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NEW VERSION OF THE DANIELL-STONE-RIESZ REPRESENTATION THEOREM

HEINZ KÖNIG

Abstract. The traditional representation theorems after Daniell-Stone and Riesz were in a kind of separate existence until Pollard-Topsøe 1975 and Topsøe 1976 were the first to put them under common roofs. In the same spirit the present article wants to obtain a unified representation theorem in the context of the author's work in measure and integration. It is an *inner* theorem like the previous ones. The basis is the recent comprehensive inner Daniell-Stone theorem, so that in particular there are no a priori assumptions on the *additive* behaviour of the data.

Dedicated to the Memory of Helmut H. Schaefer

1. Introduction

The present article wants to put an adequate unified result on top of the collection of representation theorems of Daniell-Stone and Riesz type obtained in the context of the author's work in measure and integration described in [12][15]. We shall concentrate on the *inner* development which turned out to be more profound than the outer one. We recall that its basic concepts are the *inner* $\bullet$ *premeasures* $\vartheta : \mathfrak{S} \to [0,\infty[$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in a nonvoid set X and their *inner* $\bullet$ *extensions* ($\bullet = \star\sigma\tau$ with $\star =$ finite, $\sigma =$ sequential, $\tau =$ nonsequential), and that its basic devices are the *inner* $\bullet$ *envelopes* $\vartheta_\bullet : \mathfrak{P}(X) \to [0,\infty]$ of the isotone set functions $\vartheta : \mathfrak{S} \to [0,\infty[$ with $\vartheta(\varnothing) = 0$. We shall often make free use of the concepts and results set up so far.

The basis is the inner Daniell-Stone representation procedure in [12] section 7. It assumed a function system $E \subset [0,\infty[^X$ with $0 \in E$ and an isotone functional $I : E \to [0,\infty[$ with $I(0) = 0$, for the most part such that E is positive-homogeneous and a lattice under the pointwise max and min operations $\vee\wedge$ which is *Stonean*: $f \in E \Rightarrow f \wedge t, (f-t)^+ \in E$ for $0 < t < \infty$. But we emphasize that there are no a priori assumptions relative to the *additive* behaviour of E and I. Then

$$\mathrm{Inn}(E) := \{\,[f \geqq t] : f \in E \text{ and } 0 < t < \infty\}$$

is a lattice in X with $\varnothing \in \mathrm{Inn}(E)$. On $\mathrm{Inn}(E)$ one defines the *inner sources* of I to be the isotone set functions $\varphi : \mathrm{Inn}(E) \to [0,\infty[$ with $\varphi(\varnothing) = 0$ which represent I via the *Choquet integral*: $I(f) = ⨍ f d\varphi$ for all $f \in E$. The inner sources fulfil $I_\star(\chi_.) \leqq \varphi \leqq I^\star(\chi_.)$ on $\mathrm{Inn}(E)$, with the usual

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crude envelopes $I_\star$ and $I^\star$ of I. For $\bullet = \sigma\tau$ one defines I to be an *inner* $\bullet$ *preintegral* iff there exist inner sources of I which are inner $\bullet$ premeasures. The subsequent *inner* $\bullet$ *representation theorem* [12] 7.6 characterized these inner $\bullet$ preintegrals and presented their basic properties. It is in terms of the *inner* $\bullet$ *envelopes* $I_\bullet$ of I and their satellites. We recall that I *Stonean* means that $I(f) = I(f \wedge t) + I((f-t)^+)$ for all $f \in E$ and $0 < t < \infty$.

1.1 INNER $\bullet$ REPRESENTATION THEOREM. *Assume that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a positive-homogeneous Stonean lattice and* $I : E \to [0,\infty[$ *with* $I(0) = 0$ *isotone. Then for* $\bullet = \sigma\tau$ *one has the equivalences*

I is an inner $\bullet$ preintegral

$\iff$ *I is supermodular and Stonean and downward $\bullet$ continuous; and $I(v) \leqq I(u) + I_\bullet(v-u)$ for all $u \leqq v$ in E*

$\iff$ *I is supermodular and Stonean and downward $\bullet$ continuous at 0; and $I(v) \leqq I(u) + I_\bullet^v(v-u)$ for all $u \leqq v$ in E.*

In this case there is a unique inner source of I which is an inner $\bullet$ premeasure, and it is in fact $\varphi = I^\star(\chi_.)|\mathrm{Inn}(E)$. This φ fulfils $I_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$. Moreover the members of $E_\bullet$ are measurable $\mathfrak{C}(\varphi_\bullet)$.

In the sequel we want to expand the above Daniell-Stone representation theorem so that it comprises the Riesz representation theorem in its recent comprehensive versions. For this purpose we assume besides E and I an *additional* lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in X. The aim is to characterize those isotone functionals $I : E \to [0,\infty[$ with $I(0) = 0$ for which there exist inner $\bullet$ premeasures $\vartheta : \mathfrak{S} \to [0,\infty[$ (or rather a *unique* one) which represent I: $I(f) = ⨍ f d\vartheta_\bullet$ for all $f \in E$, a formulation which makes sense for arbitrary $\mathfrak{S}$. In this context it is quite clear that one cannot expect substantial results without adequate connections between the two lattices $\mathrm{Inn}(E)$ and $\mathfrak{S}$. Justified by previous particular situations and by success, we shall impose the relations

$$(\bullet) \quad \mathfrak{S} \subset (\mathrm{Inn}(E))_\bullet \quad \text{and} \quad \mathrm{Inn}(E) \subset \mathfrak{S}\top\mathfrak{S}_\bullet,$$

with $\top$ the transporter; in the terms of [12] section 4 this means that $\mathfrak{S}$ and $\mathrm{Inn}(E)\perp := \{[f < t] : f \in E \text{ and } 0 < t < \infty\}$ form a $\bullet$ *complemental couple.* Besides we need a simple formulation which expresses that the functional I is *concentrated on* $\mathfrak{S}$. There are three related candidates, which we present with some obvious implications.

