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Abstract

We study the nonlinear power growth variational problem

$$J_\alpha[w] := \int_\Omega \left[\frac{1}{\alpha}|\nabla w|^\alpha - fw\right] \mathrm{d}x \to \min$$

and establish directly computable estimates for the deviation from exact solutions. In the case of superquadratic growth, these estimates are given in terms of the energy norm, in the subquadratic case we pass to estimates for the solution of the dual variational problem. Various boundary conditions are included in our considerations.

1 Introduction

A quantitative analysis of solutions of boundary-value problems associated with PDE's is inevitably connected with their approximative solutions that can be obtained by various numerical methods. The latters usually construct a sequence of approximate solutions defined by solving finite-dimensional problems that arise if the original equation is projected on a certain finite-dimensional subspace. If such a projection and the subsequent numerical procedure are correctly performed, then the respective approximate solution can be used to analyse the quantitative behavior of the desired exact one.

However, such an analysis merely is consistent if the difference between the exact and the approximative solutions is explicitely estimated.

In the last decades, many efforts were focused on the methods able to estimate the quality of approximate solutions. Numerical analysts applying finite element methods have developed several approaches to the *a posteriori* error indication (see, e.g., [AiOd], [BaRe], [BaRa], [Ve]) that are usually based on the Galerkin orthogonality of an approximate solution considered and the particular features of the discretization applied. Typically, such estimates are used to answer the question how to use the information contained in an approximative solution computed on a subspace V_k in order to construct another subspace V_{k+1} of higher dimensionality such that the extra degrees of freedom can be stated in subdomains with maximal error.

However, the problem in question can be stated more generally. Namely, we also need computable estimates of the difference between exact solutions and their approximations that are valid for the *whole energy class* of comparison functions and provide *guaranteed* upper bounds of such differences. Clearly, such estimates should be derived by purely functional methods using approximatively the same techniques as used in the classical PDE theory for establishing existence and regularity properties. Such type of estimates (called *functional type a posteriori estimates* or *deviation estimates*) have been obtained

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for some classes of linear and nonlinear boundary-value problems (see, e.g., [Re1], [Re2] and the references quoted therein).

In the present paper, we derive deviation estimates for a class of power growth variational problems and justify their properties.

Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain. We start our considerations by fixing some given Dirichlet boundary data u_0: $\Omega \to \mathbb{R}$ of class $V := W^1_\alpha(\Omega)$, $1 < \alpha$. We then are interested in the minimization problem

$$J_\alpha[w] := \int_\Omega \Big[\frac{1}{\alpha}|\nabla w|^\alpha - f w\Big]\,\mathrm{d}x \to \min \tag{$\mathcal{P}$}$$

among all comparison functions w of class $u_0 + V_0$, where

$$V_0 := \overset{\circ}{W}{}^1_\alpha(\Omega)\,, \qquad u_0 + V_0 := \{w \in V :\, w = w_0 + u_0,\, w_0 \in V_0\}\,.$$

Moreover, f is assumed to be of class $L^{\alpha^*}(\Omega)$, with the number $\alpha^* = \alpha/(\alpha-1)$ being conjugate to α. Here and in the following, we refer to the standard notion of Sobolev and Lebesgue spaces as introduced, for instance, in [Ad]. The direct method of the calculus of variations and the convexity of J_α immediately give the existence of a unique solution $u \in u_0 + V_0$ and we have the Euler equation for smooth solutions

$$\operatorname{div}\big(|\nabla u|^{\alpha-2}\nabla u\big) + f = 0 \quad \text{ in } \Omega$$

together with its weak form (see, e.g., [LaUr])

$$\int_\Omega \big(|\nabla u|^{\alpha-2}\nabla u\cdot\nabla w - f w\big)\,\mathrm{d}x = 0 \quad \text{ for any } w \in V_0\,. \tag{1.1}$$

The dual variational problem associated to the original problem ($\mathcal{P}$) is the maximizing problem

$$I_{\alpha^*}[y^*] \to \max \quad \text{ in } Y^* := L^{\alpha^*}(\Omega;\mathbb{R}^n)\,, \tag{$\mathcal{P}^*$}$$

where

$$I_{\alpha^*}[y^*] := \begin{cases} \displaystyle\int_\Omega \Big[\nabla u_0\cdot y^* - \frac{1}{\alpha^*}|y^*|^{\alpha^*} - f u_0\Big]\,\mathrm{d}x & \text{if } \ y^* \in Q^*_f, \\ -\infty & \text{if } \ y^* \notin Q^*_f\,, \end{cases}$$

and where

$$Q^*_f := \Big\{y^* \in Y^* :\, \int_\Omega y^*\cdot\nabla w\,\mathrm{d}x = \int_\Omega f w\,\mathrm{d}x \ \text{ for all } w \in V_0\Big\}.$$

Some earlier a posteriori estimates for variational problems based on duality were considered in [Mi], [MoMy], [GaGrZa]). However, they were related with the necessity to approximate the set Q^*_f (or a similar set) exactly what technically is a very complicated task. In our approach exposed further, this difficulty is avoided.

By standard results of the convex analysis (see, for instance, [ET]) it is well known that the problem ($\mathcal{P}^*$) is uniquely solvable and if the solution is denoted by p^*, then we have

$$\inf\nolimits_{u_0+V_0}\mathcal{P} = J_\alpha[u] = I_{\alpha^*}[p^*] = \sup\nolimits_{Y^*}\mathcal{P}^*\,, \tag{1.2}$$

$$p^* = |\nabla u|^{\alpha-2}\nabla u \quad \text{ a.e. in } \Omega\,, \tag{1.3}$$

$$\nabla u = |p^*|^{\alpha^*-2}p^* \quad \text{ a.e. in } \Omega\,. \tag{1.4}$$

Now let us first consider the case $\alpha \geq 2$ and suppose that $v \in u_0 + V_0$ is a certain given function viewed as an approximation of u. Note that we do not assume that v possesses some specific features coming from the method of its derivation (e.g. Galerkin orthogonality or particular approximation properties). Then our aim is to obtain explicitely computable upper bounds of the type

$$\|\nabla(u - v)\|_{\alpha,\Omega} \leq \mathcal{M}(v, f, \alpha, \Omega)\,, \tag{1.5}$$

which has to fullfil the natural property

$$\mathcal{M}(v_k, f, \alpha, \Omega) \to 0 \quad \text{whenever} \;\; v_k \to u \;\; \text{in } V\,. \tag{1.6}$$

If an estimate of this kind is valid for any function from the energy space, then it presents an computable measure for the deviation from the exact solution. Note that one should carefully try to avoid any "over-estmation" in order to get an estimate, which can be used for a reliable verification of approximative solutions obtained by various numerical methods.

