A Representation Theorem of Infinite Dimensional Algebras and Applications to Language Theory.

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Abstract: We assign to each c.f. grammar G an infinite dimensionale algebra $\mathcal{A}_R(G)$ over a semiring R. From a representation φ of $\mathcal{A}_R(G)$ in $R<Z^{(*)}>$, when $Z^{(*)}$ is a certain polycyclic monoid, we derive easily the theorems of Shamir-Nivat-Salomaa, Chomsky-Schützenberger, Greibach about a hardest c.f. languages and Greibach N.F. LL(k) und LR(k) languages get an easy algebraic characterisation, which generalises to non deterministic LL and LR-languages, which are linear in time and space too.

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<u>Introduction:</u> Let X be a set and X* be the free monoid generated by X. The empty word is $1 \in X^*$ and $|u|$ means the length of $u \in X^*$. For monoids M and semirings R is R<M> the semiring of the finite sums

$$p = \sum_{m \in M} \alpha_m \cdot m \qquad \text{where } \alpha_m \in R.$$

We write often $\alpha_m = \langle p,m \rangle$. Only for finite many elements $m \in M$ it holds $\langle p,m \rangle \neq 0$. We always assume that R has a multiplicativ unit, which we identify with $1 \in M$.

Of special importance for our theory is the syntactique monoid $X^{(*)}$ of the Dyck language D(X) over X. This monoid called polycyclic monoid by Perrot [Pe] can be defined as follows too: One takes an bijectiv equivalent set X' to X such that $X \cap X' = \emptyset$. The bijection be $x \rightarrow \bar{x}$. We take further a symbol $0 \notin X \cup X'$ and form $(X \cup X' \cup 0)^*$; then we take the quotient of this free monoid by the relation system

$$x \cdot \bar{x} = 1, \; x \cdot \bar{y} = 0, \; 0 \cdot z = z \cdot 0 = 0 \text{ for } x,y \in X, z \in X \cup X' \cup \{0\}.$$

For $\bar{x}$ we write too x^{-1} and $x^{1} = x$.

We further make use of context free grammars G=(X,T,P,S) with $X \cap T = \emptyset$, $P \subset X \times X^2 \cup X \times T$ and $S \in X$. From this it follows that we have no ε-productions and $1 \notin L(G)$, if L(G) is the language generated by G. We assume further G to be free from superfluous variables. This means, that for $x \in X$ there axists derivations f and g such that

$$S \xrightarrow{f} u x v \xrightarrow{g} w \text{ and } w \in T^*.$$

The last assumption about G is, that S does not appear in the right side of any production $q \in P$.

It is usual to write P too as an equation system

$$x = \sum \alpha_{x,u} \cdot u \text{ for } x \in X$$

and $\alpha_{x,u} = 1$ if $(x,u) \in P$ and $\alpha_{x,u} = 0$ in all other cases.

Schützenberger has shown, that this makes sense in the following way: The equation system can be solved by a system of formal power series. L(G) can be looked at as the support of the power series belonging to S. The coefficient of the word w in the series gives the multiplicity of w relativ to G, this means the number of essentially different derivations of w from S.

We assign to the grammar an equation system in a dual way by writing the quadratic terms on the left side and the corresponding linear terms as sums on the right side. This means that we study equation systems of the form

$$x \cdot y = \sum_{z \in X} \alpha^{z}_{x,y} \cdot z, \quad t = \sum_{z \in X} \alpha^{z}_{t} \cdot z$$

with $\alpha^{z}_{x,y}$, $\alpha^{z}_{t} \in \{0,1\}$ and

$$\alpha^{z}_{x,y} = 1 \iff (z,xy) \in P,$$

$$\alpha^{z}_{t} = 1 \iff (z,t) \in P.$$

These relations are similar to the multiplication rules of finite dimensional algebras over a ring R. In general such an equation system does not define an associativ algebra. But with a simple trick we get an associativ algebra from this idea.

We assign to G a new alphabet $\overline{X}$ by setting

$$X_l = \{(x,l) \mid (z,xy) \in P\},$$
$$X_r = \{(y,r) \mid (z,xy) \in P\},$$
$$\overline{X} = X_l \cup X_r.$$

For (x,l) resp. (y,r) we write often shorter x_l resp. y_r.
Now we define the grammar $\overline{G}=(\overline{X},T,\overline{P},S_r)$
with

$$\overline{P} = \{(x_i,y_l z_r) \mid (x,yz)\in P, x\neq S, i\in\{l,r\}\}$$
$$\cup \{(S_r,x_l z_r) \mid (S,xz)\in P\}$$
$$\cup \{(x_i t) \mid i\in\{l,r\},\ (x,t)\in P\}.$$

Obviously $L(G)=L(\overline{G})$ holds.
We assign to G now the following equation system

$$x\cdot y=\sum \alpha^z_{x,y}\cdot z \qquad \text{for } x\in X_l, y\in Y_r \qquad (\mathcal{R}_G)$$

and

$$\alpha^z_{x,y} = \begin{cases} 1 \text{ if } (z,xy)\in\overline{P} \\ 1 \text{ if } z=S_r, x=S_l, y=S_r \\ 0 \text{ in all other cases.}\end{cases}$$

