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Projective Limits via Inner Premeasures and the true Wiener Measure

Heinz König

Dedicated to Professor Gustave Choquet

Abstract. The paper continues the author's work in measure and integration, which is an attempt at unified systematization. It establishes projective limit theorems of the Prokhorov and Kolmogorov types in terms of inner premeasures. Then it specializes to obtain the (one-dimensional) Wiener measure on the space of real-valued functions on the positive halfline as a probability measure defined on an *immense domain*: In particular the subspace of continuous functions will be *measurable* of full measure - and not merely of full *outer* measure, as the usual projective limit theorems permit to conclude.

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The present paper wants to continue the author's chain of contributions to measure and integration. This is an attempt at unified systematization, with the particular aim to incorporate the topological theory into the abstract one. The basic idea is to develop and to convert the classical extension method due to Carathéodory into a few different procedures. These procedures are parallel to each other, but diversified in two respects: On the one hand as to their basic *inner* or *outer* character, and on the other hand as to their *discrete*, *sequential* or *nonsequential* limit behaviour. Since 1996 there are the book [11] (cited as MI) and a series of subsequent papers, and the recent survey article [15]. A number of topics has been treated with unified results which extend and improve the former ones in both of the conventional theories. A typical example is the formation of products in MI chapter VII and [13].

The present paper will be devoted to the formation of *projective limits*, like the formation of products in the *inner* context. This is a topic of particular importance, and in fact considered to be a crucial one. We quote a statement from Dellacherie-Meyer [4] of 1978 pp.65/66.

> If abstract measure theory ... is compared to the theory of Radon measures ..., it may seem that the latter is superior to the former on four counts. These are, by order of decreasing importance,
> - the existence of a good theorem on *inverse* (*projective*) *limits* of measures,
> - the existence of some reasonable topologies ... on the space of measures,
> - the possibility of passing to the limit along uncountable increasing families of lsc (:=lower semicontinuous) functions,
> - the removal of certain σ finiteness restrictions.
>
> The notion of a ... Radon measure has a counterpart in abstract theory: the notion of inner regular measure with respect to a compact paving. This notion seems to have some applications, but not of great importance.

A similar statement is in the Introduction of Schwartz [20]. We agree with the order of the four topics above, and in particular with position one for the projective limits. But otherwise it must be added that the statement is outdated with the appearance of our systematization. There is an abundance of topics which support this claim, and we think that the present article should be of particular emphasis.

In the sequel we shall obtain projective limit theorems in the spirit of our systematization, of the Prokhorov type in section 4 and of the Kolmogorov type in section 5. Before that we need to recall and to develop the infrastructure of our enterprise in sections 1-3. The extent of these sections comes from their obvious novelty compared with the usual procedures, but not at all from added complications. The two sections 4 and 5 contain the former respective results, but will be much more comprehensive. Section 4 will illuminate the nature of the Prokhorov condition (Π) as an equivalent to inner regularity, but not at all related to downward continuity. It seems that in projective limit theorems there is no source for continuity other than compactness (or perfectness), much in contrast to the situation of product measures.

At last section 6 will reveal how comprehensive the projective limit versions in sections 4 and 5 are: We specialize to obtain the (one-dimensional) Wiener measure as the maximal inner τ (:=nonsequential) extension of a simple and natural inner τ premeasure of mass one on the space $\mathbb{R}^{[0,\infty[}$ of all real-valued functions on $[0,\infty[$. Its domain is immense compared with the usual product σ algebra, the members of which are of a certain *countable* type. In particular this domain contains the subspace $\mathrm{C}([0,\infty[,\mathbb{R})$ of continuous functions as a member of full measure - while so far this subspace was but a creature of outer measure one and inner measure zero. This puts a final end to the possible (though somewhat bizarre) view that Wiener measure could equally well be considered as concentrated on the complement of $\mathrm{C}([0,\infty[,\mathbb{R})$. In section 6 we do not work with the usual probabilistic notions like stochastic processes and their modifications. To be sure, the proof of our main theorem furnishes at the same time the usual theorem on the existence of continuous modifications [1] 39.3, but this result turns up well before our new weapon, that powerful inner τ lift, comes into action.

The author thinks that the present results will have quite some influence on the probabilistic concepts around stochastic processes. He also plans to devote another paper to the familiar set-theoretical construction of projective limits, like in Bourbaki [3] chapter III section 7, and to its implications in the present context.

1. Recollections and Complements on the Inner Extension Theories

We adopt the terms of MI and [15] but shall recall the most basic and less obvious notions and facts. Let X be a nonvoid set. For $S \subset X$ the complement will be denoted S'. For a set function $\theta : \mathfrak{P}(X) \to [0,\infty]$ with $\theta(\varnothing) = 0$ we recall the *Carathéodory class*

$$\mathfrak{C}(\theta) = \{A \subset X : \theta(M) = \theta(M \cap A) + \theta(M \cap A') \text{ for all } M \subset X\}.$$

$\mathfrak{C}(\theta)$ turns out to be an algebra, and $\theta|\mathfrak{C}(\theta)$ to be a content.

The extension theories come in three parallel versions marked $\bullet = \star\sigma\tau$, where $\star$ stands for *finite*, σ for *sequential* or countable, and τ for *nonsequential* or arbitrary. For a nonvoid set system $\mathfrak{S}$ in X we define $\mathfrak{S}_\bullet$ and $\mathfrak{S}^\bullet$ to consist of the intersections and unions of its nonvoid $\bullet$ subsystems. In the sequel let $\mathfrak{S}$ be a lattice of subsets in X with $\varnothing \in \mathfrak{S}$. We restrict ourselves to the inner situation.

The fundamental definitions are for an isotone set function $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$. We define an *inner $\bullet$ extension* of φ to be an extension $\alpha : \mathfrak{A} \to [0,\infty]$ of φ which is a content on a ring, and such that moreover $\mathfrak{S}_\bullet \subset \mathfrak{A}$ with

$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$), and
α is inner regular $\mathfrak{S}_\bullet$.

We define φ to be an *inner $\bullet$ premeasure* iff it admits inner $\bullet$ extensions. The subsequent *inner extension theorem* characterizes those φ which are inner $\bullet$ premeasures, and then describes all inner $\bullet$ extensions of φ. The theorem is in terms of the *inner $\bullet$ envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$ of φ, defined to be

$$\varphi_\bullet(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A\},$$

where $\mathfrak{M} \downarrow\subset A$ means that $\mathfrak{M}$ is downward directed with intersection contained in A. We also need their *satellites* $\varphi_\bullet^B : \mathfrak{P}(X) \to [0,\infty]$ with $B \subset X$, defined to be

$$\varphi_\bullet^B(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A \text{ and } M \subset B\ \forall M \in \mathfrak{M}\}.$$

We recall that $\varphi_\bullet$ is inner regular $\mathfrak{S}_\bullet$. Moreover $\varphi = \varphi_\bullet|\mathfrak{S}$ iff φ is downward $\bullet$ continuous, and $\varphi_\bullet(\varnothing) = 0$ iff φ is downward $\bullet$ continuous at $\varnothing$.

Theorem 1.1 (Inner Extension Theorem). *Assume that* $\varphi : \mathfrak{S} \to [0,\infty[$ *is isotone with* $\varphi(\varnothing) = 0$. *Then* φ *is an inner* $\bullet$ *premeasure iff*

φ *is supermodular and downward* $\bullet$ *continuous at* $\varnothing$, *and*
$\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$.

In this case $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is an inner* $\bullet$ *extension of* φ, *and a measure on a* σ *algebra when* $\bullet = \sigma\tau$. *All inner* $\bullet$ *extensions of* φ *are restrictions of* Φ. *Moreover we have the* localization principle *which reads*

for $A \subset X$: $S \cap A \in \mathfrak{C}(\varphi_\bullet)$ *for all* $S \in \mathfrak{S} \Longrightarrow A \in \mathfrak{C}(\varphi_\bullet)$.

Thus we have $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$. It is plain that the members of $\mathfrak{S}_\bullet$ are the most basic measurable subsets. We also recall a special case of particular importance: $\mathfrak{S}$ is called $\bullet$ *compact* iff each nonvoid $\bullet$ subsystem $\mathfrak{M} \subset \mathfrak{S}$ fulfils $\mathfrak{M} \downarrow \varnothing \Rightarrow \varnothing \in \mathfrak{M}$. It is obvious that in this case the above functions φ are all downward $\bullet$ continuous at $\varnothing$.

The most natural example is that X is a Hausdorff topological space with $\mathfrak{S} = \mathrm{Comp}(X)$. For an isotone set function $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ then the conditions $\bullet = \star\sigma\tau$ in 1.1 are identical, and if fulfilled produce the same $\varphi_\bullet$ and hence $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. In this case φ is called a *Radon premeasure* and Φ the *maximal Radon measure* which comes from φ. The localization principle implies that $\mathfrak{C}(\varphi_\bullet) \supset \mathrm{Bor}(X)$.

So far the direct recollections of MI and [15]. We continue with a few simple facts which we will be of constant use. As before let X be a nonvoid set and $\mathfrak{S}$ be a lattice of subsets in X with $\varnothing \in \mathfrak{S}$.

Remark 1.2. Let $\psi : \mathfrak{S}_\bullet \to [0,\infty[$ be isotone with $\psi(\varnothing) = 0$. If $\psi|\mathfrak{S}$ is downward $\bullet$ continuous at $\varnothing$, then ψ is downward $\bullet$ continuous at $\varnothing$ as well.

Proof. Let $\mathfrak{M} \subset \mathfrak{S}_\bullet$ be nonvoid $\bullet$ with $\mathfrak{M} \downarrow \varnothing$. Then from MI 6.6 = [15] 2.1.Inn) there exists $\mathfrak{N} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{N} \downarrow \varnothing$ such that each $N \in \mathfrak{N}$ contains some $M \in \mathfrak{M}$. Thus each $N \in \mathfrak{N}$ fulfils $\inf\limits_{M \in \mathfrak{M}} \psi(M) \leqq \psi(N)$, so that we obtain $0 \leqq \inf\limits_{M \in \mathfrak{M}} \psi(M) \leqq \inf\limits_{N \in \mathfrak{N}} \psi(N) = 0$. □

Remark 1.3. Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$. i) If φ is an inner $\star$ premeasure and downward $\bullet$ continuous at $\varnothing$, then φ is an inner $\bullet$ premeasure. In view of MI 6.32 the converse need not be true, but there is a partial converse in ii) below.

ii) Assume that $\mathfrak{S} = \mathfrak{S}_\bullet$. If φ is downward $\bullet$ continuous, then $\varphi_\bullet = \varphi_\star$. Hence if φ is an inner $\bullet$ premeasure, then φ is an inner $\star$ premeasure.

Proof. i) Combine $\varphi_\star(A) \leqq \varphi_\bullet^B(A)$ for $A \subset B \in \mathfrak{S}$ with $\star$ and $\bullet$ in 1.1. ii) The first assertion is MI 6.5.iv) = [15] 2.2.4.Inn). The second one then follows from $\bullet$ and $\star$ in 1.1.

The subsequent remark has been announced without proof in [15] 3.8.Inn).

Remark 1.4. The inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ and the inner $\bullet$ premeasures $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ are in one-to-one correspondence via $\phi = \varphi_\bullet|\mathfrak{S}_\bullet$ and $\varphi = \phi|\mathfrak{S}$. Moreover then $\varphi_\bullet = \phi_\bullet = \phi_\star$.

Proof. i) Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure and $\phi := \varphi_\bullet|\mathfrak{S}_\bullet$. Then $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ, and hence an inner $\bullet$ extension of ϕ. Thus ϕ is an inner $\bullet$ premeasure. Next we have $\varphi_\bullet = \phi_\bullet$, since this holds true on $\mathfrak{S}_\bullet$ and both sides are inner regular $\mathfrak{S}_\bullet$. At last $\phi_\bullet = \phi_\star$ from 1.3.ii) above. ii) Let $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ be an inner $\bullet$ premeasure and $\varphi := \phi|\mathfrak{S}$. Then $\Phi = \phi_\bullet|\mathfrak{C}(\phi_\bullet)$ is an inner $\bullet$ extension of ϕ, and hence an inner $\bullet$ extension of φ. Thus φ is an inner $\bullet$ premeasure, and 1.1 asserts that $\varphi_\bullet = \phi_\bullet$ on $\mathfrak{C}(\phi_\bullet)$. In particular $\varphi_\bullet|\mathfrak{S}_\bullet = \phi$. $\square$

Next we recall the fundamental downward $\bullet$ continuity assertions MI 6.7 = [15] 2.8.2.Inn) and MI 6.27 = [15] 3.6.i).

Remark 1.5. Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$ and supermodular. σ) φ_σ and φ_τ are almost downward σ continuous. τ) If φ is downward τ continuous, then $\varphi_\tau|\mathfrak{S}\top\mathfrak{S}_\tau$ is almost downward τ continuous.

The subsequent lemma comes from our treatment of direct images for inner $\bullet$ premeasures in [12] section 3. It also extends [13] 2.10.

Lemma 1.6. *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure. Assume that $\mathfrak{R}$ is a lattice in X with $\varnothing \in \mathfrak{R} \subset \mathfrak{S}\top\mathfrak{S}_\bullet$ such that $\varphi_\bullet|\mathfrak{R} < \infty$ and that $\varphi_\bullet$ is inner regular $\mathfrak{R}_\bullet$. Then $\vartheta := \varphi_\bullet|\mathfrak{R}$ is an inner $\bullet$ premeasure and fulfils $\vartheta_\bullet = \varphi_\bullet$.*

Proof. i) We have $\mathfrak{R}_\bullet \subset \mathfrak{S}\top\mathfrak{S}_\bullet$ and $\varphi_\bullet|\mathfrak{R}_\bullet < \infty$, and hence 1.5 asserts that $\varphi_\bullet|\mathfrak{R}_\bullet$ is downward $\bullet$ continuous. In particular ϑ is downward $\bullet$ continuous, and hence MI 6.5.iii) = [15] 2.2.3.Inn) asserts that $\vartheta_\bullet|\mathfrak{R}_\bullet$ is downward $\bullet$ continuous. ii) From i) we have $\vartheta_\bullet = \vartheta = \varphi_\bullet$ on $\mathfrak{R}$ and hence $\vartheta_\bullet = \varphi_\bullet$ on $\mathfrak{R}_\bullet$. Thus $\vartheta_\bullet = \varphi_\bullet$ on $\mathfrak{P}(X)$, since both sides are inner regular $\mathfrak{R}_\bullet$. iii) Now $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet) = \vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ is a content on an algebra which fulfils $\mathfrak{R} \subset \mathfrak{R}_\bullet \subset \mathfrak{S}\top\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$, and hence an extension of ϑ; after i) it is an inner $\bullet$ extension of ϑ. Therefore ϑ is an inner $\bullet$ premeasure. $\square$

This terminates the plain part of the section. We continue to recall the old results MI 6.15 and 6.17 on the Carathéodory class $\mathfrak{C}(\cdot)$, which were part of the deeper foundations of the edifice built in MI chapter II (and resulted via transcription from the respective outer results MI 4.20 and 4.22). We restrict ourselves to the special case which will be needed in the sequel, that is to $\mathfrak{P} = \mathfrak{H} = \{\varnothing\}$.

Proposition 1.7. *Assume that $\xi : \mathfrak{P}(X) \to [0,\infty]$ is isotone with $\xi(\varnothing) = 0$ and supermodular. Let the nonvoid set system $\mathfrak{T}$ in X be upward directed such that $\xi|\mathfrak{T} < \infty$ and that ξ is inner regular $\sqsubset \mathfrak{T}$* (defined to consist of the subsets of the members of $\mathfrak{T}$).

1) *If $A \subset X$ fulfils $\xi(T) \leqq \xi(T \cap A) + \xi(T \cap A')$ for all $T \in \mathfrak{T}$, then $A \in \mathfrak{C}(\xi)$.*

2) *If the isotone set function $\eta : \mathfrak{P}(X) \to [0,\infty]$ fulfils $\eta|\mathfrak{T} = \xi|\mathfrak{T}$ and $\eta \leqq \xi$, then $\xi|\mathfrak{C}(\xi)$ is an extension of $\eta|\mathfrak{C}(\eta)$.*

The above proposition will be invoked several times in the sequel. At the moment we note a consequence of part 2) which is an extension of MI 18.2.

Proposition 1.8. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be lattices with $\varnothing$ in X, and assume that*

$\varphi : \mathfrak{S} \to [0,\infty[$ *is isotone with* $\varphi(\varnothing) = 0$ *and supermodular, and*
$\psi : \mathfrak{T} \to [0,\infty[$ *is isotone with* $\psi(\varnothing) = 0$.

If $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$, then

$$\varphi_\bullet = \psi_\bullet \text{ on } \mathfrak{T}_\bullet \Longrightarrow \varphi_\bullet|\mathfrak{C}(\varphi_\bullet) \text{ is an extension of } \psi_\bullet|\mathfrak{C}(\psi_\bullet);$$

and we have $\Longleftarrow$ whenever ψ is an inner $\bullet$ premeasure.

Proof. $\Rightarrow$) Follows from 1.7.2) applied to $\xi := \varphi_\bullet$ and $\eta := \psi_\bullet$ and to $\mathfrak{T}$. It suffices to note that $\varphi_\bullet = \psi_\bullet$ on $\mathfrak{T}_\bullet$ implies that $\varphi_\bullet \geqq \psi_\bullet$ on $\mathfrak{P}(X)$. $\Leftarrow$) For $T \in \mathfrak{T}_\bullet$ we have $T \in \mathfrak{C}(\psi_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$ and $\varphi_\bullet(T) = \psi_\bullet(T)$. $\square$

Our final point is on the *cut-off procedure* for an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ presented in MI 9.21. We show that the procedure can be extended from the $E \in \mathfrak{C}(\varphi_\bullet)$ to arbitrary subsets $E \subset X$. We recall that in case $\bullet = \tau$ the former procedure led to the basic decomposition theorem MI 9.24 with 9.25 = [15] 4.11.

We define a content $\alpha : \mathfrak{A} \to [0,\infty]$ on an algebra $\mathfrak{A}$ in X to *live on* $E \subset X$ iff all $A \subset E'$ fulfil $A \in \mathfrak{A}$ and $\alpha(A) = 0$. This is more than required in the usual notion of a *thick* subset $E \subset X$, for example in Fremlin [5] 132F, the definition of which is that those $A \subset E'$ which are in $\mathfrak{A}$ have $\alpha(A) = 0$.

Theorem 1.9. *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ and $E \subset X$. Define $\varphi^E : \mathfrak{S} \to [0,\infty[$ to be $\varphi^E(S) = \varphi_\bullet(S \cap E)$ for $S \in \mathfrak{S}$. Then φ^E is an inner $\bullet$ premeasure and fulfils*

1) $(\varphi^E)_\bullet(A) = \varphi_\bullet(A \cap E)$ *for all* $A \subset X$.
2) $\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}((\varphi^E)_\bullet)$.
3) $\Phi^E = (\varphi^E)_\bullet|\mathfrak{C}((\varphi^E)_\bullet)$ *lives on* E.

4) *The following are equivalent.* 4.i) $\varphi = \varphi^E$. 4.ii) $\varphi_\bullet(A) = \varphi_\bullet(A \cap E)$ *for all* $A \subset X$. 4.iii) $E \in \mathfrak{C}(\varphi_\bullet)$ *and* $\Phi(E') = 0$. 4.iv) Φ *lives on* E.

Proof. We define $\Theta : \mathfrak{P}(X) \to [0,\infty]$ to be $\Theta(A) = \varphi_\bullet(A \cap E)$ for $A \subset X$. Thus Θ is isotone with $\Theta(\varnothing) = 0$ and $\Theta|\mathfrak{S}_\bullet < \infty$.

i) We claim that

$$\Theta(A \cup B) + \Theta(A \cap B) = \Theta(A) + \Theta(B) \quad \text{for } A \in \mathfrak{C}(\varphi_\bullet) \text{ and } B \subset X.$$

In fact, we have

$$\begin{aligned}
\Theta(A \cup B) + \Theta(A \cap B) &= \varphi_\bullet((A \cup B) \cap E) + \varphi_\bullet((A \cap B) \cap E) \\
&= \Big(\varphi_\bullet\big(A \cap ((A \cup B) \cap E)\big) + \varphi_\bullet\big(A' \cap ((A \cup B) \cap E)\big)\Big) + \varphi_\bullet((A \cap B) \cap E) \\
&= \varphi(A \cap E) + \big(\varphi_\bullet(A' \cap (B \cap E)) + \varphi_\bullet(A \cap (B \cap E))\big) \\
&= \varphi_\bullet(A \cap E) + \varphi_\bullet(B \cap E) = \Theta(A) + \Theta(B).
\end{aligned}$$

ii) $\Theta|\mathfrak{S}_\bullet < \infty$ is downward $\bullet$ continuous. In fact, let $\mathfrak{M} \subset \mathfrak{S}_\bullet$ be nonvoid $\bullet$ with $\mathfrak{M} \downarrow D \in \mathfrak{S}_\bullet$. For $M \in \mathfrak{M}$ then i) furnishes

$$\begin{aligned}\Theta(M) &= \Theta(D \cup (M \setminus D)) + \Theta(D \cap (M \setminus D)) = \Theta(D) + \Theta(M \setminus D) \\ &\leqq \Theta(D) + \varphi_\bullet(M \setminus D) = \Theta(D) + \big(\varphi_\bullet(M) - \varphi_\bullet(D)\big),\end{aligned}$$

and hence the assertion.

iii) Θ is inner regular $\mathfrak{S}_\bullet$. To see this let $A \subset X$ and $c < \Theta(A) = \varphi_\bullet(A \cap E)$. Then there exists $S \in \mathfrak{S}_\bullet$ such that $S \subset A \cap E$ and $c < \varphi_\bullet(S)$. Thus on the one hand $S \subset A$, and on the other hand $S \subset E$ and hence $c < \varphi_\bullet(S) = \varphi_\bullet(S \cap E) = \Theta(S)$.

iv) We have $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$. The restriction $\vartheta := \Theta|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ^E which is isotone and modular by i), and hence a content on $\mathfrak{C}(\varphi_\bullet)$. By ii)iii) it is an inner $\bullet$ extension of φ^E. Thus φ^E is an inner $\bullet$ premeasure, and we have $\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}((\varphi^E)_\bullet)$ and $(\varphi^E)\bullet = \vartheta = \Theta$ on $\mathfrak{C}(\varphi_\bullet)$. In particular $(\varphi^E)\bullet = \Theta$ on $\mathfrak{S}_\bullet$, and hence $(\varphi^E)_\bullet = \Theta$ on $\mathfrak{P}(X)$ since both sides are inner regular $\mathfrak{S}_\bullet$ by iii). Thus we have the proved the initial assertion and 1)2).

v) To see 3) let $A \subset E'$. For $M \subset X$ then

$$\begin{aligned}(\varphi^E)_\bullet(M \cap A) + (\varphi^E)_\bullet(M \cap A') &= \varphi_\bullet(M \cap A \cap E) + \varphi_\bullet(M \cap A' \cap E) \\ &= 0 + \varphi_\bullet(M \cap E) = (\varphi^E)_\bullet(M),\end{aligned}$$

so that $A \in \mathfrak{C}((\varphi^E)_\bullet)$. Then $\Phi^E(A) = (\varphi^E)_\bullet(A) = \varphi_\bullet(A \cap E) = 0$.

vi) It remains to prove 4). We have 4.i) $\Rightarrow \Phi = \Phi^E \Rightarrow$ 4.iv) from 3). 4.iv) $\Rightarrow$ 4.iii) is obvious. 4.iii) $\Rightarrow$ 4.ii) because $\varphi_\bullet(A) = \varphi_\bullet(A \cap E) + \varphi_\bullet(A \cap E') = \varphi_\bullet(A \cap E)$ for all $A \subset X$. 4.ii) $\Rightarrow$ 4.i) is obvious. $\square$

We conclude with a pair of important properties of an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ which is such that $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ lives on $E \subset X$. We form the set system $\mathfrak{T} = \mathfrak{S} \sqcap E := \{S \cap E : S \in \mathfrak{S}\}$ with $\mathfrak{T}_\bullet = \mathfrak{S}_\bullet \sqcap E = \{S \cap E : S \in \mathfrak{S}_\bullet\} \subset \mathfrak{C}(\varphi_\bullet)$, and the set function $\psi = \varphi_\bullet|\mathfrak{T} : \mathfrak{T} \to [0, \infty[$. Then $\mathfrak{T}$ is a lattice with $\varnothing \in \mathfrak{T}$ in both X and E, and ψ is defined on a set system $\mathfrak{T}$ which is in both X and E. It is plain that at times these two rôles must not be mixed up. Thus in the latter rôles $\mathfrak{T}$ and ψ will be denoted $\mathfrak{T}_\circ$ and $\psi_\circ$. It follows that $\psi_\bullet : \mathfrak{P}(X) \to [0, \infty]$ and $(\psi_\circ)_\bullet : \mathfrak{P}(E) \to [0, \infty]$ are connected via $(\psi_\circ)_\bullet = \psi_\bullet|\mathfrak{P}(E)$.

