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MEASURE AND INTEGRATION: AN ATTEMPT AT UNIFIED SYSTEMATIZATION

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1. Introduction

The end of the 20th century marks the hundredth anniversary of the creation of measure and integration by Borel and Lebesgue. The new concepts proved to be so powerful and flexible that they soon turned into kind of foundation of mathematical analysis of the 20th century. On the surface the main reason was a small number of fundamental theorems, like the limit theorems of Beppo Levi, Fatou, and Lebesgue, and the Fubini-Tonelli theorem. Their perception and widespread application went so fast that Carathéodory in the introduction to his famous treatise [1918] declared the revolution due to Lebesgue to be complete in its main lines.

But the actual course of events over the 20th century was quite different. Mathematical analysis, like the whole of mathematics, went through a continuous chain of vivid abstractions. For the most time measure and integration were at the forefront, because each new abstraction required its proper class of measures, in order that those powerful theorems could be put into action. Thus it became clearer than before that the heart of the matter is *to produce the adequate measures and integrals*. This is always a nontrivial and often a hard problem, as it had been in the hour of birth with Lebesgue measure on the real line. Thus the emphasis shifted in direction to the fundamentals.

This means to disclose the *true basic concepts* in measure and integration, and then to develop the *true basic procedures* in order to construct their members. As a rule such procedures amount to produce true measures from more primitive data as they occur in nature, like elementary contents and elementary integrals.

The resultant situation at the end of the 20th century, viewed from the most respected textbooks, is opposite to a coherent one. One finds several theories of measure and integration which at times do not even share the most basic concepts. The main issues in favour of schism were on the one hand abstract versus topological conceptions, and on the other hand predominance of measures versus that of integrals. There were times of vehement confrontation, above all after the Intégration of Bourbaki [1952][1956][1969], which combined essential successes with bizarre extreme positions like the denial of independent existence to measures.

In this situation one of course started to express the desire for a unified and autonomous theory of measure and integration, but also to insist that this aim requires to restructure the abstract theory with the eyes on the topological theory. The traditional abstract theory on the basis of the procedures named after Carathéodory and Daniell-Stone showed certain clear deficiencies. The topological theory added a multitude of further aspects and concepts, which at times resulted in almost painful technical complications, but after all the two aspects of prime rank could be distilled. These are on the one hand *upward* and *downward continuity* of set functions and functionals, and besides σ (:=*sequential*) also τ (:=*nonsequential*) continuity (based on directed systems instead of monotone sequences), and on the other hand *outer* and *inner regularity* of set functions and functionals, defined as representations from above and below, in case of set functions as a rule from open supersets and compact subsets.

The task to unite the above two aspects with the usual concepts of traditional abstract theory turned out to be a delicate and expensive one, which shook and shakes the foundations of the edifice and took much time. On the one hand *sequential continuity* occurs in most abstract contexts under the unfortunate headline of *countable additivity*, that means degraded to an annex of simple additivity. This kind of view has caused much misfortune in the whole area, above all its subdivision into separate theories of additive and nonadditive set functions, and has to be abandoned once and for all. On the other hand *regularity* is passed over in silence in traditional abstract theory, much in contrast to the obvious fact that both the Carathéodory and the Daniell-Stone procedures (plus its nonsequential counterpart due to Bourbaki) are of outer type and produce outer regular outcomes. But in topological theory the final word is a clear hint that *inner* regularity is far more important than outer one, in virtue of the predominant position of *compactness* in topology and expressed in the notion of *Radon measures*. At last both *nonsequential continuity* and *regularity* are as a rule related to *lattices* of subsets (Lebesgue measure on the real line is upward τ continuous on the lattice of open sets and downward τ continuous on the lattice of compact sets, but not far beyond). Thus these aspects enforce that *lattices* of subsets become much more important than they are in traditional abstract theory. This adds to the fact that as a rule primitive set functions in nature come on set systems which are at most as rich as lattices.

The deliberate attempts at unification started around 1970. A decisive prelude was the short paper of Kisyński [1968], which produced the final class of Borel-Radon measures on Hausdorff topological spaces via *inner regularity.* In no time then Topsøe [1970a][1970b] realized that this procedure opens the road to unification. However, these articles and the subsequent Pollard-Topsøe [1975] and Topsøe [1976][1978] did not yet present a full systematization. They appeared less coherent than the traditional abstract theory and thus did not find access to the textbooks, except to Fremlin [1974] and Berg-Christensen-Ressel [1984]. There were related but less distinctive ideas and results in Kelley-Srinivasan [1971] and Kelley-Nayak-Srinivasan [1973], and in Ridder [1971][1973].

The monumental treatise of Fremlin [2000], now under work for several years, has the aim to present an exhaustive description of measure and integration in both the abstract and topological theories. But the main accent is not to strive for new concepts and procedures in the interest of their unification, which is the expressed intention in the work of the present author. Even so it is plain that there are overlaps in spirit, in particular in the emphasis on inner procedures. It is also plain that certain theorems are in Fremlin [2000] in more comprehensive technical versions than in the work of the present author.

The present text wants to outline the attempt at unified systematization in measure and integration developed in the author's book [1997] (cited as MI) and in a series of subsequent papers. In contrast to all work mentioned above it is based on a framework of new concepts and new tools. The *basic concepts* are the *outer* and *inner* $\bullet$ *premeasures* and their *outer* and *inner* $\bullet$ *extensions.* Here $\bullet = \star\sigma\tau$ mark three parallel theories, where $\star$ stands for *finite*, σ for *sequential* or countable, and τ for *nonsequential* or arbitrary. These concepts express in definitive terms the ideas which come from the work of Topsøe. Then quite some time later and due to the present author came the *basic tools*, first of all the *outer* and *inner* $\bullet$ *envelopes* for the relevant set functions. Their task is to unfold the basic concepts. The concepts and tools for *set functions* will then obtain their precise counterparts for *functionals.* The overall model is that of the traditional abstract development of measure and integration. The new development is not more complicated than the old one, but much more powerful and comprehensive. In particular it is for the first time ever that abstract measure and integration contain their topological counterpart as an explicit specialization.

We terminate the introduction with a short look back to the initiation of the aforementioned new basic tools. The reader is asked to concede that we postpone some obvious definitions to the main text. We start with the classical theorem on the existence of measure extensions.

THEOREM. *Let* $\varphi : \mathfrak{S} \to [0,\infty]$ *be a content on a ring* $\mathfrak{S}$ *of subsets in a nonvoid set* X*. Then* φ *can be extended to a measure* $\alpha : \mathfrak{A} \to [0,\infty]$ *on a* σ *algebra* $\mathfrak{A}$ *in* X *iff it is upward* σ *continuous.*

This theorem does not meet the actual needs in several respects. First of all the initial domain $\mathfrak{S}$ should be a lattice instead of a ring. Then the fundamental aspects from topological measure theory, on the one hand upward and downward τ continuity and on the other hand outer and inner regularity, have to be incorporated into the extension procedure. For these aims also the usual proof due to Carathéodory [1914] does not help as it stands.

We recall that this proof is based on two remarkable formations. On the one hand one defines for a set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ the so-called *outer measure* $\varphi^\circ : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi^\circ(A) = \inf\{\sum_{l=1}^{\infty}\varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } \bigcup_{l=1}^{\infty} S_l \supset A\},$$

with the usual convention $\inf\varnothing := \infty$. Thus φ° is isotone with $\varphi^\circ(\varnothing) = 0$. On the other hand one defines for a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ the so-called *Carathodory class*

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A') + \Theta(M \cap A) \ \forall M \subset X\},$$

the members of which are called *measurable* Θ. One verifies that $\Theta|\mathfrak{C}(\Theta)$ is a content on the algebra $\mathfrak{C}(\Theta)$. Then to prove the nontrivial direction of the theorem one verifies that $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ is a measure on the σ algebra $\mathfrak{C}(\varphi^\circ)$ and an extension of φ.

We shall see that the formation $\mathfrak{C}(\cdot)$ is so well chosen that it will remain of constant use for the present purposes. In contrast, the specific form φ° of the outer measure must be blamed for much of the deficiencies around the extension theorem. First of all it seems that with φ° it is not possible to extend its present proof beyond the frame of rings. In fact, this is not possible for the class of lattices $\mathfrak{S}$ such that the differences $B \setminus A$ of pairs $A \subset B$ in $\mathfrak{S}$ are countable unions of members of $\mathfrak{S}$, a class which includes the lattice of compact subsets of $\mathbb{R}^n$, and on which the theorem will be seen to persist. Moreover φ° is of an obvious *sequential* type, but it is mysterious how a *nonsequential* counterpart could look. Also φ° is of an obvious *outer regular* type, but once more it is mysterious how an *inner regular* counterpart could look. After this the suspicion comes up that the *additive* character built into the definition of φ° is not only responsible for all these flaws, but is also the site for the veneration of that unfortunate spirit of countable additivity which we insisted should be abandoned.

The situation turned around with an innocent step which the present author took in an analysis course [1969/70], thus at the same time with Kisyński [1968] and Topsøe [1970a][1970b]: He observed that the old proof of the extension theorem carries over verbatim from rings to that particular class of lattices described above (of course with an adequate notion of content), provided that instead of φ° one uses the formation $\varphi^\sigma : \mathfrak{P}(X) \to [0,\infty]$, defined for an *isotone* set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ to be

$$\varphi^\sigma(A) = \inf\{\lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } S_l \uparrow \text{ some subset } \supset A\}.$$

Thus φ^σ is isotone with $\varphi^\sigma(\varnothing) = 0$ as well. The formations φ° and φ^σ are of course close relatives, and are in fact identical for contents on rings, but can be quite different beyond (like infinite series are equivalent to infinite sequences under the condition that one can form differences).

Much later then the author realized that the formation φ^σ is superior to the former φ° in the other relevant aspects as well. Thus φ^σ has an obvious *inner* counterpart $\varphi_\sigma : \mathfrak{P}(X) \to [0,\infty]$, defined via antitone set sequences instead of isotone ones [1985]. Moreover the two sequential formations φ^σ and φ_σ have obvious *nonsequential* counterparts φ^τ and φ_τ, defined in terms of upward and downward directed set systems. To all these one has to add the well-known *finite* outer and inner envelopes $\varphi^\star$ and $\varphi_\star$. Thus we end up with the *outer envelopes* $\varphi^\bullet : \mathfrak{P}(X) \to [0,\infty]$ and the *inner envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$ for $\bullet = \star\sigma\tau$, all defined for an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$. The entire collection appeared for the first time in the author's paper [1992b].

2. The Fundamentals for Set Functions

We adopt the usual notions and notations for set systems and set functions. The present section collects from MI the relevant new ones and their first properties. Moreover we want to demonstrate that the new tools are able to contribute to the unification of the additive and non-additive theories. As an illumination we insert an extended version of the capacitability theorem due to Choquet.

Set Systems

Let X be a nonvoid set. For a nonvoid set system $\mathfrak{S}$ in X we define $\mathfrak{S}^\star \subset \mathfrak{S}^\sigma \subset \mathfrak{S}^\tau$ to consist of the unions of its nonvoid finite/countable/arbitrary subsystems, and $\mathfrak{S}_\star \subset \mathfrak{S}_\sigma \subset \mathfrak{S}_\tau$ to consist of the respective intersections. Thus in the shorthand notation $\bullet = \star\sigma\tau$ we define $\mathfrak{S}^\bullet/\mathfrak{S}_\bullet$ to consist of the unions/intersections of the nonvoid $\bullet$ subsystems of $\mathfrak{S}$. We note that $\mathfrak{S}$ is a lattice iff $\mathfrak{S}^\star = \mathfrak{S}_\star = \mathfrak{S}$, and that in this case the $\mathfrak{S}^\bullet$ and $\mathfrak{S}_\bullet$ are lattices as well.

A nonvoid set system $\mathfrak{M}$ in X is called *upward/downward directed* iff for each pair $A, B \in \mathfrak{M}$ there exists an $M \in \mathfrak{M}$ such that $A, B \subset M/A, B \supset M$. We write $\mathfrak{M} \uparrow E/\mathfrak{M} \downarrow E$ when $\mathfrak{M}$ is upward/downward directed with union/intersection $E \subset X$, and $\mathfrak{M} \uparrow\supset E/\mathfrak{M} \downarrow\subset E$ when $\mathfrak{M}$ is upward/downward directed with union $\supset E$/intersection $\subset E$. We note that a nonvoid *finite* $\mathfrak{M}$ is upward/downward directed iff the union/intersection of its members is a member of $\mathfrak{M}$.

2.1 Lemma (MI 4.6 and 6.6). *Let $\mathfrak{S}$ be a lattice in X.*

Out) *For each $\mathfrak{M} \subset \mathfrak{S}^\bullet$ nonvoid $\bullet$ with $\mathfrak{M} \uparrow A$ there exists an $\mathfrak{N} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{N} \uparrow A$ such that each $N \in \mathfrak{N}$ is contained in some $M \in \mathfrak{M}$.*

Inn) *For each $\mathfrak{M} \subset \mathfrak{S}_\bullet$ nonvoid $\bullet$ with $\mathfrak{M} \downarrow A$ there exists an $\mathfrak{N} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{N} \downarrow A$ such that each $N \in \mathfrak{N}$ contains some $M \in \mathfrak{M}$.*

• Continuous and Regular Set Functions

Let $\mathfrak{S}$ be a nonvoid set system in X. A set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ is called *isotone* iff $\varphi(A) \leqq \varphi(B)$ for all $A \subset B$ in $\mathfrak{S}$. For the remainder of the subsection we assume an isotone set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$.

One defines the set function φ to be *upward/downward σ continuous* iff $\varphi(S_l) \uparrow \varphi(A)/\varphi(S_l) \downarrow \varphi(A)$ for all sequences $(S_l)_l$ in $\mathfrak{S}$ with $S_l \uparrow / \downarrow A \in \mathfrak{S}$. One verifies at once that this is true iff $\sup\limits_{M\in\mathfrak{M}} / \inf \varphi(M) = \varphi(A)$ for all nonvoid countable $\mathfrak{M} \subset \mathfrak{S}$ with $\mathfrak{M} \uparrow / \downarrow A \in \mathfrak{S}$. Thus one defines for $\bullet = \star\sigma\tau$ the set function φ to be *upward/downward $\bullet$ continuous* iff $\sup\limits_{M\in\mathfrak{M}} / \inf \varphi(M) = \varphi(A)$ for all $\mathfrak{M} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{M} \uparrow / \downarrow A \in \mathfrak{S}$. In case $\bullet = \star$ this is always true. An important variant is *almost upward/downward $\bullet$ continuous*, defined to mean that $\sup\limits_{M\in\mathfrak{M}} / \inf \varphi(M) = \varphi(A)$ is restricted to those $\mathfrak{M} \subset \mathfrak{S}$ which have $\varphi(M) > -\infty/\varphi(M) < \infty \ \forall M \in \mathfrak{M}$. One also defines these properties *at an individual* $A \in \mathfrak{S}$ and *at a nonvoid subsystem of* $\mathfrak{S}$.

Next let $\mathfrak{M} \subset \mathfrak{S}$ be a nonvoid subsystem. The set function φ is called *outer regular* $\mathfrak{M}$ (or *from* $\mathfrak{M}$) iff

$$\varphi(A) = \inf\{\varphi(M) : M \in \mathfrak{M} \text{ with } M \supset A\} \quad \text{for all } A \in \mathfrak{S},$$

and *inner regular* $\mathfrak{M}$ (or *from* $\mathfrak{M}$) iff

$$\varphi(A) = \sup\{\varphi(M) : M \in \mathfrak{M} \text{ with } M \subset A\} \quad \text{for all } A \in \mathfrak{S},$$

with the usual conventions $\inf \varnothing := \infty$ and $\sup \varnothing := -\infty$. One also defines these properties *at an individual* $A \in \mathfrak{S}$ and *at a nonvoid subsystem of* $\mathfrak{S}$.

The • Envelopes

Let $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ be an isotone set function on a nonvoid set system $\mathfrak{S}$ in X. We define for $\bullet = \star\sigma\tau$ the *outer $\bullet$ envelopes* $\varphi^\bullet : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ and the *inner $\bullet$ envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ for φ to be

$$\begin{aligned} \varphi^\bullet(A) &= \inf\{\sup_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \uparrow\supset A\}, \\ \varphi_\bullet(A) &= \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A\}. \end{aligned}$$

One obtains at once in case $\bullet = \sigma$ the reformulations

$$\begin{aligned}\varphi^\sigma(A) &= \inf\{\lim_{l\to\infty}\varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } S_l \uparrow\supset A\},\\ \varphi_\sigma(A) &= \sup\{\lim_{l\to\infty}\varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } S_l \downarrow\subset A\},\end{aligned}$$

as considered in the introduction, and in case $\bullet = \star$ the familiar reformulations

$$\begin{aligned}\varphi^\star(A) &= \inf\{\varphi(M) : M \in \mathfrak{S} \text{ with } M \supset A\},\\ \varphi_\star(A) &= \sup\{\varphi(M) : M \in \mathfrak{S} \text{ with } M \subset A\}.\end{aligned}$$

The envelopes $\varphi^\bullet$ and $\varphi_\bullet$ are isotone, and fulfil $\varphi^\star \geqq \varphi^\sigma \geqq \varphi^\tau$ and $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$. We note some further basic properties.

2.2 Properties (MI 4.1.4) and 6.3.4), 4.5 and 6.5). *Assume that $\mathfrak{S}$ is a lattice.*

1.Out) *$\varphi^\bullet$ is outer regular $\mathfrak{S}^\bullet$.* 1.Inn) *$\varphi_\bullet$ is inner regular $\mathfrak{S}_\bullet$.*

2.Out) *$\varphi^\star|\mathfrak{S} = \varphi$; and for $A \in \mathfrak{S}$ one has $\varphi^\bullet(A) = \varphi(A) \Leftrightarrow \varphi$ is upward $\bullet$ continuous at A.* 2.Inn) *$\varphi_\star|\mathfrak{S} = \varphi$; and for $A \in \mathfrak{S}$ one has $\varphi_\bullet(A) = \varphi(A) \Leftrightarrow \varphi$ is downward $\bullet$ continuous at A.*

3.Out) *If φ is upward $\bullet$ continuous then $\varphi^\bullet|\mathfrak{S}^\bullet = \varphi_\star|\mathfrak{S}^\bullet$, and this is upward $\bullet$ continuous as well.* 3.Inn) *If φ is downward $\bullet$ continuous then $\varphi_\bullet|\mathfrak{S}_\bullet = \varphi^\star|\mathfrak{S}_\bullet$, and this is downward $\bullet$ continuous as well.*

4.Out) *If φ is upward $\bullet$ continuous and $\{A \in \mathfrak{S}^\bullet : \varphi^\bullet(A) < \infty\} \subset \mathfrak{S}$ then $\varphi^\bullet = \varphi^\star$.* 4.Inn) *If φ is downward $\bullet$ continuous and $\{A \in \mathfrak{S}_\bullet : \varphi_\bullet(A) > -\infty\} \subset \mathfrak{S}$ then $\varphi_\bullet = \varphi_\star$.*

In conclusion we want to mention another kind of envelopes for φ, but above all in order to note that they are inferior to the above ones (see the next subsection). These formations are $\varphi^{(\bullet)} := (\varphi_\star|\mathfrak{S}^\bullet)^\star$ and $\varphi_{(\bullet)} := (\varphi^\star|\mathfrak{S}_\bullet)_\star$.

