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Abstract

We consider a Stokes-type system of partial differential equations in $2D$, which describes the stationary and also slow flow of an incompressible fluid. Here the nonlinear differential operator related to the stress tensor is generated by a potential $H(\varepsilon) = h(|\varepsilon|)$ acting on symmetric (2×2)-matrices, where h is a N-function of rather general type leading to a non-uniformly elliptic problem.

In our note we discuss the regularity problem for steady flows of certain classes of generalized Newtonian fluids in two dimensions assuming that the velocity is small which means that we mainly concentrate on some variants of the classical Stokes problem. To be precise, consider a bounded open set $\Omega \subset \mathbb{R}^2$ and a system of volume forces $f : \Omega \to \mathbb{R}^2$. For a given boundary datum u_0 we then like to find a velocity field $v : \Omega \to \mathbb{R}^2$ and a pressure function $p : \Omega \to \mathbb{R}$ such that

$$(1)\qquad \begin{cases} -\operatorname{div}\left[T\left(\varepsilon(v)\right)\right] + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = u_0 & \text{on } \partial\Omega\,. \end{cases}$$

We assume that the tensor T is the gradient of a potential $H : \mathbb{S} \to [0,\infty)$ defined on the space $\mathbb{S}$ of all symmetric (2×2)- matrices, where in (1) $\varepsilon(v) := \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)$ is the symmetric gradient of v and where the operator div has to be applied "line-wise". Due to the absence of the convective term it is easy to see that (1) is reducible to a variational problem, and therefore it makes sense to study the regularity properties of local minimizers $u : \Omega \to \mathbb{R}^2$ of the variational integral

$$(2)\qquad I[w,\Omega] = \int_\Omega H(\varepsilon(w))\,dx$$

defined for solenoidal fields w from a suitable energy space, where just for notational simplicity we assume $f \equiv 0$. The choice $H(\varepsilon) = \frac{\nu}{2}|\varepsilon|^2$ for some $\nu > 0$ leads to Stokes problem which is treated in the monographs of Ladyzhenskaya [La] and Galdi [Ga1], [Ga2]. The case of p-growth potentials, i.e.

$$(3)\qquad \lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2H(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2$$

for some $p \in (1,\infty)$ and with constants $\lambda, \Lambda > 0$ has been investigated in [KMS] leading to the $C^{1,\mu}$-regularity of local minima u. In fact, Kaplický, Málek and Stará even construct globally smooth solutions of (1) in case $u_0 = 0$ including the convective term provided $p > 3/2$ (see [KMS]), Theorem 5.30). Moreover they show the existence of a solution of (1) (+ convective term) being smooth in the interior assuming that $p > 6/5$. Here we like to remark that Frehse, Málek and Steinhauer proved in [FMS] that (1) including the convective term has a weak solution $v \in W^1_p(\Omega;\mathbb{R}^2)$ for any $p > 1$ provided $u_0 = 0$ and the tensor T is monotone with $(p-1)$-growth but not necessarily the gradient of a potential H. Concerning the regularity of this weak solution nothing is known.

In the paper [BFZ1] we investigated the behaviour of local minimizers of the energy I defined in (2) replacing (3) by its anisotropic variant

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2H(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{q-2}{2}}|\sigma|^2 \tag{4}$$

with exponents $1 < p \le q$ and obtained interior $C^{1,\mu}$- regularity of local minima provided

$$q < \min(2p, p+2)\,. \tag{5}$$

Moreover, if we have (4) and (5) with $p > 6/5$ and if we include the convective term in (1) together with homogeneous boundary data, then we constructed a weak solution of (1) without interior singularities. Further extensions concerning non-autonomous anisotropic potentials $H = H(x,\varepsilon)$ are given in [BFZ2]. We wish to mention that similar regularity results for electrorheological fluids are due to Diening, Ettwein and Růžička [DER].

Of course our list of known results is not complete, and the reader will find further references in the above mentioned papers. Moreover, the textbooks [Ga1], [Ga2], [La], [MNRR], [Ru] and [FS] provide additional information concerning the mathematical and physical background of the problems under consideration.