In our analysis the right-hand side $\mathcal{M}$ of (1.5) splits into two parts with a clear physical interpretation. The first part can be viewed as a penalty for a possible violation of the duality relations (1.3) and (1.4), the second one penalizes the error in the equilibriumm equation.

The case $1 < \alpha < 2$ essentially differs from the above considerations. Here it is not possible (without a priori estimates for the exact solution) to find a natural upper bound for the norm $\|\nabla(v-u)\|_{\alpha,\Omega}$ that makes it fully controllable as it is in the case $\alpha \geq 2$. This is due to a "lack of uniform convexity at ∞". To overcome this difficulty we pass to the dual variational problem with "good" convexity properties and establish corresponding (computable) estimates for the quantity

$$\|p^* - y^*\|_{\alpha^*,\Omega}\,,$$

where y^* is any function in Y^* with α^*-summable divergence.

Our paper is organized as follows: in Section 2 we give a precise formulation and a proof of our results in the case $\alpha \geq 2$, the case $1 < \alpha < 2$ is discussed in Section 3. In Section 4 and in Section 5 we show that with some minor changes the case of mixed Dirichlet-Neumann boundary data and the case of pure Neumann boundary data can be handled as well. In the last section we collect some auxiliary results which are needed for the proofs of our main Theorems 2.1 and 3.1.

2 Estimates for the deviation in the case $\alpha \geq 2$

In this section, we are going to establish

THEOREM 2.1 *Fix $\alpha \geq 2$. With the Notation from above we have for any $v \in u_0 + V_0$, for any $y^* \in Y^*$ with α^*-summable divergence and for any $\beta > 0$*

$$\|\nabla(v - u)\|^\alpha_{\alpha,\Omega} \leq \alpha 2^{\alpha-1}\big[\mathcal{M}_1[\nabla v, y^*, \beta] + \mathcal{M}_2[y^*, \beta]\big]\,, \tag{2.1}$$

where

$$\begin{aligned}\mathcal{M}_1[\nabla v, y^*, \beta] &= D_\alpha[\nabla v, y^*] + \frac{\beta^\alpha}{\alpha}\||y^*|^{\alpha^*-2}y^* - \nabla v\|^\alpha_{\alpha,\Omega}\,,\\ \mathcal{M}_2[y^*, \beta] &= C_\alpha^{\alpha^*}(\Omega)\Big[\frac{1}{\alpha^*\beta^{\alpha^*}} + 2^{2-\alpha^*}(3-\alpha^*)\Big]\|f + \operatorname{div} y^*\|^{\alpha^*}_{\alpha^*,\Omega}\,.\end{aligned}$$

The functional D_α: $Y \times Y^* \to \mathbb{R}_0^+$,

$$D_\alpha[y, y^*] := \int_\Omega \Big[\frac{1}{\alpha}|y|^\alpha + \frac{1}{\alpha^*}|y^*|^{\alpha^*} - y \cdot y^*\Big]\,\mathrm{d}x\,,$$

is the compound functional and $C_\alpha(\Omega)$ *is the constant in Poincaré's inequality.*

REMARK 2.1 i) *It is not difficult to observe that the right-hand side of (2.1) vanishes if and only if we have almost everywhere in* Ω

$$|y^*|^{\alpha^*-2}y^* = \nabla v\,, \tag{2.2}$$

and

$$\operatorname{div} y^* + f = 0\,. \tag{2.3}$$

ii) *Since the solution of the problem ($\mathcal{P}$) is unique, the relations (2.2) and (2.3) mean that in such a case we have* $v = u$ *and* $y^* = p^*$.

iii) *The compound functional can be thought of being a certain measure of the error in the duality relations (1.3) and (1.4) since this nonnegative functional vanishes if and only if*

$$y = |y^*|^{\alpha^*-2}y^* \quad \text{and} \quad y^* = |y|^{\alpha-2}y\,.$$

The second term in $\mathcal{M}_1$ *of course measures the same quantity.* $\mathcal{M}_2[y^*, \beta]$ *is a penalty for the error in the equilibrium equation.*

iv) *In fact, instead of estimating the* L^α*-norm of* $\nabla(v-u)$ *in (2.1), the sharp estimate is given in terms of the compound functional (compare formula (2.7) given below)*

$$\begin{aligned}D_\alpha[\nabla v, p^*] &= J_\alpha[v] - J_\alpha[u] = J_\alpha[v] - I_{\alpha^*}[p^*]\\ &= \int_\Omega \Big[\frac{1}{\alpha}|\nabla v|^\alpha + \frac{1}{\alpha^*}|p^*|^{\alpha^*} - \nabla v \cdot p^*\Big]\,\mathrm{d}x\,.\end{aligned}$$

If $w \in V$ *and* $y^*_w = |\nabla w|^{\alpha-2}\nabla w$, *then just in the case* $\alpha = 2$ *the (semi-) metric*

$$d_2[v, w] := D_2[\nabla v, y^*_w]$$

coincides with the energy norm because

$$D_2[\nabla v, y^*_w] = \frac{1}{2}\|\nabla(v-w)\|^2_{2,\Omega}\,.$$

In the nonlinear case $\alpha \neq 2$ *even the symmetry is lost and distances have to be measured w.r.t. the minimizer* u:

$$d_\alpha[v] := D_\alpha[\nabla v, p^*]\,.$$

Note that $d_\alpha[v] = 0$ if and only if we have a.e. on Ω

$$\nabla v = |p^*|^{\alpha^*-2} p^* = \nabla u \, .$$

Formally the whole space V can be equipped with the (semi-) metric ("with origin u")

$$d_\alpha[v, w] := d_\alpha[v] + d_\alpha[w] \, .$$

In particular cases $d_\alpha[v]$ can explicitely estimated w.r.t. some kind of "weighted norms", for instance if $\alpha = 3$, then we have

$$d_3[v] \geq \frac{1}{3} \int_\Omega (|\nabla v| + 2|\nabla u|)|\nabla(v-u)|^2 \, \mathrm{d}x \, .$$

v) *For the special case $\alpha = \alpha^* = 2$ we have*

$$\begin{aligned} \mathcal{M}_1[\nabla v, y^*, \beta] &= \frac{1}{2}(1+\beta^2)\|y^* - \nabla v\|_{2,\Omega}^2 \, , \\ \mathcal{M}_2[y^*, \beta] &= C^2(\Omega)\Big[1 + \frac{1}{2\beta^2}\|f + \operatorname{div} y^*\|_{2,\Omega}^2\Big] \, , \end{aligned}$$

where $C(\Omega)$ is the constant in Friedrich's inequality.

vi) *It is easy to check that for a pair of sequences $v_k \to u$ in V and $y_k^* \to p^*$ in Y^* the right-hand side of (2.1) tends to zero.*

