let us be more precise: if (C1) holds, M^i denotes the set of oriented and weakly monotone mappings : $\partial B \to \Gamma_i$ in the class of $H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)(\subset H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k))$, whereas for (C2) it consists of the traces of elements in a homotopy family of $H^{1,2} \cap C^0$ maps in N with boundary images onto Γ_i - again we assume the orientation and the monotonicity condition as above -.
In particular, for both (C1) and (C2), we will observe that $M^1 \times M^2$ is sufficiently general and any element of $M^1 \times M^2$ has a unique harmonic extension in N not only of disc type but also of annulus type for all size $\rho \in (0,1)$.
Following some idea of Struwe, for each $x^i \in M^i$, we define a convex subset of $T_{x^i} H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ denoted by $\mathcal{T}_{x^i}$ which in fact serves as a tangent space for x^i.
We will also see that the harmonic extensions of $x^i \in M^i$ possess a uniform bounded Energy for $\rho \in (0, \rho_0)$. This will be used for computations in the coming Chapter.
Moreover, we consider the following functional: for $x = (x^1, x^2, \rho) \in \overline{\mathcal{M}}$,

$$\mathcal{E}(x) := \frac{1}{2} \int |d\mathcal{F}(x)|_h^2 d\omega,$$

where $\mathcal{F}(x)$ denotes the harmonic extension of annulus type or of two discs.

In Chapter 4, the differentiability of $\mathcal{E}$ is discussed. Mainly, the situation of varing topology (from an annulus to two discs) is studied, more exactly, the (uniform) continuity of $\mathcal{E}$ and the derivative of $\mathcal{E}$ (denoted by $\delta_{x^i}\mathcal{E}$) with respect to the variables x^i and $\rho(\to 0)$.

In contrast to the Euclidean case, the harmonic extension operator in N is not linear (in general, only locally defined). Moreover, for the integration by parts which is applicable for Euclidean harmonic functions in $H^{1,2}$, we need $H^{2,2}$-regularity for harmonic functions in N. Thus, we cannot use many tools which are available in the Euclidean setting.
However, from the Courant-Lebesgue Lemma, we can divide $\mathcal{F}(x^1, x^2, \rho)(\rho > 0)$ in two parts, so that each part converges, roughly speaking, to a surface of disc type in N which should be the harmonic extension of disc type for x^1 and x^2. To prove this, we first study the uniform modulus of continuity of annulus-type harmonic extensions nearby the disc-type harmonic extension, (i.e. for $\rho \in (0, \rho_0)$), and prove a uniform convergence result. Next, in local coordinate charts, we obtain local estimates for the $H^{k,2}(k \geq 2)$-norm uniformly with respect to $\rho \in (0, \rho_0)$ using an argument in [LU].
Here we may have a singularity which can however be removed from well known arguments (see [SkU], [Grü2], [Jo1]). The convergence of the Jacobi field as the derivative of the harmonic operator is proved with a similar argument.
Then it is possible to show the continuity of $\mathcal{E}$ and $\delta_{x^i}\mathcal{E}$. Since we consider the Riemannian situation, a careful analysis is needed in doing so.
Moreover, we can also apply this argument to show the continuity of $\mathcal{E}$ and $\delta_{x^i}\mathcal{E}$ with respect to variables in M^i.

With these results we define critical points of $\mathcal{E}$. We will see the equivalence between the harmonic extensions (in N) of critical points of $\mathcal{E}$ and minimal surfaces in N. In the appendix we will prove, by using the method in [St1], that the harmonic extension of a critical point of $\mathcal{E}$ is in the class $H^{2,2}$. The case of an annulus in a manifold is handled with careful calculation.
Using the $H^{2,2}$-regularity, we can apply the arguments given in the Euclidean space ([St4]) in the Riemannian case as well and prove that the harmonic extensions of critical points of $\mathcal{E}$ are conformal parametrized, so minimal surfaces.
For the converse direction we use the regularity of minimal surfaces in N from [HH].

In Chapter 5, we investigate the so called Palais-Smale condition. As it is well known, for this we fix three points on the given boundary curves (three points condition) for the case of disc type minimal surfaces, since the Dirichlet-Integral is invariant under conformal transformations. However, in the case of an annulus, we can fix only one point on the boundary, so we need to define a new setting. We extend here the idea of [St4]. In the proof of Palais-Smale condition, we investigate carefully the behavior of boundary mappings which are fixed at only one point. For this we will use the estimates given in Chapter 4 and the uniform continuity of the derivatives of $\mathcal{E}$ as $\rho \to \rho_0 \in [0, 1)$.

The basic idea of the Ljusternik-Schnirelmann theory is to investigate the topological structures of the level sets of a given function. Certain flows with respect to the gra-

dient of the function are used to deform the level sets. Thus, in order to deform level sets of $\mathcal{E}$, a suitable vector field (related to the above new setting) and a flow for the gradient vector field of $\mathcal{E}$ are computed concretely. From the definition of $\mathcal{T}_{x^i}$, we will see the construction of the flow more exactly.
Particularly, in the case of (C2), when an element in M^i is moved by the exponential map in N with respect to a direction in $\mathcal{T}_{x^i}$, we can expect to reach an element again in M^i only for a tangent vector with small length, i.e. $\leq l_i(x^i)$ for some $l_i(x^i) > 0$, since the harmonic operator is (in general) locally defined. From the compactness condition of N, we obtain the constants l_i independent of $x^i \in M^i$. Here, in fact, the property that the closure of the image of N in $\mathbb{R}^k$ is compact, is used. These constants enable us to obtain the above vector field.

The property, that the energy of annulus-type harmonic extensions ($\rho \leq \rho_0$) are greater than the energy of two disc-type harmonic extensions with uniform positive constant, is necessary for our aim. In Euclidean spaces, this holds with a uniform positive constant (depending on $\rho \leq \rho_0$) on any set of x^i where x^i is uniformly bounded. However in manifolds no results are known to the author. In Lemma 5.2.4, we will generate this result to the case of a Riemannian manifold but with a uniform positive constant (depending on $\rho \leq \rho_0$) on a certain set of x^i with more restriction than in Euclidean spaces. This somewhat weaker result should be enough to prove our claim.

Then we can follow the arguments in the critical point theory as in [St1] and in the main theorem we conclude, if there exists a minimal surface (of annulus type) whose energy is a strict relative minimum in $\mathcal{S}(\Gamma_1, \Gamma_2)$ (suitably defined for each case (C1) and (C2)), the existence of an unstable minimal surfaces of annulus type can be ensured under certain assumptions which are related to the solutions of $\mathcal{P}(\Gamma_i)$.
As corollaries we apply this main result to the three-dimensional sphere S^3 resp. the three-dimensional hyperbolic H^3, where the curvature is 1 resp. -1. In particular, in the case of H^3, the existence of a strict relatve minimal surface of annulus type guarantees an unstable minimal surface of annulus type, because of the known uniqueness result for minimal surfaces of disc type, bounded by a Jordan curve with total curvature $\leq 4\pi$ (see [LJ]).

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Chapter 2

Preliminaries

2.1 Some definitions

Let M be a 2-dimensional Riemannian manifold with metric (g_{ij}), coordinate chart (z^1, z^2) and N a n-dimensional Riemannian manifold with metric $(h_{\alpha\beta})$, coordinate chart $(y^1, \cdots, y^n)$. Furthermore N is embedded isometrically and properly into some $\mathbb{R}^k$ by η as a closed submanifold (see [Gro]).

• (N, h) is homogeneously regular if there exist positive constants C', C'' and to every $p \in N$ a local coordinate system (V, ψ) such that

(i) $\psi(V) = \mathfrak{U}(0) \subset \mathbb{R}^n$, $\psi(p) = 0$,

(ii) $\forall q \in V$, $\forall \eta \in T_q N$:

$$C'|\psi_{*,q}\eta|^2 \leq h(q)\langle \eta, \eta \rangle \leq C''|\psi_{*,q}\eta|^2, \tag{2.1}$$

where the vector $\psi_{*,q}\eta$ is the image of η by ψ.

• For $f \in C^2((M, g), (N, h))$, df is a section of $T^*M \otimes f^*TN$, i.e.

$$df = f^\alpha_{,i} dz^i \otimes \frac{\partial}{\partial y^\alpha} \circ f.$$

We use the summation convention for indices and a colon denotes the ordinary derivative with $i = 1, 2$, $\alpha = 1, \cdots, n$.

The covariant derivative of df in the bundle $T^*M \otimes f^*TN$ is denoted by ∇df:

$$\begin{aligned} \nabla df &= \nabla(f^\alpha_{,i} dz^i \otimes \frac{\partial}{\partial y^\alpha} \circ f) \\ &= (f^\alpha_{,ij} - f^\alpha_{,k}\Gamma^k_{ij} + f^\beta_{,i} f^\gamma_{,j} \Gamma^\alpha_{\beta\gamma} \circ f) dz^i \otimes dz^j \otimes \frac{\partial}{\partial y^\alpha} \circ f. \end{aligned}$$

The energy of f is defined by

$$E(f) := \frac{1}{2}\int_M |df|^2 dM_g = \frac{1}{2}\int_M g^{ij} h_{\alpha\beta} \circ f f^\alpha_{,i} f^\beta_{,j} dM_g.$$

The Euler-Lagrange equation of E for $f \in C^2((M,g),(N,h))$, called the tension field along f, is as follows:

$$\begin{aligned} \tau_h(f) &:= \langle \nabla_{\frac{\partial}{\partial z^i}} df, dz^i \rangle = g^{ij} (\nabla df)^\alpha_{ij} \\ &= g^{ij} (f^\alpha_{,ij} - f^\alpha_{,k} \Gamma^k_{ij} + f^\beta_{,i} f^\gamma_{,j} \Gamma^\alpha_{\beta\gamma} \circ f) \frac{\partial}{\partial y^\alpha} \circ f. \end{aligned} \tag{2.2}$$

And $f \in C^2((M,g),(N,h))$ is called harmonic if $\tau_h(f) = 0$.

• For $f \in C^1(M,N)$, a section of the bundle f^*TN, $V = V^\alpha \frac{\partial}{\partial y^\alpha} \circ f \in C^1(M, f^*TN)$ is called a vector field along f with the covariant derivative along f as follows:

$$\nabla^f V := (V^\alpha_{,i} + V^\gamma f^\beta_{,i} \Gamma^\alpha_{\beta\gamma}) dz^i \otimes \frac{\partial}{\partial y^\alpha} \circ f.$$

The covariant energy of V is then defined by

$$E(V) := \frac{1}{2}\int_M |\nabla^f V|^2 dM_g,$$

and let

$$D(V) := \int_M g^{ij} h_{\alpha\beta} \circ f V^\alpha_{,i} V^\beta_{,j} dM_g.$$

• Let ∇ resp. $\widetilde{\nabla}$ be the covariant derivative with respect to (N,h) resp. $\mathbb{R}^k$. For $f = (f^a)_{a=1,\cdots,k}$, the second fundamental form of η is :

$$\widetilde{\nabla} df - \nabla df = (f^a_{,i} f^b_{,j} \widetilde{\Gamma^c_{ab}} \circ f - f^a_{,i} f^b_{,j} \Gamma^c_{ab} \circ f) dz^i \otimes dz^j \otimes \frac{\partial}{\partial y^\alpha} \circ f,$$

and

$$II \circ f(df, df) := \langle \widetilde{\nabla}_{\frac{\partial}{\partial z^i}} df - \nabla_{\frac{\partial}{\partial z^i}} df, dz^i \rangle \in T^\perp_{f(\cdot)} \eta(N).$$

2.2 Spaces of functions

In this section we recall the definitions of the Sobolev spaces in $\mathbb{R}^k$ and in a Riemannian manifold (see [St1], [Ho] and [Bu]).

In this paper $d\omega$ denotes the area element in $\Omega \subset \mathbb{R}^2$ and d_0 the (1-dimensional) area element in $\partial\Omega$.

- The space $H^{\frac{1}{2},2}(\partial B, \mathbb{R}^k)$, for $B := B_1(0)$

First, we consider the trace space $H^{\frac{1}{2},2}(\partial B, \mathbb{R}^k) \cong H^{1,2}/H_0^{1,2}(B, \mathbb{R}^k)$ to be the set of equivalence classes $X|_{\partial B} \cong X + H_0^{1,2}(B, \mathbb{R}^k)$.
Let

$$D(X) := \frac{1}{2}\int_B |\nabla X|^2 d\omega, \text{ for } X \in H^{1,2}(B, \mathbb{R}^k).$$

For $X \in H^{1,2}(B, \mathbb{R}^k)$, there exists a unique (harmonic) $X_0 \in X + H_0^{1,2}(B, \mathbb{R}^k)$ with

$$D(X_0) = \inf\{D(X')|X' \in X + H_0^{1,2}(B, \mathbb{R}^k)\},$$

since D is coercive and weakly lower semi-continuous on $(X+)H_0^{1,2}(B, \mathbb{R}^k)$ which is closed with respect to the weak topology, and the uniqueness is from the weak maximum principle.

Then, the space $H^{\frac{1}{2},2}(\partial B, \mathbb{R}^k) := \{X|_{\partial B}|X \in H^{1,2}(B, \mathbb{R}^k)\}$ is a Hilbert space with the scalar product

$$\langle X|_{\partial B}, Y|_{\partial B}\rangle = \int_{\partial B} XY d_0 + \int_B \nabla X_0 \nabla Y_0 d\omega,$$

where $\frac{1}{2}\left|X|_{\partial B}\right|_{\frac{1}{2}}^2 := D(X_0)$ is in fact a semi-norm with (see [Ni] §§ 310 - 311)

$$D(X_0) = \frac{1}{16\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|X(e^{i\theta}) - X(e^{i\theta'})|^2}{\sin^2\left(\frac{\theta-\theta'}{2}\right)} d\theta d\theta'.$$

For $x \in H^{\frac{1}{2},2} \cap L^\infty(\partial B, \mathbb{R}^k)$ which is a Banach space with norm

$$\|x\|_{\frac{1}{2},2;\infty} := \|\nabla X_0\|_{L^2} + \|x\|_\infty,$$

the above X_0 is necessary harmonic, so the harmonic extension

$$\mathcal{H} : H^{\frac{1}{2},2} \cap L^\infty(\partial B, \mathbb{R}^k) \longrightarrow H^{1,2} \cap L^\infty(B, \mathbb{R}^k), \text{ with } \mathcal{H}(x) = X_0,$$

is a linear isomorphism, since

$$\|\mathcal{H}(x)\|_{1,2;\infty} := \|\nabla\mathcal{H}(x)\|_{L^2} + \|\mathcal{H}(x)\|_\infty = |x|_{\frac{1}{2}} + \|x\|_\infty =: \|x\|_{\frac{1}{2},2;\infty}.$$

- For $B := B_1(0)$, $H^{1,2} \cap C^0(B, \mathbb{R}^k)$ is a Banach space and define

$$H^{1,2} \cap C^0(B, N) := \{f \in H^{1,2} \cap C^0(B, \mathbb{R}^k)|f(B) \subset N\}.$$

For $f \in H^{1,2} \cap C^0(B,N)$ and $q \in H^{1,2} \cap C^0(B, f^*TN)$, define

$$\exp_f q : B \to N \text{ with } [\exp_f q](\cdot) := \exp_{f(\cdot)} q(\cdot).$$

Then $H^{1,2} \cap C^0(B,N)$ is an infinite dimensional Banach manifold of class C^∞ by the following maps: for $f_0 \in H^{1,2} \cap C^0(B,N)$,

$$\exp_{f_0} : W \longrightarrow H^{1,2} \cap C^0(B,N), \tag{2.3}$$

where exp is the exponential map in $(\mathbb{R}^k, r)$ for some other metric r in which (N,h) is a totally geodesic submanifold, and W is a small neighborhood of

$$\mathrm{im}(\pi_{f_0}) := \{q \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | q(\cdot) \in T_{f_0(\cdot)}N\}$$

such that $\exp_{f_0}$ is diffeomorphism onto $\exp_{f_0}(W) \ni f_0$, since $d(\exp_{f_0})_0 = Id$. Here $\pi_{f_0}(f)(\cdot) := \pi(f_0(\cdot), f(\cdot))$, π is the projection from $N \times \mathbb{R}^k$ into TN.
Since N is a submanifold of $\mathbb{R}^k$,

$$N \times \mathbb{R}^k = TN \oplus \mathrm{Nor}(N),$$

and

$$H^{1,2} \cap C^0(B, \mathbb{R}^k) = \mathrm{im}(\pi_{f_0}) \oplus \mathrm{im}(\pi_{f_0}^\perp),$$

where $\pi_{f_0}^\perp(f)(\cdot) := \pi^\perp(f_0(\cdot), f(\cdot))$, $\pi^\perp$ is the projection from $N \times \mathbb{R}^k$ into $\mathrm{Nor}(N)$. And $\{(\exp_{f_0}(W), (\exp_{f_0})^{-1})\}$ is a Banach manifold chart.

The tangent space of $f \in H^{1,2} \cap C^0(B,N)$ is canonically isomorphic to the space of vector fields along f, i.e.

$$T_f H^{1,2} \cap C^0(B,N) \cong \{V \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | V(\cdot) \in T_{f(\cdot)}N\} =: H^{1,2} \cap C^0(B, f^*TN)$$

with norm

$$\|V\| := \Big(\int_B |\nabla^f V|^2 d\omega\Big)^{\frac{1}{2}} + \|V\|_\infty \cong \Big(\int_B |dV|^2 d\omega\Big)^{\frac{1}{2}} + \|V\|_\infty, \tag{2.4}$$

where dV means the ordinary gradient in $\mathbb{R}^k$, more exactly, $d(\eta_f V)$.

• Let Γ be a Jordan curve in N which is diffeomorphic to $S^1 := \partial B$. Then N can be equipped with some metric $\tilde{h}$ such that Γ is a geodesic in $(N, \tilde{h})$. And $(N, \tilde{h})$ is embedded into $\mathbb{R}^{\tilde{k}}$ for some $\tilde{k}$ by $\tilde{\eta}$.
Repeat the above construction with the exponential map in $(N, \tilde{h})$, denoted by $\widetilde{\exp}$. Then $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_{\tilde{h}}\big)$ and $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_h\big)$ coincide as sets, and the latter is a Banch submanifold of $H^{1,2} \cap C^0(B,N)$ with

$$T_f H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)\big) = \{V \in T_f H^{1,2} \cap C^0(B,N) | V(z) \in T_{f(z)}\Gamma \text{ for all } z \in \partial B\}.$$

Note that $TN = T\Gamma \oplus (\mathrm{Nor}(\Gamma) \cap TN)$, so

$$\begin{aligned} H^{1,2} \cap C^0(B, \mathbb{R}^k) &= \mathrm{im}(\pi_{f_0}) \oplus \mathrm{im}(\pi_{f_0}^{\perp}) \\ &= \mathrm{im}(\pi_{\Gamma f_0}) \oplus \mathrm{im}(\pi_{\Gamma f_0})^{\perp}|_{\mathrm{im}(\pi_{f_0})} \oplus \mathrm{im}(\pi_{f_0}^{\perp}). \end{aligned}$$

- The space $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$ is also a Banach space with norm

$$\|u\|_{\frac{1}{2},2;0} := \|\nabla \mathcal{H}(u)\|_{L^2} + \|u\|_{C^0}.$$

Define

$$H^{\frac{1}{2},2} \cap C^0(\partial B, N) := \{u \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) \subset N\}.$$

This is also a Banach submanifold of $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$ by the exponential map with

$$T_u H^{\frac{1}{2},2} \cap C^0(\partial B, N) \cong H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TN).$$

Now define

$$\begin{aligned} H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) &:= \{u \in H^{\frac{1}{2},2} \cap C^0(\partial B, N) | u(\partial B) = \Gamma\} \\ &= \{u \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) = \Gamma\}, \end{aligned}$$

with

$$\begin{aligned} T_u H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) &:= \{\xi \in H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TN) | \xi(z) \in T_{u(z)}\Gamma, \text{ for all } z \in \partial B\} \\ &= H^{\frac{1}{2},2} \cap C^0(\partial B, u^*T\Gamma). \end{aligned}$$

Chapter 3

The setting and harmonic extensions

3.1 The setting

Let (N, h) be a connected, oriented, complete Riemannian manifold of dimension $n \geq 2$ and embedded isometrically and properly into some $\mathbb{R}^k$ as a closed submanifold by η. And let Γ_1, Γ_2 be two Jordan curves of class C^3 in N which are diffeomorphic to the unit circle S^1 with strict positive distance, i.e. $\mathrm{dist}(\Gamma_1, \Gamma_2) > 0$.

Moreover,

$$\begin{aligned} B &= \{w \in \mathbb{R}^2 \mid |w| < 1\}, \\ A_\rho &= \{w \in B \mid \rho < |w| < 1\} \qquad \text{and} \end{aligned}$$

$$C_1 = \{w \mid |w| = 1\}, \qquad C_\rho = \{w \mid |w| = \rho\} =: C_2 \,(\rho, \text{fixed}) \quad \text{with} \quad \partial A_\rho = C_1 \cup C_\rho,$$

where $\rho \in (0, 1)$.

We will investigate minimal surfaces in (N, h) which are harmonic and conformal, so we need harmonic extensions in (N, h), and the following well known facts will be used.

Theorem 3.1.1. *Let (M, g) be a compact connected $m-$dimensional Riemannian manifold with boundary Σ, and (N, h) a complete Riemannian manifold without boundary of dimension ≥ 2 . Assume that the image of $\Phi \in H^{1,2}(M, N)$ is contained in a ball $B(p, \rho)$ which lies within normal range of all of its points with*

$$0 < \rho < \pi/(2\sqrt{\kappa}), \tag{3.1}$$

$\kappa \geq 0$ being an upper bound for the sectional curvature in N. Then there is a weakly harmonic mapping $\mathcal{F} \in H^{1,2}(M, N)$ with $\mathcal{F}(M) \subset B(p, \rho)$ such that the traces of Φ and

$\mathfrak{F}$ on Σ coincide. And the weakly harmonic mapping is harmonic.
Furthermore if the trace of Φ is continuous, then $\mathfrak{F}$ is in $C^0(\overline{M}, N)$ (in fact if the trace of $\Phi \in C^k$, then $\mathfrak{F}$ is in $C^k(\overline{M}, N)$).

Proof. See [HKW3]. □

Theorem 3.1.2. *Let M, N be as in Theorem 3.1.1. Suppose that $u_i : \overline{M} \to N$ (i=1,2) are harmonic maps of class $C^0(\overline{M}, N) \cap C^2(M, N)$ and $u_i(\overline{M}) \subset B(p, r)$, where $r < \min\{\pi/(2\sqrt{\kappa}), i(p)\}$, κ is an upper bound for the sectional curvature in N and $i(p)$ is the injectivity radius of $p \in N$.*
If $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then $u_1 = u_2$.

Proof. See [JK]. □

Theorem 3.1.3. *Let $\overline{M}$ be a compact surface with $\partial M \neq \emptyset$ and (N, h) a connected homogeneously regular Riemannian manifold with $\pi_2(N) = 0$. For $\Phi \in H^{1,2} \cap C^0(\overline{M}, N)$ there exists a harmonic map $\mathfrak{F} \in C^\infty(M, N) \cap C^0(\overline{M}, N)$ which is relative homotopic to Φ (coincides with Φ on ∂M) and energy minimizing among all such maps.*
If N has nonpositive sectional curvature the solution $\mathfrak{F}$ is unique in every relative homotopy class of extensions.

Proof. See [ES], [Le], [Hm]. □

From the above Theorems we will consider two types of conditions for $(N, h)(\supset \Gamma_1, \Gamma_2)$.

First, we introduce, for $i = 1, 2$,

$$\mathfrak{X}^i_{\text{mon}} := \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \mid \text{weakly monotone onto } \Gamma_i\},$$

where 'weakly monotone onto Γ_i' means that the parametrization can be described with a weakly monotone map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $w^i(\theta + 2\pi) = w^i(\theta) \pm 2\pi$ and a diffeomorphism $\gamma^i : \partial B \to \Gamma_i$ (see **III-a)**.

I) Condition (C1): we consider the following condition from Theorem 3.1.1.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

Notation In this paper, $B(p, r)$ denotes a ball of $p \in N$ with the properties in the condition (C1).

Remark 3.1.1. *If $\Gamma_1, \Gamma_2 \subset N$ satisfy (C1), for each $x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ and $\rho \in (0,1)$ there exists $g_\rho \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p,r))$ and $g^i \in H^{1,2} \cap C^0(\overline{B}, B(p,r))$ with $g_\rho|_{C_1} = x^1$, $g_\rho|_{C_\rho}(\cdot) = x^2(\frac{\cdot}{\rho})$ and $g^i|_{\partial B} = x^i$, $i = 1,2$.*

Proof. Consider $\exp : \Omega \to B(p,r) \subset N$ with $\Omega := \exp^{-1}(B(p,r)) \subset B(0,\tilde{r})_{\mathbb{R}^n} \subset \mathbb{R}^n$ for some $\tilde{r} > 0$.
For $\widetilde{x}^i := \exp^{-1}(x^i) \in H^{\frac{1}{2},2} \cap C^0(\partial B, \widetilde{\Gamma}_i)$ with $\widetilde{\Gamma}_i := \exp^{-1}(\Gamma_i) \subset \Omega$,we have an Euclidean harmonic extension $h_\rho(\widetilde{x}^1, \widetilde{x}^2)$ of finite energy (from the choice of x^i), whose image is in $B(0,\tilde{r})_{\mathbb{R}^n}$ from the maximum principle. The map exp is a diffeomorphism and Ω is star shape, so there exists a retraction $\delta : B(0,\tilde{r})_{\mathbb{R}^n} \to \Omega$ with $\delta|_\Omega = Id$ in the class of $H^{1,2}$. Then the map $g_\rho := \exp(\delta(h_\rho(\widetilde{x}^1, \widetilde{x}^2))) : A_\rho \to \Omega$ is an $H^{1,2} \cap C^0(\overline{A_\rho}, B(p,r))$-extension with boundary x^1 and $x^2(\frac{\cdot}{\rho})$.
We may also find an $H^{1,2} \cap C^0(\overline{B}, B(p,r))$-extension. □

From Theorem 3.1.1, Theorem 3.1.2 and the above Remark, we have a unique harmonic map of annulus and of disc type in $B(p,r) \subset N$ for a given boundary mapping in the class of $H^{\frac{1}{2},2} \cap C^0$. Among these boundary parametrizations we take mappings with same orientation as follows:

Definition

(i) Let $i = 1,2$, and define

$$\begin{aligned} M^i &:= \{x^i \in \mathcal{X}^i_{\text{mon}} \,|\, \text{orientation preserving}\} \\ &= \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \,|\, x^i \text{ is weakly monotone, orientation preserving}\}, \end{aligned}$$

(ii) and for $x^i \in M^i$,

$$\mathcal{F}_\rho(x^1, x^2)(\text{resp. } \mathcal{F}^i(x^i)) : A_\rho(\text{resp. } B) \to B(p,r) \subset N, \;\; i = 1,2$$

denote the unique harminic extension of annulus resp. of disc type in $B(p,r) \subset N$.

Then M^i is complete with respect to the norm $\|\cdot\|_{\frac{1}{2},2;0}$, since the norm $\|\cdot\|_{C^0}$ preserves the weakly monotonicity and the orientation.

Now we investigate another alternative condition for (N,h).

II) Condition (C2): we consider the following condition from Theorem 3.1.3.

(C2) N is compact with nonpositive sectional curvature.

A compact Riemannian manifold is homogeneously regular and the condition of non-positive sectional curvature for N implies $\pi_2(N) = 0$.

II-a) In order to define the trace space M^i, $i = 1, 2$ in the case of (C2), we need some preparation.

First, we consider for $\rho \in (0, 1)$:

$$\tilde{G_\rho} := \{f \in H^{1,2} \cap C^0(\overline{A_\rho}, N) |\, f|_{C_i} \text{ is continuous and weakly monotone onto } \Gamma_i\}.$$

Then we may take a continuous homotopy class from $\tilde{G_\rho}$, denoted by $\tilde{F_\rho}$ in $H^{1,2}(A_\rho, N)$, i.e. every two elements f, g in $\tilde{F_\rho}$ are continuous homotopic (not necessarily relative), denoted by $f \sim g$, more exactly:

$$f \sim g \quad \Leftrightarrow \quad \begin{array}{l} \text{there exists a continuous mapping } H : [0,1] \times \overline{A_\rho} \to N \\ \text{with } H(0, \cdot) = f(\cdot), H(1, \cdot) = g(\cdot). \end{array}$$

In addition to the above choice, we demand the following property: for any $\rho, \sigma \in (0, 1)$,

$$\tilde{F_\rho} \sim \tilde{F_\sigma} \Leftrightarrow \tilde{f}(r, \theta) = f(\tau_\sigma^\rho(r), \theta) \text{ for some } \tilde{f} \in \tilde{F_\sigma}, \; f \in \tilde{F_\rho},$$

where τ_σ^ρ is some diffeomorphism from the interval $[\sigma, 1]$ onto the interval $[\rho, 1]$. Clearly, letting $\tilde{F_\rho}$ fixed as above, for any $\sigma \in (0, 1)$, we can find $\tilde{F_\sigma}$ with $\tilde{F_\rho} \sim \tilde{F_\sigma}$, for example, choosing $\tau_\sigma^\rho(r) := \frac{(1-\rho)r+\rho-\sigma}{1-\sigma}$. And $\tilde{F_\rho} \sim \tilde{F_\sigma}$ is an equivalence relation.

Now we consider a homotopy family and take its trace space as follows:

Definition

(i) Let

$$\tilde{\mathcal{S}}(\Gamma_1, \Gamma_2) := \{f \in \tilde{F_\rho} \,|\, 0 < \rho < 1\}, \tag{3.2}$$

(ii) and define

$$\begin{aligned} \widetilde{M^1} &:= \{x^1(\cdot) = f|_{C_1}(\cdot) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_1) \,|\; f \in \tilde{\mathcal{S}}(\Gamma_1, \Gamma_2)\}, \\ \widetilde{M^2} &:= \{x^2(\cdot) = f|_{C_\rho}(\cdot\rho) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_2) \,|\; f \in \tilde{\mathcal{S}}(\Gamma_1, \Gamma_2)\}, \end{aligned}$$

with a subspace topology of $H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ for $\rho \in (0, 1)$.

Then $\widetilde{M^i} \subset \mathcal{X}^i_{\text{mon}}$, $i = 1, 2$. We will now see some properties of $\widetilde{M^i}, i = 1, 2$.

Notation For $x^1 \in \widetilde{M^1}, x^2 \in \widetilde{M^2}$ there exists a unique $\mathbb{R}^k$-harmonic extension on A_ρ with $x^1(\cdot)$ on C_1 and $x^2(\frac{\cdot}{\rho})$ on C_ρ. This extension will be denoted by $\mathcal{H}_\rho(x^1, x^2)$. Also, $\mathcal{H}(x)$ means the $\mathbb{R}^k$-harmonic extension of disc type for $x \in H^{\frac{1}{2},2}(\partial B, \mathbb{R}^k)$.

Lemma 3.1.1. *Let $x \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$, then the sequence of the Dirichlet-Integrals of $\mathcal{H}_\rho(x,0)$ converges to that of $\mathcal{H}(x)$ as $\rho \to 0$, i.e.*

$$\int_{A_\rho} |d\mathcal{H}_\rho(x,0)|^2 d\omega \to \int_B |d\mathcal{H}(x)|^2 d\omega \quad as \quad \rho \to 0,$$

uniformly on any bounded set of x.

Proof. See [St4], Lemma 4.2. □

Lemma 3.1.2. *(i) For each $x_0^i \in \widetilde{M^i}$, $i = 1, 2$, there exists $\varepsilon(x_0^i) > 0$ such that*

$$if\ x^i \in \mathcal{X}^i_{mon}\ with\ \|x^i - x_0^i\|_{\frac{1}{2},2;0} < \varepsilon, \ then\ x^i \in \widetilde{M^i}.$$

(ii) $\widetilde{M^i}$ is complete with respect to $\|\cdot\|_{\frac{1}{2},2;0}$.

Proof. *(i)* Let $\|x^i - x_0^i\|_{\frac{1}{2},2;0} < \varepsilon$ with $\varepsilon > 0$ to be determined later on.
From the definition of $\widetilde{M^i}$ there exists a $f_\rho \in \tilde{F}_\rho$ with $f_\rho|_{C_1}(\cdot) = x_0^1(\cdot)$ and $f_\rho|_{C_\rho}(\cdot) = y^2(\frac{\cdot}{\rho})$ for some $y^2 \in \widetilde{M^2}$.
Considering submanifold coordinate neighbourhoods for $N(\overset{\eta}{\hookrightarrow} \mathbb{R}^k)$, we may take a finite covering of $f_\rho((\overline{A_\rho}))$, and by projection we obtain a smooth map $r : \mathcal{N}_\delta(f_\rho(\overline{A_\rho})) \to N$ with $r|_{\mathcal{N}_\delta(f_\rho(\overline{A_\rho}))\cap N} = Id$, for some $\delta > 0$, where $\mathcal{N}_\delta(f_\rho(\overline{A_\rho}))$ is δ-neighbourhood of $f_\rho(\overline{A_\rho})$ in $\mathbb{R}^k$.
Since $\|\mathcal{H}_\rho(x^1 - x_0^1, 0)\|_{C^0} < \varepsilon$, the map $f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0)$ is from A_ρ into $\mathcal{N}_\delta(f_\rho(\overline{A_\rho}))$ for $\varepsilon < \delta$. Then we can consider the map $r(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0)) : A_\rho \to N$ and compute as follows:

$$\begin{aligned}
&\int_{A_\rho} |dr(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0))|^2 d\omega \\
&\quad = \int_{A_\rho} |dr(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0))(df_\rho + d\mathcal{H}_\rho(x^1 - x_0^1, 0))|^2 d\omega \\
&\quad \le C(\|f_\rho\|_{C^0}, \|\mathcal{H}_\rho(x^1 - x_0^1, 0)\|_{C^0}, r) \left(\int_{A_\rho} |df_\rho|^2 d\omega + \int_{A_\rho} |d\mathcal{H}_\rho(x^1 - x_0^1, 0)|^2 d\omega \right) \\
&\quad \le C(\|f_\rho\|_{C^0}, \varepsilon, N) \left(\int_{A_\rho} |df_\rho|^2 d\omega + C(\rho) \int_B |d\mathcal{H}(x^1 - x_0^1)|^2 d\omega \right) \quad \text{(by Lemma 3.1.1)} \\
&\quad \le C(\|f_\rho\|_{C^0}, \varepsilon, N) \left(\int_{A_\rho} |df_\rho|^2 d\omega + \|x^1 - x_0^1\|_{\frac{1}{2},2;0} \right) \\
&\quad \le C(\|f_\rho\|_{C^0}, \|f_\rho\|_{1,2}, \varepsilon, N),
\end{aligned}$$

so we have an $H^{1,2}$ extension $r(f_\rho+\mathcal{H}_\rho(x^1-x^1_0,0))$ for x^1, since $r(f_\rho+\mathcal{H}_\rho(x^1-x^1_0,0))|_{C_1} = x^1$.

Now consider a homotopy map $H(t,\cdot) := (1-t)\mathcal{H}_\rho(x^1-x^1_0,0) : [0,1]\times A_\rho \to \mathbb{R}^k$ with $H(0,\cdot) = \mathcal{H}_\rho(x^1-x^1_0,0)(\cdot)$ and $H(1,\cdot)=0.$, $\|H\|_{C^0} < \varepsilon$.
Let $G : [0,1]\times A_\rho \to N$ with $G(t,\cdot) = f_\rho(\cdot)$ for all $t\in[0,1]$.
Then $r(G+H) : [0,1]\times A_\rho \to N$ is a homotopy between f_ρ and $r(f_\rho+\mathcal{H}_\rho(x^1-x^1_0,0))$ in N.
Hence $r(f_\rho+\mathcal{H}_\rho(x^1-x^1_0,0))(\sim f_\rho)$ is an element of $\tilde{F}_\rho$ for each $\rho\in(0,1)$ and we have

$$x^1 \in \widetilde{M^1}, \quad \text{if} \quad \|x^1-x^1_0\|_{\frac{1}{2},2;0} < \varepsilon, \quad \text{for some} \quad \varepsilon < \delta,$$

Similarly, we can prove that $x^2 \in \widetilde{M^2}$ if $\|x^2-x^2_0\|_{\frac{1}{2},2;0} < \varepsilon'$ for some small $\varepsilon' > 0$.

(ii) A cauchy sequence $\{x^i_n\} \subset M^i$ converges to $x^i \in H^{\frac{1}{2},2}\cap C^0(\partial B;\Gamma_i)$, and for some n, $\|x^i_n - x^i\|_{C^0} < \varepsilon$. Considering $\mathcal{H}_\rho(x^1-x^1_n,0)$ and $g_\rho \in F_\rho$ with boundary x^1_n on C_1 and 0 on the other boundary, we can find a homotopy in N between g_ρ and $r(g_\rho+\mathcal{H}_\rho(x^1-x^1_0,0))$ as in *(i)*. We may also apply this argument for x^2.
Note that x^i is weakly monotone, and hence $x^i \in M^i$. □

II-b) Disc type extension in N and the definition of M^i.

We now consider all the possible $H^{1,2}\cap C^0$-extensions of disc type in N, as follows:

$$\mathcal{S}(\Gamma_i) := \{X \in H^{1,2}\cap C^0(\overline{B},N) | X|_{\partial B} \text{ is weakly monotone onto } \Gamma_i\}.$$

And we assume that $\mathcal{S}(\Gamma_i)$ is not empty for each $i=1,2$. This implies, Γ_i can be shrunk to a point in N.

Then we observe the following properties.