Inspired by Marcellini's work on variational problems with energy densities of nonstandard growth (see, e.g. [Ma1], [Ma2], [Ma3], [MP]) we now introduce a class of potentials H which not necessarily satisfy (3) or the anisotropic variant (4) together with (5). Suppose that

$$H(\varepsilon) = h(|\varepsilon|), \varepsilon \in \mathbb{S}\,, \tag{6}$$

for a function $h : [0,\infty) \to [0,\infty)$ of class C^2 such that the following assumptions (A1 - 4) hold:

$$\text{(A1)} \qquad \begin{cases} h \text{ is strictly increasing and convex together with } h''(0) > 0 \\ \text{and } \lim\limits_{t\downarrow 0} \frac{h(t)}{t} = 0\,; \end{cases}$$

$$\text{(A2)} \qquad \begin{cases} \text{there exists a constant } \overline{k} > 0 \text{ such that } h(2t) \le \overline{k}\, h(t) \text{ for all} \\ t \ge 0\,; \end{cases}$$

(A3) $\qquad \begin{cases} \text{for an exponent } \alpha \geq 0 \text{ and a constant } a \geq 0 \text{ we have} \\ \frac{h'(s)}{s} \leq h''(s) \leq a(1+s^2)^{\frac{\alpha}{2}} \frac{h'(s)}{s} \quad \forall\, s \geq 0\,; \end{cases}$

(A4) $\qquad \displaystyle\int_1^\infty \frac{h^{-1}(t)}{t^{3/2}}\, dt < \infty, h^{-1}$ denoting the inverse function.

Let us give some comments on our hypotheses:

i) We have $h(0) = h'(0) = 0$, and since h is convex, h' must be an increasing function with $h'(t) > 0$ for all $t > 0$: otherwise it would follow $h' = 0$ on a certain interval $[0, t_0]$ contradicting the first part of (A1).

ii) From $h'(s)\,\frac{1}{s} \leq h''(s)$ we get that

(7) $$h(t) \geq \frac{1}{2}\, h''(0) t^2 \quad \forall\, t \geq 0\,.$$

Moreover, $t \mapsto \frac{h'(t)}{t}$ is an increasing function.

(A1) together with (7) implies that h is a N-function in the sense of [Ad, Section 8.2], and (A2) states that h has the ($\Delta 2$)-property.

iii) It is easy to see that (A2) gives the existence of a number $c_1 > 0$ and of an exponent $\overline{m} \geq 2$ such that

(8) $$h(t) \leq c_1(t^{\overline{m}} + 1)$$

holds for all $t \geq 0$. Since h is convex, (8) implies

(9) $$h'(t) \leq c_2(t^{\overline{m}-1} + 1)\,.$$

Note that (8) does not follow from (A1) and (A3): these conditions also hold for certain functions with exponential growth.

iv) Since h is a N-function and since we have Korn's inequality in Orlicz-Sobolev spaces (see [MM], Remark 5, and [Ko1], [Ko2]), we say that a mapping u with $\operatorname{div} u = 0$ from the local Orlicz-Sobolev class $W^1_{h,\mathrm{loc}}(\Omega; \mathbb{R}^2)$ is a local minimizer of the functional I from (2) iff $I[u, \widetilde{\Omega}] \leq I[v, \widetilde{\Omega}]$ holds for all $v \in W^1_{h,\mathrm{loc}}(\Omega; \mathbb{R}^2)$ such $\operatorname{div} v = 0$ and $\operatorname{spt}(u - v) \subset \widetilde{\Omega}$, where $\widetilde{\Omega}$ is any subdomain of Ω with compact closure in Ω.

v) From (6) we get for all ε, $\sigma \in \mathbb{S}^2$

$$\min\left\{\frac{h'(|\varepsilon|)}{|\varepsilon|}, h''\left(|\varepsilon|\right)\right\} |\sigma|^2 \leq D^2 H(\varepsilon)(\sigma, \sigma) \leq \max\left\{\frac{h'(|\varepsilon|)}{|\varepsilon|}, h''(|\varepsilon|)\right\} |\sigma|^2,$$

so that by (A3)

$$
\frac{h'(|\varepsilon|)}{|\varepsilon|}\,|\sigma|^2 \le D^2H(\varepsilon)(\sigma,\sigma) \le a\left(1+|\varepsilon|^2\right)^{\frac{\alpha}{2}}\frac{h'(|\varepsilon|)}{|\varepsilon|}\,|\sigma|^2\,. \tag{10}
$$

The first inequality in (10) combined with i) gives the strict convexity of H, and from (9) it follows

$$
D^2H(\varepsilon)(\sigma,\sigma) \le c_3\left(1+|\varepsilon|^2\right)^{\frac{m-2}{2}}|\sigma|^2\,, \tag{11}
$$

where $m = \overline{m} + \alpha$.

vi) According to [Ad], Theorem 8.35, we deduce from (A4) that $W^1_{h,\mathrm{loc}}(\Omega;\mathbb{R}^2)$ is a subspace of $C^0 \cap L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$. Note that on account of (A4) $h(t)$ must grow faster than t^2 as $t \to \infty$. It is easy to see that from (A1) and (A2) we get that $h(t) \le th'(t) \le \overline{k}h(t)$. Therefore we can replace (A4) by the equivalent condition

$$
\int_1^\infty \frac{dt}{h(t)^{1/2}} < \infty\,.
$$

Let us now state our main result:

THEOREM 1. *Let (6) and (A1-4) hold with $\alpha < 2$. If u locally minimizes the energy I from (2) within the class $\{v \in W^1_{h,\mathrm{loc}}(\Omega;\mathbb{R}^2) : \operatorname{div} v = 0\}$, then u is in the space $C^{1,\mu}(\Omega;\mathbb{R}^2)$ for any $\mu < 1$.*

REMARK 1. *If we go back to problem (1) and include the convective term in the first line of (1), then - under the assumptions of Theorem 1 concerning H - we can modify the arguments leading to Theorem 4.1 of [BFZ1] in the spirit of the proof of Theorem 1 presented below in order to get the existence of a weak solution of (1) at least in the case $u_0 = 0$ being smooth in the interior of Ω.*

REMARK 2. *Given numbers $2 < p < q < \infty$ it is easy to construct functions h "alternating" between t^p and t^q which means that (4) holds exactly for these choices of p and q. At the same time $t \mapsto h'(t)/t$ is increasing and also gives an upper bound for $h''(t)$ in the sense of the second inequality from (A3). Thus Theorem 1 shows regularity of local minimizers which from [BFZ1] can only be deduced if (5) holds, i.e. if we require $q < p + 2$.*

Proof of Theorem 1

In order to be not too technical, we present a formal proof whose details can be made precise by working with the following regularisation: we fix a disc D compactly contained in Ω and consider the mollification $(u)_\rho$ with small radius $\rho > 0$ of our local minimizer. We further let

$$\delta := \delta(\rho) := \left[1 + \rho^{-1} + \|\varepsilon\left((u)_\rho\right)\|^{2m}_{L^m(D)}\right]^{-1} ,$$

where $m \geq 2$ is taken from (11).

With $H_\delta(\varepsilon) := \delta\left(1+|\varepsilon|^2\right)^{\frac{m}{2}} + H(\varepsilon), \varepsilon \in \mathbb{S}$, we then denote by u_δ the unique minimizer of $\int_D H_\delta\left(\varepsilon(w)\right)\, dx$ among all functions $w \in (u)_\rho + \overset{\circ}{W}{}^1_m(D;\mathbb{R}^2)$ such that $\operatorname{div} w = 0$. For the properties of the functions u_δ we refer to [BF]. Now, dropping the index δ, we have on account of (4.10) from [BF]

$$\int_\Omega \eta^2 \partial_k\sigma : \varepsilon(\partial_k u)\, dx \leq -2 \int_\Omega \eta\partial_k\tau : \left(\nabla\eta \odot \partial_k[u - Qx]\right)\, dx\,, \tag{12}$$

where $\sigma := DH\left(\varepsilon(u)\right)$ and $\tau := \sigma - p\,\mathbf{1}$ for a suitable pressure function p, i.e. $\nabla p = \operatorname{div}\sigma$. In (12) η denotes a cut-off function from $C^1_0(\Omega)$ such that $0 \leq \eta \leq 1$, and "$\odot$" is the symmetric product of vectors, whereas " $:$ " is the scalar product of matrices. Here and in what follows we always take the sum w.r.t. $k = 1, 2$. Finally, Q represents an arbitrary (2×2) -matrix not necessarily symmetric. We have

$$|\nabla\tau| \leq c|\nabla\sigma| \tag{13}$$

and $\left(\Phi := D^2H\left(\varepsilon(u)\right)\left(\partial_k\varepsilon(u), \partial_k\varepsilon(u)\right)^{1/2}\right)$

$$\begin{aligned} |\nabla\sigma|^2 = \partial_k\sigma : \partial_k\sigma &= D^2H\left(\varepsilon(u)\right)\left(\partial_k\varepsilon(u), \partial_k\sigma\right) \\ &\leq D^2H\left(\varepsilon(u)\right)\left(\partial_k\varepsilon(u), \partial_k\varepsilon(u)\right)^{1/2}\; D^2H\left(\varepsilon(u)\right)\left(\partial_k\sigma, \partial_k\sigma\right)^{1/2} \\ &\leq \Phi\left|D^2H\left(\varepsilon(u)\right)\right|^{1/2}|\nabla\sigma|\,, \end{aligned}$$

hence by (10)

$$|\nabla\sigma| \leq c\,\Phi\varphi, \quad \varphi := \left(h'\big(\varepsilon(u)\big)/|\varepsilon(u)|\right)^{1/2}\left(1+|\varepsilon(u)|^2\right)^{\alpha/4}. \tag{14}$$