Proof of Theorem 2.1. In the case $\alpha \geq 2$ we have uniform convexity by the following inequality (see [So]), extensions of such inequalities to spaces of tensor-valued functions can be found in [MoMy]:

$$\int_\Omega \Big[\Big|\frac{y_1 + y_2}{2}\Big|^\alpha + \Big|\frac{y_1 - y_2}{2}\Big|^\alpha\Big] \, \mathrm{d}x \leq \frac{1}{2}\|y_1\|_{\alpha,\Omega}^\alpha + \frac{1}{2}\|y_2\|_{\alpha,\Omega}^\alpha \quad \text{for all } \; y_1, \, y_2 \in Y \, . \tag{2.4}$$

In fact, we have from (2.4) for any $v \in u_0 + V_0$

$$\begin{aligned} J_\alpha[v] + J_\alpha[u] - 2J_\alpha\Big[\frac{u+v}{2}\Big] &= \frac{1}{\alpha}\int_\Omega \Big[|\nabla v|^\alpha + |\nabla u|^\alpha - 2\Big[\frac{|\nabla u + \nabla v|}{2}\Big]^\alpha\Big] \, \mathrm{d}x \\ &\geq \frac{2}{\alpha}\int_\Omega \Big[\frac{1}{2}|\nabla v|^\alpha + \frac{1}{2}|\nabla u|^\alpha - \Big[\frac{|\nabla u + \nabla v|}{2}\Big]^\alpha\Big] \, \mathrm{d}x \\ &\geq \frac{2}{\alpha}\int_\Omega \Big|\frac{\nabla(v-u)}{2}\Big|^\alpha \, \mathrm{d}x \\ &= \frac{1}{\alpha 2^{\alpha-1}}\|\nabla(v-u)\|_{\alpha,\Omega}^\alpha \, . \end{aligned}$$

On the other hand, from the minimality of u, i.e. from $J_\alpha[u] \leq J_\alpha[(u+v)/2]$, we obtain

$$J_\alpha[v] + J_\alpha[u] - 2J_\alpha\Big[\frac{u+v}{2}\Big] \leq J_\alpha[v] - J_\alpha[u] \, .$$

Therefore we arrive at the estimate

$$\|\nabla(v-u)\|_{\alpha,\Omega}^\alpha \leq \alpha 2^{\alpha-1}\Big[J_\alpha[v] - J_\alpha[u]\Big] \, . \tag{2.5}$$

Using the inf-sup relation (1.2), the inequality (2.5) turns into

$$\begin{aligned} \|\nabla(v-u)\|^{\alpha}_{\alpha,\Omega} &\leq \alpha 2^{\alpha-1}\Big[J_\alpha[v]-I_{\alpha^*}[p^*]\Big] \\ &\leq \alpha 2^{\alpha-1}\Big[J_\alpha[v]-I_{\alpha^*}[q^*]\Big] \end{aligned} \tag{2.6}$$

being valid for any $v \in u_0+V_0$ and for any $q^* \in Q^*_f$. Moreover we have

$$J_\alpha[v]-I_{\alpha^*}[q^*] = \int_\Omega \Big[\frac{1}{\alpha}|\nabla v|^\alpha + \frac{1}{\alpha^*}|q^*|^{\alpha^*} - \nabla u_0\cdot q^* - f(v-u_0)\Big]\,\mathrm{d}x$$

and since $q^* \in Q^*_f$ and $(v-u_0)\in V_0$ we also have

$$\int_\Omega f(v-u_0)\,\mathrm{d}x = \int_\Omega q^*\cdot\nabla(v-u_0)\,\mathrm{d}x\,.$$

As a result, it is found that

$$J_\alpha[v]-I_{\alpha^*}[q^*] = \int_\Omega \Big[\frac{1}{\alpha}|\nabla v|^\alpha + \frac{1}{\alpha^*}|q^*|^{\alpha^*} - \nabla v\cdot q^*\Big]\,\mathrm{d}x\,. \tag{2.7}$$

With the compound functional, (2.6) combined with (2.7) gives

$$\|\nabla(v-u)\|^{\alpha}_{\alpha,\Omega} \leq \alpha 2^{\alpha-1} D_\alpha[\nabla v, q^*] \quad \text{for any } \ v\in u_0+V_0\,,\ \ q^*\in Q^*_f\,. \tag{2.8}$$

REMARK 2.2 i) *The estimate (2.8) shows that for any $v \in u_0+V_0$ the upper bound of the deviation is defined by the value*

$$\alpha 2^{\alpha-1}\inf_{q^*\in Q^*_f} D_\alpha[\nabla v, q^*]\,.$$

ii) *In particular, if $v=u$, then we set $q^*=p^*$ and observe that both sides of (2.8) are equal to zero.*

iii) *It is also true that the deviation majorant given in (2.8) has the property (1.6) (see also Remark 2.1, iv)). Indeed, if $\{v_k\}$ is a sequence converging to u in V, then*

$$\inf_{q^*\in Q^*_f} D_\alpha[\nabla v_k, q^*] \leq D_\alpha[\nabla v_k, p^*] \to 0\,.$$

iv) *Practically the approximation can be computed by selecting a finite dimensional space $V_h \subset \{u_0+V_0\}$ and finding the quantity*

$$\min_{v_h\in V_h} D_\alpha[\nabla v_h, q^*]\,.$$

v) *Approximations of the dual problem are often obtained by means of approximations of the problem $(\mathcal{P})$, which is an unconstrained minimization problem and, therefore, much simpler from the technical point of view. Then one may try to find a suitable candidate q^* in (2.8) with the help of the duality relation $q^*=|\nabla v|^{\alpha-2}\nabla v$. However, in general q^* then does not satisfy the condition $\operatorname{div} q^*+f=0$: it is an essential drawback of the estimate (2.8) that it requires the function q^* to be in the set Q^*_f which is defined by a differential relation. In practice, finding such functions leads to serious difficulties.*

In order to avoid the drawback mentioned above, we take an arbitrary function $y^* \in Y^*$ which will be a substitute for $q^* \in Q^*_f$. We estimate

$$J_\alpha[v] - I_{\alpha^*}[q^*] = D_\alpha[\nabla v, y^*] + \int_\Omega \Big[\frac{1}{\alpha^*}|q^*|^{\alpha^*} - \frac{1}{\alpha^*}|y^*|^{\alpha^*} + \nabla v \cdot (y^* - q^*)\Big]\,\mathrm{d}x\,.$$