Theorem 1.10. *Let $\varphi : \mathfrak{S} \to [0, \infty[$ be an inner $\bullet$ premeasure such that $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ lives on $E \subset X$, and let $\Phi|E$ be the restriction of Φ to $\mathfrak{C}(\varphi_\bullet) \sqcap E = \{A \in \mathfrak{C}(\varphi_\bullet) : A \subset E\}$. In the above notations then*

1) *ψ is an inner $\bullet$ premeasure which fulfils $\varphi_\bullet = \psi_\bullet$ and hence $\Phi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$.*
2) *$\psi_\circ$ is an inner $\bullet$ premeasure which fulfils $\Phi|E = (\psi_\circ)_\bullet|\mathfrak{C}((\psi_\circ)_\bullet)$.*

Proof of 1). 1.0) We first show that for each nonvoid $\bullet$ and downward directed $\mathfrak{N} \subset \mathfrak{T}_\bullet$ there exists an $\mathfrak{M} \subset \mathfrak{S}_\bullet$ of the same kind such that $\mathfrak{N} = \mathfrak{M} \sqcap E = \{M \cap E : M \in \mathfrak{M}\}$. In fact, for each $N \in \mathfrak{N}$ fix some $F(N) \in \mathfrak{S}_\bullet$ with $N = F(N) \cap E$, and then form $f(N) := \bigcap_{R \in \mathfrak{N}, R \supset N} F(R) \in \mathfrak{S}_\bullet$. Thus

$$f(N) \cap E = \bigcap_{R \in \mathfrak{N}, R \supset N} F(R) \cap E = \bigcap_{R \in \mathfrak{N}, R \supset N} R = N \quad \text{for } N \in \mathfrak{N}.$$

Moreover $A \subset B$ in $\mathfrak{N}$ implies that $f(A) \subset f(B)$. Thus $\mathfrak{M} := \{f(N) : N \in \mathfrak{N}\} \subset \mathfrak{S}_\bullet$ is as required.

1.i) $\varphi_\bullet|\mathfrak{T}_\bullet = \Phi|\mathfrak{T}_\bullet < \infty$ is downward $\bullet$ continuous. In fact, take $\mathfrak{N} \subset \mathfrak{T}_\bullet$ nonvoid $\bullet$ with $\mathfrak{N} \downarrow B \in \mathfrak{T}_\bullet$ and then $\mathfrak{M} \subset \mathfrak{S}_\bullet$ nonvoid $\bullet$ as in 1.0), so that $\mathfrak{M} \downarrow A \in \mathfrak{S}_\bullet$ with $A \cap E = B$. The fact that Φ lives on E implies that $\inf\limits_{N \in \mathfrak{N}} \Phi(N) = \inf\limits_{M \in \mathfrak{M}} \Phi(M) = \Phi(A) = \Phi(B)$.

1.ii) Φ is inner regular $\mathfrak{T}_\bullet$. To see this let $A \in \mathfrak{C}(\varphi_\bullet)$ and $c < \Phi(A)$. Then there exists $S \in \mathfrak{S}_\bullet$ with $S \subset A$ and $c < \Phi(S)$. Hence $T := S \cap E \in \mathfrak{T}_\bullet$ with $T \subset A$ and $c < \Phi(S) = \Phi(S \cap E) = \Phi(T)$.

1.iii) We see from 1.i)ii) that Φ is an inner $\bullet$ extension of ψ. Thus ψ is an inner $\bullet$ premeasure, and we have $\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\psi_\bullet)$ and $\psi_\bullet = \varphi_\bullet$ on $\mathfrak{C}(\varphi_\bullet)$. From $\mathfrak{S}_\bullet, \mathfrak{T}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$ we obtain $\psi_\bullet = \varphi_\bullet$ on $\mathfrak{P}(X)$.

Proof of 2). 2.i) In view of 1) and $(\psi_\circ)_\bullet = \psi_\bullet|\mathfrak{P}(E)$ the inner extension theorem 1.1 shows that $\psi_\circ$ is an inner $\bullet$ premeasure.

2.ii) We have $\Phi|E = \varphi_\bullet|\{A \in \mathfrak{C}(\varphi_\bullet) : A \subset E\}$. Now 1.9.4) asserts that $\varphi_\bullet(M) = \varphi_\bullet(M \cap E)$ for all $M \subset X$. Hence for $A \subset E$ we have $A \in \mathfrak{C}(\varphi_\bullet)$ iff $\varphi_\bullet(M) = \varphi_\bullet(M \cap A) + \varphi_\bullet(M \cap A')$ for all $M \subset E$, which in view of $M \cap A' = M \cap E \cap A' = M \cap (E \setminus A)$ and of the above says that $A \in \mathfrak{C}((\psi_\circ)_\bullet)$. Therefore $\Phi|E = (\psi_\circ)_\bullet|\mathfrak{C}((\psi_\circ)_\bullet)$. $\square$

2. The Transplantation Theorem

The present section is a continuation of MI section 18. We want to establish a further transplantation theorem for inner $\star$ premeasures. The main intermediate step is an extension theorem for finite contents, which is based on the well-known extension method due to Łoś-Marczewski [18]. It is a close relative of Lipecki [17] theorem 1 (and subsequent work of this author). The present proof will be in the spirit of MI section 18.

We start to recall the basic result of Łoś-Marczewski [18] theorem 1 in the version of MI 18.29. For $\mathfrak{A}$ a ring in the nonvoid set X and $E \subset X$ we form

$$\text{the lattice } \mathfrak{A}[E] := \{M \cup (N \cap E) : M, N \in \mathfrak{A}\} \text{ and}$$
$$\text{the ring } \mathfrak{A}(E) := \{(M \cap E') \cup (N \cap E)\} : M, N \in \mathfrak{A}\}.$$

Thus $\mathfrak{A} \subset \mathfrak{A}[E] \subset \mathfrak{A}(E)$ and $\mathfrak{A}(E) = \mathrm{R}(\mathfrak{A}[E])$, where $\mathrm{R}(\cdot)$ denotes the generated ring. Also note that $\mathfrak{A}(E)$ is upward enclosable $\mathfrak{A}$.

Proposition 2.1. *Let $\alpha : \mathfrak{A} \to [0, \infty[$ be a content with the $\star$ envelopes $\alpha_\star, \alpha^\star : \mathfrak{P}(X) \to [0, \infty]$. Define $\xi, \eta : \mathfrak{A}(E) \to [0, \infty[$ to be*

$$\xi(S) = \alpha_\star(S \cap E) + \alpha^\star(S \cap E'),$$
$$\eta(S) = \alpha_\star(S \cap E') + \alpha^\star(S \cap E).$$

Then ξ and η are contents and fulfil $\xi = \alpha_\star$ and $\eta = \alpha^\star$ on $\mathfrak{A}[E]$, in particular $\xi = \eta = \alpha$ on $\mathfrak{A}$.

In the sequel we use the notation

$$\mathfrak{P} \sqcap \mathfrak{Q} := \{P \cap Q : P \in \mathfrak{P} \text{ and } Q \in \mathfrak{Q}\} \quad \text{and} \quad \mathfrak{P} \sqcap E := \{P \cap E : P \in \mathfrak{P}\}$$

for nonvoid $\mathfrak{P}, \mathfrak{Q} \subset \mathfrak{P}(X)$, and $E \subset X$ as before. We fix in X a ring $\mathfrak{A}$ and a nonvoid set system $\mathfrak{M}$ which is *totally ordered* under inclusion $\subset$. The extension theorem in question reads as follows.

Theorem 2.2. *Let $\alpha : \mathfrak{A} \to [0, \infty[$ be a content. Then there exists a unique content $\beta : \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M}) \to [0, \infty[$ such that $\beta = \alpha_\star$ on $\mathfrak{A} \sqcap \mathfrak{M}$, that is*

$$\beta(A \cap M) = \alpha_\star(A \cap M) \quad \textit{for all } A \in \mathfrak{A} \textit{ and } M \in \mathfrak{M}.$$

The uniqueness assertion is clear from the classical uniqueness theorem contained in MI 3.1.$\star$), since $\mathfrak{A} \sqcap \mathfrak{M}$ is stable under $\cap$. Thus to be shown is the existence assertion. For its proof we can assume that $X \in \mathfrak{M}$. Then $\mathfrak{A} \subset \mathfrak{A} \sqcap \mathfrak{M} \subset \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M})$, and the desired content β is an extension of α.

Proof of the existence assertion. We first assume that $n := \operatorname{card}(\mathfrak{M}) < \infty$. In case $n = 1$ we have $\mathfrak{M} = \{X\}$ and $\mathfrak{A} = \mathfrak{A} \sqcap \mathfrak{M} = \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M})$, so that $\beta = \alpha$ does it. For the induction step $1 \leqq n \Rightarrow n+1$ assume that $\mathfrak{M} = \{E_0, E_1, \cdots, E_n\}$ with $E_0 = X \supset E_1 \supset \cdots \supset E_n$, and put $\mathfrak{N} = \{E_0, \cdots, E_{n-1}\}$. We claim that

$$\mathfrak{A} \subset \mathfrak{A} \sqcap \mathfrak{N} \subset \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{N}) =: \mathfrak{B} \subset \mathfrak{B}[E_n] \subset \mathfrak{B}(E_n) = \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M}).$$

To be shown is the last $=$. It rests upon the well-known formula

$$\mathrm{R}(\mathfrak{P}) \sqcap E = \mathrm{R}(\mathfrak{P} \sqcap E) \quad \text{for nonvoid } \mathfrak{P} \subset \mathfrak{P}(X) \text{ and } E \subset X,$$

which for $E \subset E_{n-1}$ implies that

$$\begin{aligned} \mathfrak{B} \sqcap E = \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{N}) \sqcap E &= \mathrm{R}((\mathfrak{A} \sqcap \mathfrak{N}) \sqcap E) \\ &= \mathrm{R}(\mathfrak{A} \sqcap E) = \mathrm{R}(\mathfrak{A}) \sqcap E = \mathfrak{A} \sqcap E. \end{aligned}$$

Thus $\mathfrak{B}(E_n) = \mathrm{R}(\mathfrak{B}[E_n]) = \mathrm{R}\big(\mathfrak{B} \cup (\mathfrak{B} \sqcap E_n)\big)$ is the ring generated by $\mathfrak{B}$ and $\mathfrak{B} \sqcap E_n = \mathfrak{A} \sqcap E_n$, and hence in fact the ring generated by $(\mathfrak{A} \sqcap \mathfrak{N}) \cup (\mathfrak{A} \sqcap E_n) = \mathfrak{A} \sqcap \mathfrak{M}$.

By the induction hypothesis there exists a content $\beta : \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{N}) = \mathfrak{B} \to [0, \infty[$ such that $\beta = \alpha_\star$ on $\mathfrak{A} \sqcap \mathfrak{N}$, in particular $\beta = \alpha$ on $\mathfrak{A}$. Then by 2.1 there exists a content $\xi : \mathfrak{B}(E_n) = \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M}) \to [0, \infty[$ such that $\xi = \beta_\star$ on $\mathfrak{B}[E_n]$, in particular $\xi = \beta$ on $\mathfrak{B}$. To be shown is $\xi = \alpha_\star$ on $\mathfrak{A} \sqcap \mathfrak{M} = (\mathfrak{A} \sqcap \mathfrak{N}) \cup (\mathfrak{A} \sqcap E_n)$.

To see this we note on the one hand that on $\mathfrak{A} \sqcap \mathfrak{N} \subset \mathfrak{B}$ in fact $\xi = \beta = \alpha_\star$. On the other hand $\mathfrak{A} \sqcap E_n \subset \mathfrak{A}[E_n] \subset \mathfrak{B}[E_n]$, so that on $\mathfrak{A} \sqcap E_n$ we have $\xi = \beta_\star$ and thus have to prove that $\beta_\star = \alpha_\star$, which amounts to $\beta_\star \leqq \alpha_\star$, since β is an extension of α and hence $\beta_\star \geqq \alpha_\star$. Now in order to prove $\beta_\star(S) \leqq \alpha_\star(S)$ for an $S \in \mathfrak{A} \sqcap E_n$, we look at the subsets $B \in \mathfrak{B}$ with $B \subset S$. We have $B \in \mathfrak{B} \sqcap E_{n-1} = \mathfrak{A} \sqcap E_{n-1} \subset \mathfrak{A} \sqcap \mathfrak{N}$, and hence $\beta(B) = \alpha_\star(B) \leqq \alpha_\star(S)$. It follows that $\beta_\star(S) \leqq \alpha_\star(S)$ as claimed. This finishes the induction step and hence the case of finite $\mathfrak{M}$.

At last we assume that $\mathfrak{M}$ is an infinite totally ordered set system in X with $X \in \mathfrak{M}$. For each finite $\mathfrak{P} \subset \mathfrak{M}$ with $X \in \mathfrak{P}$ the above furnishes a unique content $\beta_{\mathfrak{P}} : \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{P}) \to [0, \infty[$ such that $\beta_{\mathfrak{P}} = \alpha_\star$ on $\mathfrak{A} \sqcap \mathfrak{P}$. In case $\mathfrak{P} \subset \mathfrak{Q}$ it is clear that $\beta_{\mathfrak{P}} = \beta_{\mathfrak{Q}} | \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{P})$. Now $\mathfrak{A} \sqcap \mathfrak{M}$ and $\mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M})$ are the unions of the $\mathfrak{A} \sqcap \mathfrak{P}$

and the $\mathrm{R}(\mathfrak{A}\sqcap\mathfrak{P})$ for all finite $\mathfrak{P}\subset\mathfrak{M}$ with $X\in\mathfrak{P}$. It follows that the $\beta_{\mathfrak{P}}$ for all these $\mathfrak{P}$ combine to furnish the required unique content $\beta:\mathrm{R}(\mathfrak{A}\sqcap\mathfrak{M})\rightarrow[0,\infty[$ such that $\beta=\alpha_\star$ on $\mathfrak{A}\sqcap\mathfrak{M}$. □

After this we turn to the domain of *transplantation theorems* for inner $\star$ premeasures. In the sequel we fix a pair of lattices $\mathfrak{S}$ and $\mathfrak{T}$ with $\varnothing$ in X such that $\mathfrak{S}$ is upward enclosable $\mathfrak{T}$. We assume an inner $\star$ premeasure $\psi:\mathfrak{T}\rightarrow[0,\infty[$ and want to know whether and when it fulfils

($\exists$) there exists an inner $\star$ premeasure $\varphi:\mathfrak{S}\rightarrow[0,\infty[$
such that $\Phi=\varphi_\star|\mathfrak{C}(\varphi_\star)$ is an extension of $\Psi=\psi_\star|\mathfrak{C}(\psi_\star)$;
after 1.8 this is equivalent to $\varphi_\star|\mathfrak{T}=\psi$.

These φ can be viewed as the *transplants* of ψ onto $\mathfrak{S}$. In MI section 18 the requirement that $\mathfrak{S}$ be upward enclosable $\mathfrak{T}$ turned out to be an adequate one. We recall the former main result MI 18.10.

Theorem 2.3. *Let $\psi:\mathfrak{T}\rightarrow[0,\infty[$ be an inner $\star$ premeasure. If $\vartheta:\mathfrak{S}\rightarrow[0,\infty[$ is isotone with $\vartheta(\varnothing)=0$ and supermodular such that $\vartheta_\star|\mathfrak{T}=\psi$, then there exists an inner $\star$ premeasure $\varphi:\mathfrak{S}\rightarrow[0,\infty[$ with $\varphi\geqq\vartheta$ such that $\varphi_\star|\mathfrak{T}=\psi$.*

In MI section 18 there were several important consequences, of which we emphasize MI 18.18: *If ψ satisfies the* Marczewski condition $(\psi_\star|\mathfrak{S})_\star|\mathfrak{T}=\psi$, *then it fulfils* ($\exists$). In fact, this is obvious from 2.3 applied to $\vartheta:=\psi_\star|\mathfrak{S}$. A more involved consequence of 2.3 is the new transplantation theorem which follows. We recall from MI section 1 for nonvoid set systems $\mathfrak{M}$ and $\mathfrak{N}$ in X the *transporter* $\mathfrak{M}\top\mathfrak{N}:=\{A\subset X: A\cap M\in\mathfrak{N}\text{ for all }M\in\mathfrak{M}\}$.

Theorem 2.4. *Let $\psi:\mathfrak{T}\rightarrow[0,\infty[$ be an inner $\star$ premeasure. Then*

$$\inf_{S\in\mathfrak{S}}\psi_\star(S')=0\Longrightarrow\psi\textit{ fulfils }(\exists)\quad\textit{when }\mathfrak{T}\subset\mathfrak{S}\top\mathfrak{S},\ \textit{and}$$
$$\inf_{S\in\mathfrak{S}}\psi_\star(S')=0\Longleftarrow\psi\textit{ fulfils }(\exists)\quad\textit{when }\psi_\star(X)<\infty.$$

Proof. Let (I) denote the condition $\inf\limits_{S\in\mathfrak{S}}\psi_\star(S')=0$, and (II) denote the condition $\inf\limits_{V\in\mathfrak{T}\top\mathfrak{S}}\psi_\star(V')=0$. It is obvious that (I) $\Rightarrow$ (II) when $\mathfrak{T}\subset\mathfrak{S}\top\mathfrak{S}$, because $\mathfrak{T}\subset\mathfrak{S}\top\mathfrak{S}$ is identical with $\mathfrak{S}\subset\mathfrak{T}\top\mathfrak{S}$.

We prove the first assertion, in that we deduce (II) $\Rightarrow$ ($\exists$) from the above 2.2. In view of 2.3 it suffices to produce a set function $\vartheta:\mathfrak{S}\rightarrow[0,\infty[$ which is isotone with $\vartheta(\varnothing)=0$ and supermodular and which fulfils $\vartheta_\star|\mathfrak{T}=\psi$. This is done as follows. On the one hand let $\Psi=\psi_\star|\mathfrak{C}(\psi_\star)$ and $\mathfrak{A}:=[\Psi<\infty]$, so that $\alpha:=\Psi|\mathfrak{A}=\psi_\star|\mathfrak{A}$ is a finite content on the ring $\mathfrak{A}\supset\mathfrak{T}$. On the other hand (II) furnishes a sequence $\varnothing=V_0\subset V_1\subset\cdots\subset V_n\subset\cdots$ in $\mathfrak{T}\top\mathfrak{S}$ such that $\psi_\star(V_n')\downarrow 0$, and thus the totally ordered set system $\mathfrak{M}:=\{V_n': n=0,1,2,\cdots\}$ with $X\in\mathfrak{M}$. Hence we obtain from 2.2 a content $\beta:\mathrm{R}(\mathfrak{A}\sqcap\mathfrak{M})\rightarrow[0,\infty[$ which fulfils $\beta(A\cap V_n')=\alpha_\star(A\cap V_n')=\psi_\star(A\cap V_n')$ for all $A\in\mathfrak{A}$ and $n\geqq 0$. In particular β is an extension of α and hence an extension of ψ.

After this we define $\vartheta = \beta_\star|\mathfrak{S}$. Then $\vartheta : \mathfrak{S} \to [0,\infty[$, because each $S \in \mathfrak{S}$ is contained in some $T \in \mathfrak{T}$ where $\beta(T) = \psi(T) < \infty$. It is clear that ϑ is isotone with $\vartheta(\varnothing) = 0$ and supermodular. At last we fix $T \in \mathfrak{T}$ and prove that $\vartheta_\star(T) = \psi(T)$.

1) For $S \in \mathfrak{S}$ with $S \subset T$ we have $\vartheta(S) = \beta_\star(S) \leqq \beta(T) = \psi(T)$. Hence $\vartheta_\star(T) \leqq \psi(T)$.

2) We have $T \in \mathfrak{T} \subset \mathfrak{A} \subset \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M})$ and $T \cap V_n' \in \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M})$, and hence $T \cap V_n = T \setminus (T \cap V_n') \in \mathrm{R}(\mathfrak{A} \sqcap \mathfrak{M})$ with

$$\beta(T \cap V_n) = \beta(T) - \beta(T \cap V_n') = \psi(T) - \psi_\star(T \cap V_n') \uparrow \psi(T)$$

for $n \to \infty$. But $T \cap V_n \in \mathfrak{S}$ since $V_n \in \mathfrak{T}\top\mathfrak{S}$, and hence $\beta(T \cap V_n) = \beta_\star(T \cap V_n) = \vartheta(T \cap V_n) \leqq \vartheta_\star(T)$. It follows that $\vartheta_\star(T) \geqq \psi(T)$. Thus we have proved $\vartheta_\star|\mathfrak{T} = \psi$ and hence the first assertion.