1.2 REMARK. For $E \subset [0,\infty[^X$ with $0 \in E$ (without further assumptions) and $I : E \to [0,\infty[$ isotone with $I(0) = 0$, and a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in X, consider the properties

[↓] for any $f \in E$ and $\varepsilon > 0$ there exists $S \in \mathfrak{S}$ such that all $u \in E$ with $u \leqq f$ and $u|S = 0$ fulfil $I(u) \leqq \varepsilon$,

[↑] for any $f \in E$ and $\varepsilon > 0$ there exists $S \in \mathfrak{S}$ such that all $u \in E$ with $u \leqq f$ and $u|S = f|S$ fulfil $I(f) \leqq I(u) + \varepsilon$,

[↕] for any $f \in E$ and $\varepsilon > 0$ there exists $S \in \mathfrak{S}$ such that all $u, v \in E$ with $u \leqq v \leqq f$ and $u|S = v|S$ fulfil $I(v) \leqq I(u) + \varepsilon$.

Then [↕] $\Rightarrow$ [↓] and [↕] $\Rightarrow$ [↑]. Moreover [↓] $\Rightarrow$ [↕] and [↑] $\Rightarrow$ [↕] whenever $u \leqq v$ in E implies that $v - u \in E$ and $I(v-u) = I(v) - I(u)$.

After this we can formulate the present main theorem.

1.3 THEOREM. *Assume that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a positive-homogeneous Stonean lattice, that* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *is a lattice in* X*, and that* $\operatorname{Inn}(E)$ *and* $\mathfrak{S}$ *fulfil* $(\bullet)$ *for some* $\bullet = \sigma\tau$. *Let* $I : E \to [0,\infty[$ *with* $I(0) = 0$ *be isotone. Then one has the equivalences*

There exists an inner $\bullet$ *premeasure* $\vartheta : \mathfrak{S} \to [0,\infty[$ *such that* $I(f) = ⨍ f d\vartheta_\bullet$ *for all* $f \in E$

$\Longleftrightarrow$ *I is an inner* $\bullet$ *preintegral and fulfils* $[\uparrow]$

$\Longleftrightarrow$ *I is an inner* $\bullet$ *preintegral and fulfils* $[\updownarrow]$.

In this case there is a unique such ϑ*, and it is in fact* $\vartheta = I^\star(\chi_.)|\mathfrak{S}$. *Moreover* ϑ *and the inner* $\bullet$ *premeasure* $\varphi = I^\star(\chi_.)|\operatorname{Inn}(E)$ *from* 1.1 *fulfil* $\vartheta_\bullet = \varphi_\bullet$.

It follows that the maximal inner $\bullet$ extension $\Theta = \vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ of ϑ and the maximal inner $\bullet$ extension $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ of φ are identical, with the identical domain $\mathfrak{C}(\vartheta_\bullet) = \mathfrak{C}(\varphi_\bullet)$. Hence after the final assertion of 1.1 the members of E and even of $E_\bullet$ are measurable $\mathfrak{C}(\vartheta_\bullet)$. Thus the representation of I in 1.3 means that the functions $f \in E$ are integrable Θ and fulfil $I(f) = \int f d\Theta$.

The above main theorem has several important specializations which will be treated in section 3. Then section 4 will be devoted to the comparison with the respective results in the earlier literature (where the author apologizes in advance for possible omissions). But first of all we turn to the proof.

2. PROOF OF THE MAIN THEOREM

We assume as before that $E \subset [0,\infty[^X$ with $0 \in E$ is a positive-homogeneous Stonean lattice, that $\mathfrak{S}$ is a lattice with $\varnothing \in \mathfrak{S}$ in X, and that $\operatorname{Inn}(E)$ and $\mathfrak{S}$ fulfil $(\bullet)$ for some $\bullet = \sigma\tau$.

2.1 PROPOSITION. *Let* $I : E \to [0,\infty[$ *be isotone with* $I(0) = 0$. *Then one has the equivalence*

There exists an inner $\bullet$ *premeasure* $\vartheta : \mathfrak{S} \to [0,\infty[$ *such that* $I(f) = ⨍ f d\vartheta_\bullet$ *for all* $f \in E$

$\Longleftrightarrow$ *I is an inner* $\bullet$ *preintegral, and its* $\varphi = I^\star(\chi_.)|\operatorname{Inn}(E)$ *satisfies* $[\bullet]$ $\varphi_\bullet$ *is inner regular* $\mathfrak{S}_\bullet$ (note that $\mathfrak{S}_\bullet \subset (\operatorname{Inn}(E))_\bullet$).

In this case there is a unique such ϑ*, and it is in fact* $\vartheta = I^\star(\chi_.)|\mathfrak{S}$. *Moreover* $\vartheta_\bullet = \varphi_\bullet$.

The proof will be based on the lemma [14] 1.6 which follows.

2.2 LEMMA. *Let* $\varphi : \mathfrak{S} \to [0,\infty[$ *be an inner* $\bullet$ *premeasure on the lattice* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *in* X. *Assume that* $\mathfrak{R}$ *is a lattice in* X *with* $\varnothing \in \mathfrak{R} \subset \mathfrak{S}\top\mathfrak{S}_\bullet$ *such that* $\varphi_\bullet|\mathfrak{R} < \infty$ *and that* $\varphi_\bullet$ *is inner regular* $\mathfrak{R}_\bullet$. *Then* $\vartheta := \varphi_\bullet|\mathfrak{R}$ *is an inner* $\bullet$ *premeasure and fulfils* $\vartheta_\bullet = \varphi_\bullet$.

Proof of 2.1. $\Leftarrow$) We invoke 2.2 for $\varphi : \operatorname{Inn}(E) \to [0,\infty[$ and $\mathfrak{S}$. The assumptions are fulfilled in view of $(\bullet)$ and $[\bullet]$. It follows that $\vartheta := \varphi_\bullet|\mathfrak{S}$ is an inner $\bullet$ premeasure and $\vartheta_\bullet = \varphi_\bullet$. For $f \in E$ hence $I(f) = ⨍ f d\varphi = ⨍ f d\varphi_\bullet = ⨍ f d\vartheta_\bullet$.

$\Rightarrow$) We invoke 2.2 for $\vartheta : \mathfrak{S} \to [0,\infty[$ and $\operatorname{Inn}(E)$. The assumptions are fulfilled in view of $(\bullet)$, and since $I(f) = \int_{0\leftarrow}^{\to\infty} \vartheta_\bullet([f \geqq t])dt < \infty$ and hence $\vartheta_\bullet([f \geqq t]) < \infty$ for $f \in E$ and $0 < t < \infty$. It follows that $\varphi := \vartheta_\bullet|\operatorname{Inn}(E)$

is an inner $\bullet$ premeasure and $\varphi_\bullet = \vartheta_\bullet$. For $f \in E$ hence $I(f) = \fint f d\vartheta_\bullet = \fint f d\varphi_\bullet = \fint f d\varphi$. Thus I is an inner $\bullet$ preintegral, and $\varphi_\bullet = \vartheta_\bullet$ implies ($\bullet$). Moreover the uniqueness of ϑ is clear.