Lemma 3.1.3. *(i) For $X^1 \in \mathcal{S}(\Gamma_1)$ and $X^2 \in \mathcal{S}(\Gamma_2)$, there exists $f_\rho \in H^{1,2}\cap C^0(A_\rho,N)$ such that $f_\rho|_{C_1}(\cdot) = X^1|_{\partial B}(\cdot)$ and $f_\rho|_{C_\rho}(\cdot) = X^2|_{\partial B}(\frac{\cdot}{\rho})$, for $\rho\in(0,1)$.*

(ii) Moreover, there exists $\rho_0 \in (0,1)$ and a uniform positive constant C such that for some $f_\rho \in H^{1,2}\cap C^0(A_\rho,N)$, with $f_\rho|_{C_\rho}(\cdot) = X^2|_{\partial B}(\frac{\cdot}{\rho})$

$$E(f_\rho) \le C, \text{ for all } \rho \le \rho_0. \tag{3.3}$$

Proof. *(i)* Let $i=1,2$. Since $X^i \in H^{1,2}\cap C^0(\overline{B},N)$, for given $\varepsilon>0$, there exists $\sigma_i > 0$ such that

$$\operatorname{osc}_{B_{\sigma_i}} X^i < \varepsilon.$$

Consider $T(s,\theta) := (\frac{1}{s}\rho, \theta)$ in polar coordinate, where $\rho > 0$ is so small that $\frac{\rho}{\sigma_2} < \sigma_1$. This is a conformal transformation of $B \backslash B_{\sigma_2}$ onto $B_{\frac{\rho}{\sigma_2}} \backslash B_\rho$.

Let $\mathcal{H} : B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}} \to \mathbb{R}^k$ harmonic with

$$\mathcal{H}|_{\partial B_{\sigma_1}} = X^1|_{\partial B_{\sigma_1}} - X^1(0), \quad \mathcal{H}|_{\partial B_{\frac{\rho}{\sigma_2}}} = X^2|_{\partial B_{\sigma_2}} - X^2(0),$$

then $\|\mathcal{H}\|_{C^0} < \varepsilon$.
Now let $g \in H^{1,2} \cap C^0(B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, N)$ with $X^1(0)$ on ∂B_{σ_1} and $X^2(0)$ on $\partial B_{\frac{\rho}{\sigma_2}}$. Such a g exists, since N is a (path-)connected Riemannian manifold.
Then using the argument and notation in the proof of Lemma 3.1.2, $r \circ (g + \mathcal{H})$ is in $H^{1,2} \cap C^0(B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, N)$ with boundary $X^1|_{\partial B_{\sigma_1}}$ and $X^2|_{\partial B_{\sigma_2}}$.

Now define

$$f_\rho := \begin{cases} X^1|_{B \backslash B_{\sigma_1}} & , \text{ on } B \backslash B_{\sigma_1}, \\ r \circ (g + \mathcal{H}) & , \text{ on } B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, \\ X^2(T^{-1}(\cdot)) & , \text{ on } B_{\frac{\rho}{\sigma_2}} \backslash B_\rho. \end{cases} \tag{3.4}$$

Then f has all the desired properties.

(ii) This assertion follows from the above construction, since $\frac{\rho}{\sigma_2} < \sigma_1$, $\rho \le \rho_0$ for some $\rho_0 > 0$ and by Lemma 3.1.1. □

By assumption ($\mathcal{S}(\Gamma_i) \neq \emptyset$), for given $\Gamma_i \in N$ we have an annulus type extension like the above (3.4), and we take homotopy classes as defined in **II-a)** which include such an extension. We denote this setting by the same notations as in **II-a)** but without 'tilde'. Thus, repeating the construction in **II-a)** we define:

Definition

(i) Let

$$\mathcal{S}(\Gamma_1, \Gamma_2) := \{ f \in F_\rho \,|\, 0 < \rho < 1 \}, \tag{3.5}$$

(ii) and define

$$\begin{aligned} M^1 &:= \{ f|_{C_1}(\cdot) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_1) \,|\, \text{orientation preserving}, \;\; f \in \mathcal{S}(\Gamma_1, \Gamma_2) \}, \\ M^2 &:= \{ f|_{C_\rho}(\cdot \rho) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_2) \,|\, \text{orientation preserving}, \;\; f \in \mathcal{S}(\Gamma_1, \Gamma_2) \}, \end{aligned}$$

with a subspace topology of $H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$.

Now applying once again the arguments in Lemma 3.1.2 and Lemma 3.1.1, we notice that: Let $x^i \in M^i$ possess the extensions $f_\rho \in \mathcal{S}(\Gamma_1, \Gamma_2)$ with a uniform bounded energy for $\rho \leq \rho_0$, i.e. satisfy the property in Lemma 3.1.3 *(ii)*.
Then the elements in a ε-neighborhood of x^i in $\mathcal{X}^i_{\text{mon}}$ also possess such extensions for some $\varepsilon(x^i) > 0$. Hence, the set of the elements x^i's which possess annulus type extensions with uniform energy with respect to $\rho \leq \rho_0$ is an open subset of $\mathcal{X}^i_{\text{mon}}$. On the other hand, applying the argument *(ii)* in Lemma 3.1.2, we see that this set is also closed. Thus, this is a non-empty (from the choice of M^i) connceted component of $\mathcal{X}^i_{\text{mon}}$. This implies, this set must be the same as M^i, since M^i is also a connected component in $\mathcal{X}^i_{\text{mon}}$. Hence we obtain the following property.

Remark 3.1.2. *For each $x^i \in M^i, i = 1, 2$, there exist $f_\rho \in \mathcal{S}(\Gamma_1, \Gamma_2)$ and $C > 0$ with $E(f_\rho) \leq C$ for all $\rho \leq \rho_0$ for some $\rho_0 \in (0, 1)$. Clearly, this result also holds for $x^i \in M^i$ in the case of (C1).*

We will now discuss disc-type extensions for $x^i \in M^i$. We will make use of the following Lemmata.

Lemma 3.1.4. *Let (N, h) be a homogeneously regular manifold and u an absolutely continuous map on $\partial B_r(x_0)$ into $N \ni x_0$ with*

$$\int_0^{2\pi} |u'(\theta)|_h^2 d\theta \leq \frac{C'}{\pi}. \tag{3.6}$$

Then there exists $f \in H^{1,2}(B_r(x_0), N) \cap C^0(\overline{B_r(x_0)}, N)$ with $f|_{\partial B_r(x_0)} = u$ and

$$E_{B_r(x_0)}(f) \leq \frac{C''}{C'} \int_0^{2\pi} |u'(\theta)|_h^2 d\theta,$$

where C'', C' are the constants from the homogeneously regularity(see (2.1)).

Proof. See [Mo2] Lemma 9.4.8 b). □

Lemma 3.1.5 (From the Courant-Lebesgue Lemma). *Let $f \in H^{1,2}(A_\rho, (N, h)), 0 < \rho < 1, (N, h)$ is a Riemannian manifold. Then for each $\delta \in (\rho, 1)$ there exits a $\tau \in (\delta, \sqrt{\delta})$ with*

$$\int_0^{2\pi} \left| \frac{\partial f(\tau, \theta)}{\partial \theta} \right|_h^2 d\theta \leq \frac{4E(f)}{\ln \frac{1}{\delta}}.$$

Proof. $f \in H^{1,2}(A_\rho, (N,h))$ and we compute: with $|\cdot| := |\cdot|_h$, by Fubini's Theorem,

$$\begin{aligned} 2E(f) &= \int_{A_\rho} |df|^2\, d\omega = \int_0^{2\pi}\int_\rho^1 \left(\left|\frac{\partial f}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial f}{\partial \theta}\right|^2 \right) r dr d\theta \\ &\geq \int_\rho^1 \frac{1}{r}\int_0^{2\pi} \left|\frac{\partial f}{\partial \theta}\right|^2 d\theta dr \geq \int_\delta^{\sqrt{\delta}} \frac{1}{r}\int_0^{2\pi} \left|\frac{\partial f}{\partial \theta}\right|^2 d\theta dr \\ &\geq \int_\delta^{\sqrt{\delta}} \frac{1}{r} \mathrm{essinf}_{\delta \leq r \leq \sqrt{\delta}} \left(\int_0^{2\pi} \left| \frac{\partial f(r,\theta)}{\partial \theta}\right|^2 d\theta \right) dr \\ &\geq \int_\delta^{\sqrt{\delta}} \frac{1}{r}\int_0^{2\pi} \left| \frac{\partial f(\tau,\theta)}{\partial \theta}\right|^2 d\theta dr \qquad\qquad (\text{for some } \tau \in (\delta, \sqrt{\delta})) \\ &\geq \frac{1}{2}\ln\frac{1}{\delta}\int_0^{2\pi} \left| \frac{\partial f(\tau,\theta)}{\partial \theta}\right|^2 d\theta, \end{aligned}$$

since $E(f) < \infty$. □

For $x^i \in M^i$, from Remark 3.1.2 and the choice of $\mathcal{S}(\Gamma_1, \Gamma_2)$, we can find $f_\rho \in H^{1,2}(A_\rho, N)$ with boundary x^i such that $E(f_\rho) \leq C$ for all $\rho \leq \rho_0$. Then from Lemma 3.1.5, we can choose $\tau \in (\delta, \sqrt{\delta})$ such that $f_\rho(\tau, \cdot) : \partial B_\tau \to N$ is absolutely continuous with $\int_0^{2\pi} |f'_\rho(\theta)|_h^2 d\theta \leq \frac{C'}{\pi}$ for some $\rho \leq \rho_0$. By Lemma 3.1.4, we have $g_\tau \in H^{1,2}(B_\tau, N)$ with boundary $f|_{\partial B_\tau}$.
Together with g_τ and $f|_{B \setminus B_\tau}$, we obtain a map $X \in H^{1,2}(B, N)$ with boundary x^1.
To prove the existence of a map in $H^{1,2}(B, N)$ with boundary x^2, we consider the conformal transformation $T(re^{i\theta}) = \rho \frac{1}{re^{i\theta}}$ for $re^{i\theta} \in A_\rho$ (or $T(r,\theta) = (\frac{\rho}{r}, \theta)$ which preserves the orientation). Since the Dirichlet-Integral is invariant under conformal mappings, we have $\tilde{f} \in H^{1,2}(A_\rho, N)$ with $\tilde{f}|_{C^1} = x^2$.

Moreover, the harmonic extension of disc type for each $x^i \in M^i$ in N is unique, independently of the choice of a homotopy class $\mathcal{S}(\Gamma_1, \Gamma_2)$, because of the following well-known fact.

Lemma 3.1.6. $\pi_2(N) = 0 \Leftrightarrow$ *Any* $h_0, h_1 \in C^0(B, N)$ *with* $h_0|_{\partial B} = h_1|_{\partial B}$ *are homotopic.*

On the other hand, using the construction (3.4) and by the above Lemma we can easily check that the traces of elements in $\mathcal{S}(\Gamma_i)$ are included in M^i, i.e. we can find homotopic mappings of type (3.4).

Hence we have the following results.

Remark 3.1.3. *(i) For $x^i \in M^i$, there exists a unique harminic extension of disc type and a unique harmonic extension of annulus type defined on A_ρ for each $\rho \in (0,1)$.*

(ii) The elements of M^i are actually the traces of $f \in \mathcal{S}(\Gamma_i)$ (up to the condition of preserving orientation).

III) Now let (N, h) and $\Gamma_i, i = 1, 2$ satisfy (C1) or (C2).

III-a) We will introduce a kind of tangent space of x^i in M^i.

First, let us consider a diffeomorphism of class C^3,

$$\gamma^i : \partial B \to \Gamma_i, i = 1, 2,$$

and the projection map $\pi : \mathbb{R} \to \mathbb{R}/2\pi (\approx \partial B)$.
For any given $y^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ which is weakly monotone (of degree 1) and oriented, there exists a weakly monotone map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $w^i(\theta + 2\pi) = w^i(\theta) + 2\pi, \ \theta \in \mathbb{R}$ such that

$$y^i(\pi(\theta)) = \gamma^i(\cos(w^i(\theta)), \sin(w^i(\theta))) =: \gamma^i \circ w^i(\cdot). \tag{3.7}$$

We note that $w^i = \tilde{w^i} + Id$ for some $\tilde{w^i} \in C^0(\partial B, \mathbb{R})$, and in the last term of (3.7), w^i is actually considered as a map of ∂B. Roughly speaking, w^i can be considered as a map in $C^0(\partial B, \partial B)$ and then w^i is unique for given y^i, whereas $w^i \in C^0(\mathbb{R}, \mathbb{R})$ is unique up to $2\pi l$, $l \in Z$. And whether w^i is in $C^0(\partial B, \partial B)$ or $C^0(\mathbb{R}, \mathbb{R})$, it will be determined according to a given situation, simply denoting $y^i = \gamma^i \circ w^i$.

We define further

$$W^i_{\mathbb{R}^k} := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) \mid \text{weakly monotone}, w^i(\theta + 2\pi) = w^i(\theta) + 2\pi; D(\mathcal{H}(\gamma^i \circ w^i)) < \infty\},$$

where D is the Dirichlet -Integral and $\mathcal{H}$ is the disc-type Harmonic extension in $\mathbb{R}^k$.

From the condition $w^i(\theta + 2\pi) = w^i(\theta) + 2\pi$, $W^i_{\mathbb{R}^k}$ is convex. For further details, we refer to [St1].

Now we define for $x^i \in M^i$, considering $w - w^i$ as a tangent vector along $\tilde{w^i}$ denoted by $(w - w^i)\frac{d}{d\theta} \circ \tilde{w^i}$

$$\mathcal{T}_{x^i} = \{d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w^i}) \mid w \in W^i_{\mathbb{R}^k} \text{ and } \gamma^i \circ w^i = x^i\}.$$

Then, $\mathcal{T}_{x^i}$ is convex in $T_{x^i}H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$, since $W^i_{\mathbb{R}^k}$ is convex.

Since γ^i is geodesic in $(N, \tilde{h})$, for $\xi = d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w^i}) \in \mathcal{T}_{x^i}$,

$$\widetilde{\exp}_{x^i}\xi = \widetilde{\exp}_{x^i}(d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w^i})) = \gamma^i(w^i + w - w^i) = \gamma^i(w). \tag{3.8}$$

In case of (C1), $\widetilde{\exp}_{x^i}\xi \in M^i$ for $\xi \in \mathcal{T}_{x^i}$.

For the case (C2), let us recall the proof of Lemma 3.1.2. For some small $\delta > 0$, there exists a retraction r from δ-neighborhood of Nin $\mathbb{R}^k$ onto N, since N is compact. Together with the argument in the proof of Lemma 3.1.2, this implies that there exists some $\delta > 0$, independent of $x^i_0 \in M^i$, such that for $x^i \in \mathcal{X}^i_{\mathrm{mon}}$ with preserving orientation

$$\text{if } \|x^i - x^i_0\| < \delta, \quad \text{then } x^i \in M^i.$$

Moreover, from (3.8) there exists $l_i > 0$, $i = 1, 2$, depending on γ^i, such that for any $x^i \in M^i$,

$$\widetilde{\exp}_{x^i}\xi \in M^i, \quad \text{if } \|\xi\|_{\mathcal{T}_{x^i}} < l_i.$$

Definition

(i) First,

$$W^i_{\mathbb{R}^k} := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) | \text{weakly monotone}, w^i(\cdot + 2\pi) = w^i(\cdot) + 2\pi; D(\mathcal{H}(\gamma^i \circ w^i)) < \infty\},$$

(ii) and we define for $x^i \in M^i$

$$\begin{aligned} \mathcal{T}_{x^i} &:= \{d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w^i}) \mid w \in W^i_{\mathbb{R}^k} \text{ and } \gamma^i \circ w^i = x^i\}, \\ \mathcal{T}^0_{x^i} &:= \{\xi \in \mathcal{T}_{x^i} \mid \|\xi\|_{\frac{1}{2},2;0} < l_i\}, \end{aligned}$$

then $\widetilde{\exp}_{x^i}\xi \in M^i$ for any $\xi \in \mathcal{T}^0_{x^i}$.

We want to notice the following observation.

Remark 3.1.4. *From Lemma 3.1.2, M^i is a non-empty connected component of $\mathcal{X}^i_{mon}$. But the set $\mathcal{X}^{i,ori}_{mon}$ (orientation preserving) is path connected, since $W^i_{\mathbb{R}^k}$ is convex. Hence, also for (C2), we have actually*

$$M^i := \{H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \mid \textit{weakly monotone and orientation preserving}\}.$$

And this means, for any $\xi^i \in \mathcal{T}_{x^i}$ we can expect that $\widetilde{\exp}_{x^i}\xi^i \in M^i$. Thus, the above l_i, specially in the definition of the ciritical points of $\mathcal{E}$ (see (4.20) in section 4.2), is independent of $x^i \in M^i$ hence the compactness condition for N (in the case of (C2)) is not necessary.
For the existence of harmonic extensions, the condition of 'homogeneously regular' is enough, and the homogeneously regularity of N implies, $i(N) > 0$. Thus, it seems that we can extend our results to the case of 'homogeneously regular' in (C2).

III-b) Together with the definitions in **I** and **II**, we have the following setting for both (C1) and (C2).

Definition and Notation

(i) With the product topology let

$$\mathcal{M} := M^1 \times M^2 \times (0,1)$$

and by x we denote an element of $\mathcal{M}$, i.e.

$$x := (x^1, x^2, \rho), \quad x^i \in M^i \ \ (i = 1,2), \ \ \rho \in (0,1)$$

with a convex set

$$\mathcal{T}_x\mathcal{M} = \mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}.$$

(ii) For $x^i \in M^i$, let $\mathcal{F}_\rho(x^1, x^2)$ be the unique solution of the following Dirichlet Problem:

$$\begin{aligned} \tau_h(\mathcal{F}_\rho(x^1,x^2)) &= 0 \text{ in } A_\rho \\ \mathcal{F}_\rho(x^1,x^2)(e^{i\theta}) &= x^1(e^{i\theta}) \text{ on } C_1 \\ \mathcal{F}_\rho(x^1,x^2)(\rho e^{i\theta}) &= x^2(e^{i\theta}) \text{ on } C_\rho(=\partial B_\rho), \end{aligned} \tag{3.9}$$

where $\tau_h(\cdot)$ is the tension field in (N,h) like (2.2).

And we define harmonic extension operator $\mathcal{F}_\rho$:

$$\begin{aligned} \mathcal{F}_\rho : M^1 \times M^2 &\longrightarrow H^{1,2} \cap C^0(\overline{A_\rho}, N) \\ (x^1, x^2) &\longmapsto \mathcal{F}_\rho(x^1, x^2). \end{aligned}$$

Furthermore for $x = (x^1, x^2, \rho) \in \mathcal{M}$,

$$\mathcal{F}(x) = \mathcal{F}(x^1, x^2, \rho) := \mathcal{F}_\rho(x^1, x^2).$$

(iii) Define

$$\begin{aligned} \mathcal{E} : \mathcal{M} &\longrightarrow \mathbb{R} \\ x &\longmapsto E(\mathcal{F}(x)), \end{aligned}$$

where

$$E(\mathcal{F}(x)) := \frac{1}{2}\int_{A_\rho} |d\mathcal{F}_\rho(x^1, x^2)|_h^2 d\omega. \tag{3.10}$$

(iv) **Note that:** but $\mathcal{M}$ is not closed, so we also need

$$\partial\mathcal{M} := M^1 \times M^2 \times \{0\} \ni x := (x^1, x^2, 0),$$

with

$$\mathcal{T}_x\partial\mathcal{M} := \mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \quad \text{for } x \in \partial\mathcal{M}.$$

And

$$\overline{\mathcal{M}} := \mathcal{M} \cup \partial\mathcal{M}.$$

(v) Let $\mathcal{F}^i(x^i)$ be the unique solution of the following Dirichlet Problem:

$$\begin{aligned} \tau_h(\mathcal{F}^i(x^i)) &= 0 \text{ in } B \\ \mathcal{F}^i(x^i)(e^{i\theta}) &= x^i(e^{i\theta}) \text{ on } \partial B, \end{aligned} \tag{3.11}$$

and we define harmonic extension operator $\mathcal{F}^i$:

$$\begin{aligned} \mathcal{F}^i : M^i &\longrightarrow H^{1,2} \cap C^0(\overline{B}, N) \\ x^i &\longmapsto \mathcal{F}^i(x^i). \end{aligned}$$

(vi) Finally for $x = (x^1, x^2, 0) \in \partial\mathcal{M}$,

$$\begin{aligned} \mathcal{E}(x) &= E(\mathcal{F}^1(x^1)) + E(\mathcal{F}^2(x^2)) \\ &= \frac{1}{2}\int_B |d\mathcal{F}^1(x^1)|_h^2 d\omega + \frac{1}{2}\int_B |d\mathcal{F}^2(x^2)|_h^2 d\omega. \end{aligned}$$

We now have a well defined map $\mathcal{E} : \mathcal{M} \cup \partial\mathcal{M} \longrightarrow \mathbb{R}$ together with (3.10) for $x = (x^1, x^2, \rho) \in \mathcal{M}$.

3.2 Harmonic extension operators

Here we will discuss the derivatives of $\mathcal{F}_\rho$ and $\mathcal{F}^i$ with respect to the variables in $T_{x^i}H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ denoted by $T_{x^i}H^{\frac{1}{2},2} \cap C^0$, noting that the harmonic extension exists at least locally in $H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ from Lemma 3.1.2.

Recall the operators: for fixed $\rho \in (0,1)$,

$$\begin{array}{rcl} \mathcal{F}_\rho : M^1 \times M^2 & \longrightarrow & H^{1,2} \cap C^0(\overline{A_\rho}, N) \\ x = (x^1, x^2) & \longmapsto & \mathcal{F}_\rho(x^1, x^2), \end{array}$$

and for $i = 1, 2$

$$\begin{array}{rcl} \mathcal{F}^i : M^i & \longrightarrow & H^{1,2} \cap C^0(\overline{B}, N) \\ x^i & \longmapsto & \mathcal{F}^i(x^i). \end{array}$$

Consider a 2-parameter variation f_{st} such that $f(0,t)$ are harmonic and $f(s,0)|_{\partial M} = f|_{\partial M}$ with

$$\frac{\partial f_{st}}{\partial s}\Big|_{s,t=0} = v, \quad \frac{\partial f_{st}}{\partial t}\Big|_{s,t=0} = w, \text{ then}$$

$$\frac{\partial^2 E(f_{st})}{\partial s \partial t}\Big|_{s,t=0} = \int \langle \nabla^f v, \nabla^f w \rangle - \langle tr\, R(v, df)df, w \rangle d\omega = 0.$$

Hence, a Jacobi field which is a vector field along a harmonic mapping f as a weak solution of

$$\int \langle \nabla \mathbf{J}, \nabla X \rangle + \langle tr\, R(\mathbf{J}, df)df, X \rangle d\omega = 0, \text{ for all } X \in H^{1,2} \cap L^\infty(\cdot, f^*TN),$$

is a natural candidate of derivative of harmonic operators $\mathcal{F}_\rho$ and $\mathcal{F}^i$.

For $\xi^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^i)^*T\Gamma_i)$, a weak Jacobi field along $\mathcal{F} := \mathcal{F}_\rho(x^1, x^2)$ with boundary $\xi^1, \xi^2(\frac{\cdot}{\rho})$ has the following minimality property:

$$\int_{A_\rho} |\nabla^{\mathcal{F}} \mathbf{J}_{\mathcal{F}}|^2 - \langle trR(\mathbf{J}_{\mathcal{F}}, df)df, \mathbf{J}_{\mathcal{F}} \rangle d\omega \le \int_{A_\rho} |\nabla^{\mathcal{F}} X|^2 - \langle trR(X, df)df, X \rangle d\omega,$$

for all $X \in H^{1,2}(A_\rho, \mathcal{F}^*TN)$ with $X|_{C_i} = \xi^i$. The analogous property holds for Jacobi fields along the harmonic extension on B.

Then we have the following property of the weak Jacobi fields.

Lemma 3.2.1. *(i) The above weak Jacobi field with boundary $\eta \in T_{x^i}H^{\frac{1}{2},2} \cap C^0$ along a harmonic $\mathcal{F}$ with boundary x^i is well defined in the class $H^{1,2}$ and continuous until the boundary with*

$$\begin{aligned} \|\mathbf{J}_{\mathcal{F}}\|_0 &\leq \|\mathbf{J}_{\mathcal{F}}|_{\partial M}\|_0, \\ \|\mathbf{J}_{\mathcal{F}}\|_{1,2;0} &\leq C(N, \|f\|_{1,2:0})\|\mathbf{J}_{\mathcal{F}}|_{\partial M}\|_{\frac{1}{2},2;0}, \end{aligned}$$

(ii) let f_t be a variation of harmonic mappings with $\frac{d}{dt}(f_t|_{\partial M}) \in (f_0|_{\partial M})^*TN$ then

$$\frac{f_t - f_0}{t} \longrightarrow \mathbf{J}_{\mathcal{F}_\rho} \quad \text{in } H^{1,2} \cap C^0(A_\rho, \mathbb{R}^k) \quad \text{as } t \to 0,$$

for $M = A_\rho$ or $M = B$.

Proof. In the case of B, it follows from the results in [Ho]. And the similar arguments can be applied to the case of A_ρ. □

Now we can talk about the differentiabilty of the harmonic extension operators.

Lemma 3.2.2. *The operators $\mathcal{F}_\rho, \mathcal{F}^i$ are partially differentiable in x^1, x^2 with respect to variations in $T_{x^1}H^{\frac{1}{2},2} \cap C^0$ resp. $T_{x^2}H^{\frac{1}{2},2} \cap C^0$ with the following derivatives: at a point $x_0 = (x_0^1, x_0^2, \rho)$ for $\rho \in (0,1)$ resp. $x_0 = (x_0^1, x_0^2, 0)$,*

$$\begin{aligned} D_{x^1}\mathcal{F}_\rho(x_0) : H^{\frac{1}{2},2} \cap C^0(\partial B, (x_0^1)^*T\Gamma_1) &\longrightarrow H^{1,2} \cap C^0(A_\rho, \mathcal{F}_\rho^*TN) \\ \xi &\longmapsto \mathbf{J}_{\mathcal{F}_\rho}(\xi, 0), \end{aligned}$$

$$\begin{aligned} D_{x^2}\mathcal{F}_\rho(x_0) : H^{\frac{1}{2},2} \cap C^0(\partial B, (x_0^1)^*T\Gamma_2) &\longrightarrow H^{1,2} \cap C^0(A_\rho, \mathcal{F}_\rho^*TN) \\ \xi &\longmapsto \mathbf{J}_{\mathcal{F}_\rho}(0, \xi(\frac{\cdot}{\rho})) \end{aligned}$$

resp.

$$\begin{aligned} D_{x^i}\mathcal{F}^i(x_0) : H^{\frac{1}{2},2} \cap C^0(\partial B, (x_0^i)^*T\Gamma_i) &\longrightarrow H^{1,2} \cap C^0(B, (\mathcal{F}^i)^*TN) \\ \xi &\longmapsto \mathbf{J}_{\mathcal{F}^i}(\xi), \end{aligned}$$

where $\mathcal{F}_\rho := \mathcal{F}_\rho(x_0^1, x_0^2)$ resp. $\mathcal{F}^i := \mathcal{F}^i(x_0^i)$ and $\mathbf{J}_{\mathcal{F}_\rho}(\dagger, \ddagger)$ is a Jacobi field along $\mathcal{F}_\rho$ with boundary $\dagger$ on C_1 and $\ddagger$ on C_ρ resp. $\mathbf{J}_{\mathcal{F}^i}(\xi)$ is a Jacobi field along $\mathcal{F}^i$ with boundary ξ on ∂B.
The derivatives are also continuous with respect to x^1, x^2.

Proof. We must show that:

letting $\mathcal{F}_\rho := \mathcal{F}_\rho(x_0^1, x_0^2)$, $D_{x^1}\mathcal{F}_\rho = \mathbf{J}_{\mathcal{F}_\rho}$ is continuous with respect to $x^1 \in M^1$ in the sense that for $\eta^1 \in T_{x^1}H^{\frac{1}{2},2} \cap C^0, \eta^2 \in T_{x^2}H^{\frac{1}{2},2} \cap C^0$,

$$L_n\left(\mathbf{J}_{\mathcal{F}_\rho(x_n^1,x_0^2)}(d\widetilde{\exp}_{x_0^1,\xi_n}(\eta^1),\eta^2)\right) \longrightarrow \mathbf{J}_{\mathcal{F}_\rho}(\eta^1,\eta^2) \text{ in } H^{1,2}\cap C^0(A_\rho, \mathcal{F}_\rho^*TN)$$
$$\text{as} \quad \xi_n \longrightarrow \xi \quad \text{in } H^{\frac{1}{2},2}\cap C^0(\partial B, (x_0^1)^*T\Gamma_1),$$

where $x_n^1 = \widetilde{\exp}_{x_0^1}\xi_n$, for some $\xi_n \in T_{x_0^1}H^{\frac{1}{2},2} \cap C^0$ and $\exp_{\mathcal{F}_\rho}\phi_n = \mathcal{F}_\rho(x_n^1, x_0^2)$, for some $\phi_n \in H^{1,2} \cap C^0(A_\rho, \mathcal{F}_\rho^*TN)$ such that $\xi_n \to 0$ and $\mathcal{F}_\rho(x_n^1, x_0^2) \to \mathcal{F}_\rho(x_0^1, x_0^2)$, note that $x_n^1 \to x_0^1$ in M^1. And $L_n := (d\exp_{\mathcal{F}_\rho,\phi_n})^{-1}$.

These can be proved with a similar argument to the proof of Lemma 4.1.1 (B), (C). We can also use the proof in [Ho].

Similarly, it can be proved in the case of x^2 with respect to variable in $T_{x^2}H^{\frac{1}{2},2} \cap C^0$ and in the case of $D_{x^i}\mathcal{F}^i, i = 1, 2$. □

Chapter 4

The variational problem

4.1 Differentiability of $\mathcal{E}$ on $\overline{\mathcal{M}}$

Let

$$\mathcal{E} : \overline{\mathcal{M}} \longrightarrow \mathbb{R}$$

be as in Chapter 3.

Lemma 4.1.1. *We have,*

(A) *$\mathcal{E}$ is continuously partially differentiable in x^1, x^2 with respect to variations in $T_{x^1}M^1$, $T_{x^2}M^2$ and the derivatives are continuous in $M^1 \times M^2$,*

(B) *$\mathcal{E}$ is continuous with respect to $\rho \in [0,1)$, uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ which is independent of $x_0^i \in M^i$, $i = 1, 2$,*

(C) *and the partial derivatives in x^1, x^2 are also continuous with respect to $\rho \in [0,1)$, uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$, independent of $x_0^i \in M^i$, $i = 1, 2$,*

(D) *$\mathcal{E}$ is differentiable with respect to $\rho \in (0,1)$.*

Proof. Here and in the sequel, the continuity will be understood in the sense of sub-sequence.

(A) The Dirichlet-Integral functional is in C^∞, so by Lemma 3.2.2 $\mathcal{E}$ is continuously partially differentiable and have continuous partial derivatives on $M^1 \times M^2$.

Computation of derivatives:

• Let $x = (x^1, x^2, \rho) \in \mathcal{M}$, $\xi^1 \in \mathcal{T}_{x^1}$. By Lemma 3.1.2, $\widetilde{\exp}_{x^1}(t\xi^1) \in M^1$, $0 \le t \le t_0$ for some small $t_0 > 0$. Thus,

$$\begin{aligned}
\langle \delta_{x^1}\mathcal{E}, \xi^1\rangle &:= \frac{d}{dt}\Big|_{t=0} \mathcal{E}(\widetilde{\exp}_{x^1}(t\xi^1), x^2, \rho) \\
&:= \frac{1}{2}\int_{A_\rho} \frac{d}{dt}\Big|_{t=0} \left| d\mathcal{F}_\rho \left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right)\right|_h^2 d\omega \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla_{\frac{d}{dt}} d\mathcal{F}_\rho \left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right)\big|_{t=0}\rangle_h d\omega \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla D_{x^1}\mathcal{F}_\rho(x^1,x^2)(\xi^1)\rangle_h d\omega \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \qquad \text{(by Lemma 3.2.2)}, \qquad (4.1)
\end{aligned}$$

since with $\mathcal{F}_\rho(t) := \mathcal{F}_\rho\left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right)$,

$$\begin{aligned}
&\nabla_{\frac{d}{dt}}\left(\mathcal{F}^\alpha_{\rho,i}(t)dx^i \otimes \frac{\partial}{\partial y^\alpha}\circ \mathcal{F}_\rho(t)\right) \\
&= \frac{\partial}{\partial t}\mathcal{F}^\alpha_{\rho,i}(t)dx^i \otimes \frac{\partial}{\partial y^\alpha}\circ\mathcal{F}_\rho(t) + \mathcal{F}^\alpha_{\rho,i}(t)dx^i\otimes\frac{\partial y^\beta}{\partial t}\circ\mathcal{F}_\rho(t)\nabla_{\frac{\partial}{\partial y^\beta}}\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}_\rho(t) \\
&= \frac{\partial}{\partial x^i}\mathcal{F}^\alpha_{\rho,t}(t)dx^i\otimes\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}_\rho(t) + \mathcal{F}^\beta_{\rho,t}(t)dx^i\otimes\frac{\partial\mathcal{F}^\alpha_\rho}{\partial x^i}(t)\nabla_{\frac{\partial}{\partial y^\alpha}}\frac{\partial}{\partial y^\beta}\circ\mathcal{F}_\rho(t) \\
&= \frac{\partial}{\partial x^i}\left(\mathcal{F}^\alpha_{\rho,t}\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}_\rho(t)\right)\otimes dx^i \\
&= \nabla\frac{d}{dt}\mathcal{F}_\rho(t) = \nabla\frac{d}{dt}\mathcal{F}_\rho\left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right). \\
&(= \nabla\left(D_{x^1}\mathcal{F}_\rho(x^1,x^2)(\xi^1)\right) \qquad \text{if} \quad t=0)
\end{aligned}$$

And for $\xi^2 \in \mathcal{T}_{x^2}$ by Lemma 3.2.2,

$$\begin{aligned}
\langle \delta_{x^2}\mathcal{E}, \xi^2\rangle &= \int_{A_\rho}\langle d\mathcal{F}_\rho(x^1,x^2), \nabla_{\frac{d}{dt}} d\mathcal{F}_\rho\left(x^1, \widetilde{\exp}_{x^2}(t\xi^2)\right)\big|_{t=0}\rangle_h d\omega \\
&= \int_{A_\rho}\langle d\mathcal{F}_\rho(x^1,x^2), \nabla\left(D_{x^2}\mathcal{F}_\rho(x^1,x^2)(\xi^2)\right)\rangle_h d\omega \\
&= \int_{A_\rho}\langle d\mathcal{F}_\rho(x^1,x^2), \nabla\mathbf{J}_{\mathcal{F}_\rho}(0, \xi^2(\frac{\cdot}{\rho}))\rangle_h d\omega. \qquad (4.2)
\end{aligned}$$

• Similarly, for $x = (x^1, x^2, 0) \in \partial\mathcal{M}$,

$$\begin{aligned}
\langle \delta_{x^1}\mathcal{E}, \xi^1\rangle &:= \frac{d}{dt}\Big|_{t=0} \mathcal{E}(\widetilde{\exp}_{x^1}(t\xi^1), x^2) \\
&:= \frac{d}{dt}\Big|_{t=0} \left[E\left(\mathcal{F}^1(\widetilde{\exp}_{x^1} t\xi^1)\right) + E\left(\mathcal{F}^1(x^2)\right)\right] \\
&= \int_B \langle d\mathcal{F}^1(x^1), \nabla\left(D\mathcal{F}^1(x^1)(\xi^1)\right)\rangle_h d\omega \\
&= \int_B \langle d\mathcal{F}^1(x^1), \nabla \mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega, \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
\langle \delta_{x^2}\mathcal{E}, \xi^2\rangle &= \int_B \langle d\mathcal{F}^2(x^2), \nabla\left(D\mathcal{F}^2(x^2)(\xi^2)\right)\rangle_h d\omega \\
&= \int_B \langle d\mathcal{F}^2(x^2), \nabla \mathbf{J}_{\mathcal{F}^2}(\xi^2)\rangle_h d\omega. \qquad (4.4)
\end{aligned}$$

(B) The continuity of $\mathcal{E}$ as $\rho \to \rho_0$ is now to prove. We will discuss only the case that $\rho_0 = 0$, i.e.:

$$\int_{A_\rho} \left|d\mathcal{F}_\rho(x^1, x^2)\right|_h^2 d\omega \longrightarrow \int_B \left|d\mathcal{F}^1(x^1)\right|_h^2 d\omega + \int_B \left|d\mathcal{F}^2(x^2)\right|_h^2 d\omega, \quad \rho \to 0$$

uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ which is independent of $x_0^i \in M^i$.

The proof for the case $\rho_0 \in (0, 1)$ is similar and somewhat easier.

We will prove the above assertion in several steps.

B-I) Two maps from $\mathcal{F}_\rho(x^1, x^2)$.

Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ and $\mathcal{F}^i := \mathcal{F}^i(x^i), i = 1, 2$.
$\mathcal{F}_\rho \in H^{1,2}(A_\rho, N)$, so by Lemma 3.1.5 for each δ with $0 < \rho < \delta < 1$, there exists $\nu \in (\delta, \sqrt{\delta})$ such that

$$\begin{aligned}
\int_0^{2\pi} \left|\frac{\partial\mathcal{F}_\rho(\nu,\theta)}{\partial\theta}\right|_h d\theta &\le \sqrt{2\pi}\left(\int_0^{2\pi} \left|\frac{\partial\mathcal{F}_\rho(\nu,\theta)}{\partial\theta}\right|_h^2 d\theta\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}\left(\frac{4E(\mathcal{F}_\rho)}{\ln\frac{1}{\delta}}\right)^{\frac{1}{2}} \le \frac{C}{\sqrt{|\ln\delta|}}, \qquad (4.5)
\end{aligned}$$

where C is independent of $\rho \le \rho_0$ for some $\rho_0 \in (0, 1)$ from Remark 3.1.2 and the choice of M^i.

