GHOSTS IN TOMOGRAPHY

- THE NULL SPACE OF THE RADON TRANSFORM -

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A 79/19

Abstract

Let $\omega_o,\ldots,\omega_{p-1} \in S^1$ be p distinct directions and let R denote the RADON transform. The functions f with support in the unit disc and with $Rf(s,\omega_j) = 0$ for $s \in [-1,1]$ and $j = 0,\ldots, p-1$ are given. "Ghosts" are constructed, i. e. functions in the null space that pretend not existent tumors. Methods to avoid such ghosts are indicated.

1 Introduction

Let $\Omega := \{x \in \mathbb{R}^2 : |x| \leq 1\}$ be the unit circle and S^1 its boundary. The RADON transform of a function with support in Ω is defined as

$$Rf(s,\omega) = g(s,\omega) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s\omega + t\omega^{\perp})\, dt \qquad (1.1)$$

where $s \in \mathbb{R}$ and $\omega, \omega^{\perp} \in S^1$ with $\omega = (\cos\varphi, \sin\varphi)$, $\omega^{\perp} = (-\sin\varphi, \cos\varphi)$.

The function f is uniquely determined if g is known for all s and for all $\varphi \in [\Phi_1, \Phi_2]$ with some $0 \leq \Phi_1 < \Phi_2 \leq \pi$. In the applications as for example in medical x-ray diagnosis only finitely many projections are known, say for $0 = \varphi_0 < \varphi_1 < \ldots < \varphi_{p-1} < \pi$. In this case the function f cannot be achieved exactly, see LUDWIG, [9] , SMITH-SOLMON-WAGNER, [14]. This means that there exist functions $f \not\equiv 0$ such that $Rf(s,\omega_j) = 0$ for all s and for $j = 0,\ldots, p-1$.

In this paper the null space is determined depending on the directions ω_j. As a consequence in the case of equidistributed angles special functions in the null space are constructed that are looking like tumors, so-called ghosts. Some typical properties of the ghosts make it possible to recognize them and so to avoid them.

In § 2 we give an inversion formula which enables us to construct the null space, see § 3. Finally § 4 contains practical considerations.

2 An Inversion Formula

With $H_o^\alpha(\Omega)$ we denote the $\tilde{H}^\alpha(\Omega)$ spaces of TRIEBEL, [15] 4.3.2. Let $f \in H_o^\alpha(\Omega)$ with $\alpha \geq -1/2$, then the RADON transform $g = Rf$ is defined for almost all s and is an element of $H^{\alpha+1/2}(Z)$ with $Z = \mathbb{R} \times S^1$, see NATTERER,[11], theorem 3.2 for $\alpha > 1/2$ and LOUIS, [8], theorem 2.1 for $\alpha \geq -1/2$. The support of $g(\cdot,\omega)$ lies in $[-1,1]$ for all φ and g satisfies the conditions of HELGASON, [3], and LUDWIG, [9]:

i) $g(-s,-\omega) = g(s,\omega)$ on Z (2.1)

ii) for all $m \in \mathbb{N}_o$

$$\int s^m \, g(s,\omega) \, ds \tag{2.2}$$

is a homogeneous polynomial of degree $\leq m$ in ω.

Let P be any real polynomial of degree m, then it follows from condition (2.2) that

$$\int P(s) \, g(s,\omega) \, ds = \sum_{\ell=-m}^{m} c_\ell^m \, e^{i\ell\varphi} \tag{2.3}$$

with $c_\ell^m \in \mathbb{C}$ and $c_{-\ell}^m = \bar{c}_\ell^m$ where the bar denotes the complex conjugation.

In the following we use the CHEBYSHEV polynomials of the second kind, U_m, which are orthogonal on the interval $[-1,1]$ with respect to the weight function $(1-s^2)^{1/2}$:

$$\int_{-1}^{1} (1-s^2)^{1/2} U_m(s) U_n(s) \, ds = \frac{\pi}{2} \, \delta_{mn} . \tag{2.4}$$

where δ_{mn} denotes the KRONECKER symbol, see [1], page 774.