Here and in the sequel c stands for a positive constant not depending on the approximation. Using (13) and (14) on the r.h.s. of (12), we find

$$\fint_{B_R(x_0)} \Phi^2\, dx \leq c\,\frac{1}{R} \fint_{B_{2R}(x_0)} \varphi\Phi|\nabla u - Q|\, dx\,, \tag{15}$$

where $B_{2R}(x_0) \Subset \Omega$, provided η has support in $B_{2R}(x_0)$, $\eta \equiv 1$ on $B_R(x_0)$ and $|\nabla \eta| \le c/R$. Letting $\gamma := 4/3$ we apply Hölder's and Sobolev-Poincaré's inequality to the r.h.s. of (15) to obtain (with $Q := \fint_{B_{2R}(x_o)} \nabla u \, dx$)

$$
\begin{aligned}
&\frac{1}{R} \fint_{B_{2R}(x_0)} \varphi \Phi |\nabla u - Q| \, dx \\
&\le \frac{1}{R} \left(\fint_{B_{2R}(x_0)} (\varphi\Phi)^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2R}(x_0)} |\nabla u - Q|^4 \, dx \right)^{1/4} \\
&\le c \left(\fint_{B_{2R}(x_0)} (\varphi\Phi)^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2R}(x_0)} |\nabla^2 u|^\gamma \, dx \right)^{1/\gamma} .
\end{aligned}
$$

Using $|\nabla^2 u| \le c |\nabla \varepsilon(u)|$ we deduce from (15)

$$
\fint_{B_R(x_0)} \Phi^2 \, dx \le c \left(\fint_{B_{2R}(x_0)} (\varphi\Phi)^\gamma \, dx \right)^{1/\gamma} \left(\fint_{B_{2r}(x_0)} |\nabla \varepsilon(u)|^\gamma \, dx \right)^{1/\gamma} . \tag{16}
$$

¿From the first inequality in (10) we get

$$
|\nabla \varepsilon(u)| \le \Phi \left(|\varepsilon(u)| \Big/ h'\left(|\varepsilon(u)|\right) \right)^{1/2} ,
$$

and if we observe the validity of

$$
\left(\frac{h'(t)}{t} \right)^{1/2} \left(1 + t^2\right)^{\frac{\alpha}{4}} \ge c \left(\frac{t}{h'(t)} \right)^{1/2} \tag{17}
$$

for all $t \ge 0$ and with a constant $c > 0$ (in case $t << 1$ (17) follows from $h'(0) = 0$, $h''(0) > 0$, whereas for "large" t we estimate $\frac{h'(t)}{t}(1+t^2)^{\frac{\alpha}{4}} \ge c \frac{h(t)}{t^2} t^{\frac{\alpha}{2}} \overset{(7)}{\ge} c t^{\alpha/2} \ge c$) we see that (16) together with (17) gives the estimate

$$
\left(\fint_{B_R(x_0)} \Phi^2 \, dx \right)^{1/2} \le c \left(\fint_{B_{2R}(x_0)} (\varphi\Phi)^\gamma \, dx \right)^{1/\gamma} . \tag{18}
$$

In order to continue we first derive a local L^2-bound for the function Φ in terms of the energy and then use this information to show that certain auxiliary functions belong to

$W^1_{2,\mathrm{loc}}$ (uniformly w.r.t. the approximation) which finally will enable us to handle the function φ so that we can exploit (18).

Step 1: a local L^2-bound for Φ

From (12) - (14) (choosing $Q=0$) we deduce

$$\int\limits_\Omega \eta^2\Phi^2\,dx \le c\int\limits_\Omega \eta|\nabla\eta|\Phi\varphi|\nabla u|\,dx\,,$$

hence by Young's inequality

$$\int\limits_\Omega \eta^2\Phi^2\,dx \le c\int\limits_\Omega |\nabla\eta|^2\varphi^2|\nabla u|^2\,dx\,. \tag{19}$$

