On the Smoothing Property of the Bayes
Approach to Picture Reconstruction

Frank Natterer

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Fachbereich 10 der Universität
des Saarlandes - Angewandte
Mathematik und Informatik
6600 Saarbrücken
BRD

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Abstract. The paper is concerned with the reconstruction of a picture density f from a finite number of its strip integrals. It is shown that the Bayes estimate for f is in the Sobolev space $H^{3/2}$ if the isotropic exponential model for f is used. As picture densities are far from being in $H^{3/2}$ (the isotropic exponential model e.g. implies, roughly speaking, only $f \in H^{1/2}$), the Bayes approach can be expected to smooth out peaks and high contrasts. This drawback of the Bayes approach was in fact observed in a comparative numerical test carried out by Herman and Lent [6].

## § 1 The Bayes Approach to Picture Reconstruction

The picture reconstruction problem we are dealing with arises e.g. in computerized tomography and nondestructive testing. It requires the computation of a picture density f from a finite number of its line integrals. See [1] for a discussion of this problem and [2] for applications.

We always assume the picture to be of finite extent, i.e. f=0 outside the unit disk $\Omega$. In order to avoid certain purely mathematical difficulties we consider strip integrals along strips $L_k$, $k=1,\dots,n$, rather than line integrals. Putting

$$R_k f = \int_{L_k} f\,dx\ , \qquad R = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}$$

we thus want to solve the underdetermined system Rf=g where the vector g is made up of the projection data.

Numerous suggestions have been made for the solution of Rf=g, see [3] for a survey. In the present paper we will study a statistical approach to its solution which has been suggested in [4] and [5].

In order to describe this approach we start out from to families f,g of random variables which are jointly normally distributed with mean values $\bar{f},\bar{g}$ and covariance

$$\begin{pmatrix} F & P \\ P^+ & G \end{pmatrix}$$

where F and G are the covariances of f,g, respectively, and P is the covariance of f and g. The Bayes estimate $f_B$ for f if g is known is then given by

$$f_B = E(f\ g) = \bar{f}+P\quad G^{-1}(g-\bar{g})$$

see [7], p.28. If g is a linear function of f, g=Rf, say, then (f,g) is jointly normal if f is, and the covariance of (f,g) is

$$\begin{pmatrix} F & FR^+ \\ RF & RFR^+ \end{pmatrix},$$

see [7], p.19. Thus we see that the Bayes estimate $f_B$ for the solution of $Rf=g$, $g$ known, is given by

$$f_B = \bar{f}+FR^+(RFR^+)^{-1}(g-R\bar{f}).$$

In order to apply Bayesian estimation to our picture reconstruction problem we think of the density function $f$ as a family of random variables $(f(x))_{x\in\Omega}$ which are normally distributed with mean value $\bar{f}(x)$. The interrelation between two pixels $x,x'$ is modelled by a covaricance operator with kernel

$$F(x,x') = E(f(x)-\bar{f}(x))(f(x')-\bar{f}(x')).$$

$R:L_2 \longrightarrow \mathbb{R}^n$ is the operator defined by the strip integrals and $R^+: \mathbb{R}^n \longrightarrow L_2$ is its adjoint which is easily seen to be

$$R^+q = \sum_{k=1}^{n} q_k \chi_k$$

with $\chi_k$ the characteristic function of $L_k$. With

$$\eta_k(x) = \int_{L_k} F(x,x')dx',$$

$$S_{k,\ell} = (RFR^+)_{k,\ell}$$

$$= \int_{L_k} \int_{L_\ell} F(x,x')dx\, dx',$$

the Bayes estimate assumes the form

$$(1.1) \qquad f_B = \bar{f} + \sum_{k=1}^{n} q_k \eta_k, \qquad q = S^{-1}(g-R\bar{f})$$

An iterative method for the computation of $f_B$ has been implemented in [6]. If the strip geometry is invariant with respect to rotation then a noniterative implementation along the lines of [8] is possible, too.

## § 2 The Bayes estimate and reproducing kernel Hilbert spaces

It is well known (see[9]) that the Bayes approach is equivalent to a completely deterministic model: Let H be that Hilbert space $\subseteq L_2$ which has the reproducing kernel F, i.e.

$$\forall u \in H \qquad (u, F(x,\cdot))_H = u(x).$$

Assume the functionals $R_k$ to be continuous on H and $\bar{f} \in H$. Define $f_H$ to be that solution of $Rf=g$ for which $\|f-\bar{f}\|_H$ is as small as possible. We show that $f_H = f_B$.