The expansion of g in terms of the CHEBYSHEV polynomials is

$$g(s,\omega) = (1-s^2)^{1/2} \sum_{m=0}^{\infty} q_m(\varphi) \, U_m(s) \tag{2.5}$$

with expansion coefficients

$$q_m(\varphi) = \frac{2}{\pi} \int_{-1}^{1} U_m(s) \, g(s,\omega) \, ds$$

$$= \sum_{\ell=-m}^{m} c_\ell^m \, e^{i\ell\varphi} \tag{2.6}$$

because of (2.3). For $\alpha \geq -1/2$ this expansion is convergent because of $g \in L_2(Z)$. From the symmetry (2.1) of g and the fact that the U_m are even (odd) for m even (odd) we conclude

$$c_\ell^m = 0 \text{ for } \ell + m \text{ odd.} \tag{2.7}$$

This leads to the following representation of g:

$$g(s,\omega) = (1-s^2)^{1/2} \sum_{m=0}^{\infty} U_m(s) \sum_{\ell=-m}^{m} c_\ell^m e^{i\ell\varphi}. \tag{2.8}$$

In order to get an expansion of f with the coefficients c_ℓ^m defined above we consider the ZERNIKE polynomials:

$$V_{\pm\ell,k}(x) = e^{\pm i\ell\varphi} r^\ell Q_{\ell,k}(r^2) \tag{2.9}$$

where $\ell, k \in \mathbb{N}_0$, $x = r(\cos\varphi, \sin\varphi)$ and $Q_{\ell,k}$ is a polynomial of degree k in t with

$$\int_0^1 Q_{\ell,k}(t)\, t^{\ell+m}\, dt = 0 \text{ for } m = 0,1,\ldots,k-1$$

and

$$Q_{\ell,k}(1) = 1. \tag{2.10}$$

The $V_{\ell,k}$ fulfill

$$\int_\Omega V_{\ell,k} \overline{V_{\ell',k'}}\, dx = \pi(|\ell|+2k+1)^{-1} \delta_{\ell\ell'} \delta_{kk'} ,$$

see MARR, [7], lemma 1.

LEMMA 1

The RADON transform of the ZERNIKE polynomials is

$$RV_{\ell,k}(s,\omega) = 2(m+1)^{-1} (1-s^2)^{1/2} U_m(s) e^{i\ell\varphi} \tag{2.11}$$

for $\ell \in Z$, $k \in \mathbb{N}_0$ and with $m = 2k+|\ell|$.

PROOF

This result follows from theorem 1 in MARR, [10]. In § 2 in [10] it is shown that the $V_{\ell,k}$ form an orthogonal system of polynomials with total degree $2k+|\ell|$. Furthermore the boundary fuction of $V_{\ell,k}$ is $e^{i\ell\varphi}$ because of (2.10), see also [6], [7].

Using (2.8) and (2.11) we finally have the following expansion of g :

$$g(s,\omega) = \frac{1}{2} \sum_{m=0}^{\infty} (m+1) \sum_{\ell=-m}^{m} c_\ell^m \, RV_{\ell,(m-|\ell|)/2}(s,\omega) . \qquad (2.12)$$

Conclusion (2.7) guarantees that in (2.12) only $V_{\ell,(m-|\ell|)/2}$ appear where $(m-|\ell|)/2$ is integer. Now because of the linearity of the RADON transform we have the following inversion formula.

THEOREM 2

Let $f \in L_2(\Omega)$ and let the coefficients c_ℓ^m be defined by (2.6).

Then

$$f(x) = \frac{1}{2} \sum_{m=0}^{\infty} (m+1) \sum_{\ell=-m}^{m} c_\ell^m \, V_{\ell,(m-|\ell|)/2}(x) . \qquad (2.13)$$

PROOF :