The proof of the second assertion is much simpler. Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\star$ premeasure such that $\varphi_\star|\mathfrak{T} = \psi$. i) We have $\psi_\star = (\varphi_\star|\mathfrak{T})_\star \leqq \varphi_\star$. ii) For each $S \in \mathfrak{S}$ there is a $T \in \mathfrak{T}$ with $S \subset T$, so that $\varphi(S) \leqq \varphi_\star(T) = \psi(T) \leqq \psi_\star(X)$. Hence $\varphi_\star(X) \leqq \psi_\star(X) < \infty$. From i)ii) we obtain for each $\varepsilon > 0$ an $S \in \mathfrak{S}$ with $\varphi(S) > \varphi_\star(X) - \varepsilon$ and hence with $\psi_\star(S') \leqq \varphi_\star(S') = \varphi_\star(X) - \varphi(S) < \varepsilon$. Thus we have proved (I). $\square$

Remark 2.5. The first assertion in 2.4 need not be true without the assumption that $\mathfrak{T} \subset \mathfrak{S}\top\mathfrak{S}$. As an example let X be a compact Hausdorff topological space with $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Bor}(X)$, so that $\psi : \mathfrak{T} \to [0,\infty[$ can be an arbitrary finite content. Then the assumption $\inf\limits_{S\in\mathfrak{S}} \psi_\star(S') = 0$ is true for the trivial reason that $X \in \mathfrak{S}$, but $(\exists)$ is not true unless ψ is inner regular $\mathfrak{S}$, that means is a Borel-Radon measure. There is a simple example for $X = [0,1]$ in [14] 1.4.

In conclusion we note that the idea of the new transplantation theorem 2.4 came from Fremlin [5] theorem 416O, which is kind of a Radon measure transplantation result like Henry's well-known theorem [5] 416M = MI 18.22. It reads as follows: *Let $\alpha : \mathfrak{A} \to [0,\infty[$ be a content on an algebra $\mathfrak{A}$ in the Hausdorff topological space X. Assume that*

1) *α is inner regular $\mathfrak{A} \cap \mathrm{Cl}(X)$, and*

2) $\alpha(X) = \sup\{\alpha^\star(K) : K \in \mathrm{Comp}(X)\}$.

Then α can be extended to an (of course finite) *Radon measure.* The result can be reformulated so as to fall under the first assertion in 2.4 for $\mathfrak{S} = \mathrm{Comp}(X)$ and a lattice $\mathfrak{T} \subset \mathrm{Cl}(X) \subset \mathfrak{S}\top\mathfrak{S}$ with $\varnothing, X \in \mathfrak{T}$. But the proof in [5] is quite different and an involved combination of abstract and topological pieces.

3. Direct and Inverse Images

The present section is another preparation for the final sections. We fix a map $H : X \to Y$ between nonvoid sets X and Y. We start with the basics on direct and inverse images of contents and measures under H. Then we pass to the direct and inverse images of inner premeasures. Part of the present section updates and extends the earlier [12] sections 2 and 3.

Our first remark presents a most useful computation rule, while the next one introduces the system $\mathrm{Sat}H \subset \mathfrak{P}(X)$ of the *saturated* subsets of X. The proofs are routine.

Remark 3.1. $H\big(A \cap H^{-1}(B)\big) = H(A) \cap B$ for all $A \subset X$ and $B \subset Y$. In particular $H\big(H^{-1}(B)\big) = B \cap H(X)$.

Remark 3.2. Define $\mathrm{Sat}H := H^{-1}(\mathfrak{P}(Y)) = \{A \subset X : A = H^{-1}(H(A))\} \subset \mathfrak{P}(X)$. Then $\mathrm{Sat}H$ is stable under arbitrary unions and intersections and under complement formation. Moreover

$$H\big(\bigcap_{M \in \mathfrak{M}} M\big) = \bigcap_{M \in \mathfrak{M}} H(M) \quad \text{for all nonvoid } \mathfrak{M} \subset \mathrm{Sat}H.$$

For an algebra $\mathfrak{A}$ in X we define the direct image

$$\overrightarrow{H}\mathfrak{A} := \{B \subset Y : H^{-1}(B) \in \mathfrak{A}\} \subset \mathfrak{P}(Y),$$

which is an algebra in Y. It must not be confused with the set system $H(\mathfrak{A}) := \{H(A) : A \in \mathfrak{A}\}$. Then for a content $\alpha : \mathfrak{A} \to [0, \infty]$ on $\mathfrak{A}$ we define the direct image $\overrightarrow{H}\alpha : \overrightarrow{H}\mathfrak{A} \to [0, \infty]$ to be $\overrightarrow{H}\alpha(B) = \alpha(H^{-1}(B))$ for $B \in \overrightarrow{H}\mathfrak{A}$. Thus $\overrightarrow{H}\alpha$ is a content on $\overrightarrow{H}\mathfrak{A}$ and lives on $H(X) \subset Y$. We note that

$$\overrightarrow{H}\mathfrak{A} = \overrightarrow{H}(\mathfrak{A} \cap \mathrm{Sat}H) \quad \text{and} \quad \overrightarrow{H}\alpha = \overrightarrow{H}(\alpha|\mathfrak{A} \cap \mathrm{Sat}H).$$

Next for an algebra $\mathfrak{B}$ in Y we define the inverse image

$$\overleftarrow{H}\mathfrak{B} := H^{-1}(\mathfrak{B}) = \{H^{-1}(B) : B \in \mathfrak{B}\} \subset \mathrm{Sat}H \subset \mathfrak{P}(X),$$

which is an algebra in X. Then let $\beta : \mathfrak{B} \to [0, \infty]$ be a content on $\mathfrak{B}$ which lives on $H(X) \subset Y$. For $A \in \overleftarrow{H}\mathfrak{B}$ and the $B \in \mathfrak{B}$ with $A = H^{-1}(B)$ we have $H(A) = H(H^{-1}(B)) = B \cap H(X) \in \mathfrak{B}$ and $\beta(H(A)) = \beta(B)$. Thus we can define the inverse image $\overleftarrow{H}\beta : \overleftarrow{H}\mathfrak{B} \to [0, \infty]$ to be $\overleftarrow{H}\beta(A) = \beta(H(A)) = \beta(B)$ for $A \in \overleftarrow{H}\mathfrak{B}$ and the $B \in \mathfrak{B}$ with $A = H^{-1}(B)$. Then $\overleftarrow{H}\beta$ is a content on $\overleftarrow{H}\mathfrak{B}$.

Both times the same holds true for σ algebras and measures. The next assertion compares the two kinds of images. The proof is routine.

Comparison 3.3. For each pair of contents

$\alpha : \mathfrak{A} \to [0, \infty]$ on an algebra $\mathfrak{A}$ in X, and
$\beta : \mathfrak{B} \to [0, \infty]$ on an algebra $\mathfrak{B}$ in Y which lives on $H(X) \subset Y$

we have the equivalences

$$\alpha \text{ is an extension of } \overleftarrow{H}\beta \iff \overrightarrow{H}\alpha \text{ is an extension of } \beta;$$
$$\alpha|\mathfrak{A} \cap \mathrm{Sat}H \text{ is a restriction of } \overleftarrow{H}\beta \iff \overrightarrow{H}\alpha \text{ is a restriction of } \beta.$$

The remainder of the section will be devoted to the direct and inverse images of inner $\bullet$ premeasures. We start with a few preparations for pairs of isotone set functions $\xi : \mathfrak{P}(X) \to [0, \infty]$ and $\eta : \mathfrak{P}(Y) \to [0, \infty]$ which are related via $\eta = \xi\big(H^{-1}(\cdot)\big)$. The first remark is for illustration and will not be needed below; for the Choquet integral see MI section 11 and [15] section 5.

Remark 3.4. For each pair of isotone set functions $\xi : \mathfrak{P}(X) \to [0,\infty]$ and $\eta : \mathfrak{P}(Y) \to [0,\infty]$ with $\xi(\varnothing) = \eta(\varnothing) = 0$ the following are equivalent.

1) $\eta = \xi(H^{-1}(\cdot))$.

2) $\int hd\eta = \int (h \circ H)d\xi$ for all $h \in [0,\infty]^Y$.

Proof. 1)⇒2) For $0 < t < \infty$ we have $[h \circ H \geqq t] = \{x \in X : h(H(x)) \geqq t\} = \{x \in X : H(x) \in [h \geqq t]\} = H^{-1}([h \geqq t])$ and hence $\xi\big([h \circ H \geqq t]\big) = \eta\big([h \geqq t]\big)$. Thus the definition of the Choquet integral furnishes

$$\int (h \circ H)d\xi = \int_{0\leftarrow}^{\to\infty} \xi\big([h \circ H \geqq t]\big)dt = \int_{0\leftarrow}^{\to\infty} \eta\big([h \geqq t]\big)dt = \int hd\eta.$$

2)⇒1) Let $B \subset Y$ and $A = H^{-1}(B) \subset X$. Then $h = \chi_B$ implies that $h \circ H = \chi_A$, so that $\int hd\eta = \int (h \circ H)d\xi$ reads $\eta(B) = \xi(A) = \xi(H^{-1}(B))$. □

Lemma 3.5. *Let $\xi : \mathfrak{P}(X) \to [0,\infty]$ be isotone and $\eta = \xi(H^{-1}(\cdot))$, so that $\eta : \mathfrak{P}(Y) \to [0,\infty]$ is isotone as well. 1) Let $\mathfrak{P}$ be a nonvoid set system in X such that ξ is inner regular $\mathfrak{P}$. Then η is inner regular $H(\mathfrak{P})$. 2) Let $\mathfrak{Q}$ be a nonvoid set system in Y such that $\xi|H^{-1}(\mathfrak{Q})$ is almost downward • continuous. Then $\eta|\mathfrak{Q}$ is almost downward • continuous.*

This lemma has a routine proof. Next we put our former result 1.7.1) on the Carathéodory class $\mathfrak{C}(\cdot)$ into action.

Lemma 3.6. *Let $\xi : \mathfrak{P}(X) \to [0,\infty]$ be isotone with $\xi(\varnothing) = 0$ and supermodular, and put $\eta = \xi(H^{-1}(\cdot))$, so that $\eta : \mathfrak{P}(Y) \to [0,\infty]$ is of the same kind. Assume that the nonvoid set system $\mathfrak{T} \subset \mathrm{Sat}H$ in X is upward directed such that $\xi|\mathfrak{T} < \infty$ and that ξ is inner regular $\sqsubset \mathfrak{T}$. Then $\vec{H}\mathfrak{C}(\xi) = \mathfrak{C}(\eta)$ and hence $\vec{H}(\xi|\mathfrak{C}(\xi)) = \eta|\mathfrak{C}(\eta)$.*

Proof. i) For $B \subset Y$ we have $B \in \mathfrak{C}(\eta)$

$$\begin{aligned}
&\Leftrightarrow\ \eta(N) = \eta(N \cap B) + \eta(N \cap B') \quad \forall N \subset Y \\
&\Leftrightarrow\ \xi\big(H^{-1}(N)\big) = \xi\big(H^{-1}(N) \cap H^{-1}(B)\big) + \xi\big(H^{-1}(N) \cap (H^{-1}(B))'\big) \quad \forall N \subset Y \\
&\Leftrightarrow\ \xi(M) = \xi\big(M \cap H^{-1}(B)\big) + \xi\big(M \cap (H^{-1}(B))'\big) \quad \forall M \in \mathrm{Sat}H.
\end{aligned}$$

In particular $B \in \vec{H}\mathfrak{C}(\xi)$, which means $H^{-1}(B) \in \mathfrak{C}(\xi)$, implies that $B \in \mathfrak{C}(\eta)$. ii) Now assume $\mathfrak{T} \subset \mathrm{Sat}H$ as above. For $B \in \mathfrak{C}(\eta)$ we see from i) that

$$\xi(T) = \xi\big(T \cap H^{-1}(B)\big) + \xi\big(T \cap (H^{-1}(B))'\big) \quad \forall T \in \mathfrak{T}.$$

Thus 1.7.1) applied to ξ furnishes $H^{-1}(B) \in \mathfrak{C}(\xi)$ or $B \in \vec{H}\mathfrak{C}(\xi)$. □

Lemma 3.7. *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner • premeasure on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in X. Put $\eta = \varphi_\bullet\big(H^{-1}(\cdot)\big)$ and assume that $\eta|H(\mathfrak{S}) < \infty$. Then $\vec{H}\mathfrak{C}(\varphi_\bullet) = \mathfrak{C}(\eta)$, so that $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ fulfils $\vec{H}\Phi = \eta|\mathfrak{C}(\eta)$.*

Proof. 3.6 can be applied to $\xi := \varphi_\bullet$ and $\mathfrak{T} := H^{-1}(H(\mathfrak{S}))$, because $\varphi_\bullet(T) = \varphi_\bullet\big(H^{-1}(H(S))\big) = \eta(H(S)) < \infty$ for $T = H^{-1}(H(S))$ with $S \in \mathfrak{S}$, and since $\varphi_\bullet$ is inner regular $\mathfrak{S}_\bullet \subset (\sqsubset \mathfrak{T})$. □

Proposition 3.8. *Let $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y be lattices with $\varnothing$ such that $H(\mathfrak{S})$ is upward enclosable $\mathfrak{T}$. Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$, and assume that $\psi := \varphi_\bullet(H^{-1}(\cdot))|\mathfrak{T} < \infty$. Then*

1) $\overrightarrow{H}\Phi$ *is an extension of* $\psi_\star|\mathfrak{C}(\psi_\star)$.

2) *Assume that* $(\leftarrow)\, H^{-1}(\mathfrak{T}) \subset \mathfrak{S}\top\mathfrak{S}_\bullet$ *and hence* $(\Leftarrow)\, H^{-1}(\mathfrak{T}_\bullet) \subset \mathfrak{S}\top\mathfrak{S}_\bullet$. *Then ψ is downward $\bullet$ continuous, and $\overrightarrow{H}\Phi$ is an extension of $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$ and an extension of $\psi_\bullet|\mathfrak{T}_\bullet$* (but ψ need not be an inner $\bullet$ premeasure).

3) *Assume that moreover $(\Rightarrow)\, H(\mathfrak{S}_\bullet) \subset \mathfrak{T}_\bullet$. Then ψ is an inner $\bullet$ premeasure with $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ which fulfils $\psi_\bullet = \varphi_\bullet(H^{-1}(\cdot))$ and $\overrightarrow{H}\Phi = \Psi$.*

Proof. We put $\eta := \varphi_\bullet(H^{-1}(\cdot))$, so that $\eta : \mathfrak{P}(Y) \to [0,\infty]$ is isotone with $\eta(\varnothing) = 0$ and supermodular. We have $\eta|\mathfrak{T} = \psi < \infty$ and $\eta|\overrightarrow{H}\mathfrak{C}(\varphi_\bullet) = \overrightarrow{H}\Psi$.

i) By assumption $\eta|H(\mathfrak{S}) < \infty$. Thus 3.7 asserts that $\overrightarrow{H}\Phi = \eta|\mathfrak{C}(\eta)$.

ii) η and $\mathfrak{T}$ fulfil in 1.7 the assumptions for ξ and $\mathfrak{T}$, because 3.5.1) implies that η is inner regular $H(\mathfrak{S}_\bullet) \subset (\sqsubset \mathfrak{T})$. Then $\psi_\star$ fulfils in 1.7.2) the assumptions for η, because $\psi_\star = \psi = \eta$ on $\mathfrak{T}$ and hence $\psi_\star \leqq \eta$ on $\mathfrak{P}(Y)$. Thus 1.7.2) asserts that $\eta|\mathfrak{C}(\eta)$ is an extension of $\psi_\star|\mathfrak{C}(\psi_\star)$. From i)ii) we obtain 1).

iii) From $(\Leftarrow)$ and 1.5 we see that $\varphi_\bullet|H^{-1}(\mathfrak{T}_\bullet)$ is almost downward $\bullet$ continuous. Thus 3.5.2) asserts that $\eta|\mathfrak{T}_\bullet$ is almost downward $\bullet$ continuous and hence downward $\bullet$ continuous. In particular $\psi = \eta|\mathfrak{T}$ is downward $\bullet$ continuous, and hence MI 6.5.iii) = [15] 2.2.3.Inn) asserts that $\psi_\bullet|\mathfrak{T}_\bullet$ is downward $\bullet$ continuous. It follows that $\psi_\bullet|\mathfrak{T}_\bullet = \eta|\mathfrak{T}_\bullet$ and hence $\psi_\bullet = (\psi_\bullet|\mathfrak{T}_\bullet)_\star = (\eta|\mathfrak{T}_\bullet)_\star$.

iv) Now 1) applied to $\eta|\mathfrak{T}_\bullet$ instead of $\eta|\mathfrak{T} = \psi$ asserts that $\overrightarrow{H}\Phi$ is an extension of $\psi_\bullet|\mathfrak{C}(\psi_\bullet)$. Moreover $H^{-1}(\mathfrak{T}_\bullet) \subset \mathfrak{S}\top\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$ shows that $\mathfrak{T}_\bullet \subset \overrightarrow{H}\mathfrak{C}(\varphi_\bullet)$, and iii) asserts that on $\mathfrak{T}_\bullet$ we have $\psi_\bullet = \eta = \overrightarrow{H}\Phi$. Thus we obtain 2).

v) Under the assumption $(\Rightarrow)$ we see from 3.5.1) that η is inner regular $\mathfrak{T}_\bullet$. Thus $\eta|\mathfrak{T}_\bullet = \psi_\bullet|\mathfrak{T}_\bullet$ implies that $\eta = \psi_\bullet$. In particular $\overrightarrow{H}\Phi = \eta|\overrightarrow{H}\mathfrak{C}(\varphi_\bullet)$ is inner regular $\mathfrak{T}_\bullet$ and hence an inner $\bullet$ extension of ψ. Therefore ψ is an inner $\bullet$ premeasure, and $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ is an extension of $\overrightarrow{H}\Phi$ and hence $= \overrightarrow{H}\Phi$. □

Example 3.9. Let $X = Y = \mathbb{N}$, and H be the identity map of $\mathbb{N}$. Let $\mathfrak{S}$ consist of the finite subsets of $\mathbb{N}$, and $\mathfrak{T}$ consist of $\varnothing$ and of the $\{1,\cdots,n\}$ with $n \in \mathbb{N}$. Then $\varphi := \mathrm{card}|\mathfrak{S}$ is an inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ which is an obvious example for the final assertion in 3.8.2).

The above proposition contains in 3) the main theorem on direct images of inner $\bullet$ premeasures, which we reproduce in view of its importance.

Theorem 3.10. *Let $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y be lattices with $\varnothing$ such that*

$(\Rightarrow)$ $H(\mathfrak{S}_\bullet) \subset \mathfrak{T}_\bullet$, *and*
$(\Leftarrow)$ $H^{-1}(\mathfrak{T}_\bullet) \subset \mathfrak{S}\top\mathfrak{S}_\bullet$.

Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$, and assume that $\psi := \varphi_\bullet(H^{-1}(\cdot))|\mathfrak{T} < \infty$. Then $\psi : \mathfrak{T} \to [0,\infty[$ is an inner $\bullet$ premeasure with $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ which fulfils $\psi_\bullet = \varphi_\bullet(H^{-1}(\cdot))$ and $\overrightarrow{H}\Phi = \Psi$.

We call $\psi : \mathfrak{T} \to [0,\infty[$ the *direct image* of φ under H and write $\psi = \overrightarrow{H}\varphi$. Note that $\psi = \overrightarrow{H}\varphi$ of course depends on the prescribed $\mathfrak{T}$, while $\psi_\bullet$ and hence Ψ do not.

Example 3.11. The most natural example is that X and Y are Hausdorff topological spaces with $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Comp}(Y)$. Then φ and ψ are Radon premeasures on X and Y. The assumptions $(\Rightarrow)(\Leftarrow)$ stand for the condition that the map H be continuous. Of course $\mathfrak{S} = \mathfrak{S}_\bullet$ and $\mathfrak{T} = \mathfrak{T}_\bullet$, but the distinction in $(\Rightarrow)(\Leftarrow)$ will be relevant beyond the example.

In fact, we need a word on the conditions $(\Rightarrow)(\Leftarrow)$ in 3.10. One could think that in place of these conditions we should rather have required

$(\rightarrow)$ $H(\mathfrak{S}) \subset \mathfrak{T}$ or the weaker $H(\mathfrak{S}) \subset \mathfrak{T}_\bullet$, and
$(\leftarrow)$ $H^{-1}(\mathfrak{T}) \subset \mathfrak{S}\top\mathfrak{S}$ or the weaker $H^{-1}(\mathfrak{T}) \subset \mathfrak{S}\top\mathfrak{S}_\bullet$.

However, it is not clear how to succeed with $(\rightarrow)(\leftarrow)$: It is well-known and has been used before that $H^{-1}(\mathfrak{T}_\bullet) = (H^{-1}(\mathfrak{T}))_\bullet$, so that the weaker $(\leftarrow)$ implies $(\Leftarrow)$ and hence is equivalent to $(\Leftarrow)$. But the relation $H(\mathfrak{S}_\bullet) \subset (H(\mathfrak{S}))_\bullet$, which would do the same for $(\rightarrow)$ and $(\Rightarrow)$, does not hold true but under severe restrictions [12] 3.3. We present the most useful positive result on this relation, which is a fortified version of [12] 3.4 and has the same proof.

Remark 3.12. Assume that the nonvoid set system $\mathfrak{S}$ in X is $\bullet$ compact (each nonvoid $\mathfrak{M} \subset \mathfrak{S}$ fulfils $\varnothing \in \mathfrak{M}_\bullet \Rightarrow \varnothing \in \mathfrak{M}_\star$), and that $H^{-1}(\{b\}) \in \mathfrak{S}\top\mathfrak{S}_\bullet$ for all $b \in Y$. Then

$$H(\bigcap_{M\in\mathfrak{M}} M) = \bigcap_{M\in\mathfrak{M}} H(M) \quad \text{for all } \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ downward directed}.$$

Thus if $\mathfrak{S}$ is stable under $\cap$ then $H(\mathfrak{S}_\bullet) \subset (H(\mathfrak{S}))_\bullet$.

We turn to the main theorem on inverse images of inner $\bullet$ premeasures.

Theorem 3.13. *Assume that the lattices $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y with $\varnothing$ fulfil $\mathfrak{S}_\bullet = H^{-1}(\mathfrak{T}_\bullet)$. Let $\psi : \mathfrak{T} \to [0,\infty[$ be an inner $\bullet$ premeasure such that $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ lives on $H(X) \subset Y$. Then $\varphi := \psi_\bullet(H(\cdot))|\mathfrak{S}$ is an inner $\bullet$ premeasure with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ which fulfils*

1) *$\varphi_\bullet(A) = \psi_\bullet\big((H(A'))'\big)$ for all $A \subset X$.*
2) *$\varphi_\bullet(A) = \psi_\bullet(H(A))$ for $A \in \mathrm{Sat}H$, equivalent to $\varphi_\bullet(H^{-1}(\cdot)) = \psi_\bullet$.*
3) *$\overrightarrow{H}\Phi = \Psi$, equivalent to $\Phi|\big(\mathfrak{C}(\varphi_\bullet) \cap \mathrm{Sat}H\big) = \overleftarrow{H}\Psi$ by* 3.3.
4) *$\varphi_\bullet = \Phi_\star = (\overleftarrow{H}\Psi)_\star$.*
5) *$H(\mathfrak{C}(\varphi_\bullet)) \subset \mathfrak{C}(\psi_\bullet)$.*

We call $\varphi : \mathfrak{S} \to [0,\infty[$ the *inverse image* of ψ under H and write $\varphi = \overleftarrow{H}\psi$. Note that $\varphi = \overleftarrow{H}\psi$ of course depends on the prescribed $\mathfrak{S}$, while $\varphi_\bullet$ and hence Φ do not. We see that there are simple relations between

Φ the maximal inner $\star$ extension of the inverse image $\varphi = \overleftarrow{H}\psi$ of ψ, and

$\overleftarrow{H}\Psi$ the inverse image of the maximal inner $\star$ extension Ψ of ψ,
but that the two need not be equal.