We generalize $\mathcal{R}_G$, because our proofs will not become harder by this, to the following situation: X_l and X_r are any to alphabets with $X_l\cap X_r=\emptyset$. We put $\overline{X}=X_l\cup X_r$. Further there are given two mappings

$$\delta' : X_l\times X_r \rightarrow R\langle\overline{X}^*\rangle$$

and

$$\eta' : T \rightarrow R\langle\overline{X}^*\rangle$$

with

$$\delta'(x,y)= \sum_{z\in\overline{X}} \alpha^z_{x,y}\cdot z \qquad \text{for } (x,y)\in X_l\times X_r,$$
$$\eta'(t) = \sum_{z\in X} \alpha^z_t\cdot z \qquad \text{for } t\in T.$$

We extend δ' on $\overline{X}^*$ by defining

$$\delta(u) = \begin{cases} u \text{ for } u\in X_r^*\cdot X_l^* \\ \delta'(x,y) \text{ for } u=xy\in X_l\cdot X_r \\ u_1\delta(xy)u_2 \text{ for } u_1\in X_r^*\cdot X_l^* \text{ and } xy\in X_l\cdot X_r.\end{cases}$$

Now we extended δ linear onto $R<\overline{X}^*>$. η is the corresponding extension of η' onto $R<T^*>$.
The equation system

$$xy = \delta(xy) \text{ for } xy \in X_1 \cdot X_r \qquad (\mathfrak{N})$$

is the generalisation of the system $(\mathfrak{N}_G)$.

We assign now an assoziativ algebra $\mathcal{A}_R(\delta)$ to $(\mathfrak{N})$. For this reason we iterate δ and finitely form the transitiv closure δ^* of δ. This means it holds $\delta \circ \delta^* = \delta^*$.

Now one easily proofs

LEMMA 1: $\delta^*(uv) = \delta^*(\delta^*(u) \cdot \delta^*(v))$.

Proof: The proof is given by induction on the length $|uv|$ of uv. For $|uv| \leq 2$ there is nothing to proof. The lemma obviously holds for $uv \in X_r^* \cdot X_1^*$ too. Let be $uv \notin X_r^* \cdot X_1^*$, this means that there exists a decomposition

$$uv = w_1 xy w_2 \text{ such that } w_1 \in X_r^* \cdot X_1^* \text{ and } xy \in X_1 \cdot X_r.$$

We have then

$$\delta(uv) = w_1 (\Sigma \alpha_{xy}^z \cdot z) w_2.$$

Each of the words of that decomposition has length $< n$ such that we are allowed to apply the induction hypothesis.

We discuss two cases:

Case 1: xy is totally part of u or part of v.
We assume the first situation: $u = u_1 xy u_2$.
Then we have

$$\delta(uv) = u_1 \cdot \left(\Sigma\, \alpha_{xy}^z \cdot z\right) \cdot u_2 \cdot v.$$

By induction we conclude

$$\delta^*(u_1(\Sigma\alpha^z_{x,y}\cdot z)u_2v) = \delta^*(\delta^*(u_1(\Sigma\alpha^z_{x,y}\cdot z)u_2)\cdot\delta^*(v))$$

$$= \delta^*(\delta^*(u)\cdot\delta^*(v)).$$

Therefore our lemma holds in this case.

<u>Case 2:</u> $u = u_1x$, $v = yv_1$ and $u_1x \in X_r^*\cdot X_l^*$, $x \in X_l$, $y \in X_r$.

Then we have

$$\delta(uv) = u_1(\sum_{\bar{X}} \alpha^z_{x,y}\cdot z)v_1$$

$$= (u_1(\sum_{X_r} \alpha^z_{x,y}\cdot z)v_1 + u_1((\sum_{X_l} \alpha^z_{x,y}\cdot z)v_1).$$

We apply to this expression the induction hypothesis as it is indicated by the brackets:

$$\delta^*(uv) = \delta^*(\delta^*(u_1\sum_{X_l} \alpha^z_{x,y}\cdot z)\cdot\delta^*(v_1))$$

$$+ \delta^*(\delta^*(u_1)\cdot\delta^*((\sum_{X_r} \alpha^z_{x,y}\cdot z)\cdot v_1))$$

$$= \delta^*(\delta^*(u_1)(\sum_{X_l} \alpha^z_{x,y}\cdot z)\cdot\delta^*(v_1))$$

$$+ \delta^*(\delta^*(u_1)(\sum_{X_r} \alpha^z_{x,y}\cdot z)\cdot\delta^*(v_1))$$

$$= \delta^*(\delta^*(u_1)\cdot xy\cdot\delta^*(v_1)$$

$$= \delta^*(\delta^*(u_1x)\cdot\delta^*(yv_1)).$$

The last relation holds because of

$$\delta^*(u_1x) = u_1x \quad \text{and} \quad \delta^*(yv_1) = y\delta^*(v_1).$$

This proofs case 2 and our lemma 1 has been proofed.

