Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 144

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Saarbrücken 2005

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 144
submitted: July 14, 2005
(updated: June 1, 2006)

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To appear in POSITIVITY

THE LEBESGUE DECOMPOSITION THEOREM FOR ARBITRARY CONTENTS

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Horst Tietz in alter Freundschaft gewidmet

Abstract. The decomposition theorem named after Lebesgue asserts that certain set functions have canonical representations as sums of particular set functions called the *absolutely continuous* and the *singular* ones with respect to some fixed set function. The traditional versions are for the *bounded* measures with respect to some fixed measure on a σ algebra, in final form due to Hahn 1921, and for the *bounded* contents with respect to some fixed content on an algebra, due to Bochner-Phillips 1941 and Darst 1962. Then came the version for *arbitrary* measures, due to R.A.Johnson 1967 and N.Y.Luther 1968. The unpleasant fact with these versions is that each one requires its particular notions of *absolutely continuous* and *singular* constituents. It seems mysterious how a common roof for all of them could look, and therefore how a *universal version for arbitrary contents* could be achieved - and all that while several abstract extensions of particular versions appeared in the subsequent decades, for example due to de Lucia-Morales 2003. After these decades now the present article claims to arrive at the final aim in the original context of arbitrary contents. The article will be based on the author's new *difference formation* for arbitrary contents 1999. This difference formation even furnishes simple *explicit formulas* for the two constituents.

1. Introduction and Previous Results

Let X be a nonvoid set. The present article considers both

the class of contents $\alpha,\cdots:\mathfrak{A}\to[0,\infty]$ on an algebra $\mathfrak{A}$ in X, and
the class of measures $\alpha,\cdots:\mathfrak{A}\to[0,\infty]$ on a σ algebra $\mathfrak{A}$ in X,

with the latter case marked with *meas*. We shall meet fundamental discrepancies between *contents* and *measures*, but also between *arbitrary* members and *finite* members in each of the two classes.

We start to recall for $\alpha,\beta:\mathfrak{A}\to[0,\infty]$ the formations $\alpha\wedge\beta,\alpha\vee\beta:\mathfrak{A}\to[0,\infty]$, defined to be

$$\alpha\wedge\beta(A)=\inf\{\alpha(A\setminus T)+\beta(T):T\in\mathfrak{A}\text{ with }T\subset A\},$$
$$\alpha\vee\beta(A)=\sup\{\alpha(A\setminus T)+\beta(T):T\in\mathfrak{A}\text{ with }T\subset A\},$$

which are the lattice minimum and maximum of α and β in the respective class of set functions $\mathfrak{A}\to[0,\infty]$, equipped with the pointwise order $\leqq$. These facts and other basic properties of $\alpha\wedge\beta$ and $\alpha\vee\beta$ are listed in [10] section 1. The formations are in common use for *finite* contents and

2000 *Mathematics Subject Classification.* 28A12.
Key words and phrases. Lebesgue decomposition theorem for contents and measures, absolutely continuous and singular in different versions, difference formation for arbitrary contents.

measures, also for signed ones, in the context of vector lattices = Riesz spaces, but the author is surprised to note that apart from that they almost never appear in the textbooks on measure and integration. We recall some of its properties.

1.1 PROPERTIES. 1) *The operations* $\wedge$ *and* $\vee$ *are commutative and associative.*

2) $(\alpha+\beta)\wedge\tau \leqq \alpha+\beta\wedge\tau$.

3) *In the* meas *case: For* $A\in\mathfrak{A}$ *there exist* $P,Q\subset A$ *in* $\mathfrak{A}$ *such that* $\alpha\wedge\beta(A)=\alpha(A\setminus P)+\beta(P)$ *and* $\alpha\vee\beta(A)=\alpha(A\setminus Q)+\beta(Q)$.

We come to the traditional concepts of absolute continuity and singularity for set functions $\alpha,\vartheta:\mathfrak{A}\to[0,\infty]$. For ϑ *absolutely continuous* with respect to α these concepts are

$$\vartheta\ll\alpha:\alpha(T)=0\Rightarrow\vartheta(T)=0 \text{ for all } T\in\mathfrak{A}; \text{ and}$$

$$\vartheta\,\mathrm{AC}\,\alpha:\forall\varepsilon>0\ \exists\delta>0 \text{ such that } \alpha(T)\leqq\delta\Rightarrow\vartheta(T)\leqq\varepsilon \text{ for all } T\in\mathfrak{A}.$$

One has the implications

$$\vartheta\ll\alpha\longleftarrow\vartheta\,\mathrm{AC}\,\alpha \quad\text{and}\quad \vartheta\ll\alpha\underset{\text{meas }\vartheta<\infty}{\Longrightarrow}\vartheta\,\mathrm{AC}\,\alpha,$$

for example from [9] 24.1; here and in the sequel we use simple arrows to denote obvious implications. For ϑ *singular* with respect to α the concepts are

$$\vartheta\wedge\alpha=0\ ; \text{ and}$$

$$\vartheta\perp\alpha:\ \exists\,T\in\mathfrak{A} \text{ such that } \alpha(T)=0 \text{ and } \vartheta(T')=0.$$

From 1.1.3) we see that

$$\vartheta\wedge\alpha=0\longleftarrow\vartheta\perp\alpha \quad\text{and}\quad \vartheta\wedge\alpha=0\underset{\text{meas}}{\Longrightarrow}\vartheta\perp\alpha.$$

It is obvious that the combination $\vartheta\ll\alpha$ & $\vartheta\perp\alpha$ implies that $\vartheta=0$. Thus in the *meas* case the combination $\vartheta\ll\alpha$ & $\vartheta\wedge\alpha=0$ implies that $\vartheta=0$. It is important to note that this is not true for *contents*. We present a typical example.

1.2 EXAMPLE. 0) We recall the Hahn-Banach type result [11] 1.3: *Let* $\mathfrak{S}$ *be a lattice in* X *with* $\varnothing\in\mathfrak{S}$. *Then each isotone and modular set function* $\varphi:\mathfrak{S}\to[0,\infty]$ *with* $\varphi(\varnothing)=0$ *can be extended to a content* $\Phi:\mathfrak{P}(X)\to[0,\infty]$ *with* $\Phi(X)=\sup\varphi$. 1) Let $X=[0,1[$ and $\mathfrak{A}=\mathrm{Bor}(X)$, and $\alpha=\mathrm{Leb}|\mathfrak{A}$ be the Borel-Lebesgue measure. From 0) we obtain for each $0<c\leqq\infty$ a content $\vartheta:\mathfrak{A}\to[0,\infty]$ which fulfils

$$\vartheta(S)=0 \text{ for all } S\in\mathfrak{A} \text{ such that } \alpha\big(S\cap]s,1[\big)=0 \text{ for some } 0<s<1,$$

and $\vartheta(X)=c$. 2) It is obvious that $\vartheta\ll\alpha$. On the other side we have

$$\text{for } 0<s<1:\vartheta\wedge\alpha(X)\leqq\vartheta([0,s])+\alpha(]s,1[)=0+(1-s),$$

which for $s\uparrow1$ furnishes $\vartheta\wedge\alpha(X)=0$ and hence $\vartheta\wedge\alpha=0$.

We turn to the Lebesgue decomposition theorem. Its assertion for $\alpha,\vartheta:\mathfrak{A}\to[0,\infty]$ is that there exist decompositions $\vartheta=\varphi+\psi$ into $\varphi:\mathfrak{A}\to[0,\infty]$ *absolutely continuous* α and $\psi:\mathfrak{A}\to[0,\infty]$ *singular* α in some sense, and that this representation is *unique* either without or under certain additional conditions on φ and ψ. We quote the two traditional versions of the theorem.