In order to show the convergence of the series on the right of (2.13) we calculate the expansion coefficients of f in terms of the ZERNIKE polynomials

$$d_\ell^m = \frac{m+1}{\pi} (f, V_{\ell,(m-|\ell|)/2}) .$$

Because of $V_{\ell,k}(x) = R^* KRV_{\ell,k}(x)$ where R^* is the adjoint operator of R and

$$Ku(s,\omega) = \frac{1}{4\pi} \, i \, \mathcal{H} \, \frac{\partial}{i \partial s} \, u(s,\omega)$$

where $\mathcal{H}$ is the HILBERT transform, see [9], we come to

$$d_\ell^m = \frac{m+1}{\pi} (Rf, KRV_{\ell,(m-|\ell|)/2}) .$$

With (2.11) and

$$\mathcal{H} \left((1-s^2)^{1/2} \, U_m(s) \right) = T_{m+1}(s)$$

where T_m is the CHEBYSHEV polynomial of the first kind, see ABRAMOWITZ-STEGUN, [1], formula 22.13.4, we get

$$KRV_{\ell,(m-|\ell|)/2}(s,\omega) = \frac{1}{2\pi} U_m(s) e^{i\ell\varphi}$$

where we have used $T'_{m+1}(s) = (m+1) U_m(s)$. Finally the d_ℓ^m can be calculated as

$$\begin{aligned} d_\ell^m &= \frac{m+1}{\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} Rf(s,\omega) \overline{KRV_{\ell,(m-|\ell|)/2}(s,\omega)} \, ds \, d\varphi \\ &= \frac{m+1}{2\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} g(s,\omega) U_m(s) \, ds \, e^{-i\ell\varphi} \, d\varphi \\ &= \frac{m+1}{4\pi} \sum_{k=-m}^{m} \int_0^{2\pi} c_k^m e^{i(k-\ell)\varphi} \, d\varphi \\ &= \frac{m+1}{2} c_\ell^m \end{aligned}$$

where we have used (2.6). This completes the proof.

The direct application of this inversion formula leads to an algorithm with $O(p^2(N^2+q))$ arithmetic operations, where q is the number of positions s where $g(s,\omega_j)$ is known and N^2 is the number of points x where the density is evaluated. Therefore this method cannot compete with the backprojection method for example, which needs only $O(pN^2)$ operations, see RAMACHANDRAN-LAKSHMINARAYANAN, [13]. The advantage of this formula consits in the characterization of the RADON transform.

3 The Null Space of the RADON Transform

Let $\omega_j \in S^1$, $j = 0,\dots,p-1$ be p distinct directions. For $f \in L_2(\Omega)$, $\mathrm{supp}(f) \subset \bar{\Omega}$, the functions $Rf(\cdot,\omega_j)$ are in L_2, so we can define the null space of the RADON transform for the projections in the directions ω_j as

$$\mathcal{N} = \{f \in L_2(\Omega): Rf(s,\omega_j) = 0 \text{ for almost all } s \text{ and } j=0,\dots,p-1\}$$

Using the representation (2.5) for a function g we can characterize the functions $f \in \mathcal{N}$. If $f \in \mathcal{N}$ then $g = Rf$ fulfills

$$g(s,\omega_j) = (1-s^2)^{1/2} \sum_{m=0}^{\infty} q_m(\varphi_j)\, U_m(s) = 0 \qquad (3.1)$$

for almost all s and $j = 0,\dots, p-1$.

Because of the linear independence of the CHEBYSHEV polynomials this is equivalent with

$$q_m(\varphi_j) = 0 \text{ for } j = 0,\dots,p-1 \text{ and for all } m \in \mathbb{N}_0. \qquad (3.2)$$

The q_m are trigonometric polynomials with m+1 coefficients:

$$q_m(\varphi) = \sum_{\ell=0}^{[m/2]} (a_\ell^m \cos \alpha_\ell^m \varphi + b_\ell^m \sin \alpha_\ell^m \varphi) \qquad (3.3)$$

with $\alpha_\ell^m = \begin{cases} 2\ell & \text{if m is even} \\ 2\ell+1 & \text{if m is odd} \end{cases}$

and a_ℓ^m, $b_\ell^m \in \mathbb{R}$.

Now (3.2) is a system of p linear homogeneous equations with m+1 unknowns and the rank of the matrix is min(p,m+1). Therefore the q_m vanish identically for $m < p$. This leads to the following result.