Now we define the operation 'o' on $R<\overline{X}^*>$ by setting

$$u \circ v := \delta^*(uv).$$

It follows from this

$$(u\circ v)\circ w=\delta^*(\delta^*(uv)\cdot w)=\delta^*(\delta^*(uv)\delta^*(w))=\delta^*(uvw),$$

$$u\circ(v\circ w)=\delta^*(u\cdot\delta^*(vw))=\delta^*(\delta^*(u)\delta^*(vw))=\delta^*(uvw).$$

Therefore the following theorem holds.

Theorem 1: $\mathcal{A}_R(\delta) := (R<\overline{X}^*>,+,\circ)$

is an assoziative algebra and

$$\delta^*: (R<\overline{X}^*>,+,\cdot) \rightarrow (R<\overline{X}^*>,+,\circ)$$

is an algebra homomorphism.

For the algebra we so assigned to our grammar G we write $\mathcal{A}_R(G)$. We extend this algebra to include the terminals too. For this reason we use the defined mapping η and extend η onto $(\overline{X}\cup T)^*$ by setting $\eta(x)=x$ for $x\in\overline{X}$.
Now for $u,v\in(\overline{X}\cup T)^*$ we define

$$u\circ v=\delta^*(\eta(uv)).$$

The assoziativ algebra we get by this construction we call $\bar{\mathcal{A}}_R(G)$.

For $u_1\circ u_2\circ\ldots\circ u_n$ we write again $u_1u_2\ldots u_n$.
In a case that it is not clear which product we mean we write

$$u_1u_2\ldots u_n \quad [\mathcal{A}_R(G)]$$

if the product is in $\mathcal{A}_R(G)$. Analogously we proceed with other algebras.
The following concerns the questions

How are the algebras $\mathcal{A}_R(G)$ structured?
Which information contains $\mathcal{A}_R(G)$ about $L(G)$?
How is the structure of $\mathcal{A}_R(G)$, if G is determistic?

The follwing section is dedicated to the first question.

A representation theorem for $\mathcal{A}_R(\delta)$

We are going to show, that for each algebra $\mathcal{A}_R(\delta)$ there exist a non trivial representation $\varphi: \mathcal{A}_R(\delta) \rightarrow R\langle X^{(*)}\rangle$. We will show that the algebra $R\langle X^{(*)}\rangle$ for our algebras and for the finite dimensional algebras plays a similar role as the matrix ring in the finite dimensional case. It is clear that $R\langle X^{(*)}\rangle$ is a special exemplar of our algebras $\mathcal{A}_R(\delta)$. The following lemma showes that $R\langle X^{(*)}\rangle$ has a very simple algebraic structure.

LEMMA 2: $\mathcal{A}_D = R\langle X^{(*)}\rangle$ contains only trivial two sided ideals.

Ideals $\mathfrak{a}$ of $\mathcal{A}_D$ here are considered to be trivial, if there exists an ideal $\mathfrak{a}'$ of R such that $\mathfrak{a} = \mathfrak{a}'\langle X^{(*)}\rangle$.

Proof of LEMMA 2: Let $\mathfrak{a} \subset \mathcal{A}_D$ be an two sided ideal, that means that $\mathcal{A}_D \mathfrak{a} \mathcal{A}_D \subset \mathfrak{a}$ holds. We study several cases.

1) Let be $\alpha \in R$ and $\alpha \cdot \bar{u}v$ with $u,v \in X^*$ in $\mathfrak{a}$. Then it follows $\alpha \in \mathfrak{a}$.

2) $p = \alpha\bar{u}v + q \in \mathfrak{a} \Rightarrow p' = \alpha + q' \in \mathfrak{a}.\ q' = uqv.$

3) $p = \alpha + \beta\bar{u}v + q \in \mathfrak{a}.$

   a) $u\bar{v} = 0 \Rightarrow up\bar{v} = \beta + q.$
      $up\bar{v}$ has one summand fewer then p.

   b) $u\bar{v} \neq 0$. We may assume $u\bar{v} = u' \in X^*,\ u' \neq 1$.
      We have
      $up\bar{v} = \alpha u' + \beta + uq\bar{v}.$
      chose $y \in X$, $y \neq$ last letter of u'.
      $up\bar{v}\bar{y} = \beta\bar{y} + uq\bar{v}\bar{y}$, this means one summand fewer.

4) From 1),2) and 3) it follows:
   $\langle p,u\rangle = \alpha,\ p \in \mathfrak{a} \Rightarrow \alpha \in \mathfrak{a}.$
   Let be $\mathfrak{a}' = \mathfrak{a} \cap R$, then therefore it holds $\mathfrak{a} = \mathfrak{a}'\langle X^{(*)}\rangle$, what we have claimed.

We show now, that each finite dimensional algebra $\mathcal{A}$ over R has a non trivial representation in $\mathcal{A}_D$.