2.3 Properties (MI 6.10 and 6.11). *Assume that $\mathfrak{S}$ is a lattice.*

1.Out) *$\varphi^{(\bullet)} = \varphi_\star$ on $\mathfrak{S}^\bullet$, and hence $\varphi^{(\bullet)}|\mathfrak{S} = \varphi$.* 1.Inn) *$\varphi_{(\bullet)} = \varphi^\star$ on $\mathfrak{S}_\bullet$, and hence $\varphi_{(\bullet)}|\mathfrak{S} = \varphi$.*

2.Out) *$\varphi^\star \geqq \varphi^{(\bullet)} \geqq \varphi^\bullet$; and $\varphi^{(\bullet)} = \varphi^\bullet \Leftrightarrow \varphi$ is upward $\bullet$ continuous.* 2.Inn) *$\varphi_\star \leqq \varphi_{(\bullet)} \leqq \varphi_\bullet$; and $\varphi_{(\bullet)} = \varphi_\bullet \Leftrightarrow \varphi$ is downward $\bullet$ continuous.*

The Capacitability Theorem

We first recall the *Suslin sets.* Let $\mathbb{N}^\infty := \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ consist of all finite sequences of natural numbers, while as usual $\mathbb{N}^\mathbb{N}$ consists of all infinite sequences of natural numbers. The basic procedure in a nonvoid set X is to form for each family $(A(\lambda))_{\lambda\in\mathbb{N}^\infty}$ of subsets the so-called *kernel*

$$A = \bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} \bigcap_{n\in\mathbb{N}} A\big((\alpha(1),\cdots,\alpha(n))\big).$$

Then for a nonvoid set system $\mathfrak{S}$ in X one defines $\mathfrak{S}^\#$ to consist of the kernels A of all families $(A(\lambda))_{\lambda\in\mathbb{N}^\infty}$ in $\mathfrak{S}$. The members of $\mathfrak{S}^\#$ are the *Suslin sets for* $\mathfrak{S}$. As a rule the set system $\mathfrak{S}^\#$ is of enormous size, expressed in the formulas

$\mathfrak{S}^\sigma, \mathfrak{S}_\sigma \subset \mathfrak{S}^\#$ and $\mathfrak{S}^{\#\#} = \mathfrak{S}^\#$. Thus $\mathfrak{S}^\#$ is stable under countable unions and intersections. It can be a problem that $\mathfrak{S}^\#$ need not be stable under complement formation; but at least one notes that $\mathfrak{A} := \{A \subset X : A, A' \in \mathfrak{S}^\#\} \subset \mathfrak{S}^\#$ is a σ algebra.

After this we consider an isotone set function $\Phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ which is upward σ continuous. If $\mathfrak{S}$ is a lattice in X then Φ is called a *Choquet capacity for* $\mathfrak{S}$ iff moreover $\Phi|\mathfrak{S}_\sigma$ is downward σ continuous; after 2.1.Inn) it suffices to require that $\Phi(S_l) \downarrow \Phi(A)$ for all sequences $(S_l)_l$ in $\mathfrak{S}$ with $S_l \downarrow A \in \mathfrak{S}_\sigma$. Then the famous capacitability theorem due to Choquet reads as follows.

2.4 Theorem. *Let $\Phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ be a Choquet capacity for the lattice $\mathfrak{S}$ in X. Then $\Phi(A) \leqq (\Phi|\mathfrak{S}_\sigma)_\star(A)$ and hence $\Phi(A) = (\Phi|\mathfrak{S}_\sigma)_\star(A)$ for all $A \in \mathfrak{S}^\#$, that is Φ is inner regular $\mathfrak{S}_\sigma$ at $\mathfrak{S}^\#$.*

It turned out that the idea of proof in Choquet [1959] can be transferred to the new inner σ envelope. This leads to the result which follows.

2.5 Theorem (MI 10.5). *Let $\Phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ be isotone and upward σ continuous. If $\mathfrak{S}$ is a lattice in X then $\Phi(A) \leqq (\Phi|\mathfrak{S})_\sigma(A)$ for all $A \in \mathfrak{S}^\#$.*

The two results are connected via the next remark in case $\bullet = \sigma$. It combines 2.1.Inn) and 2.3.2.Inn) with routine manipulations.

2.6 Remark. *Let $\Phi : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ be isotone, and $\mathfrak{S}$ be a lattice in X. Then $(\Phi|\mathfrak{S}_\bullet)_\star \leqq (\Phi|\mathfrak{S})_{(\bullet)} \leqq (\Phi|\mathfrak{S})_\bullet$; and $(\Phi|\mathfrak{S}_\bullet)_\star = (\Phi|\mathfrak{S})_{(\bullet)} = (\Phi|\mathfrak{S})_\bullet$ iff $\Phi|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous.*

It follows that the new theorem 2.5 contains the familiar Choquet theorem 2.4, and is in fact a drastic extension because it assumes no connection between Φ and $\mathfrak{S}$.

We want to note that the new theorem 2.5 becomes false when instead of $(\Phi|\mathfrak{S})_\sigma$ one takes its variant $(\Phi|\mathfrak{S})_{(\sigma)}$, and a fortiori when one takes $(\Phi|\mathfrak{S}_\sigma)_\star$. This fact underlines the privileged position of the new envelopes.

2.7 Example. Let $X = \mathbb{R}$ and $\mathfrak{S} = \mathrm{Cl}(X)$, and let $E \subset X$ consist of the irrational numbers. Define $\Phi : \mathfrak{P}(X) \to \{0, 1\}$ to be $\Phi(A) = 0$ when A is contained in some $\mathfrak{S}^\sigma$ subset of E, and $\Phi(A) = 1$ otherwise. Thus Φ is isotone and upward σ continuous. We claim that $E \in \mathfrak{S}^\#$ and has $\Phi(E) = 1$ and $(\Phi|\mathfrak{S}_\sigma)_\star(E) = (\Phi|\mathfrak{S})_{(\sigma)}(E) = 0$. 1) We have $S \in \mathfrak{S} \Rightarrow S' \in \mathrm{Op}(X) \subset \mathfrak{S}^\sigma \subset \mathfrak{S}^\#$. Thus $\mathfrak{S} \subset \mathfrak{A} := \{A \subset X : A, A' \in \mathfrak{S}^\#\}$ and hence $\mathrm{Bor}(X) \subset \mathfrak{A} \subset \mathfrak{S}^\#$. In particular $E \in \mathfrak{S}^\#$. 2) The Baire category theorem implies that $E \notin \mathfrak{S}^\sigma$, because E' is countable. Thus $\Phi(E) = 1$. 3) We have $(\Phi|\mathfrak{S})_{(\sigma)}(E) = ((\Phi|\mathfrak{S})^\star|\mathfrak{S}_\sigma)_\star(E) = (\Phi|\mathfrak{S})_\star(E) = 0$. $\square$

Modular Set Functions

Let $\mathfrak{S}$ be a lattice in X. A set function $\varphi : \mathfrak{S} \to [0, \infty]$ is called *modular* iff

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \quad \text{for all } A, B \in \mathfrak{S},$$

and *sub/supermodular* iff

$$\varphi(A \cup B) + \varphi(A \cap B) \leqq / \geqq \varphi(A) + \varphi(B) \quad \text{for all } A, B \in \mathfrak{S}.$$

We place these notions behind the capacitability theorem, in order to emphasize the non-additive character of that theorem.

We recall that on a *ring* $\mathfrak{S}$ one defines $\varphi : \mathfrak{S} \to [0, \infty]$ to be a *content* iff $\varphi \not\equiv \infty$ and $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ for disjoint $A, B \in \mathfrak{S}$. It is equivalent to require that φ be isotone with $\varphi(\varnothing) = 0$ and modular. It is a little problem how to extend this notion to a *lattice* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$. While the present author tends to vote for the latter option, it must be noted that there are other choices in the literature. Thus Halmos [1950] section 53 requires φ to be isotone with $\varphi(\varnothing) = 0$, and to fulfil $\varphi(A \cup B) \leqq \varphi(A) + \varphi(B)$ for all $A, B \in \mathfrak{S}$ with $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ for disjoint $A, B \in \mathfrak{S}$. Here is an example: Let X be a set with at least three elements, and fix $c \in X$. Let $\mathfrak{S}$ consist of $\varnothing$ and of the finite subsets $S \subset X$ with $c \in S$. Then define $\varphi : \mathfrak{S} \to [0, \infty[$ to be $\varphi(S) = \sqrt{\operatorname{card}(S)}$. One verifies that φ has the last two properties but is not modular.

We conclude with some further basic properties. We note that the assumptions are more restrictive than in 2.2. Part 0) is a useful recapitulation.

2.8 PROPERTIES (MI 4.1.5) and 6.3.5), 4.7 and 6.7). *Let $\varphi : \mathfrak{S} \to [0, \infty]$ be an isotone set function with $\varphi(\varnothing) = 0$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$.*

0.Out) *$\varphi^\bullet : \mathfrak{P}(X) \to [0, \infty]$ with $\varphi^\bullet(\varnothing) = 0$.* 0.Inn) *$\varphi_\bullet : \mathfrak{P}(X) \to [0, \infty]$; and $\varphi_\bullet(\varnothing) = 0 \Leftrightarrow \varphi$ is downward $\bullet$ continuous at $\varnothing$.*

1.Out) *φ is submodular $\Rightarrow \varphi^\bullet$ is submodular.* 1.Inn) *φ is supermodular $\Rightarrow \varphi_\bullet$ is supermodular.*

2.Out) *If φ is submodular then φ^σ and φ^τ are upward σ continuous.* 2.Inn) *If φ is supermodular then φ_σ and φ_τ are almost downward σ continuous.*

Part 2) in its actual form comes as a surprise, because there are no assumptions of equal kind on φ itself. We note that the proof of 2.Out) will be simpler than in MI 4.7 when conducted as in the identical situation for functionals in [1998b] 3.4.Out).

3. The Outer and Inner $\bullet$ Extension Theorems

The outer and inner theories will be parallel in all essentials, except those typical little peculiarities which must be expected from the traditional context. For historical reasons the outer version looks more familiar, but the inner version appears to be the superior one at quite some important places. The development will be almost uniform in $\bullet = \star\sigma\tau$.

The Outer Situation

Let $\varphi : \mathfrak{S} \to [0, \infty]$ be an isotone set function with $\varphi(\varnothing) = 0$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in a nonvoid set X. We define an *outer $\bullet$ extension of* φ to be a content $\alpha:\mathfrak{A}\to[0,\infty]$ on a ring $\mathfrak{A}$ which is an extension of φ, and moreover fulfils $\mathfrak{S}\subset\mathfrak{S}^\bullet\subset\mathfrak{A}$ with the properties that

$\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous, and
α is outer regular $\mathfrak{S}^\bullet$.

One concludes at once that in this case φ is modular and upward $\bullet$ continuous, and that $\alpha=\varphi^\bullet|\mathfrak{A}$. In fact, α and $\varphi^\bullet$ are equal on $\mathfrak{S}$ by 2.2.2.Out), they are both upward $\bullet$ continuous on $\mathfrak{S}^\bullet$ by assumption and 2.2.3.Out) and hence coincide on $\mathfrak{S}^\bullet$, and they are both outer regular $\mathfrak{S}^\bullet$ at $\mathfrak{A}$ by assumption and 2.2.1.Out) and hence coincide on $\mathfrak{A}$.

After this we define φ to be an *outer* $\bullet$ *premeasure* iff it admits outer $\bullet$ extensions. The immediate problems are then to *characterize the outer* $\bullet$ *premeasures*, and for an outer $\bullet$ premeasure to *describe all its outer* $\bullet$ *extensions.* Both questions will be answered with the outer $\bullet$ extension theorem which follows.

We have to insert two notations. For nonvoid set systems $\mathfrak{P}$ and $\mathfrak{Q}$ in X we define the *transporter* $\mathfrak{P}\top\mathfrak{Q}:=\{A\subset X:P\in\mathfrak{P}\Rightarrow A\cap P\in\mathfrak{Q}\}$. And for a nonvoid set system $\mathfrak{M}$ we put $\mathfrak{M}\perp:=\{M':M\in\mathfrak{M}\}$.

3.1 Outer $\bullet$ Extension Theorem (MI 5.11 with 5.1 and 5.4). *Assume that* $\varphi:\mathfrak{S}\to[0,\infty]$ *is an isotone set function with* $\varphi(\varnothing)=0$ *on a lattice* $\mathfrak{S}$ *with* $\varnothing\in\mathfrak{S}$. *Then the following are equivalent.*

1) φ *is an outer* $\bullet$ *premeasure.*

2) φ *is submodular and upward* $\bullet$ *continuous; and* $\varphi(B)\geqq\varphi(A)+\varphi^\bullet(B\setminus A)$ *for all* $A\subset B$ *in* $\mathfrak{S}$. *Furthermore*

($\bullet$) $\quad\varphi^\bullet(A)=\sup\{\varphi^\bullet(A\cap S):S\in[\varphi<\infty]\}$ *for all* $A\in[\varphi^\bullet<\infty]$.

3) φ *is submodular and upward* $\bullet$ *continuous; and* $\varphi^\bullet(B)\geqq\varphi(A)+\varphi^\bullet(B\setminus A)$ *for all* $A\subset B$ *with* $A\in\mathfrak{S}$ *and* $B\in\mathfrak{S}^\bullet$.

In this case $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ *is an outer* $\bullet$ *extension of* φ, *and all outer* $\bullet$ *extensions of* φ *are restrictions of* $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$. *For* $\bullet=\sigma\tau$ *it is a measure on the* σ *algebra* $\mathfrak{C}(\varphi^\bullet)$.

Moreover $[\varphi<\infty]\top\mathfrak{C}(\varphi^\bullet)\subset\mathfrak{C}(\varphi^\bullet)$; *in particular* $[\varphi<\infty]\top\mathfrak{S}^\bullet\subset\mathfrak{C}(\varphi^\bullet)$.

We add at once that condition ($\bullet$) is superfluous for $\bullet=\star\sigma$, because in case $\bullet=\star$ it is obvious and in case $\bullet=\sigma$ it follows from 2.8.2.Out) when φ is submodular. But in case $\bullet=\tau$ it cannot be dispensed with (MI 4.11).

The prominent rôle of $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ as the unique maximal outer $\bullet$ extension of φ (and the same fact in the inner situation) emphasize the fundamental nature of the Carathéodory formation $\mathfrak{C}(\cdot)$, a nature which has been doubted in the literature at several places. In this connection we note that in the traditional abstract context the respective fact is false: The Carathéodory extension $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ of an upper σ continuous content $\varphi:\mathfrak{S}\to[0,\infty]$ on a ring $\mathfrak{S}$ need not be a maximal measure extension of φ.

3.2 Addendum. i) *If* φ *is upward* τ *continuous then* $\varphi^\tau|\mathfrak{S}\top\mathfrak{S}^\tau$ *is upward* τ *continuous* (this is obvious because 2.2.1.Out) implies that the $A\in\mathfrak{S}\top\mathfrak{S}^\tau$

with $\varphi^\tau(A) < \infty$ are in $\mathfrak{S}^\tau$). ii) ([1998a] 1.6) *If φ is an outer τ premeasure then $\varphi^\tau|(\mathfrak{S}\top\mathfrak{S}^\tau)\perp$ is almost downward τ continuous.*

3.3 Specialization. *Assume that $\mathfrak{S}$ is a ring. Then one has in* 3.1 *the equivalent condition*

3) *φ is modular and upward $\bullet$ continuous.* The case $\bullet = \sigma$ contains the traditional measure extension theorem.

In fact, the condition is necessary. So assume that it is fulfilled, and fix $A \subset B$ with $A \in \mathfrak{S}$ and $B \in \mathfrak{S}^\bullet$. Then $B \setminus A \in \mathfrak{S}^\bullet$ since $\mathfrak{S}$ is a ring. For $S \in \mathfrak{S}$ with $S \subset B \setminus A$ we have $\varphi^\bullet(B) \geqq \varphi(A \cup S) = \varphi(A) + \varphi(S)$. It follows that $\varphi^\bullet(B) \geqq \varphi(A) + \varphi^\bullet(B \setminus A)$, and thus the previous condition 3). $\square$

3.4 Example. We conclude with the simplest possible example. Let $\mathfrak{S}$ consist of the finite subsets of a nonvoid set X, and let $\varphi : \mathfrak{S} \to [0, \infty[$ be $\varphi = 0$. After 3.3 φ is an outer $\bullet$ premeasure for all $\bullet = \star\sigma\tau$. For $A \subset X$ we have

$\varphi^\star(A) = 0$ when A is finite, and $= \infty$ otherwise;
$\varphi^\sigma(A) = 0$ when A is countable, and $= \infty$ otherwise;
$\varphi^\tau(A) = 0$ for all A.

Each time the whole $\varphi^\bullet$ is an outer $\bullet$ extension of φ, and hence is the unique maximal outer $\bullet$ extension on $\mathfrak{C}(\varphi^\bullet) = \mathfrak{P}(X)$.

The Inner Situation

For the inner counterpart one has to be aware from the traditional context that measures as a rule are *not* downward σ continuous but *almost* downward σ continuous. It turned out that the adequate conclusion is to start the inner development from set functions with *finite* values, at least for $\bullet = \sigma\tau$.

Thus let $\varphi : \mathfrak{S} \to [0, \infty[$ be an isotone set function with $\varphi(\varnothing) = 0$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$. We define an *inner $\bullet$ extension of φ* to be a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A}$ which is an extension of φ, and moreover fulfils $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{A}$ with the properties that

$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$), and
α is inner regular $\mathfrak{S}_\bullet$.

As before one concludes that in this case φ is modular and downward $\bullet$ continuous, and that $\alpha = \varphi_\bullet|\mathfrak{A}$. Next we define φ to be an *inner $\bullet$ premeasure* iff it admits inner $\bullet$ extensions. We are faced with the immediate problems and shall obtain the basic answers as before.

However, the two theories are somewhat distinct in two points. On the one hand the trouble with condition ($\bullet$) in the outer situation does not occur in the inner one. On the other hand the inner situation involves the *two conditions*

φ downward $\bullet$ continuous, equivalent to $\varphi_\bullet|\mathfrak{S} = \varphi$, and
φ downward $\bullet$ continuous at $\varnothing$, equivalent to $\varphi_\bullet(\varnothing) = 0$,

which turn out to be quite different. As a rule it is much easier to confirm the second condition. In fact, there is a frequent case where the second condition is obvious while the first one need not even be fulfilled: A lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ is called $\bullet$ *compact* iff each $\mathfrak{M} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{M} \downarrow \varnothing$ has $\varnothing \in \mathfrak{M}$, a notion of an obvious topological flavour. In this case of course all set functions φ under consideration are downward $\bullet$ continuous at $\varnothing$.

Thus for the inner $\bullet$ extension theorem there is vital interest in an equivalent statement which contains the second condition above instead of the first one. To this end we form certain *satellites* of the inner $\bullet$ envelopes $\varphi_\bullet$ of φ: For $B \in \mathfrak{S}$ we define $\varphi_\bullet^B : \mathfrak{P}(X) \to [0,\infty[$ to be

$$\varphi_\bullet^B(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A \\ \text{and } M \subset B \text{ for all } M \in \mathfrak{M}\}.$$

These satellites are isotone and fulfil $\sup\limits_{B\in\mathfrak{S}} \varphi_\bullet^B = \varphi_\bullet$; moreover $\varphi_\star(A) \leqq \varphi_{(\bullet)}(A) \leqq \varphi_\bullet^B(A)$ when $A \subset B$. After this we can formulate the inner main theorem.