THEOREM 3

Let $f \in \mathcal{N}$. Then

$$f(x) = \sum_{m=p}^{\infty} \sum_{\ell=-m}^{m} c_\ell^m V_{\ell,(m-|\ell|)/2}(x)$$
$$= \sum_{m=p}^{\infty} \sum_{\ell=-m}^{m} c_\ell^m e^{i\ell\varphi} r^{|\ell|} Q_{|\ell|,(m-|\ell|)/2}(r^2) \qquad (3.4)$$

with $x = r(\cos\varphi, \sin\varphi)$
where

$$q_m(\varphi_j) = \sum_{\ell=-m}^{m} c_\ell^m e^{i\ell\varphi_j} = 0 \text{ for } j = 0,\dots,p-1.$$

From theorem 3 it follows that the expansion of a function $f \in \mathcal{N}$ in terms of ZERNIKE polynomials starts with a polynomial of total degree p.

In the special case of equidistributed angles $\varphi_j = j\pi/p$, $j=0,\dots,p-1$, this result bears more information. For $m \geq p$ the homogeneous system of linear equations (3.2) has the m-p+1 linearly independent solutions

$$q_m(\varphi) = \sin p\varphi \sum_{\ell=0}^{m-p} (a_\ell^{m-p} \cos \ell\varphi + b_\ell^{m-p} \sin \ell\varphi) \qquad (3.5)$$

$$\text{with } b_0^{m-p} = 0 \text{ and } a_\ell^{m-p} = b_\ell^{m-p} = 0 \text{ for } \ell+m-p \text{ odd} \qquad (3.6)$$

and a_ℓ^{m-p}, $b_\ell^{m-p} \in \mathbb{R}$ arbitrary otherwise.

With $c_\ell = a_\ell^{m-p} - i\, b_\ell^{m-p}$ and $c_{-\ell} = \bar{c}_\ell$ these polynomials can be represented as

$$q_m(\varphi) = \frac{1}{4i} \sum_{\ell=0}^{m-p} \{c_\ell (e^{i(p+\ell)\varphi} - e^{-(p+\ell)\varphi}) + c_{-\ell}(e^{i(p-\ell)\varphi} - e^{-i(p-\ell)\varphi})\}.$$

For $x = r(\cos\varphi, \sin\varphi)$ this leads with (2.8) and theorem 2 to

$$f(x) = \frac{1}{4i}\sum_{m=p}^{\infty}\frac{m+1}{2}\sum_{\ell=0}^{m-p}\{(c_\ell e^{i(p+\ell)\varphi} - c_{-\ell} e^{-i(p+\ell)\varphi}) r^{p+\ell} Q_{p+\ell,(m-p-\ell)/2}(r^2)$$

$$+ (c_{-\ell} e^{i(p-\ell)\varphi} - c_\ell e^{-i(p-\ell)\varphi}) r^{|p-\ell|} Q_{|p-\ell|,(m-|p-\ell|)/2}(r^2)\}$$

$$= \sin p\varphi \sum_{m=p}^{\infty}\sum_{\ell=0}^{m-p}(a_\ell^{m-p}\cos\ell\varphi + b_\ell^{m-p}\sin\ell\varphi)\cdot P_{m,\ell}^1(r)$$

$$+\cos p\varphi \sum_{m=p+1}^{\infty}\sum_{\ell=1}^{m-p}(b_\ell^{m-p}\cos\ell\varphi - a_\ell^{m-p}\sin\ell\varphi)\cdot P_{m,\ell}^2(r) \tag{3.7}$$

with

$$P_{m,\ell}^{1,2}(r) = r^{p+\ell} Q_{p+\ell,(m-p-\ell)/2}(r^2) \pm r^{|p-\ell|} Q_{|p-\ell|,(m-|p-\ell|)/2}(r^2) \tag{3.8}$$

with degree $P_{m,\ell}^{1,2} = m$. Here we have multiplied the coefficients which can be chosen arbitrarily by $\frac{4}{m+1}$ and again denoted by a_ℓ^{m-p}, b_ℓ^{m-p}. The second sum in (3.6) starts with p+1 because for m=p the inner sum is $b_o^{m-p}\cos 0 - a_o^{m-p}\sin 0 = 0$, see (3.6)

In the first sum in (3.7) we get for $\ell=0$

$$a_o^{m-p} P_{m,o}^1(r) = 2\, a_o^{m-p} r^p Q_{p,(m-p)/2}(r^2)$$

and with m=p+2k

$$2a_o^{2k} Q_{p,k}(r^2) r^p$$

Finally we come to the following result.