Proof. 0) We see from 1.9.4) that $\psi_\bullet(B) = \psi_\bullet(B \cap H(X))$ for all $B \subset Y$. We invoke 1.10 in order to realize that the members of $\mathfrak{T}$ and hence the members of $\mathfrak{T}_\bullet$ can be assumed to be contained in $H(X)$. In fact, we see from 1.10.1) that the lattice $\mathfrak{D} := \mathfrak{T} \sqcap H(X)$ has $\mathfrak{D}_\bullet = \mathfrak{T}_\bullet \sqcap H(X)$, and that $\delta := \psi_\bullet|\mathfrak{D}$ is an inner $\bullet$ premeasure $\delta : \mathfrak{D} \to [0,\infty[$ which fulfils $\delta_\bullet = \psi_\bullet$. It follows that $\mathfrak{S}_\bullet = H^{-1}(\mathfrak{D}_\bullet)$ and that $\varphi = \delta_\bullet(H(\cdot))|\mathfrak{S}$. Thus the theorem for $\psi : \mathfrak{T} \to [0,\infty[$ is identical with that for $\delta : \mathfrak{D} \to [0,\infty[$, where the members of $\mathfrak{D}$ and of $\mathfrak{D}_\bullet$ are indeed $\subset H(X)$. The additional assumption thus achieved implies that $H(\mathfrak{S}_\bullet) = H(H^{-1}(\mathfrak{T}_\bullet)) = \mathfrak{T}_\bullet \sqcap H(X) = \mathfrak{T}_\bullet$. This will be important in part iii) below.

i) We define $\xi : \mathfrak{P}(X) \to [0,\infty]$ to be $\xi(A) = \psi_\bullet\big((H(A'))'\big)$ for $A \subset X$. We claim that $\xi(A) = \psi_\bullet(H(A))$ for $A \in \mathrm{Sat}H$. In fact, for $A = H^{-1}(B)$ with $B \subset Y$ we have $A' = H^{-1}(B')$ and hence $H(A') = B' \cap H(X)$ or $(H(A'))' = B \cup (H(X))'$, so that $(H(A'))' \cap H(X) = B \cap H(X) = H(H^{-1}(B)) = H(A)$, which proves the present claim.

ii) ξ is inner regular $\mathfrak{S}_\bullet = H^{-1}(\mathfrak{T}_\bullet)$. In fact, let $A \subset X$ and $c < \xi(A) = \psi_\bullet\big((H(A'))'\big)$. Then there exists $T \in \mathfrak{T}_\bullet$ with $T \subset (H(A'))'$ and $c < \psi_\bullet(T)$. From $H\big(H^{-1}(T) \cap A'\big) = T \cap H(A') = \varnothing$ after 3.1 we have $H^{-1}(T) \cap A' = \varnothing$ or $H^{-1}(T) \subset A$, and from i) we obtain $\xi(H^{-1}(T)) = \psi_\bullet(H(H^{-1}(T))) = \psi_\bullet(T \cap H(X)) = \psi_\bullet(T) > c$.

iii) From i) we have $\xi|\mathfrak{S}_\bullet < \infty$. We claim that $\xi|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous. To see this fix $\mathfrak{M} \subset \mathfrak{S}_\bullet$ nonvoid $\bullet$ with $\mathfrak{M} \downarrow A \in \mathfrak{S}_\bullet$. From the last assertion in 3.2 then $H(\mathfrak{M}) \downarrow H(A)$. Now after 0) we can assume that $H(\mathfrak{S}_\bullet) = \mathfrak{T}_\bullet$. Since $\psi_\bullet|\mathfrak{T}_\bullet$ is downward $\bullet$ continuous it follows from $\mathfrak{S}_\bullet = H^{-1}(\mathfrak{T}_\bullet) \subset \mathrm{Sat}H$ and i) that

$$\inf_{M \in \mathfrak{M}} \xi(M) = \inf_{M \in \mathfrak{M}} \psi_\bullet(H(M)) = \psi_\bullet(H(A)) = \xi(A).$$

iv) We know that $\overleftarrow{H}\Psi$ is a content on the algebra $\overleftarrow{H}\mathfrak{C}(\psi_\bullet) = H^{-1}\big(\mathfrak{C}(\psi_\bullet)\big) \subset \mathrm{Sat}H$ in X. From the definition

$$\overleftarrow{H}\Psi(A) = \Psi(H(A)) = \psi_\bullet(H(A)) = \xi(A) \quad \text{for } A \in \overleftarrow{H}\mathfrak{C}(\psi_\bullet)$$

we see that $\overleftarrow{H}\Psi = \xi|\overleftarrow{H}\mathfrak{C}(\psi_\bullet)$. Now $\mathfrak{S} \subset \mathfrak{S}_\bullet = H^{-1}(\mathfrak{T}_\bullet) \subset \overleftarrow{H}\mathfrak{C}(\psi_\bullet)$. Thus $\overleftarrow{H}\Psi$ is an extension of $\xi|\mathfrak{S}_\bullet < \infty$, and in particular an extension of $\varphi = \xi|\mathfrak{S}$. By ii)iii) hence $\overleftarrow{H}\Psi$ is an inner $\bullet$ extension of φ. Thus φ is an inner $\bullet$ premeasure, and we have $\varphi_\bullet = \overleftarrow{H}\Psi = \xi$ on $\overleftarrow{H}\mathfrak{C}(\psi_\bullet)$ and in particular on $\mathfrak{S}_\bullet$. Once more from ii) it follows that $\varphi_\bullet = \xi$ on $\mathfrak{P}(X)$. Thus we have proved 1)2).

v) Assertion 3) follows from 2) combined with 3.7.

vi) It is obvious that $\varphi_\bullet = \Phi_\star$. Thus to be shown in 4) is $\Phi_\star = (\overleftarrow{H}\Psi)_\star$. In fact, $(\overleftarrow{H}\Psi)_\star(A)$ for $A \subset X$ is by definition

$$= \sup\{\Psi(B) : B \in \mathfrak{C}(\psi_\bullet) \text{ with } H^{-1}(B) \subset A\}$$
$$= \sup\{\Psi(T) = \psi_\bullet\big(H(H^{-1}(T))\big) : T \in \mathfrak{T}_\bullet \text{ with } H^{-1}(T) \subset A\}$$
$$= \sup\{\psi_\bullet(H(S)) = \varphi_\bullet(S) : S \in \mathfrak{S}_\bullet \text{ with } S \subset A\} = \varphi_\bullet(A) = \Phi_\star(A).$$

vii) To prove 5) we fix $M \in \mathfrak{C}(\varphi_\bullet)$. For $A \subset X$ then

$$\begin{aligned}\psi_\bullet((H(A))') &= \varphi_\bullet(A') = \varphi_\bullet(A' \cap M) + \varphi_\bullet(A' \cap M') \\ &= \psi_\bullet\big((H(A \cup M'))'\big) + \psi_\bullet\big((H(A \cup M))'\big) \\ &= \psi_\bullet\big((H(A))' \cap (H(M'))'\big) + \psi_\bullet\big((H(A))' \cap (H(M))'\big) \\ &\leqq \psi_\bullet\big((H(A))' \cap H(M)\big) + \psi_\bullet\big((H(A))' \cap (H(M))'\big) \leqq \psi_\bullet((H(A)')),\end{aligned}$$

where we used $(H(M'))' \cap H(X) \subset H(M)$ and that $\psi_\bullet$ is supermodular. Thus

$$\psi_\bullet\big((H(A))'\big) = \psi_\bullet\big((H(A))' \cap H(M)\big) + \psi_\bullet\big((H(A))' \cap (H(M))'\big).$$

Now let $B \subset Y$ and put $A := H^{-1}(B') \subset X$. From $(H(A))' \cap H(X) = B \cap H(X)$ we conclude that $\psi_\bullet\big((H(A))' \cap N\big) = \psi_\bullet(B \cap N)$ for any subset $N \subset Y$. Therefore $\psi_\bullet(B) = \psi_\bullet(B \cap H(M)) + \psi_\bullet(B \cap (H(M))')$ for all $B \subset Y$, so that $H(M) \in \mathfrak{C}(\psi_\bullet)$. □

4. The Prokhorov Type Theorem

The present section will be devoted to the principal results. The section is under the assumption formulated in 4.1 below.

Assumption 4.1. *Let I be a nonvoid index set, equipped with an order $\leqq$ under which I is upward directed.*

a) *For each $p \in I$ let Y_p be a nonvoid set, and for each pair $p \leqq q$ in I let $H_{pq} : Y_p \leftarrow Y_q$. For each $p \in I$ let $\mathfrak{T}_p$ be a lattice in Y_p with $\varnothing \in \mathfrak{T}_p$. For each pair $p \leqq q$ in I assume that*

($\rightarrow$) $H_{pq}(\mathfrak{T}_q) \subset (\mathfrak{T}_p)_\bullet$, *and*
($\leftarrow$) $H_{pq}^{-1}(\mathfrak{T}_p) \subset \mathfrak{T}_q \top (\mathfrak{T}_q)_\bullet$ *and hence* ($\Leftarrow$) $H_{pq}^{-1}((\mathfrak{T}_p)_\bullet) \subset \mathfrak{T}_q \top (\mathfrak{T}_q)_\bullet$.

We shall have as a rule that H_{pp} is the identity map of Y_p, and that $H_{pr} = H_{pq} \circ H_{qr}$ for $p \leqq q \leqq r$ in I.

b) *For each $p \in I$ let $\psi_p : \mathfrak{T}_p \to [0, \infty[$ be an inner $\bullet$ premeasure with $\Psi_p = (\psi_p)_\bullet | \mathfrak{C}((\psi_p)_\bullet)$. For each pair $p \leqq q$ in I assume that $\psi_p = (\psi_q)_\bullet(H_{pq}^{-1}(\cdot))|\mathfrak{T}_p$.*

A) *Let X be a nonvoid set, and for each $p \in I$ let $H_p : Y_p \leftarrow X$. For each pair $p \leqq q$ in I assume that $H_p = H_{pq} \circ H_q$. Let $\mathfrak{S}$ be a lattice in X with $\varnothing \in \mathfrak{S}$. For each $p \in I$ assume that*

($\rightarrow$) $H_p(\mathfrak{S}) \subset (\mathfrak{T}_p)_\bullet$, *and*
($\leftarrow$) $H_p^{-1}(\mathfrak{T}_p) \subset \mathfrak{S} \top \mathfrak{S}_\bullet$ *and hence* ($\Leftarrow$) $H_p^{-1}((\mathfrak{T}_p)_\bullet) \subset \mathfrak{S} \top \mathfrak{S}_\bullet$.

There is no part B) in the assumption. It will rather be the aim of the present section to establish an appropriate B).

Aim 4.2. B) *There exists an inner* $\bullet$ *premeasure* $\varphi : \mathfrak{S} \to [0,\infty[$ *such that* $\psi_p = \varphi_\bullet(H_p^{-1}(\cdot))|\mathfrak{T}_p$ *for all* $p \in I$. Each such φ will be named a *solution.*

wB) *There exists an inner* $\star$ *premeasure* $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ *such that* $(\psi_p)_\bullet = \phi_\star(H_p^{-1}(\cdot))$ *on* $(\mathfrak{T}_p)_\bullet$ *for all* $p \in I$. Each such ϕ will be named a *weak solution.*

The relation between these two concepts follows at once from the simple facts 1.2-1.5: For the set functions $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ we have from 1.2 and 1.3 the equivalence

$$\phi \text{ inner } \bullet \text{ premeasure} \iff \phi \text{ inner } \star \text{ premeasure and } \phi|\mathfrak{S} \text{ downward } \bullet \text{ continuous at } \varnothing.$$

Then 1.4 asserts that these $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ are in one-to-one correspondence with the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ via both $\varphi = \phi|\mathfrak{S}$ and $\phi = \varphi_\bullet|\mathfrak{S}_\bullet$. For a couple φ and ϕ moreover $\varphi_\bullet = \phi_\bullet = \phi_\star$ and hence $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet) = \phi_\star|\mathfrak{C}(\phi_\star)$. As to B) and wB) we note for $p \in I$ the equivalence

$$\psi_p = \varphi_\bullet(H_p^{-1}(\cdot))|\mathfrak{T}_p \iff (\psi_p)_\bullet|(\mathfrak{T}_p)_\bullet = \phi_\star(H_p^{-1}(\cdot))|(\mathfrak{T}_p)_\bullet.$$

Here $\Longleftarrow$ is clear, and we have $\Longrightarrow$ because $\phi_\star(H_p^{-1}(\cdot))|(\mathfrak{T}_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))|(\mathfrak{T}_p)_\bullet$ is almost downward $\bullet$ continuous in view of 1.5 with $H_p^{-1}((\mathfrak{T}_p)_\bullet) \subset \mathfrak{S}\top\mathfrak{S}_\bullet$ and 3.5.2).

Consequence 4.3. *The solutions* $\varphi : \mathfrak{S} \to [0,\infty[$ *are in one-to-one correspondence with the particular weak solutions* $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ *of which the restrictions* $\phi|\mathfrak{S}$ *are downward* $\bullet$ *continuous at* $\varnothing$, *via both* $\varphi = \phi|\mathfrak{S}$ *and* $\phi = \varphi_\bullet|\mathfrak{S}_\bullet$. *For a couple* φ *and* ϕ *moreover* $\varphi_\bullet = \phi_\bullet = \phi_\star$ *and hence* $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet) = \phi_\star|\mathfrak{C}(\phi_\star)$.

We shall see that there is quite a difference between solutions and weak solutions. The most pleasant particular situation is of course that $\mathfrak{S}$ is $\bullet$ compact, where the two notions are identical.

We start with two preliminaries. The first point is to note that 3.8 leads to certain fortifications of the basic relations in b) and B)wB).

Remark 4.4. Let $p \leqq q$ in I. Then the direct image $\vec{H}_{pq}\Psi_q$ is an extension of Ψ_p (note that this contains the assumption $\psi_p = (\psi_q)_\bullet(H_{pq}^{-1}(\cdot))|\mathfrak{T}_p$). In particular $\Psi_p(Y_p) = \vec{H}_{pq}\Psi_q(Y_p) = \Psi_q(Y_q)$. Thus the value $C := \Psi_p(Y_p) \in [0,\infty]$ is independent of $p \in I$. Moreover if $(\Rightarrow)\, H_{pq}((\mathfrak{T}_q)_\bullet) \subset (\mathfrak{T}_p)_\bullet$ then $(\psi_p)_\bullet = (\psi_q)_\bullet(H_{pq}^{-1}(\cdot))$ and $\vec{H}_{pq}\Psi_q = \Psi_p$.

Proposition 4.5. *Let* $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ *be a weak solution with* $\Phi = \phi_\star|\mathfrak{C}(\phi_\star)$. *For* $p \in I$ *then* $\vec{H}_p\Phi$ *is an extension of* Ψ_p (note that this contains the assumption $(\psi_p)_\bullet = \phi_\star((H_p^{-1}(\cdot))$ on $(\mathfrak{T}_p)_\bullet$). *In particular* $C = \Psi_p(Y_p) = \vec{H}\Phi(Y_p) = \Phi(X)$. *Moreover if* $(\Rightarrow)\, H_p(\mathfrak{S}_\bullet) \subset (\mathfrak{T}_p)_\bullet$ *then* $(\psi_p)_\bullet = \phi_\star(H_p^{-1}(\cdot))$ *and* $\vec{H}_p\Phi = \Psi_p$.

Proofs. In 4.4 the assertions follow from 3.8.2)3) applied to the lattices $\mathfrak{T}_q$ in Y_q and $\mathfrak{T}_p$ in Y_p under the map H_{pq} and to the set functions ψ_q and ψ_p. In 4.5 the assertions follow from 3.8.1)3) applied to the lattices $\mathfrak{S}_\bullet$ in X and $(\mathfrak{T}_p)_\bullet$ in Y_p under the map H_p and to the set functions ϕ and $(\psi_p)_\bullet|(\mathfrak{T}_p)_\bullet$, but in case $\star$. One has to note that $\big((\psi_p)_\bullet|(\mathfrak{T}_p)_\bullet\big)_\star = (\psi_p)_\bullet$. $\square$

The second point is to introduce the so-called PROKHOROV condition into the present situation 4.1. This is the fundamental condition which dominates the traditional treatment in the concrete situations based on Radon measures. It is due to Prokhorov [19].

Lemma 4.6. *Assume that $p \in I$ satisfies $\inf\limits_{S\in\mathfrak{S}} \Psi_p\big((H_p(S))'\big) = 0$. Then Ψ_p lives on $H_p(X) \subset Y_p$. Moreover $C = \Psi_p(Y_p) < \infty$.*

Proof. For $S \in \mathfrak{S}$ we have $H_p(S) \in (\mathfrak{T}_p)_\bullet \subset \mathfrak{C}((\psi_p)_\bullet)$. Thus for $A \subset Y_p$ it follows that

$$\begin{aligned}(\psi_p)_\bullet(A) &= (\psi_p)_\bullet(A \cap H_p(S)) + (\psi_p)_\bullet(A \cap (H_p)S))')\\ &\leqq (\psi_p)_\bullet(A \cap H_p(X)) + (\psi_p)_\bullet((H_p(S))') \quad \text{for } S \in \mathfrak{S},\end{aligned}$$

so that the present assumption implies that $(\psi_p)_\bullet(A) = (\psi_p)_\bullet(A \cap H_p(X))$. From 1.9.4) the first assertion follows. The second assertion is obvious. $\square$

After this we define the Prokhorov condition to be

$$(\Pi) \qquad \inf_{S\in\mathfrak{S}} \sup_{p\in I} \Psi_p\big((H_p(S))'\big) = 0.$$

Thus (Π) is the uniform fortification of the condition $\inf\limits_{S\in\mathfrak{S}} \Psi_p\big((H_p(S))'\big) = 0$ for the individual $p \in I$ which appears in 4.6. Therefore (Π) implies that Ψ_p lives on $H_p(X) \subset Y_p$ for all $p \in I$, and that $C < \infty$. We prove that *the existence of a weak solution $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ with $\Phi(X) = C < \infty$ enforces that condition (Π) is fulfilled.* This statement can be fortified as follows.

Proposition 4.7. *Assume that $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ is isotone with $\phi(\varnothing) = 0$ and supermodular, and that*

$$(\psi_p)_\bullet \leqq \phi_\star(H_p^{-1}(\cdot)) \text{ on } (\mathfrak{T}_p)_\bullet \text{ and hence on } \mathfrak{P}(Y_p) \text{ for each } p \in I.$$

If moreover $\phi_\star(X) < \infty$ then condition (Π) is fulfilled.

Proof. For fixed $\varepsilon > 0$ there exists an $S \in \mathfrak{S}$ with $\phi(S) \geqq \phi_\star(X) - \varepsilon$. Since $\phi_\star$ is supermodular we have $\phi_\star(S') + \phi(S) \leqq \phi_\star(X)$ and hence $\phi_\star(S') \leqq \varepsilon$. Now for $p \in I$ and $P := (H_p(S))'$ we have $\varnothing = H_p(S) \cap P = H_p\big(S \cap H_p^{-1}(P)\big)$ from 3.1 and hence $S \cap H_p^{-1}(P) = \varnothing$ or $H_p^{-1}(P) \subset S'$. Thus the assumption furnishes $(\psi_p)_\bullet\big((H_p(S))'\big) = (\psi_p)_\bullet(P) \leqq \phi_\star(H_p^{-1}(P)) \leqq \phi_\star(S') \leqq \varepsilon$. $\square$

After these preliminaries we head for the main results. These are the converse assertion that condition (Π) implies the existence of weak solutions, and another fortification of their properties. *For the subsequent development up to* 4.9 *and* 4.10 *we assume that Ψ_p lives on $H_p(X) \subset Y_p$ for all $p \in I$.*

For $p \in I$ we have on the one hand the inverse image $\overleftarrow{H}_p \Psi_p$ of Ψ_p. On the other hand we form the lattice $\mathfrak{S}_p := H_p^{-1}(\mathfrak{T}_p)$ with $\varnothing$ in X, so that $(\mathfrak{S}_p)_\bullet = H_p^{-1}((\mathfrak{T}_p)_\bullet)$. Then after 3.13 we have the inverse image $\varphi_p := (\psi_p)_\bullet(H_p(\cdot))|\mathfrak{S}_p$ of ψ_p. Thus $\varphi_p : \mathfrak{S}_p \to [0,\infty[$ is an inner $\bullet$ premeasure with $\Phi_p = (\varphi_p)_\bullet|\mathfrak{C}((\varphi_p)_\bullet)$ which fulfils

3.13.1) $(\varphi_p)_\bullet(A) = (\psi_p)_\bullet\big((H_p(A'))'\big)$ for $A \subset X$,

3.13.2) $(\varphi_p)_\bullet(A) = (\psi_p)_\bullet(H_p(A))$ for $A \in \mathrm{Sat}H_p$ or $(\varphi_p)_\bullet(H_p^{-1}(\cdot)) = (\psi_p)_\bullet$,

3.13.3) $\overrightarrow{H}_p \Phi_p = \Psi_p$ or $\Phi_p|\big(\mathfrak{C}((\varphi_p)_\bullet) \cap \mathrm{Sat}H_p\big) = \overleftarrow{H}_p \Psi_p$.

In particular $\Phi_p(X) = C$. From 4.1 one cannot expect simple inclusions between the $\mathfrak{S}_p$ and $(\mathfrak{S}_p)_\bullet$ for different $p \in I$. But from 1.8 we obtain the basic fact which follows.

Lemma 4.8. *Let $p \leqq q$ in I. Then Φ_q is an extension of Φ_p. It follows that $(\varphi_p)_\bullet \leqq (\varphi_q)_\bullet$.*

In particular $\mathfrak{C}(\varphi_p)_\bullet) \subset \mathfrak{C}(\varphi_q)_\bullet)$. For later use we also note the obvious fact that $\mathrm{Sat}H_p \subset \mathrm{Sat}H_q$.