3.5 Inner $\bullet$ Extension Theorem (MI 6.31 with 6.18 and 6.21). *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is an isotone set function with $\varphi(\varnothing) = 0$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$. Then the following are equivalent.*

1) *φ is an inner $\bullet$ premeasure.*

2) *φ is supermodular and downward $\bullet$ continuous; and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

3) *φ is supermodular and downward $\bullet$ continuous at $\varnothing$; and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

In this case $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ, and all inner $\bullet$ extensions of φ are restrictions of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. For $\bullet = \sigma\tau$ it is a measure on the σ algebra $\mathfrak{C}(\varphi_\bullet)$.

Moreover $\mathfrak{S}\top\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$; in particular $\mathfrak{S}\top\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$.

We add at once that in 2) one cannot substitute downward $\bullet$ continuous at $\varnothing$ for downward $\bullet$ continuous (MI 6.36).

3.6 Addendum (MI 6.27 and 6.28). i) *If φ is supermodular and downward τ continuous then $\varphi_\tau|\mathfrak{S}\top\mathfrak{S}_\tau$ is almost downward τ continuous.* ii) *If φ is an inner τ premeasure then $\varphi_\tau|(\mathfrak{S}\top\mathfrak{S}_\tau)\perp$ is upward τ continuous.*

3.7 Specialization. *Assume that $\mathfrak{S}$ is a ring. Then one has in* 3.5 *the equivalent condition*

3) *φ is modular and downward $\bullet$ continuous at $\varnothing$.*

In fact, the condition is necessary. So assume that it is fulfilled, and fix $A \subset B$ in $\mathfrak{S}$. From $B \setminus A \in \mathfrak{S}$ then $\varphi(B \setminus A) \leqq \varphi_\bullet^B(B \setminus A)$ and hence $\varphi(B) = \varphi(A) + \varphi(B \setminus A) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$, and thus the previous condition 3). $\square$

Some Further Remarks

We start with a note on the comparison with other authors. There is a detailed account in the bibliographical annex of MI section 7. We have said that the $\bullet$ envelopes $\varphi^\bullet$ and $\varphi_\bullet$ for $\bullet = \sigma\tau$ do not appear elsewhere. In these cases $\bullet = \sigma\tau$ one often works with $\varphi^\star$ and $\varphi_\star$ instead. Thus one formulates in the outer $\bullet$ context 3.1 condition 2) with $\varphi^\star$ instead of $\varphi^\bullet$ (this time but for $\bullet = \sigma$ and hence without ($\bullet$)), and in the inner $\bullet$ context 3.5 conditions 2)3) with $\varphi_\star$ instead of $\varphi_\bullet$ and its satellites. It is plain that these modifications are then *sufficient* conditions for the respective 1), but they cease to be *equivalent* conditions (MI 5.12 and 6.32). This also applies to Fremlin [2000] section 413.

It will become clear that to abstain from the new $\bullet = \sigma\tau$ envelopes in favour of the old $\bullet = \star$ ones could have severe consequences. In fact, it would mean to loose the adequate access to some of the most pronounced achievements of the present development, like the *true* classical measure extension theorem 4.1 below, and the basic results on the formation of products and representation of functionals in sections 6 and 7.

We continue with a few complements to the previous subsections. The first point is a useful note on outer and inner $\bullet$ premeasures. It has a routine proof.

3.8 Remark. *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$.*

Out) *The outer $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty]$ and the outer $\bullet$ premeasures $\psi : \mathfrak{S}^\bullet \to [0,\infty]$ are in one-to-one correspondence via $\psi = \varphi^\bullet|\mathfrak{S}^\bullet$ and $\varphi = \psi|\mathfrak{S}$. Moreover then $\varphi^\bullet = \psi^\bullet = \psi^\star$.*

Inn) *The inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ and the inner $\bullet$ premeasures $\psi : \mathfrak{S}_\bullet \to [0,\infty[$ are in one-to-one correspondence via $\psi = \varphi_\bullet|\mathfrak{S}_\bullet$ and $\varphi = \psi|\mathfrak{S}$. Moreover then $\varphi_\bullet = \psi_\bullet = \psi_\star$.*

3.9 Remark. *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$.*

Out) *Let $\varphi : \mathfrak{S} \to [0,\infty]$ be an outer $\bullet$ premeasure with $\Phi = \varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$. Then $\Phi^\star = \varphi^\bullet$.*

Inn) *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. Then $\Phi_\star = \varphi_\bullet$.*

In fact, we have for example in the outer case $\Phi^\star = \Phi = \varphi^\bullet$ on $\mathfrak{C}(\varphi^\bullet)$, and both sides are outer regular $\mathfrak{C}(\varphi^\bullet)$. $\square$

The last point is a remarkable connection between the outer and inner $\bullet$ extension theories for $\bullet = \star\sigma$; the case $\bullet = \tau$ is not realistic here. We note that there will be another and more important such connection in 4.6 below.

3.10 Theorem (MI 7.5). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$. Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is an outer and inner $\bullet$ premeasure, where $\bullet = \star\sigma$. Then $\mathfrak{C}(\varphi^\bullet) = \mathfrak{C}(\varphi_\bullet) =: \mathfrak{C}$. Moreover $\varphi^\bullet(A) = \varphi_\bullet(A)$ for all $A \in \mathfrak{C}$ which are contained in some member of $\mathfrak{S}^\bullet$.*

We conclude with a brief account of the framework built in MI in order to prove the outer and inner $\bullet$ extension theorems. It looks more comprehensive than needed, but we shall soon see the reason. The basic idea is the use of *isotone set functions with values in $\overline{\mathbb{R}}$*, thus as it seems of an odd kind of set functions.

We need some preparations. On $\overline{\mathbb{R}}$ one has the usual order $\leqq$ and the two extended additions $\dot{+}$ and $\underset{\cdot}{+}$ which are associative and commutative with $\infty \dot{+} (-\infty) = \infty$ and $\infty \underset{\cdot}{+} (-\infty) = -\infty$. On the domain side one has to refrain from the usual rôle of $\varnothing$. Thus one can retain lattices, of course without to require that they contain $\varnothing$, but one has to abandon rings in favour of so-called *ovals*, defined to be stable under $A, B, C \mapsto (A \cap C') \cup (B \cap C)$. One defines a $\dot{+}/\underset{\cdot}{+}$ *content* to be an isotone set function $\alpha : \mathfrak{A} \to \overline{\mathbb{R}}$ on an oval $\mathfrak{A}$ which is modular under $\dot{+}/\underset{\cdot}{+}$ and attains at least one finite value. Then a basic tool is the bijection $\mathfrak{P}(X) \to \mathfrak{P}(X)$ which maps $M \mapsto M'$. It carries lattices into lattices, ovals into ovals, also algebras into algebras, but not rings into rings. We form the transposed operation which maps a set function $\varphi : \mathfrak{M} \to \overline{\mathbb{R}}$ to the set function $\varphi\perp : \mathfrak{M}\perp \to \overline{\mathbb{R}}$, defined to be $\varphi\perp(M) = -\varphi(M')$. It carries isotone into isotone, sub/supermodular under $\dot{+}$ into super/submodular under $\underset{\cdot}{+}$, and hence $\dot{+}$ contents into $\underset{\cdot}{+}$ contents. So far the preparations.

On this basis we form what we call here the *odd outer* and *inner situations.* For the *outer* situation let $\varphi : \mathfrak{S} \to]-\infty, \infty]$ be an isotone set function $\not\equiv \infty$ on a lattice $\mathfrak{S}$ (we do *not* assume $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ since this would lead to serious technical problems which would outdo all possible profits). We define the *odd outer* $\bullet$ *extensions* of φ to be the $\dot{+}$ contents which extend φ and otherwise have the same properties as before, and then as before define φ to be an *odd outer* $\bullet$ *premeasure* iff it admits odd outer $\bullet$ extensions. Then one proves for this *odd outer situation* the appropriate version of the outer $\bullet$ extension theorem (MI 5.5 with 5.1 and 5.4). The unique point of no routine is to find the adequate extension of the Carathodory formation $\mathfrak{C}(\cdot)$.

For the *inner* situation one assumes $\varphi : \mathfrak{S} \to [-\infty, \infty[$ isotone $\not\equiv -\infty$ on a lattice $\mathfrak{S}$, and defines the inner counterparts of the odd concepts above. One observes that the *upside-down transformation* discussed above maps the set functions $\varphi : \mathfrak{S} \to]-\infty, \infty]$ for the outer situation onto the set functions $\varphi\perp : \mathfrak{S}\perp \to [-\infty, \infty[$ for the inner situation, moreover the odd outer $\bullet$ extensions of φ onto the odd inner $\bullet$ extensions of $\varphi\perp$, and hence the odd outer $\bullet$ premeasures φ on $\mathfrak{S}$ onto the odd inner $\bullet$ premeasures $\varphi\perp$ on $\mathfrak{S}\perp$. It is not hard to pursue the upside-down transformation to the point that the odd outer $\bullet$ extension theorem passes into an odd inner $\bullet$ extension theorem of the same shape (MI 6.22 with 6.18 and 6.21). There is but one little problem with the notion of transporter, because it does not behave well under the upside-down transformation. Thus no effort ab ovo is required. The two theorems are of *complete similitude*, and it can be said that they are *equal.*

After this we return to the context of the present article. It is plain that the present outer and inner $\bullet$ extension theorems 3.1 and 3.5 are *not* of complete similitude, and that there is no immediate explanation. But after the above excursion we see that the *present* outer and inner $\bullet$ premeasures φ are those among the *odd* ones which live on lattices $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and fulfil $\varphi(\varnothing) = 0$. This explains first of all that the present inner situation requires $\varphi < \infty$. Also the other deviations receive the simple explanation that they have been *produced under the involved cut-off.* In particular condition ($\bullet$) in 3.1, when formulated within the odd outer $\bullet$ theorem and then transferred to the odd inner $\bullet$ theorem, attains a form which makes it disappear after that cut-off.

What remains is the question why we did not retain the more comprehensive framework of MI. The simple answer is that the basic notions in that part of MI do not seem to fit (maybe not yet) the actual concepts in mathematical analysis and its applications. So this point must be left to the future.

4. Consequences and Applications

We present a few consequences and applications of the outer and inner $\bullet$ extension theorems which can be done without the Choquet integral, to be introduced in the next section. They are all of prime importance. We include some of their proofs from the previous results, in order to show how simple these proofs are.

The Specialization to Rings Extended

The present result extends the specialization 3.3 from rings to the more realistic class of lattices referred to in the introduction.

4.1 Theorem (MI 7.12). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ such that $B \setminus A \in \mathfrak{S}^\sigma$ for all $A \subset B$ in $\mathfrak{S}$. Assume that $\varphi : \mathfrak{S} \to [0, \infty]$ is isotone with $\varphi(\varnothing) = 0$, and modular and upward σ continuous. Then*

1) *φ is an outer σ premeasure.*

2) *If $\varphi < \infty$ then φ is an inner σ premeasure.*

Proof. 1) Fix $A \subset B$ in $\mathfrak{S}$, and let $(S_l)_l$ be a sequence in $\mathfrak{S}$ with $S_l \uparrow B \setminus A$. Then $\varphi(A) + \varphi(S_l) = \varphi(A \cup S_l) \uparrow \varphi(B)$ and hence $\varphi(A) + \varphi^\sigma(B \setminus A) \leqq \varphi(A) + \lim\limits_{l \to \infty} \varphi(S_l) = \varphi(B)$. The result follows from 3.1. 2) In the same situation we have $\varphi(B) = \varphi(A) + \lim\limits_{l \to \infty} \varphi(S_l) \leqq \varphi(A) + \varphi_\star(B \setminus A) \leqq \varphi(A) + \varphi_\sigma(B \setminus A)$. Moreover we know from 1) that $\varphi^\sigma | \mathfrak{C}(\varphi^\sigma)$ is a measure which extends φ, so that $\varphi < \infty$ is downward σ continuous. The result follows from 3.5. $\square$

The above theorem is much more useful than the former specializations 3.3 and 3.7, and a fortiori than the classical measure extension theorem.

4.2 Examples. i) The result leads with little effort to the Lebesgue measure on $X = \mathbb{R}^n$. Let $\mathfrak{S}$ consist of the finite unions of compact intervals, and let $\varphi : \mathfrak{S} \to [0, \infty[$ be the elementary content. It is obvious that $\mathfrak{S}$ fulfils the assumption, and it is simple via compactness to see that φ is upward σ continuous. Then it suffices to note that $\mathfrak{S}^\sigma$ contains the open subsets and $\mathfrak{S}_\sigma$ consists of the compact subsets of $\mathbb{R}^n$.

ii) Let X be a topological space, and let $\mathrm{CCl}(X) := \{[f \geqq 0] : f \in \mathrm{C}(X, \mathbb{R})\}$ denote the lattice of its so-called zero subsets. It is obvious that $\mathfrak{S} = \mathrm{CCl}(X)$ fulfils the assumption. Thus the theorem permits to extend certain elementary contents on $\mathrm{CCl}(X)$ to certain Baire measures on X (MI 8.6).

Despite its power the basic idea of the theorem does not appear in the literature, as far as the author is aware, and the two examples either, except in MI and his related work. The explanation is that the decisive implication 4.1.1) requires the outer σ envelope φ^σ and is not accessible via $\varphi^\star$ (MI 5.12), and that φ^σ does not appear elsewhere.

Radon Premeasures

The present subsection deals with the site where the actual $\bullet$ extension theories came into existence. We fix a Hausdorff topological space X and consider the lattice $\mathfrak{S} = \mathrm{Comp}(X)$ of its compact subsets. For the sequel it is decisive that $\mathfrak{S}$ at the same time fulfils $\mathfrak{S} = \mathfrak{S}_\bullet$ and is $\bullet$ compact in the sense of the last section, for all $\bullet = \star\sigma\tau$.

In fact, let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$ and supermodular. By the definitions the first of the above properties shows that

i) φ inner $\bullet$ premeasure $\Leftrightarrow$ φ inner $\star$ premeasure and downward $\bullet$ continuous,

while by 3.5.3) the second of the properties implies that

ii) φ inner $\bullet$ premeasure $\Leftrightarrow$ $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.

Therefore to be an inner $\bullet$ premeasure, called $\bullet$ for short, obeys the implications

$$\star \Leftarrow \sigma \Leftarrow \tau \text{ in view of i), and } \quad \star \Rightarrow \sigma \Rightarrow \tau \text{ in view of ii),}$$

and hence is in fact independent of $\bullet = \star\sigma\tau$. Thus combined with 2.2.4.Inn) we obtain the result which follows.

4.3 Theorem. *For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ on $\mathfrak{S} = \mathrm{Comp}(X)$ the following are equivalent.*

1$\bullet$) (for the individual $\bullet = \star\sigma\tau$) *$\varphi$ is an inner $\bullet$ premeasure.*

2) *φ is supermodular; and $\varphi(B) \leqq \varphi(A) + \varphi_\star(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

In this case φ is of course downward τ continuous. Moreover $\varphi_\star = \varphi_\sigma = \varphi_\tau$. These set functions will be called the *Radon premeasures* on X.

The above theorem 4.3 is the initial result of Kisyński [1968], when one adds an obvious observation on the domain of the maximal inner $\bullet$ extension $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ of φ: From the last statement in 3.5 we see that $\mathrm{Cl}(X) \subset \mathfrak{S}\top\mathfrak{S} \subset \mathfrak{C}(\varphi_\bullet)$ and hence $\mathrm{Bor}(X) = \mathrm{A}\sigma(\mathrm{Cl}(X)) \subset \mathfrak{C}(\varphi_\bullet)$, so that Φ comprises a Borel measure on X. Thus the Radon premeasures on X are in one-to-one correspondence with their inner $\bullet$ extensions to $\mathrm{Bor}(X)$, that is to those measures $\alpha : \mathrm{Bor}(X) \to [0,\infty]$ which have $\alpha|\mathrm{Comp}(X) < \infty$ and are inner regular $\mathrm{Comp}(X)$. Moreover one obtains $\mathfrak{S}\top\mathrm{Bor}(X) \subset \mathfrak{S}\top\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$, so that the maximal domain also contains the class $\mathrm{LocBor}(X) := \mathrm{Comp}(X)\top\mathrm{Bor}(X)$ of the *local Borel subsets* of X. In contrast to $\mathrm{Bor}(X)$ this class can be written $\mathrm{LocBor}(X) = \mathrm{Comp}(X)\top\mathrm{A}\sigma(\mathrm{Comp}(X))$ in terms of $\mathfrak{S} = \mathrm{Comp}(X)$ (MI 1.20).

In the literature one sometimes restricts oneself to *locally finite* Radon premeasures $\varphi : \mathfrak{S} \to [0,\infty[$, defined to mean that each $S \in \mathfrak{S}$ is contained in some open $T \in \mathrm{Op}(X)$ with $\Phi(T) < \infty$. One reason is perhaps that the concept started in the frame of *locally compact* Hausdorff spaces X, where of course all Radon premeasures are locally finite. There was no restriction to local finiteness in Kisyński [1968], in contrast to Bourbaki [1952][1956][1969]. The restriction was first removed from the systematic context in Berg-Christensen-Ressel [1984]. We want to present a simple example of a Radon premeasure which is not locally finite.

4.4 Example. 1) We equip $X = \mathbb{N} \cup \{\infty\}$ with a Hausdorff topology which is non-discrete but such that all compact subsets of X are finite [1993]: The system $\mathrm{Op}(X)$ consists i) of all $U \subset \mathbb{N}$, and ii) of all those $U \subset X$ with $\infty \in U$ such that $U' \subset \mathbb{N}$ is *small* in the sense that $\sum_{n \in U'} 1/n < \infty$. Thus it is a non-discrete Hausdorff topology on X. Then if $K \subset X$ is infinite we take an infinite subset $E \subset K \cap \mathbb{N}$ such that $\sum_{n \in E} 1/n < \infty$, and find that E' and the $\{x\}$ $\forall x \in E$ form an open cover of K which has no finite subcover. 2) Now define $\varphi : \mathfrak{S} = \mathrm{Comp}(X) \to [0, \infty[$ to be $\varphi(S) = \mathrm{card}(S \cap \mathbb{N})$. Then φ is a Radon premeasure on X. It has $\varphi_\bullet(A) = \mathrm{card}(A \cap \mathbb{N})$ for all $A \subset X$, and therefore $\mathfrak{C}(\varphi_\bullet) = \mathfrak{P}(X)$. It is of course not locally finite.

The Radon premeasures on X can also be characterized as the isotone set functions $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ on $\mathfrak{S} = \mathrm{Comp}(X)$ which are *modular* and *downward τ continuous* (MI 9.6). The relevant ideas are due to Choquet [1953/54] section 26.6 and Bourbaki [1969] section 3.1. See also [2000b] 3.3.

We conclude with the notion of *support*. Let $\varphi : \mathfrak{S} \to [0, \infty[$ be an inner τ premeasure on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in a nonvoid set X. Since $(\mathfrak{S} \top \mathfrak{S}_\tau)\perp$ is stable under unions, and since after 3.6.ii) the restriction $\Phi|(\mathfrak{S} \top \mathfrak{S}_\tau)\perp$ of $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$ is upward τ continuous, the union V of all $U \in (\mathfrak{S} \top \mathfrak{S}_\tau)\perp$ with $\Phi(U) = 0$ is of the same kind. We define $V' \in \mathfrak{S} \top \mathfrak{S}_\tau$ to be the *support* of φ, denoted $\mathrm{supp}(\varphi)$.