THEOREM 4

Let the φ_j, $j = 0,\dots,p-1$ be equidistributed angles and let $f \in \mathcal{N}$. With $x = r(\cos\varphi, \sin\varphi)$ the function f has the representation

$$f(x) = \sin p\varphi \cdot \{r^p \sum_{\ell=0}^{\infty} d_\ell \; Q_{p,\ell}(r^2)$$

$$+ \sum_{m=p+1}^{\infty} \sum_{\ell=1}^{m-p} (a_\ell^{m-p} \cos \ell\varphi + b_\ell^{m-p} \sin \ell\varphi) \cdot P_{m,\ell}^{1}(r^2)\} \qquad (3.9)$$

$$+ \cos p\varphi \sum_{m=p+1}^{\infty} \sum_{\ell=1}^{m-p} (b_\ell^{m-p} \cos \ell\varphi - a_\ell^{m-p} \sin \ell\varphi) P_{m,\ell}^{2}(r^2)$$

with $P_{m,\ell}^{1,2}$ from (3.8), $a_\ell^{m-p} = b_\ell^{m-p} = 0$ for $\ell+m-p$ odd

and $d_\ell,\ a_\ell^{m-p},\ b_\ell^{m-p} \in \mathbb{R}$.

Now the following result is easily deduced from the theorems above. It has first been shown by HAMAKER-SOLMON [2] in order to compute the angles between the null spaces of the X-rays.

COROLLARY 5

$\mathcal{N}$ is a closed linear subspace of $L_2(\Omega)$.

PROOF

The projection $P_j^m : L_2(\Omega) \to \mathbb{R}$ with $P_j^m f = q_m(\varphi_j)$ and $q_m(\varphi_j)$ from (2.6) is continous for all j and m:

$$|q_m(\varphi_j)| = \frac{2}{\pi} \left| \int_{-1}^{1} U_m(s) g(s,\omega_j)\, ds \right| .$$

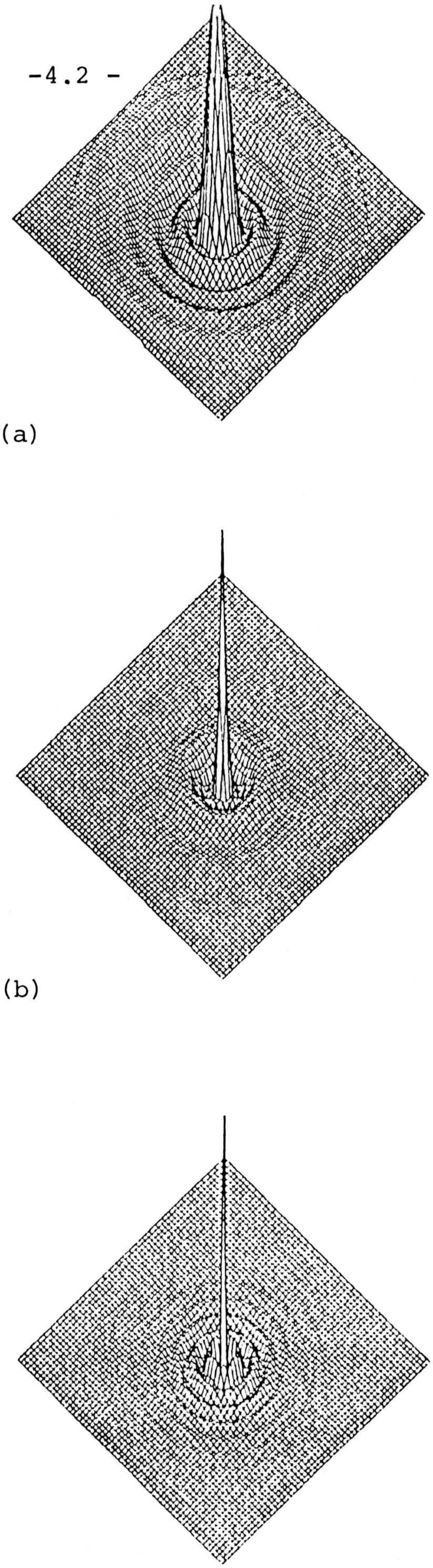
$$\leq \frac{2}{\pi} \|U_m\|_{L_2} \|g(\cdot,\omega_j)\|_{L_2} \leq C_m \|g\|_{H^{1/2}}$$