1.3 TRADITIONAL MEASURE THEOREM. *Assume that α and ϑ are measures with $\vartheta < \infty$. Then $\vartheta = \varphi + \psi$ for a unique pair of measures φ and ψ with $\varphi \ll \alpha \Leftrightarrow \varphi \operatorname{AC} \alpha$ and $\psi \wedge \alpha = 0 \Leftrightarrow \psi \perp \alpha$. Moreover $\varphi \wedge \psi = 0$.*

1.4 TRADITIONAL CONTENT THEOREM. *Assume that α and ϑ are contents with $\vartheta < \infty$. Then $\vartheta = \varphi + \psi$ for a unique pair of contents φ and ψ with $\varphi \operatorname{AC} \alpha$ and $\psi \wedge \alpha = 0$. Moreover $\varphi \wedge \psi = 0$.*

Thus in 1.3 there are four equivalent possible combinations of conditions for φ and ψ. However, in 1.4 the combination $\varphi \operatorname{AC} \alpha$ and $\psi \wedge \alpha = 0$ is the unique possible one: 1) Under the combination $\varphi \ll \alpha$ and $\psi \wedge \alpha = 0$ the existence of decompositions remains true, because $\varphi \operatorname{AC} \alpha$ implies $\varphi \ll \alpha$. But there need not be uniqueness. In fact, in example 1.2 there are nonzero ϑ such that both $\vartheta \ll \alpha$ and $\vartheta \wedge \alpha = 0$. This leads to the two decompositions $\vartheta = \vartheta + 0 = 0 + \vartheta$, which are both of the considered kind. 2) The existence of decompositions need not be true under the combination $\varphi \ll \alpha$ and $\psi \perp \alpha$, and hence not under the combination $\varphi \operatorname{AC} \alpha$ and $\psi \perp \alpha$ as well, as the next example will show.

1.5 EXAMPLE. 1) Let $X = \mathbb{N}$ and $\mathfrak{A}$ consist of its finite and cofinite subsets. Define $\alpha, \vartheta : \mathfrak{A} \to [0, \infty[$ to be $\alpha(A) = 0$ for A finite and $\alpha(A) = 1$ for A cofinite, and $\vartheta(A) = \sum_{n \in A} f(n)$ for some $f : X \to]0, \infty[$ with $\sum_{n \in X} f(n) < \infty$. 2) Assume that $\vartheta = \varphi + \psi$ with contents $\varphi \ll \alpha$ and $\psi \perp \alpha$. Then $\psi(E') = 0$ for some finite $E \subset X$. For all finite $A \subset E'$ this implies that $\varphi(A) = \psi(A) = 0$ and hence $\vartheta(A) = 0$, which contradicts the definition of ϑ.

The traditional content theorem 1.4 is reproduced in the author's book [9] 24.2. There it is noted that $\varphi = \lim_{t \uparrow \infty} \vartheta \wedge (t\alpha)$, a well-known formation from the context of vector lattices = Riesz spaces; see for example [2] chapter II.

The final one of the previous results is the version for *arbitrary* measures $\alpha, \vartheta : \mathfrak{A} \to [0, \infty]$ and for decompositions $\vartheta = \varphi + \psi$ with arbitrary measures $\varphi, \psi : \mathfrak{A} \to [0, \infty]$. The result is due to Johnson [8], with an important uniqueness supplement due to Luther [14]. In this situation one has to face the problem that none of the four possible combinations in 1.3 does work, even when one admits *contents* φ and ψ, as the next example will show. In particular the example disproves the decomposition theorem in the 1994 textbook of Doob [5] p.148.

1.6 EXAMPLE. 1) Let X be an uncountable set and $\mathfrak{A}$ consist of its countable and cocountable subsets. Define $\alpha, \vartheta : \mathfrak{A} \to [0, \infty]$ to be $\alpha(A) = 0$ for A countable and $\alpha(A) = 1$ for A cocountable, and $\vartheta = \#|\mathfrak{A}$ to be the cardinality. Thus α and ϑ are measures. 2) Assume that $\vartheta = \varphi + \psi$ with contents $\varphi \ll \alpha$ and $\psi \wedge \alpha = 0$. For countable A then $\varphi(A) = 0$ and hence $\vartheta(A) = \psi(A)$, which at once implies that $\vartheta = \psi$ on $\mathfrak{A}$. It follows that $\psi \wedge \alpha = \vartheta \wedge \alpha = \alpha$ since $\alpha \leqq \vartheta$, and hence $\psi \wedge \alpha \neq 0$, which is a contradiction.

The example shows that the decomposition $\vartheta = \varphi + \psi$ must be effected under some combination of conditions which (at least for *measures* φ and ψ) is weaker than the previous combination $\varphi \ll \alpha$ and $\psi \wedge \alpha = 0$. Since it is hard to see how the condition $\varphi \ll \alpha$ could be weakened, this ought to be done on the part of ψ. The answer due to Johnson [8] is the condition named *singular in the sense* J and defined to be

$$\vartheta\,\mathrm{singJ}\,\alpha:\ \forall A\in\mathfrak{A}\ \exists T\subset A \text{ in } \mathfrak{A} \text{ such that } \alpha(T)=0 \text{ and } \vartheta(T)=\vartheta(A).$$

We have the obvious implications

$$\vartheta\perp\alpha\longrightarrow\vartheta\,\mathrm{singJ}\,\alpha \quad\text{and}\quad \vartheta\perp\alpha\underset{\vartheta<\infty}{\longleftarrow}\vartheta\,\mathrm{singJ}\,\alpha.$$

With this condition the theorem reads as follows.

1.7 Full Measure Theorem. *Let α and ϑ be measures. Then $\vartheta=\varphi+\psi$ for certain pairs of measures φ and ψ with $\varphi\ll\alpha$ and $\psi\,\mathrm{singJ}\,\alpha$. In all these pairs the measure ψ is the same and fulfils $\psi\,\mathrm{singJ}\,\varphi$, and there is a unique pair in which the measure φ fulfils $\varphi\,\mathrm{singJ}\,\psi$.*

In view of the definition of singJ the relation $\psi\,\mathrm{singJ}\,\varphi$ is an obvious consequence of $\psi\,\mathrm{singJ}\,\alpha$ combined with $\varphi\ll\alpha$. In the theorem the union $\psi\,\mathrm{singJ}\,\varphi$ & $\varphi\,\mathrm{singJ}\,\psi$ appears to attain the place of the previous relation $\varphi\wedge\psi=0$. Moreover it is clear that in case $\vartheta<\infty$ the full measure theorem 1.7 specializes to the former traditional measure theorem 1.3.

After this we are confronted with the task to handle the full situation of *arbitrary contents* $\alpha,\vartheta:\mathfrak{A}\to[0,\infty]$. We shall see that in this situation the decomposition $\vartheta=\varphi+\psi$ will require entirely new conditions on the contents $\varphi,\psi:\mathfrak{A}\to[0,\infty]$. The new concepts for *absolutely continuous* and *singular* will be presented in sections 2 and 3.

For the moment we only want to note that the combination $\varphi\ll\alpha$ and $\psi\,\mathrm{singJ}\,\alpha$ of the full measure theorem 1.7 will not work in the future full content situation, because it does not even work in the previous traditional content situation 1.4. In fact, assume in example 1.5.1) that $\vartheta=\varphi+\psi$ with contents $\varphi\ll\alpha$ and $\psi\,\mathrm{singJ}\,\alpha$. Since $\psi\,\mathrm{singJ}\,\alpha$ in case $\psi<\infty$ implies $\psi\perp\alpha$ as we know, it follows from 1.5.2) that this is not possible.