Proof. We show that 1.8 can be applied to $\varphi_q : \mathfrak{S}_q \to [0,\infty[$ and $\varphi_p : \mathfrak{S}_p \to [0,\infty[$. Thus we have to prove that 1) $\mathfrak{S}_q$ is upward enclosable $\mathfrak{S}_p$, and 2) $(\varphi_q)_\bullet = (\varphi_p)_\bullet$ on $(\mathfrak{S}_p)_\bullet$. 1) Let $B \in \mathfrak{S}_q$, so that $B = H_q^{-1}(Q)$ for some $Q \in \mathfrak{T}_q$. Then $H_{pq}(Q) \in (\mathfrak{T}_p)_\bullet$ and hence $H_{pq}(Q) \subset$ some $P \in \mathfrak{T}_p$. It follows that

$$B = H_q^{-1}(Q) \subset H_q^{-1}\big(H_{pq}^{-1}(H_{pq}(Q))\big) \subset H_q^{-1}(H_{pq}^{-1}(P)) = H_p^{-1}(P) \in \mathfrak{S}_p.$$

2) Let $A \in (\mathfrak{S}_p)_\bullet = H_p^{-1}((\mathfrak{T}_p)_\bullet)$, so that $A = H_p^{-1}(P) = H_q^{-1}(H_{pq}^{-1}(P))$ for some $P \in (\mathfrak{T}_p)_\bullet$. Then

$$\begin{aligned}(\varphi_q)_\bullet(A) &= (\psi_q)_\bullet\big(H_q(H_q^{-1}(H_{pq}^{-1}(P))\big)\\ &= (\psi_q)_\bullet\big(H_{pq}^{-1}(P) \cap H_q(X)\big) = (\psi_q)_\bullet(H_{pq}^{-1}(P)),\\ (\varphi_p)_\bullet(A) &= (\psi_p)_\bullet(H_p(H_p^{-1}(P)) = (\psi_p)_\bullet(P \cap H_p(X)) = (\psi_p)_\bullet(P)),\end{aligned}$$

and the two final terms are equal in view of 4.4. Thus 1.8 asserts that Φ_q is an extension of Φ_p. □

We conclude from 4.8 that $\mathfrak{A} := \bigcup_{p\in I} \mathfrak{C}((\varphi_p)_\bullet)$ is an algebra in X, and that there is a unique content $\alpha : \mathfrak{A} \to [0,\infty]$ such that $\alpha|\mathfrak{C}((\varphi_p)_\bullet) = \Phi_p$ for all $p \in I$. In particular $\alpha(X) = C$. The definition implies that

$$\alpha_\star = \sup_{p\in I} (\Phi_p)_\star = \sup_{p\in I} (\varphi_p)_\bullet.$$

Next we define $\mathfrak{T}$ to be the lattice generated by $\bigcup_{p\in I} (\mathfrak{S}_p)_\bullet$. Thus $\mathfrak{T}$ consists of the nonvoid-finite unions of the nonvoid-finite intersections of members of $\bigcup_{p\in I} (\mathfrak{S}_p)_\bullet$. We list its relevant properties.

i) $\mathfrak{T} \subset \bigcup_{p\in I} \big(\mathfrak{C}(\varphi_p)_\bullet \cap \mathrm{Sat}H_p\big) \subset \mathfrak{A}$. This follows from 4.8.

ii) $\mathfrak{T} \subset \mathfrak{S}\top\mathfrak{S}_\bullet = \mathfrak{S}_\bullet\top\mathfrak{S}_\bullet$. This follows from $(\Leftarrow)$ in A).

iii) $\mathfrak{S}$ and hence $\mathfrak{S}_\bullet$ are upward enclosable $\mathfrak{T}$. In fact, for $S \in \mathfrak{S}$ we have $H_p(S) \in (\mathfrak{T}_p)_\bullet$ from $(\rightarrow)$ in A) and hence $S \subset H_p^{-1}(H_p(S)) \in H_p^{-1}((\mathfrak{T}_p)_\bullet) = (\mathfrak{S}_p)_\bullet \subset \mathfrak{T}$.
iv) α is inner regular $\mathfrak{T}$ and hence an inner $\star$ extension of $\alpha|\mathfrak{T} < \infty$. This is clear from the definition of α.

We see from iv) that $\psi := \alpha|\mathfrak{T}$ is an inner $\star$ premeasure $\psi : \mathfrak{T} \to [0,\infty[$ and that $\Psi = \psi_\star|\mathfrak{C}(\psi_\star)$ is an extension of α. Thus $\mathfrak{A} \subset \mathfrak{C}(\psi_\star)$ and $\psi_\star = \alpha$ on $\mathfrak{A}$. It follows that $\psi_\star = \alpha_\star$ and hence $\Psi = \alpha_\star|\mathfrak{C}(\alpha_\star)$. Then we use iii) to see that 1.8 can be applied in case $\star$ to the lattices $\mathfrak{S}_\bullet$ and $\mathfrak{T}$, and to any inner $\star$ premeasure $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ with $\Phi = \phi_\star|\mathfrak{C}(\phi_\star)$ and the above $\psi : \mathfrak{T} \to [0,\infty[$. In view of ii) we have $\mathfrak{T} \subset \mathfrak{S}_\bullet \top \mathfrak{S}_\bullet \subset \mathfrak{C}(\phi_\star)$, so that $\Longrightarrow$ in 1.8 reads that

$$\Phi \text{ extension of } \psi \Longrightarrow \Phi \text{ extension of } \Psi.$$

We combine this with the obvious implications

$$\Phi \text{ extension of } \Psi = \alpha_\star|\mathfrak{C}(\alpha_\star) \Longrightarrow \Phi \text{ extension of } \alpha \Longrightarrow$$
$$\Phi \text{ extension of } \alpha|\bigcup_{p\in I}\big(\mathfrak{C}(\varphi_p)_\bullet\big) \cap \mathrm{Sat}H_p\big) \Longrightarrow \Phi \text{ extension of } \alpha|\mathfrak{T} = \psi,$$

where the last $\Longrightarrow$ follows from i). The result is that all these assertions are equivalent. Now the third assertion means that Φ is an extension of $\Phi_p|\big(\mathfrak{C}((\varphi_p)_\bullet) \cap \mathrm{Sat}H_p\big) = \overleftarrow{H}_p\Psi_p$ for all $p \in I$, where 3.13.3) has been used. Equivalent by 3.3 is that $\overrightarrow{H}_p\Phi$ is an extension of Ψ_p for all $p \in I$, and hence by 4.5 that ϕ is a weak solution. Thus we have proved what follows.

Theorem 4.9. *Assume that Ψ_p lives on $H_p(X) \subset Y_p$ for all $p \in I$, so that the inverse images $\varphi_p : \mathfrak{S}_p \to [0,\infty[$ are defined, and likewise their combination $\alpha : \mathfrak{A} \to [0,\infty]$ which fulfils*

$$\alpha_\star(A) = \sup_{p\in I}(\varphi_p)_\bullet(A) = \sup_{p\in I}(\psi_p)_\bullet\big((H_p(A'))'\big) \quad \textit{for all } A \subset X.$$

Then each inner $\star$ premeasure $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ with $\Phi = \phi_\star|\mathfrak{C}(\phi_\star)$ fulfils

$$\phi \textit{ is a weak solution} \iff \Phi \textit{ is an extension of } \alpha_\star|\mathfrak{C}(\alpha_\star).$$

In this case $\alpha_\star \leqq \phi_\star$.

We continue to invoke the new transplantation theorem 2.4 in order to obtain the main result. Because of iii)ii) the implication $\Longrightarrow$ in 2.4 can be applied to the lattices $\mathfrak{S}_\bullet$ and $\mathfrak{T}$, and to the above inner $\star$ premeasure $\psi : \mathfrak{T} \to [0,\infty[$. Now on the one hand the assumption in $\Longrightarrow$ requires that $\inf\{\psi_\star(S') : S \in \mathfrak{S}_\bullet\} = \inf\{\psi_\star(S') : S \in \mathfrak{S}\}$ be $= 0$. To appreciate this we recall that our previous formulas combine to

$$\psi_\star(A') = \alpha_\star(A') = \sup_{p\in I}(\varphi_p)_\bullet(A') = \sup_{p\in I}(\psi_p)_\bullet\big((H_p(A))'\big) \ \forall A \subset X.$$

Thus the assumption in question is identical with the Prokhorov condition (Π). On the other hand the conclusion (∃) in $\Longrightarrow$ asserts that there exists an inner $\star$ premeasure $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ such that $\Phi = \phi_\star|\mathfrak{C}(\phi_\star)$ is an extension of $\Psi = \alpha_\star|\mathfrak{C}(\alpha_\star)$. In view of 4.9 this means that ϕ is a weak solution. Thus 2.4 leads at once to the main result which follows. We need to recall that condition (Π) implies the overall assumption that Ψ_p lives on $H_p(X) \subset Y_p$ for all $p \in I$.

Theorem 4.10. *Assume that* (Π) *is fulfilled. Then there exists at least one weak solution* $\phi : \mathfrak{S}_\bullet \to [0, \infty[$.

Example 4.16 at the end of the section will show that condition (Π) *does not enforce the existence of solutions* $\varphi : \mathfrak{S} \to [0, \infty[$. At last we want to obtain a *uniqueness* assertion. We need a simple remark and two more utensils.

Remark 4.11. Let $p \leqq q$ in I. i) For $A \subset X$ we have $H_p^{-1}(H_p(A)) \supset H_q^{-1}(H_q(A))$. ii) For $A \subset X$ with $H_p(A) \in \mathfrak{C}((\psi_p)_\bullet)$ we have $(\psi_p)_\bullet(H_p(A)) \geqq (\psi_q)_\bullet(H_q(A))$.

Proof. i) We have $H_p^{-1}(H_p(A)) = H_q^{-1}\big(H_{pq}^{-1}(H_{pq}(H_q(A)))\big) \supset H_q^{-1}(H_q(A))$. ii) We have $H_{pq}^{-1}(H_p(A)) \in \mathfrak{C}((\psi_q)_\bullet)$ with $\Psi_p(H_p(A)) = \Psi_q(H_{pq}^{-1}(H_p(A)))$, and $H_{pq}^{-1}(H_p(A)) \supset H_q(A)$. □

Next we define the set function $\vartheta : \mathrm{dom}(\vartheta) \to [0, \infty]$ as follows. Its domain consists of the subsets $A \subset X$ such that there exists $u \in I$ with $H_p(A) \in \mathfrak{C}((\psi_p)_\bullet)$ for all $p \geqq u$ in I. From 4.11.ii) then $(\psi_p)_\bullet(H_p(A)) \geqq (\psi_q)_\bullet(H_q(A))$ for $u \leqq p \leqq q$ in I. We are entitled to define

$$\vartheta(A) = \inf_{p \in I, p \geqq u} (\psi_p)_\bullet(H_p(A)) =: \lim_{p \uparrow I} (\psi_p)_\bullet(H_p(A)),$$

because the infimum in question does not depend on the individual $u \in I$. In particular $\mathfrak{S} \subset \mathrm{dom}(\vartheta)$ and $\vartheta|\mathfrak{S} < \infty$. Moreover 3.13.5) implies that $\mathfrak{A} \subset \mathrm{dom}(\vartheta)$ when Ψ_p lives on $H_p(X) \subset Y_p$ for all $p \in I$. We note some properties.

Remark 4.12. i) Each weak solution $\phi : \mathfrak{S}_\bullet \to [0, \infty[$ fulfils $\phi_\star(A) \leqq \vartheta(A)$ for all $A \in \mathrm{dom}(\vartheta)$. In particular $\phi \leqq \vartheta$ on $\mathfrak{S}$.

ii) Assume that Ψ_p lives on $H_p(X) \subset Y_p$ for all $p \in I$. Then

$$\alpha_\star(A) \leqq \vartheta(A) \quad \text{and} \quad C = \vartheta(A) + \alpha_\star(A') \quad \text{for all } A \in \mathrm{dom}(\vartheta).$$

In particular in case $C < \infty$ one has $\alpha_\star(A) = \vartheta(A)$ for $A \in \mathrm{dom}(\vartheta) \cap \mathfrak{C}(\alpha_\star)$.

Proof. i) For $A \in \mathrm{dom}(\vartheta)$ let as in the definition for $p \geqq u$ in I be $H_p(A) \in \mathfrak{C}((\psi_p)_\bullet)$, so that $H_p^{-1}(H_p(A)) \in \mathfrak{C}(\phi_\star)$ and $\phi_\star\big(H_p^{-1}(H_p(A))\big) = (\psi_p)_\bullet(H_p(A))$ after 4.5. This implies that $\phi_\star(A) \leqq \vartheta(A)$.

ii) For $A \in \mathrm{dom}(\vartheta)$ let once more be $H_p(A) \in \mathfrak{C}((\psi_p)_\bullet)$ for $p \geqq u$ in I. Then

$$(\varphi_p)_\bullet(A) = (\psi_p)_\bullet((H_p(A'))') = (\psi_p)_\bullet\big((H_p(A'))' \cap H_p(X)\big) \leqq (\psi_p)_\bullet(H_p(A)),$$
$$C = (\psi_p)_\bullet(H_p(A)) + (\psi_p)_\bullet((H_p(A))') = (\psi_p)_\bullet(H_p(A)) + (\varphi_p)_\bullet(A').$$

These relations and the monotone dependence on $p \geqq u$ of the terms involved furnish the two assertions. □

The other concept to be introduced is the *uniqueness condition*

(UC•) for each $S \in \mathfrak{S}$ there exists a nonvoid $\bullet$ subset $K \subset I$

such that $\bigcap_{p \in K} H_p^{-1}(H_p(S)) = S$.

In case $\bullet = \tau$ one can take $K = I$. In view of 4.11.i) one can take K in case $\bullet = \sigma$ to be an isotone sequence $p(1) \leqq \cdots \leqq p(n) \leqq \cdots$ in I, and in case $\bullet = \star$ to be $K = \{p\}$ for some $p \in I$. Thus K can be assumed to be upward directed. Then 4.11.i) implies that $\{H_p^{-1}(H_p(S)) : p \in K\} \downarrow S$.

Remark 4.13. Assume that i) $\mathfrak{S}$ is $\bullet$ compact,
ii) $H_p^{-1}(\{b\}) \in \mathfrak{S}\top\mathfrak{S}_\bullet$ for all $b \in Y_p$ and $p \in I$,
iii) for each $S \in \mathfrak{S}$ there exists a nonvoid $\bullet$ subset $K \subset I$ such that $\bigcap_{p\in K} H_p^{-1}(\{H_p(a)\}) \subset S$ for all $a \in S$.
Then (UC$\bullet$) is fulfilled.

Let us note that in case $\bullet = \tau$ condition iii) is satisfied when the product map $(H_p)_{p\in I} : X \to \prod_{p\in I} Y_p$ is injective.

Proof. Fix $S \in \mathfrak{S}$ and then a nonvoid $\bullet$ subset $K \subset I$ as in iii). We can assume as before that K is upward directed. We claim that for this K condition (UC$\bullet$) is satisfied. In fact, let $u \in X$ such that $u \in H_p^{-1}(H_p(S))$ or $H_p(u) \in H_p(S)$ for all $p \in K$. To be shown is $u \in S$. For $p \in K$ note that $A_p := \{x \in S : H_p(x) = H_p(u)\} = H_p^{-1}(\{H_p(u)\}) \cap S$ is nonvoid and $\in \mathfrak{S}_\bullet$ by ii). Since $\mathfrak{S}_\bullet$ is $\bullet$ compact it follows that $\{A_p : p \in K\} \downarrow$ some nonvoid $A \subset S$. For each $a \in A \subset S$ we obtain $u \in \bigcap_{p\in K} H_p^{-1}(\{H_p(a)\})$ and hence $u \in S$ in view of iii). $\square$

With condition (UC$\bullet$) we obtain the uniqueness assertion which follows.

Proposition 4.14. *Assume that* (UC$\bullet$) *is fulfilled. Then each solution* $\varphi : \mathfrak{S} \to [0,\infty[$ *must be* $\varphi = \vartheta$. *We have even* $\Phi(A) = \vartheta(A)$ *for all those* $A \in \mathfrak{S}_\bullet$ *which fulfil* $H_p(A) \in (\mathfrak{T}_p)_\bullet$ *for* $p \geqq$ *some* $u \in I$.

Proof. Let $A \in \mathfrak{S}_\bullet$ with its $u \in I$ as above, and fix $\mathfrak{M} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{M} \downarrow A$. There exists a nonvoid $\bullet$ subset $K \subset I$ such that $\bigcap_{p\in K} H_p^{-1}(H_p(M)) = M$ for all $M \in \mathfrak{M}$, which implies that $\bigcap_{p\in K} H_p^{-1}(H_p(A)) = A$. We can assume as before that K is upward directed. Thus $\{H_p^{-1}(H_p(A)) : p \in K\} \downarrow A$. We can also assume that the $p \in K$ are $\geqq u$, so that $H_p(A) \in (\mathfrak{T}_p)_\bullet \subset \mathfrak{C}((\psi_p)_\bullet)$ and hence

$$\Psi_p(H_p(A)) = \Phi\big(H_p^{-1}(H_p(A))\big) < \infty \quad \text{with } H_p^{-1}(H_p(A)) \in \mathfrak{S}\top\mathfrak{S}_\bullet.$$

The infimum under $u \leqq p \in I$ on the left side is $= \vartheta(A)$. On the right side it is on the one hand $\geqq \Phi(A)$, and on the other hand $\leqq$ the infimum under $p \in K$ which is $\Phi(A)$ from 1.5, so that it is $= \Phi(A)$. $\square$

We have to realize that the above uniqueness assertion on the basis of (UC$\bullet$) *does not refer to the weak solutions* $\phi : \mathfrak{S}_\bullet \to [0,\infty[$, *but is restricted to the solutions* $\varphi : \mathfrak{S} \to [0,\infty[$.

Example 4.15. The most natural example for the situation 4.1 is that the Y_p and X are Hausdorff topological spaces with $\mathfrak{T}_p = \mathrm{Comp}(Y_p)$ and $\mathfrak{S} = \mathrm{Comp}(X)$, and that the maps H_{pq} and H_p are continuous. Then the ψ_p are Radon premeasures on the Y_p such that $\psi_p = (\psi_q)_\bullet(H_{pq}^{-1}(\cdot))|\mathfrak{T}_p$ and hence $\overrightarrow{H}_{pq}\Psi_q = \Psi_p$ for $p \leqq q$.

We are of course in the case $\bullet = \tau$. The above 4.10 and its converse 4.7 combined with 4.3 and 4.5 then assert that condition (Π) is equivalent to $C < \infty$ plus the existence of at least one Radon premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ on X which fulfils $\psi_p = \varphi_\bullet(H_p^{-1}(\cdot))|\mathfrak{T}_p$ and hence $\vec{H}_p\Phi = \Psi_p$ for all $p \in I$. Furthermore 4.14 with 4.13 assert that φ is unique whenever the product map $(H_p)_{p\in I} : X \to \prod_{p\in I} X_p$ is injective, and that in this case $\varphi(S) = \inf_{p\in I} \psi_p(H_p(S))$ for $S \in \mathrm{Comp}(X)$. These results, or at least the last-mentioned case of uniqueness, are for instance in Kisyński [9] section 3, Bourbaki [2] section 4.2, Schwartz [20] section I.10, and Fremlin [5] 418M.

We turn to the final example announced above. The example covers both cases $\bullet = \sigma\tau$. We use on $\mathbb{R}$ the Lebesgue premeasure $\lambda : \mathrm{Comp}(\mathbb{R}) \to [0,\infty[$, and on the unit circle $\mathbb{S} = \{s \in \mathbb{C} : |s| = 1\}$ the arc length premeasure $\gamma : \mathrm{Comp}(\mathbb{S}) \to [0,\infty[$ normalized to $\gamma(\mathbb{S}) = 1$. The connection is via the map $H : \mathbb{R} \to \mathbb{S}$ defined to be $H(x) = \exp(2\pi i x)$: For each $c \in \mathbb{R}$ the restricted map $H|[c, c+1]$ transforms the restricted Lebesgue premeasure $\lambda|[c, c+1]$ into γ in the direct image sense of 3.10 and 3.11, that is

$$\gamma(K) = \lambda\big(\{x \in [c, c+1] : \exp(2\pi i x) \in K\}\big) \quad \text{for } K \in \mathrm{Comp}(\mathbb{S}).$$

We recall the well-known fact that γ is invariant under the maps $h_m : \mathbb{S} \to \mathbb{S}$ defined to be $h_m(s) = s^m$ for $m \in \mathbb{N}$, that is $\gamma(K) = \gamma(h_m^{-1}(K))$ for $K \in \mathrm{Comp}(\mathbb{S})$.

Example 4.16. Let $I = \{p \in \mathbb{Z} : p \geqq 0\}$ with the usual total order $\leqq$. For $p \in I$ let $Y_p = \mathbb{S}$ with $\mathfrak{T}_p = \mathrm{Comp}(\mathbb{S})$ and $\psi_p = \gamma$. For $p \leqq q$ in I define $H_{pq} : Y_p \leftarrow Y_q$ to be $H_{pq}(s) = s^{2^{q-p}}$. Then $\psi_p = \psi_q(H_{pq}^{-1}(\cdot))|\mathfrak{T}_p$ as mentioned above. On the other side let $X = \mathbb{R}$. For $p \in I$ define $H_p : Y_p \leftarrow X$ to be $H_p(x) = \exp(2^{-p}2\pi i x)$, so that $H_p = H_{pq} \circ H_q$ for $p \leqq q$ in I. Note that the H_p are surjective.

For $p \in I$ let $\mathfrak{S}_p := H_p^{-1}(\mathfrak{T}_p) = H_p^{-1}\big(\mathrm{Comp}(\mathbb{S})\big)$, so that $\mathfrak{S}_p$ is a lattice in $X = \mathbb{R}$ with $\varnothing, X \in \mathfrak{S}_p$ and $\mathfrak{S}_p = (\mathfrak{S}_p)_\bullet$. $\mathfrak{S}_p$ consists of the closed subsets of $\mathbb{R}$ which are periodic with period 2^p. Thus $\mathfrak{S}_p \subset \mathfrak{S}_q$ for $p \leqq q$ in I. Therefore $\mathfrak{S} := \bigcup_{p\in I} \mathfrak{S}_p$ is a lattice in $X = \mathbb{R}$ with $\varnothing, X \in \mathfrak{S}$. It is clear that $(\to)\, H_p(\mathfrak{S}) \subset \mathrm{Comp}(\mathbb{S}) = \mathfrak{T}_p$, and obvious that $(\leftarrow)\, H_p^{-1}(\mathfrak{T}_p) = \mathfrak{S}_p \subset \mathfrak{S} = \mathfrak{S}\top\mathfrak{S} \subset \mathfrak{S}\top\mathfrak{S}_\bullet$. Thus the situation fulfils 4.1, and we have $C = 1$. Moreover condition (Π) is fulfilled for the trivial reasons that $X \in \mathfrak{S}$ and $H_p(X) = Y_p$. However, note that $\mathfrak{S}_\bullet$ is a more complicated formation.