By the convexity of $|\cdot|^{\alpha^*}$ we have

$$\frac{1}{\alpha^*}|q^*|^{\alpha^*} - \frac{1}{\alpha^*}|y^*|^{\alpha^*} \le |q^*|^{\alpha^*-2}q^* \cdot (q^* - y^*)\,,$$

hence we have established

$$\begin{aligned} J_\alpha[v] - I_{\alpha^*}[q^*] \;\le\;& D_\alpha[\nabla v, y^*] + \int_\Omega \big(|y^*|^{\alpha^*-2}y^* - \nabla v\big)\cdot(q^* - y^*)\,\mathrm{d}x \\ &+ \int_\Omega \big(|q^*|^{\alpha^*-2}q^* - |y^*|^{\alpha^*-2}y^*\big)\cdot(q^* - y^*)\,\mathrm{d}x\,. \end{aligned} \tag{2.9}$$

In order to find an upper bound for the right-hand side of (2.9) we claim that

$$\int_\Omega \Big[\frac{q^*}{|q^*|^{2-\alpha^*}} - \frac{y^*}{|y^*|^{2-\alpha^*}}\Big]\cdot(q^* - y^*)\,\mathrm{d}x \le 2^{2-\alpha^*}(3-\alpha^*)\int_\Omega |q^* - y^*|^{\alpha^*}\,\mathrm{d}x\,. \tag{2.10}$$

In fact (2.10) is an immediate consequence of Corollary 6.1. Here we note that the claim is trivial if either $q^* = 0$ or $y^* = 0$. With (2.9) and (2.10) we arrive at the estimate

$$\begin{aligned} J_\alpha[v] - I_{\alpha^*}[q^*] \;\le\;& D_\alpha[\nabla v, y^*] + \||y^*|^{\alpha^*-2}y^* - \nabla v\|_{\alpha,\Omega}\|q^* - y^*\|_{\alpha^*,\Omega} \\ &+ 2^{2-\alpha^*}(3-\alpha^*)\|q^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega}\,. \end{aligned} \tag{2.11}$$

Keeping (2.11) in mind, we now need to select a proper function q^* in order to estimate the norm $\|q^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega}$. To do so, let us assume that $\operatorname{div} y^*$ is of class $L^{\alpha^*}(\Omega)$. Then we choose $q^* \in Q^*_f$ as the projection of y^* on Q^*_f, i.e.

$$\|q^* - y^*\|_{\alpha^*,\Omega} = \inf_{\tilde q^* \in Q^*_f}\|\tilde q^* - y^*\|_{\alpha^*,\Omega}\,.$$

Here we note that

$$\inf_{\tilde q^* \in Q^*_f}\frac{1}{\alpha^*}\|\tilde q^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega} = -\sup_{\eta^* \in Q^*_{\tilde f}}\Big[-\frac{1}{\alpha^*}\|\eta^*\|^{\alpha^*}_{\alpha^*,\Omega}\Big]\,, \tag{2.12}$$

where

$$Q^*_{\tilde f} := \Big\{\eta^* \in Y^* : \int_\Omega \eta^*\cdot\nabla w\,\mathrm{d}x = \int_\Omega \tilde f w\,\mathrm{d}x \;\text{ for all }\; w \in V_0\Big\}$$

and where $\tilde f := f + \operatorname{div} y^*$. The problem on the right-hand side of (2.12) is similar to the problem $(\mathcal{P}^*)$ and, as a consequence, the duality relation

$$\sup_{\eta^* \in Q^*_{\tilde f}}\Big[-\frac{1}{\alpha^*}\|\eta^*\|^{\alpha^*}_{\alpha^*,\Omega}\Big] = \inf_{w \in V_0}\int_\Omega \Big[\frac{1}{\alpha}|\nabla w|^\alpha - \tilde f w\Big]\,\mathrm{d}x \tag{2.13}$$

holds (the only formal difference to (1.2) is that on the right-hand side we minimize w.r.t. V_0, i.e. no terms involving boundary data appear on the left-hand side). We have

$$\int_\Omega \left[\frac{1}{\alpha}|\nabla w|^\alpha - \tilde{f}w\right] \mathrm{d}x \geq \frac{1}{\alpha}\|\nabla w\|_{\alpha,\Omega}^\alpha - \|\tilde{f}\|_{\alpha^*,\Omega}\|w\|_{\alpha,\Omega}$$

and on V_0 by Poincaré's inequality

$$\|w\|_{\alpha,\Omega} \leq C_\alpha(\Omega)\|\nabla w\|_{\alpha,\Omega}\,.$$

Summing up it is shown that

$$\int_\Omega \left[\frac{1}{\alpha}|\nabla w|^\alpha - \tilde{f}w\right] \mathrm{d}x \geq \inf_{t\geq 0}\left[\frac{1}{\alpha}t^\alpha - \|\tilde{f}\|_{\alpha^*,\Omega}C_\alpha(\Omega)t\right],$$

whose right-hand side attains the lower bound at

$$t_0 = \left[\|\tilde{f}\|_{\alpha,\Omega}C_\alpha(\Omega)\right]^{\frac{1}{\alpha-1}},$$

and inserting this we arrive at

$$\int_\Omega \left[\frac{1}{\alpha}|\nabla w|^\alpha - \tilde{f}w\right] \mathrm{d}x \geq -\frac{1}{\alpha^*}\left[\|\tilde{f}\|_{\alpha^*,\Omega}C_\alpha(\Omega)\right]^{\alpha^*}. \tag{2.14}$$

By (2.12), (2.13) and (2.14) we conclude that

$$\inf_{\tilde{q}^*\in Q_f^*}\frac{1}{\alpha^*}\|\tilde{q}^* - y^*\|_{\alpha^*,\Omega}^{\alpha^*} \leq \frac{1}{\alpha^*}C_\alpha^{\alpha^*}(\Omega)\|f + \operatorname{div} y^*\|_{\alpha^*,\Omega}^{\alpha^*}$$

$$\inf_{\tilde{q}^*\in Q_f^*}\frac{1}{\alpha^*}\|\tilde{q}^* - y^*\|_{\alpha^*,\Omega}^{\alpha^*} \leq \frac{1}{\alpha^*}C_\alpha^{\alpha^*}(\Omega)\|f + \operatorname{div} y^*\|_{\alpha^*,\Omega}^{\alpha^*}$$

and, therefore,

$$\|q^* - y^*\|_{\alpha^*,\Omega} \leq C_\alpha(\Omega)\|f + \operatorname{div} y^*\|_{\alpha^*,\Omega}\,. \tag{2.15}$$