$$\leq C_m' \|f\|_{L_2} .$$

Now let $\{f_\ell\}_{\ell \in \mathbb{N}}$ be a sequence in $\mathcal{N}$ with $f_\ell \to f \in L_2(\Omega)$. Because of the continuity of P_j^m we have $P_j^m f = 0$ and so $f \in \mathcal{N}$.

4 Practical Aspects

The most important application of the RADON transform lies in the medical X-ray diagnosis. The computed tomography was originally used in the evaluation of disorders in the brain and has become a well established radiological technique for the whole body. Thereby the total attenuation of the X-ray beam between the source and the detector provides us with an estimate of the line integrals of the linear attenuation coefficient and so with the density of the scanned tissue. It is evident that only finitely many projections can be realized and that therefore the density is not uniquely determined. The question arises whether or not there are pictures whose projections coincide with that of the original density and which pretend not existent tumors, so-called "ghosts". At the first glance this seems to be impossible because the null space consists in the radial direction of polynomials of high order and in the angular direction by oscillating terms. But the figures 1-2 show the contrary: the projection of the characteristic function of the circle around 0 with small radius R on the null space looks like a ghost after normalizing such that the maximal value is equal to one. Figure 1 shows 3-D pictures of these functions with $R = 2.5 \cdot 10^{-3}$ and for $p = 10,30,90$ in the circle with radius 0.16; figure 2 shows a cut of these functions along the positive x-axis.



(a)

(b)

(c)

Figure 1. "Ghosts" for p=10(a), p=30(b), p=90(c)

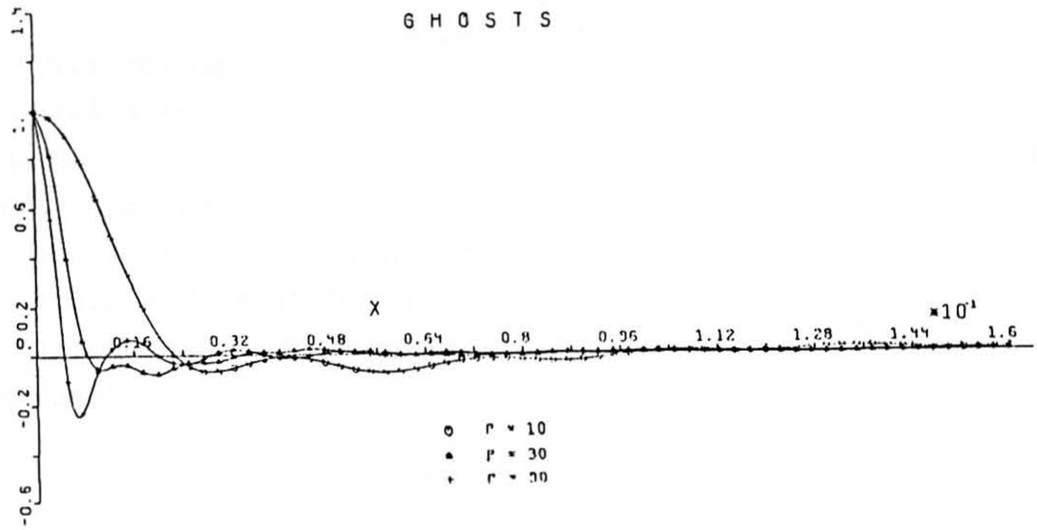


Figure 2. Cut of the functions from figure 1 along the positive x-axis.

Typical for the functions in the null space are the following facts. The H^s norm grows exponentially with s, see figure 3. This is caused by the oscillatory nature of these functions.