2. The New Concept for Absolutely Continuous

The new concept for contents $\alpha,\vartheta:\mathfrak{A}\to[0,\infty]$ centers around the condition $\vartheta\wedge(t\alpha)\uparrow\vartheta$ for $t\uparrow\infty$. First of all we recall from [13] 3.2 the basic fact that

$$\vartheta\ll\alpha\longleftarrow\vartheta\wedge(t\alpha)\uparrow\vartheta\ \text{ for } t\uparrow\infty \quad\text{and}\quad \vartheta\ll\alpha\underset{\text{meas}}{\Longrightarrow}\vartheta\wedge(t\alpha)\uparrow\vartheta\ \text{ for } t\uparrow\infty.$$

We want to reformulate the condition in order that it looks close to $\vartheta\,\mathrm{AC}\,\alpha$. Thus we define

$$\vartheta\,\mathrm{ac}\,\alpha\ :\forall A\in\mathfrak{A} \text{ with } \lim_{t\uparrow\infty}\vartheta\wedge(t\alpha)(A)<\infty \text{ and } \forall\varepsilon>0\ \exists\delta>0$$
$$\text{such that } \alpha(T)\leqq\delta\ \Rightarrow\ \vartheta(T)\leqq\varepsilon \text{ for all } T\subset A \text{ in } \mathfrak{A}.$$

Then we have in fact the connection which follows.

2.1 Theorem. *Let α and ϑ be contents. Then $\vartheta\wedge(t\alpha)\uparrow\vartheta$ for $t\uparrow\infty\iff\vartheta\,\mathrm{ac}\,\alpha$.*

Proof of $\Rightarrow$. Assume that $\vartheta\,\mathrm{ac}\,\alpha$ is false for $A\in\mathfrak{A}$ with $\lim\limits_{t\uparrow\infty}\vartheta\wedge(t\alpha)(A)=\vartheta(A)<\infty$ and $\varepsilon>0$. Thus there exist $T_l\subset A$ in $\mathfrak{A}$ with $\alpha(T_l)\to 0$ and

$\vartheta(T_l) > \varepsilon$ for $l \in \mathbb{N}$, and we can of course assume that $\alpha(T_l) < \infty$. Now fix $t > 0$ with $\vartheta \wedge (t\alpha)(A) > \vartheta(A) - \varepsilon/2$. For $l \in \mathbb{N}$ then

$$t\alpha(T_l) + \vartheta(A \setminus T_l) > \vartheta(A) - \varepsilon/2 \quad \text{or} \quad \vartheta(T_l) < t\alpha(T_l) + \varepsilon/2.$$

It follows that $\vartheta(T_l) < \varepsilon$ for l sufficiently large, and thus a contradiction.

Proof of $\Leftarrow$. Fix $A \in \mathfrak{A}$ and put $C := \lim\limits_{t\uparrow\infty} \vartheta \wedge (t\alpha)(A) \leqq \vartheta(A)$. To be shown is $C \geqq \vartheta(A)$, so that we can assume that $C < \infty$. Fix $\varepsilon > 0$, and then for each $t > 0$ some $P_t \subset A$ in $\mathfrak{A}$ with

$$\vartheta(P_t) + t\alpha(A \setminus P_t) < \vartheta \wedge (t\alpha)(A) + \varepsilon \leqq C + \varepsilon.$$

From the assumption $\vartheta \operatorname{ac} \alpha$ we obtain $\delta > 0$ such that $\alpha(T) \leqq \delta \Rightarrow \vartheta(T) \leqq \varepsilon$ for all $T \subset A$ in $\mathfrak{A}$. Thus for $t \geqq (C + \varepsilon)/\delta$ we have $\vartheta(A \setminus P_t) \leqq \varepsilon$, which combined with $\vartheta(P_t) \leqq C + \varepsilon$ furnishes $\vartheta(A) \leqq C + 2\varepsilon$. It follows that $\vartheta(A) \leqq C$. □

We combine these results and an obvious observation to obtain

$$\vartheta \ll \alpha \longleftarrow \vartheta \operatorname{ac} \alpha \longleftarrow \vartheta \operatorname{AC} \alpha \quad \text{and} \quad \vartheta \ll \alpha \underset{\text{meas}}{\Longrightarrow} \vartheta \operatorname{ac} \alpha \underset{\vartheta<\infty}{\longrightarrow} \vartheta \operatorname{AC} \alpha.$$

Thus the new condition $\vartheta \operatorname{ac} \alpha$ has a kind of intermediate position.

2.2 Remark. Instead of $\vartheta \operatorname{ac} \alpha$ one could think to use the simpler variant

$$\begin{aligned} \vartheta \operatorname{aco} \alpha : \forall A \in \mathfrak{A} \text{ with } \vartheta(A) < \infty \text{ and } \forall \varepsilon > 0 \ \exists \delta > 0 \\ \text{such that } \alpha(T) \leqq \delta \Rightarrow \vartheta(T) \leqq \varepsilon \text{ for all } T \subset A \text{ in } \mathfrak{A}. \end{aligned}$$

But $\vartheta \operatorname{aco} \alpha$ does not fulfil 2.1: The implication $\vartheta \operatorname{ac} \alpha \Rightarrow \vartheta \operatorname{aco} \alpha$ is obvious, but the converse implication $\vartheta \operatorname{ac} \alpha \Leftarrow \vartheta \operatorname{aco} \alpha$ is false.

Proof. As in example 1.5.1) let $X = \mathbb{N}$ and $\mathfrak{A}$ consist of its finite and cofinite subsets. Define $\alpha, \vartheta : \mathfrak{A} \to [0, \infty]$ to be $\alpha(A) = \sum\limits_{l \in A} 2^{-l}$, and $\vartheta(A) = 0$ for A finite and $\vartheta(A) = \infty$ for A cofinite. For $t > 0$ then

$$\vartheta(\{1, \cdots, n\}) + t\alpha(\{n + 1, \cdots\}) = t2^{-n} \quad \text{for } n \in \mathbb{N},$$

and hence $\vartheta \wedge (t\alpha)(X) = 0$, so that $\vartheta \wedge (t\alpha) = 0$. Thus 2.1 shows that $\vartheta \operatorname{ac} \alpha$ is not true. But $\vartheta \operatorname{aco} \alpha$ is fulfilled, because ϑ attains but the values 0 and ∞. □

We add two important properties of the new concept.

2.3 Proposition. *Let α and ϑ be contents. Then $\xi := \lim\limits_{t\uparrow\infty} \vartheta \wedge (t\alpha)$ is a content $\leqq \vartheta$ which fulfils $\xi \wedge (t\alpha) = \vartheta \wedge (t\alpha)$ for all $t > 0$. Thus $\xi \operatorname{ac} \alpha$.*

Proof. For $t > 0$ we have on the one hand $\xi \wedge (t\alpha) \leqq \vartheta \wedge (t\alpha)$, and on the other hand $\vartheta \wedge (t\alpha) \leqq \xi$ and $\leqq t\alpha$, and hence $\vartheta \wedge (t\alpha) \leqq \xi \wedge (t\alpha)$. □

2.4 Proposition. *Assume that $\vartheta \wedge \alpha(A) = 0$ for some $A \in \mathfrak{A}$. If the content $\varphi : \mathfrak{A} \to [0, \infty]$ fulfils $\varphi \operatorname{ac} \alpha$ and $\varphi(A) < \infty$, then $\vartheta \wedge \varphi(A) = 0$.*

Proof. Fix $\varepsilon > 0$, and then $\delta > 0$ from $\varphi \operatorname{ac} \alpha$ for A and ε, that is $\alpha(T) \leqq \delta \Rightarrow \varphi(T) \leqq \varepsilon$ for all $T \subset A$ in $\mathfrak{A}$. Now let $t \geqq \varepsilon/\delta$, and from $\vartheta \wedge (t\alpha)(A) = 0$ take $T \subset A$ in $\mathfrak{A}$ such that $\vartheta(A \setminus T) + t\alpha(T) \leqq \varepsilon$. Then

$$\vartheta \wedge \varphi(A) \leqq \vartheta(A \setminus T) + \varphi(T) \leqq 2\varepsilon.$$

It follows that $\vartheta \wedge \varphi(A) = 0$. □

We conclude with a certain weak point of the new concept.