We have $R_k f = (\eta_k, f)_H$ with some $\eta_k \in H$ which can be determined by putting $f = F(t,\cdot)$:

$$\eta_k(x) = (\eta_k, F(x,\cdot))_H = R_k F(x,\cdot) = \int_{L_k} F(x,x')dx'.$$

Using Lagrangian multipliers $q_k$ we have to look for the stationary points of

$$\|f-\bar{f}\|_H^2 - \sum_{k=1}^{n} q_k(\eta_k, f).$$

It follows that

$$(2.1) \qquad f_H = \bar{f} + \sum_{k=1}^{n} q_k \eta_k$$

where q is determined by $Rf_H = g$ or

$$(2.2) \qquad \sum_{k=1}^{n} q_k R\eta_k = g - R\bar{f}.$$

But

$$R_\ell \eta_k = \int_{L_\ell} \eta_k(x)dx = \int_{L_\ell}\int_{L_k} F(x,x')\, dx\, dx' = S_{k,\ell},$$

which shows that (2.1), (2.2) are identical to (1.1), hence $f_H = f_B$.

## § 3 The Smoothing Property of the Bayes approach

We have seen in § 2 that the Bayes estimate $f_B$ for the solution of $Rf=g$ is that element of H which among all solutions to this equations is closest to $\bar{f}$ in the sense of H. We conclude that $\|f_B-\bar{f}\|_H < \infty$ if the equations $Rf=g$ are consistent. In fact $\|f_B-\bar{f}\|_H$ is likely to be comparatively small since it is obtained by a minimization process. Thus we can get some qualitative information on $f_B-\bar{f}$ simply by looking at the norm in the Hilbert space H which depends only on the function F.

It is easy to compute the norm in H if F is given by the isotropic exponential model (see [13], p.21), i.e.

$$F(x,x') = F(x-x') = e^{-\lambda|x-x'|}, \quad \lambda>0, \tag{3.1}$$

where $|\cdot|$ is the euclidean distance. We show that H is basically the Sobolev space $H^{3/2}$; see [10] for the Sobolev spaces $H^\alpha$. We use the inner product

$$(u,v)_{H^\alpha} = \int(\lambda^2+|\xi|^2)^\alpha \, \hat{u}(\xi)\bar{\hat{v}}(\xi)\, d\xi$$

where $\hat{u}$ is the Fourier transform of u.

The Fourier transform of F is

$$\hat{F}(\xi) = \frac{1}{2\pi}\int e^{-ix\xi}\, e^{-\lambda|x|}\, dx$$

$$= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} r\, e^{-ir\rho\cos(\varphi-\psi)-\lambda r}\, dr\, d\varphi$$

where $r,\varphi$ and $\rho,\psi$ are polar coordinates in the x and $\xi$ plane, respectively. Using the formula

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ir\rho\cos\varphi}\, d\varphi = J_0(r\rho)$$

with $J_0$ the Bessel function of order 0 (see[11], formula 9.1.21 on p. 360) we obtain

$$\hat{F}(\xi) = \int_0^\infty r\, e^{-\lambda r}\, J_o(r\rho)\, dr$$

$$= \lambda(\lambda^2+\rho^2)^{-3/2}$$

where formula 4.432 of [12], p.236 has been used. Thus

$$(u,F(x,\cdot))_{H^{3/2}} = \int (\lambda^2+|\xi|^2)^{3/2}\, \hat{u}(\xi) e^{ix\xi}\, \bar{\hat{F}}(\xi)\, d\xi$$

$$= \lambda \int \hat{u}(\xi) e^{ix\xi}\, d\xi$$

$$= 2\pi\, \lambda\, u(x)$$

by the Fourier inversion formula. Thus we see that $\frac{1}{2\pi\lambda}$ F is a reproducing kernel for $H^{3/2}$, hence basicall $H=H^{3/2}$.