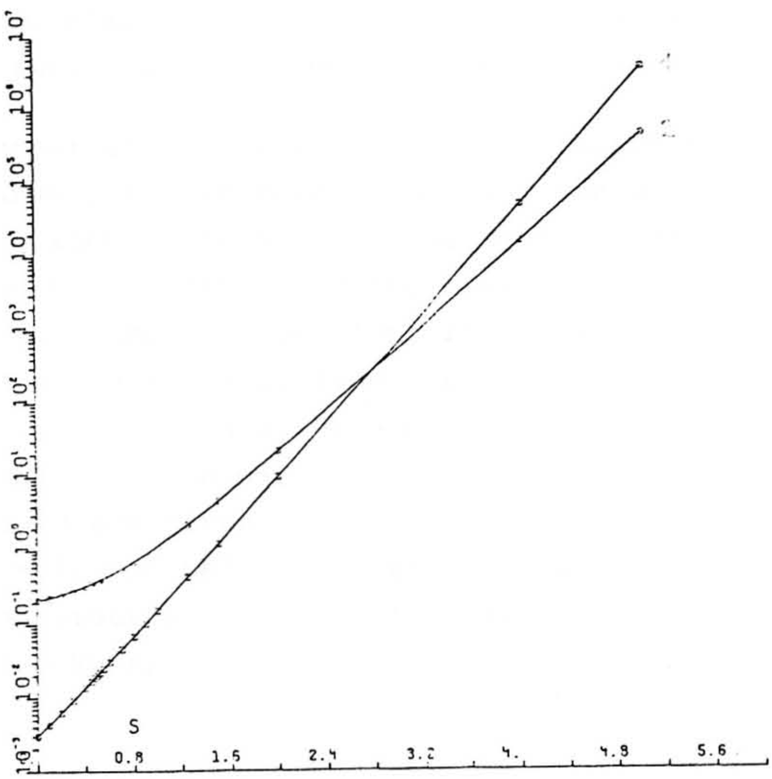


Figure 3. H^s norm of the ghost of figure 1c) (1) and of a phantom of a tumor in the brain (2).

For this reason the regularization method of TYCHONOV and PHILLIPS provides a remedy against ghosts because usual densities do not show this behaviour for small s, see figure 3, where the second graph is the H^s norm of the reconstruction of object 2 in [8] with filtered backprojection (p=30, q=128). The original is defined as 1 for $|x| < 0.3$ and $|x-(0.05,0)| > 0.225$, as 0.3 for $|x-(0.15,0)| < 0.05$ and 0 otherwise. The regularization method consits in minimizing the functional

$$J_\alpha(z) = \frac{1}{p} \sum_{j=0}^{p-1} ||Rz(\cdot,\omega_j)-g(\cdot,\omega_j)||_{L_2}^2 + \alpha^2 ||z||_{H^s}^2$$

with suitable s and α, where g is measured. The regularization term takes care that the influence of the null space is absorbed.

A further property that is shared by the ghosts is that the width of the peak is about 1/p. This agrees with the first root of the first polynomial in the expansion (3.9) which is of order 1/2p, see figure 4. Consequently the resolution in the reconstruction must not be to high. The distance of two points where the density is evaluated should be at least 1/p. This coincides with the result of KOWALSKI-WAGNER, [5], see formula (18). In addition to that filtering helps to overcome this problem, see [5].

Finally another consequence of the uncertainty in the recon - struction should be discussed. The null space is responsible for two well known artefacts in the reconstruction : the arte - facts in radial direction and the ring-like artefacts. This can be seen by the simplest functions in (3.9). Disturbations caused by $c \cdot r^p \cdot \sin p\varphi$ go radially from a point of the picture. Setting $a_p^p = c/2$ in (3.9) and all the other coefficients equal to zero we obtain the function $c \cdot Q_{o,2p}(r^2)$. Graphs for $c \cdot Q_{o,2p}(r^2)$ for p = 10, 30, 90 are given in figure 4, c is chosen such that $c \cdot Q_{o,2p}(0) = 1$. The relative extrema of these functions show the same behaviour as the ring-like artefacts. For pictures of artefacts see KOWALSKI-WAGNER, [5].