It remains to prove that $\vartheta = I^\star(\chi_.)|\mathfrak{S}$. We invoke [11] 3.2.Inn), in the old notations $\mathrm{Inn}(E) = \geqq (E)$ and $\mathfrak{t}(E) := \{A \subset X : \chi_A \in E\}$, to obtain

$$(\mathrm{Inn}(E))_\bullet = \mathrm{Inn}(E_\bullet) = \mathfrak{t}(E_\bullet) = \{A \subset X : \chi_A \in E_\bullet\},$$

so that $A \in \mathfrak{S} \subset (\mathrm{Inn}(E))_\bullet$ implies that $\chi_A \in E_\bullet$. It follows that

$$\vartheta(A) = \vartheta_\bullet(A) = \varphi_\bullet(A) = I_\bullet(\chi_A) = I^\star(\chi_A),$$

with the last two equalities from the inner $\bullet$ representation theorem 1.1 combined with [12] 5.10.6.Inn). $\square$

2.3 Proposition. *Let $I : E \to [0, \infty[$ be an inner $\bullet$ preintegral and $\varphi = I^\star(\chi_.)|\mathrm{Inn}(E)$. Then* [$\bullet$] *from* 2.1 *is equivalent to both* [$\uparrow$] *and* [$\updownarrow$].

The two propositions 2.1 and 2.3 combine to furnish the main theorem 1.3. We also note an obvious consequence of 2.3 which confirms the relation of 1.3 with 1.1.

2.4 Addendum to 1.3. *Let I be an inner $\bullet$ preintegral. Then in case $\mathfrak{S} = \mathrm{Inn}(E)$ the conditions* [$\downarrow$][$\uparrow$][$\updownarrow$] *are fulfilled.*

Proof of 2.3. [$\updownarrow$] $\Rightarrow$ [$\uparrow$] is obvious. Thus we have to prove [$\uparrow$] $\Rightarrow$ [$\bullet$] and [$\bullet$] $\Rightarrow$ [$\updownarrow$].

Proof of [$\uparrow$] $\Rightarrow$ [$\bullet$]. To be shown is that $\varphi_\bullet$ is inner regular $\mathfrak{S}_\bullet \subset (\mathrm{Inn}(E))_\bullet$ on $(\mathrm{Inn}(E))_\bullet$. Thus fix $A \in (\mathrm{Inn}(E))_\bullet = \mathrm{Inn}(E_\bullet) = \mathfrak{t}(E_\bullet)$ and hence with $\chi_A \in E_\bullet$, as in the proof of 2.1, and $\varepsilon > 0$. i) Take an $f \in E$ with $\chi_A \leqq f$, and pass to $f \wedge 1$ in order to achieve $\chi_A \leqq f \leqq 1$. From [$\uparrow$] we obtain an $S \in \mathfrak{S}$ such that all $u \in E$ with $u \leqq f$ and $u|S = f|S$ fulfil $I(f) \leqq I(u) + \varepsilon$.

ii) Now let $v \in E$ with $v \geqq \chi_S$. Then $f \wedge v \in E$ with $f \wedge v \leqq f$ and $f \wedge v|S = f|S$, so that $I(f) \leqq I(f \wedge v) + \varepsilon$. Thus $I(f \vee v) \leqq I(v) + \varepsilon$ since I is modular [13] 1.4(M). Here we have on the left $f \vee v \in E$ with

$$I(f \vee v) = \fint (f \vee v) d\varphi = \fint (f \vee v) d\varphi_\bullet \geqq \fint \chi_{A \cup S} d\varphi_\bullet = \varphi_\bullet(A \cup S) = \Phi(A \cup S)$$

for $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$, since $A, S \in (\mathrm{Inn}(E))_\bullet \subset \mathfrak{C}(\varphi_\bullet)$. It follows that

$$\Phi(A \cup S) \leqq \inf\{I(v) : v \in E \text{ with } v \geqq \chi_S\} + \varepsilon = I^\star(\chi_S) + \varepsilon = I_\bullet(\chi_S) + \varepsilon,$$

since $S \in \mathfrak{S} \subset (\mathrm{Inn}(E))_\bullet = \mathrm{Inn}(E_\bullet) = \mathfrak{t}(E_\bullet)$ so that $\chi_S \in E_\bullet$ as above, and hence $I^\star(\chi_S) = I_\bullet(\chi_S)$ from 1.1 and [12] 5.10.6.Inn). Thus we obtain $\Phi(A \cup S) \leqq \varphi_\bullet(S) + \varepsilon = \Phi(S) + \varepsilon$ and hence $\Phi(A) \leqq \Phi(A \cap S) + \varepsilon$, since all terms are $< \infty$, or $\varphi_\bullet(A) \leqq \varphi_\bullet(A \cap S) + \varepsilon$. Now $A \in (\mathrm{Inn}(E))_\bullet \subset \mathfrak{S} \top \mathfrak{S}_\bullet$ and $S \in \mathfrak{S}$ furnish $A \cap S \in \mathfrak{S}_\bullet$, and the assertion follows.

Proof of [$\bullet$] $\Rightarrow$ [$\updownarrow$]. i) Fix $f \in E$, so that

$$I(f) = \fint f d\varphi = \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq t]) dt < \infty,$$

and $\varepsilon > 0$, and then $0 < a < b < \infty$ such that

$$\int\limits_{0\leftarrow}^{a} \varphi([f \geqq t])dt \leqq \frac{\varepsilon}{4} \quad \text{and} \quad \int\limits_{b}^{\to\infty} \varphi([f \geqq t])dt \leqq \frac{\varepsilon}{4}.$$

From $[\bullet]$ applied to $[f \geqq a] \in \operatorname{Inn}(E)$ we obtain $D \in \mathfrak{S}_\bullet$ with

$$D \subset [f \geqq a] \quad \text{and} \quad \varphi([f \geqq a]) \leqq \varphi_\bullet(D) + \frac{\varepsilon}{2(b-a)}.$$

Then fix $S \in \mathfrak{S}$ with $S \supset D$.