We now construct two mappings from $\mathcal{F}_\rho$: define

$$f_\nu : A_\nu \longrightarrow N \qquad \text{with} \quad f_\nu(\omega) := \mathcal{F}_\rho(\omega), \;\; \omega \in A_\nu,$$

and

$$g_{\frac{\rho}{\nu}} : A_{\frac{\rho}{\nu}} \longrightarrow N \qquad \text{with} \quad g_{\frac{\rho}{\nu}}(re^{i\theta}) := \mathcal{F}_\rho(T(re^{i\theta})), \;\; re^{i\theta} \in A_{\frac{\rho}{\nu}}, \tag{4.6}$$

where $T(r,\theta) = (\frac{\rho}{r}, \theta)$ in polar coordinate which maps $A_{\frac{\rho}{\nu}}$ onto $B_\nu \backslash B_\rho$ and is conformal. Choosing $\delta \in (\rho, 1)$ and $\nu \in (\delta, \sqrt{\delta})$ with the property (4.5) such that $\frac{\rho}{\nu} \to 0$, $\nu \to 0$ as $\rho \to 0$, for example, $\delta := \sqrt{\rho}$, we have the following: letting $\nu' := \frac{\rho}{\nu}$

- f_ν is harmonic map on A_ν into N with $f_\nu|_{\partial B} = x^1$,
- $g_{\nu'}$ is harmonic map on $A_{\nu'}$ into N with $g_{\nu'}|_{\partial B} = x^2$,
- $\operatorname{osc}_{\partial B_\nu} f_\nu$, $\operatorname{osc}_{\partial B_{\nu'}} g_{\nu'} \to 0$ as $\rho \to 0$,
- T is conformal, so by the conformal invariance of the Dirichlet-Integral we have

$$E(\mathcal{F}_\rho) = E(\mathcal{F}_\rho|_{A_\nu}) + E(\mathcal{F}_\rho|_{B_\nu \backslash B_\rho}) = E(f_\nu) + E(g_{\nu'})$$

  for each $\rho > 0$ with $\nu, \nu' \to 0$ as $\rho \to 0$.

B-II) The uniform convergence of $\{f_\nu\}$, $\{g_{\nu'}\}$.

We first investigate the modulus of continuity of harmonic maps $\{h_\nu\}$, defined on A_ν into N, which converge uniformly (C^0-norm) on ∂B with $E(h_\nu) \leq L$ for some $L > 0$, independent of $\nu \leq \nu_0$ for some $\nu_0 \in (0,1)$. We will discuss only the case (C2), because the argument in the case of (C2) can clearly be applied to the case (C1):

Let $G_R := \overline{B_R(z)} \subset A_\nu$ for $\nu \leq \tilde{\nu_0}$. If $z \in \partial B$, consider $G_R := \overline{B_R(z)} \cap \overline{A_\nu}$.
Given $\varepsilon > 0$, by the Courant-Lebesgue Lemma, there exists $\delta > 0$, independent of $\nu \leq \nu_0$, such that

$$\text{Length of}\, h_\nu|_{\partial G_\delta} \leq \min\{\frac{\varepsilon}{4}, \frac{i(N)}{4}\}.$$

By assumption, $i(N) > 0$. Then $h_\nu|_{\partial G_\delta} \subset B(q,s)$ for some $q \in N, s \leq \min\{\frac{\varepsilon}{2}, \frac{i(N)}{2}\}$. We observe, h_ν is continuous on ∂G_δ, and there exists an $H^{1,2}$-extension of disc type X, whose image is in $B(q,s)$ with $X|_{\partial B_\delta} = h_\nu|_{\partial B_\delta}$ from the same argument as in the proof of Remark 3.1.1. Thus, by Theorem 3.1.1, there exists a harmonic extension h' with $h'(G_\delta) \subset B(q,s) \subset B(q, \frac{\varepsilon}{2})$. From Lemma 3.1.6, h' is homotopic to h on G_δ, and from the energy minimizing property of harmonic maps, $h_\nu|_{G_\delta} = h'$. This implies that

$$|h_\nu(z') - h_\nu(z)|_h < \varepsilon, \;\; \text{if} \;\; |z' - z| < \delta \;\; \text{for all} \;\; \nu \leq \nu_0.$$

Hence, the functions h_ν with $\nu \leq \nu_0$ have the same modulus of continuity.

Moreover, if the above mappings are with the same boundary image, the mappings are C^0-uniform bounded on each relative compact domain, since these are with the same modulus of continuity from the above argument and uniformly bounded with respect to L^2-norm (the energies are uniformly bounded and from the Poincaré inequality).

Now considering our maps $\{\mathcal{F}_\rho,\, \rho \leq \rho_0\}$ in $\mathbb{R}^k$, we may obtain a locally (including ∂B) uniform convergent subsequence by the Arzela-Ascoli theorem.

Then we have that

- the functions f_ν resp. $g_{\nu'}$ have the same modulus of continuity for all $\rho \in (0, \rho_0)$, $\rho_0 \in (0,1)$ fixed, note that the modulus of continuity of $g_{\nu'}$ is also controlled by the map T, however $|T(z)| = |\frac{\rho}{z}| \leq |\frac{\rho_0}{z}|$,
- and some subsequences denoted again by f_ν $g_{\nu'}$ are locally uniform convergent.

B-III) The convergence of $\{f_\nu\}$, $\{g_{\nu'}\}$ to $\mathcal{F}^i$.

Recall that our mappings are continuous, so by localizing in domain and image, the solutions of the Dirichlet Problems (3.9), (3.11) may be regarded as weak solutions f of the following elliptic systems in local coordinate chart of N:

$$\nabla_i \nabla_i f^\alpha = -\Gamma^\alpha_{\beta\gamma} \nabla_i f^\beta \nabla_i f^\gamma =: G^\alpha(\cdot, f(\cdot), \nabla f(\cdot)). \tag{4.7}$$

Denoting the above uniform convergent subsequence again by f_ν and $g_{\nu'}$, and letting $\nu_0 := \nu(\rho_0)$, $\nu_0' := \nu'(\rho_0)$,

- we can assume the same coordinate charts for the image of $\{f_\nu\}_{\nu \leq \nu_0}$ and $\{g_{\nu'}\}_{\nu' \leq \nu_0'}$, hence the same weak solution system for (4.7).

Moreover, since $h_{\alpha\beta}$ and $\Gamma^\alpha_{\beta\gamma}$ of N are smooth,

- all the structural constants of the weak systems as in Lemma 4.1.2 are independent of $\rho \leq \rho_0$.

Now from Lemma 4.1.2(B), for each $z \in B\backslash\{0\}$ there exists $B_R(z) \subset\subset B\backslash\{0\}$ on which $f_{\nu(\rho)}$ are uniform bounded with repect to $H^{4,2}$-norm for all $\rho \geq \rho_0$, for some $\rho_0 \in (0,1)$. Note that $\nu(\rho) \to 0$ as $\rho \to 0$, from the construction. By covering argument, we can apply this argument to compact subsets of $B\backslash\{0\}$.
Hence we consider $K^\sigma_\sigma = \{\sigma \leq |z| \leq 1-\sigma\}$ with $\sigma > 0$. Then it holds, for some $C \in \mathbb{R}$

$$\left\| f_\nu|_{K^\sigma_\sigma} \right\|_{H^{4,2}} \leq C \quad \text{for all } \nu \in (0, \nu(\rho_0)),$$

and by the Sobolev's embedding Theorem for some sequence $\{\rho_i\} \subset (0,1)$,

$$\lim_{\rho_i \to 0} f_{\nu(\rho_i)}|_{K^\sigma_\sigma} = f' \qquad \text{in } C^2(K^\sigma_\sigma, \mathbb{R}^n),$$

with $\tau_h(f') = 0$ in K^σ_σ.

Now letting $\sigma := \frac{1}{n}$, choose sequence $\{f_{\nu(\rho_{n,i})}\}$ as above such that $\{\rho_{n+1,i}\}$ is a subsequence of $\{\rho_{n,i}\}$. Then by diagonalizing we have a subsequence $\{f_{\nu(\rho_{n,n})}\}, n \geq n_0$ which converges to f' locally with C^2-norm, so f' is harmonic on $B\backslash(\partial B \cup \{0\})$.

On the other hand $f_\nu|_{\partial B} = x^1$ for all ν, and f_ν converge uniformly to f' in a compact neighborhood of ∂B. Thus, f' is continuous on $\overline{B}\backslash\{0\}$ with $f'|_{\partial B} = x^1$.
We also notice that from the construction, $\text{osc}_{\partial B_r} f' \to 0$ as $r \to 0$.

For each compact subset K of $B\backslash(\partial B \cup \{0\})$,

$$\int_K |df'|^2 d\omega = \lim_{\rho_i \to 0} \int_K |df_{\nu(\rho_i)}|^2 \leq L$$

with L, independent of K, hence $f' \in H^{1,2}(B\backslash\{0\}, N)$ and f' can be extended to the whole disc B as a weakly harmonic map from Lemma 4.1.3 (see also [SkU], [Grü2]).

Thus, f' can be considerd as a weakly harmonic and $f' \in C^0(\overline{B}, N) \cap C^2(B, N)$ with $f'|_{\partial B} = x^1$, and from the uniqueness property we obtain, $f' = \mathfrak{F}^1(x^1)$.

We have the same result for $g_{\nu'}$, hence

$$\left\| f_{\nu(\rho_i)} - \mathfrak{F}^1(x^1) \right\|_{(C^2;K)} \to 0, \qquad \left\| g_{\nu'(\rho_i)} - \mathfrak{F}^2(x^2) \right\|_{(C^2;K)} \to 0 \quad \text{as } \rho_i \to 0,$$

for each compact region K in $B\backslash(\partial B \cup \{0\})$.

B-IV) The convergence of energy.

Consider $\eta \circ f$, denoted again by $f := (f^a)_{a=1,\cdots,k} \in H^{1,2}(M, \mathbb{R}^k)$. Since $\eta : N \to \mathbb{R}^k$ is isometric, for $f := (f^\alpha)_{\alpha=1,\cdots,n} \in H^{1,2}(M, N)$,

$$\int_M |d(f^\alpha)|_h^2 d\omega = \int_M |d(f^a)|_{\mathbb{R}^k}^2 d\omega.$$

Note that: letting M a compact manifold with boundary and $f : M \to N \overset{\eta}{\hookrightarrow} \mathbb{R}^k, f = (f^a)$ harmonic in $(\eta(N), h)$, for any $\psi \in H_0^{1,2} \cap C^0(M, \mathbb{R}^k)$,

$$\int_M (\langle df, d\psi \rangle - \langle II \circ f(df, df), \psi \rangle) dM = 0, \tag{4.8}$$

where II is the second fundamental form from η.

Letting $K_\sigma = \{\sigma \le |z| \le 1\}$ with $\sigma > 0$, fixed, and for $\nu \in (0,\sigma)$ consider harmonic maps in $\mathbb{R}^k$:

$$\begin{aligned} H_\nu : K_\sigma \to \mathbb{R}^k \qquad &\text{with} \quad H_\nu|_{\partial K_\sigma} = f_\nu|_{\partial K_\sigma}, \\ \widetilde{H}_\nu : K_\sigma \to \mathbb{R}^k \qquad &\text{with} \quad \widetilde{H}_\nu|_{\partial K_\sigma} = \mathcal{F}^1|_{\partial K_\sigma}. \end{aligned}$$

Also let $H : B \to \mathbb{R}^k$ be the harmonic map with $H|_{\partial B} = H_\nu|_{\partial B} = \widetilde{H}_\nu|_{\partial B} = x^1$, then both of $\{H_\nu\}, \{\widetilde{H}_\nu\}$ have the same modulus of continuity until ∂B, and

$$\|H_\nu - H\|_{C^0;K_\sigma} \to 0, \quad \|\widetilde{H}_\nu - H\|_{C^0;K_\sigma} \to 0 \quad \text{as } \nu \to 0.$$

And for $X_\nu := (f_\nu - \mathcal{F}^1) + (H_\nu - \widetilde{H}_\nu) \in H_0^{1,2} \cap C^0(K_\sigma, \mathbb{R}^k)$,

$$\|X_\nu\|_{(C^0;K_\sigma)} \le \|f_\nu - \mathcal{F}^1\|_{C^0;K_\sigma} + \|H_\nu - H\|_{C^0;K_\sigma} + \|H - \widetilde{H}_\nu\|_{C^0;K_\sigma} \to 0 \quad \text{as } \nu \to 0.$$

We compute then

$$\begin{aligned} &\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), d(f_\nu - \mathcal{F}^1)\rangle d\omega \\ &= \int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), dX_\nu\rangle d\omega - \int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), d(H_\nu - \widetilde{H}_\nu)\rangle d\omega \\ &= I + II. \end{aligned}$$

By (4.8),

$$\begin{aligned} |I| &\le \left|\int_{K_\sigma} \langle II \circ f_\nu(df_\nu, df_\nu), X_\nu\rangle d\omega\right| + \left|\int_{K_\sigma} \langle II \circ (d\mathcal{F}^1, d\mathcal{F}^1), X_\nu\rangle d\omega\right| \\ &= C\|X_\nu\|_{(C^0;K_\sigma)} \to 0, \qquad \text{as } \nu \to 0, \end{aligned} \tag{4.9}$$

because $\|f_\nu\|_{H^{1,2}} < \infty$, uniformly with respect to $\nu \le \nu_0$, and C is independent of ν.

Since $H_\nu - \widetilde{H}_\nu$ is harmonic in $\mathbb{R}^k$,

$$\begin{aligned} |II| &\le \left|\int_{\partial K_\sigma} \langle f_\nu - \mathcal{F}^1, \partial_r(H_\nu - \widetilde{H}_\nu)\rangle d\omega\right| \\ &\le \int_{\partial K_\sigma} \left|\partial_r(H_\nu - \widetilde{H}_\nu)\right| d\omega \|f_\nu - \mathcal{F}^1\|_{C^0;K_\sigma} \to 0 \quad \text{as } \nu \to 0. \end{aligned} \tag{4.10}$$

Thus, we have $\int_{K_\sigma} |d(f_\nu - \mathcal{F}^1)|^2 d\omega \to 0$ and by Minkowski's inequality it follows that

$$\int_{K_\sigma} |df_\nu|^2 \, d\omega \to \int_{K_\sigma} \left|d\mathcal{F}^1\right|^2 d\omega, \;\text{ as } \nu \to 0.$$

This holds for each $K_\sigma, \sigma \in (0,1]$. Thus, since $\int_{B_\sigma} |d\mathcal{F}^1|^2 d\omega \to 0$, we have that

$$\int_{A_\nu} |df_\nu|^2 \, d\omega \to \int_B \left|d\mathcal{F}^1\right|^2 d\omega \quad \text{as } \nu \to 0.$$

Similarly we also have that

$$\int_{A_{\nu'}} |dg'_\nu|^2 \, d\omega \to \int_B \left|d\mathcal{F}^2\right|^2 d\omega \quad \text{as } \nu' \to 0.$$

From the construction, ν and ν' goes to 0 as $\rho \to 0$, so we have

$$\int_{A_\rho} |d\mathcal{F}_\rho|^2 \, d\omega = \int_{A_\nu} |df_\nu|^2 \, d\omega + \int_{A'_\nu} |df_\nu|^2 \, d\omega \to \int_B \left|d\mathcal{F}^1\right|^2 d\omega + \int_B \left|d\mathcal{F}^2\right|^2 d\omega.$$

Now for the uniform convergence on $\mathcal{N}_\varepsilon(x_0^i)$, we recall the proof of Lemma 3.1.2 and replace $f(\overline{A_\rho})$ by $\overline{B(p,r)}$((C1)) and N (compact in (C2)). Then we have

$$\|\mathcal{F}_\rho(x^1,x^2)\|_{H^{1,2}} \leq C, \quad \text{uniformly on } \mathcal{N}_\varepsilon(x_0^i),$$

where the constant C is dependent on x_0^i, but ε is independent of x_0^i. And the convergence in (4.9), (4.10) is uniformly on $\mathcal{N}_\varepsilon(x_0^i)$.
Thus, we can ensure the existence of $\varepsilon > 0$ such that the above convergence is uniformly on $\mathcal{N}_\varepsilon(x_0^i) = \{x^i \in M^i : \|x^i - x_0^i\|_{\frac{1}{2},2;0} < \varepsilon\}$ for each $x_0^i \in M^i$, $i = 1,2$.

(C) We must show that: for $(x^1, x^2) \in M^1 \times M^2$,

$$\langle \delta_{x^i}\mathcal{E}_\rho, \xi^i\rangle \;\longrightarrow\; \langle \delta_{x^i}\mathcal{E}, \xi^i\rangle \quad \text{as } \rho \to 0, \text{ uniformly on } \mathcal{N}_\varepsilon(x_0^i) \subset M^i,$$

where $\xi^i \in \mathcal{T}_{x^i}$, $i = 1,2$.

It suffices to show the above assertion for $i = 1$ by the invariance of the Dirichlet-Integral under the conformal mappings.

We know that

$$\begin{aligned}
\langle \delta_{x^1}\mathcal{E}_\rho, \xi^1\rangle &= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \\
&= \int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega + \int_{B_{\nu(\rho)}\setminus B_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \\
&= \int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega + \int_{A_{\nu'}} \langle dg_{\nu'}, \nabla \mathbf{J}_{g_{\nu'}}(0, \zeta_{\nu'})\rangle d\omega,
\end{aligned}$$

where $g_{\nu'}(\cdot) = \mathcal{F}_\rho \circ T(\cdot) : A_{\nu'} \to N$, and $\zeta_{\nu'}(\nu' e^{i\theta}) = \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)(\nu e^{i\theta}) : \partial B_{\nu'} \to (g_{\nu'}|_{\partial B_{\nu'}})^*(T\Gamma_2)$, with $\nu' := \frac{\rho}{\nu(\rho)}$
We notice that $\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \circ T$ is also Jacobi-Field along $g_{\nu'}$ by the conformal property of T and the definition of Jacobi-Field, so it can be denoted by $\mathbf{J}_{g_{\nu'}}(0, \zeta_{\nu'})$.

The proof is divided into several steps.

C-I) First, letting $V_\nu := \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)|_{A_\nu} = v_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ f_\nu \in H^{1,2}(A_\nu, f_\nu^* TN)$, we will show the following uniform boundedness with respect to $\nu \in (0, \nu_0)$, for some $\nu_0 \in (0,1)$:

$$\|DV_\nu\|_2^2 := \int_{A_\nu} h_{\alpha\beta} \circ f_\nu v_{\nu,i}^\alpha v_{\nu,i}^\beta d\omega \leq C. \tag{4.11}$$

By computation we see that for $V \in H^{1,2} \cap C^0(A_\rho, f^*TN), f \in H^{1,2} \cap C^0(A_\rho, N)$,

$$\|DV_\nu\|_2^2 \leq CE(V_\nu) + C(N, \|V_\nu\|_0, \|f_\nu\|_0, E(f_\nu)).$$

And from Lemma 3.2.1 $\|V_\nu\|_{C^0} \leq \|\xi_\nu^1\|_{C^0}$, so we need to show only that

$$E(V_\nu) := \int_{A_\nu} \left|\nabla^{f_\nu} V_\nu\right|^2 d\omega \leq C, \quad \nu \in (0, \nu_0).$$

For $0 < \nu < \nu_0$, we define another vector field along f_ν as follows:
letting $x_\nu^\alpha(z) := v_{2\nu_0}^\alpha(\tau_{\nu_0}^{2\nu_0}(z))$, $\nu_0 \leq |z| \leq 1$ (see section 3.1 for the definition of $\tau_{\nu_0}^{2\nu_0}$) and $x_\nu^\alpha(z) := 0$, $\nu \leq |z| \leq \nu_0$, we have

$$X_\nu = x_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ f_\nu \in H^{1,2}(A_\nu, f_\nu^* TN),$$

with $\|DX_\nu\|_2^2 \leq C(\nu_0, N)\|DV_{2\nu_0}\|_2^2$ for all $\nu \leq \nu_0$.

Then by the minimality property of Jacobi-Field and from the Young's inequality,

$$
\begin{aligned}
&\int_{A_\nu} \left(\left| \nabla^{f_\nu}(V_\nu) \right|^2 - \langle trR(df_\nu, V_\nu)df_\nu, V_\nu \rangle \right) d\omega \\
\leq\ & \int_{A_\nu} \left(\left| \nabla^{f_\nu}(X_\nu) \right|^2 - \langle trR(df_\nu, X_\nu)df_\nu, X_\nu \rangle \right) d\omega \\
=\ & 2^{-1} \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\alpha_{\nu,i} x^\beta_{\nu,i} d\omega + \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\alpha_{\nu,i} x^\gamma_\nu f^\delta_{,i} \Gamma^\beta_{\gamma\delta} \circ f_\nu d\omega \\
& +2^{-1} \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\gamma_\nu x^\lambda_\nu f^\delta_{,i} f^\mu_{,i} \Gamma^\alpha_{\gamma\delta} \circ f_\nu \Gamma^\beta_{\lambda\mu} \circ f_\nu d\omega - \int_{A_\nu} \langle trR(df_\nu, X_\nu)df_\nu, X_\nu \rangle d\omega \\
\leq\ & 2^{-1} \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\alpha_{\nu,i} x^\beta_{\nu,i} d\omega + \varepsilon \int_{A_\nu} |x^\alpha_{\nu,i} \frac{\partial}{\partial y^\alpha} \circ f_\nu|^2_h d\omega + \varepsilon^{-1} \int_{A_\nu} |x^\gamma_\nu f^\delta_{,i} \Gamma^\beta_{\gamma\delta} \circ f_\nu \frac{\partial}{\partial y^\beta} \circ f_\nu|^2_h d\omega \\
& +2^{-1} \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\gamma_\nu x^\lambda_\nu f^\delta_{,i} f^\mu_{,i} \Gamma^\alpha_{\gamma\delta} \circ f_\nu \Gamma^\beta_{\lambda\mu} \circ f_\nu d\omega - \int_{A_\nu} \langle trR(df_\nu, X_\nu)df_\nu, X_\nu \rangle d\omega \\
\leq\ & C(N)\|DV_{2\nu_0}\|_2^2 + C(N,\varepsilon)\|DV_{2\nu_0}\|_2^2 + C(\|f_\nu\|_0, \|V_{2\nu_0}\|_0, \varepsilon)E(f_\nu) \\
& +C(\|f_\nu\|_0, \|X_\nu\|_0)E(f_\nu) + C(\|f_\nu\|_0, \|X_\nu\|_0, N)E(f_\nu) \\
\leq\ & C(N, \varepsilon, \|f_\nu\|_0, E(f_\nu), \|V_{2\nu_0}\|_0, \|DV_{2\nu_0}\|_2^2).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\int_{A_\nu} \langle trR(df_\nu, V_\nu)df_\nu, V_\nu \rangle) d\omega &\leq C(N, \|f_\nu\|_0, E(f_\nu), \|V_\nu\|_0) \\
&\leq C(N, \|f_\nu\|_0, E(f_\nu), \|\xi^1\|_0).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{A_\nu} \left| \nabla^{f_\nu}(V_\nu) \right|^2 d\omega &= \int_{A_\nu} (\left| \nabla^{f_\nu}(V_\nu) \right|^2 - \langle trR(df_\nu, V_\nu)df_\nu, V_\nu \rangle) d\omega \\
&\quad + \int_{A_\rho} \langle trR(df_\nu, V_\nu)df_\nu, V_\nu \rangle) d\omega \\
&\leq C, \text{ independent of } \nu \in (0, \nu_0).
\end{aligned}
$$

We have proved (4.11), and this means that $\{(v^\alpha_\nu) | \nu \leq \nu_0\}_{\alpha=1,\cdots,n}$ has the same modulus of continuity from the similar argument as in B-II) with Lemma 3.2.1.

C-II) The convergence of Jacobi fields.

Recall that a Jacobi field $V := v^\alpha \frac{\partial}{\partial y^\alpha} \circ f$ along a harmonic mapping $f : M \to N$, $M \subset \mathbb{R}^2$ is a weak solution of

$$
tr\, \nabla^f \nabla^f V + tr\, R(V, df)df = 0,
$$

where

$$\begin{aligned} tr\,\nabla^f\nabla^f V = & V^\alpha_{,ii} + (V^\gamma f^\beta_{,i}\Gamma^\alpha_{\beta\gamma}\circ f)_{,i} \\ & -\Gamma^k_{ii}(V^\alpha_{,k} + V^\gamma f^\beta_{,k}\Gamma^\alpha_{\beta\gamma}\circ f) + f^\nu_{,i}\Gamma^\alpha_{\rho\nu}(V^\rho_{,i} + V^\gamma f^\beta_{,i}\Gamma^\rho_{\beta\gamma}\circ f). \end{aligned}$$

With the same charts as in *B)*, $(v^\alpha_{\nu(\rho)}) \in \mathbb{R}^n$, $\nu \le \nu_0$ are weak solutions of the above system with a uniform bounded energy and the same modulus of continuity on $K_\sigma = \{\sigma \le |z| \le 1\}$ with $\sigma > 0$ for small ρ.

Now, from the Jacobi field equation and Lemma 4.1.2 *(B)*, we obtain a local $H^{4,2}$-norm bound uniformly for $\{(V^\alpha_{\nu(\rho)}|\rho \le \rho_0\}$, since $\|V\|_{C^0}$ is also uniformly bounded by $\|\gamma^1\|_{C^0}$. Thus, we have a C^2-convergent subsequence of $\{(V^\alpha_\nu)\}_\nu$.
By exhausting the domain K and diagonalizing the convergent subsequence we get a Jacobi field along $\mathcal{F}^1|_{B\setminus\{0\}}$ which will be extended to B from Lemma 4.1.3, noting again that $\|V\|_{C^0}$ is uniformly bounded. From the uniqueness of Jacobi fields for a given boundary, we have the following: writing $w^\alpha\frac{\partial}{\partial y^\beta}\circ\mathcal{F}^1 := \mathbf{J}_{\mathcal{F}^1}(\xi^1)$,

$$\|(v^\alpha_\nu(z)) - (w^\alpha(z))\|_{C^0;K_\sigma} \to 0,\ \ \nu(\text{and}\ \ \rho) \to 0,$$

also

$$\|(v^\alpha_\nu(z)) - (w^\alpha(z))\|_{C^2;K} \to 0,\ \ \nu \to 0 \quad \text{for any compact}\ \ K \subset B\setminus(\partial B \cup \{0\}).$$

C-III) The convergence of derivatives.

Considering K_σ as above, we denote $f_\nu|_{K_\sigma}$ by f_ν and $\mathcal{F}^1|_{K_\sigma}$ by $\mathcal{F}^1$.
Note that $\exp_{\mathcal{F}^1} : \mathcal{U}(0) \to H^{1,2}\cap C^0(K_\sigma, N)$ is a diffeomorphism on some neighborhood $\mathcal{U}(0) \in H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$ including 0, since $d(\exp_{\mathcal{F}^1})_0 = Id$.
Moreover, $\|f_\nu - \mathcal{F}^1|_{K_\sigma}\|_{H^{1,2}\cap C^0} \to 0$ as $\nu \to 0$, so there exists $\xi_\nu \in H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$ for $\nu > 0$, small enough such that

$$\exp_{\mathcal{F}^1}\xi_\nu = f_\nu \quad \text{with}$$

$$d\exp_{\mathcal{F}^1,\xi_\nu} : H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN) \longrightarrow H^{1,2}\cap C^0(K_\sigma, f_\nu^*TN),$$

note that $T_{\xi_\nu}T_{\mathcal{F}^1}H^{1,2}\cap C^0(K_\sigma, N) \cong T_{\mathcal{F}^1}H^{1,2}\cap C^0(K_\sigma, N) = H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$.

We observe that

$$d\exp_{\mathcal{F}^1} : \tilde{\mathcal{U}}(0) \to \{\text{Linear maps from}\ \ \tilde{\mathcal{U}}(0)\ \ \text{to}\ \ H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)\}$$

depends smoothly on $\xi_\nu \in \tilde{\mathcal{U}}(0) \subset T_{\xi_\nu}T_{\mathcal{F}^1}H^{1,2} \cap C^0(K_\sigma, N)$. Thus, $d\exp_{\mathcal{F}^1,\xi_\nu} \to Id$ in $H^{1,2}\cap C^0(K_\sigma)$, since $\xi_\nu \to 0$ in $H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$ as $\nu \to 0$.

Hence letting

$$W_\nu := w_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1 := d\exp^{-1}_{\mathcal{F}^1,\xi_\nu}(V_\nu) \in H^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN),$$

from (I), we obtain that

$$\|w_\nu^\alpha(z) - w^\alpha(z)\|_{C^0;K_\sigma} \to 0, \text{ as } \nu \to 0.$$

Now we observe that

$$\int_{K_\sigma} |d\exp_{\mathcal{F}^1,\xi_\nu}(d\mathcal{F}^1) - df_\nu|^2 d\omega \to 0, \tag{4.12}$$

since $d\mathcal{F}^1 \to df_\nu$ in L^2 and $d\exp_{\mathcal{F}^1,\xi_\nu} \to Id$ in C^0.

Next we observe that

$$\int_{K_\sigma} |d\exp_{\mathcal{F}^1,\xi_\nu}(\nabla^{\mathcal{F}^1} W_\nu) - \nabla^{f_\nu} V_\nu|^2 d\omega \to 0 \text{ as } \nu \to 0, \tag{4.13}$$

where $\nabla^{\mathcal{F}^1} W_\nu = (w^\alpha_{\nu,i} + w^\gamma_\nu (\mathcal{F}^1)^\beta_{,i}\Gamma^\alpha_{\beta\gamma}(\mathcal{F}^1))dz^i \otimes \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1$, because:

$$d\exp_{\mathcal{F}^1,\xi_\nu}(W_\nu) = V_\nu, \ \ \mathcal{F}^1 \to f_\nu \text{ in } H^{1,2}\cap C^0(K_\sigma,\cdot), \ \ d\exp_{\mathcal{F}^1,\xi_\nu} \to Id \text{ in } C^0(K_\sigma,\cdot)$$

and from the fact that $d\exp_{\mathcal{F}^1,\xi_\nu} \to Id \quad \text{in } H^{1,2}\cap C^0$,

$$\partial_i(d\exp_{\mathcal{F}^1,\xi_\nu}) \to \partial_i(Id) = 0 \text{ in } L^2(K_\sigma,\cdot),$$

hence with $\partial_i(d\exp_{\mathcal{F}^1,\xi_\nu}((w^\alpha_\nu))) = \partial_i(d\exp_{\mathcal{F}^1,\xi_\nu})(w^\alpha_\nu) + d\exp_{\mathcal{F}^1,\xi_\nu}(w^\alpha_{\nu,i})$,

$$\|d\exp_{\mathcal{F}^1,\xi_\nu}(w^\alpha_{\nu,i}) - (v^\alpha_{\nu,i})\|_{L^2} = \|d\exp_{\mathcal{F}^1,\xi_\nu}(w^\alpha_{\nu,i}) - \partial_i(d\exp_{\mathcal{F}^1,\xi_\nu}((w^\alpha_\nu)))\|_{L^2} \to 0,$$

as $\nu \to 0$.

Now, we write

$$d\exp_{\mathcal{F}^1,\xi_\nu}(d\mathcal{F}^1) = df_\nu + X_\nu, \ \ d\exp_{\mathcal{F}^1,\xi_\nu}(\nabla^{\mathcal{F}^1} W_\nu) = \nabla^{f_\nu} V_\nu + Y_\nu,$$

with $X_\nu, Y_\nu \in H^{1,2}\cap C^0(K_\sigma, T^*M \otimes f_\nu^* TN)$. And from (4.12), (4.13),

$$\int_{K_\sigma} |X_\nu|^2 d\omega, \quad \int_{K_\sigma} |Y_\nu|^2 d\omega \to 0 \text{ as } \nu \to 0.$$

From Gauss Lemma,

$$\langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} W_\nu \rangle = \langle df_\nu + X_\nu, \nabla^{f_\nu} V_\nu + Y_\nu \rangle.$$

By Hölder inequality,

$$\left| \int_{K_\sigma} \langle df_\nu + X_\nu, Y_\nu \rangle_h d\omega \right| \leq (\|df_\nu\|_{L^2(K_\sigma,N)} + \|X_\nu\|_{L^2(K_\sigma,N)}) \|Y_\nu\|_{L^2(K_\sigma,N)} \to 0,$$

because $E(V_\nu)$ is uniformly bounded for $\nu \leq \nu_0$,

$$\left| \int_{K_\sigma} \langle X_\nu, \nabla^{f_\nu} V_\nu \rangle_h d\omega \right| \leq \|\nabla^{f_\nu} V_\nu\|_{L^2(K_\sigma,N)} \|X_\nu\|_{L^2(K_\sigma,N)} d\omega \to 0.$$

Hence, we have that

$$\int_{K_\sigma} \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} W_\nu \rangle d\omega = \int_{K_\sigma} \langle df_\nu, \nabla^{f_\nu} V_\nu \rangle d\omega + o(1), \text{ as } \nu \to 0.$$

Then,

$$\begin{aligned} &\int_{K_\sigma} \Big(\langle df_\nu, \nabla^{df_\nu} V_\nu \rangle_h - \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1) \rangle_h \Big) d\omega \\ &\quad = \int_{K_\sigma} \Big(\langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} W_\nu \rangle_h - \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1) \rangle_h \Big) d\omega + o(1) \\ &\quad = \int_{K_\sigma} \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} W_\nu - \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1) \rangle_h d\omega + o(1) \\ &\quad \leq E(d\mathcal{F}^1) \|\nabla^{\mathcal{F}^1} W_\nu - \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1)\|_{L^2;K_\sigma} + o(1). \end{aligned} \tag{4.14}$$

For the estimate of the latter term, letting $W := \mathbf{J}_{\mathcal{F}^1}(\xi^1)$ consider $A_\nu := a_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1$, $A := a^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1$ such that $d\eta\big(a_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1\big)$, $d\eta\big(a^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1\big) \in H^{1,2} \cap C^0(K_\sigma, \mathbb{R}^k)$ are harmonic in $\mathbb{R}^k$ with $A_\nu|_{\partial K_\sigma} = W_\nu|_{\partial K_\sigma}$ and $A|_{\partial K_\sigma} = W|_{\partial K_\sigma}$.
Clearly, $\|d\eta(A_\nu - A)\|_{H^{1,2} \cap C^0(K_\sigma, \mathbb{R}^k)} \to 0$ as $\nu \to 0$. Then we have a test vector field

$$W_\nu - W - A_\nu + A \in H_0^{1,2} \cap C^0(K_\sigma, (\mathcal{F}^1)^* TN).$$

And since W resp. V_ν is Jacobi-Field along $\mathcal{F}^1|_{K_\sigma}$ resp. $f_\nu|_{K_\sigma}$, with $L_\nu := d\exp_{\mathcal{F}^1,\xi_\nu}$

$$\int_{K_\sigma} \langle \nabla^{\mathcal{F}^1}(W_\nu - W), \nabla^{\mathcal{F}^1}(W_\nu - W - A_\nu + A)\rangle_h d\omega \tag{4.15}$$

$$\begin{aligned}
&= \int_{K_\sigma} \langle \nabla^{\mathcal{F}^1} W_\nu, \nabla^{\mathcal{F}^1}(W_\nu - W - A_\nu + A)\rangle_h d\omega \\
&\qquad - \int_{K_\sigma} \langle trR \circ \mathcal{F}^1(W, d\mathcal{F}^1)d\mathcal{F}^1, W_\nu - W - A_\nu + A\rangle_h d\omega \\
&= \int_{K_\sigma} \{\langle \nabla^{\mathcal{F}^1} W_\nu, \nabla^{\mathcal{F}^1}(W_\nu - W - A_\nu + A)\rangle_h \\
&\qquad -\langle trR \circ \mathcal{F}^1(W, d\mathcal{F}^1)d\mathcal{F}^1, W_\nu - W - A_\nu + A\rangle_h \\
&\qquad -\langle \nabla^{f_\nu} V_\nu, \nabla^{f_\nu}(L_\nu(W_\nu - W - A_\nu + A))\rangle_h \\
&\qquad +\langle trR \circ f_\nu(V_\nu, df_\nu)df_\nu, (L_\nu(W_\nu - W - A_\nu + A))\rangle_h\} d\omega \\
&= \int_{K_\sigma} \{\langle \nabla^{\mathcal{F}^1} W_\nu, \nabla^{\mathcal{F}^1}(W_\nu - W - A_\nu + A)\rangle_h \\
&\qquad -\langle trR \circ \mathcal{F}^1(W, d\mathcal{F}^1)d\mathcal{F}^1, W_\nu - W - A_\nu + A\rangle_h \\
&\qquad -\langle \nabla^{\mathcal{F}^1} L_\nu^{-1}(V_\nu), \nabla^{\mathcal{F}^1}(W_\nu - W - A_\nu + A)\rangle_h \\
&\qquad +\langle trR \circ f_\nu(V_\nu, df_\nu)df_\nu, (L_\nu(W_\nu - W - A_\nu + A))\rangle_h\} d\omega + o(1) \\
&= \int_{K_\sigma} \{-\langle trR \circ \mathcal{F}^1(W, d\mathcal{F}^1)d\mathcal{F}^1, W_\nu - W - A_\nu + A\rangle_h \\
&\qquad +\langle L_\nu^{-1}(trR \circ f_\nu(V_\nu, df_\nu)df_\nu), W_\nu - W - A_\nu + A\rangle_h\} d\omega + o(1),
\end{aligned} \tag{4.16}$$

note that $\exp_{\mathcal{F}^1} \xi_\nu = f_\nu$, $L_\nu^{-1}(V_\nu) = W_\nu$, $L_\nu(W_\nu - W - A_\nu + A) \in H_0^{1,2} \cap C^0(K_\sigma, (f_\nu)^*TN)$.

On the other hand

$$\|W_\nu - W - A_\nu + A\|_{C^0;K_\sigma} \to 0 \qquad \text{as } \nu \to 0,$$

and

$$\|\mathcal{F}^1\|_{1,2;0}, \|W\|_{C^0}, \|f_\nu\|_{1,2;0}, \|V_\nu\|_{C^0} < C \quad \text{uniformly on } \nu \in (0, \nu_0).$$

Therefore, (4.16) converges to 0, as $\nu \to 0$.