For $p \in I$ let as above $\varphi_p : \mathfrak{S}_p \to [0,\infty[$ be the inverse image of $\psi_p = \gamma$ under H_p. We also recall the content $\alpha : \mathfrak{A} \to [0,\infty]$ which this time fulfils

$$\alpha_\star(A) = \sup_{p\in I} \gamma_\bullet\big((H_p(A'))'\big) \quad \text{for all } A \subset X = \mathbb{R}.$$

We use this formula in order to prove that *each bounded subset* $A \subset \mathbb{R}$ *is in* $\mathfrak{C}(\alpha_\star)$ *and has* $\alpha_\star(A) = 0$. In view of $\alpha(X) = 1$ it follows that *the content* $\alpha_\star|\mathfrak{C}(\alpha_\star)$ *is not upward* σ *continuous.*

For the proof fix a bounded subset $A \subset \mathbb{R}$. We show that 1) $\alpha_\star(A) = 0$ and 2) $\alpha_\star(A') = 1$. Then 1.7.1) applied to $\xi := \alpha_\star$ and $\mathfrak{T} := \{X\}$ asserts that $A \in \mathfrak{C}(\alpha_\star)$ and hence the full claim. 1) is obvious since $H_p(A') = \mathbb{S}$ and hence $(H_p(A'))' = \varnothing$. 2) Fix $c > 0$ such that $A \subset [-c, c]$ and hence

$$H_p(A) \subset H_p([-c,c]) = \{\exp(2\pi i t) : |t| \leqq 2^{-p}c\}.$$

For the $p \in I$ with $2^{-p}2c < 1$ it follows that

$$\gamma_\bullet\big((H_p(A))'\big) \geqq 1 - \gamma\big(H_p([-c,c])\big) = 1 - 2^{-p}2c,$$

and hence $\alpha_\star(A') = 1$ as claimed.

Now 4.9 asserts for each weak solution $\phi : \mathfrak{S}_\bullet \to [0,\infty[$ that $\Phi = \phi_\star|\mathfrak{C}(\phi_\star)$ is an extension of $\alpha_\star|\mathfrak{C}(\alpha_\star)$. Therefore Φ is not upward σ continuous as well. Thus 4.3 tells us that *there are no solutions* $\varphi : \mathfrak{S} \to [0,\infty[$. $\square$

5. The Kolmogorov Type Theorem

In this section we consider the specialization of the above situation 4.1 which corresponds to the traditional situation named after Kolmogorov [10] chapter III section 4. For recent presentations we refer to Bauer [1] section 35 and Stromberg [21] chapter 7. In the present context the situation is as follows.

Let T be an infinite index set, and let I consist of the nonvoid finite subsets of T. On I one defines the order $\leqq$ to be the inclusion $\subset$, so that I under $\leqq$ is upward directed.

For each $t \in T$ let Y_t be a nonvoid set. For $p \in I$ one forms $Y_p := \prod_{t\in p} Y_t$. For each pair $p \leqq q$ in I let $H_{pq} : Y_p \leftarrow Y_q$ be the canonical projection. The H_{pq} are surjective and fulfil $H_{pr} = H_{pq} \circ H_{qr}$ for $p \leqq q \leqq r$ in I. Next one forms $X := \prod_{t\in T} Y_t$. For each $p \in I$ let $H_p : Y_p \leftarrow X$ be the canonical projection. The H_p are surjective and fulfil $H_p = H_{pq} \circ H_q$ for $p \leqq q$ in I. In particular for $t \in T$ the $H_{\{t\}} =: H_t$ are the canonical projections $H_t : Y_t \leftarrow X$.

Then for each $t \in T$ let $\mathfrak{T}_t$ be a lattice in Y_t with $\varnothing, Y_t \in \mathfrak{T}_t$ and hence $\mathfrak{T}_t \top \mathfrak{T}_t = \mathfrak{T}_t$, and with $\{b\} \in \mathfrak{T}_t$ for all $b \in Y_t$. We assume $\mathfrak{T}_t$ to be $\bullet$ *compact.* For $p \in I$ one forms

$$\mathfrak{T}_p := \{\prod_{t\in p} T_t : T_t \in \mathfrak{T}_t \text{ for } t \in p\}^\star,$$

that is $\mathfrak{T}_p = (\sqcap_{t\in p} \mathfrak{T}_t)^\star$ in the sense of [13] section 2 (note that $\mathfrak{M}^\star$ is defined to consist of the unions of the nonvoid finite subsets of $\mathfrak{M}$). Thus $\mathfrak{T}_p$ is a lattice in Y_p with $\varnothing, Y_p \in \mathfrak{T}_p$ and hence $\mathfrak{T}_p \top \mathfrak{T}_p = \mathfrak{T}_p$, and with $\{b\} \in \mathfrak{T}_p$ for all $b \in Y_p$. From well-known facts [13] 2.5-2.6 one knows that $\mathfrak{T}_p$ is $\bullet$ *compact.* In particular $\mathfrak{T}_{\{t\}} = \mathfrak{T}_t$ for $t \in T$. For $p \leqq q$ in I we have

($\leftarrow$) $H_{pq}^{-1}(\mathfrak{T}_p) \subset \mathfrak{T}_q$ and hence ($\Leftarrow$) $H_{pq}^{-1}((\mathfrak{T}_p)_\bullet) \subset (\mathfrak{T}_q)_\bullet$,
($\rightarrow$) $H_{pq}(\mathfrak{T}_q) = \mathfrak{T}_p$ and hence ($\Rightarrow$) $H_{pq}((\mathfrak{T}_q)_\bullet) \subset (\mathfrak{T}_p)_\bullet$ from 3.12.

Next one forms

$$\mathfrak{S} := \{\prod_{t\in T} T_t : T_t \in \mathfrak{T}_t \text{ for all } t \in T \text{ and } T_t = Y_t \text{ for almost all } t \in T\}^\star,$$

where as usual *almost all* means *all aside from a finite number*, that is $\mathfrak{S} = (\underset{t\in T}{\times} \mathfrak{T}_t)^\star$ in the sense of [13] section 2. Thus $\mathfrak{S}$ is a lattice in X with $\varnothing, X \in \mathfrak{S}$ and hence $\mathfrak{S}\top\mathfrak{S} = \mathfrak{S}$, and as before $\mathfrak{S}$ is $\bullet$ *compact.* For $p \in I$ we have

$(\leftarrow)$ $H_p^{-1}(\mathfrak{T}_p) \subset \mathfrak{S}$ and hence $(\Leftarrow)$ $H_p^{-1}((\mathfrak{T}_p)_\bullet) \subset \mathfrak{S}_\bullet$,

$(\rightarrow)$ $H_p(\mathfrak{S}) = \mathfrak{T}_p$ and hence $(\Rightarrow)$ $H_p(\mathfrak{S}_\bullet) \subset (\mathfrak{T}_p)_\bullet$ from 3.12.

At last note that for each $S \in \mathfrak{S}$ there exists $p \in I$ such that $H_p^{-1}(H_p(S)) = S$.

Thus we have a)A) in assumption 4.1. The present form of part b) will be the assumption in the theorem which follows.

Theorem 5.1. *Let $(\psi_p)_{p\in I}$ be a family of inner $\bullet$ premeasures $\psi_p : \mathfrak{T}_p \to [0,\infty[$ with $\Psi_p = (\psi_p)_\bullet|\mathfrak{C}((\psi_p)_\bullet)$. For $p \leqq q$ in I assume that*

$$\psi_p\big(\prod_{t\in p} T_t\big) = \psi_q\big(\prod_{t\in q} T_t\big) \quad \textit{for } T_t \in \mathfrak{T}_t\ \forall t \in p \textit{ and } T_t = Y_t\ \forall t \in q \setminus p.$$

Then there exists a unique inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ such that for all $p \in I$

$$\psi_p(\prod_{t\in p} T_t) = \varphi(\prod_{t\in T} T_t) \quad \textit{for } T_t \in \mathfrak{T}_t\ \forall t \in p \textit{ and } T_t = Y_t\ \forall t \in T \setminus p.$$

We have

$$\varphi(S) = \min_{p\in I} \Psi_p(H_p(S)) \qquad \textit{for } S \in \mathfrak{S}, \textit{ and even}$$

$$\Phi(A) = \inf_{p\in I} \Psi_p(H_p(A)) \qquad \textit{for } A \in \mathfrak{S}_\bullet.$$

Furthermore we have $(\psi_p)_\bullet = \varphi_\bullet(H^{-1}(\cdot))$ and $\vec{H}_p\Phi = \Psi_p$ for all $p \in I$.

Proof. i) We claim that $\psi_p = \psi_q(H_{pq}^{-1}(\cdot))|\mathfrak{T}_p$ for $p \leqq q$ in I. In fact, for an $A \in \mathfrak{T}_p$ of the form $A = \prod_{t\in p} T_t$ with $T_t \in \mathfrak{T}_t$ $\forall t \in p$ this relation coincides with the assumption. For a finite union of such particular $A \in \mathfrak{T}_p$ it then follows from the folklore formula MI 2.5.1) applied to the set functions ψ_p and $\psi_q(H_{pq}^{-1}(\cdot))|\mathfrak{T}_p$. Thus we have part b) in assumption 4.1 and hence all of 4.1.

ii) Condition (Π) is satisfied for the trivial reasons that $X \in \mathfrak{S}$ and $H_p(X) = Y_p$. Since $\mathfrak{S}$ is $\bullet$ compact we conclude from 4.10 and 4.3 that there exists at least one solution $\varphi : \mathfrak{S} \to [0,\infty[$. The equivalence with the respective formulation in the theorem is seen as in i) above. Next 4.14 can be applied, because as noted above we have (UC$\star$) and hence (UC$\bullet$). In case $S \in \mathfrak{S}$ we can write *min* instead of *inf*, because for $p \leqq q$ in I with $H_p^{-1}(H_p(S)) = S$ we have $H_{pq}^{-1}(H_p(S)) = H_q(S)$ and hence $\Psi_p(H_p(S)) = \Psi_q\big(H_{pq}^{-1}(H_p(S))\big) = \Psi_q(H_q(S))$. At last the two final assertions are contained in 4.5. $\square$

It is a matter of routine to liberate the situation from the assumption that $Y_t \in \mathfrak{T}_t$ for all $t \in T$. This will be done in the sequel. We have to supplement the situation as follows.

For each $t \in T$ let $\mathfrak{K}_t$ be a lattice in Y_t with $\varnothing \in \mathfrak{K}_t$, and with $\{b\} \in \mathfrak{K}_t$ for all $b \in Y_t$. We assume $\mathfrak{K}_t$ to be $\bullet$ *compact.* Then $\mathfrak{T}_t := \mathfrak{K}_t \cup \{Y_t\}$ is a lattice in Y_t as it has been before; in particular note that $\bullet$ *compactness* carries over from $\mathfrak{K}_t$ to $\mathfrak{T}_t$. For $p \in I$ one forms

$$\mathfrak{K}_p := \{\prod_{t\in p} K_t : K_t \in \mathfrak{K}_t \text{ for } t \in p\}^\star,$$

that is $\mathfrak{K}_p = (\prod_{t\in p} \mathfrak{K}_t)^\star$ in the sense of [13] section 2. Thus $\mathfrak{K}_p$ is a lattice in Y_p with $\varnothing \in \mathfrak{K}_p$, and with $\{b\} \in \mathfrak{K}_p$ for all $b \in Y_p$, and as before $\mathfrak{K}_p$ is $\bullet$ *compact.* In particular $\mathfrak{K}_{\{t\}} = \mathfrak{K}_t$ for $t \in T$. We also retain the lattice $\mathfrak{T}_p$ in Y_p. The basic relations between the two lattices are $\mathfrak{K}_p \subset \mathfrak{T}_p \subset \mathfrak{K}_p \top \mathfrak{K}_p$. At last we retain the lattice $\mathfrak{S}$ in X, with no companion this time.

For the sequel we need another little remark.

Remark 5.2. 1) Let $\vartheta : \mathfrak{S} \to [0, \infty[$ on a lattice $\mathfrak{S}$ be isotone and downward $\bullet$ continuous. Then $\vartheta_\bullet = \vartheta_\star$ on $\mathfrak{S}\top\mathfrak{S}$.

2) For some $p \in I$ let $\vartheta : \mathfrak{K}_p \to [0, \infty[$ be isotone with $\vartheta(\varnothing) = 0$ and downward $\bullet$ continuous. For each system of $T_t \in \mathfrak{T}_t$ $\forall t \in p$ then

$$\vartheta_\bullet\big(\prod_{t\in p} T_t\big) = \sup\{\vartheta\big(\prod_{t\in p} K_t\big) : K_t \in \mathfrak{K}_t \text{ with } K_t \subset T_t \ \forall t \in p\}.$$

Proof. 1) We fix $A \in \mathfrak{S}\top\mathfrak{S}$ and have to prove $\vartheta_\bullet(A) \leqq \vartheta_\star(A)$. Let $S \in \mathfrak{S}_\bullet$ with $S \subset A$, and $\mathfrak{M} \subset \mathfrak{S}$ nonvoid $\bullet$ with $\mathfrak{M} \downarrow S$. For $M \in \mathfrak{M}$ then $M \cap A \in \mathfrak{S}$ and $S \subset M \cap A \subset A$. It follows that $\vartheta_\bullet(S) \leqq \vartheta_\bullet(M \cap A) = \vartheta(M \cap A) \leqq \vartheta_\star(A)$, and hence $\vartheta_\bullet(A) \leqq \vartheta_\star(A)$.

2) It is obvious that $\geqq$. To see $\leqq$ let $A \in \mathfrak{K}_p$ with $A \subset \prod_{t\in p} T_t$. Then there exist $K_t \in \mathfrak{K}_t$ $\forall t \in p$ with $A \subset \prod_{t\in p} K_t \subset \prod_{t\in p} T_t$. In case $A \neq \varnothing$ it follows that $K_t \subset T_t$ for all $t \in p$ and hence $\vartheta(A) \leqq \vartheta\big(\prod_{t\in p} K_t\big) \leqq$ the second member. In view of 1) the assertion follows. $\square$

After this the above theorem attains the form which follows.

Theorem 5.3. *Let $(\varphi_p)_{p\in I}$ be a family of inner $\bullet$ premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ with $\Phi_p = (\varphi_p)_\bullet|\mathfrak{C}((\varphi_p)_\bullet) < \infty$. For $p \leqq q$ in I assume that*

$$\varphi_p\big(\prod_{t\in p} K_t\big) = \sup\{\varphi_q\big(\prod_{t\in q} K_t\big) : K_t \in \mathfrak{K}_t \ \forall t \in q \setminus p\} \ \textit{ for } K_t \in \mathfrak{K}_t \ \forall t \in p,$$

which after 5.2.2) *is equivalent to*

$$(\circ) \quad \varphi_p\big(\prod_{t\in p} K_t\big) = (\varphi_q)_\bullet\big(\prod_{t\in q} K_t\big) \ \textit{ for } K_t \in \mathfrak{K}_t \ \forall t \in p \textit{ and } K_t = Y_t \ \forall t \in q \setminus p.$$

Then there exists a unique inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ such that for all $p \in I$

$$\varphi_p(\prod_{t\in p} K_t) = \varphi(\prod_{t\in T} K_t) \ \textit{ for } K_t \in \mathfrak{K}_t \ \forall t \in p \textit{ and } K_t = Y_t \ \forall t \in T \setminus p,$$

and $\Phi_p(Y_p) = \varphi(X)$. *We have*

$$\varphi(S) = \min_{p \in I} \Phi_p(H_p(S)) \qquad \textit{for } S \in \mathfrak{S}, \textit{ and even}$$

$$\Phi(A) = \inf_{p \in I} \Phi_p(H_p(A)) \qquad \textit{for } A \in \mathfrak{S}_\bullet.$$

Furthermore we have $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ *and* $\overrightarrow{H}_p \Phi = \Phi_p$ *for all* $p \in I$.

For the purpose of reduction to 5.1 we conclude for fixed $p \in I$ from 1.6 applied to $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ and $\mathfrak{T}_p$ and from the above 5.2.1) that $\psi_p := (\varphi_p)_\bullet|\mathfrak{T}_p = (\varphi_p)_\star|\mathfrak{T}_p$ is an inner $\bullet$ premeasure $\psi_p : \mathfrak{T}_p \to [0,\infty[$ which fulfils $(\psi_p)_\bullet = (\varphi_p)_\bullet$. In particular $\Psi_p = (\psi_p)_\bullet|\mathfrak{C}((\psi_p)_\bullet)$ is $= \Phi_p$.

Proof. i) For $p \in I$ we see from 5.2.2) that

$$\psi_p\big(\prod_{t \in p} T_t\big) = \sup\{\varphi_p\big(\prod_{t \in p} K_t\big) : K_t \in \mathfrak{K}_t \text{ with } K_t \subset T_t \ \forall t \in p\} \quad \text{for } T_t \in \mathfrak{T}_t \ \forall t \in p.$$

ii) We claim that the inner $\bullet$ premeasures ψ_p for $p \in I$ fulfil the assumption in 5.1. In fact, let $p \leqq q$ in I, and fix $T_t \in \mathfrak{T}_t$ for $t \in p$ and $T_t = Y_t$ for $t \in q \setminus p$. For each system of $K_t \in \mathfrak{K}_t$ with $K_t \subset T_t \ \forall t \in p$ we have by assumption

$$\varphi_p\big(\prod_{t \in p} K_t\big) = \sup\{\varphi_q\big(\prod_{t \in q} K_t\big) : K_t \in \mathfrak{K}_t \ \forall t \in q \setminus p\}.$$

We form on either side the supremum over all these systems $(K_t)_{t \in p}$. Then i) asserts that this supremum is

$$= \psi_p\big(\prod_{t \in p} T_t\big) \text{ on the left, and } = \psi_q\big(\prod_{t \in q} T_t\big) \text{ on the right .}$$

Thus we obtain the present assertion.

iii) After this theorem 5.1 asserts that there exists a unique inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ which is as formulated at that place. It is clear from the above that the former properties

$$\psi_p(\prod_{t \in p} T_t) = \varphi(\prod_{t \in T} T_t) \ \text{ for } T_t \in \mathfrak{T}_t \ \forall t \in p \text{ and } T_t = Y_t \ \forall t \in T \setminus p,$$

asserted for all $p \in I$, are equivalent to the present ones for the φ_p and Φ_p, for all $p \in I$ as well. The further assertions persist. □

Theorem 5.4. *The assertion of* 5.3 *defines a one-to-one correspondence between the families* $(\varphi_p)_{p \in I}$ *of inner* $\bullet$ *premeasures* $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ *with* $\Phi_p = (\varphi_p)_\bullet|\mathfrak{C}((\varphi_p)_\bullet) < \infty$ *which fulfil* $(\circ)$ *for all* $p \leqq q$ *in* I, *and*

the inner $\bullet$ *premeasures* $\varphi : \mathfrak{S} \to [0,\infty[$ *with* $\varphi(X) < \infty$ *such that* $\varphi_\bullet(H_p^{-1}(\cdot)) : \mathfrak{P}(Y_p) \to [0,\infty[$ *is inner regular* $(\mathfrak{K}_p)_\bullet$ *for each* $p \in I$.

In view of the main result in the final section we define these particular $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(X) = 1$ to be the *Wiener* $\bullet$ *premeasures* for the present situation, that is for the family $(\mathfrak{K}_t)_{t \in T}$ with $\mathfrak{S}$, and their $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ to be the respective *Wiener measures.* The fundamental case will be $\bullet = \tau$.

Proof. Define Δ to consist of all families $(\varphi_p)_{p \in I}$ as described above, and Σ to consist of all $\varphi : \mathfrak{S} \to [0,\infty[$ as described above.

1) By 5.3 each $(\varphi_p)_{p\in I}$ in Δ produces an inner $\bullet$ premeasure $\varphi:\mathfrak{S}\to[0,\infty[$ which in view of $(\varphi_p)_\bullet=\varphi_\bullet(H_p^{-1}(\cdot))$ for $p\in I$ is a member of Σ.

2) Let $\varphi:\mathfrak{S}\to[0,\infty[$ be a member of Σ. For the first two steps we fix $p\in I$. 2.i) From 3.10 applied to $H_p:X\to Y_p$ with $\mathfrak{S}$ and $\mathfrak{T}_p$, for which the assumptions $(\Rightarrow)(\Leftarrow)$ are fulfilled after the initial part of this section, and to φ, we obtain an inner $\bullet$ premeasure $\psi_p:\mathfrak{T}_p\to[0,\infty[$ such that $(\psi_p)_\bullet=\varphi_\bullet(H_p^{-1}(\cdot))<\infty$. 2.ii) Then from 1.6 applied to $\psi_p:\mathfrak{T}_p\to[0,\infty[$ and to $\mathfrak{K}_p\subset\mathfrak{T}_p$ we obtain an inner $\bullet$ premeasure $\varphi_p:\mathfrak{K}_p\to[0,\infty[$ such that $(\varphi_p)_\bullet=(\psi_p)_\bullet=\varphi_\bullet(H_p^{-1}(\cdot))<\infty$.

2.iii) We claim that the family $(\varphi_p)_{p\in I}$ fulfils $(\circ)$ for all $p\leqq q$ in I, and hence is a member of Δ. In fact, for $K_t\in\mathfrak{K}_t$ for all $t\in p$ and $K_t=Y_t$ for all $t\in q\setminus p$ we have $H_q^{-1}(\prod_{t\in q}K_t)=H_p^{-1}(\prod_{t\in p}K_t)$ and hence

$$(\varphi_q)_\bullet\big(\prod_{t\in q}K_t\big)=(\varphi_p)_\bullet\big(\prod_{t\in p}K_t\big)=\varphi_p\big(\prod_{t\in p}K_t\big).$$

3) It remains to prove that the two maps $(\varphi_p)_{p\in I}\mapsto\varphi$ obtained in 1) and $\varphi\mapsto(\varphi_p)_{p\in I}$ obtained in 2) are invers to each other. 3.i) For the composition $(\varphi_p)_{p\in I}\mapsto\varphi\mapsto(\tilde\varphi_p)_{p\in I}$ we see from 1) and 2.ii) that $(\varphi_p)_\bullet=\varphi_\bullet(H_p^{-1}(\cdot))=(\tilde\varphi_p)_\bullet$ and hence $\varphi_p=\tilde\varphi_p$ for $p\in I$. 3.ii) For the composition $\varphi\mapsto(\varphi_p)_{p\in I}\mapsto\tilde\varphi$ we see from 2.ii) and 1) that

$$\varphi_\bullet(H_p^{-1}(\cdot))=(\varphi_p)_\bullet=\tilde\varphi_\bullet(H_p^{-1}(\cdot))\ \text{ on } \mathfrak{P}(Y_p) \text{ for } p\in I.$$

Now each $S\in\mathfrak{S}$ is of the form $S=A\times\prod_{t\in T\setminus p}Y_t=H_p^{-1}(A)$ for some $p\in I$ and $A\subset Y_p$. It follows that $\varphi=\tilde\varphi$. □

We conclude with an important specialization and the comparison with the traditional counterpart of the present development.