Combining (2.6), (2.11) and (2.15) we obtain the required estimate

$$\begin{aligned}\|\nabla(v-u)\|_{\alpha,\Omega}^\alpha \;\leq\; & \alpha 2^{\alpha-1}\Big[D_\alpha[\nabla v, y^*] + C_\alpha(\Omega)\||y^*|^{\alpha^*-2}y^* - \nabla v\|_{\alpha,\Omega}\|f + \operatorname{div} y^*\|_{\alpha^*,\Omega} \\ & + 2^{2-\alpha^*}(3-\alpha^*)C_\alpha^{\alpha^*}(\Omega)\|f + \operatorname{div} y^*\|_{\alpha^*,\Omega}^{\alpha^*}\Big],\end{aligned} \tag{2.16}$$

where y^* is an arbitrary function in Y^* with α^*-summable divergence. Finally (2.16) together with Young's inequality (a, $b \in \mathbb{R}^n$, $\beta > 0$)

$$ab \leq \frac{\beta^\alpha}{\alpha}|a|^\alpha + \frac{1}{\alpha^*\beta^{\alpha^*}}|b|^{\alpha^*}$$

gives the theorem. □

3 Estimates for the deviation in the case $\alpha < 2$

The counterpart of Theorem 2.1 in the case $\alpha < 2$ reads as

THEOREM 3.1 *Fix $\alpha < 2$, i.e. we have $\alpha^* > 2$. If $\alpha^* \in (2,3)$ or if $\alpha^* \geq 4$, then let $\kappa = (\alpha^*-2)/2$, otherwise let $\kappa = \alpha^* - 2$. With the notation from above we have for any $v \in u_0 + V_0$, for any $y^* \in Y^*$ with α^*-summable divergence and for any $\beta > 0$*

$$\|p^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega} \leq \mathcal{M}_1[\nabla v, y^*, \beta] + \mathcal{M}_2[y^*, \beta]\,, \tag{3.1}$$

where

$$\begin{aligned}
\mathcal{M}_1[\nabla v, y^*, \beta] &= \alpha^* 4^{\alpha^*-1}\Big[D_\alpha[\nabla v, y^*] + \frac{\beta}{2}\||y^*|^{\alpha^*-2}y^* - \nabla v\|^2_{\alpha,\Omega}\Big]\,,\\
\mathcal{M}_2[y^*, \beta] &= \alpha^* 4^{\alpha^*-1}\Big[\frac{1}{2\beta} + \kappa(\|y^*\|_{\alpha,\Omega} + e)^{\alpha^*-2} + (\kappa+1)\|y^*\|^{\alpha^*-2}_{\alpha^*,\Omega} + \kappa\|y^*\|^{\alpha^*-3}_{\alpha^*,\Omega} e\Big] e^2\\
&\quad + 2^{\alpha^*-1} e^{\alpha^*}\,.
\end{aligned}$$

Here the compound functional D_α is given as above and e is chosen such that for the orthogonal projection q^* of y^* on the set Q^*_f we have (compare (2.15)

$$\|q^* - y^*\|_{\alpha^*,\Omega} \leq C_\alpha(\Omega)\|f + \operatorname{div} y^*\|_{\alpha^*,\Omega} =: e\,.$$

REMARK 3.1 i) *We recall Remark 2.1*

ii) *Since y^* is a given function, the value e is directly computable and can be used in the upper bound of $\|p^* - y^*\|$.*

Proof of Theorem 3.1. Passing to the dual problem $(\mathcal{P}^*)$ we consider the uniformly convex functional

$$\big[-I_{\alpha^*}\big][y^*] = \int_\Omega \Big[-\nabla u_0 y^* + \frac{1}{\alpha^*}|y^*|^{\alpha^*} + f u_0\Big]\,\mathrm{d}x\,.$$

Similar to the last section we have for all $q^* \in Q^*_f$ ($p^* \in Q^*_f$ denoting the dual solution)

$$\begin{aligned}
&\big[-I_{\alpha^*}\big][q^*] + \big[-I_{\alpha^*}\big][p^*] - 2\big[-I_{\alpha^*}\big]\Big[\frac{p^*+q^*}{2}\Big]\\
&= \frac{1}{\alpha^*}\int_\Omega \Big[|q^*|^{\alpha^*} + |p^*|^{\alpha^*} - 2\Big[\frac{|q^*+p^*|}{2}\Big]^{\alpha^*}\Big]\,\mathrm{d}x\\
&= \frac{1}{\alpha^* 2^{\alpha^*-1}}\|q^* - p^*\|^{\alpha^*}_{\alpha^*,\Omega}\,,
\end{aligned}$$

where we made use of (2.4) with α replaced by α^*. Instead of (2.5) we then get

$$\|q^* - p^*\|^{\alpha^*}_{\alpha^*,\Omega} \leq \alpha^* 2^{\alpha^*-1}\Big[I_{\alpha^*}[p^*] - I_{\alpha^*}[q^*]\Big] \quad \text{for all } q^* \in Q^*_f\,. \tag{3.2}$$

In (3.2) we estimate

$$I_{\alpha^*}[p^*] = J_\alpha[u] \leq J_\alpha[v] \quad \text{for all } v \in u_0 + V_0$$

and as above one arrives at

$$\|p^* - q^*\|^{\alpha^*}_{\alpha^*,\Omega} \leq \alpha^* 2^{\alpha^*-1} D_\alpha[\nabla v, q^*] \quad \text{for all } v \in u_0 + V_0\,. \tag{3.3}$$

REMARK 3.2 i) *With (3.3) the desirable upper bound for the deviation from p^* in the dual energy norm is found.*

ii) *We recall Remark 2.2.*

Now, in order to eliminate the constraint $q^* \in Q^*_f$, let y^* be a function in Y^*. Then

$$\|p^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega} \leq 2^{\alpha^*-1}\big[\|p^* - q^*\|^{\alpha^*}_{\alpha^*,\Omega} + \|y^* - q^*\|^{\alpha^*}_{\alpha^*,\Omega}\big]$$

yields using (3.3)

$$\|p^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega} \leq \alpha^* 4^{\alpha^*-1} D_\alpha[\nabla v, q^*] + 2^{\alpha^*-1}\|y^* - q^*\|^{\alpha^*}_{\alpha^*,\Omega} \,. \tag{3.4}$$

Exactly as before (recall (2.9)) we have

$$\begin{aligned} D_\alpha[\nabla v, q^*] \;\leq\;& D_\alpha[\nabla v, y^*] + \int_\Omega \big(|y^*|^{\alpha^*-2}y^* - \nabla v\big)\cdot(q^* - y^*)\,\mathrm{d}x \\ &+ \int_\Omega \big(|q^*|^{\alpha^*-2}q^* - |y^*|^{\alpha^*-2}y^*\big)\cdot(q^* - y^*)\,\mathrm{d}x \end{aligned} \tag{3.5}$$

and, as a substitude for (2.10) we claim that

$$\begin{aligned} &\int_\Omega \big[|q^*|^{\alpha^*-2}q^* - |y^*|^{\alpha^*-2}y^*\big](q^* - y^*)\,\mathrm{d}x \\ \leq\;& \Bigg[\kappa\big[\|y^*\|_{\alpha^*,\Omega} + \|q^* - y^*\|_{\alpha^*,\Omega}\big]^{\alpha^*-2} + (\kappa+1)\|y^*\|^{\alpha^*-2}_{\alpha^*,\Omega} \\ &+\kappa\|y^*\|^{\alpha^*-3}_{\alpha^*,\Omega}\|q^* - y^*\|_{\alpha^*,\Omega}\Bigg]\|q^* - y^*\|^2_{\alpha^*,\Omega} \,. \end{aligned} \tag{3.6}$$