Moreover, by (2.4) and since $\|d\eta(A_\nu - A)\|_{C^0} \to 0$, we have

$$\int_{K_\sigma} |\nabla^{\mathcal{F}^1}(A_\nu - A)|_h^2 d\omega \to 0, \;\; \nu \to 0.$$

Also note that $\int_{K_\sigma} |\nabla^{\mathcal{F}^1}(W_\nu - W)|_h^2 d\omega < C$, $\nu \in (0, \nu_0)$, so in (4.15)

$$\|\nabla^{\mathcal{F}^1} W_\nu - \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1)\|_{L^2;K_\sigma} \to 0, \;\; \text{as } \nu \to 0,$$

and (4.14) converges to 0 for each $\sigma \in (0,1)$. And by letting $\sigma \to 0$, we have

$$\int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1,0)\rangle_h d\omega \to \int_B \langle d\mathcal{F}^1(x^1), \nabla \mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega, \quad \rho \to 0,$$

note that $\int_{B_\sigma} \langle d\mathcal{F}^1(x^1), \nabla \mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega \to 0$ as $\rho \to 0$.

In a similar way, we also have

$$\int_{A_{\nu'(\rho)}} \langle dg_{\nu'}, \nabla \mathbf{J}_{g_{\nu'}}(0,\zeta_{\nu'})\rangle d\omega \to \int_B \langle d\mathcal{F}^2(x^2), \nabla \mathbf{J}_{\mathcal{F}^2}(0)\rangle_h d\omega = 0, \quad \rho \to 0,$$

since $\mathbf{J}_{\mathcal{F}^2}(0) = 0 \in H^{1,2}(B, (\mathcal{F}^2)^*TN)$.

Hence, it follows

$$\langle \delta_{x^i}\mathcal{E}_\rho, \xi^i\rangle \;\longrightarrow\; \langle \delta_{x^i}\mathcal{E}, \xi^i\rangle \quad \text{as } \rho \to 0,$$

uniformly on $\mathcal{N}_\varepsilon(x_0^i), i = 1,2$, which is clearly the same as in *(B)*, since all convergence in *(C)* is unifomly on $\mathcal{N}_\varepsilon(x_0^i), i = 1,2$ and the constant C unifomly bounded on $\mathcal{N}_\varepsilon(x_0^i), i = 1,2$.

In this manner, we may also show that $\delta_{x^1}\mathcal{E}_\rho, \delta_{x^2}\mathcal{E}_\rho$ are continuous with respect to $\rho \in (0,1)$ and uniformly on $\mathcal{N}_\varepsilon(x_0^i)$. And we have proved *(C)*.

(D) The differential form and the proof of that are the same as in [St4].
Choose $x^1 \in M^1, x^2 \in M^2$ and $\mathcal{F}_a$ means $\mathcal{F}_a(x^1,x^2)$ for $0 < a < 1$. Consider

$$\tau_\rho^\sigma(r) = \frac{\sigma - \rho + (1-\sigma)r}{1-\rho}, \text{ resp. } \tau_\sigma^\rho(r) = \frac{\rho - \sigma + (1-\rho)r}{1-\sigma},$$

which maps the interval $[\rho,1]$ resp.$[\sigma,1]$ onto the interval $[\sigma,1]$ resp.$[\rho,1]$ for $\rho,\sigma \in (0,1)$. For $\mathcal{F}_\rho \in H^{1,2}(A_\rho, N)$, in polar coordinate (r,θ), define

$$\mathcal{F}_\rho \circ \tau_\rho^\sigma(r,\theta) := \mathcal{F}_\rho\big(\tau_\rho^\sigma(r),\theta\big).$$

Then $\mathcal{F}_\rho \circ \tau_\rho^\sigma \in H^{1,2}(A_\rho, N)$, and by the minimality of Dirichlet integral of the harmonic mapping

$$E(\mathcal{F}_\sigma) \le E(\mathcal{F}_\rho \circ \tau_\rho^\sigma).$$

Thus, for any $\sigma, \rho \in (0,1)$,

$$E(\mathcal{F}_\sigma) - E(\mathcal{F}_\rho) \le E(\mathcal{F}_\rho \circ \tau_\sigma^\rho) - E(\mathcal{F}_\rho),$$

and letting $\rho \to \rho_0^+, \rho \to \rho_0^-$

$$\begin{aligned}\lim_{\rho\to\rho_0}\frac{E(\mathfrak{F}_\rho)-E(\mathfrak{F}_{\rho_0})}{\rho-\rho_0} &= \lim_{\rho\to\rho_0}\frac{E(\mathfrak{F}_{\rho_0}\circ\tau_\rho^{\rho_0})-E(\mathfrak{F}_{\rho_0}\circ\tau_{\rho_0}^{\rho_0})}{\rho-\rho_0}\\ &= \frac{d}{d\rho}|_{\rho=\rho_0}E(\mathfrak{F}_{\rho_0}\circ\tau_\rho^{\rho_0}),\end{aligned}$$

note that $\tau_{\rho_0}^{\rho_0}$ =Id.

Letting $s := \tau_\sigma^\rho(r)$, $r := \tau_\rho^\sigma(s)$,

$$\begin{aligned}&2\frac{d}{d\sigma}|_{\sigma=\rho}E(\mathfrak{F}_\rho\circ\tau_\sigma^\rho)\\ &= \frac{d}{d\sigma}|_{\sigma=\rho}\int_0^{2\pi}\int_\sigma^1\left[|\partial_r(\mathfrak{F}_\rho\circ\tau_\sigma^\rho)|^2+\frac{1}{r^2}|\partial_\theta(\mathfrak{F}_\rho\circ\tau_\sigma^\rho)|^2\right]rdrd\theta\\ &= \frac{d}{d\sigma}|_{\sigma=\rho}\int_0^{2\pi}\int_\sigma^1\left[|(\partial_r\mathfrak{F}_\rho)\circ\tau_\sigma^\rho|^2\left(\frac{1-\rho}{1-\sigma}\right)^2+\frac{1}{r^2}|(\partial_\theta\mathfrak{F}_\rho)^2\circ\tau_\sigma^\rho|^2\right]rdrd\theta\\ &= \frac{d}{d\sigma}|_{\sigma=\rho}\int_0^{2\pi}\int_\rho^1\left[|\partial_r\mathfrak{F}_\rho|^2\left(\frac{1-\rho}{1-\sigma}\right)^2+\frac{1}{(\tau_\rho^\sigma(s))^2}|\partial_\theta\mathfrak{F}_\rho|^2\right]\tau_\rho^\sigma(s)\frac{1-\sigma}{1-\rho}dsd\theta\\ &= \frac{d}{d\sigma}|_{\sigma=\rho}\int_0^{2\pi}\int_\rho^1\left[|\partial_r\mathfrak{F}_\rho|^2\frac{1-\rho}{1-\sigma}\tau_\rho^\sigma(s)+|\partial_\theta\mathfrak{F}_\rho|^2\left(\frac{1-\rho}{1-\sigma}\tau_\rho^\sigma(s)\right)^{-1}\right]dsd\theta\\ (4.17)\quad &= \int_0^{2\pi}\int_\rho^1\left[|\partial_r\mathfrak{F}_\rho|^2-\frac{1}{r^2}|\partial_\theta\mathfrak{F}_\rho|^2\right]\frac{1}{1-\rho}drd\theta\\ &= \frac{\partial}{\partial t}|_{t=\rho}\mathcal{E}(x^1,x^2,t).\end{aligned}$$

This brings to an end our proofs for Lemma 4.1.1. □

The following Lemma is due to [LU] and [Jo1].

Lemma 4.1.2. *(A) Let $f \in H^{1,2}\cap C^0(\Omega,\mathbb{R}^n)$, Ω is open in $\mathbb{R}^m$, a solution of following system*

$$\int_\Omega a^{ij}(x)\partial_i f^\alpha(x)\partial_j\varphi^\alpha dx = \int_\Omega G^\alpha(x,f(x),\nabla f(x))dx,$$

for all $\varphi \in H_0^{1,2}\cap L^\infty(\Omega,\mathbb{R}^n)$ with the following structure conditions:

- *$a^{ij} \in C^2(\Omega)$, is symmetric with respect to $i,j = 1,\cdots,m$, measurable and*

  $|a^{ij}(x)|$, $|\nabla a^{ij}(x)| \le C$, $\quad a^{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2$ *for all* $\xi \in \mathbb{R}^m$, *for almost all* $x \in \Omega$,

- *$G(x,f,p) = (G^1, \cdots, G^n)$ is measurable in x and continuous in f and p,*
- *for all $(x,f,p) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$,*

$$\begin{aligned} |G(x,f,p)| &\le C + C|p|^2 \qquad |\partial_x G(x,f,p)| \le C + C|p|^3 \\ |\partial_f G(x,f,p)| &\le C + C|p|^2 \qquad |\partial_p G(x,f,p)| \le C + C|p|. \end{aligned}$$

Then for given $\varepsilon \in (0,1)$, $x \in \Omega$, there exists $R(\varepsilon, x) > 0$ with

$$\int_{B(x,R)} \left(|\nabla f|^q |\nabla^2 f|^2 + |\nabla f|^q + |\nabla^2 f| \right) dx \le C + C\|\nabla f\|_{L^2(\Omega)}, \tag{4.18}$$

and

$$\|f\|_{H^{3,2}(B(z,R))} \le C + C\|\nabla f\|_{L^2(M)},$$

where ε is actually $\sup_{y \in B(x,R)} |f(y) - f(x)|$ and the constant C depends on the structural constants and R, ε, namely the modulus of continuity of f. In (4.18), C is also dependent on $q \in [1, \infty)$.

(B) Let $f \in H^{1,2} \cap C^0(M,N)$, M is open in $\mathbb{R}^2$, be a harmonic map in (N,h). Given $\varepsilon \in (0,1)$ and $z \in \Omega$, then for some $R(\varepsilon)$,

$$\|f\|_{H^{s,2}(B(z,R))} \le C(s) + C(s)\|\nabla f\|_{L^2(M)}, \quad \text{for } s \in [1, \infty),$$

where the constant C depend on the modulus of continuity of f and s.

Proof. See [Jo1] section 8.5. □

Lemma 4.1.3. *Suppose that $f \in H^{1,2}(B\backslash\{0\}, \mathbb{R}^n)$ satisfies*

$$\int_{B\backslash\{0\}} \nabla f(z) \nabla \phi(z) d\omega = \int_{B\backslash\{0\}} g(z, f(z), \nabla f(z)) \phi(z) d\omega, \ \text{for all } \phi \in H_0^{1,2} \cap L^\infty(B\backslash\{0\}, \mathbb{R}^n),$$

with $|g(z,f,p)| \le C + C|p|^2$ for some constant $C \in \mathbb{R}$ for all $(z,f,p) \in B\backslash\{0\} \times \mathbb{R}^n \times \mathbb{R}^{2n}$. Then we have that

$$\int_B \nabla f(z) \nabla \phi(z) d\omega = \int_B g(z, f(z), \nabla f(z)) \phi(z) d\omega, \ \text{for all } \phi \in H_0^{1,2} \cap L^\infty(B, \mathbb{R}^n).$$

Proof. See [Jo1] Lemma 8.4.5. □

4.2 Critical points of $\mathcal{E}$

For given Jordan curves $\Gamma_1, \Gamma_2, \Gamma$ in (N, h) with $\operatorname{dist}(\Gamma_1, \Gamma_2) > 0$, we define

- Problem $\mathcal{P}(\Gamma_1, \Gamma_2)$:
  for some $\rho \in (0, 1)$, a map $X \in H^{1,2}(A_\rho, N) \cap C^0(\overline{A_\rho}, N)$ is a solution of Plateau's Problem $\mathcal{P}(\Gamma_1, \Gamma_2)$ (i.e. an annulus type minimal surface in (N, h), bounded by $\Gamma_1 \cup \Gamma_2$) if

  (A1) $\tau_h(X) = 0$,

  (A2) $|X_u|_h^2 - |X_v|_h^2 = \langle X_u, X_v \rangle_h = 0$,

  (A3) $X|_{C_i}$ is a weakly monotone map onto $\Gamma_i, i = 1, 2$.

- Problem $\mathcal{P}(\Gamma)$:
  a map $X \in H^{1,2}(B, N) \cap C^0(\overline{B}, N)$ is a solution of Plateau's Problem $\mathcal{P}(\Gamma)$ (i.e. a disc type minimal surface in (N, h), bounded by Γ) if

  (B1) $\tau_h(X) = 0$,

  (B2) $|X_u|_h^2 - |X_v|_h^2 = \langle X_u, X_v \rangle_h = 0$,

  (B3) $X|_{\partial B}$ is a weakly monotone map onto Γ.

Notation

(i) In order to define critical points of $\mathcal{E}$, recall the definition l_i in section 3.1, $i = 1, 2$. This implies that

$$\widetilde{\exp}_{x^i} \xi^i \in M^i, \quad \text{for } \|\xi^i\|_{\frac{1}{2},2;0} \leq l_i. \tag{4.19}$$

(ii) Denoting the minimum of the injectivity radii of $p \in \Gamma_i$ with respect to the metric $\tilde{h}$ by $i_{\tilde{h}}(\Gamma_i)$, we can also require that $l_i \leq \{1, i_{\tilde{h}}(\Gamma_i)\}$, otherwise, we may take $\min\{1, i_{\tilde{h}}(\Gamma_i)\}$ denoted again by l_i. Clearly, $l_i > 0$.

Now we define for $x = (x^1, x^2, \rho) \in \overline{\mathcal{M}}$, $i = 1, 2$,

$$\begin{aligned} g_i(x) &:= \sup_{\substack{\xi^i \in \mathcal{T}_{x^i} \\ \|\xi^i\| < l_i}} (-\langle \delta_{x^i} \mathcal{E}, \xi^i \rangle), \\ g_3(x) &:= \begin{cases} |\rho \cdot \partial_\rho \mathcal{E}| & , \quad \rho > 0 \\ 0 & , \quad \rho = 0, \end{cases} \\ g(x) &:= \Sigma_{j=1}^3 g_j(x). \end{aligned} \tag{4.20}$$

And $x \in \overline{\mathcal{M}}$ is a critical point of $\mathcal{E}$ if $g(x) = 0$.

Now we investigate the continuity of g.

Lemma 4.2.1. *(i) For any $x \in \overline{\mathcal{M}}$, $g_j(x) \geq 0$, $j = 1, 2, 3$. Thus, $g(x) = 0$, if and only if $g_j(x) = 0$, for all $j = 1, 2, 3.$.*

(ii) g_j is continuous, and specially $g_j(x^1, x^2, \rho) \to g_j(x^1, x^2, \rho_0)$ as $\rho \to \rho_0$, uniformly on $\mathcal{N}_\varepsilon(x^i)$, for some small $\varepsilon > 0$, $j = 1, 2, 3$, $i = 1, 2$.

Proof. *(i)* Let $i = 1, 2$. If $g_i(x) = \delta < 0$, then $\langle \delta_{x^i}\mathcal{E}, \xi^i \rangle > 0$ for all $\xi^i \in \mathcal{T}_{x^i}$ with $\|\xi^i\| < l_i$. Since $\mathcal{T}_{x^i}$ is convex, $\langle \delta_{x^i}\mathcal{E}, t\xi^i \rangle = t\delta > 0$, for all $t \in [0, 1]$. This is a contradiction. Therefore, $g_i(x) \geq 0$. Clearly, $g_3(x) \geq 0$.

(ii) The uniform convergence of g_i with respect to $\rho \to \rho_0 \in [0, 1)$ on $\mathcal{N}_\varepsilon(x^i)$ follows immediately from the uniform convergence of $\delta_{x^i}\mathcal{E}$ (see Lemma(4.1.1),*(C)*).

Let us take a sequence $\{x_n\} = \{(x^1_n, x^2_n, \rho_n)\} \subset \overline{\mathcal{M}}$ which converges strongly to $x = (x^1, x^2, \rho)$. From the above we have

$$g_i(x^1_n, x^2_n, \rho_n) \to g_i(x^1_n, x^2_n, \rho), \text{ uniformly on } \{n \geq n_0\}.$$

Now, let $\tilde{x}_n := (x^1_n, x^2_n, \rho)$ and $\widetilde{\exp}_{x^i_n}\xi^i_n = x^i$. Observe that $d\widetilde{\exp}_{x^i_n,\xi^i_n} \to Id$ in $H^{\frac{1}{2},2} \cap C^0$, hence for some t_0, independent of $n \geq n_0$,

$$\|t_0 d\widetilde{\exp}_{x^i_n,\xi^i_n}(\eta^i_n)\|_{\mathcal{T}_{x^i}} < l_i \text{ if } \|\eta^i_n\|_{\mathcal{T}_{x^i_n}} < l_i,$$

here we also note that $\mathcal{T}_{x^i}$ is convex with 0.

Then by Lemma(4.1.1)*(A)*, for given $\delta > 0$ there exist $t_0(\delta)$ and $n_0(\delta)$ as above such that for each $\|\eta^i_n\|_{\mathcal{T}_{x^i_n}} < l_i$ with $n \geq n_0(\delta)$,

$$\begin{aligned} -\langle \delta_{x^i}\mathcal{E}(\tilde{x}_n), \eta^i_n \rangle &\leq -\langle \delta_{x^i}\mathcal{E}(x), d\widetilde{\exp}_{x^i_n,\xi^i_n}(\eta^i_n) \rangle + \delta \\ &\leq -\langle \delta_{x^i}\mathcal{E}(x), t_0 d\widetilde{\exp}_{x^i_n,\xi^i_n}(\eta^i_n) \rangle + 2\delta \leq g_i(x) + 2\delta. \end{aligned}$$

This implies, $g_i(\tilde{x}_n) \leq g_i(x) + 2\delta$. On the other hand, we obtain that $g_i(x) \leq g_i(\tilde{x}_n) + 2\delta$, so $g_i(x^1_n, x^2_n, \rho) \to g_i(x^1, x^2, \rho)$ as $n \to \infty$.

Together with the above uniform convergence on $\mathcal{N}_\varepsilon(x^i)$ as $\rho_n \to \rho$, we conclude that

$$g_i(x_n) \to g_i(x) \text{ as } x_n \to x, \text{ for } i = 1, 2.$$

The continuity and uniform continuity of g_3 is clear from the form of $\frac{\partial}{\partial \rho}\mathcal{E}$. □

Proposition 4.2.1. $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times [0,1)$ *is a critical point of* $\mathcal{E}$ *if and only if* $\mathcal{F}_\rho(x^1, x^2)$ *(for* $\rho \in (0,1)$*), resp.* $\mathcal{F}^i(x^i)$ *is a solution of* $\mathcal{P}(\Gamma_1, \Gamma_2)$*, resp.* $\mathcal{P}(\Gamma_i), i = 1, 2$.

Proof. (I) Let $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times [0,1)$ be a critical point of $\mathcal{E}$. From Theorem 3.1.1, $\mathcal{F}$ is continuous until the boundary.
We must show that $\mathcal{F}_\rho(x^1, x^2)$(for $\rho > 0$), and $\mathcal{F}^i(x^i)$ is conformal. We will show this only for $\mathcal{F}_\rho(x^1, x^2)$, because the proof in the case of $\mathcal{F}^i(x^i)$ is similar to the case of $\mathcal{F}_\rho(x^1, x^2)$ and still easier.

For $x \in \mathcal{M}$, a critical poinf of $\mathcal{E}$, we have that $\mathcal{F}_\rho(x^1, x^2)$ belongs to the class $H^{2,2}(A_\rho, \mathbb{R}^k)$, which will be proved in Appendix. Thus, we can compute: for $\xi^1 \in \mathcal{T}_{x^1}$

$$
\begin{aligned}
\langle \delta_{x^1}\mathcal{E}(x), \xi^1 \rangle &= \frac{1}{2}\int_{A_\rho} \frac{d}{dt}\Big|_{t=0} |d\mathcal{F}_\rho(\widetilde{\exp}_{x^1} t\xi^1, x^2)|_h^2 d\omega \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla_{\frac{d}{dt}} d\mathcal{F}_\rho(\widetilde{\exp}_{x^1} t\xi^1, x^2)\big|_{t=0}\rangle_h d\omega \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla D_{x^1}\mathcal{F}_\rho(x^1, x^2)(\xi^1)\rangle_h d\omega \\
&= \int_{A_\rho} \langle \frac{\partial}{\partial z^i}\mathcal{F}_\rho(x^1, x^2), \nabla_{\frac{\partial}{\partial z^i}} \mathbf{J}_{\mathcal{F}_\rho(x^1,x^2)}(\xi^1, 0)\rangle_h d\omega \\
&= \int_{A_\rho} \operatorname{div}(v_1, v_2) d\omega \qquad (4.21)\\
&= \int_{\partial B} \langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho(x^1, x^2)\vec{n}, \xi^1\rangle_h d\omega, \qquad (4.22)
\end{aligned}
$$

where $\frac{\partial}{\partial z^i}\mathcal{F}_\rho(x^1, x^2) = (\mathcal{F}_\rho)_i^\alpha \frac{\partial}{\partial y^\alpha}$, $(v_1, v_2) = \left(\left\langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho, \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\right\rangle_h, \left\langle \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\right\rangle_h\right)$, with $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$.

We get the equality (4.21) as follows: with $\mathbf{J}_\rho := \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)$,

$$
\begin{aligned}
\operatorname{div}(v_1, v_2) &= \frac{\partial}{\partial z^i}\langle \frac{\partial}{\partial z^i}\mathcal{F}_\rho, \mathbf{J}_\rho\rangle_h \\
&= \langle \nabla_{\frac{\partial}{\partial z^1}} \frac{\partial}{\partial z^1}\mathcal{F}_\rho, \mathbf{J}_\rho\rangle_h + \langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^1}}\mathbf{J}_\rho\rangle_h + \langle \nabla_{\frac{\partial}{\partial z^2}} \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \mathbf{J}_\rho\rangle_h \langle \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^2}}\mathbf{J}_\rho\rangle_h \\
&= \underbrace{\langle \nabla_{\frac{\partial}{\partial z^1}} \frac{\partial}{\partial z^1}\mathcal{F}_\rho + \nabla_{\frac{\partial}{\partial z^2}} \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \mathbf{J}_\rho\rangle_h}_{(=0, \text{ since } \mathcal{F}_\rho \text{ is harmonic})} + \langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^1}}\mathbf{J}_\rho\rangle_h + \langle \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^2}}\mathbf{J}_\rho\rangle_h \\
&= \Sigma_{i=1}^2 \langle \frac{\partial}{\partial z^i}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^i}}\mathbf{J}_\rho\rangle_h.
\end{aligned}
$$

Let

$$w^i_\varepsilon := w^i \circ (Id + \varepsilon\eta), \;\; x^i = \gamma^i \circ w^i, \;\; \eta \in C^1(\mathbb{R}/2\pi),$$

where $Id : \mathbb{R} \to \mathbb{R}$ and $Id + \varepsilon\eta$ is weakly monotone increasing with $|\varepsilon| < \varepsilon_0$ for some small $\varepsilon_0 > 0$.

There exist $\{f_n\} \subset C^\infty(\overline{B}, \mathbb{R}^k)$ with $f_n \to \mathcal{F}_\rho$ in $H^{2,2}$, $f_n|_{\partial B} \to \mathcal{F}_\rho|_{\partial B}$ in L^2, also $\frac{d}{d\theta} f_n|_{\partial B} \to \frac{d}{d\theta}\mathcal{F}_\rho|_{\partial B}$ in L^2. Since γ^1 is a diffeomorphism, we have a vector field along w^1, denoted by $\frac{dw^1}{d\theta} : \partial B \to (w^1)^* T(\partial B)$ with

$$\frac{d}{d\theta}\mathcal{F}_\rho|_{\partial B} = \frac{d}{d\theta}\gamma^1(w^1) = d\gamma^1(\frac{d}{d\theta}w^1) \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^1)^* T\Gamma_1).$$

And from the chain rule, $\frac{d}{d\varepsilon}w^1_\varepsilon = \eta \frac{d}{d\theta} w_1$.
We observe that $w^1_\varepsilon \in W^1_{\mathbb{R}^k}$ for $|\varepsilon| \le \varepsilon_0$, so $d\gamma^1(w^1_\varepsilon - w^1) \in \mathcal{T}_{x^1}$, and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, d\gamma^1(w^1_\varepsilon - w^1)\rangle d\theta = \int \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, d\gamma^1(\frac{d}{d\varepsilon}w^1_\varepsilon)\rangle d\theta = \int \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, d\gamma^1(\eta \frac{d}{d\theta} w_1)\rangle d\theta.$$

On the other hand, since $g_1(x) = 0$, by (4.22),

$$\begin{aligned} \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, d\gamma^1(w^1_\varepsilon - w^1)\rangle d\theta &\ge 0, \\ \lim_{\varepsilon\to 0-} \frac{1}{\varepsilon}\int \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, d\gamma^1(w^1_\varepsilon - w^1)\rangle d\theta &\le 0. \end{aligned}$$

Therefore we have

$$\int_{\partial B} \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, d\gamma^1(\frac{d}{d\theta}w^1)\rangle_h \eta \, d\theta = \int_{\partial B} \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, d\gamma^1(\frac{d}{d\varepsilon}w^1_\varepsilon)\rangle_h \, d\theta = 0,$$

for all $\eta \in C^1(\partial B)$, and

$$\langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, d\gamma^1(\frac{d}{d\theta}w^1)\rangle_h = \langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, \frac{\partial \mathcal{F}_\rho}{\partial\theta}\rangle_h|_{\partial B} \equiv 0.$$

From $g_2(x) = 0$, we also obtain that

$$\langle \frac{d\mathcal{F}_\rho}{d\vec{n}}, \, \frac{\partial \mathcal{F}_\rho}{\partial\theta}\rangle_h|_{\partial B_\rho} \equiv 0.$$

Now consider the well known holomorphic function for $z = u + iv = re^{i\theta} \in A_\rho$,

$$\Phi(z) = r^2 \Big|\frac{\partial}{\partial r}\mathcal{F}_\rho\Big|^2_h - \Big|\frac{\partial}{\partial\theta}\mathcal{F}_\rho\Big|^2_h - 2ir\langle \frac{\partial}{\partial r}\mathcal{F}_\rho, \frac{\partial}{\partial\theta}\mathcal{F}_\rho\rangle_h \in H^{1,2}(A_\rho).$$

From the above, Φ has imaginary part vanishing on ∂A_ρ, so Φ is real constant by Cauchy-Riemann equation.
But from the form of $\frac{\partial}{\partial\rho}\mathcal{E}$, since $g_3(x) = 0$,

$$0 = \frac{\partial}{\partial\rho}\mathcal{E} = \text{const.Real}(\Phi).$$

Thus, $\Phi \equiv 0$ and $\mathcal{F}_\rho$ satisfies the conformal property (B2).

(II) Let $\mathcal{F} := \mathcal{F}_\rho(x)$(resp. $\mathcal{F}^i(x^i)$) be a minimal surface of annulus (resp.disc) type. By [HH], $\mathcal{F} \in C^1(\overline{A_\rho}, N)$ (resp.$C^1(\overline{B}, N)$). Thus, from the conformal property

$$\frac{d\mathcal{F}}{d\vec{n}} \cdot \frac{d}{d\theta}x^i \equiv 0,$$

and the computation (4.22) says that $g_1(x) = 0, g_2(x) = 0$, by (4.17) also $g_3(x) = 0$. $\square$

Chapter 5

Unstable minimal surfaces

5.1 The Palais-Smale Condition

By the conformal invarianc of E, the Palais-Smale Condition((PS) condition) cannot be satisfied for some function sequence(cf. [St1] Lemma I.4.1).
Hence, we need the following normalized subsets as in [St4] (with some change):

Letting $P_k^i \in \Gamma_i$ fixed, $k = 1, 2, 3$, for each $i = 1, 2$,

$$M^{i*} = \{x^i \in M^i : x^i(\cos\frac{2\pi k}{3}, \sin\frac{2\pi k}{3}) = P_k^i \in \Gamma_i, \ \ k = 0,\ 1,\ 2\},$$

define

$$\begin{aligned} \mathcal{M}^* &= \{x = (x^1, x^2, \rho) \in \mathcal{M} : x^1(0) = P_1^1 \in \Gamma_1\} \\ \partial\mathcal{M}^* &= \{x = (x^1, x^2, 0) \in \partial\mathcal{M} : x^i \in M^{i*}\}, \end{aligned}$$

$$\overline{\mathcal{M}^*} = \mathcal{M}^* \cup \partial\mathcal{M}^*.$$

And

$$\mathcal{T}^*_{x^i} := \mathcal{T}_{x^i} \ \ , i = 1, 2,$$

$$\mathcal{T}_x\partial\mathcal{M}^* = \mathcal{T}^*_{x^1} \times \mathcal{T}^*_{x^2}, \ \text{ for } \ x = (x^1, x^2, 0) \in \partial\mathcal{M}^*,$$

$$\mathcal{T}_x\mathcal{M}^* = \{\xi = (\xi^1, \xi^2, \tilde{\rho}) \in \mathcal{T}_x\mathcal{M} \,|\, \xi^1(1) = 0\}, \ \text{ for } \ x = (x^1, x^2, \rho) \in \mathcal{M}^*.$$

Note and Definition

(i) To avoid complication in the sequel, we give some more explanation for the above $\partial\mathcal{M}^*$. We will consider an element $x^i \in M^{i*}$ as a class which consists of $y^i \in M^i$ with $T|_{C^1}(y^i) = x^i$ for some conformal transformation of disc onto itself. We notice that such a conformal may exist uniquely, namely a fractional linear transformation determined uniquely by given three points, since y^i is weakly monotone. In other words, we classify M^i in such a way that each class posesses only one element from $x^i \in M^{i*}$, if necessary, denoted by $[x^i] \in M^{i*}$, with $\|[x^i]\| = \|x^i\|, i = 1, 2$.

(ii) For $\xi \in \mathcal{T}^0_{x^i} \subset \mathcal{T}^*_{x^i}(= \mathcal{T}_{x^i})$, we may calculate:

$$\widetilde{\exp}_{[x^i]}\xi := [\widetilde{\exp}_{x^i}\xi] := [\tilde{x}^i] \in M^{i*}, \tag{5.1}$$

where $\widetilde{\exp}_{x^i}\xi \in M^i$, so $T(\widetilde{\exp}_{x^i}\xi) = \tilde{x}^i \in M^{i*}$, since T is a conformal map of B. We will write simply $\widetilde{\exp}_{x^i}\xi = \tilde{x}^i \in M^{1*}$.
Then this correspondence from M^{i*} into M^{i*} is continuous, since

$$\|T(x^i) - T(y^i)\| \to 0 \text{ as } \|x^i - y^i\| \to 0.$$

(iii) For $[x] \in \partial\mathcal{M}^*$ with $x^i \in M^{i*}$, $g([x]) := g(x)$.

Now we give a topology for our set.

Definition (Topology of $\mathcal{M}^*$)

- A neighborhood $\mathcal{U}_\varepsilon(x_0)$ of $x_0 = (x_0^1, x_0^2, 0) \in \partial\mathcal{M}^*$ consists of all $x = (x^1, x^2, \rho) \in \overline{\mathcal{M}^*}$ such that $\rho < \varepsilon$ and for each $i = 1, 2$,

$$\inf_{\{\text{all } \sigma\}} \|\mathcal{F}^i(x^i) \circ \sigma - \mathcal{F}^i(x^i)\|_{1,2} < \varepsilon,$$

  where σ is a conformal diffeomorphism of B,

- A sequence $\{x_n = (x_n^1, x_n^2, \rho_n)\} \subset \mathcal{M}^*$ converges strongly to $x = (x^1, x^2, \rho) \in \mathcal{M}^*$, if $x_n^i \to x^i$ strongly in $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$, $i = 1, 2$, and $\rho_n \to \rho$,

- A sequence $\{x_n = (x_n^1, x_n^2, \rho_n)\} \subset \overline{\mathcal{M}}^*$ converges strongly to $x = (x^1, x^2, 0) \in \partial\mathcal{M}^*$, if for any $\varepsilon > 0$ all but finitely many of x_n lie in $\mathcal{U}_\varepsilon(x)$.

Remark 5.1.1. *(i) For $x \in \mathcal{M}^*$, the value of $g_i(x)(i = 1, 2)$ in (4.20) does not change, even if we use $\mathcal{T}_x\mathcal{M}^*$ instead of $\mathcal{T}_x\mathcal{M}$.*

(ii) With the above topology, the mapping (5.1) and $g_j, j = 1, 2, 3$, are continuous and uniformly continuous as $\rho \to \rho_0 \in [0, 1)$ on some ε-neighborhood of (x^1, x^2).

(iii) For $\xi = (\xi^1, \xi^2, 0) \in \mathcal{T}_x\mathcal{M}^*$ *resp.* $\mathcal{T}_x\partial\mathcal{M}^*$*, with* $\|\xi^i\|_{\frac{1}{2},2;0} \leq l_i$*,*

$$\widetilde{\exp}_x\xi \in \mathcal{M}^* \ \ resp\ \ \partial\mathcal{M}^*.$$

Proof. *(i)* The difference between $\mathcal{M}$ and $\mathcal{M}^*$ is only the rotation under which the Dirichlet integral is invariant. And the rotation changes smoothly when the maps in $\mathcal{M}$ change smoothly. Thus, for given $\xi^i \in \mathcal{T}^0_{x^i}$, we can find a tangent vector $(\eta^1, \eta^2) \in \mathcal{T}_x\mathcal{M}^*$ such that

$$\langle \delta_{x^i}\mathcal{E}, \eta^i\rangle = \langle \delta_{x^i}\mathcal{E}, \xi^i\rangle. \tag{5.2}$$

Moreover, the rotation is independent of angle variable, so $\|\eta^i\|_{\frac{1}{2},2;0} = \|\xi^i\|_{\frac{1}{2},2;0}$.
(ii) This follows from the invariance of Dirichlet integral under the conformal maps, definition of the topology and Lemma 4.2.1.
(iii) It's clear from the above. □

Proposition 5.1.1 (Palais-Smale condition). *Suppose,* $\{x_n\}$ *is a sequence in* $\overline{\mathcal{M}^*}$ *such that* $\mathcal{E}(x_n) \to \beta$, $g(x_n) \to 0$*, as* $n \to \infty$*. Then there exists a subsequence of* $\{x_n\}$ *which converges strongly to a critical point of* $\mathcal{E}$ *in* $\overline{\mathcal{M}^*}$*.*

Proof. We prove this for the case that $\{x_n\} \subset \mathcal{M}^*$ with $0 < \rho_n < 1$, $\mathcal{E}(x_n) \to \beta$, $g_j(x_n) \to 0$. In the case that $\{x_n\} \subset \partial\mathcal{M}^*$, the proof is similar.
We may also suppose that $\rho_n \to \rho$ as $n \to \infty$.

Note that: the above ρ cannot be 1, i.e. $0 \leq \rho < 1$, because for any $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ there exists some $\theta \in [0, 2\pi)$ such that

$$0 < \mathrm{dist}(\Gamma_1, \Gamma_2) \leq \int_\rho^1 |\nabla\mathcal{F}_\rho(r,\theta)|_h dr \leq \left[\frac{1-\rho}{\rho}\int_\rho^1 |\nabla\mathcal{F}_\rho(r,\theta)|_h^2 r dr\right]^{\frac{1}{2}}, \tag{5.3}$$

so, for all $x \in \mathcal{M}$, $\frac{\rho}{1-\rho} \leq c\mathcal{E}(x)$. For further details, see [St4] Lemma 4.10.

For $x^i \in M^i$ letting $\mathcal{F}_\rho = \eta \circ \mathcal{F}_\rho(x^1, x^2)$ in $\mathbb{R}^k$, $\mathcal{H}_\rho = \mathcal{H}_\rho(x^1, x^2)$ and $\mathcal{H}^i = \mathcal{H}(x^i), i = 1, 2,$ we have

$$\int_{A_\rho} |d\mathcal{F}_\rho|^2 d\omega \geq \int_{A_\rho} |d\mathcal{H}_\rho|^2 d\omega \geq C(\rho)\int_B |d\mathcal{H}^i|^2 d\omega.$$

Thus, by [St1] Proposition II.2.2, for subsequence $\{w^i_n\}$ we have either

$$\|w^i_n - 2\pi j^i(n) - w^i\|_{C^0} \to 0 \quad \text{for some integers } j^i(n),$$

or

$$x_n^i = \gamma^i \circ w_n^i \to \text{const.} = a_i \in \Gamma_i \quad \text{in } L^1(\partial B).$$

We have to distinguish several cases.

(case 1): Suppose that $\rho \in (0,1)$ and

$$\|w_n^i - 2\pi j_n^i - w^i\|_{C^0} \to 0 \quad \text{for some integers } j_n^i, \;\; i = 1,2.$$

Then

$$\|x_n^i - x^i\|_{C^0} \to 0 \quad \text{with } x^i \in M^i, \;\; i = 1,2.$$

Letting

$$\mathcal{H}_n := \mathcal{H}_\rho(x_n^1, x_n^2), \;\; \mathcal{H} := \mathcal{H}_\rho(x^1, x^2), \;\; \mathcal{F}_n := \mathcal{F}_\rho(x_n^1, x_n^2), \;\; \mathcal{F} := \mathcal{F}_\rho(x^1, x^2), \;\; \text{in} \mathbb{R}^k,$$

$$\begin{aligned}
&\int_{A_\rho} \langle d(\mathcal{H}_n - \mathcal{H}), d(\mathcal{H}_n - \mathcal{H})\rangle d\omega \\
&\quad = \int_{A_\rho} \langle d\mathcal{H}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega - \underbrace{\int_{A_\rho} \langle d\mathcal{H}, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega}_{\star} \\
&\quad = \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1),
\end{aligned}$$

since $\mathcal{H}_n - \mathcal{H}$ is harmonic in $\mathbb{R}^k$ and $\star = o(1)$ (see [St4]).

We compute for $i = 1,2$,

$$\begin{aligned}
&\gamma^i(w_n^i(\theta)) - \gamma^i(w^i(\theta)) \\
&\quad = d\gamma^i(w_n^i(\theta))(w_n^i(\theta) - w^i(\theta)) - \int_{w^i(\theta)}^{w_n^i(\theta)} \int_{w'}^{w^i(\theta)} d^2\gamma^i(w'')dw''dw' \\
&\quad =: I_n^i + II_n^i.
\end{aligned}$$

Clearly, $I_n^i, \;\; II_n^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$ and

$$\mathcal{H}_\rho(x_n^1 - x^1, x_n^2 - x^2) = \mathcal{H}_\rho(I_n^1 + II_n^1, I_n^2 + II_n^2) = \mathcal{H}_\rho(I_n^1, I_n^2) + \mathcal{H}_\rho(II_n^1, II_n^2).$$

By [St3] (3.9),

$$\|II_n^i\|_{\frac{1}{2},2;\infty} \le C\|w_n^i - w^i\|_\infty(|w_n^i|_{\frac{1}{2}} + |w^i|_{\frac{1}{2}}),$$

which converges to 0, so $\|\mathcal{H}(II_n^i)\|_{1,2;\infty} \to 0$, as $n \to \infty, i = 1,2$.