2.5 Remark. Let $\alpha, \vartheta, \eta : \mathfrak{A} \to [0, \infty]$ be contents with $\vartheta \leqq \eta$. It is then obvious that

$$\eta \ll \alpha \Rightarrow \vartheta \ll \alpha \quad \text{and} \quad \eta \operatorname{AC} \alpha \Rightarrow \vartheta \operatorname{AC} \alpha.$$

But the implication $\eta \operatorname{ac} \alpha \Rightarrow \vartheta \operatorname{ac} \alpha$ need not be true.

Proof. Take X and $\mathfrak{A}$ and α, ϑ from example 1.2, and define η to be $\eta(A) = 0$ when $\alpha(A) = 0$ and $\eta(A) = \infty$ when $\alpha(A) > 0$. Then $\vartheta \leqq \eta$, because $\vartheta(A) > 0$ implies that $\alpha(A) > 0$ and hence $\eta(A) = \infty$. Now

$$\eta = \lim_{t \uparrow \infty} t\alpha = \lim_{t \uparrow \infty} \eta \wedge (t\alpha),$$

so that $\eta \operatorname{ac} \alpha$. But we know that $\vartheta \wedge \alpha = 0$, so that $\vartheta \operatorname{ac} \alpha$ is not true. □

3. The New Concept for Singular

In the present full content situation the appropriate condition for contents $\alpha, \vartheta : \mathfrak{A} \to [0, \infty]$ will be seen to be

$$\vartheta \operatorname{sing} \alpha : \vartheta \text{ is inner regular } [\vartheta \wedge \alpha = 0].$$

We emphasize the appearance of *inner regular*. This fact is in the spirit of the author's book [9] and subsequent work [12].

In the sequel our procedure will be based on the difference formation $\beta \setminus \alpha : \mathfrak{A} \to [0, \infty]$ for contents $\alpha, \beta : \mathfrak{A} \to [0, \infty]$ defined in [10]. The definition is

$$\beta \setminus \alpha(A) = \sup\{\beta(T) - \alpha(T) : T \subset A \text{ in } \mathfrak{A} \text{ with } \alpha(T) < \infty\}.$$

We start to recall its relevant properties from [10] section 1, with some further properties which have routine proofs.

3.1 Properties. 1) *$\beta \setminus \alpha : \mathfrak{A} \to [0, \infty]$ is a content. If β is upward σ continuous then $\beta \setminus \alpha$ is upward σ continuous as well. Thus if $\mathfrak{A}$ is a σ algebra and β is a measure then $\beta \setminus \alpha$ is a measure as well.*

2) $\beta \setminus \alpha = \beta \setminus (\alpha \wedge \beta)$.

3) $(\vartheta \setminus \alpha) \setminus \beta = \vartheta \setminus (\alpha + \beta)$.

4) $\alpha \leqq \beta \Rightarrow \vartheta \setminus \alpha \geqq \vartheta \setminus \beta$.

5) *$\beta = \alpha \wedge \beta + (\beta \setminus \alpha)$ and $\alpha \vee \beta = \alpha + (\beta \setminus \alpha)$. Thus $\alpha \vee \beta + \alpha \wedge \beta = \alpha + \beta$. In particular $\beta = \alpha + (\beta \setminus \alpha)$ when $\alpha \leqq \beta$.*

6) *$\beta \setminus \alpha$ is inner regular $[\alpha < \infty] := \{A \in \mathfrak{A} : \alpha(A) < \infty\}$.*

We mention a remarkable consequence of 3.1.1)5): If $\alpha, \beta : \mathfrak{A} \to [0, \infty]$ are measures with $\alpha \leqq \beta$, then there exists a measure $\tau : \mathfrak{A} \to [0, \infty]$ such that $\alpha + \tau = \beta$. This is of course obvious when $\alpha < \infty$, but the full assertion was an involved earlier result of Jean Guillerme [6]. Later then the present difference formation offered the immediate answer $\tau := \beta \setminus \alpha$. In fact, the result of Guillerme was the motivation for its development.

On the basis of the difference formation the present author introduced in [10] for contents $\alpha, \vartheta : \mathfrak{A} \to [0, \infty]$ the *symmetric* condition

$$\vartheta \operatorname{SING} \alpha : \vartheta = \vartheta \setminus \alpha \text{ and } \alpha = \alpha \setminus \vartheta.$$

The condition $\vartheta \operatorname{SING} \alpha$ turned out to be responsible for the connection with the concepts of *signed contents and measures* developed in that former paper. But we shall see that it is not involved in the present context. This is also true for the one-sided condition $\vartheta = \vartheta \setminus \alpha$. We note the implications

$$\vartheta \wedge \alpha = 0 \longrightarrow \vartheta \,\mathrm{SING}\, \alpha \longrightarrow \vartheta = \vartheta \setminus \alpha \underset{\vartheta<\infty}{\longrightarrow} \vartheta \wedge \alpha = 0,$$

of which the middle one is obvious and the two others follow from 3.1.5).

We shall later need the next two examples. Then the final proposition obtains some basic relations.

3.2 EXAMPLE. 1) Let $X =]0,1]$ and $\mathfrak{A} = \mathrm{Bor}(X)$, and $\alpha = \mathrm{Leb}|\mathfrak{A}$ be the Borel-Lebesgue measure. Define $\vartheta : \mathfrak{A} \to [0,\infty]$ to be $\vartheta(A) = \int\limits_A 1/x\,dx$. 2) We claim that there is no decomposition $\vartheta = \varphi + \psi$ into contents $\varphi \,\mathrm{AC}\, \alpha$ and $\psi = \psi \setminus \alpha$. In fact, in that case we conclude as follows. i) From $\varphi \,\mathrm{AC}\, \alpha$ it follows that $\varphi < \infty$. Thus $\psi = \vartheta \setminus \varphi$, and 3.1.1) implies that ψ is a measure. ii) For $0 < \delta < 1$ the subsets $A \subset [\delta, 1]$ in $\mathfrak{A}$ have $\psi(A) < \infty$, and in view of $\psi = \psi \wedge \alpha + (\psi \setminus \alpha)$ in 3.1.5) have $\psi \wedge \alpha(A) = 0$. Thus $\psi \wedge \alpha = 0$ since ψ is a measure. iii) It follows that $\psi \perp \alpha$, that is there exists $T \in \mathfrak{A}$ such that $\psi(T) = 0$ and $\alpha(T') = 0$, which implies that $\vartheta(T') = 0$ and hence $\psi(T') = 0$. Thus $\psi = 0$ and hence $\vartheta = \varphi < \infty$, which is a contradiction. □

3.3 EXAMPLE. 1) Let $\alpha : \mathfrak{A} \to [0,\infty[$ be a finite measure $\neq 0$, and define $\vartheta : \mathfrak{A} \to [0,\infty]$ to be $\vartheta(A) = 0$ when $\alpha(A) = 0$ and $\vartheta(A) = \infty$ when $\alpha(A) > 0$, so that ϑ is a measure as well. 2) We have on the one hand $\vartheta \setminus \alpha = \vartheta - \alpha = \vartheta$. On the other hand $\vartheta \wedge (t\alpha) = t\alpha \uparrow \vartheta$ for $t \uparrow \infty$, so that $\vartheta \,\mathrm{ac}\, \alpha$. Then $\vartheta|[\vartheta \wedge \alpha = 0] = 0$ shows that $\vartheta \,\mathrm{sing}\, \alpha$ does not hold true.