Let be Z a finite basis of $\mathcal{A}$ over R and $\mathcal{A}$ being given by the relations

$$x\cdot y = \sum_{z\in Z} \alpha^{z}_{x,y} \cdot z, \quad \alpha^{z}_{x,y} \in R.$$

We define

$$\varphi : \mathcal{A} \longrightarrow \mathcal{A}_D \quad \text{by defining}$$

$$\varphi(y) := \sum_{z,u\in Z} \bar{z}\cdot\alpha^{u}_{z,y}\cdot u \quad \text{for } y\in Z.$$

This defines φ uniquely. ($\bar{z}$ is the inverse of z in $Z^{(*)}$).

<u>Theorem 2:</u> φ is an algebra homomorphism. If $\mathcal{A}$ contains a multiplicativ unit, then φ is injectiv.

<u>Proof:</u> It is sufficient to show, that the relation

$$\varphi(y_1)\cdot\varphi(y_2) = \varphi(y_1 y_2) \text{ holds for } y_1, y_2 \in Z.$$

We calculate straight forward and get

$$\varphi(y_1)\cdot\varphi(y_2) = \sum_{z_1,u_1,z_2,u_2} \bar{z}_1\, \alpha^{u_1}_{z_1,y_1}\cdot u_1 \cdot \bar{z}_2\cdot \alpha^{u_2}_{z_2,y_2}\cdot u_2$$

$$= \sum_{z_1,u_1,u_2} \bar{z}_1\, \alpha^{u_1}_{z_1,y_1}\, \alpha^{u_2}_{u_1,y_2}\cdot u_2$$

$$= \sum_{z_1,u_2} \bar{z}_1 \left(\sum_{u_1} \alpha^{u_1}_{z_1,y_1}\, \alpha^{u_2}_{u_1,y_2}\right)\cdot u_2$$

Now we apply $(z_1 y_1)y_2 = z_1(y_1\cdot y_2)$ and R being element wise commutable with Z we get further

$$= \sum_{z_1,u_2} \bar{z}_1\left(\sum_{u_1} \alpha^{u_1}_{y_1,y_2}\cdot\alpha^{u_2}_{z_1,u_1}\right)\cdot u_2 = \sum_{u_1} \alpha^{u_1}_{y_1,y_2}\, \varphi(u_1)$$

$$= \varphi(y_1\cdot y_2).$$

This proofs are the first part of our theorem.

Let be

$$u = \sum_{y \in Z} \beta_y \cdot y \quad \text{and } \varphi(u) = 0.$$

then it follows

$$\varphi(u) = \sum_{x,z \in Z} \bar{z} \cdot x \sum_{y \in Z} \beta y \ \alpha^x_{z,y} \cdot x = 0,$$

and therefore we have

$$\sum_{y \in Z} \alpha^x_{z,y} \ \beta_y = 0 \qquad \text{for } x,z \in Z. \qquad (*)$$

Let be now $v \in \mathcal{A}$,

$$v = \sum_{y \in Z} \gamma_y \cdot y.$$

We form

$$v \cdot u = \sum_{y_1, y_2} \gamma_{y_1} \cdot \beta_{y_2} \ y_1 y_2 = \sum_{y_1, x} \gamma_{y_1} \left(\sum_{y_2} \alpha^x_{y_1, y_2} \ \beta_{y_2} \right) \cdot x$$

Because of (*) it holds also

$$v \cdot u = 0 \quad \text{for all } v \in \mathcal{A}.$$

We chose $v = 1$ and have $u = 0$

This proves the second part of our theorem.

Without proof we give for the case of matrix rings another representation.

Theorem 3: Let $\mathcal{A}$ be a finite dimensional ring of quadratic matrices $(a_{z,y})_{z,y \in Z}$, then

$$\varphi(a) = \sum_{z,y \in Z} \bar{z} a_{z,y} \cdot y$$

is a monomorphism from $\mathcal{A}$ into $\mathcal{A}_D$.

Now we come to the main result of this section. To construct the representation $\varphi : \mathcal{A}_R(\delta) \rightarrow \mathcal{A}_D$ we first define a suitable alphabet for $\mathcal{A}_D$.

For $u \in \overline{X}$ and $x \in X_r$ (remember $\overline{X} = X_1 \cup X_r$), we define

$$[u:x] = \begin{cases} 0 \text{ if for all } w \in \overline{X}^* \text{ it holds } <\delta^*(uw), x> = 0, \\ 1 \quad \text{for } u = x \\ \text{free variable in all other cases.} \end{cases}$$

Clearly it follows from $[u:x] \neq 0$ and $u \in X_r$ that $u=x$.
We set

$$Z = \{[u:x] \mid [u:x] \neq 1,0;\ u \in \overline{X},\ x \in X_r\}$$

and $\mathcal{A}_D = R<Z^{(*)}>$.