We can observe that: with C, independent of n,

$$\int_{A_\rho} |d\mathcal{H}_\rho(II_n^1, II_n^2)|^2 d\omega \leq C(\rho)(\|\mathcal{H}(II_n^1)\|^2_{1,2;\infty} + \|\mathcal{H}(II_n^2)\|^2_{1,2;\infty}).$$

Since $\|\mathcal{H}(II_n^i)\|_{1,2;\infty} \to 0$, it holds $\int_{A_\rho} |d\mathcal{H}_\rho(II_n^1, II_n^2)|^2 d\omega \to 0 \quad$ as $n \to \infty$, so

$$\begin{aligned}\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega &= \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1)\\ &= \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_\rho(I_n^1, I_n^2))\rangle d\omega + o(1).\end{aligned}$$

Now we consider $\xi_n^i := -I_n^i = d\gamma^i(w_n^i(\theta))(w^i(\theta) - w_n^i(\theta))$ as an element of $\mathcal{T}_{x_n^i}$, and let $\mathbf{J}_n^1 := \mathbf{J}_{\mathcal{F}_\rho}(\xi_n^1, 0)$, $\mathbf{J}_n^2 := \mathbf{J}_{\mathcal{F}_\rho}(0, \xi_n^2)$. Then

$$\begin{aligned}&\int_{A_\rho} \langle d\mathcal{F}_n, d\mathcal{H}_\rho(I_n^1, I_n^2)\rangle d\omega\\
&= \int_{A_\rho} \langle d\mathcal{F}_n, d\mathcal{H}_\rho(I_n^1, 0)\rangle d\omega + \langle d\mathcal{F}_n, \mathcal{H}_\rho(0, I_n^2)\rangle d\omega\\
&= \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^2\rangle d\omega\\
&\quad + \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_\rho(I_n^1, 0) + \mathbf{J}_n^1)\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_\rho(0, I_n^2) + \mathbf{J}_n^2)\rangle d\omega\\
&= \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^2\rangle d\omega\\
&\qquad + \int_{A_\rho} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\rho(I_n^1, 0) + \mathbf{J}_n^1\rangle d\omega\\
&\qquad\quad + \int_{A_\rho} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\rho(0, I_n^2) + \mathbf{J}_n^2\rangle d\omega\\
&\leq g_i(x_n^1, x_n^2, \rho)\|\xi_n^i\|_{\frac{1}{2},2;\infty} + C(\|\mathcal{F}_n\|_{1,2;0})\|\xi_n^i\|_\infty\\
&\leq g_i(x_n^1, x_n^2, \rho)\|\xi_n^i\|_{\frac{1}{2},2;\infty} + C(\|\mathcal{F}_n\|_{1,2;0})\|x_n^i - x^i\|_\infty\\
&\leq Cg_i(x_n)\|\xi_n^i\|_{\frac{1}{2},2;\infty} + C(\|\mathcal{F}_n\|_{1,2;0})\|x_n^i - x^i\|_\infty,\end{aligned}$$

where C is independent of $n \geq n_0$, for some n_0, because: from assumption, $\|x_n^i - x^i\|_{C^0} \to 0$ with $x^i \in M^i$. And applying the argument in Remark 3.1.1, Remark 3.1.2 and Lemma 4.2.1, we obtain the convergence of $g_i(x_n^1, x_n^2, \rho_{n'}))$ as $\rho_{n'} \to \rho$ uniformly on $\{x_n^i | n \geq n_0\}$.

Also note that $\|\xi_n^i\|$ are uniformly bounded, since from [St1]

$$\|d\gamma^i(w_n^i)(w_n^i - w^i)\|_{\frac{1}{2}} \leq \|d\gamma^i(w_n^i)\|_{\frac{1}{2}}\|w_n^i - w^i\|_\infty + \|d\gamma^i(w_n^i)\|_\infty \|w_n^i - w^i\|_{\frac{1}{2}}.$$

Therefore $\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega \to 0 \quad \text{as } n \to \infty$, and

$$x_n^i \to x^i \quad \text{strongly in } H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k), \;\; i = 1, 2.$$

And since $\lim_{n\to\infty} \|x_n^i - x^i\|_{C^0} = 0$, x^i is also an element of $M^{i,*}$, i.e. for $0 < \rho < 1$,

$$x_n = (x_n^1, x_n^2, \rho_n) \to x = (x^1, x^2, \rho) \quad \text{in } \mathcal{M}^*.$$

(case 2): Suppose that $\rho \in (0, 1)$ and

$$\|x_n^1 - x^1\|_{C^0} \to 0,$$

$$x_n^2 = \gamma_2 \circ w_n^2 \to \text{const.} = a_2 \in \Gamma_2 \quad \text{in } L^1(\partial B, \mathbb{R}^k).$$

I) First, we claim that $\mathcal{F} := \mathcal{F}_\rho(\gamma^1 \circ w^1, a^2)$ is conformal.

I-a) By assumption, it holds,

$$\|I_n^1\|_{C^0} = \|d\gamma^1(w_n)(w_n^1 - w^1)\|_{C^0} \to 0.$$

Letting

$$\mathcal{H}_n := \mathcal{H}_\rho(x_n^1, x_n^2), \mathcal{H} := \mathcal{H}_\rho(x^1, a^2), \;\; \mathcal{F}_n := \mathcal{F}_\rho(x_n^1, x_n^2), \mathcal{F} := \mathcal{F}_\rho(x^1, a^2),$$

for fixed $\sigma \in (\rho, 1)$,

$$\begin{aligned}
\int_{A_\sigma} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega &= \int_{A_\sigma} \langle d\mathcal{H}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + \int_{A_\sigma} -\langle d\mathcal{H}, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega \\
&= \int_{A_\sigma} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + \int_{A_\sigma} \langle d(\mathcal{F}_n - \mathcal{H}_n), d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1) \\
&= \int_{A_\sigma} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega \\
&\quad - \int_{\partial B} \langle \frac{d}{d\vec{n}}(\mathcal{H}_n - \mathcal{H}), 0\rangle d\omega + \int_{\partial B_\sigma} \langle \frac{d}{d\vec{n}}(\mathcal{H}_n - \mathcal{H}), \mathcal{F}_n - \mathcal{H}_n\rangle d\omega + o(1) \\
&= \int_{A_\sigma} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1) \\
&= \int_{A_\sigma} \langle d\mathcal{F}_n, d\mathcal{H}_\sigma(I_n^1 + II_n^1, \mathcal{H}_n|_{\partial B_\sigma} - \mathcal{H}|_{\partial B_\sigma})\rangle d\omega + o(1),
\end{aligned}$$

considering $\mathbf{J}_\rho(\xi_n^1, 0)|_{\partial B_\sigma} =: l_n$ with $\mathbf{J}_\rho = d\eta(\mathbf{J}_\rho(\xi_n^1, 0))$, where η is the isometric imbedding of N into $\mathbb{R}^k$,

$$\begin{aligned}
&= \int_{A_\sigma} \langle d\mathcal{F}_n, d\mathcal{H}_\sigma(I_n^1, l_n)\rangle d\omega + \int_{A_\sigma} \langle d\mathcal{F}_n, d\mathcal{H}_\sigma(II_n^1, -l_n + \mathcal{H}_n|_{\partial B_\sigma} - \mathcal{H}_{\partial B_\sigma})\rangle d\omega + o(1) \\
&= \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\sigma} \langle d\mathcal{F}_n, d\mathcal{H}_\sigma(I_n^1, l_n) + d\mathbf{J}_n^1\rangle d\omega + o(1) \\
&= \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\sigma} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\sigma(I_n^1, l_n) + \mathbf{J}_n^1\rangle d\omega + o(1) \\
&\leq \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + C\|\mathcal{F}_n\|_{1,2}(\|\mathcal{H}_\sigma(I_n^1, l_n)\|_{C^0} + \|\mathbf{J}_n^1\|_{C^0}) + o(1) \\
&= \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + o(1).
\end{aligned}$$

This holds for each $\sigma \in (\rho, 1)$, thus

$$\begin{aligned}
\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega &= \lim_{\sigma\to\rho} \int_{A_\sigma} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega \\
&\leq \lim_{\sigma\to\rho} \left(\int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + o(1) \right) \\
&= \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + o(1) \\
&\leq g_1(x_n^1, x_n^2, \rho)\|\xi_n^1\|_{\frac{1}{2},2;0} + o(1) \\
&\leq Cg_1(x_n^1, x_n^2, \rho_n)\|\xi_n^1\|_{\frac{1}{2},2;0} + o(1), \qquad \text{for large } n \geq n_0 \\
&\longrightarrow 0 \quad \text{as } n \to \infty,
\end{aligned}$$

from Lemma 4.2.1 and the uniform boundedness of $\|\xi_n^1\|_{\frac{1}{2},2;0}$.
And $\int_B |d\mathcal{H}(x_n^1 - x^1)|^2 d\omega \leq \int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega + o(1)$, $o(1) \to 0$ uniformly for $n \geq n_0$ as $\rho \to 0$, so from the above

$$x_n^1 \longrightarrow x^1 \quad \text{in } H^{\frac{1}{2},2} \cap C^0. \tag{5.4}$$

I-b) Now letting $x_n^2 = \gamma_2 \circ w_n^2$ and $a_2 = \gamma^2 \circ w^2$, let us see the behavior of $\mathcal{F}_{\rho_n}(x_n^1, x_n^2)|_{\partial B_{\rho_n}}$.

There must exist $\theta_0 \in [0, 2\pi]$ such that

$$\Big|\lim_{\theta\to\theta_0+} w^2(\theta) - \lim_{\theta\to\theta_0-} w^2(\theta)\Big| = 2\pi. \tag{5.5}$$

By the Courant-Lebesgue Lemma, for given $\varepsilon > 0$ there exists $r_n \in (\delta, \sqrt{\delta})$ such that with $B_{r_n} := B_{r_n}(\theta_0)$

$$\operatorname{osc}_{A_{\rho_n} \cap \partial B_{r_n}} \mathcal{F}_{\rho_n}(x_n^1, x_n^2) \leq C\, \frac{\mathcal{E}(x_n^1, x_n^2, \rho_n)}{\ln(\delta^{-1})} \leq \frac{C}{\ln(\delta^{-1})} < \varepsilon, \tag{5.6}$$

for some small $\delta := \delta(\varepsilon) > 0$. Let $Y_n^2 := \Gamma_2 \backslash \mathcal{F}_{\rho_n}(B_{r_n}) \cup \mathcal{F}_{\rho_n}(\partial B_{r_n})$, then by (5.5) and (5.6) we have

$$\text{dist}(Y_n^2, a_2) \to 0 \text{ as } n \to \infty. \tag{5.7}$$

(because: x_n^2 is monotone for all n, by (5.5) the length of $\Gamma_2 \cap \mathcal{F}_{\rho_n}(B_{r_n})$ or $\Gamma_2 \backslash \mathcal{F}_{\rho_n}(B_{r_n})$ converges to 0, but by (5.6) the latter must converge to 0)

We next recall the argument in the proof of Lemma 4.1.1, *(B)* and apply it for $\{\mathcal{F}_{\rho_n}|_{A_{\rho_n} \backslash B_{r_n}}\} =: \tilde{\mathcal{F}_n}$, then $\tilde{\mathcal{F}_n}|_K$ converges to $\mathcal{F}|_K$ in C^2 by (5.4), (5.7), for compact $K \subset\subset (A_{\rho_n} \backslash B_{r_n})$ for large n.

I-c) Now, we investigate the behavior of Jacobi fields.

For large $n \geq n_0$, we have that $\widetilde{\exp}_{x^1}\xi_n^1 = x_n^1$ for some $\xi_n^1 \in \mathcal{T}_{x^1}$, with

$$\|d\widetilde{\exp}_{x^1,\xi_n^1}\phi^1\| < l_1 \text{ for } \phi^1 \in \mathcal{T}_{x^1} \text{ with } \|\phi^1\| < l_1,$$

and letting $(v_n^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}_{\rho_n}) := \mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\xi_n^1}\phi^1, 0)$,

$$\int_{A_{\rho_n}} h_{\alpha\beta} \circ \mathcal{F}_{\rho_n} v_{n,i}^\alpha v_{n,i}^\beta d\omega \leq C, \text{ independent of } n \geq n_0,$$

because $d\widetilde{\exp}_{x^1,\xi_n^1} \to Id$ in $H^{\frac{1}{2},2} \cap C^0$. Thus, once again from the Courant-Lebesgue Lemma and $v_n^\alpha|_{\partial B_{\rho_n}} \equiv 0$, for some $\tilde{r_n} \in (\sqrt{\delta}, \sqrt{\sqrt{\delta}})$,

$$\int_{\partial(B_{\tilde{r_n}} \cap A_{\rho_n})} h_{\alpha\beta} \circ \mathcal{F}_{\rho_n} \partial_\theta v_n^\alpha \partial_\theta v_n^\beta d\theta \leq \frac{C}{|\ln \delta|} \text{ and } \|(v_n^\alpha)\|_{C^0(B_{\tilde{r_n}(\theta_0)} \cap A_{\rho_n})} \leq \frac{C}{|\ln \delta|}.$$

Hence, from Lemma 3.2.1, $E(\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\xi_n^1}\phi^1, 0)|_{B_{\tilde{r_n}}})$ is less than $\frac{C}{|\ln \delta|}$, small enough on B_{r_n}, since $r_n \leq \tilde{r_n}$. Now we choose the above δ so small that $\frac{C}{|\ln \delta|} \leq \varepsilon := \frac{1}{n}, n \geq n_0$.

I-d) Letting $\mathcal{F}_{\rho_n} := \mathcal{F}_{\rho_n}(x_n^1, x_n^2)$, by Hölder inequality

$$\begin{aligned}
0 &= \lim_{n\to\infty} g^1(x_n^1, x_n^2, \rho_n) \\
&\geq \lim_{n\to\infty} \left(- \int_{A_{\rho_n} \backslash B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\xi_n^1}\phi^1, 0)\rangle d\omega - \int_{B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\xi_n^1}\phi^1, 0)\rangle d\omega \right) \\
&= \lim_{n\to\infty} \left(- \int_{A_{\rho_n} \backslash B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\xi_n^1}\phi^1, 0)\rangle d\omega - o(1) \right) \\
&= - \int_{A_\rho} \langle d\mathcal{F}, d\mathbf{J}_{\mathcal{F}}(\phi^1, 0)\rangle d\omega.
\end{aligned}$$

Then, from (A.3), $\mathbf{A}(\mathcal{F})(\Delta_{-h}\Delta_h\mathcal{F}|_{\partial A_\rho}) \geq 0$. Since $\mathcal{F}|_{\partial B_\rho} \equiv a_2$, the similar computation as in the proof of Theorem A.1 delivers that $\mathcal{F}_\rho(x^1, a_2) \in H^{2,2}(A_\rho, N)$.

Now we can handle the same computation as in Proposition 4.2.1, and it follows that

$$\langle \frac{d\mathcal{F}}{d\vec{n}}, \frac{\partial \mathcal{F}}{\partial \theta} \rangle_h|_{\partial B} \equiv 0.$$

And $\langle \frac{d\mathcal{F}}{d\vec{n}}, \frac{\partial \mathcal{F}}{\partial \theta} \rangle_h|_{\partial B_\rho} \equiv 0$, so as a consequence, the holomorphic function for $z = re^{i\theta} \in A_\rho$,

$$\Phi(z) = r^2\Big|\frac{\partial}{\partial r}\mathcal{F}\Big|_h^2 - \Big|\frac{\partial}{\partial \theta}\mathcal{F}\Big|_h^2 - 2ir\Big\langle \frac{\partial}{\partial r}\mathcal{F}, \frac{\partial}{\partial \theta}\mathcal{F}\Big\rangle_h$$

is real constant.

I-e) We must now show that $\frac{\partial}{\partial \rho}\mathcal{E}(x^1, a_2, \rho) = 0$ for the conformal property of $\mathcal{F}$ from the from of $\frac{\partial}{\partial \rho}\mathcal{E}$ in Lemma 4.1.1. We will use the idea in [St4].

Suppose that $\frac{\partial}{\partial \rho}\mathcal{E}(x^1, a_2, \rho) \neq 0$, then for some $\delta > 0$,

$$\left|\int_0^{2\pi}\int_{\rho+\delta}^1 \left[\left|\frac{\partial}{\partial r}\mathcal{F}\right|_h^2 - \frac{1}{r^2}\left|\frac{\partial}{\partial \theta}\mathcal{F}\right|_h^2\right] \frac{1}{1-\rho-\delta} dr d\theta\right| = c > 0.$$

For large $n \geq n_o$, $\rho_n < \rho + \delta$ and $r_n < \delta$ in (5.6), since r_n converges to 0. Hence, we have that

$$\int_{A_{\rho+\delta}} |d(\mathcal{F}_{\rho_n} - \mathcal{F})|_h^2 d\omega \to 0 \quad \text{as } n \to \infty.$$

And for some $C \in \mathbb{R}$, independent of n, which are large enough,

$$\left|\int_0^{2\pi}\int_{\rho+\delta}^1 \left[\left|\frac{\partial}{\partial r}\mathcal{F}_{\rho_n}\right|_h^2 - \frac{1}{r^2}\left|\frac{\partial}{\partial r}\mathcal{F}_{\rho_n}\right|_h^2\right] \frac{1}{1-\rho-\delta} dr d\theta\right| = C > 0,$$

Now let

$$\widetilde{\mathcal{F}}_n^\sigma := \begin{cases} \mathcal{F}_{\rho_n} \circ \tau_\sigma^{\rho+\delta} & , \text{ on } A_\sigma \\ \mathcal{F}_{\rho_n}(\frac{\rho+\delta}{\sigma}r, \theta) & , \text{ on } A_{\frac{\sigma\rho_n}{\rho+\sigma}} \backslash A_\sigma, \end{cases}$$

where $\tau_\sigma^{\rho+\delta}$ is from $[\sigma, 1]$ onto $[\rho+\delta, 1]$. Then

$$\begin{aligned}
&2\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^\sigma)|_{\sigma=\rho+\delta} \\
&= \frac{d}{d\sigma}|_{\sigma=\rho+\delta}\int_{A_\sigma}|\nabla(\mathcal{F}_{\rho_n}\circ\tau_\sigma^{\rho+\delta})|^2 d\omega + \frac{d}{d\sigma}|_{\sigma=\rho+\delta}\int_{A_{\frac{\sigma\rho_n}{\rho+\sigma}}\setminus A_\sigma}|\nabla\mathcal{F}_{\rho_n}(\frac{\rho+\delta}{\sigma}r,\theta)|^2 d\omega \\
&= \frac{d}{d\sigma}|_{\sigma=\rho+\delta}\int_{A_\sigma}|\nabla(\mathcal{F}_{\rho_n}\circ\tau_\sigma^{\rho+\delta})|^2 d\omega + \underbrace{\frac{d}{d\sigma}|_{\sigma=\rho+\delta}\int_{A_{\rho_n}\setminus A_{\rho+\delta}}|\nabla\mathcal{F}_{\rho_n}(r,\theta)|^2 d\omega}_{=0} \\
&= \int_0^{2\pi}\int_{\rho+\delta}^1\left[|\partial_r\mathcal{F}_{\rho_n}|^2 - \frac{1}{r^2}|\partial_\theta\mathcal{F}_{\rho_n}|^2\right]\frac{1}{1-\rho-\delta}dr d\theta.
\end{aligned}$$

Letting $\mathcal{F}_n^\sigma := \mathcal{F}(x_n^1, x_n^2, \sigma)$ and $\rho_n + h = \frac{(\rho+\delta+l)\rho_n}{\rho+\delta}$, so $\frac{\rho_n}{h} = \frac{\rho+\delta}{l}$, for $h, l > 0$

$$\lim_{h\to 0}\frac{\rho_n(E(\mathcal{F}_n^{\rho_n+h}) - E(\mathcal{F}_n^{\rho_n}))}{h} \le \lim_{l\to 0}\frac{(\rho+\delta)(E(\widetilde{\mathcal{F}}_n^{\rho+\delta+l}) - E(\mathcal{F}_n^{\rho_n}))}{l},$$

since $\widetilde{\mathcal{F}}_n^{\rho+\delta} = \mathcal{F}_n^{\rho_n} = \mathcal{F}_{\rho_n}$. For $h, l > 0$, we have the inverse inequality, so

$$\rho_n|g_3(x_n)| = \left|\rho_n\left[\frac{d}{d\sigma}E(\mathcal{F}_n^\sigma)|_{\sigma=\rho_n}\right]\right| = \left|(\rho+\delta)\left[\frac{d}{d\sigma}E(\tilde{\mathcal{F}}_n^\sigma)|_{\sigma=\rho+\delta}\right]\right| \ge \frac{\rho}{2}C > 0,$$

contradicting the assumption that $g_3(x_n) \to 0$. And $\mathcal{F}_\rho(x^1, x^2)$ is conformal.

II) We now have a harmonic, conformal map $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2) \in H^{1,2}\cap C^0(\overline{A_\rho}, N)$, and we will see that $\mathcal{F}$ must be a constant map from the argument as in [Jo1] Theorem 8.2.3.

Consider the complex plane with positive imaginary part, i.e.

$$\mathbb{C}^+ = \{\theta + ir | r > 0\} = \{(\theta, r) | r > 0\}.$$

Let

$$\mathcal{F}((r+\rho)e^{i\theta}) =: \widetilde{X}(\theta, r), \quad \text{well defined on } \mathbb{R}\times[0, 1-\rho]$$

with $\widetilde{X}(\theta, 0) = \mathcal{F}(\rho e^{i\theta}) \equiv a_2$, image in $N \overset{\text{isometry}}{\hookrightarrow} \mathbb{R}^k$ and $\frac{\partial^m\widetilde{X}}{\partial\theta^m}|_{\{r=0\}} \equiv 0$ for each m. Choosing such a local coordinate chart in a neighborhood of a_2 that a_2 is corresponded to $0 \in \mathbb{R}^n$, we may assume that $\widetilde{X}(\theta, 0) = 0$.
Since $\mathcal{F}$ is conformal and harmonic, $\mathcal{F}|_{A_\rho\cup\partial B_\rho} \in C^\infty$(by Theorem 3.1.1), and we get the following by simple computation:

$$\frac{\partial^m}{\partial\theta^m}\widetilde{X} \equiv \frac{\partial^m}{\partial r^m}\widetilde{X} \equiv 0 \quad \text{on } \{r = 0\}, \qquad m \in \mathbb{N}.$$

For some $\rho_0 \leq \rho$, let

$$\Omega := \{\theta + ir | \theta \in \mathbb{R},\ \ r \in [0, 1-\rho_0)\}, \quad \Omega^- := \{\theta + ir | \theta \in \mathbb{R},\ \ -r \in [0, 1-\rho_0)\}.$$

We expand $\widetilde{X}$ to $\Omega \cup \Omega^- =: \widetilde{\Omega}$ by reflection, i.e.

$$\widetilde{X}(\theta + ir) := \widetilde{X}(\theta - ir) \quad \text{for } (\theta + ir) \in \Omega^-.$$

Then, $\widetilde{X} \in C^\infty(\widetilde{\Omega}, N)$, and from the harmonicity of $\mathcal{F}$ and the construction of $\widetilde{X}$,

$$|\widetilde{X}_{z\bar{z}}| \leq C|\widetilde{X}_z|, \quad \text{for some constant } C,$$

where $\partial_z := \frac{1}{2}(\partial_\theta - i\partial_r)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_\theta + i\partial_r)$.
Furthermore,

$$\frac{\partial^m}{\partial \theta^m}\widetilde{X}(0) = \frac{\partial^m}{\partial r^m}\widetilde{X}(0) = 0, \quad \text{for all } m \in \mathbb{N},$$

and

$$\lim_{z=(\theta,r)\to 0} \widetilde{X}(z)|z|^{-m} = 0, \quad \text{for all } m \in \mathbb{N}.$$

Hence $\widetilde{X}$ is constant in $\widetilde{\Omega}$ from the Hartman-Wintmer Lemma (see [Jo1]). Repeating this finitely many times, we get $\mathcal{F} \equiv a_2$ on $\overline{A_\rho}$. But this cannot occur, because we have assumed that $\text{dist}(\Gamma_1, \Gamma_2) > 0$. Therefore we may exclude this case.

(case3): Suppose that $x^i = \gamma_i \circ x_n^i \to$const.$=: a_i \in \Gamma_i$ in $L^1(\partial B, \mathbb{R}^k), i = 1, 2$.

Then $\Phi(\mathcal{F})$ is real constant, where $\mathcal{F} := \mathcal{F}_\rho(a_1, a_2)$. Similarly to the second case, supposing $\frac{d}{d\rho}E(\mathcal{F}) \neq 0$, we have for some fixed $t, \delta > 0$ and large $n \geq n_0$,

$$\left| \int_0^{2\pi} \int_{\rho+\delta}^{1-t} \left[\left| \frac{\partial}{\partial r}\mathcal{F}_{\rho_n} \right|_h^2 - \frac{1}{r^2} \left| \frac{\partial}{\partial r}\mathcal{F}_{\rho_n} \right|_h^2 \right] \frac{1}{1-\rho-\delta} dr d\theta \right| = C > 0.$$

Letting

$$\widetilde{\mathcal{F}}_n^\sigma := \begin{cases} \mathcal{F}_{\rho_n} & , \quad \text{on } A_{1-t}, \\ \mathcal{F}_{\rho_n} \circ \tau_{(1-t)\sigma}^{\rho+\delta} & , \quad \text{on } A_\sigma \backslash A_{1-t}, \\ \mathcal{F}_{\rho_n}(\frac{\rho+\delta}{\sigma} r, \theta) & , \quad \text{on } A_{\frac{\sigma\rho_n}{\rho+\sigma}} \backslash A_\sigma, \end{cases}$$

we have

$$2\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^\sigma)|_{\sigma=\rho+\delta} = \int_0^{2\pi} \int_{\rho+\delta}^{1-t} \left[|\partial_r \mathcal{F}_{\rho_n}|^2 - \frac{1}{r^2}|\partial_\theta \mathcal{F}_{\rho_n}|^2 \right] \frac{1-t}{1-t-\rho-\delta} dr d\theta.$$

And because $\widetilde{\mathcal{F}}_n^{\rho+\delta} = \mathcal{F}_{\rho_n}$,

$$\rho_n |g_3(x_n)| = |\rho_n \frac{d}{d\sigma}E(\mathcal{F}_{\rho_n})|_{\sigma=\rho_n}| = |(\rho+\delta)\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^{\rho+\delta})| \geq C > 0,$$

contradicting the assumption, $g_3(x_n) \to 0$. Thus, $\mathcal{F}_\rho(a_1, a_2)$ is also conformal.
From the same argument as in **(case3)**, we can also exclude this case.

(case4): Suppose that $\rho = 0$.

For a conformal diffeomorphism τ_n^i of B, it holds $\mathcal{F}^i(x_n^i) \circ \tau_n^i = \mathcal{F}^i(\widetilde{x_n^i})$ with $\widetilde{x_n^i} \in M^{i*}$, $i = 1, 2$. And $\widetilde{x_n^i} \to x^i \in M^{i*}$ uniformly for some subsequence.
For given $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that for $n \geq n_0$(subsequence), $(x_n^1, x_n^2, \rho_n) \in \mathcal{N}_\delta(\widetilde{x_n^1}, \widetilde{x_n^2}, 0)$ and $|g(x_n^1, x_n^2, \rho_n) - g(\widetilde{x_n^1}, \widetilde{x_n^2}, 0)| < \varepsilon$.
Hence $g(\widetilde{x_n^1}, \widetilde{x_n^2}, 0) \to 0$ as $n \to \infty$. And for some subsequence,

$$\mathcal{F}^i(\widetilde{x_n^i}) \to \mathcal{F}^i(\widetilde{x^i}) \text{ in } H^{1,2}.$$

Clearly, $\{x_n\}$ converges strongly to $(\widetilde{x_n^1}, \widetilde{x_n^2}, 0) \in \partial\mathcal{M}^*$ with $g(\widetilde{x_n^1}, \widetilde{x_n^2}, 0) = 0$. □

5.2 Unstable minimal surfaces of annulus type

We need some Lemmata as in [St4] for our case.

Lemma 5.2.1. *For any $\delta > 0$, there exists a uniformly bounded, continuous vectorfield $e_\delta : M^1 \times M^2 \times [0,1) \to TM^1 \times TM^2 \times \mathbb{R}$, with locally Lipschitz continuity on $\mathcal{M}$ and $\partial\mathcal{M}$(separably) with the following properties,*

(i) for $\beta \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for any $x \in \mathcal{M}(\rho) := \{x = (x^1, x^2, \rho) \in \mathcal{M}\}$ with $\mathcal{E}(x) \leq \beta, 0 < \rho < \varepsilon$ it holds that $y_\delta(x) = (\widetilde{\exp}_{x^1} e_\delta^1(x^1), \widetilde{\exp}_{x^2} e_\delta^2(x^2), \rho + e_\delta^3(\rho)) \in \mathcal{M}(\rho)$, namely $e_\delta^3(\rho) = 0$ (e_δ is parallel to $\partial\mathcal{M}$ near $\partial\mathcal{M}$),

(ii) for any such $\beta, \mathcal{E}, x$ as above and any pair $T = (\tau^1, \tau^2)$ of conformal transformations of B,

$$y_\delta(x \circ T) = y_\delta(x) \circ T,$$

where by definition $x \circ T$ satisfies

$$\mathcal{F}^i\left((x \circ T)^i\right) = \mathcal{F}^i(x^i) \circ T, \quad i = 1, 2,$$

(iii) for any $x \in \overline{\mathcal{M}}$,

$$\langle d\mathcal{E}(x), e_\delta(x)\rangle_{\mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}} \leq \delta - g(x),$$

(iv)

$$y_\delta(x) \in \mathcal{M}^* \text{ resp. } \partial\mathcal{M}^*, \text{ for all } x \in \mathcal{M}^* \text{ resp. } \partial\mathcal{M}^*.$$

Proof. Let $i = 1, 2$. First, consider $x = (x^1, x^2, 0)$ in $\partial\mathcal{M}^*$. By Remark 5.1.1, we can take a vector $e^*_\delta(x) := (e^*_{1\delta}, e^*_{2\delta}) \in \mathcal{T}_x\partial\mathcal{M}^*$ such that

$$-\langle d\mathcal{E}(x), e^*_\delta(x)\rangle_{\mathcal{T}_{x^1}\times\mathcal{T}_{x^2}} \geq g(x) - \delta \text{ and } |e^*_{i\delta}(x)| \leq l_i - \delta.$$

By continuity there exists a neighborhood $\mathcal{U}(x)$ in $\partial\mathcal{M}^*$ with

$$-\langle d\mathcal{E}(y), d\widetilde{\exp}_{x,\xi}\big(e^*_\delta(x)\big)\rangle_{\mathcal{T}_{y^1}\times\mathcal{T}_{y^2}} \geq g(x) - \delta \text{ and } |e^*_{i\delta}(x)| \leq l_i - \delta,$$

where $\widetilde{\exp}_x\xi := (\widetilde{\exp}^1_x\xi^1, = \widetilde{\exp}^2_x\xi^2) = y \in \mathcal{U}(x),\ \ \xi \in \mathcal{T}^*_{x^1} \times \mathcal{T}^*_{x^2}$, $\mathcal{U}(x)$ is so small that $\|d\widetilde{\exp}_{x,\xi}\| \leq A$ when $\widetilde{\exp}_x\xi \in \mathcal{U}(x)$ for a fixed constant A, and let

$$e^x_\delta := d\widetilde{\exp}_{x,\xi}\big(e^*_\delta(x)\big) \in \mathcal{T}^*_{y^1} \times \mathcal{T}^*_{y^2}.$$

Choosing a locally finite refinement $\{\mathcal{U}(x_l)\}_{l\in I}$ of $\partial\mathcal{M}^*$ and a Lipschitzian partition of unity $\{\Psi^1_l\}_{l\in I}$, subordinate to $\{\mathcal{U}(x_l)\}_{l\in I}$ (note that $\partial\mathcal{M}^*$ is a metric space), define a vector for $x \in \partial\mathcal{M}^*$,

$$e_\delta := \Sigma_{l\in I}\Psi^1_l(x)e^{x_l}_\delta(x).$$

Now, in order to extend this definition to $\partial\mathcal{M}$, consider conformal mapping (from two unit discs onto themselves) $T := (\tau^1, \tau^2)$ such that $x = \tilde{x} \circ T$, for some $\tilde{x} \in \partial\mathcal{M}^*$ and let

$$e_\delta(x) = e_\delta(\tilde{x} \circ T) := e_\delta(\tilde{x}) \circ T = \Sigma_{l\in I}\Psi^1_l(\tilde{x})e^{x_l}_\delta(\tilde{x}) \circ T,$$

and $(i), (ii)$ hold. Note that x^i is weakly monotone, so such T and $\tilde{x}$ exists uniquely, hence $e_\delta(x)$ is well defined.
We know that g is uniformly continuous as $\rho \to 0$ on a set of uniform bounded x^i's, and if $\mathcal{E}(x) \leq \beta$ then the norm of x^i is also uniform bounded by some $C(\beta) \in \mathbb{R}$.
Thus, for any $\beta > 0$, there exists $\varepsilon > 0$ such that

$$-\langle d\mathcal{E}(x^1, x^2, \rho), \big(e_\delta(x^1, x^2, 0), 0\big)\rangle_{\mathcal{T}_{x^1}\times\mathcal{T}_{x^2}\times\mathbb{R}} \geq g(x) - \delta,$$

for all $(x^1, x^2, \rho) \in \mathcal{M}$ with $\rho < \varepsilon$ and $\mathcal{E}(x^1, x^2, \rho) < \beta$, and let $e^1_\delta(x) = \big(e_\delta(x^1, x^2, 0), 0\big)$ for such $x = (x^1, x^2, \rho)$ which satisfies the condition (i) and is continuous near $\partial\mathcal{M}^*$ with respect to the given topology.

On the other hand, we can also choose $e^{2*}_\delta(x) = (e^{2*}_{j\delta}(x))_{j=1,2,3}$ for $x \in \mathcal{M}$ with $|e^{2*}_{i\delta}(x)| \leq l_i - \delta$, $|e^{2*}_{3\delta}(x)| \leq 1 - \delta$, $\rho + e^{2*}_{3\delta}(x) \in (0, 1)$ and

$$-\langle d\mathcal{E}(x), e^{2*}_\delta(x)\rangle_{\mathcal{T}_{x^1}\times\mathcal{T}_{x^2}\times\mathbb{R}} \geq g(x) - \delta,$$

specially for $x \in \mathcal{M}^*$, $e^{2*}_\delta(x) \in \mathcal{T}_x\mathcal{M}^*$ by Remark 5.1.1.
Like the above construction, for $y \in \mathcal{U}(x)$, define $e^{x2*}_\delta(y) := d\widetilde{\exp}_{x,\xi}\big(e^{2*}_\delta(x)\big) \in \mathcal{T}^*_{y^1} \times$

$\mathcal{T}^*_{y^2} \times \mathbb{R}$. And with the locally finite refinement and Lipschitz continuous partition of unity, we obtain

$$e^2_\delta := \Sigma_{l\in I}\Psi^2_l(x)e^{x_l 2*}_\delta(x)$$

satisfying (iii).

For $\beta > 0$, let Ψ^3_1, Ψ^3_2 be a Lipschitzian partition of unity on $\overline{\mathcal{M}}$, subordinate to

$$\left\{\{x \in (x^1, x^2, \rho) \in \mathcal{M} \,|\, \rho > \frac{\varepsilon}{2}\},\ \{x \in (x^1, x^2, \rho) \in \overline{\mathcal{M}} \,|\, \rho < \varepsilon\}\right\},$$

where ε is related to β as above. Define

$$e_{\delta,\beta}(x) := \Sigma_{i=1,2}\Psi^3_i(x)e^i_\delta(x).$$

Finally, letting $\Psi^4_k \in C^\infty$ be a Lipschitzian partition of unity, subordinate to $\{(k-2,k)\}_{k\in\mathbb{Z}}$, define

$$e_\delta(x) := \Sigma_{k\in\mathbb{Z}}\Psi^4_k\big(\mathcal{E}(x)\big)e_{\delta,\beta}(x),$$

with $\|e_{i\delta}(x)\| \le l_i - \delta$ which satisfies $(i) \sim (iv)$(Remark 5.1.1) and (separably)locally Lipschitz, since $\mathcal{E}$ is in C^1 in the sense of Lemma 4.1.1. □

Lemma 5.2.2. *For a given a vector field $f : \overline{\mathcal{M}} \to TM^1 \times TM^2 \times \mathbb{R}$ which is locally Lipschitz continuous with the properties in Lemma 5.2.1, there exists a unique flow $\Phi : [0,\infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$ satisfying*

$$\Phi(0,x) = x,\ \ \frac{\partial}{\partial t}\Phi(t,x) = f\left(\Phi(t,x)\right), \quad x \in \overline{\mathcal{M}^*}.$$

Proof. We use the Euler's method.
Let's first define $\Phi^{(m)} : [0,\infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$, $m \ge m_0$ as follows:

$$\begin{aligned} \Phi^{(m)}(0,x) &:= x \\ \Phi^{(m)}(t,x) &:= \widetilde{\exp}_{\Phi^{(m)}(\frac{[mt]}{m},x)}\Big(\frac{mt-[mt]}{m}f\big(\Phi^{(m)}(\frac{[mt]}{m},x)\big)\Big),\ \ t > 0 \end{aligned} \tag{5.8}$$

where $[\tau]$ denotes the largest integer which is smaller than $\tau \in \mathbb{R}$. This is well defined from the convexity of $\mathcal{T}_{x^i}$, $x^i \in M^i$ and from Lemma 5.2.1 *(iv)*.
Recalling a map $w^i \in C^0(\mathbb{R},\mathbb{R})$ with $x^i = \gamma^i \circ w^i$, $x^i \in M^i$, $i = 1,2$ as in section 3.1, let

$$W^i := \{w^i \in C^0(\mathbb{R},\mathbb{R}) : \gamma^i \circ w^i = x^i \text{ for some } x^i \in M^i\},$$

and

$$W := W^1 \times W^2 \times [0,\infty).$$

For a given x^i, the above map w^i is unique up to $2\pi l, l \in \mathbb{Z}$. Note that $w^i(\theta+2\pi) = w^i(\theta)+2\pi$, so the tangent vectors of w^i in W^i are continuous maps on $\mathbb{R}/2\pi$.
We define $\widetilde{f^i}(w^i) := (d\gamma^i)^{-1}(f^i(x^i)) \in C^0(\mathbb{R}/2\pi, \mathbb{R})$, then from (3.8), it satisfies

$$\widetilde{\exp}_{x^i} l f^i(x^i) = \gamma^i \circ (w^i + l\widetilde{f^i}(w^i)), \quad 0 \le l \le 1.$$

Furthermore letting

$$(w^1, w^2, \rho) =: w, \;\; \gamma := (\gamma^1, \gamma^2, Id), \;\; \widetilde{f} := (\widetilde{f}^1, \widetilde{f}^2, f^3) \text{ and } \gamma(w) := (\gamma^1 \circ w^1, \gamma^2 \circ w^2, \rho),$$

there exists $\widetilde{\Phi}^{(m)}(t,w) \in W$ such that $\Phi^{(m)}(t,x) = \gamma(\widetilde{\Phi}^{(m)}(t,w))$, so we can rewrite (5.8) as follows:

$$\gamma \circ \widetilde{\Phi}^{(m)}(t,w) = \gamma \circ \Big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w) + \frac{mt-[mt]}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w)\big)\Big),$$

and

$$\widetilde{\Phi}^{(m)}(t,w) = \widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w) + \frac{mt-[mt]}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w)\big) + 2\pi l, \;\; l \in Z,$$

where $\widetilde{\Phi}^{(m)}(t,w)$ is considered as a map from ∂B to $\mathbb{R}$, i.e. as $\pi \circ \widetilde{\Phi}^{(m)}(t,w)$ for each $(t,w) \in [0,\infty) \times W$.