Example 5.5. The most natural example is that Y_t for $t\in T$ is a Hausdorff topological space with $\mathfrak{K}_t=\mathrm{Comp}(Y_t)$. We equip Y_p for $p\in I$ with the product topology. We are then led to assume that $\bullet=\tau$, because one has $(\mathfrak{K}_p)_\tau=\mathrm{Comp}(Y_p)$ from MI 21.3.2) and [13] 2.4.2). We recall from 1.4 the one-to-one correspondence between the inner τ premeasures $\varphi_p:\mathfrak{K}_p\to[0,\infty[$ and the Radon premeasures ϕ_p on Y_p via $(\varphi_p)_\tau=(\phi_p)_\tau$. Thus 5.3 and 5.4 produce a *one-to-one correspondence between the families $(\phi_p)_{p\in I}$ of Radon premeasures ϕ_p on Y_p with $\Phi_p(Y_p)=1$ which fulfil $(\circ)$ for all $p\leqq q$ in I, and the Wiener τ premeasures $\varphi:\mathfrak{S}\to[0,\infty[$ for the present situation.* We emphasize that this result reaches beyond topological measure theory, because $\mathfrak{S}$ does not appear as a set system which comes from some Hausdorff topology on X. We also note that the final assertion $\vec{H}_p\Phi=\Phi_p$ in 5.3 implies that

$$R\in\mathrm{Bor}(Y_p)\subset\mathfrak{C}((\varphi_p)_\tau)\Longrightarrow R\times\big(\prod_{t\in T\setminus p}Y_t\big)=H_p^{-1}(R)\in\mathfrak{C}(\varphi_\tau)\quad\text{for } p\in I.$$

After this we turn to the *traditional situation* cited at the outset of the section. Here one considers T with I and the Y_t for $t\in T$ with the Y_p for $p\in I$ and X as before. Then one assumes a family of σ algebras $\mathfrak{B}_t$ in Y_t for $t\in T$ and forms their usual product σ algebras $\mathfrak{B}_p$ in Y_p and

$$\mathfrak{A} = \mathrm{A}\sigma\big(\{\prod_{t\in T} B_t : B_t \in \mathfrak{B}_t \ \forall t \in T \text{ and } B_t = Y_t \text{ for almost all } t \in T\}\big) \text{ in } X.$$

We recall that for T uncountable the formation $\mathfrak{A}$ appears to be too small, because its members $A \in \mathfrak{A}$ are of *countable type* in the sense that $A = R \times \prod_{t\in T\setminus C} Y_t$ for some nonvoid countable $C \subset T$ and some $R \subset \prod_{t\in C} Y_t$. In this frame the desired counterpart of the above 5.3 would read as follows: *If $(\theta_p)_{p\in I}$ is a family of probability measures θ_p on $\mathfrak{B}_p$ which for $p \leqq q$ in I fulfils*

$$\theta_p\big(\prod_{t\in p} B_t\big) = \theta_q\big(\prod_{t\in q} B_t\big) \ \text{ for } B_t \in \mathfrak{B}_t \ \forall t \in p \text{ and } B_t = Y_t \ \forall t \in q \setminus p,$$

then there exists a unique probability measure θ on $\mathfrak{A}$ such that for $p \in I$

$$\theta_p\big(\prod_{t\in p} B_t\big) = \theta\big(\prod_{t\in T} B_t\big) \ \text{ for } B_t \in \mathfrak{B}_t \ \forall t \in p \text{ and } B_t = Y_t \ \forall t \in T \setminus p.$$

However, this statement is not true as it stands. But it is true in the special case that the Y_t for $t \in T$ are Polish topological spaces with $\mathfrak{B}_t = \mathrm{Bor}(Y_t)$. The finite product spaces Y_p are then Polish as well with $\mathfrak{B}_p = \mathrm{Bor}(Y_p)$. This fact and the further well-known particularities of the Polish spaces show that the present special case is an immediate outcome of the situation considered in 5.5 above: In fact, in view of the inner extension theorem 1.1 the families $(\theta_p)_{p\in I}$ of the present kind are in one-to-one correspondence with the families $(\varphi_p)_{p\in I}$ and $(\phi_p)_{p\in I}$ in 5.5. Thus from $(\theta_p)_{p\in I}$ the result in 5.5 produces the Wiener τ premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ with its Wiener measure $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. Its domain $\mathfrak{C}(\varphi_\tau)$ has been seen to contain the present $\mathfrak{A}$, and we obtain the measure θ as the restriction of Φ to $\mathfrak{A}$.

But the fundamental point is that θ can be a rather poor restriction of Φ, in that its domain $\mathfrak{A}$ can be much smaller than the comprehensive $\mathfrak{C}(\varphi_\tau)$ and refuse even the most important requirements. In fact, it can happen that some subset $E \subset X$ of utmost importance turns out to be *thick* with respect to θ, that is

$$\theta^\star(E) := \inf\{\theta(A) : A \in \mathfrak{A} \text{ with } A \supset E\} = 1, \ \text{ but has } \theta_\star(E) = 0,$$

so that E cannot be a member of $\mathfrak{A}$, whereas in our new approach one has $E \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(E) = 1$, so that Φ lives on E. The most prominent example will be the topic of the final section.

If in such situation one wants to pass to a probability measure on E, then in the traditional frame one has to form the so-called *contraction* $\theta_E := \theta^\star|\mathfrak{A} \sqcap E$ of θ onto E, that is a formation defined on a domain $\mathfrak{A} \sqcap E$ which is in essence *outside* the former $\mathfrak{A}$. In contrast, in the new frame one can form the *restriction* $\Phi|E$ of Φ to the domain $\mathfrak{C}(\varphi_\tau) \sqcap E = \{A \in \mathfrak{C}(\varphi_\tau) : A \subset E\}$ which is *contained* in the former domain $\mathfrak{C}(\varphi_\tau)$, and one has the pleasant properties listed in 1.10.

The final section below will be under a more special assumption, which is the usual one in probabilistic context. For T with I as before one assumes that $Y_t = Y$ for $t \in T$, so that $Y_p = Y^p$ for $p \in I$ and $X = Y^T$. In the traditional frame one then assumes $\mathfrak{B}_t = \mathfrak{B}$ for $t \in T$, so that $\mathfrak{B}_p = \mathrm{A}\sigma(\mathfrak{B}^p)$ with the usual product set system $\mathfrak{B}^p = \mathfrak{B} \times \cdots \times \mathfrak{B}$ and $\mathfrak{A}$ as before. In the present new situation we assume $\mathfrak{K}_t = \mathfrak{K}$ for $t \in T$, so that $\mathfrak{K}_p = (\mathfrak{K}^p)^\star$ and $\mathfrak{S}$ as before. Of course the most important special case is that Y is a Polish topological space with $\mathfrak{B} = \mathrm{Bor}(Y)$ and $\mathfrak{K} = \mathrm{Comp}(Y)$.

6. The true Wiener Measure

We assume the situation described at the end of the last section with $T = [0,\infty[$ and $Y = \mathbb{R}$ with $\mathfrak{B} = \mathrm{Bor}(\mathbb{R})$ and $\mathfrak{K} = \mathrm{Comp}(\mathbb{R})$, so that $X = \mathbb{R}^T = \mathbb{R}^{[0,\infty[}$ with $\mathfrak{A}$ and $\mathfrak{S}$ as before. Thus the members of X are the one-dimensional paths $x = (x_t)_{t\in T} : T = [0,\infty[\to \mathbb{R}$. We also assume that $\bullet = \tau$.

We fix a family $(\varphi_p)_{p\in I}$ of inner τ premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ with $\Phi_p(\mathbb{R}^p) = 1$ which fulfil $(\circ)$ for all $p \leqq q$ in I, and its Wiener τ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ and Wiener measure $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. We recall that $\overrightarrow{H_p}\Phi = \Phi_p$ implies that the projection $H_p : X \to \mathbb{R}^p$ is measurable with respect to $\mathfrak{C}(\varphi_\tau)$ and $\mathfrak{C}((\varphi_p)_\tau) \supset \mathrm{Bor}(\mathbb{R}^p)$. The present main theorem then reads as follows.

Theorem 6.1. *Assume that there are real numbers $\alpha,\beta > 0$ and $c > 0$ such that the projections $H_t : X \to \mathbb{R}\ \forall t \in T$ fulfil*

$$\int |H_s - H_t|^\alpha d\Phi \leqq c|s-t|^{1+\beta} \quad \text{for all } s,t \in T.$$

Fix $0 < \gamma \leqq 1$ with $\gamma < \beta/\alpha$, and define for real $M > 0$ the function class

$$E(\gamma, M) := \{x \in X : |x_0| \leqq M \text{ and } |x_u - x_v| \leqq M2^{(u\vee v)(1-\gamma)}|u-v|^\gamma\ \forall u,v \in T\}.$$

Then $E(\gamma,M) \in \mathfrak{S}_\tau$ and hence $E(\gamma) := \bigcup_{M>0} E(\gamma,M) \in (\mathfrak{S}_\tau)^\sigma \subset \mathfrak{C}(\varphi_\tau)$ with

$$\Phi(E(\gamma)) = \lim_{M\uparrow\infty} \Phi(E(\gamma,M)) = 1.$$

The assumption in 6.1 is the usual one in the theorem on the existence of so-called continuous modifications, like for instance in Bauer [1] 39.3. But the assertion is a drastic improvement: The traditional result in terms of the measure θ on $\mathfrak{A}$ is $\theta^\star(\mathrm{C}(T,\mathbb{R})) = 1$, and of course $\theta_\star(\mathrm{C}(T,\mathbb{R})) = 0$, and one obtains via contraction the traditional Wiener measure $\theta_{\mathrm{C}(T,\mathbb{R})} := \theta^\star|(\mathfrak{A} \sqcap \mathrm{C}(T,\mathbb{R}))$. In sharp contrast, the present situation concludes from $E(\gamma) \subset \mathrm{C}(T,\mathbb{R})$ that the subspace $\mathrm{C}(T,\mathbb{R})$ is a member of $\mathfrak{C}(\varphi_\tau)$ with $\Phi(\mathrm{C}(T,\mathbb{R})) = 1$. Also note the occurrence of the small subsystem $\mathfrak{S}_\tau$ of $\mathfrak{C}(\varphi_\tau)$, which after all is the most basic system of measurable sets, and the almost global character of the Hölder classes $E(\gamma,M)$, connected with the particular *bound of increase at infinity* contained in their definition.

The technical problems with the proof will be finished off with the lemma below. It is modelled after the standard procedure, like for instance in the proof of Stromberg [21] 8.2. We present the details both for the sake of completeness and because we need some peculiarities. We retain the assumption of 6.1. For fixed $0 < \gamma \leqq 1$ with $\gamma < \beta/\alpha$ we form

$$\delta := \frac{1}{2}(\beta - \alpha\gamma) > 0 \quad \text{and} \quad \lambda := \frac{2+\delta}{2+2\delta}, \text{ so that } 0 < \lambda < 1,$$
$$b := \frac{2^\gamma + 1}{2^\gamma - 1} > 0 \quad \text{and} \quad B := b\Big(\frac{2}{1-\lambda}\Big)^{1-\gamma}.$$

Let $\mathbb{D} \subset T$ consist of the dyadic rationals $\geqq 0$. Moreover let

$$\begin{aligned} \mathbb{D}(n) &:= \{t \in \mathbb{D} : 2^n t \in \mathbb{Z} \text{ and } t \leqq n\} \text{ and} \\ \mathbb{E}(n) &:= \{(s,t) \in \mathbb{D}(n) \times \mathbb{D}(n) : 0 < t - s \leqq 2^{-n\lambda}\} \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

Thus $\operatorname{card}(\mathbb{E}(n)) \leqq n2^{2n-n\lambda}$. Then define

$$\begin{aligned} A_n &:= \bigcap_{(s,t)\in\mathbb{E}(n)} [|H_s - H_t| \leqq |s-t|^\gamma] \in \mathfrak{C}(\varphi_T) \quad \text{for } n \in \mathbb{N}, \\ A &:= \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n \in \mathfrak{C}(\varphi_T). \end{aligned}$$

The lemma in question reads as follows.

Lemma 6.2. i) $\Phi(A) = 1$. ii) *Fix $x = (x_t)_{t\in T} \in A$, and choose $m \in \mathbb{N}$ such that $x \in \bigcap_{n=m}^{\infty} A_n$ and $m \geqq \frac{1+\lambda}{1-\lambda}$. Then*

$$|x_u - x_v| \leqq B2^{m\lambda(1-\gamma)}2^{(u\vee v)(1-\gamma)}|u-v|^\gamma \quad \textit{for } u,v \in \mathbb{D}.$$

Proof of 6.2. i) For $n \in \mathbb{N}$ and $(s,t) \in \mathbb{E}(n)$ we have

$$\begin{aligned} \Phi\big([|H_s - H_t| > |s-t|^\gamma]\big) &\leqq \int \Big(\frac{|H_s - H_t|}{|s-t|^\gamma}\Big)^\alpha d\Phi \\ &\leqq c|s-t|^{1+\beta-\alpha\gamma} = c|s-t|^{1+2\delta} \leqq c2^{-n\lambda(1+2\delta)}, \end{aligned}$$

and hence $\Phi(A'_n) \leqq cn2^{2n-n\lambda-n\lambda(1+2\delta)} = cn2^{2n(1-\lambda-\lambda\delta)} = cn2^{-n\delta}$, because one computes that $1-\lambda-\lambda\delta = -\delta/2$. It follows for $m \in \mathbb{N}$ that

$$A' \subset \bigcup_{n=m}^{\infty} A'_n \quad \text{and hence} \quad \Phi(A') \leqq \sum_{n=m}^{\infty} cn2^{-n\delta}.$$

Thus $\Phi(A') = 0$ or $\Phi(A) = 1$.

ii) The proof of this part is more involved. ii.0) First note for $n \geqq m$ that $x \in A_n$ and hence $|x_s - x_t| \leqq |s-t|^\gamma$ for all $(s,t) \in \mathbb{E}(n)$.

ii.1) We fix $0 < a < \infty$ and put $M := m + [a]$, with $[a]$ the integer part of a. Thus $M \in \mathbb{N}$ with $M \geqq m$ and $M \geqq a$. Then we fix $u, v \in \mathbb{D}$ with $0 \leqq u, v \leqq a$ and $0 < v - u \leqq 2^{-M\lambda}$. We claim that $|x_u - x_v| \leqq b|u-v|^\gamma$.

ii.1.1) There is a unique $n \in \mathbb{N}$ with $n \geqq M$ and $2^{-(n+1)\lambda} < v-u \leqq 2^{-n\lambda}$, and then there are unique integers i and j with $i - 1 < 2^n u \leqq i$ and $j \leqq 2^n v < j+1$. We have $i \geqq 0$, and

$$j - i + 2 > 2^n(v-u) > 2^{n-(n+1)\lambda} = 2^{n(1-\lambda)-\lambda} \geqq 2 \;\text{ implies that } j > i.$$

ii.1.2) We put $s = i2^{-n}$ and $t = j2^{-n}$. Then $0 \leqq u \leqq s < t \leqq v \leqq a \leqq M \leqq n$ and $0 < t - s \leqq v - u \leqq 2^{-n\lambda}$. Hence $(s,t) \in \mathbb{E}(n)$. From ii.0) therefore $|x_s - x_t| \leqq |s-t|^\gamma$.

ii.1.3) Next we estimate $|x_u - x_s|$. We have $s - 2^{-n} < u \leqq s$ or $0 \leqq s - u < 2^{-n}$ and $s - u \in \mathbb{D}$. Thus

$$s - u = \sum_{k=n+1}^{n+p} \varepsilon_k 2^{-k} \quad \text{with } p \in \mathbb{N} \text{ and } \varepsilon_k \in \{0,1\}.$$

We put $s(0) := s$ and

$$s(l) := s - \sum_{k=n+1}^{n+l} \varepsilon_k 2^{-k} \quad \text{for } 1 \leqq l \leqq p,$$

so that $s = s(0) \geqq s(1) \geqq \cdots \geqq s(p) = u$. For $0 \leqq l \leqq p$ we have $2^{n+l}s(l) \in \mathbb{Z}$ and $0 \leqq s(l) \leqq s \leqq n \leqq n+l$, that is $s(l) \in \mathbb{D}(n+l)$. Moreover $s(l-1) - s(l) = \varepsilon_{n+l}2^{-(n+l)}$, so that either $s(l-1) - s(l) = 0$ or $0 < s(l-1) - s(l) = 2^{-(n+l)} < 2^{-(n+l)\lambda}$ and hence $(s(l), s(l-1)) \in \mathbb{E}(n+l)$. In view of ii.0) we have in both cases $|x_{s(l)} - x_{s(l-1)}| \leqq |s(l) - s(l-1)|^\gamma \leqq 2^{-(n+l)\gamma}$. It follows that

$$|x_u - x_s| \leqq \sum_{l=1}^{p} 2^{-(n+l)\gamma} < \frac{2^{-n\gamma}}{2^\gamma - 1}.$$

ii.1.4) The same idea furnishes $|x_v - x_t| \leqq 2^{-n\gamma}/(2^\gamma - 1)$.

ii.1.5) From ii.1.2)3)4) and ii.1.1) we obtain

$$|x_u - x_v| < |u-v|^\gamma + \frac{2\,2^{-n\gamma}}{2^\gamma - 1} < \Big(1 + \frac{2\,2^{-n\gamma+(n+1)\lambda\gamma}}{2^\gamma - 1}\Big)|u-v|^\gamma,$$

which in view of $(n+1)\lambda - n = -n(1-\lambda) + \lambda \leqq -m(1-\lambda) + \lambda < 0$ is $< b|u-v|^\gamma$. This proves ii.1).

ii.2) After these preparations we prove assertion ii). We fix $u, v \in \mathbb{D}$ with $u < v =: a$, and put $M := m + [a]$ as in ii.1). In view of ii.1) we can assume that $a > 2^{-M\lambda}$. Then there is a unique $r \in \mathbb{N}$ with $2^{r-1} < a2^{M\lambda} \leqq 2^r$. The points $u(l) := u + l2^{-r}(v-u)$ $\forall 0 \leqq l \leqq 2^r$ are $\in \mathbb{D}$ with $u = u(0) < u(1) < \cdots < u(2^r) = v$ and fulfil $0 < u(l) - u(l-1) = 2^{-r}(v-u) \leqq 2^{-r}a \leqq 2^{-M\lambda}$ for $1 \leqq l \leqq 2^r$. Thus ii.1) asserts that

$$|x_{u(l)} - x_{u(l-1)}| \leqq b|u(l) - u(l-1)|^\gamma = b2^{-r\gamma}|u-v|^\gamma \quad \text{for } 1 \leqq l \leqq 2^r,$$

and hence $|x_u - x_v| \leqq b2^{r(1-\gamma)}|u-v|^\gamma$. Now

$$2^r < 2a2^{M\lambda} \leqq 2a2^{m\lambda}2^{a\lambda} \leqq \frac{2}{1-\lambda}2^{a(1-\lambda)}2^{m\lambda}2^{a\lambda} = \frac{2}{1-\lambda}2^{m\lambda}2^a,$$

because $z \leqq 2^z$ for $z \geqq 0$. It follows that $|x_u - x_v| \leqq B2^{m\lambda(1-\gamma)}2^{a(1-\gamma)}\ |u-v|^\gamma$, which in view of $a = u \vee v$ is the assertion ii). $\square$

Proof of 6.1. As before let $0 < \gamma \leqq 1$ with $\gamma < \beta/\alpha$. For $M > 0$ and $0 \in U \subset T = [0,\infty[$ we form the function sets

$$E(\gamma, M, U) := \{x \in X : |x_0| \leqq M \text{ and } |x_u - x_v| \leqq M2^{(u\vee v)(1-\gamma)}|u-v|^\gamma\ \forall u,v \in U\},$$

so that $E(\gamma, M) = E(\gamma, M, T)$. We collect their relevant properties.

1) $E(\gamma, M, U)$ is increasing in M and decreasing in U. Moreover

$$E(\gamma, M, U) = \bigcap_{p \in I, 0 \in p \subset U} E(\gamma, M, p).$$

2) $E(\gamma, M, U) \in \mathfrak{S}_\tau$. In fact, in view of 1) it suffices to prove $E(\gamma, M, p) \in \mathfrak{S}_\tau$ for $0 \in p \in I$. We have

$$E(\gamma, M, p) = \{z = (z_t)_{t \in p} \in \mathbb{R}^p : |z_0| \leqq M \text{ and}$$
$$|z_u - z_v| \leqq M 2^{(u \vee v)(1-\gamma)} |u - v|^\gamma \ \forall u, v \in p\} \times \mathbb{R}^{T \setminus p}.$$

The first factor is a closed subset of $\mathbb{R}^p$, and bounded since $|z_u - z_0| \leqq M 2^{u(1-\gamma)} u^\gamma$ $\forall u \in p$, and hence compact, that is in $\mathrm{Comp}(\mathbb{R}^p) = (\mathfrak{K}_p)_\tau$. Thus the product set $E(\gamma, M, p)$ is in $\mathfrak{S}_\tau$.

3) Assume that U is dense in T. Then

$$\Phi(E(\gamma, M, U \cup p)) = \Phi(E(\gamma, M, U)) \quad \text{for } p \in I.$$

In fact, it suffices to prove $\Phi(E(\gamma, M, U \cup \{t\})) = \Phi(E(\gamma, M, U))$ for $t \in T \setminus U$. To this end we use

$$\int |H_s - H_t|^\alpha d\Phi \leqq c|s - t|^{1+\beta} \quad \text{for all } s \in T.$$

We fix a sequence $(s(l))_{l \geqq 1}$ in U with $\sum_{l=1}^{\infty} |s(l) - t|^{1+\beta} < \infty$. Then there exists a subset $R \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(R) = 1$ such that $H_{s(l)} \to H_t$ pointwise on R, that is $x_{s(l)} \to x_t$ for all $x \in R$. Since for $x \in E(\gamma, M, U)$ we have

$$|x_s - x_{s(l)}| \leqq M 2^{(s \vee s(l))(1-\gamma)} |s - s(l)|^\gamma \quad \text{for } s \in U,$$

it follows for $x \in E(\gamma, M, U) \cap R$ that

$$|x_s - x_t| \leqq M 2^{(s \vee t)(1-\gamma)} |s - t|^\gamma \quad \text{for } s \in U.$$

Therefore $E(\gamma, M, U) \cap R \subset E(\gamma, M, U \cup \{t\})$, and hence the assertion.

4) Assume that U is dense in T. Then for each $x \in E(\gamma, M, U)$ there exists $y \in E(\gamma, M, T)$ such that $x_t = y_t$ for all $t \in U$. In fact, it is obvious that

$$y = (y_t)_{t \in T} : y_t = \lim_{s \in U, s \to t} x_s \text{ for } t \in T$$

exists and is as required.

5) Assume that U is dense in T. Then

$$E(\gamma, M, U \cup p) \subset H_p^{-1}\big(H_p(E(\gamma, M, T))\big) \quad \text{for all } p \in I.$$

This is an obvious consequence of 4).