We have (recall $\alpha<2$)

$$\begin{aligned}\varphi^2|\nabla u|^2 &= \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}(1+|\varepsilon(u)|^2)^{\frac{\alpha}{2}}|\nabla u|^2\\ &\le c\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}\left[|\varepsilon(u)|^2+1\right]|\nabla u|^2\\ &\le ch'\left(|\varepsilon(u)|\right)|\varepsilon(u)||\nabla u|^2\end{aligned}$$

a.e. on $[|\varepsilon(u)|\ge 1]$. From (A1) it follows

$$h(t)\ge\int_{t/2}^t h'(s)ds\ge h'(t/2)t/2\,,$$

hence

$$th'(t)\le\overline{k}h(t)$$

on account of (A2). This implies

$$\varphi^2|\nabla u|^2\le ch\left(|\varepsilon(u)|\right)|\nabla u|^2$$

on $[|\varepsilon(u)|\ge 1]$. On $[|\varepsilon(u)\le 1]$ we just observe $\varphi^2|\nabla u|^2\le c|\nabla u|^2$ so that by(7)

$$\varphi^2|\nabla u|^2\le c\left[1+|\nabla u|^2\right]h(|\nabla u|)\,.$$

W.r.t. (19) this shows

$$\int\limits_\Omega \eta^2\Phi^2\,dx\le c\int\limits_\Omega |\nabla\eta|^2\left(1+|\nabla u|^2\right)h\left(|\nabla u|\right)\,dx\,, \tag{20}$$

and it therefore remains to bound $|\varepsilon(u)|^2 h\left(|\varepsilon(u)|\right)$ in $L^1_{\rm loc}$ (in terms of the energy) which by (20) then leads to the desired $L^1_{\rm loc}$-bound for Φ^2 on account of Korn's inequality now applied in the space generated by $t \mapsto t^2 h(t)$. For the following calculations we observe that we actually work with a regularisation which means that we have enough smoothness to perform the steps. Moreover, from (A4) we deduce uniform L^∞-bounds for the functions u_δ. Now, with η as usual, we have

$$
\begin{aligned}
\int\limits_\Omega \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx &= \frac{1}{2}\int\limits_\Omega \eta^2 (\partial_j u^i + \partial_i u^j)\varepsilon(u)_{ij} h(|\varepsilon(u)|)\, dx \\
&= \int\limits_\Omega \partial_j u^i \left[\eta^2 \varepsilon(u)_{ij} h\left(|\varepsilon(u)|\right)\right]\, dx = -\int\limits_\Omega u^i \partial_j \left[\eta^2 \varepsilon(u)_{ij} h\left(|\varepsilon(u)|\right)\right]\, dx \\
&\le c\Big[\int\limits_\Omega \eta |\nabla\eta| |\varepsilon(u)| h(|\varepsilon(u)|)\, dx + \int\limits_\Omega \eta^2 |\nabla\varepsilon(u)| h(|\varepsilon(u)|)\, dx \\
&\qquad + \int\limits_\Omega \eta^2 |\nabla\varepsilon(u)| |\varepsilon(u)| h'(|\varepsilon(u)|)\, dx\Big] =: c[T_1 + T_2 + T_3]\,.
\end{aligned}
$$

By Young's inequality we see for all $\beta > 0$

$$
T_1 \le \beta \int\limits_\Omega \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx + c(\beta)\int\limits_\Omega |\nabla\eta|^2 h(|\varepsilon(u)|)\, dx,
$$

and if β is small enough we get

$$
\int\limits_\Omega \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx \le c\left[\int\limits_\Omega |\nabla\eta|^2 h(|\varepsilon(u)|)\, dx + T_2 + T_3\right]. \tag{21}
$$

In a similar way (using $0 = h(0) \ge h(t) - t h'(t)$) we find

$$
\begin{aligned}
T_2 &= \int\limits_\Omega \eta^2 |\nabla\varepsilon(u)| h^{1/2}(|\varepsilon(u)|) |\varepsilon(u)|^{-1} |\varepsilon(u)| h(|\varepsilon(u)|)^{1/2}\, dx \\
&\le \beta \int\limits_\Omega \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx + c(\beta)\int_\omega \eta^2 |\nabla\varepsilon(u)|^2 h(|\varepsilon(u)|) |\varepsilon(u)|^{-2}\, dx \\
&\le \beta \int\limits_\Omega \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx + c(\beta)\int\limits_\Omega \eta^2 |\nabla\varepsilon(u)|^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}\, dx \\
&\overset{(10)}{\le} \beta \int\limits_\Omega \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx + c(\beta)\int_\omega \eta^2 \Phi^2\, dx\,,
\end{aligned}
$$

and since $th'(t) \le \overline{k}h(t)$, we obtain the same bound for T_3. Thus (21) together with the above estimates implies

$$\int\limits_{\Omega} \eta^2 |\varepsilon(u)|^2 h(|\varepsilon(u)|)\, dx \le c \left[\int\limits_{\Omega} |\nabla\eta|^2 h(|\varepsilon(u)|)\, dx + \int\limits_{\Omega} \eta^2 \Phi^2\, dx \right] . \tag{22}$$