3.4 PROPOSITION. *Let α and ϑ be contents. Then*

$$\vartheta \,\mathrm{sing}\, \alpha \underset{\mathrm{meas}}{\Longrightarrow} \vartheta \,\mathrm{singJ}\, \alpha \quad \textit{and} \quad \vartheta \,\mathrm{sing}\, \alpha \longleftarrow \vartheta \,\mathrm{singJ}\, \alpha,$$
$$\vartheta \,\mathrm{SING}\, \alpha \Longrightarrow \vartheta \,\mathrm{sing}\, \alpha \Longrightarrow \vartheta = \vartheta \setminus \alpha.$$

Proof of the first assertion. The implication $\Leftarrow$ is clear from the definitions. We prove $\Rightarrow$ for measures α and ϑ. For fixed $A \in \mathfrak{A}$ there exist $T_l \subset A$ in $\mathfrak{A}$ with $\vartheta \wedge \alpha(T_l) = 0$ for $l \in \mathbb{N}$ and $\vartheta(T_l) \to \vartheta(A)$. We can of course assume that $T_l \uparrow$ and hence $T_l \uparrow$ some $T \subset A$ in $\mathfrak{A}$. Then $\vartheta \wedge \alpha(T) = 0$ and $\vartheta(T) = \vartheta(A)$. From 1.1.3) we obtain $S \subset T$ in $\mathfrak{A}$ such that $\alpha(S) = 0$ and $\vartheta(T \setminus S) = 0$. It follows that $\vartheta(S) = \vartheta(A)$, so that S is as required.

Proof of the second assertion. We start with the first implication $\Rightarrow$. On the one hand 3.1.6) asserts that $\vartheta = \vartheta \setminus \alpha$ is inner regular $[\alpha < \infty]$. On the other hand we have

$$\alpha = \alpha \wedge \vartheta + (\alpha \setminus \vartheta) = \alpha \wedge \vartheta + \alpha \quad \text{from 3.1.5)},$$

which implies that $[\alpha < \infty] \subset [\alpha \wedge \vartheta = 0]$. Thus ϑ is inner regular $[\vartheta \wedge \alpha = 0]$.

We turn to the proof of the second implication $\Rightarrow$. Fix $A \in \mathfrak{A}$ and $c < \vartheta(A)$. To be shown is $\vartheta \setminus \alpha(A) > c$. By assumption there exists $N \subset A$ in $\mathfrak{A}$ with $\vartheta \wedge \alpha(N) = 0$ and $\vartheta(N) > c$. Now 3.1.2) furnishes

$$\begin{aligned} &\vartheta \setminus \alpha(A) = \vartheta \setminus (\vartheta \wedge \alpha)(A) \\ &= \sup\{\vartheta(T) - \vartheta \wedge \alpha(T) : T \subset A \text{ in } \mathfrak{A} \text{ with } \vartheta \wedge \alpha(T) < \infty\} \geqq \vartheta(N) > c, \end{aligned}$$

and hence the assertion. □

4. The Main Theorem

4.1 Full Content Theorem. *Let* $\alpha,\vartheta:\mathfrak{A}\to[0,\infty]$ *be contents. Then there exists a unique decomposition* $\vartheta=\varphi+\psi$ *into contents* $\varphi,\psi:\mathfrak{A}\to[0,\infty]$ *which fulfils*

$$\varphi\,\mathrm{ac}\,\alpha \quad \textit{and} \quad \psi\,\mathrm{sing}\,\alpha \quad \textit{and in addition} \quad \psi\,\mathrm{sing}\,\varphi \quad \textit{and} \quad \varphi\,\mathrm{sing}\,\psi.$$

Let $\xi:=\lim\limits_{t\uparrow\infty}\vartheta\wedge(t\alpha)$*, so that* 2.3 *asserts that* $\xi\leqq\vartheta$ *is a content* $\xi:\mathfrak{A}\to[0,\infty]$ *with* $\xi\,\mathrm{ac}\,\alpha$*. Then* $\varphi=\xi\setminus(\vartheta\setminus\xi)=\vartheta\setminus(\vartheta\setminus\xi)$ *and* $\psi=\vartheta\setminus\xi$*.*

The above theorem will come with several complements. In the final section it will be seen that the new conditions $\varphi\,\mathrm{ac}\,\alpha$ and $\psi\,\mathrm{sing}\,\alpha$ are the natural and canonical ones, and that at the same time the previous theorems are all direct consequences.

4.2 Lemma. *For each pair of contents* $\alpha,\beta:\mathfrak{A}\to[0,\infty]$ *one has*

$$\beta|[\beta<\infty] \textit{ inner regular } [\alpha<\infty]$$
$$\Longrightarrow\ \alpha\wedge\beta|[\beta<\infty] \textit{ inner regular } [\alpha+\beta<\infty].$$

Proof of 4.2. Fix $B\in[\beta<\infty]$ and $c<\alpha\wedge\beta(B)$, and then $\varepsilon>0$ with $c+\varepsilon<\alpha\wedge\beta(B)$. By assumption there exists $A\subset B$ in $[\alpha<\infty]$ such that $\beta(A)>\beta(B)-\varepsilon$ or $\beta(B\setminus A)<\varepsilon$. Now for $T\subset A$ in $\mathfrak{A}$ we have

$$\begin{aligned} c+\varepsilon<\alpha\wedge\beta(B)&\leqq\alpha(T)+\beta(B\setminus T)\\ &=\alpha(T)+\beta(A\setminus T)+\beta(B\setminus A)<\alpha(T)+\beta(A\setminus T)+\varepsilon, \end{aligned}$$

and hence $c<\alpha(T)+\beta(A\setminus T)$. Thus $c\leqq\alpha\wedge\beta(A)$. In view of $A\in[\alpha+\beta<\infty]$ the assertion follows. □

We turn to the proof of the full content theorem. 1) $(\vartheta\setminus\xi)\wedge\alpha=0$ on $[\xi<\infty]$. It suffices to prove that

$$(\vartheta\setminus\xi)\wedge\alpha\leqq\frac{1}{t}\,\vartheta\wedge(t\alpha)\quad\text{for all } t>0.$$

To see this fix $A\in\mathfrak{A}$ and $t>0$ with $c:=\vartheta\wedge(t\alpha)(A)<\infty$, and take $P_l\subset A$ in $\mathfrak{A}$ with $\vartheta(A\setminus P_l)+t\alpha(P_l)\to c$ for $l\uparrow\infty$, and of course $<\infty$ for $l\in\mathbb{N}$. We put

$$\begin{aligned} \delta_l:&=\vartheta(A\setminus P_l)+t\alpha(P_l)-c\quad\text{with } 0\leqq\delta_l<\infty \text{ and } \delta_l\to 0\\ &=\Big(\vartheta(A\setminus P_l)-\vartheta\wedge(t\alpha)(A\setminus P_l)\Big)+\Big(t\alpha(P_l)-\vartheta\wedge(t\alpha)(P_l)\Big), \end{aligned}$$

where these two brackets are both $\geqq 0$ and hence $\leqq\delta_l$. From 3.1.4) we obtain

$$\begin{aligned} (\vartheta\setminus\xi)\wedge\alpha(A)&\leqq\vartheta\setminus\xi(A\setminus P_l)+\alpha(P_l)\leqq\vartheta\setminus(\vartheta\wedge(t\alpha))(A\setminus P_l)+\alpha(P_l)\\ &=\Big(\vartheta(A\setminus P_l)-\vartheta\wedge(t\alpha)(A\setminus P_l)\Big)+\frac{1}{t}\Big(t\alpha(P_l)-\vartheta\wedge(t\alpha)(P_l)\Big)\\ &\quad+\frac{1}{t}\,\vartheta\wedge(t\alpha)(P_l)\leqq(1+1/t)\delta_l+(1/t)c. \end{aligned}$$

For $l\uparrow\infty$ the assertion follows.