For $z \in \overline{X}$ we define

$$\varphi'(z) = \sum_{\substack{y,v,u,x \\ [y:x] \in Z}} \alpha^u_{y,v} \overline{[y:x]}[u:x][z:v]$$

<u>Theorem 4:</u> There exists an uniquely defined extension of φ' to an algebra homomorphism $\varphi : \mathcal{A}_R(\delta) \rightarrow \mathcal{A}_D$

<u>Proof:</u> $\overline{X}$ generates $\mathcal{A}_R(\delta)$ and therefore there exists not more as one homomorphic extension of φ' onto $\mathcal{A}_R(\delta)$. To show that such an extension exists, it is sufficient to show, that for the linear extension φ of φ' it holds

$$\varphi(z_1) \cdot \varphi(z_2) = \varphi(z_1 z_2) \text{ holds for } z_1 \in X_1,\ z_2 \in X_r.$$

By straight forward calculation one gets

$$\varphi(z_1) \cdot \varphi(z_2) =$$

$$\sum_{\substack{y_1,v_1,u_1,x_1, \\ y_2,v_2,u_2,x_2,}} \alpha^{u_1}_{y_1,v_1} \overline{[y_1:x_1]}[u_1:x_1][z_1:v_1]\alpha^{u_2}_{y_2,v_2} \overline{[y_2:x_2]}[u_2:x_2][z_2:v_2]$$

$$= \sum_{\substack{y_1, v_1, u_1, x_1 \\ v_2, u_2}} \alpha_{y_1, v_1}^{u_1} \alpha_{z_1, v_2}^{u_2} \overline{[y_2 : x_1]} [u_1 : x_1][u_2 : v_1][u_2 : v_2].$$

For $z_2 \neq v_2$ we have $[z_2 : v_2] = 0$ becauce $z_2 \in X_r$. Therefore there remain only the cases $z_2 = v_2$, that means $[z_2 : v_2] = 1$.
We use the commutativity of R and have

$$\varphi(z_1) \cdot \varphi(z_2) = \sum_{u_2} \alpha_{z_1, z_2}^{u_2} \cdot \sum \alpha_{y_1, v_1}^{u_1} \overline{[y_1 : x_1]} [u_1 : x_1][u_2 : v_1]$$

$$= \sum \alpha_{z_1 z_2}^{u_2} \varphi(u_2) = \varphi(z_1 \cdot z_2).$$

Historical remark: Nivat uses in his thesis a homomorphism which formally looks like our homomorphism φ. But ψ is a mapping

$$\psi : R\langle X^* \rangle \longrightarrow R\langle H(B) \rangle,$$

where H(B) is the free half group generated by B. The main difference comes from the different domains of φ and ψ. Nivat uses ψ to proof the representation theorem of Shamir. But he needs for this proof the normal form theorem of Greibach, which follows as the theorem of Shamir from the existence of φ. The reason is, that $\mathcal{A}_R(G)$ contains a lot of information over G, but R<X*> not at all. More detailed informations over this subject the reader may find in the book [Sa] of Saloma.

As we will show later one can derive from φ a representation of L(G) by a grammar in Greibach normal form. The size of the grammar corresponds to the size of φ. We define

$$|\mathcal{A}_R(\delta)| = \sum_{x, y, z \in \overline{X}} |\alpha_{x,y}^{z}|$$

with

$$|\alpha| = \begin{cases} 1 \in \mathbb{N} & \text{for } \alpha \neq 0 \\ 0 \in \mathbb{N} & \text{else} \end{cases}$$

For $p \in \mathcal{A}$ we put

$$|p| = \sum_{w \in Z^{(*)}} |\langle p,w \rangle| .$$

We define as size $|\varphi|$ of φ

$$|\varphi| = \sum_{z \in \overline{X}} |\varphi(z)| .$$

One easily proofs

LEMMA 3: $|\varphi| \leq |\mathcal{A}_D| \cdot |\overline{X}|^2$,
where $|\overline{X}|$ is the number of elements of $\overline{X}$.

3. Invariants of the Transformation $G \rightarrow \overline{G}$.

We return to grammars and study which properties of G remain unchanged when passing from G to $\overline{G}$ as we did in section 1.

The set of derivations of words into other words using G we call $\mathcal{F}$. If $f \in \mathcal{F}$ then $Q(f)$ is the word on which the derivation starts and $Z(f)$ is the result of the derivation f. If $f,g \in \mathcal{F}$ and $Q(f) = Z(g)$, then $f \circ g$ is the derivation, which one gets by first applying g and then applying f. Obviously $Q(f \circ g) = Q(g)$ and $Z(f \circ g) = Z(f)$ and "o" is assoziativ. The empty derivation belonging to the word w is 1_w. We have $1_{Z(f)} \circ f \circ 1_{Q(f)} = f$. In the case $Q(f) = w$, $Z(f) = v$ we write too

$$w \xrightarrow{f} v.$$

If we have

$$w_1 \xrightarrow{f_1} v_1 \quad \text{and} \quad w_2 \xrightarrow{f_2} v_2$$

we may form the derivation

$$w_1 \cdot w_2 \xrightarrow{f_1 \times f_2} v_1 \cdot v_2 .$$

This leads to an further assoziativ operation on $\mathcal{F}$. The unit belonging to 'x' is 1_1.