Then for $t \in (\frac{k}{m}, \frac{k+1}{m}], k \in \mathbb{Z}$,

$$\begin{aligned} \widetilde{\Phi}^{(m)}(t,w) &= \widetilde{\Phi}^{(m)}(0,w) + \Sigma_{j=1}^k \int_{\frac{j-1}{m}}^{\frac{j}{m}} \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{j-1}{m}, w))ds + \int_{\frac{k}{m}}^t \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{k}{m}, w)) \\ &= \widetilde{\Phi}^{(m)}(0,w) + \int_0^t \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ms]}{m}, w))ds. \end{aligned} \tag{5.9}$$

We observe that for some bounded subset A in M^i, if $x^i, y^i \in A$ with $\gamma^i \circ w^i = x^i, \gamma^i \circ v^i = y^i$

$$\|\widetilde{f^i}(w^i) - \widetilde{f^i}(v^i)\|_{H^{\frac{1}{2},2}\cap C^0(\partial B, \mathbb{R}^k)} \le C(\gamma^i)L(A)\|w^i - v^i\|_{H^{\frac{1}{2},2}\cap C^0(\partial B, \mathbb{R}^2)}, \tag{5.10}$$

where $L(A)$ is the Lipschitz constant of f on A.

From now on, we will not distinguish the norm in $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$ and $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^2)$, simply denoting $\|\cdot\|$.

From (5.9), for any $T > 0$ and $G > 0$, there exists a constant $C(T,G)$ with

$$\|\Phi^{(m)}(\cdot, w)\|_{L^\infty([0,T]\times W_K, \overline{\mathfrak{M}})} \le C(T,G) \;\text{ for }\; w \in W \;\text{ with }\; \|w\|_W \le G.$$

Let L_1, L_2 be the Lipschitz constants of f in $\{x \in \mathcal{M} \,|\, \|x\| \leq C(T,G)\}$ resp. $x \in \partial\mathcal{M} \,|\, \|x\| \leq C(T,G)\}$. Then from Lemma 5.2.1,*(i)*, letting $L := \max\{C(\gamma^i)L_1, C(\gamma^i)L_2\}$,

$$\begin{aligned}
\|\widetilde{\Phi}^{(m)}(t,w) - \widetilde{\Phi}^{(n)}(t,w)\| &= \left\| \int_0^t \Big(\widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ms]}{m}, w)) - \widetilde{f}(\widetilde{\Phi}^{(n)}(\frac{[ns]}{n}, w)) \Big) ds \right\| \\
&\leq \int_0^t \underbrace{\left\| \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ms]}{m}, w)) - \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ns]}{n}, w)) \right\|}_{=:\|I\|} ds \\
&\qquad + \int_0^t \left\| \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ns]}{n}, w)) - \widetilde{f}(\widetilde{\Phi}^{(n)}(\frac{[ns]}{n}, w)) \right\| ds \\
&\leq \int_0^t \|I\| ds + \int_0^t L \|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)} ds.
\end{aligned}$$

To estimate $\|I\|$, let us assume that $\frac{m}{n} < 1$ without loss of generality. Then

$$\widetilde{\Phi}^{(m)}\big(\frac{[ns]}{n}, w\big) = \begin{cases} \widetilde{\Phi}^{(m)}\big(\frac{[ms]}{m}, w\big) + \frac{p}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[\frac{m}{n}[ns]]}{m}, w)\big) \\ \text{or} \\ \widetilde{\Phi}^{(m)}\big(\frac{[ms]-1}{m}, w\big) + \frac{p}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[\frac{m}{n}[ns]]}{m}, w)\big), \end{cases}$$

where $0 \leq p = \frac{m}{n}[ns] - \big[\frac{m}{n}[ns]\big] \leq 1$.

For the first case, i.e. $\frac{[\frac{m}{n}[ns]]}{m} = \frac{[ms]}{m}$, from (5.10)

$$\begin{aligned}
\|I\| &\leq L \left\| \widetilde{\Phi}^{(m)}\big(\frac{[ms]}{m}, w\big) - \widetilde{\Phi}^{(m)}\big(\frac{[ns]}{n}, w\big) \right\| \\
&\leq L \left\| \frac{p}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[\frac{m}{n}[ns]]}{m}, w)\big) \right\| \\
&\leq L\frac{1}{m}C(f)
\end{aligned}$$

For the second case, i.e. $\big[\frac{m}{n}[ns]\big] = \frac{[ms]-1}{m}$,

$$\begin{aligned}
\|I\| &\leq L \left\| \widetilde{\Phi}^{(m)}\big(\frac{[ms]}{m}, w\big) - \widetilde{\Phi}^{(m)}\big(\frac{[ms]-1}{m}, w\big) - \frac{p}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[\frac{m}{n}[ns]]}{m}, w)\big) \right\| \\
&\leq L \left\| \frac{1}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[\frac{m}{n}[ns]]}{m}, w)\big) - \frac{p}{m}\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[\frac{m}{n}[ns]]}{m}, w)\big) \right\| \\
&\leq L\frac{2}{m}C(f),
\end{aligned}$$

thus for each $t \in [0,T]$, $\|I\| \leq L(\frac{2}{m} + \frac{2}{n})C(f)$ in general.

Hence,

$$\begin{aligned}
&\|\widetilde{\Phi}^{(m)}(t,w) - \widetilde{\Phi}^{(n)}(t,w)\| \\
\le\ & \int_0^t L(\frac{2}{m} + \frac{2}{n})C(f) + \int_0^t L\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)} ds \\
\le\ & tL(\frac{2}{m} + \frac{2}{n})C(f) + tL\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)}.
\end{aligned}$$

And

$$\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)} \le tL(\frac{2}{m}+\frac{2}{n})C(f) + tL\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)}.$$

Note that the constants are independent of $w \in W$ with $\|w\| \le G$. And M^i is complete. Thus, choosing $t \le \min\{T, \frac{1}{2L}\}$, $\{\widetilde{\Phi}^{(m)}\}$ converges to some function $\widetilde{\Phi}$, uniformly on $[0,t] \times \{w \in W : \|w\| \le G\}$ as $m \to \infty$.

Clearly, $\frac{\partial}{\partial t}\widetilde{\Phi}(m)(t,w) = \widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w)\big)$ if $t \notin Z$.
And $\widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w)\big)$ converges to $\widetilde{f}\big(\widetilde{\Phi}^{(m)}(t,w)\big)$ uniformly on $[0,t] \times \{w \in W : \|w\| \le G\}$ because $\frac{[mt]}{m}$ converges uniformly to t and f is locally Lipschitz.
Moreover, for $t \in Z$ $\lim_{m\to\infty} \Big|\lim_{s\to t+} \frac{\partial}{\partial t}\widetilde{\Phi}(m)(t,w) - \lim_{s\to t-} \frac{\partial}{\partial t}\widetilde{\Phi}(m)(t,w)\Big| = 0$.
Thus, for a given $w \in W$ whose norm is clearly finite, we may conclude that $\widetilde{\Phi}(t,w)$ is differentiable with respect to t with

$$\frac{\partial}{\partial t}\widetilde{\Phi}(t,w) = \widetilde{f}\big(\widetilde{\Phi}(t,w)\big).$$

Similary, we may also have that $\widetilde{\Phi}(t,w)$ depends continuously on the initial datum.

By the uniformly convergence of $\widetilde{\Phi}^{(m)}(\cdot,w)$, we get

$$\Phi(t,w) := \gamma \circ \widetilde{\Phi}(t,w) \in \overline{\mathcal{M}}^*, \text{ for some time } t \in [0,t_0].$$

From the uniform boundednes of f, we get the flow Φ for each $x \in \overline{\mathcal{M}}$ with,

$$\frac{\partial}{\partial t}\Phi(t,w) = d\gamma\Big(\widetilde{f}\big(\widetilde{\Phi}^{(m)}(t,w)\big)\Big) = f\big(\Phi(t,w)\big),$$

and from (5.3) it cannot reach $\{x = (x^1, x^2, 1)\}$ in finite time, so we have the flow $\Phi : [0,\infty) \times \overline{\mathcal{M}}^* \to \overline{\mathcal{M}}^*$. □

We now introduce the following well known Proposition and Lemma. For the completeness of this paper the proofs will also be introduced with exact computation.

Proposition 5.2.1. *Let* $y_0, y_1 \in \overline{\mathcal{M}^*}$, *and*

$$P := \{p \in C^0([0,1], \overline{\mathcal{M}^*}) \mid p(0) = y_0,\, p(1) = y_1\}.$$

And suppose that

$$\beta := \inf_{p \in P} \sup_{0 \leq t \leq 1} \mathcal{E}\big(p(t)\big) > \max\{\mathcal{E}(y_0), \mathcal{E}(y_1)\} = \beta_0$$

Then β *is a critical value of* $\mathcal{E}$.

Proof. Supposing that β is not critical value, we have constants $\delta, \varepsilon > 0$ such that $g(x) \geq 2\delta$ for $x \in \overline{\mathcal{M}^*}$ with $|\mathcal{E}(x) - \beta| < 2\varepsilon$ by Proposition 5.1.1(Palais-Smale condition). We may assume that $2\varepsilon < \beta - \beta_0$, $\varepsilon < \delta$. Choose $\theta \in C^\infty(\mathbb{R})$ with $\theta(s) \equiv 1$ if $|s - \beta| < \varepsilon$, $\theta(s) \equiv 0$ if $|s - \beta| > 2\varepsilon$ and let

$$\widetilde{e}_\delta(x) = \theta\left(\mathcal{E}(x)\right) e_\delta(x),$$

with the flow $\Phi : [0, \infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$, such that $\frac{\partial}{\partial t}\Phi(t,x) = \widetilde{e}_\delta(x)$.
For $|\mathcal{E}(\Phi(t,x)) - \beta| < 2\varepsilon$,

$$\frac{d}{dt}\mathcal{E}\left(\Phi(t,x)\right) = \langle d\mathcal{E}(\Phi(t,x)), \widetilde{e}_\delta(\Phi(t,x))\rangle = \theta(\Phi(t,x))(\delta - g(\Phi(t,x)) \leq -\delta,$$

otherwise, $\frac{d}{dt}\mathcal{E}\left(\Phi(t,x)\right) = 0$ from the definition of $\widetilde{e}_\delta$, so $\mathcal{E}\left(\Phi(t,x)\right)$ is non-increasing with respect to $t \in [0, \infty)$.
Hence for $\mathcal{E}(x) \leq \beta + \varepsilon$, we have either $\mathcal{E}\left(\Phi(2,x)\right) \leq \beta - \varepsilon$ or $|\mathcal{E}\left(\Phi(t,x)\right) - \beta| \leq \varepsilon$ for all $t \in [0,2]$. For the latter case, by the definition of θ,

$$\begin{aligned}
\mathcal{E}\left(\Phi(2,x)\right) &\leq \int_0^2 \frac{d}{dt}\mathcal{E}\left(\Phi(t,x)\right) dt + \mathcal{E}(x) \\
&\leq \int_0^2 \langle d\mathcal{E}(\Phi(t,x)), \widetilde{e}_\delta(\Phi(t,x))\rangle\, dt + \mathcal{E}(x) \\
&\leq \int_0^2 \left(\delta - g(\Phi(t,x))\right) dt + \beta - \varepsilon \\
&\leq \beta + \varepsilon - 2\delta \leq \beta - \varepsilon.
\end{aligned}$$

Choosing $p \in P$ such that $\sup_{0 \leq t \leq 1} \mathcal{E}(p(t)) < \beta + \epsilon$, let $\tilde{p}(t) := \Phi(2, p(t)), t \in [0,1]$. It holds, $2\varepsilon < \beta - \mathcal{E}(y_j), j = 0, 1$, so from the definition of θ, we get $\Phi(2, p(j)) = y_j$, and hence $\tilde{p} \in P$. But by the above argument

$$\sup_{0 \leq t \leq 1} \mathcal{E}(\tilde{p}(t)) \leq \beta - \varepsilon,$$

which contradicts the definition of β. □

Lemma 5.2.3. *Let $x_0 \in \mathcal{M}^*$ (resp.$\partial\mathcal{M}^*$) be a strict local minimizer of $\mathcal{E}$ in $\mathcal{M}^*$ (resp.$\partial\mathcal{M}^*$), then there exists a neighborhood $\mathcal{N}$ of x_0 in $\mathcal{M}^*$ (resp.$\partial\mathcal{M}^*$) such that*

$$\mathcal{E}(x_0) < \inf_{x\in\partial\mathcal{N}} \mathcal{E}(x).$$

Proof. Suppose that $\mathcal{E}(x_0) < \mathcal{E}(x)$ for all $x \in \mathcal{M}^*$ satisfying $0 < \|x - x_0\| < 2\varepsilon$.
Let x_m be a minimizing sequence for $\mathcal{E}$ in $\{x \in \mathcal{M}^* | \, \|x - x_0\| = \varepsilon\}$ and suppose that $\mathcal{E}(x_m) \to \mathcal{E}(x_0)(m \to \infty)$, then $g(x_m) \to 0$. Otherwise, with $g(x_m) \geq 2\delta$ and the flow Φ of e_δ,

$$\begin{aligned}\mathcal{E}(\Phi(\varepsilon', x_m)) &= \int_0^{\varepsilon'} \frac{d}{dt}\mathcal{E}\left(\Phi(t, x_m)\right) dt + \mathcal{E}\left((\Phi(0, x_m)\right) \\ &\leq \int_0^{\varepsilon'} (\delta - g(x_m))\, dt + \mathcal{E}(x_m) \\ &\leq -\varepsilon'\delta + \mathcal{E}(x_m),\end{aligned}$$

contradicting the minimal property of $\mathcal{E}$ at x_0(note: Φ depends continuously on the initial datum, and the above result holds for arbitrary $\varepsilon' > 0$, finally $\mathcal{E}$ is invariant under conformal mappings).
And Lemma 5.1.1 implies, a subsequence $x_m \to y$ strongly for some $y \in \mathcal{M}^*$, so $\|y - x_0\| = \varepsilon$, $\mathcal{E}(y) = \mathcal{E}(x_0)$, which contradicts the strict minimality of x_0. Hence there exists no such a minimizing sequence, and we have a $\mathcal{N}$ with the property in this Lemma. For the case of $\partial\mathcal{M}^*$, it can be proved similarly. □

In the following Lemma we have a somewhat weaker result than the corresponding Lemma 4.15 in [St4]. But this result is enough for our aim.

Lemma 5.2.4. *Let $\mathcal{F}^i(x_0^i)$ be a solution of $\mathcal{P}(\Gamma_i)$, i.e. a minimal surface of disc type spanning Γ_i in N, for some $x_0^i \in M^i$, $i = 1, 2$.*
And suppose that $d := dist(\mathcal{F}^1(x_0^1), \mathcal{F}^2(x_0^2)) > 0$. Then there exists $\varepsilon > 0$, $\rho_0 \in (0, 1)$ and $C > 0$, (which are dependent on $\mathcal{E}(x_0^1, x_0^2, 0)$) such that for $x^i \in M^i$ with $\|x^i - x_0^i\|_{\frac{1}{2},2;0} =: s(x^i) < \varepsilon$,

$$\mathcal{E}(x^1, x^2, \rho) \geq \mathcal{E}(x^1, x^2, 0) + \frac{Cd^2}{|\ln\rho|},$$

for all $\rho \in (0, \rho_0)$.

Proof.(I) Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$. We take σ_1 and δ, dependent on ρ, such that $\sqrt{\rho} < \delta < \sigma_1 < \sqrt{\sqrt{\rho}}$. Considering the conformal map $T(re^{i\theta}) := \rho\frac{1}{re^{i\theta}}$ on $B_\delta \backslash B_\rho$, we define: writing $\frac{\rho}{\delta} =: \sigma_2$,

$$\mathcal{F}_\rho|_{A_{\sigma_1}} =: f_{\sigma_1} : A_{\sigma_1} \to N, \text{ and } \mathcal{F}_\rho|_{B_\delta\backslash B_\rho}(T^{-1}) =: g_{\sigma_2} : A_{\sigma_2} \to N.$$

Then

$$\begin{aligned} E(\mathcal{F}_\rho) &= E(\mathcal{F}_\rho|_{A_\sigma}) + E(\mathcal{F}_\rho|_{B_\sigma \backslash B_\delta}) + E(\mathcal{F}_\rho|_{B_\delta \backslash B_\rho}) \\ &= E(f_{\sigma_1}) + E(\mathcal{F}_\rho|_{B_\sigma \backslash B_\delta}) + E(g_{\sigma_2}). \end{aligned} \tag{5.11}$$

(II) For the estimate of the first and the third term we take $a_i \in N, i = 1,2$ with

$$\min_{a \in N} E(\mathcal{F}_{\sigma_i}(x^i, a)) = \mathcal{F}_{\sigma_i}(x^i, a_i) =: \mathcal{F}^i_{\sigma_i},$$

and write

$$f_{\sigma_1} = \mathcal{F}^1_{\sigma_1} + X^1(0, -a_1 + \mathcal{F}_\rho|_{\partial B_{\sigma_1}}) \text{ and } g_{\sigma_2} = \mathcal{F}^2_{\sigma_2} + X^2(0, -a_2 + \mathcal{F}_\rho|_{\partial B_{\sigma_2}}),$$

where $X^i := X^i(0, -a_i + \mathcal{F}_\rho|_{\partial B_{\sigma_i}})$ is clearly in $H^{1,2}(A_{\sigma_i}, \mathbb{R}^k)$ with boundary 0 on ∂B and $-a_1 + \mathcal{F}_\rho|_{\partial B_{\sigma_i}}$ on ∂B_{σ_i}.

We now define

$$\widetilde{\mathcal{F}^i_{\sigma_i}} = \begin{cases} \mathcal{F}^i(x^i) =: \mathcal{F}^i & , \text{ on } B \backslash B_{\frac{1}{2}} \\ \mathcal{F}^{\frac{1}{2}}_{\sigma_i}(\mathcal{F}^i|_{\partial B_{\frac{1}{2}}}, \mathcal{F}^i(0)) & , \text{ on } B_{\frac{1}{2}} \backslash B_{\sigma_i}, \end{cases}$$

where $\mathcal{F}^{\frac{1}{2}}_{\sigma_i}$ is harmonic in N with boundary $\mathcal{F}^i|_{\partial B_{\frac{1}{2}}}$ on $\partial B_{\frac{1}{2}}$ and $\mathcal{F}^i(0)$ on ∂B_{σ_i}. Then $\widetilde{\mathcal{F}^i_{\sigma_i}} \in H^{1,2}(A_{\sigma_i}, N)$.

Letting $\mathcal{F}^1_{\sigma_i}|_{B_{\sigma_i}} \equiv a_i$, $\widetilde{\mathcal{F}^i_{\sigma_i}}|_{B_{\sigma_i}} \equiv \mathcal{F}^i(0)$, we have from the Dirichlet integral minimality of harmonic maps and by the choice of a_i,

$$E(\mathcal{F}^i) \le E(\mathcal{F}^i_{\sigma_i}) \le E(\widetilde{\mathcal{F}^i_{\sigma_i}}),$$

thus

$$0 \le E(\widetilde{\mathcal{F}^i_{\sigma_i}}) - E(\mathcal{F}^i) \le E(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i).$$

We now estimate the last term.

$$2E(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i) = \int_{B_{\frac{1}{2}} \backslash B_{\sigma_i}} |\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i)|^2 d\omega + \int_{B_{\sigma_i}} |\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i)|^2 d\omega = a + b.$$

It is easy to see that $b \le C|\sigma_i|^2$, since $\mathcal{F}^i$ is regular on $B_{\frac{1}{2}}$.
To estimate the term a, we observe that $\widetilde{\mathcal{F}^i_{\sigma_i}}|_{B_{\frac{1}{2}} \backslash B_{\sigma_i}} \in H^{2,2}$, since $\widetilde{\mathcal{F}^i_{\sigma_i}}|_{\partial B_{\frac{1}{2}}}$ is regular and contant on ∂B_{σ_i} (see the proof of Theorem A.1). Thus, we can compute that

$$\begin{aligned} \int_{B_{\frac{1}{2}} \backslash B_{\sigma_i}} |\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^1)|^2 d\omega &= \int_{B_{\frac{1}{2}} \backslash B_{\sigma_i}} |\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^1)|^2 d\omega \\ &= \int_{\partial B_{\frac{1}{2}}} \langle \nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i)\vec{n}, \widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i \rangle d_0 + \int_{\partial B_{\sigma_i}} \langle \nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i)\vec{n}, \widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i \rangle d_0 \\ &\le C\|\mathcal{F}^i(0) - \mathcal{F}^i|_{\partial B_{\sigma_i}}\|_0 \sigma_i \le C|\sigma_i|^2. \end{aligned}$$

Here $d_o = rd\theta$ and C only depends on $E(\mathcal{F}^i(x^i))$.

Now from Remark 5.2.1

$$
\begin{aligned}
(5.12)\quad E(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i) &\leq E(\mathcal{F}^i_{\sigma_i}) - E(\mathcal{F}^i) + o_s(1) \\
&\leq E(\widetilde{\mathcal{F}^i_{\sigma_i}}) - E(\mathcal{F}^i) + o_s(1) \leq E(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i) + o_s(1) \leq C|\sigma_i|^2 + o_s(1),
\end{aligned}
$$

where $o_s(1) \to 0$ as $\|x^i - x^i_0\|_{\frac{1}{2},2;0} =: s(x^i) \to 0$.
Since $E(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)|_{B_{\sigma_i}} \leq C|\sigma_i|^2 + o_s(1)$, we also have that $E(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)|_{A_{\sigma_i}} \leq C|\sigma_i|^2 + o_s(1)$.

Moreover,

$$
\begin{aligned}
\left|\int_{A_{\sigma_i}} \nabla(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)\nabla X^i d\omega\right| &\leq \left(\int_{A_{\sigma_i}} |\nabla(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)|^2 d\omega\right)^{\frac{1}{2}} \left(\int_{A_{\sigma_i}} |\nabla(X)|^2 d\omega\right)^{\frac{1}{2}} \\
&\leq C\sigma_i \left(\int_{A_{\sigma_i}} |\nabla(f_{\sigma_1}(\text{or } \ g_{\sigma_2}) - \mathcal{F}^i_{\sigma_i})|^2 d\omega\right)^{\frac{1}{2}} \\
&\leq C\sigma_i.
\end{aligned}
$$

Now we will estiamte $|a_i - \mathcal{F}^i(0)|$. By Hölder inequality,

$$
\begin{aligned}
|a_i - \mathcal{F}^i(0)|^2 &= \left|\int_{\sigma_i}^1 \partial_r(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i_{\sigma_i})dr\right|^2 \leq \left(\int_{\sigma_i}^1 \left|\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i_{\sigma_i})\right| dr\right)^2 \\
&\leq (1-\sigma_i)\int_{\sigma_i}^1 \left|\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i_{\sigma_i})\right|^2 dr \leq \frac{1-\sigma_i}{\sigma_i}\int_{A_{\sigma_i}} \left|\nabla(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i_{\sigma_i})\right|^2 d\omega \\
&\leq \frac{1-\sigma_i}{\sigma_i}(E(\widetilde{\mathcal{F}^i_{\sigma_i}} - \mathcal{F}^i) + E(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)) \\
(5.13)\qquad &\leq C\sigma_i.
\end{aligned}
$$

From the above we have that

$$
\begin{aligned}
\left|\int_{A_{\sigma_i}} \langle \nabla\mathcal{F}^i_{\sigma_i}, \nabla X^i\rangle d\omega\right| &\leq \left|\int_{A_{\sigma_i}} \langle \nabla\mathcal{F}^i, \nabla X^i\rangle d\omega\right| + C\sigma_i \\
&= \left|\int_{\partial B_{\sigma_i}} \langle \nabla\mathcal{F}^i\vec{n}, X\rangle d_0\right| + C\sigma_i \\
&= \|\nabla\mathcal{F}^i|_{\partial B_{\sigma_i}}\|\|(-a_i + \mathcal{F}_\rho|_{\partial B_{\sigma_i}})\|\sigma_i + C\sigma_i \\
&= C\sigma_i.
\end{aligned}
$$

Now we can compute that

$$
\begin{aligned}
E(f_{\sigma_1}) \text{ or } E(g_{\sigma_2}) &= \int_{A_{\sigma_i}} \langle \nabla \mathcal{F}^i_{\sigma_i}, + \nabla X, \nabla \mathcal{F}^i_{\sigma_i}, + \nabla X \rangle d\omega \\
&= E(\mathcal{F}^i_{\sigma_i}) + \langle \nabla \mathcal{F}^i_{\sigma_i}, \nabla X \rangle d\omega + E(X) \\
&\geq E(\mathcal{F}^i) - C\sigma_i, \qquad (5.14)
\end{aligned}
$$

C depends on $E(\mathcal{F}^i)$.

(III) Here we will estimate the second term of (5.11).

From (5.13), with $o_\rho(1) \to 0$ as $\rho \to 0$,

$$
|a_1 - a_2| \geq \big| |\mathcal{F}^1(0) - \mathcal{F}^2(0)| - |a_1 - \mathcal{F}^1(0) + \mathcal{F}^2(0) - a_2| \big| \geq d - o_\rho(1) - o_s(1).
$$

Now letting $\mathcal{H}^a_b(f, g) :=$ the harmonic map on $B_a \backslash B_b$ in $\mathbb{R}^k$ with boundary f on ∂B_a and g on ∂B_b, it clearly holds that

$$
E(\mathcal{H}^a_b(f, g) = E(\mathcal{H}^1_{\frac{b}{a}}(f(\cdot a), g(\cdot a))).
$$

Writing $\sigma_1 =: \sigma$, $\frac{\delta}{\sigma^1} =: \tau$, $\mathcal{F}_\rho|_{\partial B_{\sigma_1}} =: p$, $\mathcal{F}_\rho|_{\partial B_\delta} =: q$,

$$
\begin{aligned}
&\left| \int \langle \nabla \mathcal{H}^\sigma_\delta(a_1, a_2), \nabla \mathcal{H}^\sigma_\delta(-a_1 + p, -a_2 + q) \rangle d\omega \right| \\
&= \left| \int \langle \nabla \mathcal{H}^1_\tau(a_1, a_2), \nabla \mathcal{H}^1_\tau(-a_1 + p(\cdot\sigma), -a_2 + q(\cdot\sigma) \rangle d\omega \right| \\
&= \left| \int \langle \nabla \mathcal{H}^1_\tau(0, -a_1 + a_2), \nabla \mathcal{H}^1_\tau(-a_1 + p(\cdot\sigma), -a_2 + q(\cdot\sigma) \rangle d\omega \right| \\
&\leq \left| \int_{\partial B} \partial_r \mathcal{H}^1_\tau(0, -a_1 + a_2)[-a_1 + p(\cdot\sigma)] d_0 \right| + \left| \int_{\partial B_\tau} \partial_r \mathcal{H}^1_\tau(0, -a_1 + a_2)[-a_2 + q(\cdot\sigma)] d_0 \right| \\
&\leq \frac{2\pi}{|\ln \tau|} | - a_1 + a_2| [| - a_1 + p(\cdot\sigma)| + | - a_2 + q(\cdot\sigma)|] \\
&\leq C \frac{(o_\rho(1) + o_s(1))}{|\ln \rho|}.
\end{aligned}
$$

And

$$
\begin{aligned}
E(\mathcal{H}^\sigma_\delta(a_1, a_2)) &\geq E(\mathcal{H}^1_\rho(a_1, a_2)) = E(\mathcal{H}^1_\rho(0, -a_1 + a_2)) \\
&= E((-a_1 + a_2) \frac{\ln r}{\ln \rho}) \\
&\geq \frac{\pi d^2}{|\ln \rho|} - C \frac{(o_\rho(1) + o_s(1))}{|\ln \rho|}.
\end{aligned}
$$

So,

$$
\begin{aligned}
E(\mathcal{F}_\rho|_{B_\sigma\setminus B_\delta}) \geq E(\mathcal{H}^\sigma_\delta(p,q)) & \\
&= E(\mathcal{H}^\sigma_\delta(a_1,a_2) + H^\sigma_\delta(-a_1+p,-a_2+q)) \\
&= E(\mathcal{H}^\sigma_\delta(a_1,a_2)) + E(\mathcal{H}^\sigma_\delta(-a_1+p,-a_2+q)) \\
&\qquad + \int \langle \nabla\mathcal{H}^\sigma_\delta(a_1,a_2), \nabla\mathcal{H}^\sigma_\delta(-a_1+p,-a_2+q)\rangle d\omega \\
&\geq \frac{\pi d^2}{|\ln\rho|} - C\frac{o_\rho(1)+o_s(1)}{|\ln\rho|},
\end{aligned} \tag{5.15}
$$

where C only depends on $E(\mathcal{F}^i)$.

From (5.11), (5.14), (5.15) and the choice of σ_i,

$$
\begin{aligned}
\mathcal{E}(x^1,x^2,\rho) &\geq \mathcal{E}(x^1,x^2,0) - C\sigma_i + \frac{\pi d^2}{|\ln\rho|} - C\frac{(o_{\rho(1)+o_s(1)})}{|\ln\rho|} \\
&\geq \mathcal{E}(x^1,x^2,0) - C(\sqrt{\rho} + \sqrt{\sqrt{\rho}}) + \frac{\pi d^2}{|\ln\rho|} - C\frac{(o_\rho(1)+o_s(1))}{|\ln\rho|} \\
&\geq \mathcal{E}(x^1,x^2,0) + C\frac{d^2}{|\ln\rho|},
\end{aligned}
$$

for $\rho \leq \rho_0$, for some small $\rho_0 \in (0,1)$ and small $s(x^i)$. □

Remark 5.2.1. *With the same notations as in Lemma 5.2.4, it holds that*

$$E(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i) \leq E(\mathcal{F}^i_{\sigma_i}) - E(\mathcal{F}^i) + o_s(1).$$

Proof. Let $G^i := \mathcal{F}^i(x^i_0)$, $G^i_{\sigma_i} := \mathcal{F}_{\sigma_i}(x^i_0, a^i) = \min_{a\in N} E(\mathcal{F}_{\sigma_i}(x^i_0, a))$.
First, we observe that

$$\int_B \langle \nabla\mathcal{F}^i, \nabla(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)\rangle d\omega = \int_B \langle II \circ \mathcal{F}^i(d\mathcal{F}^i, d\mathcal{F}^i), \mathcal{F}^i_{\sigma_i} - \mathcal{F}^i\rangle,$$

and since $G^i \in H^{2,2}$,

$$0 = \int_B \langle \nabla G^i, \nabla(G^i_{\sigma_i} - G^i)\rangle d\omega = \int_B \langle II \circ G^i(dG^i, dG^i), G^i_{\sigma_i} - G^i\rangle.$$

Note that $\|\mathcal{F}^i_{\sigma_i} - G^i_{\sigma_i}\|_0 \to 0$ as $\|y^i - x^i\|_{\frac{1}{2},2;0} =: s \to 0$, because:
letting $G^i_\delta|_{B_\delta\setminus B_\sigma} \equiv a_i$, $\sigma < \delta$, we know that $E(G^i_{\sigma_i})$ is uniformly bounded for $\sigma_i \leq \sigma^0_i$.
Thus, by the Courant-Lebesgue Lemma and the argument in Lemma 4.1.1, $\{G^i_{\sigma_i}\}_{\sigma_i \leq \sigma^0_i}$

has the same modulus of continuity and we use the Arzela-Ascoli Theorem.

Moreover, $\|\mathcal{F}^i(y^i) - \mathcal{F}^i\|_{1,2;0} \to 0$, so by the Hölder inequality,

$$\left| \int_B \langle II \circ \mathcal{F}^i(d\mathcal{F}^i, d\mathcal{F}^i), \mathcal{F}^i_{\sigma_i} - \mathcal{F}^i \rangle d\omega - \int_B \langle II \circ G^i(dG^i, dG^i), G^i_{\sigma_i} - G^i \rangle d\omega \right| \leq o_s(1).$$

Hence

$$\begin{aligned} & 2E(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i) \\ & \quad = \int_B \langle \nabla \mathcal{F}^i_{\sigma_i}, \nabla(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)\rangle d\omega - \int_B \langle \nabla \mathcal{F}^i, \nabla(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)\rangle d\omega \\ & \quad \leq \int_B \langle \nabla \mathcal{F}^i_{\sigma_i}, \nabla(\mathcal{F}^i_{\sigma_i} - \mathcal{F}^i)\rangle d\omega + o_s(1) \\ & \quad \leq \int_B |\nabla \mathcal{F}^i_{\sigma_i}|^2 d\omega - \int_B \langle \nabla \mathcal{F}^i, \nabla \mathcal{F}^i_{\sigma_i} - \nabla \mathcal{F}^i \rangle d\omega - \int_B |\nabla \mathcal{F}^i|^2 d\omega + o_s(1) \\ & \quad \leq \int_B |\nabla \mathcal{F}^i_{\sigma_i}|^2 d\omega - \int_B |\nabla \mathcal{F}^i|^2 d\omega + o_s(1) \end{aligned}$$

□

From this Lemma we will need to consider the following type of condition: for solutions $X^i(x^i)$ of $\mathcal{P}(\Gamma_i), i = 1, 2,$

$$\operatorname{dist}(X^1(x^1), X^2(x^2)) > 0.$$

Now let for $\beta \in \mathbb{R}$,

$$\begin{aligned} M_\beta &= \{x \in \overline{\mathcal{M}^*} | \mathcal{E}(x) \leq \beta\}, \\ K_\beta &= \{x \in \overline{\mathcal{M}^*} | \mathcal{E}(x) = \beta,\ g(x) = 0\}. \end{aligned}$$

And we define two types of open subsets in $\overline{\mathcal{M}^*}$ as in [St3]:

$$\begin{aligned} N_{\beta,\delta} &= \{x \in \overline{\mathcal{M}^*} |\ |\mathcal{E}(x) - \beta| < \delta,\ g(x) < \delta\},\ \ \delta > 0, \\ U_{\beta,\rho} &= \{x \in \overline{\mathcal{M}^*} |\ |x - y| < \rho \text{ for some } y \in K_\beta\},\ \ \rho > 0, \end{aligned}$$

which satisfies the following property:

for a fixed $\beta \in \mathbb{R}$, each open neighborhood of N_β includes open subsets $N_{\beta,\delta}$, $U_{\beta,\rho}$ for some $\delta > 0, \rho > 0$, since $\overline{\mathcal{M}^*}$ satisfies the (P.S.) and from Lemma II.1.10 in [St1].

Finally, we also need the following Lemma from [St1] for our main Theorem.