In case of a Radon premeasure $\varphi : \mathfrak{S} = \mathrm{Comp}(X) \to [0, \infty[$ on a Hausdorff topological space X one of course wants to compare $\mathrm{supp}(\varphi)$ with the usual *closed support* $\mathrm{Supp}(\Phi)$ of $\Phi = \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$. $\mathrm{Supp}(\Phi)$ is defined to be the complement $W' \in \mathrm{Cl}(X)$ of the unique maximal $W \in \mathrm{Op}(X)$ with $\Phi(W) = 0$. One proves that $\mathrm{Supp}(\Phi) = \overline{\mathrm{supp}(\varphi)}$ (MI 9.19). We shall see that $\mathrm{supp}(\varphi)$ need not be closed.

4.5 Example. In example 4.4 we have $\mathfrak{S} \top \mathfrak{S} = \mathfrak{P}(X)$ and hence $(\mathfrak{S} \top \mathfrak{S})\perp = \mathfrak{P}(X)$. Therefore $\mathrm{supp}(\varphi) = \mathbb{N}$. But $\mathrm{Supp}(\Phi) = X$, because $U = \varnothing$ is the unique open subset of X with $\Phi(U) = 0$. Thus in this example $\mathrm{supp}(\varphi)$ seems to be the better choice.

Complemental Couples of • Premeasures

Outer and inner • premeasures and their maximal outer and inner • extensions seem, after the cut-off at $\varnothing$ described at the end of the last section, to conduct their independent lives. More and more the emphasis moves to the inner situation, and one could think that there is no immanent connection between the outer and inner situations. Yet there exists a fundamental connection of this kind, at least under certain natural and frequent additional assumptions, as the present subsection will reveal. It comprises several previous ideas and results. Also there will be further evidence of the predominance of the inner situation.

We define a couple of lattices $\mathfrak{S}$ and $\mathfrak{T}$ with $\varnothing$ in a nonvoid set X to be $\bullet$ *complemental* iff $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$ and $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T}^\bullet)\perp$. The relevant examples are as follows. 1) The pair $\mathfrak{S} = \mathrm{Cl}(X)$ and $\mathfrak{T} = \mathrm{Op}(X)$ in a topological space X, and the pair $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Op}(X)$ when X is Hausdorff. 2) An arbitrary $\mathfrak{S}$ with $\mathfrak{T} = (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$, and an arbitrary $\mathfrak{T}$ with $\mathfrak{S} = (\mathfrak{T}\top\mathfrak{T}^\bullet)\perp$. Thus for example in a Hausdorff space X the lattice $\mathfrak{S} = \mathrm{Comp}(X)$ can have the companions $\mathfrak{T} = \mathrm{Op}(X)$ and $\mathfrak{T} = (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp$, and example 4.4 shows that the two can be different.

Next assume that $\mathfrak{S}$ and $\mathfrak{T}$ form a $\bullet$ complemental couple. We define an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ to be $\bullet$ *tame for* $\mathfrak{S}$ *and* $\mathfrak{T}$ iff $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is outer regular $\mathfrak{T}^\bullet$ at $\mathfrak{S}$ (and hence at $\mathfrak{S}_\bullet$); note that $\mathfrak{T}^\bullet \subset (\mathfrak{S}\top\mathfrak{S}_\bullet)\perp \subset \mathfrak{C}(\varphi_\bullet)$. Equivalent is the much simpler condition that each $S \in \mathfrak{S}$ (and hence each $S \in \mathfrak{S}_\bullet$) be contained in some $T \in \mathfrak{T}^\bullet$ with $\Phi(T) < \infty$ ([1998a] 4.3); in this case φ could be named $\bullet$ *locally finite for* $\mathfrak{S}$ *and* $\mathfrak{T}$. Likewise one defines an outer $\bullet$ premeasure $\psi : \mathfrak{T} \to [0,\infty]$ to be $\bullet$ *tame for* $\mathfrak{S}$ *and* $\mathfrak{T}$ iff $\Psi = \psi^\bullet|\mathfrak{C}(\psi^\bullet)$ has $\Psi|\mathfrak{S} < \infty$ (and hence $\Psi|\mathfrak{S}_\bullet < \infty$) and is inner regular $\mathfrak{S}_\bullet$ at $\mathfrak{T}$ (and hence at $\mathfrak{T}^\bullet$); as above note that $\mathfrak{S}_\bullet \subset (\mathfrak{T}\top\mathfrak{T}^\bullet)\perp \subset \mathfrak{C}(\psi^\bullet)$. After this the promised fundamental connection reads as follows.

4.6 Theorem ([1998a] 4.6). *Assume that the lattices* $\mathfrak{S}$ *and* $\mathfrak{T}$ *form a* $\bullet$ *complemental couple. Then*

the inner $\bullet$ *premeasures* $\varphi : \mathfrak{S} \to [0,\infty[$ *which are* $\bullet$ *tame for* $\mathfrak{S}$ *and* $\mathfrak{T}$*, and*
the outer $\bullet$ *premeasures* $\psi : \mathfrak{T} \to [0,\infty]$ *which are* $\bullet$ *tame for* $\mathfrak{S}$ *and* $\mathfrak{T}$

are in one-to-one correspondence via each of the two maps

$\varphi \mapsto \psi := \Phi|\mathfrak{T}$ *for* $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$*, and*
$\psi \mapsto \varphi := \Psi|\mathfrak{S}$ *for* $\Psi = \psi^\bullet|\mathfrak{C}(\psi^\bullet)$*,*

which are inverse to each other. Under this correspondence we have $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\psi^\bullet) =: \mathfrak{C}$ *and* $\Phi \leqq \Psi$*, and* $\Phi = \Psi$ *on* $\mathfrak{S}_\bullet$ *and* $\mathfrak{T}^\bullet$ *and* $[\Psi < \infty]$*. Moreover* $\varphi_\bullet \leqq \psi^\bullet$*.*

We define a couple of an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ and an outer $\bullet$ premeasure $\psi : \mathfrak{T} \to [0,\infty]$ to be $\bullet$ *complemental* iff $\mathfrak{S}$ and $\mathfrak{T}$ form a $\bullet$ complemental couple, and φ and ψ are $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{T}$ and correspond to each other as described in 4.6 above.

4.7 Example. The situation $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Op}(X)$ in a Hausdorff space X is due to Schwartz [1973]. A Radon premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ is $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{T}$ iff it is locally finite in the former sense (Schwartz considers but these locally finite ones). Therefore the class of the outer $\bullet$ premeasures $\psi : \mathfrak{T} \to [0,\infty]$ which are $\bullet$ tame for $\mathfrak{S}$ and $\mathfrak{T}$ must be independent of $\bullet = \star\sigma\tau$

too. Schwartz views these entities in form of their inner and outer $\bullet$ extensions to $\mathrm{Bor}(X)$. Thus he writes that

the measures $\alpha : \mathrm{Bor}(X) \to [0,\infty]$ with $\alpha|\mathfrak{S} < \infty$ which are inner regular $\mathfrak{S}$ and locally finite, and

the measures $\beta : \mathrm{Bor}(X) \to [0,\infty]$ with $\beta|\mathfrak{S} < \infty$ which are inner regular $\mathfrak{S}$ at $\mathfrak{T}$ and outer regular $\mathfrak{T}$

are in one-to-one correspondence. Under this correspondence one has $\alpha \leqq \beta$, and $\alpha = \beta$ on $\mathfrak{S}$ and $\mathfrak{T}$ and $[\beta < \infty]$. It is nontrivial to note that $\alpha \neq \beta$ can happen even for locally compact X. The simplest example known to the author is due to Dowker ([1998a] 4.8); we note that it is related to example 6.6 below. In the textbooks one meets, in most cases restricted to locally compact spaces, sometimes both the locally finite *Borel-Radon measures* $\alpha : \mathrm{Bor}(X) \to [0,\infty]$ and their companions $\beta : \mathrm{Bor}(X) \to [0,\infty]$, for example in Bauer [1990], but the older textbooks are often confined to the latter companions alone, for example Rudin [1966] and Cohn [1980].

In conclusion we want to reformulate the last assertion in 4.6 on the two maximal $\bullet$ extensions Φ and Ψ. To this end we recall a certain *difference formation* from [1999a] section 1: For a couple of contents $\alpha, \beta : \mathfrak{A} \to [0,\infty]$ on an algebra $\mathfrak{A}$ one defines $\beta \setminus \alpha : \mathfrak{A} \to [0,\infty]$ to be

$$(\beta \setminus \alpha)(A) = \sup\{\beta(K) - \alpha(K) : K \in \mathfrak{A} \text{ with } K \subset A \text{ and } \alpha(K) < \infty\}.$$

Then $\beta \setminus \alpha$ is a content which in case $\alpha \leqq \beta$ fulfils $\alpha + (\beta \setminus \alpha) = \beta$. It is upward $\bullet$ continuous whenever β is, and has important further properties.

4.8 Proposition. *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty]$ form a $\bullet$ complemental couple with maximal $\bullet$ extensions $\Phi = \varphi_\bullet|\mathfrak{C}$ and $\Psi = \psi^\bullet|\mathfrak{C}$. Thus the difference $\delta = \Psi \setminus \Phi : \mathfrak{C} \to [0,\infty]$ is a content, and for $\bullet = \sigma\tau$ a measure, which fulfils $\Phi + \delta = \Psi$. This δ attains but the values 0 and ∞.*

In fact, this can be read from the definition of δ. The assertion speaks for itself, in that the most inferior contents and measures are certainly those which attain but the values 0 and ∞.

Quasi-Radon Measures

The class of quasi-Radon measures was introduced in Fremlin [1974] and pursued further in Fremlin [2000], in order to profit from the favourable properties of Radon measures beyond the frame of compactness. We shall see that this amounts to a certain move into the present inner τ situation.

The definition is quite technical. Let X be a topological space with $\mathfrak{U} = \mathrm{Op}(X)$, that is a nonvoid set equipped with a lattice $\mathfrak{U}$ of subsets with $\varnothing, X \in \mathfrak{U}$ which is stable under unions. A measure $\alpha : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A} \supset \mathrm{Bor}(X)$ is called *quasi-Radon* iff

i) α is *complete*: if $M \subset N \in \mathfrak{A}$ with $\alpha(N) = 0$ then $M \in \mathfrak{A}$;
ii) α is *saturated*: $[\alpha < \infty] \top \mathfrak{A} \subset \mathfrak{A}$;
iii) $\alpha|\mathfrak{U}$ is upward τ continuous;

iv) α is inner regular $\mathrm{Cl}(X)$;

v) α is *effectively locally finite*: for each $A \in \mathfrak{A}$ with $\alpha(A) > 0$ there exists a $U \in \mathfrak{U} \cap [\alpha < \infty]$ such that $\alpha(A \cap U) > 0$; one notes that it is equivalent to require: α is inner regular $\{A \in \mathfrak{A} : A \subset \text{some } U \in \mathfrak{U} \cap [\alpha < \infty]\}$.

It follows that the union of iv)v) means that α is inner regular $\{S \in \mathrm{Cl}(X) : S \subset \text{some } U \in \mathfrak{U} \cap [\alpha < \infty]\}$. After this definition our result of comparison will look somewhat simpler.

4.9 Theorem. *Let X be a topological space with $\mathfrak{U} = \mathrm{Op}(X)$. Then a measure $\alpha : \mathfrak{A} \to [0, \infty]$ on a σ algebra $\mathfrak{A} \supset \mathrm{Bor}(X)$ is quasi-Radon iff there exist a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in X such that $\mathfrak{S}$ and $\mathfrak{U}$ are τ complemental* (which means that $\mathfrak{S} \subset \mathrm{Cl}(X) \subset \mathfrak{S} \top \mathfrak{S}_\tau$)*, and an inner τ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which is τ tame for $\mathfrak{S}$ and $\mathfrak{U}$, such that $\alpha = \varphi_\tau | \mathfrak{C}(\varphi_\tau) = \Phi$. In this case*

$$\mathfrak{S} := \{S \in \mathrm{Cl}(X) : S \subset \textit{some } U \in \mathfrak{U} \cap [\alpha < \infty]\}$$

and $\varphi := \alpha | \mathfrak{S}$ are as required.

The proof is a routine application of what we have presented so far and of the later result 8.6.

At this point we want to note that part of the structure expressed in the notion of quasi-Radonness, to wit the involvement of τ complemental lattices and of τ tameness, that is after all of *local finiteness*, is not needed for certain fundamental applications of the notion in Fremlin [2000]. The most important cases in point are the formation of products and the representation of functionals, both to be discussed in later sections, but to some extent also the existence of decompositions. The rôle of local finiteness reminds one of its older rôle in the notion of Radonness. But in that case one could simply delete local finiteness from the definition, which does not seem so in the present case. In plain words: It is the present concept of inner τ premeasures and their inner τ extensions which does the essential.

Decompositions

Let $\alpha : \mathfrak{A} \to [0, \infty]$ be a measure on a σ algebra $\mathfrak{A}$. We define a *decomposition for α* to be a nonvoid and pairwise disjoint subsystem $\mathfrak{M} \subset \mathfrak{A}$ of members M with $0 < \alpha(M) < \infty$ such that

1) $\mathfrak{M} \top \mathfrak{A} \subset \mathfrak{A}$; and

2) $\alpha(A) = \sum\limits_{M \in \mathfrak{M}} \alpha(A \cap M)$ for all $A \in [\alpha < \infty]$.

Condition 1) implies that the union $H := \bigcup\limits_{M \in \mathfrak{M}} M$ is in $\mathfrak{A}$. Condition 2) is often required for *all* $A \in \mathfrak{A}$, which follows from our version when α is *semifinite*, that is inner regular $[\alpha < \infty]$. The reason for the present weaker condition is that the unavoidable finiteness assumption on α which ensures the existence of a decomposition becomes of course weaker, but also better adapted to our conception. In concrete terms, the assumptions in Fremlin [2000] 414I are the above i)ii)iii)v) for quasi-Radonness, while in the theorem below we shall weaken condition v) from $\mathfrak{A}$ to $[\alpha < \infty]$. The difference is considerable, as it can be seen in the example of Dowker ([1998a] 4.8) invoked in 4.7 above.

We need one more notion, which is related to the concept of support (MI 9.23): Let $\mathfrak{U} \subset \mathfrak{A}$ be a lattice with $\varnothing \in \mathfrak{U}$. Then $F \in \mathfrak{A}$ is called *full for* $\mathfrak{U}$ iff $0 < \alpha(F) < \infty$, and for each $U \in \mathfrak{U}$ one has $\alpha(F \cap U) = 0 \Rightarrow F \cap U = \varnothing$. Define $\mathfrak{F}(\alpha, \mathfrak{U})$ to consist of all these $F \in \mathfrak{A}$. After this the present theorem reads as follows. The proof is conventional.

4.10 Theorem. *Let the measure* $\alpha : \mathfrak{A} \to [0, \infty]$ *on the* σ *algebra* $\mathfrak{A}$ *be complete and saturated, and let* $\mathfrak{U} \subset \mathfrak{A}$ *be a lattice with* $\varnothing \in \mathfrak{U}$ *and stable under unions such that* $\alpha|\mathfrak{U}$ *is upward* τ *continuous. Assume that* α *is inner regular* $\{A \in \mathfrak{A} : A \subset \textit{some } U \in \mathfrak{U} \cap [\alpha < \infty]\}$ *at* $[\alpha < \infty]$.

1) *If* $\mathfrak{M} \subset \mathfrak{F}(\alpha, \mathfrak{U})$ *is a maximal pairwise disjoint nonvoid subsystem, then* $\mathfrak{M}$ *is a decomposition for* α.

2) *Let* $\mathfrak{K} \subset [\alpha < \infty]$ *be a lattice with* $\varnothing \in \mathfrak{K}$ *and* $\mathfrak{U} \subset (\mathfrak{K} \top \mathfrak{K})\perp$ *such that* α *is inner regular* $\mathfrak{K}$ *at* $[\alpha < \infty]$. *Then each* $A \in [0 < \alpha < \infty]$ *contains some* $K \in \mathfrak{K} \cap \mathfrak{F}(\alpha, \mathfrak{U})$. *Thus if* $\mathfrak{M} \subset \mathfrak{K} \cap \mathfrak{F}(\alpha, \mathfrak{U})$ *is a maximal pairwise disjoint nonvoid subsystem, then* $\mathfrak{M}$ *is a decomposition for* α.

The value of this theorem lies in its specializations to the present inner and outer τ situations. The inner one reaches well beyond local finiteness (MI 13.39), while the outer one carries no restriction at all.

4.11 Inner Consequence (MI 9.24). *Let* $\varphi : \mathfrak{S} \to [0, \infty[$ *be an inner* τ *premeasure with* $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$. *Assume for* $\mathfrak{U} = (\mathfrak{S} \top \mathfrak{S}_\tau)\perp$ *that* Φ *is inner regular* $\{A \in \mathfrak{C}(\varphi_\tau) : A \subset \textit{some } U \in \mathfrak{U} \cap [\Phi < \infty]\}$ *at* $\mathfrak{S}$ (and hence at $\{D \in \mathfrak{C}(\varphi_\tau) : D \subset \text{ some } S \in \mathfrak{S}\} \supset \mathfrak{S}_\tau$ and hence at $\mathfrak{C}(\varphi_\tau)$).

1) *If* $\mathfrak{M} \subset \mathfrak{F}(\Phi, \mathfrak{U})$ *is a maximal pairwise disjoint nonvoid subsystem, then* $\mathfrak{M}$ *is a decomposition for* Φ (note that Φ is semifinite).

2) *Each* $A \in [0 < \Phi < \infty]$ *contains some* $K \in \mathfrak{S}_\tau \cap \mathfrak{F}(\Phi, \mathfrak{U})$. *Thus if* $\mathfrak{M} \subset \mathfrak{S}_\tau \cap \mathfrak{F}(\Phi, \mathfrak{U})$ *is a maximal pairwise disjoint nonvoid subsystem, then* $\mathfrak{M}$ *is a decomposition for* Φ.

4.12 Outer Consequence. *Let* $\psi : \mathfrak{T} \to [0, \infty]$ *be an outer* τ *premeasure with* $\Psi = \psi^\tau | \mathfrak{C}(\psi^\tau)$. *Put* $\mathfrak{U} = \mathfrak{T}^\tau$ (and note condition (τ) in 3.1).

1) *If* $\mathfrak{M} \subset \mathfrak{F}(\Psi, \mathfrak{U})$ *is a maximal pairwise disjoint nonvoid subsystem, then* $\mathfrak{M}$ *is a decomposition for* Ψ.

2) *Each* $A \in [0 < \Psi < \infty]$ *contains some* $K \in \mathfrak{F}(\Psi, \mathfrak{U})$.

One of the most important applications of decompositions is to the Radon-Nikodým theorem. We regret that for the time being we have to refrain from this topic.

5. The Fundamentals for Functionals

The basic notions and notations for function systems and functionals correspond more or less to those for set systems and set functions. Beside them we introduce the Choquet integral for nonnegative functions, and in that connection present a fundamental theorem on sub/supermodular functionals.