Both operations are connected by the property

$$(f_1 \circ g_1) \times (f_2 \circ g_2) = (f_1 \times f_2) \circ (g_1 \times g_2)$$

if the left side is defined. ($\mathcal{F}$, $(X \cup T)^*$, Q,Z,o,x) forms a free monoidal category, which in [Ho.O] has been called free x-category and syntactical category in [Be].
The elements of $\mathcal{F}$ are trees or words over the derivation trees in the case of context free grammars. The trees of the production set P generate $\mathcal{F}$. $\overline{\mathcal{F}}$ is the category belonging to $\overline{G}$. The structure preserving mappings are called x-functors. An x-functor consists of two mappings (φ_1, φ_2), the first one is a monoid homomorphism from the monoid of the source category into the monoid of the aim category. φ_2 maps the derivation set into the derivation set.
We use further the apprevations

$$\mathrm{Mor}_{\mathcal{F}}(w,v) = \{f \in \mathcal{F} \mid Q(f) = w,\ Z(f) = v\},$$

$$\mathrm{mult}_G(w) = \mathrm{card}\ \mathrm{Mor}_{\mathcal{F}}(S,w).$$

The multiplicity of w over G tells us in how many essentially different ways w may be derived from S using G.

LEMMA 4: For $w \in T^*$ it holds

$$\mathrm{mult}_G(w) = \mathrm{mult}_{\overline{G}}(w)$$

Proof: To proof this lemma we construct the x-functor $\varphi = (\varphi_1, \varphi_2)$ from $\overline{\mathcal{F}}$ onto $\mathcal{F}$ which forgets the indices r,l in $\overline{G}$. Thus we define

$$\varphi_1(x,i) = x \quad \text{for } x \in X \text{ and } i \in \{l,r\}$$

and for $f \in \overline{P}$

$$\varphi_2(f) = f' \iff \varphi_1(Q(f)) = Q(f'),\ \varphi_1(Z(f)) = Z(f').$$

This defines uniquely an x-functor from $\overline{\mathcal{F}}$ into $\mathcal{F}$.

Obviously $\varphi_2(\overline{P}) = P$.

We show now for $x_i \in \overline{X}$, that the restriction

$$\varphi_2 | \mathrm{Mor}_{\overline{\mathcal{F}}}(x_i, (\overline{X} \cup T)^*) \longrightarrow \mathrm{Mor}_{\mathcal{F}}(x, (X \cup T)^*)$$

is bijective. From this fact our lemma follows then immediatly. The proof is by induction on the number $|f|$ of knots of the trees of f.

Our claim is true for all f such that $Q(f) = x_i$ and $|f| = 1$. Inductively we assume, that it holds for

$$\varphi_2 | \{f \in \mathrm{Mor}_{\overline{\mathcal{F}}}(x_i, (\overline{X} \cup T)^*) \,|\, |f| \leq n\} \longrightarrow$$

$$\{f \in \mathrm{Mor}_{\mathcal{F}}(x, (X \cup T)^*) \,|\, |f| \leq n\}.$$

It is clear that

$$|f| = |\varphi_2(f)| \quad \text{for } f \in \overline{\mathcal{F}}.$$

Let be $|f| = n+1$ and $Q(f) = x_i$.

We decompose

$$f = (1_u \times h \times 1_v) \circ g$$

such that $h \in \overline{P}$ and $|u|$ being minimal with this condition. This determines h uniquely.

From

$$(1_u \times h \times 1_v) \circ g = (1_u \times h \times 1_v) \circ g'$$

it follows $g = g'$, that means that g is uniquely determined by this condition. [Ho.65]

Because of

$$\varphi_2(f) = (1_{\varphi_1(u)} \times \varphi_2(h) \times 1_{\varphi_1(v)}) \circ \varphi_2(g)$$

and $|\varphi_1(u)| = |u|$ we see, that $\varphi_2(f)$ has exactly one co-image. This proofs our lemma.

Now we are going to show that the LL(k) and LR(k) properties of G do not change, when passing from G to $\overline{G}$. [Kn],[H.S.]. For this reason we introduce the following notions.

$f \in \mathcal{F}$ we call <u>u-left-prim</u> for $u \in (X \cup T)^*$,

iff from

$f = (1_u \times h) \circ g$ it follows $g = f$.

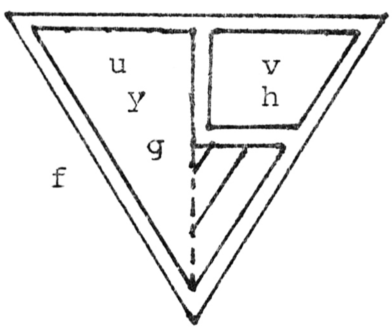
The definition <u>u-right-prim</u> is symmetric to the former definition.

One easily shows the

<u>LEMMA 5:</u> For each $f \in \mathcal{F}$, u prefic of Z(f), there exists exactly one decomposition $f = (1_u \times h) \circ g$ such that g is <u>u-left-prim</u>.