ii) Now let $u, v \in E$ with $u \leqq v \leqq f$ and $u|S = v|S$, so that in particular $u|D = v|D$. From $[v \geqq t] \subset [f \geqq t]$ for $0 < t < \infty$ we obtain

$$I(v) = \fint v d\varphi = \int\limits_{0\leftarrow}^{\to\infty} \varphi([v \geqq t])dt \leqq \int\limits_{a}^{b} \varphi([v \geqq t])dt + \frac{\varepsilon}{2}.$$

For $t \geqq a$ we have $[v \geqq t] \subset [u \geqq t] \cup \big([f \geqq a] \setminus D\big)$, and since all these sets are in $(\operatorname{Inn}(E))_\bullet \subset \mathfrak{C}(\varphi_\bullet)$ it follows that

$$\varphi([v \geqq t]) \leqq \varphi([u \geqq t]) + \frac{\varepsilon}{2(b-a)},$$

$$\int\limits_{a}^{b} \varphi([v \geqq t])dt \leqq \int\limits_{a}^{b} \varphi([u \geqq t])dt + \frac{\varepsilon}{2}.$$

These inequalities combine to furnish $I(v) \leqq \int\limits_{a}^{b} \varphi([u \geqq t])dt + \varepsilon \leqq I(u) + \varepsilon$.

□

3. Specializations of the Main Theorem

We start with the specializations of 1.1 and 1.3 under the two most important assumptions on the additive behaviour of E and I.

3.1 First Specialization of 1.1. *Assume that $E \subset [0, \infty[^X$ with $0 \in E$ is a Stonean lattice cone and $I : E \to [0, \infty[$ is isotone and additive. Then for $\bullet = \sigma\tau$ one has the equivalence*

I is an inner $\bullet$ preintegral

$\iff$ *I is downward $\bullet$ continuous at 0 and fulfils*

$I(v) \leqq I(u) + I^v_\bullet(v - u)$ for all $u \leqq v$ in E.

In this case there is a unique inner source of I which is an inner $\bullet$ premeasure, and it is in fact $\varphi = I^\star(\chi_\cdot)|\operatorname{Inn}(E)$.

3.2 Second Specialization of 1.1. *Assume that $E \subset [0, \infty[^X$ with $0 \in E$ is a Stonean lattice cone and fulfils $v - u \in E$ for all $u \leqq v$ in E, and that $I : E \to [0, \infty[$ is additive* (and hence isotone). *Then for $\bullet = \sigma\tau$ one has the equivalence*

I is an inner $\bullet$ preintegral

$\iff$ *I is downward $\bullet$ continuous at 0.*

In this case there is a unique inner source of I which is an inner $\bullet$ premeasure, and it is in fact $\varphi = I^\star(\chi_\cdot)|\operatorname{Inn}(E)$.

In fact, the first specialization is obvious, while the second one is contained in [12] 7.8.

We recall from [10] 14.6-7 that a lattice cone $E \subset [0,\infty[^X$ with $0 \in E$ fulfils the additional assumption $v - u \in E$ for all $u \leqq v$ in E iff $E = F^+ := \{ f \in F : f \geqq 0\}$ for some lattice subspace $F \subset \mathbb{R}^X$. In this case there is a unique such F, and it is in fact $F = E - E$. Moreover E is Stonean iff F is Stonean, which for a lattice subspace means that $f \in F \Rightarrow f \wedge t \in F$ for $0 < t < \infty$.

3.3 First Specialization of 1.3. *Assume that $E \subset [0,\infty[^X$ with $0 \in E$ is a Stonean lattice cone, that $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ is a lattice in X, and that* ($\bullet$) *for some $\bullet = \sigma\tau$. Let $I : E \to [0,\infty[$ be isotone and additive. Then one has the equivalences*

There exists an inner $\bullet$ premeasure $\vartheta : \mathfrak{S} \to [0,\infty[$ such that $I(f) = ⨍ f d\vartheta_\bullet$ for all $f \in E$

$\Longleftrightarrow$ *I is downward $\bullet$ continuous at 0 and fulfils $I(v) \leqq I(u) + I^v_\bullet(v-u)$ for all $u \leqq v$ in E and* [$\uparrow$]

$\Longleftrightarrow$ *I is downward $\bullet$ continuous at 0 and fulfils $I(v) \leqq I(u) + I^v_\bullet(v-u)$ for all $u \leqq v$ in E and* [$\updownarrow$].

In this case there is a unique such ϑ, and it is in fact $\vartheta = I^\star(\chi_.)|\mathfrak{S}$.

3.4 Second Specialization of 1.3. *Assume that $E \subset [0,\infty[^X$ with $0 \in E$ is a Stonean lattice cone and fulfils $v - u \in E$ for all $u \leqq v$ in E, that $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ is a lattice in X, and that* ($\bullet$) *for some $\bullet = \sigma\tau$. Let $I : E \to [0,\infty[$ be additive* (and hence isotone). *Then one has the equivalence*

There exists an inner $\bullet$ premeasure $\vartheta : \mathfrak{S} \to [0,\infty[$ such that $I(f) = ⨍ f d\vartheta_\bullet$ for all $f \in E$

$\Longleftrightarrow$ *I is downward $\bullet$ continuous at 0 and fulfils the equivalent* [$\downarrow$][$\uparrow$][$\updownarrow$].

In this case there is a unique such ϑ, and it is in fact $\vartheta = I^\star(\chi_.)|\mathfrak{S}$.

In fact, both specializations are immediate consequences of the respective previous ones combined with 1.3.

For an important addendum we next insert simple abstract versions of the USC (:=upper semicontinuous) and Dini theorems and a subsequent lemma. We recall for $\bullet = \sigma\tau$ that a nonvoid set system $\mathfrak{M}$ in X is defined to be $\bullet$ *compact* iff each $\mathfrak{T} \subset \mathfrak{M}$ with $\varnothing \in \mathfrak{T}_\bullet$ fulfils $\varnothing \in \mathfrak{T}_\star$. Moreover we put

$$\mathrm{Inn}(\mathfrak{M}) := \{f \in [0,\infty]^X : [f \geqq t] \in \mathfrak{M} \text{ for all } 0 < t < \infty\}.$$

3.5 Abstract USC Theorem. *Let $f : X \to [-\infty,\infty[$ be $\not\equiv -\infty$ such that for some real $c < \sup f$ the set system $\mathfrak{M}(c) := \{[f \geqq t] : c < t < \infty\}$ is τ compact. Then there exists $x \in X$ such that $f(x) = \sup f$* (so that in particular f is bounded above).