Function Systems

Let X be a nonvoid set. We consider nonvoid sets $E \subset [0,\infty]^X$ of nonnegative functions on X, in most cases with $0 \in E$. The important properties to be imposed on E are to be stable under multiplication by positive real numbers, called *positive-homogeneous*, to be stable under pointwise addition, the combination of the two, called *cone*, and to be stable under pointwise maximum and minimum formations $\vee$ and $\wedge$, called *lattice* for short. Moreover E is called *Stonean* iff $f \in E \Rightarrow f \wedge t, (f-t)^+ \in E$ for $0 < t < \infty$; in this connection note that $f = f \wedge t + (f-t)^+$.

We define $E^\star \subset E^\sigma \subset E^\tau$ and $E_\star \subset E_\sigma \subset E_\tau$ to consist of the pointwise suprema and infima of the nonvoid finite/countable/arbitrary subsets of E. Thus E is a lattice iff $E^\star = E_\star = E$, and in this case the $E^\bullet$ and $E_\bullet$ are lattices as well.

A nonvoid subset $M \subset [0,\infty]^X$ is called *upward/downward directed* iff for each pair $u, v \in M$ there exists a $w \in M$ such that $u, v \leqq w/u, v \geqq w$. We write $M \uparrow f / M \downarrow f$ and $M \uparrow\geqq f / M \downarrow\leqq f$ in the same sense as before. We note that a nonvoid *finite* M is upward/downward directed iff the supremum/infimum of its members is a member of M.

5.1 Lemma ([1998b] 3.1.6)). *Let E be a lattice.*

Out) *For each $M \subset E^\bullet$ nonvoid $\bullet$ with $M \uparrow f$ there exists an $N \subset E$ nonvoid $\bullet$ with $N \uparrow f$ such that each $v \in N$ is $\leqq$ some $u \in M$.*

Inn) *For each $M \subset E_\bullet$ nonvoid $\bullet$ with $M \downarrow f$ there exists an $N \subset E$ nonvoid $\bullet$ with $N \downarrow f$ such that each $v \in N$ is $\geqq$ some $u \in M$.*

Additive and Modular Functionals

Let $E \subset [0,\infty]^X$ be nonvoid, and $I : E \to [0,\infty]$ be a nonnegative functional on E. I is called *isotone* iff $I(u) \leqq I(v)$ for all $u \leqq v$ in E. If $0 \in E$ then in most cases one requires $I(0) = 0$. If E is positive-homogeneous then I is called *positive-homogeneous* iff $I(cf) = cI(f)$ for all $f \in E$ and $0 < c < \infty$. I is called

additive iff $I(u+v) = I(u) + I(v)$, and
sub/superadditive iff $I(u+v) \leqq / \geqq I(u) + I(v)$,

both times for all $u, v \in E$ such that $u + v \in E$; note that E is *not* assumed to be stable under addition. If E is a lattice then I is called

modular iff $I(u \vee v) + I(u \wedge v) = I(u) + I(v)$, and
sub/supermodular iff $I(u \vee v) + I(u \wedge v) \leqq / \geqq I(u) + I(v)$,

both times for all $u, v \in E$. If E is Stonean then I is called *Stonean* iff

$$I(f) = I(f \wedge t) + I((f-t)^+) \quad \text{for all } f \in E \text{ and } 0 < t < \infty,$$

and an isotone I is called *truncable* iff

$$I(f) = \sup\{I\big((f-a)^+ \wedge (b-a)\big) : 0 < a < b < \infty\} \quad \text{for all } f \in E.$$

In this connection we recall the relevant basic estimation.

5.2 LEMMA (MI 11.6). *For $f : X \to \overline{\mathbb{R}}$ and real numbers $a = t(0) < t(1) < \cdots < t(r) = b$ we have*

$$\begin{aligned}
\sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f>t(l)]} &\leqq \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f\geqq t(l)]} \\
&\leqq (f-a)^+ \wedge (b-a) \leqq \\
\sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f>t(l-1)]} &\leqq \sum_{l=1}^{r} (t(l) - t(l-1))\chi_{[f\geqq t(l-1)]}.
\end{aligned}$$

The next theorem is a basic result. The idea that a result of this kind could be true is due to Choquet [1953/54] 54.1, but his proof was confined to a certain special case with finite X. Then the specialization which will be needed for the Choquet integral was an independent result of Topsøe [1978] and Bassanezi-Greco [1984]. The full assertion is due to the author [1998b] 1.1 and [2001b]. It is required in the proofs of the representation theorems 7.3 and 7.6 below.

5.3 THEOREM. *Let $E \subset [0,\infty]^X$ be positive-homogeneous with $0 \in E$, and $I : E \to [0,\infty]$ be positive-homogeneous with $I(0) = 0$. Assume that E is a Stonean lattice, and that I is isotone and truncable. Then*

$$I \text{ sub/supermodular} \Longrightarrow I \text{ sub/superadditive.}$$

At last we anticipate for $I : E \to [0,\infty]$ isotone under the assumption $0 \in E$ and $I(0) = 0$ the *finite envelopes* $I^\star, I_\star : [0,\infty]^X \to [0,\infty]$, defined to be

$$\begin{aligned}
I^\star(f) &= \inf\{I(u) : u \in E \text{ with } u \geqq f\}, \\
I_\star(f) &= \sup\{I(u) : u \in E \text{ with } u \leqq f\},
\end{aligned}$$

which will be needed for the Choquet integral which comes next. $I^\star$ and $I_\star$ are isotone with $I_\star \leqq I^\star$ and $I^\star|E = I_\star|E = I$.

THE CHOQUET INTEGRAL

We fix a lattice $\mathfrak{S}$ of subsets with $\varnothing \in \mathfrak{S}$ in X, and an isotone set function $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$. We shall define after an idea of Choquet [1953/54] section 48 an *integral* $⨍ f d\varphi \in [0,\infty]$ for appropriate functions $f \in [0,\infty]^X$, which is such that in case of a measure φ on a σ algebra $\mathfrak{S}$ the formation $⨍ f d\varphi$ is defined for all $f \in [0,\infty]^X$ measurable $\mathfrak{S}$ and is the usual integral $\int f d\varphi$. Our procedure will be somewhat different from that of Greco [1982] and will feature *two* admissible function classes; they are in perfect accord with our *two* situations, the outer and the inner ones.

We define $\mathrm{LM}(\mathfrak{S})/\mathrm{UM}(\mathfrak{S})$ to consist of the functions $f \in [0,\infty]^X$ such that $[f > t]/[f \geqq t] \in \mathfrak{S}$ for all $0 < t < \infty$; these functions will be called *lower/upper measurable* $\mathfrak{S}$.

5.4 Properties (MI 11.1 and 11.4). 1) $\mathrm{LM}(\mathfrak{S})$ *and* $\mathrm{UM}(\mathfrak{S})$ *are positive-homogeneous with* $0 \in \mathrm{LM}(\mathfrak{S}), \mathrm{UM}(\mathfrak{S})$.

2.Out) *If* $\mathfrak{S} = \mathfrak{S}^\sigma$ *then* $\mathrm{LM}(\mathfrak{S})$ *is stable under addition.* 2.Inn) *If* $\mathfrak{S} = \mathfrak{S}_\sigma$ *then* $\mathrm{UM}(\mathfrak{S})$ *is stable under addition.*

3) $\mathrm{LM}(\mathfrak{S})$ *and* $\mathrm{UM}(\mathfrak{S})$ *are Stonean lattices.*

4.Out) *If* $\mathfrak{S} = \mathfrak{S}^\sigma$ *then* $\mathrm{UM}(\mathfrak{S}) \subset \mathrm{LM}(\mathfrak{S})$. 4.Inn) *If* $\mathfrak{S} = \mathfrak{S}_\sigma$ *then* $\mathrm{LM}(\mathfrak{S}) \subset \mathrm{UM}(\mathfrak{S})$. *Thus if* $\mathfrak{S} = \mathfrak{S}^\sigma = \mathfrak{S}_\sigma$ *then* $\mathrm{LM}(\mathfrak{S}) = \mathrm{UM}(\mathfrak{S}) =: \mathrm{M}(\mathfrak{S})$.

5) *For a function* $f : X \to [0, \infty[$ *with a finite number of values the following are equivalent.* i) $f \in \mathrm{LM}(\mathfrak{S})$. ii) $f \in \mathrm{UM}(\mathfrak{S})$. iii) *There exist* $A(1), \cdots, A(r) \in \mathfrak{S}$ *and real* $t_1, \cdots, t_r > 0$ *such that* $f = \sum_{l=1}^{r} t_l \chi_{A(l)}$. iv) *The same as* iii) *with* $A(1) \supset \cdots \supset A(r)$. We define $\mathrm{S}(\mathfrak{S})$ to consist of these functions; note that $\mathrm{S}(\mathfrak{S})$ is stable under addition.

After this we define the *Choquet integral* to be

$$\fint f d\varphi := \int_{0\leftarrow}^{\to\infty} \varphi([f > t])dt \quad \text{for } f \in \mathrm{LM}(\mathfrak{S}),$$
$$\fint f d\varphi := \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq t])dt \quad \text{for } f \in \mathrm{UM}(\mathfrak{S}),$$

both times as an improper Riemann integral of a monotone function $\geqq 0$. It is a simple verification that in case $f \in \mathrm{LM}(\mathfrak{S}) \cap \mathrm{UM}(\mathfrak{S})$ the two second members are equal (MI 11.7). Thus for $A \in \mathfrak{S}$ we have $\chi_A \in \mathrm{LM}(\mathfrak{S}) \cap \mathrm{UM}(\mathfrak{S})$ with $\fint \chi_A d\varphi = \varphi(A)$. In the subsequent properties we write $I : I(f) = \fint f d\varphi$ for $f \in \mathrm{LM}(\mathfrak{S})/\mathrm{UM}(\mathfrak{S})$ whenever adequate.

5.5 Properties (MI 11.8 and [1998b] 2.9). i) *In case of a measure* φ *on a* σ *algebra* $\mathfrak{S}$ *one has* $\fint f d\varphi = \int f d\varphi$ *for all* $f \in \mathrm{M}(\mathfrak{S})$.

ii) *For* $f \in \mathrm{S}(\mathfrak{S})$ *and for the representations* 5.4.5.iv) *one has*

$$\fint f d\varphi = \sum_{l=1}^{r} t_l \varphi(A(l)).$$

1) *I is isotone and positive-homogeneous with* $I(0) = 0$.

2) *I is Stonean and truncable.*

3) *I is sub/supermodular* $\Leftrightarrow$ *φ is sub/supermodular.*

4) $I|\mathrm{S}(\mathfrak{S})$ *is sub/superadditive* $\Rightarrow$ *φ is sub/supermodular.*

Assertion i) is in all textbooks, but often as a consequence of the Fubini theorem and hence restricted to σ finite measures. ii) requires a bit of work. 1)3) are clear, and 2)4) are pleasant verifications, with 4) based on ii). Now 3)4) combine with theorem 5.3 to furnish the basic fact which follows (note that 5.3 is only needed for *I Stonean*).

5.6 THEOREM. *For both* LM($\mathfrak{S}$) *and* UM($\mathfrak{S}$) *the following are equivalent.* i) *I is sub/superadditive.* ii) *$I|$S($\mathfrak{S}$) is sub/superadditive.* iii) *I is sub/supermodular.* iv) *φ is sub/supermodular.*

The present context calls for a hint at another approach to an *additive* Choquet integral: To find those pairs $u, v \in$ LM($\mathfrak{S}$)/UM($\mathfrak{S}$) for which the Choquet integral becomes additive for *all* isotone $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$. The codeword is *comonotonic*; see for example Denneberg [1994].

We conclude the subsection with the first and simplest of our theorems on integral representations of functionals. It is a version of the representation theorem in terms of the Choquet integral due to Greco [1982].

5.7 THEOREM ([1998b] 2.10). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$, and $E \subset$* LM($\mathfrak{S}$)/UM($\mathfrak{S}$) *be positive-homogeneous with $0 \in E$ and Stonean. Assume that $I : E \to [0, \infty]$ is isotone with $I(0) = 0$. Then there exists an isotone $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ which represents I, that is which fulfils $I(f) = ⨍ f d\varphi$ for all $f \in E$, iff I is Stonean and truncable. In this case an isotone $\varphi : \mathfrak{S} \to [0, \infty]$ with $\varphi(\varnothing) = 0$ represents I iff*

$$I_\star(\chi_S) \leqq \varphi(S) \leqq I^\star(\chi_S) \quad \textit{for all } S \in \mathfrak{S}.$$

• CONTINUOUS AND REGULAR FUNCTIONALS

Assume that $I : E \to [0, \infty]$ is an isotone functional on a nonvoid $E \subset [0, \infty]^X$. I is called *upward/downward • continuous* iff $\sup\limits_{u \in M} / \inf\limits_{u \in M} I(u) = I(f)$ for all $M \subset E$ nonvoid • with $M \uparrow / \downarrow f \in E$. As before this is always true in case $\bullet = \star$, and can be formulated in terms of sequences in case $\bullet = \sigma$. Likewise as before an important variant is *almost downward • continuous*, defined to mean that $\inf\limits_{u \in M} I(u) = I(f)$ is restricted to those $M \subset E$ which have $I(u) < \infty \ \forall u \in M$. One also defines these properties *at an individual $f \in E$* and *at a nonvoid subset* of E.

Next let $M \subset E$ be a nonvoid subset. The functional I is called *outer regular M* (or *from M*) iff

$$I(f) = \inf\{I(u) : u \in M \text{ with } u \geqq f\} \quad \text{for all } f \in E,$$

and *inner regular M* (or *from M*) iff

$$I(f) = \sup\{I(u) : u \in M \text{ with } u \leqq f\} \quad \text{for all } f \in E.$$

One also defines these properties *at an individual $f \in E$* and *at a nonvoid subset* of E.

It turns out that the Choquet integral enjoys these properties to the reasonable extent. 5.8 below is a wide extension of the Beppo Levi theorem.

5.8 THEOREM (MI 11.18 and 11.17). *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ in X, and $\varphi : \mathfrak{S} \to [0, \infty]$ be isotone with $\varphi(\varnothing) = 0$. Denote $I : I(f) = ⨍ f d\varphi$ on* LM($\mathfrak{S}$)/UM($\mathfrak{S}$).

Out) *φ is upward $\bullet$ continuous $\Leftrightarrow$ I is upward $\bullet$ continuous on* $\mathrm{LM}(\mathfrak{S})$*.*

Inn) *φ is almost downward $\bullet$ continuous $\Leftrightarrow$ I is almost downward $\bullet$ continuous on* $\mathrm{UM}(\mathfrak{S})$*.*

5.9 THEOREM. *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ in X, and $\Phi : \mathfrak{P}(X) \to [0,\infty]$ be isotone with $\Phi(\varnothing) = 0$. Denote $I : I(f) = ⨍ f d\Phi$ for $f \in [0,\infty]^X$.*

Out) *If Φ is outer regular $\mathfrak{S}$ then I is outer regular* $\mathrm{S}(\mathfrak{S})$ *at $\{f \in [0,\infty]^X :$ f bounded and $\Phi([f > 0]) < \infty\}$.*

Inn) *If Φ is inner regular $\mathfrak{S}$ then I is inner regular* $\mathrm{S}(\mathfrak{S})$*.*

THE $\bullet$ ENVELOPES

Assume that $I : E \to [0,\infty]$ is an isotone functional with $I(0) = 0$ on an $E \subset [0,\infty]^X$ with $0 \in E$. As before we define for $\bullet = \star\sigma\tau$ the *outer $\bullet$ envelopes* $I^\bullet : [0,\infty]^X \to [0,\infty]$ and the *inner $\bullet$ envelopes* $I_\bullet : [0,\infty]^X \to [0,\infty]$ for I to be

$$\begin{aligned} I^\bullet(f) &= \inf\{\sup_{u\in M} I(u) : M \subset E \text{ nonvoid } \bullet \text{ with } M \uparrow\geqq f\}, \\ I_\bullet(f) &= \sup\{\inf_{u\in M} I(u) : M \subset E \text{ nonvoid } \bullet \text{ with } M \downarrow\leqq f\}. \end{aligned}$$

In case $\bullet = \star$ we return to the former envelopes, and in case $\bullet = \sigma$ the definition can be reformulated in terms of sequences as before. The envelopes $I^\bullet$ and $I_\bullet$ are isotone, and fulfil $I^\star \geqq I^\sigma \geqq I^\tau$ and $I_\star \leqq I_\sigma \leqq I_\tau$. We note some further basic properties.

5.10 PROPERTIES ([1998b] 3.3-3.5). *Assume that E is a lattice.*

1) *I is positive-homogeneous $\Rightarrow$ $I^\bullet$ and $I_\bullet$ are positive-homogeneous.*

2) *Assume that E is stable under addition.* 2.Out) *I is subadditive $\Rightarrow$ $I^\bullet$ is subadditive.* 2.Inn) *I is superadditive $\Rightarrow$ $I_\bullet$ is superadditive.*

3.Out) *I is submodular $\Rightarrow$ $I^\bullet$ is submodular.* 3.Inn) *I is supermodular $\Rightarrow$ $I_\bullet$ is supermodular.*

4.Out) *$I^\bullet$ is outer regular $E^\bullet$.* 4.Inn) *$I_\bullet$ is inner regular $E_\bullet$.*

5.Out) *For $f \in E$ one has $I^\bullet(f) = I(f) \Leftrightarrow I$ is upward $\bullet$ continuous at f.* 5.Inn) *For $f \in E$ one has $I_\bullet(f) = I(f) \Leftrightarrow I$ is downward $\bullet$ continuous at f.*

6.Out) *If I is upward $\bullet$ continuous then $I^\bullet|E^\bullet = I_\star|E^\bullet$, and this is upward $\bullet$ continuous as well.* 6.Inn) *If I is downward $\bullet$ continuous then $I_\bullet|E_\bullet = I^\star|E_\bullet$, and this is downward $\bullet$ continuous as well.*

7.Out) *If I is upward $\bullet$ continuous and $\{f \in E^\bullet : I^\bullet(f) < \infty\} \subset E$ then $I^\bullet = I^\star$.* 7.Inn) *If I is downward $\bullet$ continuous and $E_\bullet = E$ then $I_\bullet = I_\star$.*

8.Out) *If I is submodular then I^σ and I^τ are upward σ continuous.* 8.Inn) *If I is supermodular then I_σ and I_τ are almost downward σ continuous.*

6. The Formation of Products

The present approach leads to a new method for the formation of products in measure and integration, which for the first time unites the traditional abstract and topological product theories. The decisive point is that one performs the explicit product formation on the level of $\bullet$ premeasures and not on that of full measures. The main theorem then presents our final outcome as an adequate $\bullet$ extension in the *inner* situation, but as a rule *not* in the outer one. We shall restrict ourselves to the case of two factors.

The Traditional Product Theories

We fix nonvoid sets X and Y. For nonvoid set systems $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y we define the *product set system* $\mathfrak{S} \times \mathfrak{T} := \{S \times T : S \in \mathfrak{S} \text{ and } T \in \mathfrak{T}\}$ in $X \times Y$.

The *abstract main theorem* asserts that for measures $\alpha : \mathfrak{A} \rightarrow [0, \infty]$ and $\beta : \mathfrak{B} \rightarrow [0, \infty]$ on σ algebras $\mathfrak{A}$ in X and $\mathfrak{B}$ in Y there exists at least one measure $\pi : \mathfrak{A} \otimes \mathfrak{B} \rightarrow [0, \infty]$ on the product σ algebra $\mathfrak{A} \otimes \mathfrak{B} := \mathrm{A}\sigma(\mathfrak{A} \times \mathfrak{B})$ in $X \times Y$ such that $\pi(A \times B) = \alpha(A)\beta(B)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, with the usual convention $0\infty := 0$. Uniqueness of the product measure cannot be claimed but on the subring of those $E \in \mathfrak{A} \otimes \mathfrak{B}$ which are contained in some member of $[\alpha < \infty]^\sigma \times [\beta < \infty]^\sigma$. Thus most textbooks develop the entire context restricted to σ *finite measures*. It seems hopeless to strive for a more comprehensive uniqueness assertion in the traditional abstract frame, that is without that the concept of regularity enters the scene.