2) $\vartheta\setminus\xi$ is inner regular $[(\vartheta\setminus\xi)\wedge\alpha=0]$, that is $(\vartheta\setminus\xi)\,\mathrm{sing}\,\alpha$. In fact, the above 1) asserts that $[\xi<\infty]\subset[(\vartheta\setminus\xi)\wedge\alpha=0]$, and $\vartheta\setminus\xi$ is inner

regular $[\xi < \infty]$ from 3.1.6). Thus 3.1.5) furnishes a first decomposition $\vartheta = \xi + (\vartheta \setminus \xi)$ of ϑ into $\xi \operatorname{ac} \alpha$ and $(\vartheta \setminus \xi) \operatorname{sing} \alpha$.

3) On both $[\xi < \infty]$ and $[\vartheta \setminus \xi < \infty]$ we have $(\vartheta \setminus \xi) \wedge \xi = 0$, and hence $\xi = \xi \setminus (\vartheta \setminus \xi)$ from 3.1.5). For the first case let $A \in \mathfrak{A}$ with $\xi(A) < \infty$. Then $(\vartheta \setminus \xi) \wedge \alpha(A) = 0$ from 1) and hence $(\vartheta \setminus \xi) \wedge \xi(A) = 0$ from 2.4. For the second case we note that

$$\begin{aligned} &\vartheta \setminus \xi | [\vartheta \setminus \xi < \infty] \text{ inner regular } [\xi < \infty] \\ &\qquad \Longrightarrow \xi \wedge (\vartheta \setminus \xi) | [\vartheta \setminus \xi < \infty] \text{ inner regular } [\vartheta < \infty], \end{aligned}$$

from lemma 4.2 applied to ξ and $\vartheta \setminus \xi$ and from 3.1.5). Since we know that $\vartheta \setminus \xi$ is inner regular $[\xi < \infty]$ and that $\xi \wedge (\vartheta \setminus \xi) = 0$ on $[\vartheta < \infty] \subset [\xi < \infty]$, it follows that $\xi \wedge (\vartheta \setminus \xi) = 0$ on $[\vartheta \setminus \xi < \infty]$.

4) For $A \in \mathfrak{A}$ we have

$$\xi \setminus (\vartheta \setminus \xi)(A) = \sup\{\xi(T) : T \subset A \text{ in } \mathfrak{A} \text{ with } \vartheta \setminus \xi(T) < \infty\}.$$

This is an immediate combination of 3.1.6) and 3).

5) $\xi \setminus (\vartheta \setminus \xi) = \vartheta \setminus (\vartheta \setminus \xi)$. In view of 3.1.6) it suffices to prove this on $[\vartheta \setminus \xi < \infty]$. Now 3.1.5) furnishes $\vartheta = (\vartheta \setminus \xi) + \vartheta \setminus (\vartheta \setminus \xi)$ besides $\vartheta = \xi + (\vartheta \setminus \xi)$. Thus on $[\vartheta \setminus \xi < \infty]$ we have $\vartheta \setminus (\vartheta \setminus \xi) = \xi$ and hence $= \xi \setminus (\vartheta \setminus \xi)$ from 3).

6) For $\varphi := \xi \setminus (\vartheta \setminus \xi) = \vartheta \setminus (\vartheta \setminus \xi)$ we have $\vartheta = \varphi + (\vartheta \setminus \xi)$ and $\varphi \operatorname{ac} \alpha$. Thus we obtain another decomposition of ϑ into $\varphi \operatorname{ac} \alpha$ and $(\vartheta \setminus \xi) \operatorname{sing} \alpha$. In fact, the first assertion follows from 3.1.5). For the second assertion fix $A \in \mathfrak{A}$. From 3) we see for $T \subset A$ in $\mathfrak{A}$ with $\vartheta \setminus \xi(T) < \infty$ that

$$\xi \wedge (t\alpha)(T) = \varphi \wedge (t\alpha)(T) \leqq \varphi \wedge (t\alpha)(A) \quad \text{for } t > 0,$$

and hence $\xi(T) \leqq \lim\limits_{t \uparrow \infty} \varphi \wedge (t\alpha)(A)$ since $\xi \operatorname{ac} \alpha$. Thus from 4) it follows that $\varphi(A) \leqq \lim\limits_{t \uparrow \infty} \varphi \wedge (t\alpha)(A)$, that is $\varphi \operatorname{ac} \alpha$.

7) The decomposition $\vartheta = \varphi + \psi$ of ϑ into $\varphi := \xi \setminus (\vartheta \setminus \xi) = \vartheta \setminus (\vartheta \setminus \xi)$ and $\psi := \vartheta \setminus \xi$ obtained in 6) fulfils $\varphi \operatorname{sing} \psi$ and $\psi \operatorname{sing} \varphi$, that is φ and ψ are inner regular $[\varphi \wedge \psi = 0]$. In fact, on both $[\xi < \infty]$ and $[\vartheta \setminus \xi < \infty]$ we have $\varphi \wedge \psi \leqq \xi \wedge (\vartheta \setminus \xi) = 0$ from 3). Thus $[\xi < \infty], [\vartheta \setminus \xi < \infty] \subset [\varphi \wedge \psi = 0]$. Now ψ is inner regular $[\xi < \infty]$ and φ is inner regular $[\vartheta \setminus \xi < \infty]$ after 3.1.6). Thus the assertion follows.

At this point the proof of the existence assertion in 4.1 is complete. We turn to the uniqueness assertion. For the parts 8)9)10) below we assume an arbitrary representation $\vartheta = \varphi + \psi$ of ϑ into contents $\varphi \operatorname{ac} \alpha$ and $\psi \operatorname{sing} \alpha$.

8) We have i) $\varphi \leqq \xi$ and ii) $\psi \geqq \vartheta \setminus \xi$, and iii) $\vartheta = \xi + \psi$. For the proof of i) note that $\vartheta \wedge (t\alpha) \geqq \varphi \wedge (t\alpha)$ for $t > 0$, which in view of $\varphi \operatorname{ac} \alpha$ implies that $\xi \geqq \varphi$. ii) From

$$\xi + (\vartheta \setminus \xi) = \vartheta = \varphi + \psi \leqq \xi + \psi \quad \text{by i)}$$

we obtain $\vartheta \setminus \xi \leqq \psi$ on $[\xi < \infty]$, and hence $\vartheta \setminus \xi \leqq \psi$ on $\mathfrak{A}$ since $\vartheta \setminus \xi$ is inner regular $[\xi < \infty]$. iii) From $\psi \operatorname{sing} \alpha$ we obtain $\psi = \psi \setminus \alpha$ after 3.4. This implies that $\psi = \psi \setminus (n\alpha)$ for $n \in \mathbb{N}$ via induction from 3.1.3) and hence $\psi = \psi \setminus (t\alpha)$ for $t > 0$ from 3.1.4). Thus 3.1.5) furnishes

$$\vartheta = \vartheta \wedge (t\alpha) + \vartheta \setminus (t\alpha) \geqq \vartheta \wedge (t\alpha) + \psi \setminus (t\alpha) = \vartheta \wedge (t\alpha) + \psi \quad \text{for } t > 0,$$

and hence $\vartheta \geqq \xi + \psi$. Combined with $\xi \geqq \varphi$ this implies that $\vartheta = \xi + \psi$.