Proof. Assume that for all $x \in X$ one has $f(x) < \sup f$, and fix an $F(x) > f(x)$ with $c < F(x) < \sup f$. Then the set system $\{[f \geqq F(x)] : x \in X\} \subset \mathfrak{M}(c)$ has $\bigcap\limits_{x\in X} [f \geqq F(x)] = \varnothing$, and hence there exist $u(1), \cdots, u(r) \in X$ such that $\bigcap\limits_{l=1}^{r} [f \geqq F(u(l))] = \varnothing$ or $X = \bigcup\limits_{l=1}^{r} [f < F(u(l))]$. Thus all $x \in X$ fulfil $f(x) < \max(F(u(1)), \cdots, F(u(r))) =: s < \sup f$, which produces the contradiction $\sup f \leqq s < \sup f$. $\square$

3.6 Abstract Dini Theorem. *Let* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *be a lattice in* X *and* τ *compact. Assume that the nonvoid* $M \subset \operatorname{Inn}(\mathfrak{S}\top\mathfrak{S}_\tau)$ *is downward directed with pointwise infimum* $= 0$*. Then*

$$\inf_{f\in M} \sup(f|S) = 0 \quad \textit{for all nonvoid } S \in \mathfrak{S}.$$

Proof. Fix a nonvoid $S \in \mathfrak{S}$ and $\varepsilon > 0$, and then for each $u \in S$ a function $f_u \in M$ with $f_u(u) < \varepsilon$. We have $[f_u \geqq \varepsilon] \in \mathfrak{S}\top\mathfrak{S}_\tau$ and hence $[f_u \geqq \varepsilon] \cap S \in \mathfrak{S}_\tau$, and it follows that $\bigcap\limits_{u\in S} [f_u \geqq \varepsilon] \cap S = \varnothing$. Since $\mathfrak{S}_\tau$ is τ compact there exist $u(1), \cdots, u(r) \in S$ such that $\bigcap\limits_{l=1}^{r} [f_{u(l)} \geqq \varepsilon] \cap S = \varnothing$ or $S \subset \bigcup\limits_{l=1}^{r} [f_{u(l)} < \varepsilon]$. Since M is downward directed there exists $f \in M$ with $f \leqq f_{u(1)}, \cdots, f_{u(r)}$. Thus $S \subset [f < \varepsilon]$ and hence $\sup(f|S) \leqq \varepsilon$. □

3.7 Lemma. *Assume that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a positive-homogeneous lattice, and that* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *is a lattice in* X *and* τ *compact. Moreover assume that* $\operatorname{Inn}(E)$ *and* $\mathfrak{S}$ *fulfil the weakened form of* (τ) *which instead of the first part requires that* $\mathfrak{S}$ *be upward enclosable* $\operatorname{Inn}(E)$*. Let* $I : E \to [0,\infty[$ *with* $I(0) = 0$ *be isotone and positive-homogeneous, and assume that* $[\updownarrow]$ *is fulfilled. Then* I *is downward* τ *continuous at* 0*.*

Proof. 0) Each $f \in E$ is bounded above on each $S \in \mathfrak{S}$. In fact, for $0 < t < \infty$ we have $[f \geqq t] \in \mathfrak{S}\top\mathfrak{S}_\tau$ from the second part of (τ) and hence $[f \geqq t] \cap S = [f|S \geqq t] \in \mathfrak{S}_\tau$. Thus 3.5 applied to $f|S$ and $c = 0$ furnishes $\sup(f|S) < \infty$.

1) Now let the nonvoid $M \subset E$ be downward directed with pointwise infimum $= 0$. To be shown is $\inf\limits_{f\in M} I(f) = 0$. We fix $f \in M$ and $\varepsilon > 0$, and from $[\updownarrow]$ then $S \in \mathfrak{S}$ such that all $u, v \in E$ with $u \leqq v \leqq f$ and $u|S = v|S$ fulfil $I(v) \leqq I(u) + \varepsilon$. At last from the assumption fix $h \in E$ with $S \subset [h \geqq 1]$, that is $h|S \geqq 1$.

2) For $v \in M$ with $v \leqq f$ we have $\sup(v|S) < \infty$ from 0) and hence $u := \big(\sup(v|S)h\big) \wedge v \in E$ with $u \leqq v$ and $u|S = v|S$, so that $I(v) \leqq I(u) + \varepsilon \leqq \sup(v|S)I(h) + \varepsilon$. Therefore

$$\begin{aligned}\inf_{v\in M} I(v) &= \inf\{I(v) : v \in M \text{ with } v \leqq f\}\\ &\leqq \inf\{\sup(v|S) : v \in M \text{ with } v \leqq f\}I(h) + \varepsilon = \inf_{v\in M} \sup(v|S)I(h) + \varepsilon.\end{aligned}$$

Thus 3.6 applied to $M \subset E \subset \operatorname{Inn}(\mathfrak{S}\top\mathfrak{S}_\tau)$ furnishes $\inf\limits_{v\in M} I(v) \leqq \varepsilon$ and hence the assertion. □

3.8 Addendum to 3.3 and 3.4 ($\bullet = \tau$). *If* I *fulfils* $[\updownarrow]$ *and* $\mathfrak{S}$ *is* τ *compact then* I *is downward* τ *continuous at* 0*.*

In fact, one verifies as usual that I is positive-homogeneous. Then the assertion follows from 3.7. For convenience we include the explicit formulation of the resultant consequences of 3.3 and 3.4.

3.9 Consequence of 3.3. *Assume that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a Stonean lattice cone, that* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *is a lattice in* X *and* τ *compact, and assume that* (τ) *is fulfilled. Let* $I : E \to [0,\infty[$ *be isotone and additive. Then one has the equivalence*

There exists an inner τ *premeasure* $\vartheta : \mathfrak{S} \to [0,\infty[$ *such that* $I(f) = ⨍ f d\vartheta_\tau$ *for all* $f \in E$

$\Longleftrightarrow$ I *fulfils* $I(v) \leqq I(u) + I_\tau^v(v-u)$ *for all* $u \leqq v$ *in* E *and* $[\updownarrow]$.

In this case there is a unique such ϑ*, and it is in fact* $\vartheta = I^\star(\chi_\cdot)|\mathfrak{S}$.