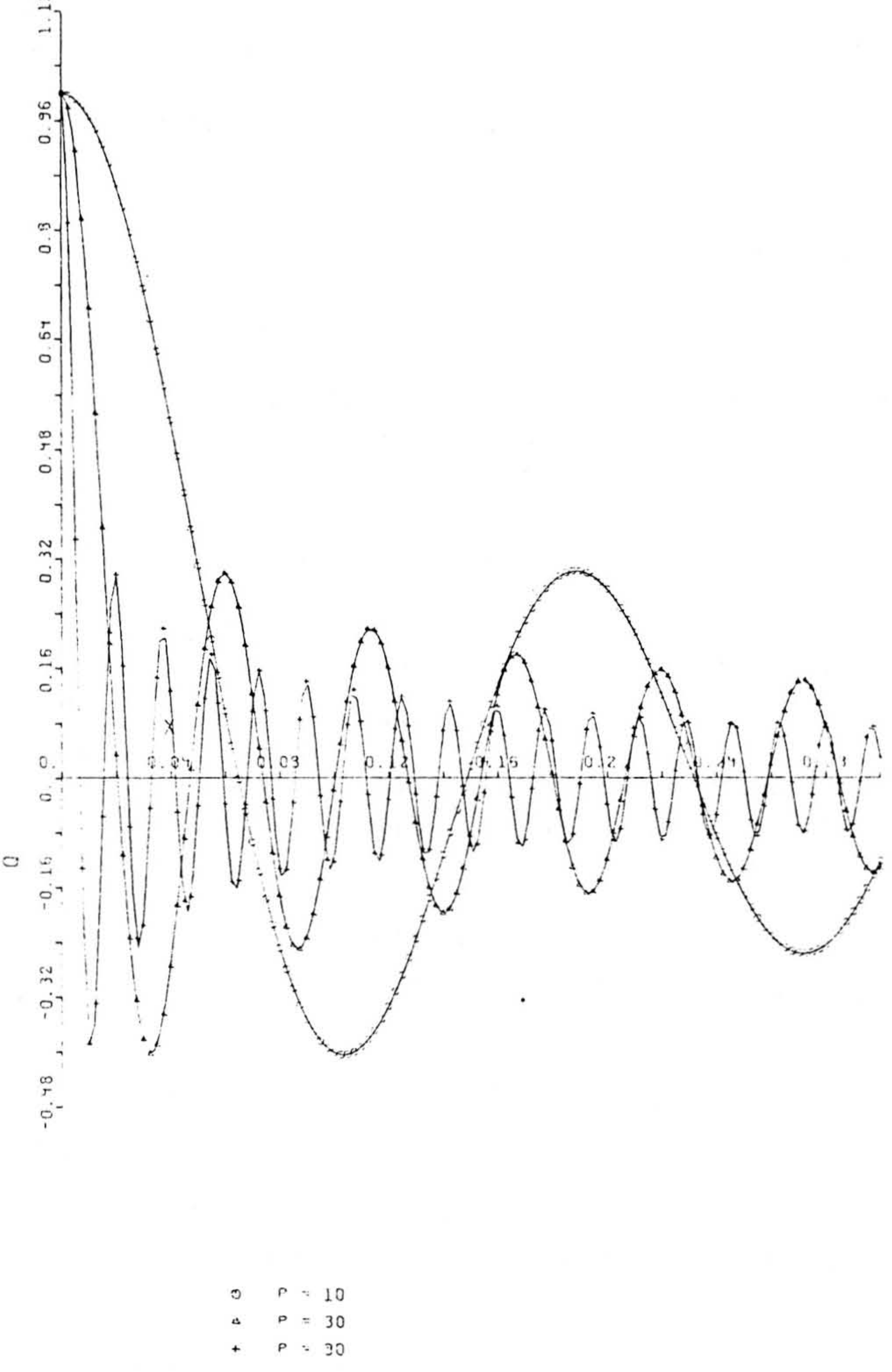


Figure 4. ZERNIKE-polynomials $Q_{o,2p}(r^2)/Q_{o,2p}(0)$ in [0,0.3] for p=10,30,90

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