The turn to the *topological* context leads into another world. In this world the fundamental theorem asserts that for *Borel-Radon* measures $\alpha : \mathrm{Bor}(X) \rightarrow [0, \infty]$ and $\beta : \mathrm{Bor}(Y) \rightarrow [0, \infty]$ on Hausdorff topological spaces X and Y there exists a *unique Borel-Radon* measure $\pi : \mathrm{Bor}(X \times Y) \rightarrow [0, \infty]$ on $X \times Y$ such that $\pi(A \times B) = \alpha(A)\beta(B)$ for all $A \in \mathrm{Bor}(X)$ and $B \in \mathrm{Bor}(Y)$. The theorem was first obtained for locally compact X and Y via combination of the Riesz representation and Stone-Weierstrass theorems, and thus became a showpiece in the development à la Bourbaki. Since in $\mathrm{Bor}(X) \otimes \mathrm{Bor}(Y) \subset \mathrm{Bor}(X \times Y)$ one has $\neq$ in most cases, it is beyond reach of the abstract approach as above.

This gap is left wide open in all textbooks in measure and integration known to the author, except in Fremlin [2000] sections 251 and 417. Here the entire context is rightly based on inner regularity. But the presentation differs from ours in that the topological part remains separated from the abstract one, in that it is under an unneeded local finiteness restriction as mentioned above, and above all in that it does not flow from an overall concept like the present $\bullet$ premeasures and $\bullet$ extensions which could break up the notorious discrepancies.

The New Product Formation

On the product set $X \times Y$ we form for $E \subset X \times Y$ the *vertical sections* $E(x) := \{y \in Y : (x,y) \in E\} \subset Y \; \forall x \in X$ and the *vertical projection*

$$\mathrm{Pr}(E) := \{y \in Y : (x,y) \in E \text{ for some } x \in X\} = \bigcup_{x \in X} E(x) \subset Y,$$

and of course the respective *horizontal* formations. Next we form for lattices $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y the *product lattice* $\mathfrak{R} := (\mathfrak{S} \times \mathfrak{T})^\star$, and note that $\mathfrak{R}$ is a ring/an algebra whenever $\mathfrak{S}$ and $\mathfrak{T}$ are rings/algebras.

6.1 Remark (MI 20.3). *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$, and let $E \in \mathfrak{R}$. Then* 1) *$E(x) \in \mathfrak{T}$ for all $x \in X$.*

2) *Assume that $\psi : \mathfrak{T} \to [0,\infty]$ is isotone with $\psi(\varnothing) = 0$. Then the function $\psi(E(\cdot)) : X \to [0,\infty]$ has a finite value set and is in* $\mathrm{LM}(\mathfrak{S}) \cap \mathrm{UM}(\mathfrak{S})$ (thus is in $\mathrm{S}(\mathfrak{S})$ except that it can attain the value ∞).

We come to the explicit product formation for set functions. It is based on the Choquet integral.

6.2 Proposition (MI 20.4-20.7). *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$, and let*

$\varphi : \mathfrak{S} \to [0,\infty]$ *be isotone with* $\varphi(\varnothing) = 0$,
$\psi : \mathfrak{T} \to [0,\infty]$ *be isotone with* $\psi(\varnothing) = 0$.

We define the product set function $\vartheta = \varphi \times \psi : \mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star \to [0,\infty]$ *to be*

$$\vartheta(E) = \int\!\!\!\!\!-\, \psi(E(\cdot))d\varphi,$$

which makes sense in view of 6.1. *Then* 1) *ϑ is isotone with $\vartheta(\varnothing) = 0$.*

2) *$\vartheta(S \times T) = \varphi(S)\psi(T)$ for all $S \in \mathfrak{S}$ and $T \in \mathfrak{T}$.*

3) *φ and ψ are finite $\Rightarrow$ ϑ is finite.*

4) *φ and ψ are modular $\Rightarrow$ ϑ is modular.*

5) *If φ and ψ are modular then ϑ is the unique isotone and modular set function $\mathfrak{R} \to [0,\infty]$ with* 2) (thus the symmetric formation yields the same result).

6) *Let $\alpha : \mathfrak{A} \to [0,\infty]$ and $\beta : \mathfrak{B} \to [0,\infty]$ on lattices $\mathfrak{A} \supset \mathfrak{S}$ and $\mathfrak{B} \supset \mathfrak{T}$ be isotone with restrictions $\alpha|\mathfrak{S} = \varphi$ and $\beta|\mathfrak{T} = \psi$. Then $(\alpha \times \beta)|(\mathfrak{S} \times \mathfrak{T})^\star = \varphi \times \psi$.*

At this point the development splits into the inner and the outer one. We start with the inner situation which is much more favourable.

6.3 Inner Proposition (MI 21.4-21.7). *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$, and let*

$\varphi : \mathfrak{S} \to [0,\infty[$ *be isotone with* $\varphi(\varnothing) = 0$,
$\psi : \mathfrak{T} \to [0,\infty[$ *be isotone with* $\psi(\varnothing) = 0$,

so that $\vartheta : \mathfrak{R} \to [0,\infty[$ is isotone with $\vartheta(\varnothing) = 0$. Assume that φ and ψ are downward $\bullet$ continuous. Then 1) *ϑ is downward $\bullet$ continuous* (the same implication holds true for downward $\bullet$ continuous at $\varnothing$).

2) *For* $E \in \mathfrak{R}_\bullet$ *one has* $E(x) \in \mathfrak{T}_\bullet$ *for all* $x \in X$. *Moreover the function* $\psi_\bullet(E(\cdot)) : X \to [0,\infty[$ *is in* $\mathrm{UM}(\mathfrak{S}_\bullet)$, *and* $\vartheta_\bullet(E) = \fint \psi_\bullet(E(\cdot))d\varphi_\bullet$.

3) $\vartheta_\bullet(A \times B) = \varphi_\bullet(A)\psi_\bullet(B)$ *for all* $A \subset X$ *and* $B \subset Y$.

After this one proves the fundamental theorem which follows.

6.4 Theorem (MI 21.9). *Assume that*

$\varphi : \mathfrak{S} \to [0,\infty[$ *is an inner* $\bullet$ *premeasure with* $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$,

$\psi : \mathfrak{T} \to [0,\infty[$ *is an inner* $\bullet$ *premeasure with* $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$.

Then $\vartheta : \mathfrak{R} \to [0,\infty[$ *is an inner* $\bullet$ *premeasure, and* $\Theta = \vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ *is an extension of* $\Phi \times \Psi$.

It is clear that this is a comprehensive abstract existence and uniqueness theorem for the formation of products. We emphasize that it contains no trace of local finiteness.

We continue to convince ourselves that the theorem contains the previous product theorem for Borel-Radon measures. Let X and Y be topological spaces. The formula $\mathrm{Op}(X \times Y) = (\mathrm{Op}(X) \times \mathrm{Op}(Y))^\tau$ is but the definition of the product topology. By manipulation with complements it follows that $\mathrm{Cl}(X \times Y) = \big((\mathrm{Cl}(X) \times \mathrm{Cl}(Y))^\star\big)_\tau$, and when X and Y are Hausdorff one concludes that

$$\mathrm{Comp}(X \times Y) = \big((\mathrm{Comp}(X) \times \mathrm{Comp}(Y))^\star\big)_\tau.$$

Now if $\varphi : \mathfrak{S} = \mathrm{Comp}(X) \to [0,\infty[$ and $\psi : \mathfrak{T} = \mathrm{Comp}(Y) \to [0,\infty[$ are Radon premeasures with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ and $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ then 6.4 asserts that $\vartheta = \varphi \times \psi : \mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star \to [0,\infty[$ is an inner τ premeasure, and $\Theta = \vartheta_\tau|\mathfrak{C}(\vartheta_\tau)$ is an extension of $\Phi \times \Psi$. But we know that $\mathfrak{R}_\tau = \mathrm{Comp}(X \times Y)$, and then 3.8.Inn) tells us that $\rho := \vartheta_\tau|\mathfrak{R}_\tau$ is an inner τ premeasure and thus a Radon premeasure on $X \times Y$ with $\vartheta_\tau = \rho_\tau = \rho_\star$ and hence with $\Theta = \rho_\bullet|\mathfrak{C}(\rho_\bullet)$. The uniqueness assertion is clear.

We turn to the outer situation (which has not been dealt with in MI). There is a partial counterpart of the inner proposition 6.3.

6.5 Outer Proposition. *Let* $\mathfrak{S}$ *and* $\mathfrak{T}$ *be lattices with* $\varnothing$, *and let*

$\varphi : \mathfrak{S} \to [0,\infty]$ *be isotone with* $\varphi(\varnothing) = 0$,

$\psi : \mathfrak{T} \to [0,\infty]$ *be isotone with* $\psi(\varnothing) = 0$,

so that $\vartheta : \mathfrak{R} \to [0,\infty]$ *is isotone with* $\vartheta(\varnothing) = 0$. *Assume that* φ *and* ψ *are upward* $\bullet$ *continuous. Then* 1) ϑ *is upward* $\bullet$ *continuous.*

2) *For* $E \in \mathfrak{R}^\bullet$ *one has* $E(x) \in \mathfrak{T}^\bullet$ *for all* $x \in X$. *Moreover the function* $\psi^\bullet(E(\cdot)) : X \to [0,\infty]$ *is in* $\mathrm{LM}(\mathfrak{S}^\bullet)$, *and* $\vartheta^\bullet(E) = \fint \psi^\bullet(E(\cdot))d\varphi^\bullet$.

3) $\vartheta^\bullet(A \times B) = \varphi^\bullet(A)\psi^\bullet(B)$ *for all* $A \subset X$ *and* $B \subset Y$, *except when the latter product is* 0∞ *or* $\infty 0$. *In this case the assertion can be false for* $\bullet = \star\sigma\tau$, *even for* φ *and* ψ *outer* $\bullet$ *premeasures and for* $A \in \mathfrak{C}(\varphi^\bullet)$ *and* $B \in \mathfrak{C}(\psi^\bullet)$.

6.6 Example. Let X be an uncountable set, and $\mathfrak{S}$ consist of the finite subsets of X. Then $\varphi = \mathrm{card}|\mathfrak{S}$ is an outer $\bullet$ premeasure with $\varphi^\bullet = \mathrm{card}$ and $\mathfrak{C}(\varphi^\bullet) = \mathfrak{P}(X)$. Let $\psi : \mathfrak{T} \to [0,\infty]$ be an outer $\bullet$ premeasure on Y such that for some $c \in Y$ one has $\psi^\bullet(\{c\}) = 0$ but $\psi^\bullet(T) > 0$ for all $T \in \mathfrak{T}^\bullet$ with $c \in T$. For example one can take $Y = \mathbb{R}$ and $\psi : \mathfrak{T} = \mathrm{Op}(Y) \to [0, \infty]$ the restriction of Lebesgue measure. Then

$$\vartheta^\bullet(E) = \,⨍\, \psi^\bullet(E(\cdot))d\varphi^\bullet = \sum_{x \in X} \psi^\bullet(E(x)) = \infty \quad \forall E \in \mathfrak{R}^\bullet \text{ with } E \supset X \times \{c\},$$

and hence $\vartheta^\bullet(X \times \{c\}) = \infty$, whereas $\varphi^\bullet(X)\psi^\bullet(\{c\}) = \infty 0 = 0$.

Even so the outer proposition 6.5 can be used for the formation of products in a more conventional spirit: Let $\alpha : \mathfrak{A} \to [0, \infty]$ and $\beta : \mathfrak{B} \to [0, \infty]$ be measures on σ algebras $\mathfrak{A}$ in X and $\mathfrak{B}$ in Y. Then $\mathfrak{R} = (\mathfrak{A} \times \mathfrak{B})^\star$ is an algebra, and $\vartheta = \alpha \times \beta : \mathfrak{R} \to [0, \infty]$ is isotone with $\vartheta(\varnothing) = 0$, and modular and upward σ continuous, and hence after 3.3 an outer σ premeasure. Thus $\Theta := \vartheta^\sigma | \mathfrak{C}(\vartheta^\sigma)$ is a measure which of course extends $\vartheta = \alpha \times \beta$. It could be named the *product* of α and β in the traditional abstract sense (and is in fact the *primitive product* of α and β in the sense of Fremlin [2000] 251C).

In the same spirit the inner proposition 6.3 can be used as well: As above $\mathfrak{R} = ([\alpha < \infty] \times [\beta < \infty])^\star$ is a ring, and $\vartheta = (\alpha|[\alpha < \infty]) \times (\beta|[\beta < \infty]) : \mathfrak{R} \to [0, \infty[$ is isotone with $\vartheta(\varnothing) = 0$, and modular and downward σ continuous, and hence after 3.7 an inner σ premeasure. Thus $\Theta = \vartheta_\sigma | \mathfrak{C}(\vartheta_\sigma)$ is a measure which extends $\vartheta = (\alpha|[\alpha < \infty]) \times (\beta|[\beta < \infty])$ (and is in fact the *c.l.d. product* of α and β in the sense of Fremlin [2000] 251F).

In conclusion we return to the inner situation, in order to present the *sectional representation theorem*, that is the nucleus of the Fubini-Tonelli theorem.

6.7 Theorem (MI 21.19). *Let $\varphi : \mathfrak{S} \to [0, \infty[$ and $\psi : \mathfrak{T} \to [0, \infty[$ be inner $\bullet$ premeasures for some $\bullet = \sigma\tau$, and $\vartheta = \varphi \times \psi : \mathfrak{R} = (\mathfrak{S} \times \mathfrak{T})^\star \to [0, \infty[$. Assume that $E \in \mathfrak{C}(\vartheta_\bullet)$ has $\mathrm{Pr}(E) \subset Y$ contained in some member of $[\Psi < \infty]^\sigma$. Then $E(x) \in \mathfrak{C}(\psi_\bullet)$ for all $x \in X$ except on some $N \in \mathfrak{C}(\varphi_\bullet)$ with $\Phi(N) = 0$. Moreover the function $\psi_\bullet(E(\cdot)) : X \to [0, \infty]$ is measurable $\mathfrak{C}(\varphi_\bullet)$, and $\Theta(E) = ⨍\psi_\bullet(E(\cdot))d\varphi_\bullet$.*

7. Integral Representations of Functionals

The present section is the precise counterpart for functionals of the outer and inner $\bullet$ extension theories for set functions in section 3. The functionals of prime rank, which correspond to the former outer and inner $\bullet$ premeasures, will be the outer and inner $\bullet$ *preintegrals*. However, in order to define the $\bullet$ preintegrals one must know what the $\bullet$ premeasures are. Thus the old dispute about predominance between set functions and functionals will end this time in favour of the *set functions*. Moreover the present development is restricted to $\bullet = \sigma\tau$; in case $\bullet = \star$ it turns out to be different (MI section 17). We start from the preparations in section 5.

Outer and Inner Sources of Functionals

Let X be a nonvoid set. We fix a set of functions $E \subset [0,\infty]^X$ which is positive-homogeneous with $0 \in E$ and a lattice (as before under the pointwise $\vee$ and $\wedge$). We shall soon be forced to assume that E is Stonean, but in the basic theorems we want to avoid the assumption that E be stable under addition. We define

$$\begin{aligned}\mathfrak{lm}(E) &:= \{[f > t] : f \in E \text{ and } 0 < t < \infty\},\\ \mathfrak{um}(E) &:= \{[f \geqq t] : f \in E \text{ and } 0 < t < \infty\},\end{aligned}$$

which are lattices of subsets with $\varnothing$ in X (former notations were $\mathfrak{lm}(E) = >(E)$ and $\mathfrak{um}(E) = \geqq(E) = \mathfrak{T}(E)$). For a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in X one reads from the definitions that

$$E \subset \mathrm{LM}(\mathfrak{S}) \Leftrightarrow \mathfrak{lm}(E) \subset \mathfrak{S} \quad \text{and} \quad E \subset \mathrm{UM}(\mathfrak{S}) \Leftrightarrow \mathfrak{um}(E) \subset \mathfrak{S}.$$

Now let $I : E \to [0,\infty]$ be an isotone functional with $I(0) = 0$. We define

an *outer source of* I to be an isotone $\varphi : \mathfrak{lm}(E) \to [0,\infty]$ with $\varphi(\varnothing) = 0$,
an *inner source of* I to be an isotone $\varphi : \mathfrak{um}(E) \to [0,\infty]$ with $\varphi(\varnothing) = 0$,

such that φ represents $I : I(f) = ⨍ f d\varphi$ for all $f \in E$. For this purpose the domains $\mathfrak{lm}(E)$ and $\mathfrak{um}(E)$ are the smallest possible ones. Under the assumption that E is *Stonean* the Greco representation theorem 5.7 with the above equivalences asserts that I *has outer/inner sources iff it is Stonean and truncable.* In this case a set function φ as above on $\mathfrak{lm}(E)/\mathfrak{um}(E)$ is an outer/inner source of I iff

$$I_\star(\chi_S) \leqq \varphi(S) \leqq I^\star(\chi_S) \quad \text{for all } S \in \mathfrak{lm}(E)/\mathfrak{um}(E).$$

Moreover we note that $I < \infty$ implies that $I^\star(\chi.) < \infty$ on $\mathfrak{lm}(E)/\mathfrak{um}(E)$. Therefore in case $I < \infty$ all outer and inner sources of I must be finite.

We come to the first step toward the representation theorems.

7.1 Outer Theorem ([1998b] 4.1). *Assume that E is Stonean. Then an isotone $I : E \to [0,\infty]$ with $I(0) = 0$ has outer sources which are upward $\bullet$ continuous iff I is Stonean and upward $\bullet$ continuous. Assume that this holds true, and put $\varphi = I_\star(\chi.)|\mathfrak{lm}(E)$. Then*

1) *φ is the unique outer source of I which is upward $\bullet$ continuous.*
2) *$\varphi^\bullet = I^\bullet(\chi.)$.*
3) *φ is submodular $\Leftrightarrow$ I is submodular.*
4) *If I is submodular then $I^\bullet(f) = ⨍ f d\varphi^\bullet$ for all $f \in [0,\infty]^X$.*

7.2 Inner Theorem ([1998b] 4.2). *Assume that E is Stonean and $\subset [0,\infty[^X$. Then an isotone $I : E \to [0,\infty[$ with $I(0) = 0$ has inner sources which are downward $\bullet$ continuous iff I is Stonean and downward $\bullet$ continuous. Assume that this holds true, and put $\varphi = I^\star(\chi.)|\mathfrak{um}(E)$. Then*

1) *φ is the unique inner source of I which is downward $\bullet$ continuous.*
2) *$\varphi_\bullet = I_\bullet(\chi.)$.*

3) *φ is supermodular* $\Leftrightarrow$ *I is supermodular.*

4) *If I is supermodular then $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$.*

The Outer Situation

Let $E \subset [0,\infty]^X$ be positive-homogeneous with $0 \in E$ and a Stonean lattice. An isotone $I : E \to [0,\infty]$ with $I(0) = 0$ is called an *outer $\bullet$ preintegral* iff it has an outer source which is an outer $\bullet$ premeasure. After 7.1 then $\varphi = I_\star(\chi.)|\mathfrak{Im}(E)$ is the unique such one. One proves the counterpart of the former outer $\bullet$ extension theorem which follows.