3.10 Consequence of 3.4. *Assume that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a Stonean lattice cone and fulfils* $v - u \in E$ *for all* $u \leqq v$ *in* E*, that* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *is a lattice in* X *and* τ *compact, and assume that* (τ) *is fulfilled. Let* $I : E \to [0,\infty[$ *be additive* (and hence isotone)*. Then one has the equivalence*

There exists an inner τ *premeasure* $\vartheta : \mathfrak{S} \to [0,\infty[$ *such that* $I(f) = ⨍ f d\vartheta_\tau$ *for all* $f \in E$

$\Longleftrightarrow$ I *fulfils the equivalent* $[\downarrow][\uparrow][\updownarrow]$.

In this case there is a unique such ϑ*, and it is in fact* $\vartheta = I^\star(\chi_\cdot)|\mathfrak{S}$.

We conclude the treatment of the abstract situation with one more point in case $\bullet = \tau$, before we turn to the topological situation. The point is to describe the first condition in (τ) as a kind of *separation* which E performs on $\mathfrak{S}$. We note that the second condition in (τ) will find a natural fulfilment in the topological situation.

3.11 Lemma. *Assume that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a positive-homogeneous lattice, and that* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *is a lattice in* X.

1) *We have* $\mathfrak{S} \subset (\mathrm{Inn}(E))_\tau \Longleftrightarrow$ *for each* $S \in \mathfrak{S}$ *there exists an* $f \in E$ *with* $f|S \geqq 1$*, and furthermore for any* $S \in \mathfrak{S}$ *and* $u \in S' := X \setminus S$ *an* $f \in E$ *with* $f|S \geqq 1 > f(u)$.

2) *We have* $\mathfrak{S} \subset (\mathrm{Inn}(E))_\tau \Longrightarrow$ *for any* $S \in \mathfrak{S}$ *and* $v \in S$ *there exists an* $f \in E$ *with* $f(v) > 0$*, and furthermore for any* $S \in \mathfrak{S}$ *and* $v \in S$ *and* $u \in S'$ *an* $f \in E$ *with* $f(v) > f(u)$*. We have* $\Longleftarrow$ *if in addition the lattice* $\mathfrak{S}$ *is* τ *compact and the* outer counterpart

$$\mathrm{Out}(E) := \{\, [f > t] : f \in E \text{ and } 0 < t < \infty\}$$

of $\mathrm{Inn}(E)$ *fulfils* $(\mathrm{Out}(E))\bot := \{\, [f \leqq t] : f \in E \text{ and } 0 < t < \infty\} \subset \mathfrak{S}\top\mathfrak{S}_\tau$.

Proof. 1) and the implication $\Rightarrow$ in 2) are obvious, thus to be shown is $\Leftarrow$ in 2). Fix $S \in \mathfrak{S}$. In case $S = \varnothing$ we have $S = \varnothing = [0 \geqq 1] \in \mathrm{Inn}(E)$ since $0 \in E$. In case $S = X$ take $f_v \in E$ with $f_v(v) > 1$ for each $v \in X$. Then $[f_v \leqq 1] \in (\mathrm{Out}(E))\bot \subset \mathfrak{S}\top\mathfrak{S}_\tau$ which in the present case is $= \mathfrak{S}_\tau$. Now $\bigcap\limits_{v\in X} [f_v \leqq 1] = \varnothing$, and hence there exist $v(1), \cdots, v(r) \in X$ with $\bigcap\limits_{l=1}^{r} [f_{v(l)} \leqq 1] = \varnothing$ or $X = \bigcup\limits_{l=1}^{r} [f_{v(l)} > 1]$. Thus $f := f_{v(1)} \vee \cdots \vee f_{v(r)} \in E$ fulfils $f > 1$, so that $S = X = [f \geqq 1] \in \mathrm{Inn}(E)$.

It remains the case $S \neq \varnothing, X$. For fixed $u \in S'$ take $f_v^u \in E$ with $f_v^u(v) > 1 > f_v^u(u)$ for each $v \in S$. Then $[f_v^u \leqq 1] \in (\mathrm{Out}(E))\bot \subset \mathfrak{S}\top\mathfrak{S}_\tau$ and hence $[f_v^u \leqq 1] \cap S \in \mathfrak{S}_\tau$. Now $\bigcap\limits_{v\in S} [f_v^u \leqq 1] \cap S = \varnothing$, so that there exist $v(1), \cdots, v(r) \in S$ with $\bigcap\limits_{l=1}^{r} [f_{v(l)}^u \leqq 1] \cap S = \varnothing$ or $S \subset \bigcup\limits_{l=1}^{r} [f_{v(l)}^u > 1]$. Thus $f^u := f_{v(1)}^u \vee \cdots \vee f_{v(r)}^u \in E$ fulfils $f^u|S > 1 > f^u(u)$. It follows that $S = \bigcap\limits_{u\in S'} [f^u \geqq 1] \in (\mathrm{Inn}(E))_\tau$. □

In the remainder of the section we approach the Riesz representation theorem. We assume that X is a Hausdorff topological space and that

$\mathfrak{S} = \mathrm{Comp}(X)$ consists of its compact subsets. We specialize the above representation theorems 3.9 and 3.10. We recall that an $f : X \to [-\infty, \infty[$ is called USC (:=upper semicontinuous) iff $[f \geqq t] \subset X$ is closed for all $t \in \mathbb{R}$.

3.12 SPECIALIZATION OF 3.9. *Let $E \subset \mathrm{USC}(X, [0, \infty[)$ with $0 \in E$ be a Stonean lattice cone. Assume that for each compact $S \subset X$ there exists an $f \in E$ with $f|S \geqq 1$, and furthermore for any compact $S \subset X$ and $u \in S'$ an $f \in E$ with $f|S \geqq 1 > f(u)$. Let $I : E \to [0, \infty[$ be isotone and additive. Then one has the equivalence*

There exists a Radon premeasure $\vartheta : \mathrm{Comp}(X) \to [0, \infty[$ such that $I(f) = ⨍ f d\vartheta_\tau$ for all $f \in E$

$\iff$ *I fulfils $I(v) \leqq I(u) + I_\tau^v(v - u)$ for all $u \leqq v$ in E and $[\updownarrow]$.*

In this case there is a unique such ϑ, and it is in fact $\vartheta = I^\star(\chi_.)|\mathfrak{S}$.