At first glance (22) does not look very promising since our goal is to bound $\int\limits_{\Omega} \eta^2\Phi^2\, dx$ from above through the l.h.s. of (22). But if we use (19) on the r.h.s. of (22) and observe that

$$\int\limits_{\Omega} |\nabla\eta|^2 h\left(|\varepsilon(u)|\right)\, dx \le c_{\mathrm{loc}} < \infty$$

for a local constant depending on η, we get

$$\int\limits_{\Omega} \eta^2|\varepsilon(u)|^2 h\left(|\varepsilon(u)|\right)\, dx \le c \left[c_{\mathrm{loc}} + \int\limits_{\Omega} |\nabla\eta|^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} \left(1 + |\varepsilon(u)|^2\right)^{\frac{\alpha}{2}} |\nabla u|^2\, dx \right] . \tag{23}$$

Let Ω^* denote a subdomain such that $\Omega^* \Subset \Omega$ and consider discs $B_r(z) \subset B_R(z)$ in Ω^*. The constant c_{loc} in (23) depends on Ω^*, and if η is chosen such that $0 \le \eta \le 1$, $\eta = 1$ on $B_r(z)$, $|\nabla\eta| \le c/(R-r)$, $\operatorname{spt}\eta \subset B_R(z)$, we deduce with the help of Korn's inequality (applied to the N-function $t \mapsto t^2h(t)$)

$$\text{l.h.s. of (23)} \;\ge c \left[\int_{B_r(z)} |\nabla u|^2 h\left(|\nabla u|\right)\, dx - c_{\mathrm{loc}} \right] , \tag{24}$$

where here and in the sequel "c" is a constant not depending on Ω^* or the approximation. For the discussion of the r.h.s. of (23) we observe that $b(t) := th(\sqrt{t})$, $t \ge 0$, is a N-function. Applying Young's inequality with $b_\beta(t) := \beta b(t)$, $\beta > 0$, we get

$$\begin{aligned} &\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} \left(1 + |\varepsilon(u)|^2\right)^{\frac{\alpha}{2}} |\nabla u|^2 \\ &\le \; b_\beta\left(|\nabla u|^2\right) + b^*_\beta\Big(\underbrace{\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} \left(1 + |\varepsilon(u)|^2\right)^{\frac{\alpha}{2}}}_{=:\xi} \Big) , \end{aligned}$$

b^*_β denoting the conjugate function for which we have $b^*_\beta(t) \le t\left[h^{-1}\left(\frac{1}{\beta}t\right)\right]^2$, h^{-1} being the inverse. It follows

$$\begin{aligned} \text{r.h.s. of (23)} \;\le\; c \Big[& c_{\mathrm{loc}} + \beta(R-r)^{-2} \int_{B_R(z)} |\nabla u|^2 h\left(|\nabla u|\right)\, dx \\ & + (R-r)^{-2} \int_{B_R(z)} \xi \left(h^{-1}\left(\frac{1}{\beta}\xi\right)\right)^2\, dx \Big] . \end{aligned} \tag{25}$$

For discussing the last integral on the r.h.s. of (25) we observe that on $[|\varepsilon(u)| \geq 1]$ (using $\alpha \leq 2$)

$$\frac{1}{\beta}\xi \leq \frac{c}{\beta}h'\left(|\varepsilon(u)|\right)|\varepsilon(u)| \leq \frac{c}{\beta}h(|\varepsilon(u)|)\,.$$

Assuming $\frac{\beta}{c} \leq 1$ the convexity of h gives ($h(0)=0$)

$$h\left(|\varepsilon(u)|\right) = h\left(\frac{\beta}{c}\frac{c}{\beta}|\varepsilon(u)|\right) \leq \frac{\beta}{c}h\left(\frac{c}{\beta}|\varepsilon(u)|\right)\,,$$

hence $\frac{1}{\beta}\xi \leq h\left(\frac{c}{\beta}|\varepsilon(u)|\right)$ and in conclusion

$$\begin{aligned}
\xi\left(h^{-1}\left(\frac{1}{\beta}\xi\right)\right)^2 &\leq \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}\left(1+|\varepsilon(u)|^2\right)^{\frac{\alpha}{2}}\frac{c^2}{\beta^2}|\varepsilon(u)|^2 \\
&\leq \frac{c}{\beta^2}h\left(|\varepsilon(u)|\right)|\varepsilon(u)|^\alpha
\end{aligned}$$

on the set $[|\varepsilon(u)| \geq 1]$. If $|\varepsilon(u)| \leq 1$, then by (A3)

$$\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} \leq h''\left(|\varepsilon(u)|\right) \leq c\,,$$

i.e. $\frac{1}{\beta}\xi \leq \frac{c}{\beta}$. From (7) we deduce $h^{-1}(s) \leq cs^{1/2}$ so that

$$\xi\left(h^{-1}\left(\frac{1}{\beta}\xi\right)\right)^2 \leq \frac{c}{\beta^2}\,,$$

and we get

$$\begin{aligned}
(26)\qquad & (R-r)^{-2}\int_{B_R(z)}\xi\left(h^{-1}\left(\frac{1}{\beta}\xi\right)\right)^2\,dx \\
&\leq c(R-r)^{-2}\beta^{-2}\int_{B_R(z)}\left[1+|\varepsilon(u)|^\alpha h\left(|\varepsilon(u)|\right)\right]\,dx\,.
\end{aligned}$$