Lemma 5.2.5. *Let $\beta \in \mathbb{R}, \bar{\varepsilon} > 0$ and N a neighborhood of K_β in $\overline{\mathcal{M}^*}$. Then there exists a $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\Phi : [0,1] \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$ such that*

$$\Phi(t,x) = x, \;\; \text{if } t = 0 \;\text{ or }\; |\mathcal{E}(x) - \beta| \geq \bar{\varepsilon},$$

and

$$\Phi(1, M_{\beta+\varepsilon} \backslash N) \subset M_{\beta-\varepsilon}, \quad \Phi(1, M_{\beta+\varepsilon}) \subset M_{\beta-\varepsilon} \cup N.$$

Proof. Choose $0 < \rho \leq 1, \;\; 0 < \delta < \delta' \leq 1$ such that

$$N \supset U_{\beta,\rho} \supset U_{\beta,\frac{\rho}{2}} \supset N_{\beta,\delta'} \supset N_{\beta,\delta}.$$

And let η be a Lipschitz function on $\overline{\mathcal{M}^*}$ with $0 \leq \eta \leq 1$, $\eta|_{N_{\beta,\delta}} \equiv 0$, $\eta|_{\overline{\mathcal{M}^*} \backslash N_{\beta,\delta'}} \equiv 1$. For $\varepsilon < \frac{1}{2}\min\{\delta, \bar{\varepsilon}\}$, choose also $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$ and

$$\varphi(s) = 1, \text{ if } |s - \beta| < \varepsilon, \quad \varphi(s) = 0, \text{ if } |s - \beta| \geq 2\varepsilon.$$

Now we define

$$\tilde{e}(x) = \varphi(\mathcal{E}(x))\eta(x)e_{\frac{\delta}{2}}(x), \;\; x \in \overline{\mathcal{M}^*},$$

satisfying the properties in Lemma 5.2.1. Thus, from Lemma 5.2.2, there exists continuous map $\Phi : [0,\infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$ with

$$\Phi(0,x) = x, \quad \frac{\partial}{\partial t}\Phi(t,x) = \tilde{e}(x), \;\; x \in \overline{\mathcal{M}^*}.$$

Clearly, from the definition of Φ(Lemma 5.2.2), $\Phi(t,x) = x$ for $|\mathcal{E}(x) - \beta| \geq \bar{\varepsilon} \geq 2\varepsilon$, since $\varphi(\mathcal{E}(\cdot)) = 0$, $\tilde{e}(\cdot) = 0$.
Furthermore

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}(\Phi(t,x)) &= \langle d\mathcal{E}(\Phi(t,x)), \tilde{e}(\Phi(t,x))\rangle \\
&= \varphi\left(\mathcal{E}\left(\Phi(t,x)\right)\right)\eta(\Phi(t,x))\langle d\mathcal{E}(\Phi(t,x)), e_{\frac{\delta}{2}}(\Phi(t,x))\rangle \\
&\leq \varphi\left(\mathcal{E}\left(\Phi(t,x)\right)\right)\eta(\Phi(t,x))\left(\frac{\delta}{2} - g\left(\Phi(t,x)\right)\right).
\end{aligned}$$

Hence from the definitions of φ, η, for $\Phi(t,x) \notin N_{\beta,\delta}$, if $|\mathcal{E}(\Phi(t,x)) - \beta| > \delta$, by $\delta > 2\varepsilon$, $\frac{\partial}{\partial t}\mathcal{E}(\Phi(t,x)) = 0$, and if $g((\Phi(t,x)) > \delta$, $\frac{\partial}{\partial t}\mathcal{E}(\Phi(t,x)) \leq -\frac{\delta}{2}$. For $\Phi(t,x) \in N_{\beta,\delta}$, clearly $\frac{\partial}{\partial t}\mathcal{E}(\Phi(t,x)) = 0$. Thus, $\mathcal{E}((\Phi(t,x))$ is non-increasing with respect to $t \in [0,\infty)$.
Therefore, for x with $\mathcal{E}(x) \leq \beta+\varepsilon$, we have either $\mathcal{E}(\Phi(1,x)) \leq \beta-\varepsilon$ or $|\mathcal{E}(\Phi(t,x))-\beta| \leq \varepsilon$ for all $t \in [0,1]$. And for the latter case,

$$\begin{aligned}
\mathcal{E}(\phi(1,x)) &= \mathcal{E}(x) + \int_0^1 \frac{d}{dt}\mathcal{E}(\Phi(t,x))dt \\
&= \mathcal{E}(x) + \int_0^1 \eta(\Phi(t,x))\left(\frac{\delta}{2} - g\left(\Phi(t,x)\right)\right) \\
&= \mathcal{E}(x) - \frac{\delta}{2}|\{t \in [0,1] | g(\Phi(t,x)) \geq \delta\}|,
\end{aligned}$$

since for $g(\Phi(t,x)) < \frac{\delta}{2}$, $\Phi(t,x) \in N_{\beta,\delta}$,

$$\leq \mathcal{E}(x) - \frac{\delta}{2}|\{t \in [0,1]|\Phi(t,x) \notin N_{\beta,\delta'}\}| .$$

Now suppose $x \notin N$ or $\Phi(1,x) \notin N$, then either $\Phi(t,x) \notin N_{\beta,\delta'}$ for all $t \in [0,1]$, or the flow $\{\Phi(t,x)|t \in [0,1]\}$ must traverse $U_{\beta,\rho} \backslash U_{\beta,\frac{\rho}{2}}$. For the latter, we have some $0 \leq t_1 \leq t_2 \leq 1$ with

$$\frac{\rho}{2} \leq |\Phi(t_2,x) - \Phi(t_1,x)| \leq \int_{t_1}^{t_2} \left| \frac{\partial}{\partial t}\Phi(t,x) \right| dt \leq A|t_2 - t_1|,$$

since $|e_{\frac{\delta}{2}(x)}| \leq A$, $x \in \overline{\mathcal{M}^*}$ for some $A > 0$. Hence $|\{t \in [0,1]|g(\Phi(t,x)) \geq \delta\}| \geq \frac{\rho}{2A}$. Choosing $\varepsilon \leq \frac{1}{2}\min\{\bar{\varepsilon}, \frac{\rho\delta}{4A}\}(\leq \frac{1}{2}\min\{\bar{\varepsilon},\delta\})$, we have

$$\mathcal{E}(\Phi(1,x)) \leq \beta + \varepsilon - \frac{\rho\delta}{4A} < \beta - \varepsilon.$$

From the above, this means the assertion in Lemma. □

For the main theorem we recall some definitions in section 3.1:

- For the condition (C1), let

$$\begin{aligned} \mathcal{S}(\Gamma_1,\Gamma_2) &= \{X \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p,r))|\, 0 < \rho < 1,\ \ X|_{C_i}\text{is weakly monotone}\}, \\ \mathcal{S}(\Gamma_i) &= \{X \in H^{1,2} \cap C^0(\overline{B}, B(p,r))|X|_{\partial B} \text{ is weakly monotone}\}. \end{aligned}$$

  Note that, if (N,h) is simply connected, complete Riemannian manifold with curvature ≤ 0, then $i(p) = \infty$ for all $p \in N$ from the 'Theorem of Hadamard'.

- For the condition (C2), taking an arbitrary homotopy family for annulus type (see (3.5)),

$$\begin{aligned} \mathcal{S}(\Gamma_1,\Gamma_2) &= \{X \in F_\rho \,|\, 0 < \rho < 1\}, \\ \mathcal{S}(\Gamma_i) &= \{X \in H^{1,2} \cap C^0(\overline{B}, N)|X|_{\partial B} \text{ is weakly monotone}\}. \end{aligned}$$

Now we can use the arguments in [St4] and [St1] and have the following existence results in manifolds. For a self contained paper we give a complete proof.

Theorem 5.2.1. *Let $\Gamma_1,\Gamma_2 \subset (N,h)$ satisfy the condition (C1) or (C2).*

(A) Letting

$$\begin{aligned} d &= \inf\{E(X)\,|X \in \mathcal{S}(\Gamma_1,\Gamma_2)\}, \\ d^* &= \inf\{E(X^1) + E(X^2)\,|X^i \in \mathcal{S}(\Gamma_i), i = 1,2\}, \end{aligned}$$

if $d < d^*$, there exists a minimal surface of annulus type bounded by Γ_1 and Γ_2.

(B) For $\mathcal{F}^1$, resp. $\mathcal{F}^2$, an absolute minimizer of E in $\mathcal{S}(\Gamma_1)$, resp. $\mathcal{S}(\Gamma_2)$, suppose that $dist(\mathcal{F}^1, \mathcal{F}^2) > 0$ and suppose furthermore there exists a strict relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$.
Then there exists a solution of $\mathcal{P}(\Gamma_1, \Gamma_2)$ which is not a relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$, i.e. an unstable annulus type minimal surface or there exists a pair of solutions to $\mathcal{P}(\Gamma_1)$, $\mathcal{P}(\Gamma_2)$ one of which does not yield an absolute minimizer of E (in $\mathcal{S}(\Gamma_1)$ or $\mathcal{S}(\Gamma_2)$).

Proof. *(A)* Clearly $d = \inf_{x \in \overline{\mathcal{M}^*}} \mathcal{E}(x)$, since $d < d^*$. Thus, for a minimizing sequence $\{x_m\} \subset \overline{\mathcal{M}^*}$ with $\mathcal{E}(x_m) \to d$ we have $g(x_m) \to 0$ as $m \to \infty$. Otherwise, $g(x_m) \geq 2\delta$ for small δ and big m, then as before in the proof of Lemma 5.2.3, using the flow $\Phi(1, x)$ related to e_δ,

$$\mathcal{E}(\Phi(1, x_m)) \leq -\delta + \mathcal{E}(x_m).$$

This is a contradiction.
Now the (P.S.) condition (Lemma 5.1.1) gives a subsequence of $\{x_m\}$ converging strongly in $\overline{\mathcal{M}^*}$ to a critical point of $\mathcal{E}$ denoted by $x \in \overline{\mathcal{M}^*}$ and by continuity, $\mathcal{E}(x) = d$, so $x \in \mathcal{M}^*$.
Hence from Proposition 4.2.1, $\mathcal{F}(x)$ is a solution of $\mathcal{P}(\Gamma_1, \Gamma_2)$.

(B) We can write that $\mathcal{F}^i := \mathcal{F}^i(x^i)$, for some $x^i \in M^{i*}$, $i = 1, 2$, moreover, for some $y \in \mathcal{M}^*$, $\mathcal{F}(y)$ is the strict relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$. Clearly, y is also a strict relative minimum of $\mathcal{E}$ in $\mathcal{M}^*$. Letting $x = (x^1, x^2, 0)$, consider

$$P = \{p \in C^0([0,1], \overline{\mathcal{M}}) | p(0) = x, p(1) = y\}.$$

By Lemma 5.2.3, 5.2.4 (ε is dependent on the energy) and the choice of x^i, also noting the uniform convergence of $\mathcal{E}(a, b, \rho)$ as $\rho \to 0$ on the sets of a, b with bounded value of $\mathcal{E}(a, b, 0)$ (see the proof of Lemma 4.1.1) and the proof of *(A)* , we have

$$\beta := \inf_{p \in P} \max_{t \in [0,1]} \mathcal{E}(p(t)) > \max\{\mathcal{E}(x), \mathcal{E}(y)\}.$$

And β is a critical value by Proposition 5.2.1. Supposing that any solution of $\mathcal{P}(\Gamma_i)$ is an absolute minimum of E in $\mathcal{S}(\Gamma_i)$ we have $K_\beta \subset \mathcal{M}^*$.

Now in order to prove the existence of a solution which is not a relative minimum we assume that all critical points of β are relative minimums of $\mathcal{E}$ in $\mathcal{M}^*$. Then there exists an open neighborhood N of K_β with

$$\overline{M_\beta} \cap N = \{x \in N | \, \mathcal{E}(x) = \beta\}.$$

Hence the two open sets, N and M_β are disjoint, so N und $M_{\beta-\varepsilon}$ are disconnected for all $\varepsilon > 0$, since $M_{\beta-\varepsilon} \subset M_\beta$.

With the above N and $\bar{\varepsilon} = \beta - \mathcal{E}(y) > 0$, choose ε and Φ as in Lemma 5.2.5. Then for some $p \in P$, $p \in M_{\beta+\varepsilon}$ and $p' = \Phi(1, p) \in P$, since $\Phi(1, x) = x, \Phi(1, y) = y$ with $p' \subset M_{\beta-\varepsilon} \cup N$.
(Note that the difference between $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^*}$ are only conformal mappings which do not influence the Dirichlet Integral.)
Since N, $M_{\beta-\varepsilon}$ are disconnected and $x, y \notin N$, we have $p' \in M_{\beta-\varepsilon}$, contradicting the definition of β.
Therefore there must be a critical point $\tilde{x} \in \mathcal{M}^*$ which is not a relative minimum of $\mathcal{E}$. And $\mathcal{F}(\tilde{x})$ is a solution of $\mathcal{P}(\Gamma_1, \Gamma_2)$ but not a relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$ from the E-minimality of harmonic extensions. □

Now we apply the main result to the case of the three-dimensional sphere S^3 and of the three-dimensional hyperbolic space H^3. We consider the case of condition (C1).
Let Γ_1, Γ_2 be Jordan curves which are diffeomorphic to S^1 and of class C^3 with $\text{dist}(\Gamma_1, \Gamma_2) > 0$, and let $\mathcal{F}^i$ denote an absolute minimizer of E in $\mathcal{S}(\Gamma_i)$, $i = 1, 2$.

Corollary 5.2.1. *Let $\Gamma_1, \Gamma_2 \subset B(p, \pi/2)$ for some $p \in S^3$, in other words Γ_1, Γ_2 are in a (three-dimensional) hemisphere.*
Suppose that dist$(\mathcal{F}^1, \mathcal{F}^2) > 0$ and there exists a strict relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$. Then there exists an unstable annulus type minimal surface or there exists a pair of solutions to $\mathcal{P}(\Gamma_1)$, $\mathcal{P}(\Gamma_2)$ one of which does not yield an absolute minimizer of E(in $\mathcal{S}(\Gamma_1)$ or $\mathcal{S}(\Gamma_2)$).

If there exists exactly one solution of $\mathcal{P}(\Gamma_i)$, $i = 1, 2$, the main theorem says, the existence of a minimal surface of annulus type whose energy is a strict relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$ ensures the existence of an unstable minimal surface of annulus type.

From [LJ], the solution of $\mathcal{P}(\Gamma_i)$ is unique in the 3-dimensional hyperbolic space H^3, if the total curvature of Γ_i is less than 4π. Therefore we have the following result for H^3.

Corollary 5.2.2. *Let Γ_1, Γ_2 possess total curvature $\leq 4\pi$ in H^3 and dist$(\mathcal{F}^1, \mathcal{F}^2) > 0$. If there exists a strict relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$, then there is an unstable minimal surface of annulus type in H^3.*

Appendix A

Regularity of critical points of $\mathcal{E}$

We will now show the regularity of critical points of $\mathcal{E}$ which is defined in Chapter 4. The idea is from [St1] and [Ho].

We will use the following Lemma from [Mo2].

Lemma A.1. *Let G be a bounded domain in $\mathbb{R}^2$. Suppose $\varphi \in H_0^{1,2}(G)$ and $\psi \in L^1(G)$ satisfies the Morrey growth condition*

$$\int_{B_r(z_0)} |\psi| d\omega \leq C_0 r^\mu, \text{ for all } B_r(z_0).$$

Then $\psi\varphi^2 \in L^1(G)$ and for all $B_r(z_0)$ there holds

$$\int_{B_r(z_0)\cap G} |\psi\varphi^2| d\omega \leq C_1 C_0 r^{\mu/2} \int_G |d\varphi|^2 d\omega$$

for some uniform constant C_1.

We also need the Poincaré inequality as follows (see [St1] Lemma 5.5):

Lemma A.2. *Let $z_0 \in \partial A_\rho$, $B_r := B_r(z_0)$, $G_r := A_\rho \cap (B_{3r} \backslash B_r)$, $K_r := A_\rho \cap (B_{2r} \backslash B_r)$ and $S_r := \partial A_\rho \cap B_{2r} \backslash B_r$. Then, for some small $r_0 > 0$, there exists a uniform constant C independent of z_0 such that for all $r \leq r_0$ and for each $\varphi \in H^{1,2}(G_r)$:*

$$\begin{aligned} \int_{G_r} |\varphi|^2 d\omega &\leq Cr^2 \int_{G_r} |d\varphi|^2 d\omega + C\left(\int_{S_r} \varphi \, d_o\right)^2, \text{ and} \\ \int_{S_r} |\varphi|^2 d_o &\leq Cr \int_{K_r} |d\varphi|^2 d\omega + \frac{C}{r}\left(\int_{S_r} \varphi \, d_o\right)^2, \end{aligned}$$

where d_o is the one-dimensional area element.

Proof. Let z_0, r fixed. Suppose by contradiction that for a sequence $\varphi_m \in H^{1,2}(G_r)$

$$1 \equiv \int_{G_r} |\varphi_m|^2 d\omega \geq mr^2 \int_{G_r} |d\varphi_m|^2 d\omega + m \left(\int_{S_r} \varphi_m \, d_o \right)^2 .$$

Then $\{\varphi_m\}$ in bounded in $H^{1,2}(G_r)$ and some subsequence, denoted again by $\{\varphi_m\}$, converges weakly to some φ in $H^{1,2}(G_r)$ but strongly in $L^2(G_r)$ by Rellich-Kondrakov. From the above assumption, $d\varphi_m \to 0$ strongly.
Thus, $\{\varphi_m\}$ converges strongly to some constant C in $H^{1,2}(G_r)$ and $\varphi_m \to C$ in $L^2(S_r)$. On the other hand, $\int_{S_r} \varphi_m \, d_o \to 0$, so $\varphi \equiv 0$ in G_r, contradicting the assumption, since $\varphi_m \to \varphi$ in L^2.

The second inequality can be proved similarly, supposing by contradiction that

$$1 \equiv \int_{S_r} |\varphi_m|^2 d_o \geq mr \int_{K_r} |d\varphi_m|^2 d\omega + \frac{m}{r} \left(\int_{S_r} \varphi_m \, d_o \right)^2$$

and applying the above result for $\int_{K_r} |\varphi_m|^2 d\omega$.

By scaling, one can see that C is independenf of z_0,r. □

Theorem A.1. *Let $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times (0,1)$ be a critical point of $\mathcal{E}$. Then $\mathcal{F}_\rho(x^1, x^2)$ is in the class of $H^{2,2}(A_\rho, N)$.*

Proof. **(I)** Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2) : A_\rho \to N \overset{\eta}{\hookrightarrow} \mathbb{R}^k$ and $\mathcal{F}_\rho \in H^{1,2}(A_\rho, \mathbb{R}^k)$. Then we have the following estimate by the young's inequality:
in polar coordinate with $\Delta\mathcal{F}_\rho$:=$\Delta_{\mathbb{R}^k}\mathcal{F}_\rho$,

$$\begin{aligned}
|\nabla^2 \mathcal{F}_\rho|^2 &= |\partial_r d\mathcal{F}_\rho|^2 + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&= \left| \Delta\mathcal{F}_\rho - \frac{1}{r^2}\partial_{\theta\theta}\mathcal{F}_\rho - \frac{1}{r}\partial_r \mathcal{F}_\rho \right|^2 + \frac{1}{r^2}\,|\partial_{\theta r}\mathcal{F}_\rho|^2 + \frac{1}{r^4}|\partial_\theta \mathcal{F}_\rho|^2 - 2\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho \partial_\theta \mathcal{F}_\rho + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&\leq C(\varepsilon)|\Delta\mathcal{F}_\rho|^2 + (1+\varepsilon)\left| \frac{1}{r^2}\partial_{\theta\theta}\mathcal{F}_\rho + \frac{1}{r}\partial_r \mathcal{F}_\rho \right|^2 + \frac{1}{r^2}|\partial_{\theta r}\mathcal{F}_\rho|^2 + \frac{1}{r^4}|\partial_\theta \mathcal{F}_\rho|^2 - 2\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho \partial_\theta \mathcal{F}_\rho \\
&\qquad -2\varepsilon\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho \partial_\theta \mathcal{F}_\rho + 2\varepsilon\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho \partial_\theta \mathcal{F}_\rho + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&\leq C(\varepsilon)|\Delta\mathcal{F}_\rho|^2 + (1+\varepsilon)\left| \frac{1}{r^2}\partial_{\theta\theta}\mathcal{F}_\rho + \frac{1}{r}\partial_r \mathcal{F}_\rho \right|^2 + \frac{1}{r^2}|\partial_{\theta r}\mathcal{F}_\rho|^2 + \frac{1}{r^4}|\partial_\theta \mathcal{F}_\rho|^2 - 2\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho \partial_\theta \mathcal{F}_\rho \\
&\qquad -2\varepsilon\frac{1}{r^3}\partial_{\theta r}\mathcal{F}_\rho \partial_\theta \mathcal{F}_\rho + \varepsilon\frac{1}{r^2}|\partial_{\theta r}\mathcal{F}_\rho|^2 + C(\varepsilon)\frac{1}{r^4}|\partial_\theta \mathcal{F}_\rho|^2 + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&\leq C(\varepsilon)|\Delta\mathcal{F}_\rho|^2 + (2+\varepsilon)\frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 + C(\varepsilon)\frac{1}{r^2}\frac{1}{r^2}|\partial_\theta \mathcal{F}_\rho|^2 \\
&\leq C(\varepsilon, \eta, A_\rho)|d\mathcal{F}_\rho|^2 + C(\varepsilon, \rho)|\partial_\theta d\mathcal{F}_\rho|^2,
\end{aligned}$$

since $\mathcal{F}_\rho$ is harmonic in $N \overset{\eta}{\hookrightarrow} \mathbb{R}^k$, i.e. $\tau_h(f) = 0$.

Therefore it suffices to show that

$$\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega \leq C < \infty,$$

where $\Delta_h d\mathcal{F}_\rho := \frac{d\mathcal{F}_\rho(r,\theta+h) - d\mathcal{F}_\rho(r,\theta)}{h}, h \neq 0$ and C is independent of h, then by a well known result in [GT] it holds that $\int_{A_\rho} |\partial_\theta d\mathcal{F}_\rho|^2 d\omega \leq C < \infty$.

(II) Since $\mathcal{F}_\rho$ is harmonic , for $X \in H_0^{1,2}(A_\rho, \mathbb{R}^k)$, we have

$$-\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), X\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX\rangle d\omega = 0.$$

This means, for $X \in H^{1,2}(A_\rho, \mathbb{R}^k)$ the above expression only depends on the boundary of X. Thus, for $\phi = (\phi^1, \phi^2) \in H^{\frac{1}{2},2} \times H^{\frac{1}{2},2}(\frac{\cdot}{\rho})$ we define

$$\text{(A.1)} \qquad \mathbf{A}(\mathcal{F}_\rho)(\phi) := -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), X\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX\rangle d\omega,$$

where X is any mapping in $H^{1,2}(A_\rho, \mathbb{R}^k)$ with $X|_{\partial A_\rho} = \phi$.

Specially for $\phi^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^i)^* T\Gamma_i), i = 1, 2$, we take $X := \mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2)$ which is tangent to N along $\mathcal{F}_\rho$, then from the definition of the second fundamental form $\langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \mathbf{J}_\rho(\phi^1, \phi^2)\rangle \equiv 0$, so

$$\begin{aligned} \text{(A.2)} \qquad \mathbf{A}(\mathcal{F}_\rho)(\phi) &= \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2)\rangle d\omega \\ &= \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, 0)\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(0, \phi^2)\rangle d\omega \\ &= \langle \partial_{x^1}\mathcal{E}, \phi^1\rangle + \langle \partial_{x^2}\mathcal{E}, \phi^2\rangle. \end{aligned}$$

Hence for a critical point $x = (x^1, x^2, \rho)$ of $\mathcal{E}$, $\mathbf{A}(\mathcal{F}_\rho)(\xi) \geq 0$, for all $\xi = (\xi^1, \xi^2) \in \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$.

(III) Recall the construction in section 3.1, namely that for $x^i \in M^i$ there exist $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $w^i(\theta + 2\pi) = w^i(\theta) + 2\pi, \theta \in \mathbb{R}$ such that

$$x^i = \gamma^i \circ e^{iw^j}, \text{ and } D(H(w^i)) < \infty,$$

where the map $H(\cdot)$ is the harmonic extension map from $B(0)$ into $\mathbb{R}$. Note that γ^i is a diffeomorphism.

With $\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial B} = \Delta_{-h}\Delta_h\gamma^1 \circ e^{iw^1}$ and $\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial B_\rho}(\cdot\rho) = \Delta_{-h}\Delta_h\gamma^2 \circ e^{iw^2(\cdot)}$,

$$\begin{aligned}\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega &= -\int_{A_\rho} \langle d\mathcal{F}_\rho, d\Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega \\ &= -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial A_\rho}).\end{aligned}$$

Denoting $\gamma^1 \circ e^{iw^1}$ and $\gamma^2 \circ e^{iw^2}$ by $\gamma^i(w^i(\theta))$ and $w^i(\cdot + h)$ resp. $w^i(\cdot - h)$ by w^i_+ resp. w^i_-, we have:

$$\begin{aligned}\Delta_{-h}\Delta_h\gamma^i(w^i) &= \Delta_{-h}\left[\frac{\gamma^i(w^i_+) - \gamma^i(w^i_-)}{h}\right] \\ &= \Delta_{-h}\left[d\gamma^i(w^i)\left(\frac{w^i_+ - w^i_-}{h}\right) + \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\right] \\ &= d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) - \frac{1}{h}\int_{w^i}^{w^i_-} d^2\gamma^i(s')ds' \cdot \Delta_h w^i_- + \Delta_{-h}\left(\frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\right) \\ &= d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) + P^i.\end{aligned}$$

Since γ^i is smooth, clearly $d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^i)^*T\Gamma_i)$.

Writing $w^i = \tilde{w}^i + Id$ for some $\tilde{w}^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R})$ and define a real valued map of $(r,\theta) \in [\rho, 1] \times \mathbb{R}$ as follows: for $i = 1$

$$T^1(w^1)(r,\theta) := H_\rho(\tilde{w}, 0)(r,\theta) + Id(r,\theta) \;\text{ with }\; Id(r,\theta) = \theta,$$

where $H_\rho(\tilde{w}, 0)$ is the harmonic extension to $A_\rho \approx [\rho, 1] \times \mathbb{R}/2\pi$ with $\tilde{w}$ on ∂B and 0 on ∂B_ρ. Then it holds that

$$T^1(w^1)(r,\theta + 2\pi) = T^1(w^1)(r,\theta) + 2\pi, \;\text{ for }\; (r,\theta) \in [\rho, 1] \times \mathbb{R},$$

and $e^{iT^1(w^1)}$ can be considered as a map from ∂B into itself.

Now define a map $S(P^1, 0)(\cdot) : A_\rho \to \mathbb{R}^k$ with the boundary P^1 (resp. 0) on C_1 (resp. C_ρ) as follows:

$$\begin{aligned}S(P^1,0)(\cdot) \;&:=\; -\frac{1}{h}\int_{T^1(w^1)(\cdot)}^{T^1(w^1_-)(\cdot)} d^2\gamma^1(s')ds' \cdot H_\rho(\Delta_h w^1_-, 0)(\cdot) \\ &\quad + \Delta_{-h}\left(\frac{1}{h}\int_{T^1(w^1)(\cdot)}^{T^1(w^1_+)(\cdot)}\int_{T^1(w^1)(\cdot)}^{s'} d^2\gamma^1(s'')ds''ds'\right).\end{aligned}$$

Similarly, a map $S(0,P^2)(\cdot): A_\rho \to \mathbb{R}^k$ with the boundary 0 (resp. P^2) on C_1 (resp. C_ρ):

$$S(0,P^2)(\cdot) \;:=\; -\frac{1}{h}\int_{T^2(w^2)(\cdot)}^{T^2(w^2_-)(\cdot)} d^2\gamma^2(s')ds' \cdot H_\rho(0,\Delta_h w^2_-)(\cdot)$$
$$+\Delta_{-h}\left(\frac{1}{h}\int_{T^2(w^2)(\cdot)}^{T^2(w^2_+)(\cdot)}\int_{T^2(w^2)(\cdot)}^{s'} d^2\gamma^2(s'')ds''ds'\right),$$

where $T^2(w^2_-)(\cdot) = H_\rho(0,\tilde{w})(\cdot) + Id(\cdot)$, and $S(0,P^2)|_{C_1} \equiv 0, S(0,P^2)|_{C_\rho}(\cdot\rho) = P^2(\cdot)$.

Clearly $S(P^1,0), S(0,P^2) \in H^{1,2}(A_\rho,\mathbb{R}^k)$, so letting

$$S(P^1,P^2) := S(P^1,0) + S(0,P^2),$$

we have a map in $H^{1,2}(A_\rho,\mathbb{R}^k)$ with boundary (P^1,P^2).

By computation, $\frac{h^2}{2}\Delta_{-h}\Delta_h w^i = \frac{1}{2}(w^i_- + w^i_+) - w^i$. And $\frac{1}{2}(w^i_- + w^i_+) \in W^i_{\mathbb{R}^k}$ which is convex. Thus, by the definition of $\mathcal{T}_{x^i}$,

$$\frac{h^2}{2}d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \in \mathcal{T}_{x^i}.$$

And $\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i)$ is in $H^{\frac{1}{2},2}$ for which $A(\mathcal{F}_\rho)$ is well defined.

From (A.1) and since x is a critical point of $\mathcal{E}$,

$$\frac{h^2}{2}A(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1),0\right) = A(\mathcal{F}_\rho)\left(\frac{h^2}{2}d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1),0\right) \geq 0,$$

so $A(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1),0\right) \geq 0$.

Similarly, for the second variation,

$$A(\mathcal{F}_\rho)\left(0, d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)(\frac{\cdot}{\rho})\right) \geq 0.$$

From now on we will omit the scaling term $(\frac{\cdot}{\rho})$ for the second variation.

Moreover, from the definition of $A(\mathcal{F}_\rho)$, clearly

$$\mathbf{A}(\mathcal{F}_\rho)(\phi^1+\xi^1,\phi^2+\xi^2) = \mathbf{A}(\mathcal{F}_\rho)(\phi^1,\phi^2) + \mathbf{A}(\mathcal{F}_\rho)(\xi^1,\xi^2),$$

if there exist $H^{1,2}$ extension of (ϕ^1, ϕ^2) and (ξ^1, ξ^2).

Hence we have that

$$\begin{aligned} &\mathbf{A}(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^i), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\right) \\ \text{(A.3)} \quad &= \mathbf{A}(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), 0\right) + \mathbf{A}(\mathcal{F}_\rho)\left(0, d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\right) \geq 0. \end{aligned}$$

Now we can compute:

$$\begin{aligned} \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega &= -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\Delta_{-h}\Delta_h \mathcal{F}_\rho|_{\partial A_\rho}) \\ &= -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho\rangle d\omega \\ &\qquad -\mathbf{A}(\mathcal{F}_\rho)(P^1, P^2) - \mathbf{A}(\mathcal{F}_\rho)\left(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\right) \\ &\leq -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(P^1, P^2) \\ &= -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho\rangle d\omega \\ &\qquad + \int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), S(P^1, P^2)\rangle d\omega - \int_{A_\rho} \langle d\mathcal{F}_\rho, dS(P^1, P^2)\rangle d\omega \\ \text{(A.4)} \quad &= -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h \mathcal{F}_\rho\rangle d\omega \\ \text{(A.5)} \quad &\quad + \int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), S(P^1, 0)\rangle d\omega + \int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), S(0, P^2)\rangle d\omega \\ \text{(A.6)} \quad &\quad - \int_{A_\rho} \langle d\mathcal{F}_\rho, dS(P^1, 0)\rangle d\omega - \int_{A_\rho} \langle d\mathcal{F}_\rho, dS(0, P^2)\rangle d\omega. \end{aligned}$$

(IV) For the estimstes of the above terms we need some preparation.

Recall that

$$\begin{aligned} II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho) &= \langle \nabla_{\frac{\partial}{\partial x^i}} d\mathcal{F}_\rho - \tilde{\nabla}_{\frac{\partial}{\partial x^i}} d\mathcal{F}_\rho, dx^i \rangle \\ &= g^{ij}(\mathcal{F}^a_{\rho,i}\mathcal{F}^b_{\rho,j}\Gamma^c_{ab} \circ \mathcal{F}_\rho - \mathcal{F}^a_{\rho,i}\mathcal{F}^b_{\rho,j}\tilde{\Gamma}^c_{ab} \circ \mathcal{F}_\rho) \otimes \frac{\partial}{\partial y^c} \circ \mathcal{F}_\rho, \end{aligned}$$

where ∇(resp. $\tilde{\nabla}$) is the covariant derivative along $\mathcal{F}_\rho$ in $\mathbb{R}^k$(resp. N) and note that $\Gamma^c_{ab}, \tilde{\Gamma}^c_{ab}$ are smooth.

Thus,

$$
\begin{aligned}
&|\Delta_h II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho)| = |\frac{1}{h}\{II\circ\mathcal{F}_{\rho,+}(\mathcal{F}_{\rho,+},\mathcal{F}_{\rho,+}) - II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho)\}|\\
&= |\frac{1}{h}\{II\circ\mathcal{F}_{\rho,+}(d\mathcal{F}_{\rho,+},d\mathcal{F}_{\rho,+}) - II\circ\mathcal{F}_\rho(d\mathcal{F}_{\rho,+},d\mathcal{F}_{\rho,+})\\
&\qquad + II\circ\mathcal{F}_\rho(d\mathcal{F}_{\rho,+},d\mathcal{F}_{\rho,+}) - II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho)\}|\\
&= |\frac{1}{h}\{(dII(\mathcal{F}_\rho)\cdot(\mathcal{F}_{\rho,+}-\mathcal{F}_\rho) + \int_0^1\int_0^t d^2II(s(\tau))|\mathcal{F}_{\rho,+}-\mathcal{F}_\rho|^2 d\tau dt)(d\mathcal{F}_{\rho,+},d\mathcal{F}_{\rho,+})\\
&\qquad + II\circ\mathcal{F}_\rho(d\mathcal{F}_{\rho,+}-d\mathcal{F}_\rho,d\mathcal{F}_{\rho,+}) + II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_{\rho,+}-d\mathcal{F}_\rho)\}|\\
&= |dII(\mathcal{F}_\rho)\cdot\Delta_h\mathcal{F}_\rho(d\mathcal{F}_{\rho,+},d\mathcal{F}_{\rho,+}) + \frac{1}{h}\int_0^1\int_0^t d^2II(s(\tau))|\mathcal{F}_{\rho,+}-\mathcal{F}_\rho|^2 d\tau dt(d\mathcal{F}_{\rho,+},d\mathcal{F}_{\rho,+})\\
&\qquad + II\circ\mathcal{F}_\rho(\Delta_h d\mathcal{F}_\rho,d\mathcal{F}_{\rho,+}) + II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,\Delta_h d\mathcal{F}_\rho)|\\
&\leq C[|\Delta_h\mathcal{F}_\rho||d\mathcal{F}_{\rho,+}|^2 + |\Delta_h d\mathcal{F}_\rho|(|d\mathcal{F}_{\rho,+}|+|d\mathcal{F}_\rho|)],
\end{aligned}
$$

where $s(\tau) := \tau\mathcal{F}_{\rho,+} + (1-\tau)\mathcal{F}_\rho$, $0\leq\tau\leq 1$ and $C = C(\|\mathcal{F}_\rho\|_{C^0(A_\rho)})$.