Proof. In (τ) the first condition results from 3.11.1), and the second one from $E \in \mathrm{USC}(X, [0, \infty[)$. $\square$

3.13 SPECIALIZATION OF 3.10. *Let $E \subset \mathrm{C}(X, [0, \infty[)$ with $0 \in E$ be a Stonean lattice cone which fulfils $v - u \in E$ for all $u \leqq v$ in E. Assume that for each $v \in X$ there exists an $f \in E$ with $f(v) > 0$, and furthermore for any $v \neq u$ in X an $f \in E$ with $f(v) \neq f(u)$. Let $I : E \to [0, \infty[$ be additive* (and hence isotone). *Then one has the equivalence*

There exists a Radon premeasure $\vartheta : \mathrm{Comp}(X) \to [0, \infty[$ such that $I(f) = ⨍ f d\vartheta_\tau$ for all $f \in E$

$\iff$ *I fulfils the equivalent $[\downarrow][\uparrow][\updownarrow]$.*

In this case there is a unique such ϑ, and it is in fact $\vartheta = I^\star(\chi_.)|\mathfrak{S}$.

Proof. In (τ) the first condition results from 3.11.2). In fact, if for some $v \neq u$ one has an $f \in E$ with $f(v) < f(u)$ then one can obtain an $F \in E$ with $F(v) > F(u)$ as follows: From the assumption there exists $h \in E$ such that $h(v) = f(u)$ and $h \leqq f(u)$, and then $F := h - h \wedge f \in E$ is as required. The second condition in (τ) is clear. $\square$

4. COMPARISON WITH THE EARLIER LITERATURE

We start with the Daniell-Stone representation theorem 1.1. The traditional Daniell-Stone theorem [10] 14.1 was of rather limited use, above all in view of its lack of regularity, and at times appeared to be bound for oblivion. But then the road to the present inner $\bullet$ representation theorem 1.1 opened in Pollard-Topsøe [16], who in theorem 4 obtained the substance of the second specialization 3.2, and in Topsøe [20], who for the first time considered an arbitrary Stonean lattice cone $E \subset [0, \infty[^X$ with $0 \in E$ as in the present first specialization 3.1. However, these papers did not yet possess the inner $\bullet$ envelopes for $\bullet = \sigma\tau$, but had to work with the traditional crude $\star$ envelopes instead. Thus in [20] theorem 3 the tightness requirement on I appeared as $I(v) \leqq I(u) + I_\star(v - u)$ for all $u \leqq v$ in E. But in contrast to the tightness condition in 3.1 this one need not be true when I is an inner $\bullet$ preintegral, as shown in the later example [10] 15.11. Therefore [20] theorem 3 does not offer an *equivalent*, but a *sufficient* condition for the desired representation. The inner (and outer) $\bullet$ formations for $\bullet = \sigma\tau$ then appeared as the main novelties in the author's book [10], and the full first specialization 3.1 was in [10] 15.9. After this the final step to the present 1.1 (and its outer counterpart) was done in [11]. The basis was the elaboration of a fundamental idea of Choquet on additive functionals in [11] section 1 and in [13].

We turn to the present main theorem 1.3, and compare Pollard-Topsøe [16] theorem 3 with the second specialization 3.4 and Topsøe [20] theorem 1 with the first specialization 3.3. In the assumptions there are several deviations which prevent a direct comparison: In place of the present initial condition $(\bullet)$ one requires the first part as a separation condition of different kind which involves I, and the second part as $\mathrm{Inn}(E) \subset \mathfrak{S}\top\mathfrak{S}$ in [20] and as $(\mathrm{Out}(E))\perp \subset \mathfrak{S}\top\mathfrak{S}$ in [16]. In the final equivalence conditions the requirement that I be downward $\bullet$ continuous at 0 takes the form that the resultant set function $\vartheta = I^\star(\chi_\cdot)|\mathfrak{S}$ be downward $\bullet$ continuous at $\varnothing$, a form called into question at once in [16], plus $\sup\limits_{n\in\mathbb{N}} I(f \wedge n) = I(f)$ for all $f \in E$. Also both theorems require $[\downarrow]$. Above all the tightness requirement on I in [20] theorem 1 is in terms of $I_\star$ as before, but a bit weaker. As before both theorems do not offer *equivalent*, but *sufficient* conditions for the desired representations, though [16] theorem 3 has an equivalent condition in the special case $\mathfrak{S} = \mathfrak{S}_\bullet$.

After this the version [10] 15.15 is of course more in the present terms. But in the condition $\mathrm{Inn}(E) \subset \mathfrak{S}\top\mathfrak{S}$ and in the assertion it replaces $\mathfrak{S}_\bullet$ with $\mathfrak{S}$, and the condition that I be an inner $\bullet$ preintegral becomes part of the initial assumption. Thus the theorem once more offers a *sufficient*, but not an *equivalent* condition for the desired representation. Therefore both 1.3 and its specializations 3.3 and 3.4 appear in the present paper for the first time.

In this connection we recall the book of Anger-Portenier [1] and their related article [2]. For the comparison we refer to the bibliographical notes [10] 15.14 and 16.13. The concrete comparison is complicated because of fundamental differences in the basic concepts (in particular we note their ubiquitous notions of *regular* functionals and of *essential integration*). As far as the author knows the work in question has not been pursued.

The next point are the consequences 3.9 of 3.3 and 3.10 of 3.4, and their specializations 3.12 and 3.13. As far as the author can see the relevant earlier contributions are all within 3.12 and 3.13.

Our unique earlier source relative to the USC situation of 3.12 is Topsøe [20]. In its last sentence the author pointed to the USC case as an important aim for application. But he did not present an explicit treatment, and later in his 1982 lecture [21] section 2 remarked that he had not been satisfied with the respective results. The full consequence 3.12 then appeared in [10] 16.11, while [12] 7.11 was content with the simpler particular case of the USC functions with compact support.

In the $\mathrm{C}(X, [0, \infty[)$ situation the present specialization 3.13 appears to be more comprehensive than the previous results known to the author. The closest to 3.13 is Berg-Christensen-Ressel [4] theorem 2.2 pp.35/36, which amounts to the restriction $E \subset \mathrm{CB}(X, [0, \infty[) :=$ the *bounded* functions in $\mathrm{C}(X,[0,\infty[)$. The same result is in Topsøe [21] with X assumed to be *completely regular*. There are of course several particular representation theorems in the familiar textbooks: The classical versions for $E :=$ the functions of *compact support* in $\mathrm{C}(X,[0,\infty[)$ and for $E :=$ the functions in $\mathrm{C}(X,[0,\infty[)$ which *vanish at infinity* are of course restricted to *locally compact* spaces X. For the versions with $E = \mathrm{CB}(X,[0,\infty[)$ and $E = \mathrm{C}(X,[0,\infty[)$ we refer to Bourbaki [5] Chap.IX section 5, Behrends [3] Chap.V section 3 and for example Elstrodt [6] Chap.VIII section 2. All these sources assume X to be *completely regular*. However, the present 3.13 shows that it suffices to require that $E \subset \mathrm{C}(X,[0,\infty[)$ contains the constants and *separates the points* of X. An example of a Hausdorff topological space X which is not completely regular but in which $\mathrm{C}(X,[0,\infty[)$ separates the points is the so-called *deleted Tychonov corkscrew* in Steen-Seebach [17] example 91.