9) Assume that in addition $\psi \operatorname{sing} \varphi$. Then i) $\varphi \geqq \xi \setminus (\vartheta \setminus \xi)$ and ii) $\psi = \vartheta \setminus \xi$. For the proof of ii) fix $A \in \mathfrak{A}$ and real $c < \psi(A)$, and then $\varepsilon > 0$ with $c + \varepsilon < \psi(A)$. Since ψ is inner regular $[\varphi \wedge \psi = 0]$ there exists $P \subset A$ in $\mathfrak{A}$ such that $\varphi \wedge \psi(P) = 0$ and $\psi(P) > c + \varepsilon$. Hence there exists $Q \subset P$ in $\mathfrak{A}$ with $\varphi(Q) < \varepsilon$ and $\psi(P \setminus Q) < \varepsilon$, so that $\psi(Q) > c$. And since ψ is inner regular $[\psi \wedge \alpha = 0]$ there exists $R \subset Q$ such that $\psi \wedge \alpha(R) = 0$ and $\psi(R) > c$. After this 1.1.2) implies that $\vartheta \wedge (t\alpha) \leqq \varphi + \psi \wedge (t\alpha)$ for all $t > 0$, in particular $\vartheta \wedge (t\alpha)(R) \leqq \varphi(R)$, and hence $\xi(R) \leqq \varphi(R) \leqq \varphi(Q) < \varepsilon < \infty$. Thus $\psi(R) = \vartheta \setminus \xi(R)$ from the above 8.iii). It follows that $c < \psi(R) = \vartheta \setminus \xi(R) \leqq \vartheta \setminus \xi(A)$ and hence $\psi(A) \leqq \vartheta \setminus \xi(A)$, which combined with 8.ii) furnishes the assertion.

For the proof of i) combine $\vartheta = \varphi + \psi = \varphi + (\vartheta \setminus \xi)$ from ii) with $\vartheta = \xi \setminus (\vartheta \setminus \xi) + (\vartheta \setminus \xi)$ from 6) to obtain $\varphi = \xi \setminus (\vartheta \setminus \xi)$ on $[\vartheta \setminus \xi < \infty]$. This implies that $\varphi \geqq \xi \setminus (\vartheta \setminus \xi)$ on $\mathfrak{A}$ since $\xi \setminus (\vartheta \setminus \xi)$ is inner regular $[\vartheta \setminus \xi < \infty]$ by 3.1.6).

10) Assume that in addition $\psi \operatorname{sing} \varphi$ and $\varphi \operatorname{sing} \psi$. Then $\varphi = \xi \setminus (\vartheta \setminus \xi)$. In fact, we see from 3.4 and 9.ii) that $\varphi = \varphi \setminus \psi = \varphi \setminus (\vartheta \setminus \xi)$, and hence that φ is inner regular $[\vartheta \setminus \xi < \infty]$. Now note that $\varphi = \xi \setminus (\vartheta \setminus \xi)$ on $[\vartheta \setminus \xi < \infty]$ from 8.i) with 3) and 9.i). Since both sides are inner regular $[\vartheta \setminus \xi < \infty]$ it follows that $\varphi = \xi \setminus (\vartheta \setminus \xi)$ on $\mathfrak{A}$. This completes the proof of 4.1. $\square$

4.3 Addendum. *Let $\vartheta = \varphi + \psi$ be an arbitrary representation of ϑ with contents $\varphi \operatorname{ac} \alpha$ and $\psi \operatorname{sing} \alpha$. Then $\varphi \leqq \xi$ and $\psi \geqq \vartheta \setminus \xi$. In case $\psi \operatorname{sing} \varphi$ we have $\xi \setminus (\vartheta \setminus \xi) \leqq \varphi \leqq \xi$ and $\psi = \vartheta \setminus \xi$. If moreover $\varphi \operatorname{sing} \psi$ then $\varphi = \xi \setminus (\vartheta \setminus \xi) = \vartheta \setminus (\vartheta \setminus \xi)$.*

4.4 Addendum. *We have $\xi \wedge (\vartheta \setminus \xi) = 0$ and hence $\xi = \xi \setminus (\vartheta \setminus \xi) = \vartheta \setminus (\vartheta \setminus \xi)$ on $[\xi < \infty]$ and on $[\vartheta \setminus \xi < \infty]$.*

4.5 Example. We return to example 1.6.1), where α and ϑ are both measures. Here $\xi = \lim_{t \uparrow \infty} \vartheta \wedge (t\alpha) = \lim_{t \uparrow \infty} t\alpha$ implies that $\xi(A) = 0$ for A countable and $\xi(A) = \infty$ for A cocountable. It follows that

$$\vartheta \setminus \xi(A) = \sup\{\vartheta(T) : T \subset A \text{ countable}\} = \vartheta(A) \quad \text{for } A \in \mathfrak{A}.$$

Now $\vartheta \setminus \xi = \vartheta$ implies that $\vartheta \setminus (\vartheta \setminus \xi) = \vartheta \setminus \vartheta = 0$. We see that there can be a big difference between the two first terms ξ and $\xi \setminus (\vartheta \setminus \xi) = \vartheta \setminus (\vartheta \setminus \xi)$ in the representations of ϑ which occur in 2)6) of the above proof of 4.1.

5. Complements to the Main Theorem

In the course of the paper we have collected for the pairs of contents $\alpha, \vartheta : \mathfrak{A} \to [0, \infty]$ three conditions $\mathrm{P} : \vartheta \, \mathrm{P} \, \alpha$ of the type *absolutely continuous* and six conditions $\mathrm{Q} : \vartheta \, \mathrm{Q} \, \alpha$ of the type *singular*. The conditions P are $\vartheta \ll \alpha$, $\vartheta \operatorname{AC} \alpha$, and $\vartheta \operatorname{ac} \alpha$, and their implications from sections 1 and 2 are

$$\vartheta \ll \alpha \longleftarrow \vartheta \operatorname{ac} \alpha \longleftarrow \vartheta \operatorname{AC} \alpha \quad \text{and} \quad \vartheta \ll \alpha \underset{\text{meas}}{\longrightarrow} \vartheta \operatorname{ac} \alpha \underset{\vartheta < \infty}{\longrightarrow} \vartheta \operatorname{AC} \alpha.$$

The conditions Q are

$$\vartheta \wedge \alpha = 0, \ \vartheta \perp \alpha, \ \vartheta \operatorname{singJ} \alpha \quad \text{and} \quad \vartheta \operatorname{sing} \alpha, \ \vartheta \operatorname{SING} \alpha, \ \vartheta = \vartheta \setminus \alpha,$$

and their implications from sections 1 and 3 are

$$
\begin{array}{llllll}
\vartheta\wedge\alpha=0 \longrightarrow \vartheta\operatorname{sing}\alpha & \underset{\text{meas}}{\longrightarrow} \vartheta\operatorname{singJ}\alpha & \underset{\vartheta<\infty}{\longrightarrow} \vartheta\perp\alpha & & \longrightarrow \vartheta\wedge\alpha=0, \\
\vartheta\wedge\alpha=0 \underset{\vartheta<\infty}{\longleftarrow} \vartheta\operatorname{sing}\alpha & \longleftarrow \vartheta\operatorname{singJ}\alpha & \longleftarrow \vartheta\perp\alpha & & \underset{\text{meas}}{\longleftarrow} \vartheta\wedge\alpha=0, \\
\vartheta\wedge\alpha=0 \longrightarrow \vartheta\operatorname{SING}\alpha & \longrightarrow \vartheta\operatorname{sing}\alpha & \longrightarrow \vartheta=\vartheta\setminus\alpha & & \underset{\vartheta<\infty}{\longrightarrow} \vartheta\wedge\alpha=0.
\end{array}
$$

The last line serves to separate the less involved conditions $\vartheta\operatorname{SING}\alpha$ and $\vartheta=\vartheta\setminus\alpha$.

5.1 Remark. 1) Each condition P of the type *absolutely continuous* fulfils $0\,\mathrm{P}\,\alpha$ for all contents α. In fact, it suffices to confirm that $0\,\mathrm{AC}\,\alpha$, and this is obvious from the definition. 2) Each condition Q of the type *singular* fulfils $0\,\mathrm{Q}\,\alpha$ and $\alpha\,\mathrm{Q}\,0$ for all contents α. In fact, it suffices to confirm that $0\perp\alpha$ and $\alpha\perp 0$, and these are identical and obvious from the definition.