Now letting

$$
-\frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_-)} d^2\gamma^1(s')ds' := \star \quad\text{and}\quad \frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_-)}\int_{T^1(w^1)}^{s'} d^2\gamma^1(s'')ds''ds' := \star\star,
$$

we have

$$
|\star| \leq C(\gamma^1)|H_\rho(\Delta_{-h}w^1,0)|,\quad |\star\star| \leq C(\gamma^1)|H_\rho(\Delta_h w^1,0)|,
$$

and

$$
\begin{aligned}
|d\star| &= \Big|-\frac{1}{h}\big[d^2\gamma^1(T^1(w^1_-))dT^1(w^1_-) - d^2\gamma^1(T^1(w^1))dT^1(w^1)\big]\Big|\\
&= \Big|-\frac{1}{h}\big[\frac{d^2\gamma^1(T^1(w^1_-)) - d^2\gamma^1(T^1(w^1))}{T^1(w^1_-)-T^1(w^1)}\big(T^1(w^1_-)-T^1(w^1)\big)dT^1(w^1_-)\\
&\qquad + d^2\gamma^1(T^1(w^1))\big(dT^1(w^1_-)-dT^1(w^1)\big)\big]\Big|\\
&\leq C(\|\gamma^1\|_{C^3})\big(|H_\rho(\Delta_{-h}w^1,0)||dH_\rho(w^1_-,0)| + |dH_\rho(\Delta_{-h}w^1,0)|\big),
\end{aligned}
$$

$$
\begin{aligned}
|d\star\star| &= \Big|d\big[\frac{1}{h}\big(\int_{T^1(w^1)}^{T^1(w^1_+)} d\gamma^1(s')ds' - \int_{T^1(w^1)}^{T^1(w^1_+)} d\gamma^1(T^1(w^1))ds'\big)\big]\Big|\\
&= \Big|\frac{1}{h}\big[\frac{d\gamma^1(T^1(w^1_+)) - d\gamma^1(T^1(w^1))}{T^1(w^1_+)-T^1(w^1)}\big(T^1(w^1_+)-T^1(w^1)\big)dT^1(w^1_+)\\
&\qquad - d^2\gamma^1(T^1(w^1))dT^1(w^1)\big(dT^1(w^1_+)-dT^1(w^1)\big)\big]\Big|\\
&\leq C(\|\gamma^1\|_{C^2})|H_\rho(\Delta_h w^1,0)|\big(|dH_\rho(\tilde{w^1}_+,0)| + |dH_\rho(\tilde{w^1},0)|\big).
\end{aligned}
$$

Using the above results, we now estimate (A.4), (A.5),(A.6) for some $C \in \mathbb{R}$ which is independent of h:

First,

$$\begin{aligned}
(A.4) &\leq \int_{A_\rho} \big|\langle \Delta_h II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_h \mathcal{F}_\rho\rangle\big| d\omega \\
&\leq C \int_{A_\rho} \big(|\Delta_h \mathcal{F}_\rho|^2 |d\mathcal{F}_{\rho,+}|^2 + |\Delta_h d\mathcal{F}_\rho|(|d\mathcal{F}_{\rho,+}| + |d\mathcal{F}_\rho|)|\Delta_h \mathcal{F}_\rho|\big) d\omega \\
&\leq C \int_{A_\rho} |d\mathcal{F}_{\rho,+}|^2 |\Delta_h \mathcal{F}_\rho|^2 d\omega \\
&\quad + \varepsilon \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + C(\varepsilon) \int_{A_\rho} (|d\mathcal{F}_{\rho,+}|^2 + |d\mathcal{F}_\rho|^2)|\Delta_h \mathcal{F}_\rho|^2 d\omega.
\end{aligned}$$

For the estimate of (A.5),

$$\begin{aligned}
&\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), S(P^1, 0)\rangle d\omega \\
&\leq \int_{A_\rho} \{|\langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), (\star) H_\rho(\Delta_h w^1_-, 0)\rangle| + |\langle \Delta_h II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), (\star\star)\rangle|\} d\omega \\
&\leq C \int_{A_\rho} |d\mathcal{F}_\rho|^2 |H_\rho(\Delta_{-h} w^1, 0)|^2 d\omega \\
&\quad + C \int_{A_\rho} \{|\Delta_h \mathcal{F}_\rho||d\mathcal{F}_{\rho,+}|^2 |H_\rho(\Delta_h w^1, 0)| + |\Delta_h d\mathcal{F}_\rho|(|d\mathcal{F}_{\rho,+}| + |d\mathcal{F}_\rho|)|H_\rho(\Delta_h w^1, 0)|\} d\omega \\
&\leq C \int_{A_\rho} |d\mathcal{F}_\rho|^2 |H_\rho(\Delta_{-h} w^1, 0)|^2 d\omega + C \int_{A_\rho} |d\mathcal{F}_{\rho,+}|^2 (|\Delta_h \mathcal{F}_\rho|^2 + |H_\rho(\Delta_h w^1, 0)|^2) d\omega \\
&\quad + \varepsilon \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + C(\varepsilon) \int_{A_\rho} (|d\mathcal{F}_{\rho,+}|^2 |H_\rho(\Delta_h w^1, 0)|^2 + |d\mathcal{F}_\rho|^2 |H_\rho(\Delta_h w^1, 0)|^2) d\omega,
\end{aligned}$$

note that $\Delta_h w^1_- = \Delta_{-h} w^1$, and we obtain a similar estimate for the second term of (A.5).
Thus, we have that

$$\begin{aligned}
(A.5) &\leq \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega \\
&\quad + C(\varepsilon) \int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho,+}|^2\big)\big(|\Delta_h \mathcal{F}_\rho|^2 + |H_\rho(\Delta_{-h} w^1, 0)|^2 + |H_\rho(0, \Delta_{-h} w^2)|^2 \\
&\qquad + H_\rho(\Delta_h w^1, 0)|^2 + |H_\rho(0, \Delta_h w^2)|^2\big) d\omega.
\end{aligned}$$

For the estimate of (A.6),

$$
\begin{aligned}
-\int_{A_\rho} \langle d\mathfrak{F}_\rho, dS(P^1,0)\rangle d\omega \le& \int_{A_\rho} \big|\langle d\mathfrak{F}_\rho, d(\star)H_\rho(\Delta_{-h}w^1,0)\rangle\big| d\omega \\
&+ \int_{A_\rho} \big|\langle d\mathfrak{F}_\rho, (\star)dH_\rho(\Delta_{-h}w^1,0)\rangle\big| d\omega + \int_{A_\rho} \big|\langle \Delta_h d\mathfrak{F}_\rho, d(\star\star)\rangle\big| d\omega \\
\le\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(\Delta_h w^1,0)|^2 d\omega \\
&+C(\varepsilon)\int_{A_\rho} \big(|d\mathfrak{F}_\rho|^2 + |dH_\rho(\tilde{w}^1_-,0)|^2 + |dH_\rho(\tilde{w}^1_+,0)|^2 + |dH_\rho(\tilde{w}^1,0)|^2\big)\cdot \\
&\qquad \big(|H_\rho(\Delta_{-h}w^1,0)|^2 + |H_\rho(\Delta_h w^1,0)|^2\big) d\omega.
\end{aligned}
$$

We get a similar estimate for the second term of (A.6):

$$
\begin{aligned}
(A.6) \ \le\ & \varepsilon C \int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(\Delta_h w^1,0)|^2 d\omega \\
&+C(\varepsilon)\int_{A_\rho} \big(|d\mathfrak{F}_\rho|^2 + |dH_\rho(\tilde{w}^1_-,0)|^2 + |dH_\rho(\tilde{w}^1_+,0)|^2 + |dH_\rho(\tilde{w}^1,0)|^2 \\
&\qquad +|dH_\rho(0,\tilde{w}^2_-)|^2 + |dH_\rho(0,\tilde{w}^2_+)|^2 + |dH_\rho(0,\tilde{w}^2)|^2\big)\cdot \\
&\big(|H_\rho(\Delta_{-h}w^1,0)|^2 + |H_\rho(\Delta_h w^1,0)|^2 + |H_\rho(0,\Delta_{-h}w^2)|^2 + |H_\rho(0,\Delta_h w^2)|^2\big) d\omega.
\end{aligned}
$$

Now gathering all the above results:

$$
\begin{aligned}
&\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega \\
&\quad \le\ \varepsilon C \int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(\Delta_h w^1,0)|^2 d\omega \\
&\qquad +C(\varepsilon)\int_{A_\rho} \big(|d\mathfrak{F}_\rho|^2 + |d\mathfrak{F}_\rho|^2 + |dH_\rho(\tilde{w}^1_-,0)|^2 + |dH_\rho(\tilde{w}^1_+,0)|^2 + |dH_\rho(\tilde{w}^1,0)|^2 \\
&\qquad\qquad +|dH_\rho(0,\tilde{w}^2_-)|^2 + |dH_\rho(0,\tilde{w}^2_+)|^2 + |dH_\rho(0,\tilde{w}^2)|^2\big)\cdot \\
&\qquad\quad \big(|\Delta_h \mathfrak{F}_\rho|^2 + |H_\rho(\Delta_{-h}w^1,0)|^2 + |H_\rho(\Delta_h w^1,0)|^2 \\
&\qquad\qquad +|H_\rho(0,\Delta_{-h}w^2)|^2 + |H_\rho(0,\Delta_h w^2)|^2\big) d\omega \\
&\quad =\ \varepsilon C \int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(\Delta_h w^1,0)|^2 d\omega + C(\varepsilon)\Xi\,.
\end{aligned} \tag{A.7}
$$

(V) On ∂B, it holds that

$$
\Delta_h(\gamma^i \circ w^i) = d\gamma^i(w^i)\Delta_h w^i + \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds',
$$

so

$$\Delta_h w^i = |d\gamma^i(w^i)|^{-2}\big[d\gamma^i(w^i)\cdot\Delta_h\mathfrak{F}_\rho - d\gamma^i(w^i)\cdot\frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\big]. \tag{A.8}$$

Using $T^i(w^i)$ at the right side of (A.8), we get a $H^{1,2}(A_\rho,\mathbb{R}^k)$- extension with boundary $\Delta_h w^i$ on C^1 and 0 on C_ρ, and by the D-minimality of the harmonic extension between the maps with the same boundary, we have

$$\begin{aligned}
&\int_{A_\rho}|dH_\rho(\Delta_h w^1,0)|^2d\omega\\
&\quad\le C\int_{A_\rho}\big[|dH_\rho(w^1,0)|\big(|\Delta_h\mathfrak{F}_\rho|+|\star\star|\big)+|d\Delta_h\mathfrak{F}_\rho|+|d\star\star|\big]^2d\omega\\
&\quad\le C\int_{A_\rho}\big\{|dH_\rho(w^1,0)|^2|\Delta_h\mathfrak{F}_\rho|^2+|dH_\rho(\Delta_h w^1,0)|^2|H_\rho(\Delta_h w^1,0)|^2+|d\Delta_h\mathfrak{F}_\rho|^2\\
&\qquad\quad+|H_\rho(\Delta_h w^1,0)|^2(|dH_\rho(\tilde{w^1}_+,0)|+|dH_\rho(\tilde{w^1},0)|)^2\\
&\qquad\quad+|dH_\rho(\tilde{w^1},0)|^2|\Delta_h w^1,0)|+|dH_\rho(\tilde{w^1},0)||\Delta_h\mathfrak{F}_\rho||d\Delta_h\mathfrak{F}_\rho|\\
&\qquad\quad+|dH_\rho(\tilde{w^1},0)||H_\rho(\Delta_h w^1,0)|(|dH_\rho(\tilde{w^1}_+,0)|+|dH_\rho(\tilde{w^1},0)|)|\Delta_h\mathfrak{F}_\rho|\\
&\qquad\quad+|dH_\rho(\tilde{w^1},0)||H_\rho(\Delta_h w^1,0)||d\Delta_h\mathfrak{F}_\rho|\\
&\qquad\quad+|dH_\rho(\tilde{w^1},0)||H_\rho(\Delta_h w^1,0)||H_\rho(\Delta_h w^1,0)|(|dH_\rho(\tilde{w^1}_+,0)|+|dH_\rho(\tilde{w^1},0)|)\\
&\qquad\quad+|d\Delta_h\mathfrak{F}_\rho||H_\rho(\Delta_h w^1,0)|(|dH_\rho(\tilde{w^1}_+,0)|+|dH_\rho(\tilde{w^1},0)|)\big\}d\omega\\
\text{(A.9)}&\quad\le C\int_{A_\rho}|d\Delta_h\mathfrak{F}_\rho|^2d\omega+C\Xi
\end{aligned}$$

by the young's inequality, and Ξ is from (A.7). Similarly, we get an estimate

$$\int_{A_\rho}|dH_\rho(0,\Delta_h w^2)|^2d\omega\le C\int_{A_\rho}|d\Delta_h\mathfrak{F}_\rho|^2d\omega+C\Xi. \tag{A.10}$$

Using the estimate (A.7) for $\int_{A_\rho}|d\Delta_h\mathfrak{F}_\rho|^2d\omega$ and from (A.9), (A.10),

$$\begin{aligned}
&\int_{A_\rho}|d\Delta_h\mathfrak{F}_\rho|^2d\omega+\int_{A_\rho}|dH_\rho(\Delta_h w^1,0)|^2d\omega+\int_{A_\rho}|dH_\rho(0,\Delta_h w^2)|^2d\omega\\
&\qquad\le\ \varepsilon C\int_{A_\rho}|d\Delta_h\mathfrak{F}_\rho|^2d\omega+\varepsilon C\int_{A_\rho}|dH_\rho(\Delta_h w^1,0)|^2d\omega+\varepsilon C\int_{A_\rho}|dH_\rho(0,\Delta_h w^2)|^2d\omega\\
&\qquad\quad+C(\varepsilon)\Xi\,.
\end{aligned}$$

Since

$$\frac{1}{2}(a^2+b^2)\le(a+b)^2\le\frac{3}{2}(a^2+b^2),\ a,b\in\mathbb{R}\ \text{ and }\ H_\rho(f,g)=H_\rho(f,0)+H_\rho(0,g) \tag{A.11}$$

for $f, g \in H^{\frac{1}{2},2}(\partial B, \mathbb{R})$, for some small $\varepsilon > 0$ in the above estimate we get finally the following inequality:

$$\begin{aligned}
&\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |dH_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\
\text{(A.12)} \quad &\leq \; C(\varepsilon) \int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho+}|^2 + |d\mathcal{F}_{\rho-}|^2 \\
&\qquad + |dH_\rho(\tilde{w^1}, \tilde{w^2}|)^2 + |dH_\rho(\tilde{w^1_+}, \tilde{w^2_+})|^2 + |dH_\rho(\tilde{w^1_-}, \tilde{w^2_-}|^2\big) \cdot \\
&\qquad \big(|\Delta_h \mathcal{F}_\rho|^2 + |H(\Delta_{-h} w^1, \Delta_{-h} w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2 \big) d\omega.
\end{aligned}$$

(VI) In several steps we will show that for each $P_0 \in \partial A_\rho$ there exist $C_0, \mu, r_0 > 0$ such that for all $r \in [0, r_0]$ it holds

$$\text{(A.13)} \quad \int_{A_\rho \cap B_r(P_0)} (|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2) d\omega \leq C_0 r^\mu \int_{A_\rho} (|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2) d\omega.$$

VI-a) Let $P_0 \in C_1$ fixed, $B_r := B_r(P_0)$, and

$$\text{(A.14)} \qquad \tilde{w^1_0} := Q^{-1} \int_{(B_{2r} \setminus B_r) \cap \partial B} \tilde{w^1} d_o, \;\; w^1_0 := \tilde{w^1_0} + Id : \mathbb{R} \to \mathbb{R},$$

where $\int_{\partial B \cap (B_{2r} \setminus B_r)} d_o := Q$,

$$\tilde{\xi}_\phi := -\big[\phi(|e^{i\theta} - P_0|)\big]^2 (\tilde{w^1} - \tilde{w^1_0}) \frac{\partial}{\partial\theta} \circ \bar{w}^1 \in H^{\frac{1}{2},2} \cap C^0(\partial B, \bar{w}^{1*} T(\partial B)),$$

where $\bar{w}^1$ means the map from ∂B into itself, and $\phi \in C^\infty$ is a non-increasing function of $|z|$ satisfying the conditions $0 \leq \phi(z) \leq 1$, $\phi \equiv 1$ if $|z| \leq 2r$, $\phi \equiv 0$ if $|z| \geq 3r$, $|d\phi| \leq \frac{C}{r}$, $|d^2\phi| \leq \frac{C}{r^2}$ for some C, fixed r.

Since $(1 - \phi^2) w^1 + \phi^2 w^1_0 \in W^1_{\mathbb{R}^k}$, $d\gamma^1(\tilde{\xi}_\phi) \in \mathcal{T}_{x^1}$, hence

$$\text{(A.15)} \qquad \mathbf{A}(\mathcal{F}_\rho)(-d\gamma^1(\tilde{\xi}_\phi), 0) \geq 0.$$

Letting $x^1_0 := \gamma^1(w^1_0)$

$$\begin{aligned}
x^1 - x^1_0 &= d\gamma^1(w^1 - w^1_0) - \int_{w^1_0}^{w^1} \int_{s'}^{w^1} d^2\gamma^1(s'') ds'' ds' \\
&= d\gamma^1(w^1 - w^1_0) - \alpha(w^1),
\end{aligned}$$

and for small $r > 0$,

$$\begin{aligned}
\mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}^0_\rho)|_{C_1}, 0) &= \mathbf{A}(\mathcal{F}_\rho)(\phi^2 d\gamma^1(w^1 - w^1_0), 0) - \mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0) \\
&\leq -\mathbf{A}(\mathcal{F}_\rho)(\phi^2 \alpha(w^1), 0),
\end{aligned}$$

where $\mathcal{F}_\rho^0(A_\rho) \equiv x_0^1 \in \Gamma_1$.

On the other hand, for small $r > 0$, $\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_\rho} \equiv 0$, so we can take $\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)$ in the definition of $\mathbf{A}(\mathcal{F}_\rho)$. Hence

$$\begin{aligned}
&\mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_1}, 0) \\
&= \int_{A_\rho} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega + \int_{A_\rho} \langle 2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho\rangle d\omega \\
&\qquad - \int_{A_\rho} \langle \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\rangle d\omega \\
&\le -\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0),
\end{aligned}$$

and

$$\begin{aligned}
\int_{A_\rho} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega &\le \int_{A_\rho} \langle \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\rangle d\omega \\
&\quad - \int_{A_\rho} \langle 2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0). \tag{A.16}
\end{aligned}$$

For the estimate of $-\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0)$, consider

$$\widetilde{\star\star} := \phi^2 \int_{w_0^1}^{T^1(w^1)} \int_{s'}^{T^1(w^1)} d^2\gamma^1(s'')ds''ds' \in H^{1,2}(A_\rho, \mathbb{R}^k)$$

with $\widetilde{\star\star}|_{C_1} = \phi^2\alpha(w^1), \widetilde{\star\star}|_{C_\rho} \equiv 0$, where $w_0^1(r,\theta) = \tilde{w_0^1} + Id(r,\theta) = \tilde{w_0^1} + \theta, \ (r,\theta) \in [\rho, 1] \times \mathbb{R}$.
By simple computation we get

$$\begin{aligned}
|\widetilde{\star\star}| &\le C(\gamma^1, x^1)\phi^2|H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2, \\
|d\widetilde{\star\star}| &\le C(\gamma^1, x^1)|H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2\phi|d\phi| + C(\gamma^1, x^1)|dH_\rho(\tilde{w^1}, 0)||H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2\phi^2,
\end{aligned}$$

and from (A.16) by the young's inequality

$$\begin{aligned}
\int_{A_\rho} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega &\le \int_{A_\rho} |d\mathcal{F}_\rho|^2|\mathcal{F}_\rho - \mathcal{F}_\rho^0|\phi^2 d\omega \\
&\quad + \frac{\varepsilon}{5}\int_{A_\rho} |d\mathcal{F}_\rho|^2\phi^2 d\omega + C(\varepsilon)\int_{A_\rho} |\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2|d\phi|^2 d\omega \\
&\quad + C\|H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}\|_{L^\infty(B_{3r})} \int_{A_\rho} \big(|d\mathcal{F}_\rho|^2\phi^2 + |H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2|d\phi|^2\big) d\omega \\
&\quad + C\|H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}\|_{L^\infty(B_{3r})} \int_{A_\rho} \big(|dH_\rho(\tilde{w^1}, 0)|^2 + |d\mathcal{F}_\rho|^2\big)\phi^2 d\omega \\
&\quad + C\int_{A_\rho} |H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2|d\mathcal{F}_\rho|^2\phi^2 d\omega.
\end{aligned}$$

Thus, for $r \in (0, r_0)$, sufficiently small, dependent on ε, C, modulus of continuity of $\mathfrak{F}_\rho - \mathfrak{F}_\rho^0$ and $H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}$ we have the following estimate:

$$\int_{A_\rho} \langle \phi^2 d\mathfrak{F}_\rho, d\mathfrak{F}_\rho \rangle d\omega \leq \varepsilon \int_{A_\rho} \big(|d\mathfrak{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2 \big) \phi^2 d\omega$$
$$+C(\varepsilon) \int_{A_\rho} \big(|\mathfrak{F}_\rho - \mathfrak{F}_\rho^0|^2 + |H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}|^2 \big) |d\phi|^2 d\omega. \tag{A.17}$$

VI-b) We will estimate $\int_{A_\rho} |dH_\rho(\tilde{w^1}, 0)|^2 \phi^2 d\omega$.

• First,

$$D\big[(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})\phi\big] = \int_{A_\rho} \big[|dH_\rho(\tilde{w^1}, 0)|^2 \phi^2 + |(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})|^2 |d\phi|^2$$
$$+2dH_\rho(\tilde{w^1}, 0)(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})\phi d\phi \big] d\omega,$$

and by the Young's inequality

$$\int_{A_\rho} |dH_\rho(\tilde{w^1}, 0)|^2 \phi^2 d\omega \leq D\big[(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})\phi\big]$$
$$+\frac{\varepsilon}{4} \int_{A_\rho} |dH_\rho(\tilde{w^1}, 0)|^2 \phi^2 d\omega + C(\varepsilon) \int_{A_\rho} \big(|H_\rho(\tilde{w^1}, 0)|^2 + |\tilde{w_0^1}|^2 \big) |d\phi|^2 d\omega. \tag{A.18}$$

• The estimate of $D\big[(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})\phi\big]$:

On C^1, $\mathfrak{F}_\rho - \mathfrak{F}_\rho^0 = d\gamma^1(w^1 - w_0^1) - \int_{w_0^1}^{w^1} \int_{s'}^{w^1} d^2\gamma^1(s'') ds'' ds'$, and $\phi|_{\partial B_{3r}(P_0)} \equiv 0$, so on $\partial(A_\rho \cap B_{3r}(P_0))$,

$$(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})\phi = |d\gamma^1(T^1(w^1))|^{-2} \big[d\gamma^1(T^1(w^1)) \cdot (\mathfrak{F}_\rho - \mathfrak{F}_\rho^0)$$
$$+d\gamma^1(T^1(w^1)) \cdot \int_{w_0^1}^{T^1(w^1)} \int_{s'}^{T^1(w^1)} d^2\gamma^1(s'') ds' \big] \phi.$$

We denote the latter map on A_ρ by Ψ.

And it holds,

$$\Delta \big[(H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1})\phi \big] = 2dH_\rho(\tilde{w^1}, 0) \cdot d\phi + (H_\rho(\tilde{w^1}, 0) - \tilde{w_0^1}) \Delta\phi =: f. \tag{A.19}$$

Note that: for a solution $\varphi \in C^2(\Omega, \mathbb{R})$ of $\Delta\varphi = f$, it holds, with a boundary data φ_0

$$D\varphi \leq D\psi - \int f(\varphi - \psi), \text{ for all } \psi \in \varphi_0 + H_0^{1,2}(\Omega).$$

Hence, by the variation characterization of the equation (A.19), we get

$$D\big[(H_\rho(\tilde{w^1},0)-\tilde{w^1_0})\phi\big] \le D(\Psi) - \int_{A_\rho\cap B_{3r}} f\big[(H_\rho(\tilde{w^1},0)-\tilde{w^1_0})\phi - \Psi\big]d\omega. \tag{A.20}$$

Letting

$$\begin{aligned}\Psi &:= \frac{d\gamma^1(T^1(w^1))\cdot(\mathcal{F}_\rho-\mathcal{F}^0_\rho) + d\gamma^1(T^1(w^1))\cdot\int_{w^1_0}^{T^1(w^1)}\int_{s'}^{T^1(w^1)} d^2\gamma^1(s'')ds'}{|d\gamma^1(T^1(w^1))|^2}\phi \\ &= \frac{\Theta}{|d\gamma^1(T^1(w^1))|^2}\phi,\end{aligned}$$

$$d[d\gamma^1(T^1(w^1))\cdot(\mathcal{F}_\rho-\mathcal{F}^0_\rho)] = d^2\gamma^1(T^1(w^1))d(T^1(w^1))(\mathcal{F}_\rho-\mathcal{F}^0_\rho) + d\gamma^1(T^1(w^1))d\mathcal{F}_\rho =: a,$$

$$d\Big(\int_{w^1_0}^{T^1(w^1)}\int_{s'}^{T^1(w^1)} d^2\gamma^1(s'')ds'd\Big) = d^2\gamma^1(T^1(w^1))dH_\rho(\tilde{w^1},0)(H_\rho(\tilde{w^1},0)-\tilde{w^1_0}) =: b,$$

$$d|d\gamma^1(T^1(w^1))|^{-2} = -2|d\gamma^1(T^1(w^1))|^{-4}\langle d^2\gamma^1(T^1(w^1)), d^1\gamma^1(T^1(w^1))\rangle dH_\rho(\tilde{w^1},0) =: c,$$

we have

$$|d\Psi|^2 = \frac{|a+b|^2\phi^2 + \Theta^2\phi^2c^2 + \Theta^2|d\phi|^2 + (a+b)c\phi^2\Theta + (a+b)\phi\Theta d\phi + \Theta^2\phi c d\phi}{|d\gamma^1(T^1(w^1))|^2},$$

and we compute further from the property of ϕ

$$\begin{aligned}\int_{A_\rho}|d\Psi|^2 d\omega \le C\int_{A_\rho}|d\mathcal{F}_\rho|^2\phi^2 d\omega + C\int_{A_\rho}\big[|\mathcal{F}_\rho-\mathcal{F}^0_\rho|^2 + |H_\rho(\tilde{w^1},0)-\tilde{w^1_0}|^2\big]|d\phi|^2 d\omega \\ + C\delta\int_{A_\rho}\big[|H_\rho(\tilde{w^1},0)-\tilde{w^1_0}|^2|d\phi|^2 + |dH_\rho(\tilde{w^1},0)|^2\phi^2\big]d\omega,\end{aligned}$$

where $\delta = \big\||\mathcal{F}_\rho-\mathcal{F}^0_\rho| + |H_\rho(\tilde{w^1},0)-\tilde{w^1_0}|\big\|_{L^\infty(A_\rho\cap B_{3r})}$.

We can also compute that

$$
\begin{aligned}
&-\int_{A_\rho\cap B_{3r}} f\big[(H_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi-\Psi\big]d\omega \\
&\leq \int_{A_\rho\cap B_{3r}} \big[2|dH_\rho(\tilde{w^1},0)d\phi|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}||d\phi|+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|^2|\Delta\phi|\phi \\
&\quad +C|dH_\rho(\tilde{w^1},0)|\phi|\mathfrak{F}_\rho-\mathfrak{F}_\rho^0||d\phi|+C|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}||\mathfrak{F}_\rho-\mathfrak{F}_\rho^0||\Delta\phi|\phi \\
&\quad +C\|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}\|(|dH_\rho(\tilde{w^1},0)|\phi|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}||d\phi|+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|^2|\Delta\phi|\phi)\big]d\omega \\
&\leq \int_{A_\rho\cap B_{3r}} \big[C(|\mathfrak{F}_\rho-\mathfrak{F}_\rho^0|^2+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|)^2(|d\phi|^2+|\Delta\phi|) \\
&\qquad\qquad +(\frac{\varepsilon}{2}+C\|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}\|_{L^\infty(A_\rho\cap B_{3r})})|dH_\rho(\tilde{w^1},0)|^2\phi^2\big]d\omega.
\end{aligned}
$$

Now the estimate of $D\big[(H_\rho(\tilde{w^1},0)-\tilde{w_0^1})\phi\big]$ follows from (A.20).

• From (A.18) and the above estimates, we get

$$
\begin{aligned}
&\int_{A_\rho}|dH_\rho(\tilde{w^1},0)|^2\phi^2d\omega \leq C\int_{A_\rho}|d\mathfrak{F}_\rho|^2\phi^2d\omega \\
&\qquad +C(\varepsilon)\int_{A_\rho}(|\mathfrak{F}_\rho-\mathfrak{F}_\rho^0|^2+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|^2)(|d\phi|^2+|\Delta\phi|)d\omega \\
&\qquad +(\frac{3\varepsilon}{4}+C\big\||\mathfrak{F}_\rho-\mathfrak{F}_\rho^0|+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|\big\|_{L^\infty(A_\rho\cap B_{3r})})\int_{A_\rho}|dH_\rho(\tilde{w^1},0)|^2\phi^2d\omega.
\end{aligned}
\tag{A.21}
$$

VI-c) From (A.17), (A.21), for $r\leq r_0$, where r_0 is dependent on ε, $C(x^1,\rho)$ and the modulus of continuity of $\mathfrak{F}_\rho-\mathfrak{F}_\rho^0$ and $H_\rho(\tilde{w^1},0)-\tilde{w_0^1}$, we get from the definition of ϕ:

$$
\begin{aligned}
&\int_{A_\rho\cap B_{3r}}(|d\mathfrak{F}_\rho|^2+|dH_\rho(\tilde{w^1},0)|^2)d\omega \leq Cr^{-2}\int_{A_\rho\cap B_{3r}\setminus B_{2r}}(|\mathfrak{F}_\rho-\mathfrak{F}_\rho^0|^2+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|^2)d\omega \\
&\qquad\qquad \leq Cr^{-2}\int_{A_\rho\cap B_{3r}\setminus B_r}(|\mathfrak{F}_\rho-\mathfrak{F}_\rho^0|^2+|H_\rho(\tilde{w^1},0)-\tilde{w_0^1}|^2)d\omega \\
&\text{(Poincaré inequality)}\quad \leq C\int_{A_\rho\cap B_{3r}\setminus B_r}\big(|d\mathfrak{F}_\rho|^2+|dH_\rho(\tilde{w^1},0)|^2\big)d\omega \\
&\qquad +Cr^{-2}\Big(\int_{\partial B\cap B_{2r}\setminus B_r}(\mathfrak{F}_\rho-\mathfrak{F}_\rho^0)d_o\Big)^2+Cr^{-2}\Big(\int_{\partial B\cap B_{2r}\setminus B_r}\big(H_\rho(\tilde{w^1},0)-\tilde{w_0^1}\big)d_o\Big)^2,
\end{aligned}
$$

where the last term is 0 from the definition of $\tilde{w_0^1}$.

On ∂B, we have

$$\mathcal{F}_\rho - \mathcal{F}_\rho^0 = d\gamma^1(w_0^1)(\tilde{w^1} - \tilde{w^1}_0) + \int_{\partial B\cap B_{2r}\setminus B_r}\int_{w_0^1}^{w^1}\int_{w_0^1}^{s'} d^2\gamma^1(s'')ds''ds',$$

so, from the estimate in integration and by the second inequality in Lemma A.2,

$$\begin{aligned}
&\int_{\partial B\cap B_{2r}\setminus B_r}(\mathcal{F}_\rho - \mathcal{F}_\rho^0)d_o \\
&= \int_{\partial B\cap B_{2r}\setminus B_r} d\gamma^1(w_0^1)(\tilde{w^1} - \tilde{w^1}_0)d_o + \int_{\partial B\cap B_{2r}\setminus B_r}\int_{w_0^1}^{w^1}\int_{w_0^1}^{s'} d^2\gamma^1(s'')ds''ds' \\
&\le C\int_{\partial B\cap(B_{2r}\setminus B_r)}|w^1 - w_0^1|^2 d_o \\
&\le Cr\int_{B\cap(B_{2r}\setminus B_r)}|dH_\rho(\tilde{w^1}, o)|^2 d\omega + \frac{C}{r}\Big(\int_{\partial B\cap B_{2r}\setminus B_r}(\tilde{w^1} - \tilde{w^1}_0)d_o\Big)^2.
\end{aligned}$$

Here the last term is again zero fromt he definition of $\tilde{w_0^1}$.

Thus,

$$\begin{aligned}
&Cr^{-2}\Big(\int_{\partial B\cap B_{2r}\setminus B_r}(\mathcal{F}_\rho - \mathcal{F}_\rho^0)d_o\Big)^2 \\
&\quad\le C\Big(\int_{B\cap(B_{2r}\setminus B_r)}|dH_\rho(\tilde{w^1}, 0)|^2 d\omega\Big)^2 \le C(x^1,\rho)\int_{B\cap(B_{2r}\setminus B_r)}|dH_\rho(\tilde{w^1}, 0)|^2 d\omega,
\end{aligned}$$

hence

$$\int_{A_\rho\cap B_r}\big(|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2\big)d\omega \le C\int_{A_\rho\cap B_{3r}\setminus B_r}\big(|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2\big)d\omega.$$

Letting $\Upsilon(r) := \int_{A_\rho\cap B_r(P_0)}\big(|d\mathcal{F}_\rho|^2 + |H_\rho(\tilde{w^1}, 0)|^2\big)d\omega$, the above inequality means that

$$\Upsilon(r) \le C\big(\Upsilon(3r) - \Upsilon(r)\big),$$

where C is independent of $r \le r_0$, for some small r_0.

Then the inequality (A.13) follows from the Iteration-lemma. And because of (A.11) there exists $r_0 > 0$ such that

$$\int_{A_\rho\cap B_r(P_0)}(|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, \tilde{w^2})|^2)d\omega \le C_0 r^\mu\int_{A_\rho}(|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, , \tilde{w^2})|^2)d\omega,$$

for some $C_0, \mu > 0$, independent of $r \leq r_0$ and $P_0 \in C_1$.

Similarly, we get the same result for $|d\mathfrak{F}_{\rho+}|^2$ (resp. $|d\mathfrak{F}_{\rho-}|^2$) and $|dH_\rho(\tilde{w_+^1}, \tilde{w_+^2})|^2$ (resp. $|dH_\rho(\tilde{w_-^1}, \tilde{w_-^2})|^2$) with $P_0 \in \partial A_\rho = C_1 \cup C_\rho$.
Thus, it holds that

$$\begin{aligned}
&\int_{A_\rho \cap B_r(P_0)} \big(|d\mathfrak{F}_\rho|^2 + |d\mathfrak{F}_{\rho+}|^2 + |d\mathfrak{F}_{\rho-}|^2 \\
&\qquad\qquad + |dH_\rho(\tilde{w^1}, \tilde{w^2})|^2 + |dH_\rho(\tilde{w_+^1}, \tilde{w_+^2})|^2 + |dH_\rho(\tilde{w_-^1}, \tilde{w_-^2})|^2\big) d\omega \\
&\quad \leq C_0 r^\mu \int_{A_\rho} \big(|d\mathfrak{F}_\rho|^2 + |d\mathfrak{F}_{\rho+}|^2 + |d\mathfrak{F}_{\rho-}|^2 \\
&\qquad\qquad + |dH_\rho(\tilde{w^1}, \tilde{w^2})|^2 + |dH_\rho(\tilde{w_+^1}, \tilde{w_+^2})|^2 + |dH_\rho(\tilde{w_-^1}, \tilde{w_-^2})|^2\big) d\omega
\end{aligned} \tag{A.22}$$

for some $C_0, \mu > 0$, independent of $r \leq r_0$ and $P_0 \in \partial A_\rho$.

(VII) Now extend $\mathfrak{F}_\rho$ to $\mathbb{R}^2 \backslash B_{\rho^2}$ by conformal reflection as follows

$$\begin{aligned}
\mathfrak{F}_\rho(z) &= \mathfrak{F}_\rho\big(\frac{z}{|z|^2}\big), \text{ if } 1 \leq |z| \\
\mathfrak{F}_\rho(z) &= \mathfrak{F}_\rho\big(\frac{z}{|z|^2}\rho^2\big), \text{ if } \rho^2 \leq |z| \leq \rho.
\end{aligned}$$

Choose $r \in \big(0, \min\{\frac{\rho-\rho^2}{2}, r_0\}\big)$, and $\varphi \in C_0^\infty\big(B_{2r}(0)\big)$ with $\varphi \equiv 1$ on $B_r(0)$.

We may cover A_ρ with balls of radius r in such a way that at most k balls of the covering intersect at any point $p \in A_\rho$, for any r as above ($\mathbb{R}^2$ is metrizable). Let B^i denote the balls of the covering with centers p_i and $\varphi_i(p) := \varphi(p - p_i)$.
Then from (A.12),

$$\begin{aligned}
&C^{-1}\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega + \int_{A_\rho} |dH_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\
&\quad \leq \Sigma_i \int_{\mathbb{R}^2 \backslash A_{\rho^2}} \big(|\Delta_h \mathfrak{F}_\rho|^2 + |H(\Delta_{-h} w^1, \Delta_{-h} w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big)\varphi_i^2 \cdot \\
&\quad \underbrace{\big(|d\mathfrak{F}_\rho|^2 + |d\mathfrak{F}_{\rho+}|^2 + |d\mathfrak{F}_{\rho-}|^2 + |dH_\rho(\tilde{w^1}, \tilde{w^2})|^2 + |dH_\rho(\tilde{w_+^1}, \tilde{w_+^2})|^2 + |dH_\rho(\tilde{w_-^1}, \tilde{w_-^2})|^2\big)}_{=:\chi} d\omega.
\end{aligned}$$

By (A.22), χ satisfies the Morrey growth condition, so apply the Morrey Lemma with

χ and $(\Delta_h \mathfrak{F}_\rho)\varphi_i$ resp.$H(\Delta_{-h}w^1, \Delta_{-h}w^2)\varphi_i$ resp. $H(\Delta_h w^1, \Delta_h w^2)\varphi_i$. Then we obtain

$$\begin{aligned}
&\int_{B_{2r}(p_i)} \chi\big(|\Delta_h \mathfrak{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big)\varphi_i^2 d\omega \\
&\quad \leq C r^{\frac{\mu}{2}} \int_{B_2 \setminus B_{\rho^2}} \chi d\omega \int_{B_{2r}(P_i)} \big(|d\Delta_h \mathfrak{F}_\rho|^2 + |dH(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |dH(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega \\
&\quad + C r^{\frac{\mu}{2}} \int_{B_2 \setminus B_{\rho^2}} \chi d\omega \int_{B_{2r}(P_i)} \big(|\Delta_h \mathfrak{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega.
\end{aligned}$$

Summing over i we get a constant C, independent of r such that

$$\begin{aligned}
&\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega + \int_{A_\rho} |dH_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\
&\quad \leq C r^{\frac{\mu}{2}} \int_{B_2 \setminus B_{\rho^2}} \big(|d\Delta_h \mathfrak{F}_\rho|^2 + |dH(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |dH(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega \\
&\quad + C r^{\frac{\mu}{2}} \int_{B_2 \setminus B_{\rho^2}} \big(|\Delta_h \mathfrak{F}_\rho|^2 + |H(\Delta_{-h}w^1, \Delta_{-h}w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega.
\end{aligned}$$

Since $d\mathfrak{F}_\rho$, $dH(w^1, w^2) \in L^2$, choosing small $r > 0$, we obtain $C \in \mathbb{R}$, independent of $|h| \leq h_0$ with

$$\int_{A_\rho} |\Delta_h d\mathfrak{F}_\rho|^2 d\omega \leq C.$$

□

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Abstract

Unstable minimal surfaces are the unstable stationary points of the Dirichlet-Integral. In order to obtain unstable solutions, the method of the gradient flow together with the minimax-principle is generally used. The application of this method for minimal surfaces in the Euclidean spacce was presented by M. Struwe in 1984. We extend this theory for obtaining unstable minimal surfaces in Riemannian manifolds. In particular, we handle minimal surfaces of annulus type, i.e. we prescribe two Jordan curves of class C^3 in a Riemannian manifold and prove the existence of unstable minimal surfaces of annulus type bounded by these given curves. We consider two types of conditions for the target manifolds (and the curves), in which the existence and the uniqueness of the harmonic extension for a given boundary parametrization are well known. As corollaries we apply our main theorem, for instance, to the case of the three-dimensional sphere S^3 resp. the three-dimensional hyperbolic space H^3 with constant curvature 1 resp. -1.

Kurzzusammenfassung

Instabile Minimalflächen sind die instabilen stationären Punkten des Dirichletschen Integrals. Um instabile Lösungen zu erhalten, wird im Allgemeinen die Gradientenfluß-Methode zusammen mit dem Minimax-Prinzip verwendet. Eine Anwendung dieser Methode auf instabile Minmalflächen im Euklidischen Raum wurde im Jahr 1984 von M. Struwe präsentiert. In der vorliegenden Arbeit wird diese Theorie auf den Fall instabiler Minimalflächen in Riemannschen Mannigfaltigkeiten verallgemeinert. Insbesondere werden instabile Minimalflächen vom Typ des Kreisrings untersucht. Es werden zwei Jordansche Kurven von der Klasse C^3 auf einer Riemannschen Mannigfaltigkeit vorgegeben, und wir beweisen die Existenz einer instabilen Minimal fläche, die von den beiden gegebenen Kurven berandet wird. Wir betrachten zwei Typen von Bedingungen an die Zielmannigfaltigkeit (und Kurven), unter denen die Existenz und die Eindeutigkeit der harmonischen Fortsetzung für eine gegebene Parametrisierung der Randkurven bekannt sind. Als Korollare, wenden wir den Hauptsatz, z.B. auf den Fall der drei dimensionalen Kugel S^3 bzw. des drei dimensionalen hyperbolischen Raums H^3 an, welche konstante Krümmung 1 bzw. -1 besitzen.