Next we choose $\beta \sim (R-r)^2$ in order to obtain from (23) - (26) (γ_1 denoting a positve exponent)

$$\begin{aligned}
(27)\qquad & \int_{B_r(z)}|\nabla u|^2 h\left(|\nabla u|\right)\,dx \leq \frac{1}{2}\int_{B_R(z)}|\nabla u|^2 h\left(|\nabla u|\right)\,dx \\
&+c(R-r)^{-\gamma_1}\int_{B_R(z)}|\varepsilon(u)|^\alpha h\left(|\varepsilon(u)|\right)\,dx + c_{\mathrm{loc}}(R-r)^{-\gamma_1}\,.
\end{aligned}$$

Up to now the strict inequality $\alpha < 2$ has not been used but due to this assumption we can apply Young's inequality to the second term on the r.h.s. of (27) with the result ($\gamma_2 > 0$ properly chosen)

$$\begin{aligned} c(R-r)^{-\gamma_1} & \int_{B_R(z)} |\varepsilon(u)|^\alpha h\left(|\varepsilon(u)|\right)\, dx \\ \leq\ & \frac{1}{4}\int_{B_R(z)} |\nabla u|^2 h\left(|\nabla u|\right)\, dx + c(R-r)^{-\gamma_2}\int_{B_R(z)} h\left(|\varepsilon(u)|\right)\, dx\, . \end{aligned}$$

Since $\int_{B_R(z)} h\left(|\varepsilon(u)|\right)\, dx \leq c_{\mathrm{loc}}$, we finally deduce from (27)

$$\int_{B_r(z)} |\nabla u|^2 h\left(|\nabla u|\right)\, dx \leq \frac{3}{4}\int_{B_R(z)} |\nabla u|^2 h\left(|\nabla u|\right)\, dx + c_{\mathrm{loc}}(R-r)^{-\gamma_2}\, ,$$

and this inequality holds for $0 < r < R$, $R - r \leq 1$, $B_R(z) \subset \Omega^*$. We therefore can apply Lemma 3.1, p.161, of [Gi] to get $|\nabla u|^2 h(|\nabla u|) \in L^1_{\mathrm{loc}}$ (uniformly with respect to the approximation). Returning to (20), the desired L^2_{loc}-bound for Φ is established.

Step 2: estimates for the function φ

We recall that $\varphi := \left(\frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|}\right)^{1/2}\left(1+|\varepsilon(u)|^2\right)^{\alpha/4}$. Let us introduce the function

$$\Psi := \int_0^{|\varepsilon(u)|}\left(\frac{h'(t)}{t}\right)^{1/2}\, dt\, .$$

Hölder's inequality implies

$$\begin{aligned} \Psi^2 & \leq |\varepsilon(u)| \int_0^{|\varepsilon(u)|} \frac{h'(t)}{t}\, dt \\ & \overset{(A3)}{\leq}\ |\varepsilon(u)| \int_0^{|\varepsilon(u)|} h''(t)\, dt = |\varepsilon(u)| h'(|\varepsilon(u)|) \\ & \leq\ c h(|\varepsilon(u)|)\, , \end{aligned}$$

hence $\Psi \in L^2_{\mathrm{loc}}$ (uniformly w.r.t. the hidden approximation parameter ρ). At the same time

$$\begin{aligned} |\nabla \Psi|^2 & \leq |\nabla \varepsilon(u)|^2 \frac{h'(|\varepsilon(u)|)}{|\varepsilon(u)|} \\ & \overset{(10)}{\leq}\ D^2 H(\varepsilon(u))\left(\partial_k \varepsilon(u), \partial_k \varepsilon(u)\right) = \Phi^2\, , \end{aligned}$$

so that we can apply the result of Step 1 in order to get $\Psi \in W^1_{2,\mathrm{loc}}$ (uniformly). Therefore, by Trudinger's inequality (Theorem 7.15 in [GT]), we can state

$$\int_{B_t(x_0)} \exp(\beta_0 \Psi^2)\, dx \leq \text{ const } (B_t(x_0))\,, \tag{28}$$