It follows that for the decompositions $\vartheta=\varphi+\psi$ of ϑ into pairs of contents $\varphi,\psi:\mathfrak{A}\to[0,\infty]$ we have eighteen combinations of conditions (P,Q): $\varphi\,\mathrm{P}\,\alpha$ and $\psi\,\mathrm{Q}\,\alpha$.

We want to sort out those combinations which are of no use. These are on the one hand the combinations (P,Q) of *non-existence* kind, defined to mean that there exists a pair α and ϑ such that ϑ has no decomposition $\vartheta=\varphi+\psi$ with $\varphi\,\mathrm{P}\,\alpha$ and $\psi\,\mathrm{Q}\,\alpha$.

On the other hand these are the combinations (P,Q) of *non-uniqueness* kind, defined to mean that there exists a pair α and $\vartheta\neq 0$ with both $\vartheta\,\mathrm{P}\,\alpha$ and $\vartheta\,\mathrm{Q}\,\alpha$. In fact, in this case ϑ has the two different decompositions $\vartheta=\vartheta+0=0+\vartheta$, each of which $\vartheta=\varphi+\psi$ fulfils $\varphi\,\mathrm{P}\,\alpha$ and $\psi\,\mathrm{Q}\,\alpha$ and moreover $\psi\,\mathrm{Q}\,\varphi$ and $\varphi\,\mathrm{Q}\,\psi$. Thus the combination (P,Q) violates the uniqueness assertion as formulated in theorem 4.1. It can be said that we have a non-uniqueness situation of maximal badness.

Now the examples in the previous sections combine to furnish the uniqueness result which follows.

5.2 Theorem. *The above combinations of conditions* (P,Q) *:* $\varphi\,\mathrm{P}\,\alpha$ *and* $\psi\,\mathrm{Q}\,\alpha$ *are all, except the combination* (P,Q) *:* $\varphi\operatorname{ac}\alpha$ *and* $\psi\operatorname{sing}\alpha$*, either of non-existence kind or of non-uniqueness kind* (or both)*.*

It follows that besides 4.1 there can be no other Lebesgue type theorem for arbitrary contents, at least within the terms which appear in the present article.

Proof. We shall make constant use of the above lists of conditions P and Q with their implications $\longleftarrow$ and $\longrightarrow$. We note that for a combination (P,Q) the non-existence behaviour passes across an arrow $\longleftarrow$ from left to right, while the non-uniqueness behaviour passes from right to left (and of course the opposite at an arrow $\longrightarrow$).

1) Example 1.5.2) asserts that the combination (P,Q) : $\varphi\ll\alpha$ and $\psi\perp\alpha$ is of non-existence kind, even with an example α and $\vartheta<\infty$. Then the final remark of section 1 notes as a consequence that the combination (P,Q) : $\varphi\ll\alpha$ and $\psi\operatorname{singJ}\alpha$ is of non-existence kind as well. It follows that all combinations (P,Q) with arbitrary P and with Q : $\psi\perp\alpha$ and Q : $\psi\operatorname{singJ}\alpha$ are of non-existence kind.

2) Example 1.6.2) asserts that the combination (P,Q) : $\varphi \ll \alpha$ and $\psi \wedge \alpha = 0$ is of non-existence kind. As a consequence the combination (P,Q) : $\varphi \ll \alpha$ and $\psi \,\mathrm{SING}\, \alpha$ is of non-existence kind as well. It follows that all combinations (P,Q) with arbitrary P and with Q : $\psi \wedge \alpha = 0$ and Q : $\psi \,\mathrm{SING}\, \alpha$ are of non-existence kind.

Thus what remains are the combinations (P,Q) with arbitrary P and with Q : $\psi = \psi \setminus \alpha$ and Q : $\psi \,\mathrm{sing}\, \alpha$.

3) Example 3.2.2) asserts that the combination (P,Q) : $\varphi \,\mathrm{AC}\, \alpha$ and $\psi = \psi \setminus \alpha$ is of non-existence kind. Hence the combination (P,Q) : $\varphi \,\mathrm{AC}\, \alpha$ and $\psi \,\mathrm{sing}\, \alpha$ is of non-existence kind as well.

4) At this point non-uniqueness enters the scene. Example 1.2.2) asserts that the combination (P,Q) : $\varphi \ll \alpha$ and $\psi \wedge \alpha = 0$ is of non-uniqueness kind. Thus the combinations (P,Q) : $\varphi \ll \alpha$ and $\psi = \psi \,\mathrm{sing}\, \alpha$ and (P,Q) : $\varphi \ll \alpha$ and $\psi = \psi \setminus \alpha$ are of non-uniqueness kind as well.

5) It remains the combination (P,Q) : $\varphi \,\mathrm{ac}\, \alpha$ and $\psi = \psi \setminus \alpha$. But this combination is of non-uniqueness kind by example 3.3.2). This completes the proof. $\square$

The final task in the present paper is to convince ourselves that the previous results quoted in section 1, which after all appear in quite different terms, are nontheless direct consequences of our main theorem 4.1 and its addenda 4.3 and 4.4. Of course this has to be deduced from the connections between the different terms, established in the previous sections and summarized above. It should be noted, however, that the new approach offers much more than the former ones: in view of its explicit formulas, thanks to the difference formation $\setminus$.

Proof of the Traditional Content Theorem 1.4. Assume that $\alpha, \vartheta : \mathfrak{A} \rightarrow [0, \infty]$ are contents with $\vartheta < \infty$. i) From 4.1 we obtain the decomposition $\vartheta = \varphi + \psi$ of ϑ into $\varphi = \xi \setminus (\vartheta \setminus \xi)$ and $\psi = \vartheta \setminus \xi$, which fulfil $\varphi \,\mathrm{ac}\, \alpha$ and $\psi \,\mathrm{sing}\, \alpha$ and hence $\varphi \,\mathrm{AC}\, \alpha$ and $\psi \wedge \alpha = 0$, and $\psi \,\mathrm{sing}\, \varphi$ and $\varphi \,\mathrm{sing}\, \psi$ and hence $\varphi \wedge \psi = 0$. Moreover $\xi \leqq \vartheta < \infty$ implies that $\varphi = \xi$ after 4.4. ii) Now let $\vartheta = \varphi + \psi$ be an arbitrary decomposition of ϑ with $\varphi \,\mathrm{AC}\, \alpha$ and $\psi \wedge \alpha = 0$. Then $\varphi \,\mathrm{ac}\, \alpha$ and $\psi \,\mathrm{sing}\, \alpha$. From 2.4 we see that $\psi \wedge \varphi = 0$, and hence that $\psi \,\mathrm{sing}\, \varphi$ and $\varphi \,\mathrm{sing}\, \psi$. Thus 4.1 furnishes the desired uniqueness assertion. $\square$

Proof of the Full Measure Theorem 1.7 (and hence of the Traditional Measure Theorem 1.3). Assume that $\alpha, \vartheta : \mathfrak{A} \rightarrow [0, \infty]$ are measures. Then φ and ψ in 4.1 are measures in view of 3.1.1), and we have $\varphi \ll \alpha$ and $\psi \,\mathrm{singJ}\, \alpha$. ii) Let $\vartheta = \varphi + \psi$ be an arbitrary decomposition of ϑ into measures $\varphi \ll \alpha$ and $\psi \,\mathrm{singJ}\, \alpha$. Then $\varphi \,\mathrm{ac}\, \alpha$ and $\psi \,\mathrm{sing}\, \alpha$, and moreover $\psi \,\mathrm{singJ}\, \varphi$ and hence $\psi \,\mathrm{sing}\, \varphi$. Now the two uniqueness assertions in 1.7 follow from 4.3. $\square$

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