Relating to the notion in this lemma, we call g the <u>u-left-prim factor</u> of f and h the <u>v-right-base</u> of f if $Z(f) = u \cdot v$. We write

$$g = \text{left-prim}\ (u,f), \quad h = \text{right-base}\ (v,f).$$



This figure should explain the definitions. We use the notions too, which we get from this definition by changing 'left' into 'right' and right into 'left'.

We give now the definition of LR(k) which is equivalent to the definition [Har.P.502] and for LL(k) equivalent to the one given by Hennie and Stearns [H.S.]. The reader should remember, that we assume G to be in Chomsky NF and G without ε-productions.

G is a LR(k) grammar resp. LL(k+1) grammar for $k = 0,1,\ldots,$ if the following holds:

For all $f,f' \in \mathcal{F}$ with $Z(f) = u \cdot v$ and $Z(f') = u \cdot v'$ we have

left-base(u,f) = left-base(u,f')

for $Q(f) = Q(f') = S$ and $\mathrm{First}_k(v) = \mathrm{First}_k(v')$

resp.:

left-prim(u,f) = left-prim(u,f') for $Q(f) = Q(f') \in X$ and $\mathrm{First}_k(v) = \mathrm{First}_k(v')$.

Remember that we assume that S never appears on the right hand side of any production. From this it follows [Har.P.525], that our LR(o) grammars produce only ALR(o)-languages i.e. strict determin. languages.

LEMMA 6: If G is a LL(k) resp.LR(k) grammar then $\overline{G}$ is a LL(k) resp. LR(k) grammar.

Proof: To proof this lemma we use the x-functor defined in the proof of lemma 4. Let be $f: x_i \rightarrow uv$ any derivation tree of $\overline{\mathcal{F}}$ and $x_i \in \overline{X}$. We define

$$h = \text{left-base}(u,f) \text{ and } g = \text{left-prime}(u,f).$$

Then we have

$$h' = \varphi_2(h) = \text{left-base}(\varphi_1(u),\varphi_2(f))$$

and

$$g' = \varphi_2(g) = \text{left-prime}(\varphi_1(u),\varphi_2(f)).$$

Now x_i and g' determine g uniquely as has been shown in lemma 4.
Now let be G a LL(k) grammar. Then g' is uniquely determined by $\varphi_1(x_i)$ and $\varphi_1(u)\cdot \text{First}_k\varphi_1(v)$ and therefore x_i and $u\cdot\text{First}_k(v)$ determine g' uniquely and so g too. This means that $\overline{G}$ is a LL(k) grammar.

Now we study the case, that G is a LR(k) grammar. By the same argumentation as before we see that h is uniquely determined by h' and Q(h). Using the LR(k) property we see that $Q(f) = S_r$ and $u\cdot\text{First}_k v$ determined h' uniquely. If we are able to show that Q(h) is uniquely determined by $u\cdot\text{First}_k v$, then it follows that $\overline{G}$ has the LR(k) property too. For this reason it is sufficient to show, that $Q(h)\in X_l^*$ holds.
Therefore let be

$$f = (h\times 1_v)\circ g,$$

where by definition of h as left-base of f the factor g is v-right-prime. Suppose $Q(h)\notin X_l^*$, then there exists a decomposition

$$Q(h) = q_1 x_l x_r q_2$$

and we have

$$Z(g) = q_1 x_l x_r q_2 v.$$

This contradicts the assumtion g to be v-right-prime.
Therefore we have $Q(h)\in X_l^*$, what we wished to show.

The last result in our proof we will use in a later part of this paper again. Therefore we formulate it as

<u>LEMMA 7:</u> If f is v-right-prime then it holds $u\in X_l^*$ for $Z(f) = u\cdot v$. If h is u-left-base of f, then is $Q(h)\in X_l^*$.
The lemma remains true if we exchange the words left and right.

4. Connections between $L(G)$, $\mathcal{A}_R(G)$ and φ

In this section we work out the general relations between $L(G)$ and $\mathcal{A}_R(G)$ and our representation φ. A first information gives the

Theorem 5: $w \in L(G) \iff \langle\eta(w),S_r\rangle \neq 0$ for $\chi(R)=0$ ($\chi(R)$=characteristic of R),

$$\mathrm{mult}_G(w) = \langle\eta(w),S_r\rangle \; [\mathcal{A}_{\mathbb{N}}(G)].$$

(remember: '[]' contains the algebra in which the relation is to be understood).