In order to verify (3.6), Corollary 6.2 is used with the result (again the claim is trivial if either $q^* = 0$ or $y^* = 0$)

$$\int_\Omega \big[|q^*|^{\alpha^*-2}q^* - |y^*|^{\alpha^*-2}y^*\big](q^* - y^*)\,\mathrm{d}x \leq I_1 + I_2 + I_3 \,,$$

where

$$\begin{aligned} I_1 &= \kappa\int_\Omega |q^*|^{\alpha^*-2}|q^* - y^*|^2\,\mathrm{d}x \,, \\ I_2 &= \kappa\int_\Omega |q^*||y^*|^{\alpha^*-3}|q^* - y^*|^2\,\mathrm{d}x \,, \\ I_3 &= \int_\Omega |y^*|^{\alpha^*-2}|q^* - y^*|^2\,\mathrm{d}x \,. \end{aligned}$$

Hölder's inequality implies

$$\begin{aligned} I_1 &\leq \kappa\|q^*\|^{\alpha^*-2}_{\alpha^*,\Omega}\|q^* - y^*\|^2_{\alpha^*,\Omega} \,, \\ I_3 &\leq \|y^*\|^{\alpha^*-2}_{\alpha^*,\Omega}\|q^* - y^*\|^2_{\alpha^*,\Omega} \,. \end{aligned} \tag{3.7}$$

Here, discussing I_1 we additionally use the fact, that $\alpha^* \geq 2$, i.e. $|\cdot|^{\alpha^*-2}$ is an increasing function, thus we have

$$\|q^*\|_{\alpha^*,\Omega}^{\alpha^*-2} \leq \Big[\|y^*\|_{\alpha^*,\Omega} + \|q^* - y^*\|_{\alpha^*,\Omega}\Big]^{\alpha^*-2}$$

and, as a consequence,

$$I_1 \leq \kappa\Big[\|y^*\|_{\alpha^*,\Omega} + \|q^* - y^*\|_{\alpha^*,\Omega}\Big]^{\alpha^*-2} \|q^* - y^*\|_{\alpha^*,\Omega}^2 \,. \tag{3.8}$$

Finally, I_2 is estimated with the help of

$$\begin{aligned}
&\int_\Omega |q^*||y^*|^{\alpha^*-3}|q^* - y^*|^2 \,\mathrm{d}x \\
\leq\ & \int_\Omega |y^*|^{\alpha^*-3}|q^* - y^*|^3 \,\mathrm{d}x + \int_\Omega |y^*|^{\alpha^*-2}|q^* - y^*|^2 \,\mathrm{d}x \\
\leq\ & \Big[\int_\Omega |y^*|^{\alpha^*} \,\mathrm{d}x\Big]^{\frac{\alpha^*-3}{\alpha^*}} \Big[\int_\Omega |q^* - y^*|^{\alpha^*} \,\mathrm{d}x\Big]^{\frac{3}{\alpha^*}} + \|y^*\|_{\alpha^*,\Omega}^{\alpha^*-2}\|y^* - q^*\|_{\alpha^*,\Omega}^2 \\
=\ & \|y^*\|_{\alpha^*,\Omega}^{\alpha^*-3}\|y^* - q^*\|_{\alpha^*,\Omega}^3 + \|y^*\|_{\alpha^*,\Omega}^{\alpha^*-2}\|y^* - q^*\|_{\alpha^*,\Omega}^2
\end{aligned} \tag{3.9}$$

and the claim (3.6) follows from (3.7), (3.8) and (3.9).

Next we choose q^* according to the statement of Theorem 3.1 such that (3.5) and (3.6) imply

$$\begin{aligned}
D_\alpha[\nabla v, q^*] \ \leq\ & D_\alpha[\nabla v, y^*] + \||y^*|^{\alpha^*-2}y^* - \nabla v\|_{\alpha,\Omega} e + \kappa(\|y^*\|_{\alpha^*,\Omega} + e)^{\alpha^*-2} e^2 \\
& + (\kappa+1)\|y^*\|_{\alpha^*,\Omega}^{\alpha^*-2} e^2 + \kappa\|y^*\|_{\alpha^*,\Omega}^{\alpha^*-3} e^3 \,.
\end{aligned}$$

This, together with (3.4) and Young's inequality completes the proof of Theorem 3.1. □

4 Mixed Dirichlet-Neumann boundary conditions

In this section we like to point out, that the above results remain (up to minor changes) valid, if we consider boundary conditions of mixed Dirichlet and Neumann type. We just consider the case $\alpha \geq 2$, the case $\alpha < 2$ is left to the reader.

Suppose now that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are nonintersecting measurable sets and where $\mathcal{H}^{n-1}(\Gamma_1) > 0$. Suppose further that $F \in L^{\alpha^*}(\Gamma_2)$ and that

$$V_0 := \big\{ w \in W^1_\alpha(\Omega) : \ w = 0 \ \text{ on } \ \Gamma_1 \big\} \,.$$

Then the variational problem under consideration is given by

$$J_\alpha[w] := \int_\Omega \Big[\frac{1}{\alpha}|\nabla w|^\alpha - fw\Big] \,\mathrm{d}x - \int_{\Gamma_2} Fw \,\mathrm{d}\mathcal{H}^{n-1} \to \min \tag{$\mathcal{P}$}$$

among all comparison functions w of class $u_0 + V_0$, i.e. in the case of a smooth solution u we now have

$$\begin{aligned} \operatorname{div}\big(|\nabla u|^{\alpha-2}\nabla u\big) + f &= 0 \quad \text{in} \quad \Omega\,, \\ u &= u_0 \quad \text{on} \quad \Gamma_1\,, \\ |\nabla u|^{\alpha-2}\nabla u \cdot \nu &= F \quad \text{on} \quad \Gamma_2\,, \end{aligned}$$

where ν denotes the outward unit normal to Γ_2. The weak formulation of the Euler equation reads as

$$\int_\Omega \big(|\nabla u|^{\alpha-2}\nabla u \cdot \nabla w - f w\big)\,\mathrm{d}x - \int_{\Gamma_2} F w \,\mathrm{d}\mathcal{H}^{n-1} = 0 \qquad \text{for any } \ w \in V_0\,.$$