We come to the conclusion that the Bayes estimate for f is in $H^{3/2}$ if the model (3.1) for f is used. But this model requires much less smoothness of f than $f\in H^{3/2}$: Using the Aronszajn norm in $H^\alpha$ for $0<\alpha<1$, see [10], p.214,

$$\|f\|^2_{H^\alpha} = \int f^2(x)dx + \int \int \frac{(f(x)-f(x'))^2}{|x-x'|^{2+2\alpha}}\, dx\, dx'$$

we obtain for a family of random variables with covariance (3.1)

$$E\|f-\bar{f}\|^2_{H^\alpha} = \int_\Omega dx + \int_\Omega \int_\Omega (1-e^{\lambda|x-x'|})\,|x-x'|^{-2-2\alpha}\, dx\, dx'$$

where $\Omega$ is the support of f, and this is finite if and only if $\alpha<\frac{1}{2}$. Thus the smoothness which is built in the model (3.1) corresponds roughly to the Sobolev space $H^{1/2}$, whereas the Bayes estimate ends up with a function in $H^{3/2}$!

The last statement contains the basic result of the present paper. It reveals a smoothing power of the Bayes approach which is, in the author's opinion, by far too strong. Whereas $f\in H^{1/2}$ is reasonable for picture densities, $f\in H^{3/2}$ is not, see [14] for a discussion of that point. Here it suffices to mention two examples: The density function of a peak of height 1 and diameter $\varepsilon$ has the norm $O(\varepsilon^{1-\alpha})$ in $H^{\alpha}$, and the $H^{\alpha}$-norm of a density function jumping by 1 across a smooth curve is $O(\varepsilon^{1/2-\alpha})$ for $\alpha>1/2$. Thus the norm in $H^{3/2}$ of these density functions tends to $\infty$ as $\varepsilon \to 0$. Therefore a reconstruction procedure which minimizes the $H^{3/2}$-norm of the density function necessarily smoothes out peaks and high contrasts. This is precisely what happend in the numerical experiments carried out in [6]: The graphs in figure 4 of that paper show that the Bayes method failed to detect peaks which have been clearly recovered by other methods and performed generally poorly in high contrast areas.

It is hoped that the present paper not only gives an explanation for the failure of the Bayes approach to picture reconstruction reported in [6], but also helps to overcome the difficulties of this approach which has been applied successfully to many problems in science.

REFERENCES

[1] SMITH, K.T. - SOLMON, D.C. - WAGNER, S.L.: Practical and Mathematical Aspects of the Problem of Reconstructing Objects from Radiographs. Bul. AMS (1977).

[2] GORDON, R. (ed.): Image Processing for 2-D and 3-D Reconstruction from Projections: Theory and Practice in Medicine and the Physical Sciences. Optical Society of America, 2000 L Street, N.W., Washington D.C. 20036.

[3] BUDINGER, T.F. - GULBERG, G.T.: Three-Dimensional Reconstruction in Nuclear Medicine Emission Imaging, IEEE Transaction on Nuclear Science 21, 2-20 (1974).

[4] HURWITZ, H.: Entropy Reduction in Bayesian Analysis of Measurements, Phys. Rev., Sect. A 12, 698-706 (1975).

[5] TASTO, M.: Reconstruction of Objects from Noisy Projections, Computer Graphics and Image Processing 6, 103-122 (1977).

[6] HERMAN, G.T. - LENT, A.: A Computer Implementation of a Bayesian Analysis of Image Reconstruction, Information and Control 31, 364-384 (1976).

[7] ANDERSON, T.W.: An Introduction to Multivariate Statistical Analysis. Wiley 1957.

[8] NATTERER, F.: Optimality Properties and a Non-Iterative Implementation of ART. Manuscript.

[9] WAHBA, G.: A Class of Approximate Solutions to Linear Operator Equations, J. Approximation Theory 9, 61-77 (1973).

[10] ADAMS, R.A.: Sobolev Spaces. Academic Press 1975.

[11] ABRAMOWITZ, M. - STEGUN, I.A.: Handbook of Mathematical Fuctions. Dover Publications 1972.

[12] RYSHIK, I.M. - GRADSTEIN, I.S.: Summen-, Produkt- und Integraltafeln. Berlin 1957.

[13] PRATT, W.K.: Digital Image Processing, Wiley 1978.

[14] NATTERER, F.: A Sobolev Space Analysis of Picture Reconstruction. Manuscript.