We come to the decisive point. i) From theorem 5.3 and 2) we obtain

$$\Phi(E(\gamma, M, T)) = \inf_{p \in I} \Phi_p\big(H_p(E(\gamma, M, T))\big).$$

From 5)3) we see that

$$\begin{aligned}\Phi_p(H_p(E(\gamma,M,T)) = &\ \varphi_\tau\big(H_p^{-1}\big(H_p(E(\gamma,M,T))\big)\big)\\ \geqq &\ \Phi\big(E(\gamma,M,\mathbb{D}\cup p)\big) = \Phi(E(\gamma,M,\mathbb{D})),\end{aligned}$$

so that $\Phi(E(\gamma,M,T)) \geqq \Phi(E(\gamma,M,\mathbb{D}))$. From 1) it follows that $\Phi(E(\gamma,M,T)) = \Phi(E(\gamma,M,\mathbb{D}))$.

ii) We see from 6.2.ii) that each $x \in A$ is contained in $E(\gamma,M,\mathbb{D})$ for some $M > 0$, that is

$$A \subset \bigcup_{M>0} E(\gamma,M,\mathbb{D}) = \bigcup_{n=1}^{\infty} E(\gamma,n,\mathbb{D}) \in (\mathfrak{S}_\tau)^\sigma \subset \mathfrak{C}(\varphi_\tau).$$

Thus 6.2.1) implies that $\Phi(\bigcup_{M>0} E(\gamma,M,\mathbb{D})) = \lim_{M\uparrow\infty} \Phi(E(\gamma,M,\mathbb{D})) = 1$. From i)ii) we obtain the assertion of theorem 6.1. □

Consequence 6.3. *Fix as above $0 < \gamma \leqq 1$ with $\gamma < \beta/\alpha$, and define $\mathfrak{U} := \{U \in \mathfrak{S}_\tau : U \subset E(\gamma,M)$ for some $M > 0\}$. Then φ_τ is inner regular $\mathfrak{U}$.*

Proof. Fix $A \subset X$ and $c < \varphi_\tau(A)$, and then $S \in \mathfrak{S}_\tau$ with $S \subset A$ and $c < \varphi_\tau(S)$. From 6.1 we obtain $\varphi_\tau(E(\gamma,M)) > 1 - (\varphi_\tau(S) - c)$ for some $M > 0$. Then $U := S \cap E(\gamma,M) \in \mathfrak{S}_\tau$ fulfils

$$\begin{aligned}1 + \varphi_\tau(U) \geqq &\ \varphi_\tau(S \cup (E(\gamma,M)) + \varphi_\tau(S \cap (E(\gamma,M))\\ = &\ \varphi_\tau(S) + \varphi_\tau(E(\gamma,M)) > 1 + c,\end{aligned}$$

and hence is as required. □

We add one more consequence with respect to topologies. One has on $X = \mathbb{R}^T$ the product topology $\mathfrak{P}$, and on $\mathrm{C}(T,\mathbb{R})$ its restriction $\mathfrak{P}|\mathrm{C}(T,\mathbb{R})$ and the topology $\mathfrak{Q}$ of uniform convergence on the compact subsets of $T = [0,\infty[$, which is Polish. For these topologies we obtain what follows.

Proposition 6.4. 1) *Φ is maximal Radon with respect to $\mathfrak{P}$.* 2) *The restriction $\Phi|\mathrm{C}(T,\mathbb{R})$ is maximal Radon with respect to $\mathfrak{P}|\mathrm{C}(T,\mathbb{R})$ and to $\mathfrak{Q}$.*

Proof. We write $\mathrm{C}(T,\mathbb{R}) =: E$ for short. i) For $\mathfrak{U}$ as defined in 6.3 we claim that

i.1) $\mathfrak{U} \subset \mathrm{Comp}(\mathfrak{P}) \subset \mathfrak{S}_\tau \subset \mathrm{Cl}(\mathfrak{P})$,

i.2) $\mathfrak{U} \subset \mathrm{Comp}(\mathfrak{Q}) \subset \mathrm{Comp}(\mathfrak{P}|E) = \{P \in \mathrm{Comp}(\mathfrak{P}) : P \subset E\}$,

where in i.2) $\mathfrak{U}$ is viewed as a set system in E. In fact, in i.1) the third $\subset$ is obvious, and the second $\subset$ follows from [13] 2.4.2). In i.2) the $=$ is obvious, and the second $\subset$ holds true since $\mathfrak{Q}$ is finer than $\mathfrak{P}|E$. As to the first $\subset$ in i.2), the classical Ascoli theorem asserts that $E(\gamma,M) \in \mathrm{Comp}(\mathfrak{Q})$ for $M > 0$. In view of $U \subset E(\gamma,M)$ for some $M > 0$ it remains to show that U is closed in $\mathfrak{Q}$. But the third $\subset$ in i.1) asserts that U is closed in $\mathfrak{P}$, that is in $\mathfrak{P}|E$, and hence in $\mathfrak{Q}$. This also proves the first $\subset$ in i.1).

ii) We see from 1.4 that $\phi := \varphi_\tau|\mathfrak{S}_\tau$ is an inner τ premeasure $\phi : \mathfrak{S}_\tau \to [0,\infty[$ which fulfils $\phi_\tau = \varphi_\tau$ and hence $\Phi = \phi_\tau|\mathfrak{C}(\phi_\tau)$.

iii) To prove 1) we combine 1.6 applied to $\phi : \mathfrak{S}_\tau \to [0,\infty[$ and to $\mathrm{Comp}(\mathfrak{P})$ with i.1). It follows via 6.3 that $\vartheta := \phi|\mathrm{Comp}(\mathfrak{P})$ is an inner τ premeasure $\vartheta : \mathrm{Comp}(\mathfrak{P}) \to [0,\infty[$ which fulfils $\vartheta_\tau = \phi_\tau$ and hence $\Phi = \vartheta_\tau|\mathfrak{C}(\vartheta_\tau)$.

iv) To prove 2) we first invoke 1.10 for $\phi : \mathfrak{S}_\tau \to [0,\infty[$ and E. Let $\mathfrak{T} = \mathfrak{S}_\tau \sqcap E = \mathfrak{T}_\tau \subset \mathfrak{C}(\phi_\tau)$ with $\mathfrak{T}_\circ$ as before, and note that $\mathrm{Comp}(\mathfrak{P}|E) \subset \mathfrak{T}_\circ$ from i). Then 1.10.1) asserts that $\psi = \phi_\tau|\mathfrak{T}$ is an inner τ premeasure $\psi : \mathfrak{T} \to [0,\infty[$ with $\psi_\tau = \phi_\tau$, and 1.10.2) asserts that $\psi_\circ$ as before is an inner τ premeasure $\psi_\circ : \mathfrak{T}_\circ \to [0,\infty[$ with $(\psi_\circ)_\tau = \psi_\tau|\mathfrak{P}(E) = \phi_\tau|\mathfrak{P}(E)$ and $\Phi|E = (\psi_\circ)_\tau|\mathfrak{C}((\psi_\circ)_\tau)$. After this we combine 1.6 applied to $\psi_\circ : \mathfrak{T}_\circ \to [0,\infty[$ and to both $\mathrm{Comp}(\mathfrak{P}|E)$ and $\mathrm{Comp}(\mathfrak{Q})$ with i.2) plus the above $\mathrm{Comp}(\mathfrak{P}|E) \subset \mathfrak{T}_\circ$. It follows via 6.3 that both $\vartheta := \psi_\circ|\mathrm{Comp}(\mathfrak{P}|E)$ and $\vartheta := \psi_\circ|\mathrm{Comp}(\mathfrak{Q})$ are inner τ premeasures which fulfil $\vartheta_\tau = (\psi_\circ)_\tau$ and hence $\Phi|E = \vartheta_\tau|\mathfrak{C}(\vartheta_\tau)$. $\square$.

The remainder of the section wants to establish the explicit connection with the usual Wiener measure situation. What follows are standard procedures, but to be transferred into the world of inner premeasures. We want to note that in the sequel we shall have $\varphi_0 = \varphi_{\{0\}} =$ the Dirac premeasure $\delta_0|\mathfrak{K}$ for $0 \in \mathbb{R}$. Therefore $N := H_0^{-1}(\{0\}) = \{x \in X : x_0 = 0\} \in \mathfrak{S}$ has $\Phi(N) = \varphi_\tau(H_0^{-1}(\{0\})) = (\varphi_0)_\tau(\{0\}) = 1$.

We start to recall the notion of *convolution*, for the sake of fun for Radon premeasures $\varphi, \psi : \mathfrak{R} = \mathrm{Comp}(X) \to [0,\infty[$ on a Hausdorff topological space X which is a group under a continuous operation $G : (u,v) \mapsto uv$ (this is less than a topological group [7] (4.20)). From MI 21.9 = [15] 6.4 we obtain the product inner τ premeasure $\varphi \times \psi : (\mathfrak{R} \times \mathfrak{R})^\star \to [0,\infty[$, with $((\mathfrak{R} \times \mathfrak{R})^\star)_\tau = \mathrm{Comp}(X \times X)$ from MI 21.3.2). Now the map $G : X \times X \to X$ with the lattices $(\mathfrak{R} \times \mathfrak{R})^\star$ and $\mathfrak{R}$ fulfils conditions $(\Rightarrow)(\Leftarrow)$ in 3.10 for $\bullet = \tau$. We assume that $\varphi_\tau(X), \psi_\tau(X) < \infty$ and hence $(\varphi \times \psi)_\tau(X \times X) = \varphi_\tau(X)\psi_\tau(X) < \infty$. Then 3.10 furnishes the image Radon premeasure $\chi = \vec{G}(\varphi \times \psi) : \mathfrak{R} \to [0,\infty[$ on X. It fulfils $\chi_\tau = (\varphi \times \psi)_\tau(G^{-1}(\cdot))$. We call $\chi = \varphi \star \psi$ the *convolution* of φ and ψ.

After this we fix a family $(\gamma_t)_{t\in T}$ of Radon premeasures $\gamma_t : \mathfrak{K} = \mathrm{Comp}(\mathbb{R}) \to [0,\infty[$ on $\mathbb{R}$ with $\Gamma_t = (\gamma_t)_\tau|\mathfrak{C}((\gamma_t)_\tau)$ such that $\Gamma_t(\mathbb{R}) = 1$ and $\gamma_0 = \delta_0|\mathfrak{K}$, which under convolution fulfils $\gamma_s \star \gamma_t = \gamma_{s+t}$ for all $s,t \in T$. We form for $p \in I$, written $p = \{t(1), \cdots, t(n)\}$ with $0 =: t(0) \leqq t(1) < \cdots < t(n)$, after [13] 1.5 the p-fold product inner τ premeasure

$$\gamma_p := \prod_{l=1}^{n} \gamma_{t(l)-t(l-1)} : \mathfrak{K}_p = (\mathfrak{K}^p)^\star \to [0,\infty[\ \text{ with } \Gamma_p = (\gamma_p)_\tau|\mathfrak{C}((\gamma_p)_\tau),$$

so that $\Gamma_p(\mathbb{R}^p) = 1$. Then define $G_p : \mathbb{R}^p \to \mathbb{R}^p$ to be the partial-sum map

$$G_p : u = (u_1, \cdots, u_n) \mapsto G_p(u) = (v_1, \cdots, v_n) \ \text{ with } \ v_l = \sum_{k=1}^{l} u_k \ \text{ for } 1 \leqq l \leqq n.$$

The map G_p is homeomorphic and hence fulfils, with the lattice $\mathfrak{K}_p$ on both sides, conditions $(\Rightarrow)(\Leftarrow)$ in 3.10. Thus 3.10 furnishes the image inner τ premeasure $\varphi_p = \vec{G}_p\gamma_p : \mathfrak{K}_p \to [0,\infty[$ with $\Phi_p = (\varphi_p)_\tau|\mathfrak{C}((\varphi_p)_\tau)$. It fulfils $(\varphi_p)_\tau = (\gamma_p)_\tau(G_p^{-1}(\cdot))$ and hence $\Phi_p(\mathbb{R}^p) = 1$. We claim that the family of these φ_p for $p \in I$ is appropriate for the application of theorem 6.1.

Proposition 6.5. *The family* $(\varphi_p)_{p\in I}$ *fulfils condition* $(\circ)$ *of theorem* 5.3. *Moreover its Wiener* τ *premeasure* $\varphi : \mathfrak{S} \to [0,\infty[$ *with* $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ *fulfils for* $s \geqq 0$ *and* $t > 0$ *the relation*

$$\varphi_\tau\big([H_{s+t} - H_s \in B]\big) = (\gamma_t)_\tau(B) \quad \textit{for all } B \subset \mathbb{R}.$$

Hence in particular $\int |H_{s+t} - H_s|^\alpha d\Phi = \int |v|^\alpha d\Gamma_t(v)$ *for* $\alpha > 0$.

Proof. 1) We recall for $p \leqq q$ in I the canonical projection $H_{pq} : \mathbb{R}^q \to \mathbb{R}^p$. It is continuous and hence fulfils, with the lattices $\mathfrak{K}_q$ and $\mathfrak{K}_p$, conditions $(\Rightarrow)(\Leftarrow)$ in 3.10.The desired $(\circ)$ is equivalent to $\varphi_p = (\varphi_q)_\tau(H_{pq}^{-1}(\cdot))|\mathfrak{K}_p$ or $\varphi_p = \overrightarrow{H}_{pq}\varphi_q$, which once more is seen via MI 2.5.1). Besides H_{pq} we consider the map $G_{pq} = G_p^{-1} \circ H_{pq} \circ G_q : \mathbb{R}^q \to \mathbb{R}^p$, which likewise is continuous and hence fulfils, with the lattices $\mathfrak{K}_q$ and $\mathfrak{K}_p$, conditions $(\Rightarrow)(\Leftarrow)$ in 3.10. We note that $\varphi_p = \overrightarrow{H}_{pq}\varphi_q$ is equivalent to $\gamma_p = \overrightarrow{G}_{pq}\gamma_q$: In fact, in view of $G_p \circ G_{pq} = H_{pq} \circ G_q$ we have

$$\begin{aligned}
(\gamma_p)_\tau(A) &= (\gamma_q)_\tau(G_{pq}^{-1}(A))\ \forall A \subset \mathbb{R}^p\\
\Leftrightarrow (\gamma_p)_\tau(G_p^{-1}(B)) &= (\gamma_q)_\tau(G_{pq}^{-1}(G_p^{-1}(B))) = (\gamma_q)_\tau(G_q^{-1}(H_{pq}^{-1}(B)))\ \forall B \subset \mathbb{R}^p\\
\Leftrightarrow (\varphi_p)_\tau(B) &= (\varphi_q)_\tau(H_{pq}^{-1}(B))\ \forall B \subset \mathbb{R}^p.
\end{aligned}$$

It is clear from the form of $(\circ)$ that it suffices to prove the equivalent conditions in the special case $q = p \cup \{s\}$ with $s \in T \setminus p$. In 2) below we shall do this for $\gamma_p = \overrightarrow{G}_{pq}\gamma_q$.

2) Thus let $p = \{t(1), \cdots, t(n)\}$ with $0 = t(0) \leqq t(1) < \cdots < t(n)$ and $q = p \cup \{s\}$ with $s \in T \setminus p$ as before. There are the three cases

(L) $t(0) \leqq s < t(1)$,

(M) $t(l-1) < s < t(l)$ for some $2 \leqq l \leqq n$,

(R) $t(n) < s$.

We first want to obtain an explicit formula for G_{pq}. To this end we write $v \in \mathbb{R}^q$ and an associate $u \in \mathbb{R}^p$ in the three cases (L)(M)(R) in the forms

$$\begin{aligned}
v &= (a, b, v_2, \cdots, v_n) & u &= (a+b, v_2, \cdots, v_n),\\
v &= (v_1, \cdots, v_{l-1}, a, b, v_{l+1}, \cdots, v_n) & u &= (v_1, \cdots, v_{l-1}, a+b, v_{l+1}, \cdots, v_n),\\
v &= (v_1, \cdots, v_n, x) & u &= (v_1, \cdots, v_n).
\end{aligned}$$

Then one notes that $H_{pq}(G_q(v)) = G_p(u)$ and hence $G_{pq}(v) = u$. It follows for $A = A_1 \times \cdots \times A_n \subset \mathbb{R}^p$ that

$$\begin{aligned}
G_{pq}^{-1}(A) &= G^{-1}(A_1) \times A_2 \times \cdots \times A_n,\\
G_{pq}^{-1}(A) &= A_1 \times \cdots \times A_{l-1} \times G^{-1}(A_l) \times A_{l+1} \times \cdots \times A_n,\\
G_{pq}^{-1}(A) &= A_1 \times \cdots \times A_n \times \mathbb{R},
\end{aligned}$$

where $G:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ denotes the addition $G(a,b)=a+b$. Next we note that in the three cases (L)(M)(R)

$$\gamma_q=\gamma_{s-t(0)}\times\gamma_{t(1)-s}\times\prod_{k=2}^{n}\gamma_{t(k)-t(k-1)},$$
$$\gamma_q=\prod_{k=1}^{l-1}\gamma_{t(k)-t(k-1)}\times\gamma_{s-t(l-1)}\times\gamma_{t(l)-s}\times\prod_{k=l+1}^{n}\gamma_{t(k)-t(k-1)},$$
$$\gamma_q=\prod_{k=1}^{n}\gamma_{t(k)-t(k-1)}\times\gamma_{s-t(n)}.$$

We know from [13] 1.2 that this product formation is associative. Thus [13] 1.3 implies for $A=A_1\times\cdots\times A_n\subset\mathbb{R}^p$ that

$$(\gamma_q)_\tau(G_{pq}^{-1}(A))=(\gamma_{s-t(0)}\times\gamma_{t(1)-s})_\tau(G^{-1}(A_1))\times\prod_{k=2}^{n}(\gamma_{t(k)-t(k-1)})_\tau(A_k),$$
$$(\gamma_q)_\tau(G_{pq}^{-1}(A))=\prod_{k=1}^{l-1}(\gamma_{t(k)-t(k-1)})_\tau(A_k)\times(\gamma_{s-t(l-1)}\times\gamma_{t(l)-s})_\tau(G^{-1}(A_l))\times$$
$$\times\prod_{k=l+1}^{n}(\gamma_{t(k)-t(k-1)})_\tau(A_k),$$
$$(\gamma_q)_\tau(G_{pq}^{-1}(A))=\prod_{k=1}^{n}(\gamma_{t(k)-t(k-1)})_\tau(A_k)\times(\gamma_{s-t(n)})_\tau(\mathbb{R}),$$

which in view of

$$(\gamma_{s-t(l-1)}\times\gamma_{t(l)-s})_\tau(G^{-1}(A_l))=(\gamma_{s-t(l-1)}\star\gamma_{t(l)-s})_\tau(A_l)=(\gamma_{t(l)-t(l-1)})_\tau(A_l)$$

in (L)(M) and $(\gamma_{s-t(n)})_\tau(\mathbb{R})=1$ in (R) boils down to

$$(\gamma_q)_\tau(G_{pq}^{-1}(A))=\prod_{k=1}^{n}(\gamma_{t(k)-t(k-1)})_\tau(A_k)=(\gamma_p)_\tau(A).$$

The result holds true in particular for $A\in\mathfrak{K}^p$, and hence for $A\in(\mathfrak{K}^p)^\star=\mathfrak{K}_p$, once more in view of MI 2.5.1). Thus we have $\gamma_p=\overrightarrow{G}_{pq}\gamma_q$ as claimed.

3) We turn to the final assertions in 6.5. Let $s\geqq 0$ and $t>0$. For $B\subset\mathbb{R}$ we have

$$\begin{aligned}[H_{s+t}-H_s\in B]&=\{x\in X:x_{s+t}-x_s\in B\}\\&=H_{\{s,s+t\}}^{-1}\big(\{(u,v)\in\mathbb{R}^{\{s,s+t\}}:v-u\in B\}\big)=H_{\{s,s+t\}}^{-1}\big(G_{\{s,s+t\}}(\mathbb{R}\times B)\big),\end{aligned}$$

and hence

$$\begin{aligned}\varphi_\tau\big([H_{s+t}-H_s\in B]\big)&=(\varphi_{\{s,s+t\}})_\tau\big(G_{\{s,s+t\}}(\mathbb{R}\times B)\big)\\&=(\gamma_{\{s,s+t\}})_\tau(\mathbb{R}\times B)=(\gamma_s)_\tau(\mathbb{R})(\gamma_t)_\tau(B)=(\gamma_t)_\tau(B).\end{aligned}$$

For $\alpha>0$ it follows via the Choquet integral

$$\begin{aligned}\int|H_{s+t}-H_s|^\alpha d\Phi&=\int_{0\leftarrow}^{\to\infty}\Phi\big([|H_{s+t}-H_s|^\alpha\geqq z]\big)dz\\&=\int_{0\leftarrow}^{\to\infty}\Gamma_t\big(\{v\in\mathbb{R}:|v|^\alpha\geqq z\}\big)dz=\int|v|^\alpha d\Gamma_t(v).\end{aligned}$$

This completes the proof of 6.5. □

At last we specialize the family $(\gamma_t)_{t\in T}$ to the Brownian convolution semigroup of the Gaussian premeasures

$$\gamma_t : \gamma_t(K) = \frac{1}{\sqrt{2\pi t}} \int\limits_K e^{-x^2/2t} dx \quad \text{for } K \in \mathfrak{K} = \mathrm{Comp}(\mathbb{R}) \text{ when } t > 0,$$

and $\gamma_0 = \delta_0 | \mathfrak{K}$. In this case one computes for $\alpha > 0$ and $t > 0$ that

$$\int |v|^\alpha d\Gamma_t(v) = t^{\alpha/2} M(\alpha) \quad \text{with } M(\alpha) = \frac{2^{1+\alpha/2}}{\sqrt{\pi}} \int_{0\leftarrow}^{\rightarrow\infty} x^\alpha e^{-x^2} dx.$$

It follows that the assumption in 6.1 is fulfilled for $\alpha > 2$ with $1 + \beta = \alpha/2$. Thus we obtain the assertion of 6.1 for the exponents $0 < \gamma < 1/2$. The measure $\Phi = \varphi_T | \mathfrak{C}(\varphi_T)$ which satisfies all this is what we call the *true Wiener measure.*

We conclude with a few further remarks on the traditional Wiener measure $\theta_{\mathrm{C}(T,\mathbb{R})} := \theta^\star | (\mathfrak{A} \sqcap \mathrm{C}(T, \mathbb{R}))$. 1) Instead of $\mathrm{C}(T, \mathbb{R})$ one often considers the smaller $\mathrm{C}_\circ(T, \mathbb{R}) = \{x \in \mathrm{C}(T, \mathbb{R}) : x_0 = 0\}$. In the present context we have $\mathrm{C}_\circ(T, \mathbb{R}) = \mathrm{C}(T, \mathbb{R}) \cap N$, where $N \in \mathfrak{S}$ with $\Phi(N) = 1$ has been defined above, and one could proceed alike.

2) The traditional Wiener measure has the domain $\mathfrak{A} \sqcap \mathrm{C}(T, \mathbb{R})$. In connection with the topologies $\mathfrak{P}|\mathrm{C}(T, \mathbb{R})$ and $\mathfrak{Q}$ on $\mathrm{C}(T, \mathbb{R})$ and with 6.4 we recall that this domain is $= \mathrm{Bor}(\mathrm{C}(T, \mathbb{R}))$ for both these topologies [1] 38.6. In connection with the Radon properties 6.4 we also refer to Fremlin [5] 454-455.

3) At last Kisyński [9] section 3 follows an alternative but not unrelated route, in that he uses his Prokhorov type theorem mentioned in 4.15 for the direct construction of the traditional Wiener measure on $\mathrm{C}([0, T], \mathbb{R})$ (to appear in corrected and augmented form). Kisyński refers to Itô-McKean [8] as a predecessor.

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