The dual maximizing problem

$$I_{\alpha^*}[y^*] \to \max \qquad \text{in } \ Y^* \tag{$\mathcal{P}^*$}$$

is defined through the modified functional

$$I_{\alpha^*}[y^*] := \begin{cases} \displaystyle\int_\Omega \Big[\nabla u_0 \cdot y^* - \frac{1}{\alpha^*}|y^*|^{\alpha^*} - f u_0\Big]\,\mathrm{d}x - \int_{\Gamma_2} F u_0 \,\mathrm{d}\mathcal{H}^{n-1} & \text{if } \ y^* \in Q^*_{f,F}\,, \\ -\infty & \text{if } \ y^* \notin Q^*_{f,F}\,, \end{cases}$$

where

$$Q^*_{f,F} := \Big\{ y^* \in Y^* : \int_\Omega y^* \cdot \nabla w \,\mathrm{d}x = \int_\Omega f w \,\mathrm{d}x + \int_{\Gamma_2} F w \,\mathrm{d}\mathcal{H}^{n-1} \ \text{ for all } w \in V_0 \Big\}.$$

For any $v \in u_0 + V_0$ and for any $q^* \in Q^*_{f,F}$ one obtains exactly as above

$$\begin{aligned} J_\alpha[v] - I_{\alpha^*}[q^*] \;&=\; \int_\Omega \Big[\frac{1}{\alpha}|\nabla v|^\alpha + \frac{1}{\alpha^*}|q^*|^{\alpha^*} - \nabla u_0 \cdot q^* - f(v-u_0)\Big]\,\mathrm{d}x \\ &\quad - \int_{\Gamma_2} F(v-u_0)\,\mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

and the definition of $Q^*_{f,F}$ gives

$$\int_{\Gamma_2} F(v-u_0)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega f(v-u_0)\,\mathrm{d}x = \int_\Omega q^* \cdot \nabla(v-u_0)\,\mathrm{d}x\,,$$

thus we have reproduced (2.7) for any $v \in u_0 + V_0$ and for any $q^* \in Q^*_{f,F}$.

In order to consider the orthogonal projection q^* of $y^* \in Y^*$ on $Q^*_{f,F}$, we assume in the following that $y^* \cdot \nu$ is of class $L^{\alpha^*}(\Gamma_2)$. Then the counterpart of (2.12) is

$$\inf_{\tilde{q}^* \in Q^*_{f,F}} \frac{1}{\alpha^*}\|\tilde{q}^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega} = - \sup_{\eta^* \in Q^*_{\tilde{f},\tilde{F}}} \Big[- \frac{1}{\alpha^*}\|\eta^*\|^{\alpha^*}_{\alpha^*,\Omega}\Big]\,,$$

where

$$Q^*_{\tilde{f},\tilde{F}} := \Big\{ \eta^* \in Y^* : \int_\Omega \eta^* \cdot \nabla w \,\mathrm{d}x = \int_\Omega \tilde{f} w \,\mathrm{d}x + \int_{\Gamma_2} \tilde{F} w \,\mathrm{d}\mathcal{H}^{n-1} \ \text{ for all } \ w \in V_0 \Big\}\,,$$

where $\tilde{f} := f + \operatorname{div} y^*$ and where $\tilde{F} = F - y^* \cdot \nu$. Using

$$\|w\|_{\alpha,\Omega} + \|w\|_{\alpha,\Gamma_2} \le C_\alpha(\Omega,\Gamma_2)\|w\|_{\alpha,\Omega}$$

(for the generalized Poincaré type inequality we refer to [Mo], Theorem 3.6.4) we obtain

$$\inf_{\tilde{q}^* \in Q^*_{f,F}} \frac{1}{\alpha^*}\|\tilde{q}^* - y^*\|^{\alpha^*}_{\alpha^*,\Omega} \le \frac{1}{\alpha^*} C_\alpha^{\alpha^*}(\Omega)\big(\|f + \operatorname{div} y^*\|^{\alpha^*}_{\alpha^*,\Omega} + \|y^* \cdot \nu - F\|^{\alpha^*}_{\alpha^*,\Gamma_2}\big)$$

and Theorem 2.1 holds in the case of mixed Dirichlet-Neumann boundary conditions with $\mathcal{M}_2$ replaced by

$$\mathcal{M}_2[y^*,\beta] = C_\alpha^{\alpha^*}(\Omega,\Gamma_2)\Big[\frac{1}{\alpha^*\beta^{\alpha^*}} + 2^{2-\alpha^*}(3-\alpha^*)\Big]\big(\|f + \operatorname{div} y^*\|^{\alpha^*}_{\alpha^*,\Omega} + \|y^* \cdot \nu - F\|^{\alpha^*}_{\alpha^*,\Gamma_2}\big)$$

and where we additionally assume that $y^* \cdot \nu \in L^{\alpha^*}(\Gamma_2)$.

5 Pure Neumann boundary conditions

The case of pure Neumann boundary conditions is easily discussed with the arguments of the last section. We have $\Gamma_1 = \emptyset$ and $\Gamma_2 = \partial\Omega$. Now V_0 is the set of functions $w \in W^1_\alpha(\Omega)$ such that

$$-\!\!\!\!\!\!\int_\Omega w \,\mathrm{d}x = 0\,.$$

With these slight changes, the formulation of the problem formally is the same as in the last section, we just mention that here

$$I_{\alpha^*}[y^*] := \begin{cases} \displaystyle\int_\Omega -\frac{1}{\alpha^*}|y^*|^{\alpha^*}\,\mathrm{d}x & \text{if} \quad y^* \in Q^*_{f,F}, \\ \quad -\infty & \text{if} \quad y^* \notin Q^*_{f,F}\,, \end{cases}$$

and that $Q^*_{f,F}$ is defined as before (w.r.t. the new space V_0). The following arguments of the last section remain unchanged if we formally let $u_0 = 0$ and if we refer to the Poincaré type inequality as given in Theorem 3.6.5 of [Mo].