7.3 Outer $\bullet$ Representation Theorem ([1998b] 5.3). *For an isotone functional $I : E \to [0,\infty]$ with $I(0) = 0$ the following are equivalent.*

1) *I is an outer $\bullet$ preintegral.*

2) *I is submodular and Stonean and upward $\bullet$ continuous; and $I(v) \geqq I(u) + I^\bullet(v-u)$ for all $u \leqq v$ in E with $u < \infty$. Furthermore*

($\bullet$) $\quad I^\bullet(f) = \sup\{I^\bullet(f \wedge u) : u \in [I < \infty]\}$ *for all* $f \in [I^\bullet < \infty]$.

3) *I is submodular and Stonean and upward $\bullet$ continuous; and $I^\bullet(v) \geqq I(u) + I^\bullet(v-u)$ for all $u \leqq v$ with $u \in E$ and $v \in E^\bullet$ such that $u < \infty$.*

In this case $\varphi = I_\star(\chi.)|\mathfrak{Im}(E)$ is the unique outer source of I which is an outer $\bullet$ premeasure. It fulfils $I^\bullet(f) = ⨍ f d\varphi^\bullet$ for all $f \in [0,\infty]^X$.

Moreover $E^\bullet \subset \mathrm{M}(\mathfrak{C}(\varphi^\bullet))$, that is the members of $E^\bullet$ are measurable $\mathfrak{C}(\varphi^\bullet)$.

We add at once that condition ($\bullet$) is superfluous for $\bullet = \sigma$, because it follows from 5.10.8.Out) when I is submodular. But in case $\bullet = \tau$ it cannot be dispensed with ([1998b] 5.4).

7.4 Addendum ([1998b] 6.1). *Assume that I is is an outer $\bullet$ preintegral, and put $\Phi = \varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$. For $f \in [0,\infty]^X$ then the following are equivalent.*

1) *f is integrable with respect to Φ.*

2) $\inf\{⨍ |f-u| d\Phi^\star : u \in \mathrm{S}([\varphi < \infty])\} = 0$.

3) $\inf\{⨍ |f-u| d\Phi^\star : u \in E$ *with* $u < \infty$ *and* $I(u) < \infty\} = 0$.

This addendum extends a fundamental equivalence from the traditional $\bullet = \sigma\tau$ representation theories (see section 8 below). We turn to the counterpart of the specialization 3.3. It deserves particular attention because it comprises all those traditional $\bullet = \sigma\tau$ representation theories. The decisive assumption is of course ii).

7.5 Specialization. *Assume that*

i) *E is stable under addition, and hence a Stonean lattice cone with $0 \in E$;*

ii) *$v - u \in E^\bullet$ for all $u \leqq v$ in E with $u < \infty$;*

and that $I : E \to [0,\infty]$ is isotone with $I(0) = 0$ and additive. *Then in* 7.3 *the equivalent condition* 3) *reduces to*

3) *I is upward $\bullet$ continuous.*

In fact, the new condition is necessary. So assume that it is fulfilled. To be shown is the last part in the previous condition 3). Thus fix $u \in E$ and $v \in E^\bullet$ with $u \leqq v$ and $u < \infty$. Then first of all $v - u \in E^\bullet$. For $h \in E$ with $h \leqq v - u$ we have $I^\bullet(v) \geqq I(u+h) = I(u) + I(h)$. It follows that $I^\bullet(v) \geqq I(u) + I_\star(v-u) = I(u) + I^\bullet(v-u)$. □

THE INNER SITUATION

As before the inner situation requires finiteness assumptions. Let $E \subset [0,\infty[^X$ be positive-homogeneous with $0 \in E$ and a Stonean lattice. An isotone $I : E \to [0,\infty[$ with $I(0) = 0$ is called an *inner $\bullet$ preintegral* iff it has an inner source which is an inner $\bullet$ premeasure. After 7.2 then $\varphi = I^\star(\chi.)|\mathfrak{um}(E)$ is the unique such one. As before one proves the counterpart of the former inner $\bullet$ extension theorem 3.5, and for the same reason as before one needs certain *satellites* of the inner $\bullet$ envelopes $I_\bullet$ of I: For $v \in E$ we define $I_\bullet^v : [0,\infty]^X \to [0,\infty]$ to be

$$I_\bullet^v(f) = \sup\{\inf_{u \in M} I(u) : M \subset E \text{ nonvoid } \bullet \text{ with } M \downarrow\leqq f \text{ and } u \leqq v \text{ for all } u \in M\}.$$

These satellites are isotone and fulfil $\sup\limits_{v\in E} I_\bullet^v = I_\bullet$; moreover we have $I_\star(f) \leqq I_\bullet^v(f)$ when $f \leqq v$.

7.6 INNER $\bullet$ REPRESENTATION THEOREM ([1998b] 5.8). *For an isotone functional $I : E \to [0,\infty[$ with $I(0) = 0$ the following are equivalent.*

1) *I is an inner $\bullet$ preintegral.*

2) *I is supermodular and Stonean and downward $\bullet$ continuous; and $I(v) \leqq I(u) + I_\bullet(v-u)$ for all $u \leqq v$ in E.*

3) *I is supermodular and Stonean and downward $\bullet$ continuous at 0; and $I(v) \leqq I(u) + I_\bullet^v(v-u)$ for all $u \leqq v$ in E.*

In this case $\varphi = I^\star(\chi.)|\mathfrak{um}(E)$ is the unique inner source of I which is an inner $\bullet$ premeasure. It fulfils $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$.

Moreover $E_\bullet \subset \mathrm{M}(\mathfrak{C}(\varphi_\bullet))$, that is the members of $E_\bullet$ are measurable $\mathfrak{C}(\varphi_\bullet)$.

7.7 ADDENDUM ([1998b] 6.2). *Assume that I is an inner $\bullet$ preintegral, and put $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. For $f \in [0,\infty]^X$ then the following are equivalent.*

1) *f is integrable with respect to Φ.*

2) $\inf\{⨍ |f-u| d\Phi^\star : u \in \mathrm{S}(\mathfrak{um}(E))\} = 0$.

3) $\inf\{⨍ |f-u| d\Phi^\star : u \in E\} = 0$.

The subsequent counterpart of the specialization 3.7 comprises the traditional $\bullet = \sigma\tau$ representation theories like the above 7.5, but it will be seen to do better.

7.8 SPECIALIZATION. *Assume that*

i) *E is stable under addition, and hence a Stonean lattice cone with $0 \in E$;*

ii) *$v - u \in E_\bullet$ for all $u \leqq v$ in E;*

and that $I : E \to [0, \infty[$ *is isotone and* additive (which implies that $I(0) = 0$). *Then in* 7.6 *the equivalent condition* 3) *reduces to*

3) I *is downward* $\bullet$ *continuous at* 0.

In fact, to be shown is the last part in the previous condition 3). Thus let $u \leqq v$ in E, and fix $M \subset E$ nonvoid $\bullet$ with $M \downarrow v - u$ such that $h \leqq v$ for all $h \in M$. For $h \in M$ then $v - u \leqq h$ and hence $I(v) \leqq I(u + h) = I(u) + I(h)$. It follows that $I(v) \leqq I(u) + \inf\limits_{h \in M} I(h) \leqq I(u) + I^v_\bullet(v - u)$. $\square$

The General Riesz Representation Theorem

The remainder of the article consists of two applications of the representation theorems of the present section. We start here with an extension of the Riesz representation theorem from locally compact to arbitrary Hausdorff topological spaces. It is based on the full inner $\bullet$ representation theorem 7.6 in the version $\bullet = \tau$. The second application will be the topic of the final section. It will be based on the two specializations 7.5 and 7.8.

Let X be a Hausdorff topological space. We define $\mathrm{CK}(X) \subset \mathrm{USCK}(X) \subset [0, \infty[^X$ to consist of the continuous and of the upper semicontinuous functions $f : X \to [0, \infty[$ which vanish outside of certain compact subsets of X. These functions have compact level sets $[f \geqq t]$ for $0 < t < \infty$.

We fix a *Stonean lattice cone* $E \subset \mathrm{USCK}(X)$ with $0 \in E$. Examples are $E = \mathrm{CK}(X)$ and $E = \mathrm{USCK}(X)$ themselves. Thus $\mathfrak{um}(E) \subset \mathrm{Comp}(X)$, and hence also $(\mathfrak{um}(E))_\tau \subset \mathrm{Comp}(X)$. We define E to be *rich* iff $(\mathfrak{um}(E))_\tau = \mathrm{Comp}(X)$. Of course $E = \mathrm{USCK}(X)$ itself is rich.

7.9 Remark (MI 16.3). $E = \mathrm{CK}(X)$ *is rich iff* X *is locally compact.*

In fact, if $E = \mathrm{CK}(X)$ is rich then for each compact $K \subset X$ there are $f \in \mathrm{CK}(X)$ and $0 < t < \infty$ with $K \subset [f \geqq t]$. Thus $K \subset [f \geqq t] \subset [f > s] \subset [f \geqq s]$ for $0 < s < t$, where $[f > s]$ is open and $[f \geqq s]$ is compact. As to the opposite direction, it is well-known for locally compact X that each compact $K \subset X$ is the intersection of the $[f \geqq 1]$ extended over the $f \in \mathrm{CK}(X)$ with $\chi_K \leqq f$. $\square$

We turn to functionals. We start with a fundamental observation.

7.10 Dini Consequence (MI 16.4). *If* E *is rich then all isotone and positive-homogeneous* $I : E \to [0, \infty[$ *are downward* τ *continuous at* 0.

In fact, let $M \subset E$ be nonvoid with $M \downarrow 0$. We can assume that there is some $P \in M$ with $u \leqq P\ \forall u \in M$. Let $K \subset X$ be compact with $[P > 0] \subset K$ and hence with $[u > 0] \subset K\ \forall u \in M$. Then in view of richness let $Q \in E$ be such that $K \subset [Q \geqq 1]$. For $u \in M$ then $u \leqq (\sup u)Q$ and hence $I(u) \leqq (\sup u)I(Q)$, with of course $\sup u = \sup(u|K)$. Thus the assertion follows from the Dini theorem. $\square$

Now assume that E be rich. An isotone $I : E \to [0, \infty[$ with $I(0) = 0$ is an inner τ preintegral iff there exists an inner τ premeasure $\varphi : \mathfrak{um}(E) \to [0, \infty[$

with $I(f) = ⨍ f d\varphi$ for all $f \in E$. After 3.8.Inn) and $(\mathfrak{um}(E))_\tau = \mathrm{Comp}(X)$ it is equivalent that there exists a Radon premeasure $\psi : \mathrm{Comp}(X) \to [0,\infty[$ with $I(f) = ⨍ f d\psi$ for all $f \in E$. In this case we see from 7.6 and 3.8.Inn) that both φ and ψ are unique: We have $\varphi = I^\star(\chi_\cdot)|\mathfrak{um}(E)$ and $\psi = \varphi_\tau|\mathrm{Comp}(X) = \varphi^\star|\mathrm{Comp}(X)$ from 2.2.3.Inn). From this one obtains for $K \in \mathrm{Comp}(X)$ that

$$\begin{aligned}\psi(K) &= \varphi^\star(K) = \inf\{\varphi(A) = I^\star(\chi_A) : A \in \mathfrak{um}(E) \text{ with } A \supset K\}\\ &= \inf\{I^\star(\chi_{[u \geqq 1]}) : u \in E \text{ with } [u \geqq 1] \supset K\}\\ &= \inf\{I(v) : u, v \in E \text{ with } [v \geqq 1] \supset [u \geqq 1] \supset K\}\\ &= \inf\{I(v) : v \in E \text{ with } v \geqq \chi_K\} = I^\star(\chi_K),\end{aligned}$$

so that $\psi = I^\star(\chi_\cdot)|\mathrm{Comp}(X)$. We also recall from 3.8.Inn) that $\varphi_\tau = \psi_\tau = \psi_\star$.

We combine all this with 7.6 in case $\bullet = \tau$ to obtain the Riesz representation theorem in the comprehensive version which follows.

7.11 Riesz Representation Theorem. *Let $E \subset \mathrm{USCK}(X)$ be a rich Stonean lattice cone with $0 \in E$. Then for an isotone and positive-linear* (:=additive and positive-homogeneous) *functional $I : E \to [0,\infty[$ there exists a Radon premeasure $\psi : \mathrm{Comp}(X) \to [0,\infty[$ such that $I(f) = ⨍ f d\psi$ for all $f \in E$ iff*

$$(0) \qquad I(v) \leqq I(u) + I_\tau^v(v-u) \quad \textit{for all } u \leqq v \textit{ in } E.$$

In this case $\psi = I^\star(\chi_\cdot)|\mathrm{Comp}(X)$ is the unique such Radon premeasure on X. It fulfils $I_\tau(f) = ⨍ f d\psi_\tau$ for all $f \in [0,\infty]^X$.

Moreover $E_\tau \subset \mathrm{M}(\mathfrak{C}(\psi_\tau))$, that is the members of E_τ are measurable $\mathfrak{C}(\psi_\tau)$.

7.12 Addendum. *Assume in addition that $v - u \in E_\tau$ for all $u \leqq v$ in E. Then each isotone and positive-linear functional $I : E \to [0,\infty[$ fulfils* (0)*, and hence fulfils the assertions of the above theorem.*

This addendum follows from 7.8. In particular it combines with 7.9 to furnish the traditional Riesz representation theorem $E = \mathrm{CK}(X)$ for locally compact X. We repeat that the new theorem contains no trace of local finiteness.

In conclusion we note that our inner $\bullet = \tau$ representation theorem 7.6 is capable of a multitude of further Riesz type representation theorems. There are quite some examples in MI section 16 and in [2000b] section 3.

8. Comparison with the Traditional Daniell-Stone and Bourbaki Procedures

The final section compares the traditional Daniell-Stone procedure and its amendment in terms of the so-called *essential* formation due to Bourbaki with the present outer and inner $\bullet$ representation theories in section 7, of which this time the specializations 7.5 and 7.8 are the adequate versions. The comparison with the essential formation requires some ad hoc work within the present theories, but the result will be remarkable.

The Traditional Daniell-Stone Procedure

For this procedure we refer to Pfeffer [1977], Floret [1981], and Leinert [1995]. Let $F \subset \mathbb{R}^X$ be a *Stonean lattice subspace* of real-valued functions on a nonvoid set X, that is a linear subspace which is stable under the pointwise formations $\vee$ and $\wedge$ and fulfils $f \in F \Rightarrow f \wedge t \in F$ for $0 < t < \infty$. We fix an *isotone* and *linear* (=:positive linear) functional $J : F \to \mathbb{R}$. We assume from the start that J is $\bullet$ *continuous* in the obvious sense, once more with $\bullet = \sigma\tau$; note that *upward* and *downward* $\bullet$ *continuous* amount to the same. This time the different models for envelope formation amount to the same, because one can at once pass from series to sequences. Thus one defines the *outer* $\bullet$ *envelopes* $J^\bullet : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ to be

$$J^\bullet(f) = \inf\{\sup_{u \in M} J(u) : M \subset F \text{ nonvoid } \bullet \text{ with } M \uparrow\geqq f\}.$$

It is then common to define $J_\bullet : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ to be $J_\bullet(f) = -J^\bullet(-f)$, which of course amounts to

$$J_\bullet(f) = \sup\{\inf_{u \in M} J(u) : M \subset F \text{ nonvoid } \bullet \text{ with } M \downarrow\leqq f\}.$$

The properties below are all obvious except the last one; for its proof we refer to the author's paper [1992a] 5.1.

8.1 Properties. 0) *$J^\bullet|F = J$.*

1) *$J^\bullet$ is isotone and positive-homogeneous with $J^\bullet(0) = 0$.*

2) *$J^\bullet$ is subadditive for the addition $\dot{+}$. Hence $J_\bullet \leqq J^\bullet$.*

3) *For $f \in \overline{\mathbb{R}}^X$ one has the equivalence*

$$\inf_{u \in F} J^\bullet(|f - u|) = 0 \Longleftrightarrow J^\bullet(f) = J_\bullet(f) \in \mathbb{R}.$$

The basic idea of the Daniell-Stone procedure is to define a function $f \in \overline{\mathbb{R}}^X$ to be $\bullet$ *integrable for J* iff it satisfies the two equivalent conditions in 8.1.3) above. We then write $J^\bullet(f) = J_\bullet(f) =: J\bullet(f) \in \mathbb{R}$. The class of these functions will be denoted $\bullet\mathrm{Int}(J)$.

After this one passes to a certain set function. One forms the set systems $\mathfrak{m} := \{A \subset X : \chi_A \in \bullet\mathrm{Int}(J)\}$ and $\mathfrak{M} := \mathfrak{m}\top\mathfrak{m}$, the members of which are called the $\bullet$*integrable* and the $\bullet$*measurable subsets for J*. Define $\beta : \mathfrak{M} \to [0, \infty]$ to be $\beta(A) = J\bullet(\chi_A)$ for $A \in \mathfrak{m}$ and $\beta(A) = \infty$ for $A \in \mathfrak{M}$ not in $\mathfrak{m}$. The traditional main theorem then reads as follows.

8.2 Theorem. *$\beta : \mathfrak{M} \to [0, \infty]$ is a measure on the σ algebra $\mathfrak{M}$. A function $f \in \overline{\mathbb{R}}^X$ is in $\bullet\mathrm{Int}(J)$ iff it is integrable with respect to β. In this case $J\bullet(f) = \int f d\beta$.*

In most of the presentations which are based on functionals one is content with the platform thus achieved, and the above theorem forms their centre of development for measure and integration in some sense or other. However, in the topological situation Bourbaki [1967] Préface and Schwartz [1973] p.16

noted that the functionals $J^\bullet$ with their descendants $\bullet\mathrm{Int}(J)$ and β be less adequate than desirable, and in order to overcome this flaw they built another level on top of the edifice in form of their *essential* construction.

In fact, the subsequent comparison with the development of the present article will make clear that these complaints are justified in full. But it will also become clear that our procedure renders the essential construction superfluous at all, and this for the simple reason that it is a procedure which runs on both the outer and the inner road from the start.

Preparations for the Comparison

We collect a few simple facts. Let X be a nonvoid set. 1) The Stonean lattice subspaces $F \subset \mathbb{R}^X$ are in one-to-one correspondence with the Stonean lattice cones $E \subset [0,\infty[^X$ such that $0 \in E$ and $v - u \in E$ for all $u \leqq v$ in E, via each of the two maps

$$\begin{aligned} &F \mapsto E := F \cap [0,\infty[^X = \{f \in F : f \geqq 0\} = \{f^+ = f \vee 0 : f \in F\}, \text{ and} \\ &E \mapsto F := E - E, \end{aligned}$$

which are inverse to each other. In fact, the unique nontrivial point is the relation

$$(v-u) \wedge t = v - (v-t)^+ \vee u \quad \text{for } u, v, t \in \mathbb{R} \text{ with } u \geqq 0,$$

which ensures that the second of these maps is well-defined.