where the positive number β_0 depends on the $W^1_2(B_t(x_0))$-norm of Ψ. Consider next a (large) number $\beta > 0$. We have on the set $[|\varepsilon(u)| \geq 1]$ (writing $\varepsilon := \varepsilon(u)$)

$$\begin{aligned} \Psi &\geq \int_{|\varepsilon|/2}^{|\varepsilon|} \left(\frac{h'(t)}{t}\right)^{1/2} dt \geq c \int_{|\varepsilon|/2}^{|\varepsilon|} \frac{h(t)^{1/2}}{t}\, dt \\ &\geq \ ch\,(|\varepsilon|/2)^{1/2} \int_{|\varepsilon|/2}^{|\varepsilon|} \frac{dt}{t} \geq ch \left(\frac{|\varepsilon|}{2}\right)^{1/2} \\ &\geq \ ch\,(|\varepsilon|)^{1/2} \geq c\,(|\varepsilon| h'(|\varepsilon|))^{1/2}\,, \end{aligned}$$

where we have made use of $th'(t) \geq h(t) \geq cth'(t)$ and the monotonicity of h. This shows

$$\varphi \leq c|\varepsilon|^{\frac{\alpha}{2}-1}\Psi\,,$$

hence

$$\varphi^2 \leq \mu\Psi^2 + c(\mu)|\varepsilon(u)|^{\alpha-2}$$

for any $\mu > 0$. On the set $[|\varepsilon(u)| \leq 1]$ it holds

$$\varphi^2 \leq \sup_{0\leq t\leq 1} \left(\frac{h'(t)}{t}(1+t^2)^{\frac{\alpha}{2}}\right) < \infty$$

so that in both cases we have

$$\varphi^2 \leq \mu\Psi^2 + c(\mu)\,. \tag{29}$$

Letting $\mu = \beta_0/\beta$ we get from (28) and (29)

$$\begin{aligned} \int_{B_t(x_0)} \exp(\beta\varphi^2)\, dx \ &\leq \ \int_{B_t(x_0)} \exp\left(\beta_0\Psi^2 + c(\beta)\right)\, dx \\ &\leq \ \text{const } \left(\beta, B_t(x_0)\right)\,, \end{aligned}$$

which means

$$\exp\left(\beta\varphi^2\right) \in L^1_{\mathrm{loc}} \quad \forall\, \beta > 0\,, \tag{30}$$

where the L^1_{loc}- norm depends on β but is independent of the approximation.

Step 3: conclusions

Letting $d := 2/\gamma = 3/2$, $f := \Phi^\gamma$, $g := \varphi^\gamma$, inequality (18) takes the form

$$\left(\fint_{B_R(x_0)} f^d \, dx \right)^{1/d} \le c \fint_{B_{2R}} f g \, dx$$

with $f \in L^d_{\mathrm{loc}}$. By (30) $\exp(\beta g^d) \in L^1_{\mathrm{loc}}$ for any $\beta > 0$, and Lemma 1.2 of [BFZ1] implies

$$\Phi^2 \ln^\beta (e + \Phi) \in L^1_{\mathrm{loc}} \tag{31}$$

for any $\beta > 0$. We claim that (31) implies

$$|\nabla \sigma|^2 \ln^\beta (e + |\nabla \sigma|) \in L^1_{\mathrm{loc}} \tag{32}$$

again for all $\beta > 0$. Assuming that (32) is true we can apply exactly the same arguments as used at the end of the proof of Theorem 1.1 in [BFZ1] to get the result of Theorem 1. Let us now discuss (32): to this purpose we recall estimate (14), i.e.

$$|\nabla \sigma| \le c \varphi \Phi \,,$$

which we combine with the inequality (see (2.12) in [BFZ1])

$$(st)^2 \ln^\omega (e + st) \le 2^\omega s^2 \ln^{\omega+2}(e + s) + c(\omega) \exp(6t)$$

valid for s, $t \ge 0$ and $\omega > 0$. This gives for $\beta > 0$

$$\begin{aligned} &\int_{B_R(x_0)} |\nabla \sigma|^2 \ln^\beta (e + |\nabla \sigma|) \, dx \\ \le\ & \int_{B_R(x_0)} (c\Phi\varphi)^2 \ln^\beta (e + c\Phi\varphi) \, dx \\ \le\ & c(\beta) \int_{B_R(x_0)} \Phi^2 \ln^{\beta+2} (e + \Phi) \, dx \\ & + c(\beta) \int_{B_R(x_0)} \exp(6c\varphi) \, dx \le c\,(\beta, B_R(x_0)) \,, \end{aligned}$$

where we have applied (30) and (31). Thus we have (32). □

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