We want to add that both Berg-Christensen-Ressel [4] and Elstrodt [6] in Chap.VIII section 2 are based on one of the initial steps Kisyński [9] and Topsøe [18][19] of the present new development in measure and integration.

At last we turn to Fremlin [7] chapter 7 and [8] section 436, both devoted to representation theorems like the present *Stonean lattice subspace* assertions. We consider the more delicate nonsequential case $\bullet = \tau$ based on the notion of *quasi-Radon* measures. There were no representation theorems like the present *Stonean lattice cone* assertions, because the concepts were set up in the respective sense, so to speak in the spirit of continuous functions rather than of USC functions, and also because Radon and quasi-Radon measures had to be locally finite. The main result is [7] theorem 72E = [8] theorem 436H in terms of quasi-Radon measures. We want to show that this theorem is contained in the present second specialization 3.2 of 1.1. However, an essential implication of the modified set-up is that Fremlin's theorem has no claim of uniqueness, in contrast to the present results (there are certain uniqueness assertions in [7] 72Fb and [8] exercise 436X(l)).

4.1 Addendum to 3.2 ($\bullet = \tau$). *Assume as before that* $E \subset [0,\infty[^X$ *with* $0 \in E$ *is a Stonean lattice cone and fulfils* $v - u \in E$ *for all* $u \leqq v$ *in* E, *and let* $\mathfrak{U}$ *be the weakest topology on* X *in which the members of* E *are continuous. Let* $I : E \to [0,\infty[$ *be additive* (and hence isotone) *and downward* τ *continuous at* 0, *so that* $\varphi := I^\star(\chi_.)|\mathrm{Inn}(E)$ *is the unique inner source of* I *which is an inner* τ *premeasure. Then* $\mathfrak{U} \subset \mathfrak{C}(\varphi_\tau)$, *and* $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ *is a quasi-Radon measure for* $\mathfrak{U}$.

Proof. For the notion of quasi-Radon measures we refer to [12] section 4. 1) Define the set system $\mathfrak{B}$ in X to consist of the subsets

$$[u \geqq t], [v \leqq 0] = [v = 0], [u \geqq t] \cup [v \leqq 0] \quad \text{for } u, v \in E \text{ and } 0 < t < \infty.$$

1.i) One notes that $\mathfrak{B}$ is stable under finite unions, and $X \in \mathfrak{B}$. Hence $\mathfrak{B}_\tau$ is stable under finite unions and arbitrary intersections, and $X, \varnothing \in \mathfrak{B}_\tau$. Thus $\mathfrak{B}_\tau$ is the class of closed subsets $\mathfrak{B}_\tau = \mathfrak{U}\perp$ for some topology $\mathfrak{U}$ on X. 1.ii) The members of E are continuous in $\mathfrak{U}$. It suffices to note that $[f \geqq t]$ and $[f \leqq t]$ are in $\mathfrak{B}_\tau$ for all $f \in E$ and $t \in \mathbb{R}$, which are obvious verifications. 1.iii) If $\mathfrak{V}$ is a topology on X in which the members of E are continuous then $\mathfrak{U} \subset \mathfrak{V}$. In fact, we have $\mathfrak{B} \subset \mathfrak{V}\perp$ and hence $\mathfrak{U}\perp = \mathfrak{B}_\tau \subset \mathfrak{V}\perp$. Thus the present $\mathfrak{U}$ is as required in 4.1.

2) We claim that $\mathfrak{B} \subset \mathrm{Inn}(E)\top(\mathrm{Inn}(E))_\sigma \subset \mathrm{Inn}(E)\top(\mathrm{Inn}(E))_\tau$. Therefore

$$\mathfrak{B}_\tau \subset \mathrm{Inn}(E)\top(\mathrm{Inn}(E))_\tau \text{ or } \mathfrak{U} \subset \Big(\mathrm{Inn}(E)\top(\mathrm{Inn}(E))_\tau\Big)\perp \subset \mathfrak{C}(\varphi_\tau).$$

In fact, fix $u, v \in E$ and $0 < t < \infty$. 2.i) By definition $[u \geqq t] \in \mathrm{Inn}(E) \subset \mathrm{Inn}(E)\top\mathrm{Inn}(E)$. 2.ii) One verifies that

$$[v \leqq 0] \cap [u \geqq t] = \bigcap_{n\in\mathbb{N}} [(u - nv)^+ \geqq t].$$

Now $(u - nv)^+ = u \vee (nv) - nv \in E$, so that the second member is in $(\mathrm{Inn}(E))_\sigma$. Thus $[v \leqq 0] \in \mathrm{Inn}(E)\top(\mathrm{Inn}(E))_\sigma$. 2.iii) From 2.i) and 2.ii) the assertion follows.

3) The definition of quasi-Radon measures for $\mathfrak{U}$ in [12] section 4 consists of five properties, of which i)ii) are clear for the $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. iii) requires that $\Phi|\mathfrak{U}$ be upward τ continuous, and thus follows from [12] 3.6.ii) combined with 2). The combination of iv)v) requires that Φ be inner regular $\{S \in \mathfrak{U}\perp : S \subset \text{some } U \in \mathfrak{U} \text{ with } \Phi(U) < \infty\}$. Now φ_τ and hence Φ are inner regular $(\mathrm{Inn}(E))_\tau \subset \mathfrak{B}_\tau = \mathfrak{U}\perp$. For fixed $S \in (\mathrm{Inn}(E))_\tau$ we have $S \subset [u \geqq t]$ for some $u \in E$ and $0 < t < \infty$. For $0 < s < t$ hence $S \subset [u \geqq t] \subset U := [u > s] \subset [u \geqq s]$ with $U \in \mathfrak{U}$ and $\Phi(U) \leqq \varphi([u \geqq s]) < \infty$. □

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