6 Auxiliary results

Let us finally prove the auxiliary estimates used throughout this paper. We start with

PROPOSITION 6.1 i) *For any fixed $s \in (0,1)$, for any fixed $s \ge 2$ and for any ξ_1, $\xi_2 > 0$ we have*

$$\big|\xi_2^s - \xi_1^s\big| \le \frac{s}{2}\big[\xi_1^{s-1} + \xi_2^{s-1}\big]|\xi_2 - \xi_1|\,.$$

ii) *For any fixed $s \in [1,2)$ and for any ξ_1, $\xi_2 > 0$ we have*

$$\big|\xi_2^s - \xi_1^s\big| \le s\big[\xi_1^{s-1} + \xi_2^{s-1}\big]|\xi_2 - \xi_1|\,.$$

Proof. Ad i). We assume w.l.o.g. that $\xi_1 < \xi_2$, we let $t := s - 1$ and observe that for the range of s in this case the function $\xi^t \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function, i.e. the secant through the points (ξ_1, ξ_1^t), (ξ_2, ξ_2^t) lies above the graph of ξ^t, $\xi_1 \le \xi \le \xi_2$. This gives

$$\begin{aligned} \int_{\xi_1}^{\xi_2} \xi^t \, \mathrm{d}t &\le \int_{\xi_1}^{\xi_2} \Big[\xi_1^t + \frac{\xi_2^t - \xi_1^t}{\xi_2 - \xi_1} (\xi - \xi_1) \Big] \, \mathrm{d}t \\ &= \frac{\xi_1^t + \xi_2^t}{2} (\xi_2 - \xi_1) \, , \end{aligned}$$

and, as a consequence,

$$\frac{1}{t+1} (\xi_2^{t+1} - \xi_1^{t+1}) \le \frac{\xi_1^t + \xi_2^t}{2} (\xi_2 - \xi_1) \, ,$$

which is the first claim.

Ad ii). With the notation from above we now note the ξ^t is a concave function. This means that the graph of ξ^t lies below the tangent through the point $(\xi_1 + \xi_2)/2, ((\xi_1 + \xi_2)/2)^t)$ and one obtains for $\xi_1 \le \xi \le \xi_2$

$$\xi^t \le \Big[\frac{\xi_1 + \xi_2}{2} \Big]^t + t \Big[\frac{\xi_1 + \xi_2}{2} \Big]^{t-1} \Big[\xi - \frac{\xi_1 + \xi_2}{2} \Big] \, .$$

Integration yields in this case

$$\frac{1}{t+1} (\xi_2^{t+1} - \xi_1^{t+1}) \le \Big[\frac{\xi_1 + \xi_2}{2} \Big]^t (\xi_2 - \xi_2) \le (\xi_1^t + \xi_2^t)(\xi_2 - \xi_1)$$

and the second claim is established as well. □

The first Corollary of Proposition 6.1 is needed in the superquadratic case:

COROLLARY 6.1 *Fix $\theta \in [0,1)$. Then for any a, $b \in \mathbb{R}^n$ we have*

$$T(a,b) := \Big| \frac{a}{|a|^\theta} - \frac{b}{|b|^\theta} \Big| |a - b| \le 2^\theta (\theta + 1) |b - a|^{2-\theta} \, .$$

Proof. Consider a, $b \in \mathbb{R}^n$, $|a|$, $|b| \neq 0$ and suppose w.l.o.g. that $|a| \le |b|$. Then we have

$$T(a,b) = \Big| \frac{a(|b|^\theta - |a|^\theta) + |a|^\theta (a - b)}{|a|^\theta |b|^\theta} \Big| |b - a|$$

and since by Proposition 6.1, i),

$$\big| |b|^\theta - |a|^\theta \big| \le \frac{\theta}{2} (|a|^{\theta-1} + |b|^{\theta-1}) |b - a|$$

we arrive at the estimate

$$\begin{aligned} T(a,b) &\le |a|^{1-\theta} \frac{\theta}{2} (|a|^{\theta-1} + |b|^{\theta-1}) \frac{|b-a|^2}{|b|^\theta} + \frac{|b-a|^2}{|b|^\theta} \\ &\le \Bigg[\frac{\theta}{2} \Big[1 + \Big| \frac{b}{a} \Big|^{\theta-1} \Big] + 1 \Bigg] \frac{|b-a|^2}{|b|^\theta} \, . \end{aligned}$$

Thus we have (recall $|a| \le |b|$ and $\theta - 1 \in [-1, 0)$)

$$T(a,b) \le (\theta+1)\frac{|b-a|^2}{|b|^\theta} \le (\theta+1)\frac{|b-a|^\theta}{|b|^\theta}|b-a|^{2-\theta}$$

and on account of

$$\frac{|b-a|^\theta}{|b|^\theta} \le \frac{(|b|+|a|)^\theta}{|b|^\theta} \le \left[1+\frac{|a|}{|b|}\right]^\theta \le 2^\theta$$

we have proved our claim. □

In order to discuss the subquadratic situation, we need

COROLLARY 6.2 *Fix $s > 0$ and let $\kappa(s) = s/2$ if $s \in (0,1)$ or if $s \ge 2$. Otherwise let $\kappa(s) = s$. Then for any a, $b \in \mathbb{R}^n$ we have*

$$T(a,b) := \big|a|a|^s - b|b|^s\big||a-b| \le \kappa(s)|a||a-b|^2\big[|a|^{s-1}+|b|^{s-1}\big] + |b|^s|a-b|^2 .$$

Proof. We just note that

$$\big|a|a|^s - b|b|^s\big||a-b| = \big|a(|a|^s - |b|^s) + (a-b)|b|^s\big||a-b|$$

and apply Proposition 6.1. □

References

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London 1975.

[AiOd] Ainsworth, M., Oden, J. T., A posteriori error estimation in finite element analysis. Wiley and Sons, New York 2000.

[BaRe] Babuška, I., Rheinboldt, W.C., Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. 15 (1978), 736–754.

[BaRa] Bangerth, W., Rannacher, R., Adaptive finite element methods for differential equations. Birkhäuser, Berlin 2003.

[ET] Ekeland, I., Temam, R., Convex analysis and variational problems. North-Holland, Amsterdam 1976.

[GaGrZa] Gaevskii, H., Gröger, K., Zacharias, K., Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie-Verlag, Berlin 1974.

[LaUr] Ladyzhenskaya, O. A., Ural'tseva, N. N., Linear and quasilinear elliptic equations. Academic Press, New York 1968.

[Mi] Mikhlin, S. G., Variational methods in mathematical physics. Pergamon, Oxford 1964.

[Mo] Morrey, C. B., Multiple integrals in the calculus of variations. Grundlehren der math. Wiss. in Einzeldarstellungen 130, Springer, Berlin-Heidelberg-New York 1966.

[MoMy] Mosolov, P., Myasnokov, V., Mechanics of rigid plastic bodies. Nauka, Moscow 1981 (in Russian).

[Re1] Repin, S., A posteriori error estimation for nonlinear variational problems by duality theory. Zap. Nauch. Sem. POMI 243 (1997), 201–214.

[Re2] Repin, S., A posteriori error estimation for variational problems with uniformly convex functionals. Math. Comp. 69 (230) (2000), 481–500.

[So] Sobolev, S. L., Some applications of functional analysis in mathematical physics (3^{rd} edition). Translations of Mathematical Monographs 90, American Mathematical Society, Providence, Rhode Island 1991 (1^{st} edition: Izdt. Leningrad Gos. Univ., Leningrad 1955).

[Ve] Verfürth, R., A review of a posteriori error estimation and adaptive mesh-refinement techniques. Wiley and Sons, Teubner, New York 1996.