2) Fix $F \subset \mathbb{R}^X$ and $E \subset [0,\infty[^X$ as in 1). Then the isotone and linear functionals $J : F \to \mathbb{R}$ are in one-to-one correspondence with the positive-linear and hence isotone functionals $I : E \to [0,\infty[$, via $J \mapsto I := J|E$. Also J is $\bullet$ continuous as defined above iff I is upward $\bullet$ continuous /downward $\bullet$ continuous/downward $\bullet$ continuous at 0 as defined earlier, notions which all coincide under the present particular form of E. Moreover we claim that in this case

$$I^\bullet = J^\bullet|[0,\infty]^X \quad \text{and} \quad I_\bullet = J_\bullet|[0,\infty]^X.$$

To see the first relation for $f \in [0,\infty]^X$, note that $I^\bullet(f) \geqq J^\bullet(f)$ is clear. To prove $I^\bullet(f) \leqq J^\bullet(f)$ let $M \subset F$ be nonvoid $\bullet$ with $M \uparrow P \geqq f$. For fixed $v \in M$ then $\{u \wedge v^+ : u \in M\} \subset F$ is nonvoid $\bullet$ with $\uparrow P \wedge v^+ = v^+$, so that $J(v^+) = \sup\limits_{u \in M} J(u \wedge v^+) \leqq \sup\limits_{u \in M} J(u)$. Since $\{v^+ : v \in M\} \subset E$ is nonvoid $\bullet$ with $\uparrow P \geqq f$, it follows that

$$I^\bullet(f) \leqq \sup_{v \in M} I(v^+) = \sup_{v \in M} J(v^+) \leqq \sup_{u \in M} J(u),$$

and hence the assertion. The proof of the second relation is similar, but much simpler.

Thus we have paved the road which permits to pass from one situation to the other.

Return to the $\bullet$ Representation Theorems

After this we return to section 7 for recollection of the specializations 7.5 and 7.8 and an important addendum.

Let $E \subset [0,\infty[^X$ be a Stonean lattice cone such that $0 \in E$ and $v - u \in E$ for all $u \leqq v$ in E. Assume that the positive-linear and hence isotone functional $I : E \to [0,\infty[$ is $\bullet$ continuous as above. Thus I is an *outer* and *inner* $\bullet$ *preintegral.* The set function $\psi = I_\star(\chi.)|\mathfrak{lm}(E)$ is the unique outer $\bullet$ premeasure on $\mathfrak{lm}(E)$ such that $I(f) = \fint f d\psi$ for all $f \in E$. It fulfils

$$I^\bullet(f) = \fint f d\psi^\bullet \quad \text{for all } f \in [0,\infty]^X,$$

and $E^\bullet \subset \mathrm{M}(\mathfrak{C}(\psi^\bullet))$. Likewise the set function $\varphi = I^\star(\chi.)|\mathfrak{um}(E)$ is the unique inner $\bullet$ premeasure on $\mathfrak{um}(E)$ such that $I(f) = \fint f d\varphi$ for all $f \in E$. It fulfils

$$I_\bullet(f) = \fint f d\varphi_\bullet \quad \text{for all } f \in [0,\infty]^X,$$

and $E_\bullet \subset \mathrm{M}(\mathfrak{C}(\varphi_\bullet))$. The addendum announced above now reads as follows.

8.3 Proposition ([1999b] 5.3). 1) *The lattices* $\mathfrak{S} := \mathfrak{um}(E)$ *and* $\mathfrak{T} := \mathfrak{lm}(E)$ *fulfil* $\mathfrak{T} \subset (\mathfrak{S}\top\mathfrak{S})\perp$ *and* $\mathfrak{S} \subset (\mathfrak{T}\top\mathfrak{T})\perp$*, and hence are* $\bullet$ *complemental.*

2) *The* $\bullet$ *premeasures* $\varphi : \mathfrak{S} \to [0,\infty[$ *and* $\psi : \mathfrak{T} \to [0,\infty]$ *are* $\bullet$ *complemental, that is they are both* $\bullet$ *tame and fulfil* $\psi = \varphi_\bullet|\mathfrak{T}$ *and* $\varphi = \psi^\bullet|\mathfrak{S}$.

For the proof of 1) we refer to the cited source, but we include the proof of 2) in an improved version. i) We know from the initial part of section 7 that $I < \infty$ implies that $\psi < \infty$. ii) Each $S \in \mathfrak{S}$ is contained in some $T \in \mathfrak{T}$. In fact, one has $[f \geqq s] \subset [f > t]$ for $f \in E$ and $0 < t < s < \infty$. iii) From i)ii) we see that $\psi^\bullet|\mathfrak{S} < \infty$. iv) 3.6.ii) implies that $\varphi_\bullet|\mathfrak{T}$ is upward $\bullet$ continuous. Likewise 3.2.ii) implies that $\psi^\bullet|\mathfrak{S}$ is almost downward $\bullet$ continuous, and hence downward $\bullet$ continuous in view of iii). v) Now the functions $f \in E$ are in $\mathrm{UM}(\mathfrak{S})$ and in $\mathrm{LM}(\mathfrak{T})$. Therefore

$$I(f) = I^\bullet(f) = \fint f d\psi^\bullet = \fint f d(\psi^\bullet|\mathfrak{S}),$$
$$I(f) = I_\bullet(f) = \fint f d\varphi_\bullet = \fint f d(\varphi_\bullet|\mathfrak{T}).$$

Thus $\psi^\bullet|\mathfrak{S}$ is an inner source of I, and since it is downward $\bullet$ continuous by iv) it follows from 7.2 that $\psi^\bullet|\mathfrak{S} = \varphi$. Likewise it follows from 7.1 that $\varphi_\bullet|\mathfrak{T} = \psi$. vi) The inner $\bullet$ premeasure φ is $\bullet$ tame. In fact, for $S \in \mathfrak{S}$ there exists a $T \in \mathfrak{T}$ with $S \subset T$ by ii), and from v)i) we obtain $\varphi_\bullet(T) = \psi(T) < \infty$. Now from 4.6 the assertion follows. $\square$

Thus we have from 4.6 and 4.8 the two maximal $\bullet$ extensions $\Phi = \varphi_\bullet|\mathfrak{C}$ and $\Psi = \psi^\bullet|\mathfrak{C}$ on the same $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\psi^\bullet) =: \mathfrak{C}$, and the measure $\delta := \Psi \setminus \Phi$ on $\mathfrak{C}$ which fulfils $\Phi + \delta = \Psi$ and attains but the values 0 and ∞. We repeat that in face of this situation the inner $\bullet$ procedure and its result Φ appear to be more profound than the outer $\bullet$ procedure and its result Ψ.

The First Step of Comparison

After this extended recollection we take up the comparison. We fix $F \subset \mathbb{R}^X$ and $J : F \to \mathbb{R}$ and their $E := F \cap [0,\infty[^X$ and $I := J|E$ as before. The comparison will use the results of the new outer and inner $\bullet$ procedures, whereas the traditional main theorem 8.2 will not be used but will be reobtained at once. The fundamental link is the relation

$$J^\bullet(f) = I^\bullet(f) = \fint f d\psi^\bullet = \fint f d\Psi^\star \quad \text{for } f \in [0,\infty]^X,$$

where 3.9.Out) has been applied.

8.4 Theorem ([1999b] 5.2). i) *A function $f \in [0,\infty]^X$ is in* $\bullet\mathrm{Int}(J)$ *iff it is integrable with respect to Ψ, that is measurable $\mathfrak{C}$ with $\int f d\Psi < \infty$. In this case $J\bullet(f) = \int f d\Psi$; we have even $J^\bullet(f) = \fint f d\Psi^\star$ for all $f \in [0,\infty]^X$.* ii) *$\beta = \Psi$ on $\mathfrak{M} = \mathfrak{C}$.*

Proof. i) A function $f \in [0,\infty]^X$ is defined to be in $\bullet\mathrm{Int}(J)$ iff $\inf\{J^\bullet(|f-u|) : u \in F\} = 0$, which in view of $|f-u^+| \leqq |f-u|$ can be written $\inf\{J^\bullet(|f-u|) : u \in E\} = 0$. After the above fundamental relation combined with 7.4 this means indeed that f is integrable with respect to Ψ. ii) A subset $A \subset X$ is defined to be in $\mathfrak{m}$ iff $\chi_A \in \bullet\mathrm{Int}(J)$, hence by i) iff $A \in \mathfrak{C}$ with $\Psi(A) < \infty$, and then $\beta(A) = J\bullet(\chi_A) = \Psi(A)$. Thus $\mathfrak{m} = [\Psi < \infty]$. This implies that $\mathfrak{M} = \mathfrak{m}\top\mathfrak{m} = [\Psi < \infty]\top[\Psi < \infty]$ fulfils $\mathfrak{C} \subset \mathfrak{M} \subset [\Psi < \infty]\top\mathfrak{C} \subset \mathfrak{C}$ and hence $\mathfrak{M} = \mathfrak{C}$. Then after all $\beta = \Psi$ from its definition. □

In view of this theorem the objections which came from Bourbaki [1967] and Schwartz [1973] can be understood only too well: The traditional Daniell-Stone procedure does not lead to the most appropriate result Φ, but to the less appropriate Ψ. For the supporters of that procedure this fact must be a disappointment, because the equivalence in 8.1.3) seemed to indicate that the method takes equal care of both outer and inner aspects. But this is not so, since as we have seen the heart of the procedure is an approximation in the seminorm $J^\bullet(|\cdot|)$ and hence of outer nature.

If for all that one wanted to retain the basic idea of the approach, that is to form first of all the class of *integrable functions with respect to the functional J itself*, before it comes to measures if at all, then it looks mysterious how the procedure could be modified in order to lead to the result Φ instead of Ψ. In fact, one would need another seminorm in place of $J^\bullet(|\cdot|)$, that is another *sublinear* functional in place of $J^\bullet$, so that $J_\bullet$ were out of consideration.

But the question found a certain positive answer in Bourbaki [1956] with the functional $[J^\bullet] : [0,\infty]^X \to [0,\infty]$, defined to be

$$[J^\bullet](f) = \sup\{J^\bullet(u) : u \in [0,\infty]^X \text{ with } u \leqq f \text{ and } J^\bullet(u) < \infty\},$$

and called the *essential upper integral* attached to $J^\bullet$, or rather to $J^\bullet|[0,\infty]^X$. One verifies that $[J^\bullet]$ is indeed isotone and sublinear. Thus it is reasonable to define a function $f \in \overline{\mathbb{R}}^X$ to be *essentially $\bullet$ integrable for J* iff

$$\inf_{u \in F} [J^\bullet](|f-u|) = 0.$$

The class of these functions will be denoted $\bullet\mathrm{EssInt}(J)$. As before one then passes to a certain set function. It is of course the hope and expectation that, in the terms of the present article, this set function will turn out to be the above measure Φ.

As before we shall not pursue this development, but rather relate it to the present outer and inner $\bullet$ procedures with the means of these procedures. This will require some ad hoc work within these procedures under the headline of an *essential formation*, which in our context stands somewhat apart and therefore will be put into a separate subsection.

The Essential Formation

Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ in X. For $\varphi : \mathfrak{S} \to [0,\infty]$ isotone with $\varphi(\varnothing) = 0$ and *submodular* we define the *essential satellite* $\varphi_\circ : \mathfrak{P}(X) \to [0,\infty]$ to be $\varphi_\circ = (\varphi|[\varphi < \infty])_\star$. Note that the assumption *submodular* implies that $[\varphi < \infty] \subset \mathfrak{S}$ is a *lattice* as well.

8.5 Properties ([1998a] 6.1). 1) *$\varphi_\circ \leqq \varphi$ on $\mathfrak{S}$, and $\varphi_\circ = \varphi$ on $[\varphi < \infty]$.*

2) *$\varphi_\circ$ is isotone with $\varphi_\circ(\varnothing) = 0$, and $\varphi_\circ|\mathfrak{S}$ is submodular.*

3) *φ is upward $\bullet$ continuous $\Rightarrow$ $\varphi_\circ$ is upward $\bullet$ continuous.*

4) *In case $\mathfrak{S} = \mathfrak{P}(X)$ we have $\mathfrak{C}(\varphi_\circ) = \mathfrak{C}(\varphi)$.*

We see that the essential formation is of a *hybrid nature*: It is a formation of *inner* type, but its formal properties are those of an *outer* formation. Therefore it has no central place in our systematic theories. For the actual purpose we need some further details in the special cases of contents and measures ([1998a] 6.2).

$\star$) Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a content on an algebra $\mathfrak{A}$. We see from 3.3 and 3.7 that α and $\alpha|[\alpha < \infty]$ are outer $\star$ premeasures and $\alpha|[\alpha < \infty]$ is an inner $\star$ premeasure. One verifies that $\alpha^\star = (\alpha|[\alpha < \infty])^\star$. Therefore 3.10 implies that $\mathfrak{C}(\alpha^\star) = \mathfrak{C}(\alpha_\circ) =: \mathfrak{C} \supset \mathfrak{A}$.

σ) Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a measure on a σ algebra $\mathfrak{A}$. We see from 3.3 and 3.7 that α and $\alpha|[\alpha < \infty]$ are outer σ premeasures and $\alpha|[\alpha < \infty]$ is an inner σ premeasure. One verifies that $\alpha^\sigma = (\alpha|[\alpha < \infty])^\sigma$ is $= \alpha^\star = (\alpha|[\alpha < \infty])^\star$, and that $(\alpha|[\alpha < \infty])_\sigma$ is $= (\alpha|[\alpha < \infty])_\star = \alpha_\circ$. Therefore $\mathfrak{C} \supset \mathfrak{A}$ is a σ algebra, and $\alpha^\star|\mathfrak{C}$ and $\alpha_\circ|\mathfrak{C}$ are measures.

At this point we insert a consequence which is needed in order to prove the above theorem 4.9 on the quasi-Radon measures of Fremlin [1974][2000]. It has a routine proof.

8.6 Consequence. *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a measure on a σ algebra $\mathfrak{A}$. Then α is complete and saturated iff $\mathfrak{C} = \mathfrak{A}$.*

The central result of the subsection is the theorem which follows. It is also in Fremlin [2000] exercise 213X(g). The result will be used in form of the specialization 8.8.ii) below.

8.7 THEOREM ([1998a] 6.3). *Let $\alpha : \mathfrak{A} \to [0,\infty]$ be a measure on a σ algebra $\mathfrak{A}$. Then $(\alpha^\star)_\circ = (\alpha_\circ|\mathfrak{C})^\star$.*

We turn to the context which will be needed for the present purpose.

8.8 THEOREM ([1998a] 6.5). *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty]$ form a $\bullet$ complemental couple of $\bullet$ premeasures with $\Phi = \varphi_\bullet|\mathfrak{C}$ and $\Psi = \psi^\bullet|\mathfrak{C}$ on $\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\psi^\bullet) =: \mathfrak{C}$, once more with $\bullet = \sigma\tau$.*

i) *We have $\Psi_\circ = \Phi_\circ = \varphi_\bullet$. In particular $\Psi_\circ|\mathfrak{C} = \Phi$.*

ii) *From* i) *and* 8.7 *we have $(\Psi^\star)_\circ = (\Phi^\star)_\circ = \Phi^\star$.*

Proof. i) We look at the restrictions of $\varphi_\bullet$ to the members of the chain $\mathfrak{S}_\bullet \subset [\Psi < \infty] \subset [\Phi < \infty] \subset \mathfrak{P}(X)$, and recall from 4.6 that $\Psi = \Phi$ on $[\Psi < \infty]$. Then the inner $\star$ envelopes of these restrictions fulfil

$$\varphi_\bullet = (\varphi_\bullet|\mathfrak{S}_\bullet)_\star \leqq (\Psi|[\Psi < \infty])_\star \leqq (\Phi|[\Phi < \infty])_\star \leqq \varphi_\bullet.$$

Thus we have the assertion. ii) then indeed follows from 8.7. $\square$

The subsection concludes with the essential formation for *functionals.* For $T : [0,\infty]^X \to [0,\infty]$ isotone and *sublinear* we define the *essential satellite* $T_\circ : [0,\infty]^X \to [0,\infty]$ to be $T_\circ = (T|[T < \infty])_\star$.

8.9 PROPERTIES ([1998a] 9.2). 1) *$T_\circ \leqq T$, and $T_\circ = T$ on $[T < \infty]$.*

2) *$T_\circ$ is isotone and sublinear.*

3) *T is upward $\bullet$ continuous $\Rightarrow$ $T_\circ$ is upward $\bullet$ continuous.*

The final result asserts that the two essential formations are related in the natural manner.

8.10 PROPOSITION ([1998a] 9.3). *Let $\Theta : \mathfrak{P}(X) \to [0,\infty]$ be isotone with $\Theta(\varnothing) = 0$ and submodular. Define $T : [0,\infty]^X \to [0,\infty]$ to be $T(f) = ⨍ f d\Theta$, so that T is isotone and sublinear. Then $T_\circ(f) = ⨍ f d\Theta_\circ$ for all $f \in [0,\infty]^X$.*

THE SECOND STEP OF COMPARISON

We fix $F \subset \mathbb{R}^X$ and $J : F \to \mathbb{R}$ and their $E := F \cap [0,\infty[^X$ and $I := J|E$ as before. After the last subsection the second step of comparison will be a matter of a few lines.

8.11 THEOREM ([1999b] section 5). *A function $f \in [0,\infty]^X$ is in* $\bullet$EssInt(J) *iff it is integrable with respect to Φ, that is measurable $\mathfrak{C}$ with $\int f d\Phi < \infty$. In this case $[J^\bullet](f) = \int f d\Phi$; we have even $[J^\bullet](f) = ⨍ f d\Phi^\star$ for all $f \in [0,\infty]^X$.*

Proof. From the former fundamental relation $J^\bullet(f) = ⨍ f d\Psi^\star$ for $f \in [0,\infty]^X$ and from the definition $[J^\bullet] = (J^\bullet|[0,\infty]^X)_\circ$ combined with 8.10 we obtain $[J^\bullet](f) = \fint f d(\Psi^\star)_\circ$ for $f \in [0,\infty]^X$. Thus 8.8.ii) furnishes

$$[J^\bullet](f) = \fint f d\Phi^\star \quad \text{for } f \in [0,\infty]^X.$$

Now a function $f \in [0,\infty]^X$ is in $\bullet\mathrm{EssInt}(J)$ iff $\inf\{[J^\bullet](|f-u|) : u \in E\} = 0$, as before in view of $|f-u^+| \leqq |f-u|$ for $u \in F$. According to 7.7 this means that f is integrable with respect to Φ. $\square$

At the end of the section it should be clear that we did not claim too much: The Bourbaki version of the Daniell-Stone procedure with its two steps

first to form $J^\bullet$ in order to obtain the measure Ψ, and
then to form $[J^\bullet]$ in order to obtain the measure Φ,

represents an enormous complication and detour when compared with the outer and inner $\bullet$ representation theories of section 7. The unique treatise after Bourbaki [1967] and Schwartz [1973] known to the author with full treatment of the second step is Anger-Portenier [1992]. We have said that as a rule the treatises on measure and integration which are based on functionals are content with the traditional first step, even though it ends with the less favourable $\beta = \Psi$. However, Leinert [1995] chapters 4 and 14 proceeds to define $\Phi := \beta_\circ|\mathfrak{M}$ and to develop some of its properties, in particular that $\Phi = J_\bullet(\chi.)|\mathfrak{M}$ and that Φ is inner regular $\mathfrak{S}_\bullet$.

We conclude to recall that the $\bullet$ representation procedures in section 7 are much more comprehensive than the earlier theories considered in the present section. In fact, our comparison did not invoke section 7 but on the level of its specializations 7.5 and 7.8.

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