Proof: As we have shown in lemma 4 we may use $\overline{G}$ instead of G. The proof is by induction on the length $|w|$ of w. For the proof we show a little more general result:

$$\mathrm{mult}_{\overline{G}}(x_i,w) = \langle\eta(w),x_i\rangle \text{ for } w\in T^*,\; x_i\in\overline{X},$$

where

$$\mathrm{mult}_{\overline{G}}(x_i,w) = \mathrm{cardMor}_{\overline{\mathcal{F}}}(x_i,w).$$

The Theorem is obvious for $|w| = 1$.
Let be $f: x_i \longrightarrow w$ a derivation and $|w|>1$. Then we may decompose

$$f = p\circ(f_1\times f_2),\; p\in\overline{P}.$$

Therefore we have

$$\mathrm{mult}_{\overline{G}}(x_i,w) = \sum_{\substack{w_1\cdot w_2=w \\ w_1\neq 1,\, w_2\neq 1 \\ \langle y_1 z_r, x_i\rangle=1}} \mathrm{mult}_{\overline{G}}(y_1,w_1)\cdot \mathrm{mult}_{\overline{G}}(z_r,w_2).$$

By the induction hypothesis is

$$\mathrm{mult}_{\overline{G}}(x_i,w) = \sum_{\substack{w_1\cdot w_2=w \\ w_1\neq 1,\, w_2\neq 1}} \langle\eta(w_1),y_1\rangle\cdot\langle\eta(w_2),z_r\rangle\cdot\langle y_1 z_r,x_i\rangle$$

$\langle\eta(w),x_i\rangle$,

what has to be proved.

In the following we use the definition

$$(u) = u + \mathcal{A}_R(G).$$

(u) is the additive residual-class of u.

<u>Corollary to THEOREM 5:</u> For $R = \mathbb{B}$ = boolean-ring with two elements we have

$$L(G) = \eta^{-1}(S_r).$$

We now study how the representation φ transforms the residual-class (S_r).

<u>LEMMA 8:</u> For $z_o z_1 \ldots z_n \in \bar{X}^*$ it holds for $z_o \neq s$

$$\langle z_o z_1 \ldots z_n, s\rangle \quad \langle \varphi(z_1 \ldots z_n), \overline{[z_o:s]}\rangle.$$

<u>Proof:</u> The proof is by induction on n.

case n = 1. Then we have

$$\langle z_o z_1, s\rangle = \alpha^s_{z_o, z_1}.$$

Because of

$$\varphi(z_1) = \sum \alpha^u_{y,v} \overline{[y:x]}[u:x][z_1:v]$$

it follows

$$\langle \varphi(z_1), \overline{[z_o:s]}\rangle = \sum_{[y:x][u:x][z_1:v]=[z_o:s]} \alpha^u_{y,v}$$

From this it follows that the sum is only to be taken over the cases

$$y = z_o,\ x = s,\ u = x,\ z_1 = v.$$

Therefore

$$\langle \varphi(z_1), \overline{[z_o:s]}\rangle = \alpha^s_{z_o, z_1},$$

what has to be shown.

Inductionstep: It holds

$$\varphi(z_1 \ldots z_n) = \sum_{u,y,v,x} \alpha^u_{y,v} \overline{[y:x]}[u:x][z_1:v]\varphi(z_2 \ldots z_n).$$

From this one derives

$$<\varphi(z_1 \ldots z_n), \overline{[z_o:s_o]}^{-1}> = \sum_{u,v} \alpha^u_{z_o,v} <[u:s_o][z_1:v]\varphi(z_2 \ldots z_n),1>$$

$$= \sum_{u,v} \alpha^u_{z_o,v} \sum_{j=2}^{n-1} <\varphi(z_2 \ldots z_j), \overline{[z_1:v]}^{-1}> \cdot <\varphi(z_{j+1} \ldots z_n), \overline{[u:s_o]}^{-1}>$$

$$+ \sum_{v} \alpha^{s_o}_{z_o,v} <\varphi(z_2 \ldots z_n), \overline{[z_1:v]}^{-1}>$$

$$+ \sum_{u} \alpha^u_{z_o,z_1} <\varphi(z_2 \ldots z_n), \overline{[u:s_o]}^{-1}>.$$

By the induction hypothesis we get from this

$$= \sum_{u,v} \alpha^u_{z_o,v} \sum_{j=1}^{n} <z_1 \ldots z_j, v><uz_{j+1} \ldots z_n, s_o>, \qquad (*)$$

where $z_{n+1} = 1$ has to be taken.

In the other hand it holds

$$<z_o z_1 \ldots z_n, z_o> = \sum_{k=o}^{n-1} \sum_{y_o,y_1} \alpha^{s_o}_{y_o,y_1} <z_o \ldots z_k, y_o><z_{k+1} \ldots z_n, y_1>. \qquad (**)$$

We proof by induction

$$<z_o \ldots z_k, y_o> = \sum_{\substack{j=1 \\ u,v}}^{k} \alpha^u_{z_o,v} <z_1 \ldots z_j, v><uz_{j+1} \ldots z_k, y_o>.$$

For $k = 1$ we have

$$<z_o\, z_1, y_o> = \sum_{u,v} \alpha^u_{z_o,v} <z_1, v><u, y_o> = \alpha^{y_o}_{z_o,z_1}.$$

Therefore our claim holds for k = 1.
We assume the claim being correct for k < n and apply this onto $(\overset{*}{*})$. We get

$$<z_o\ldots z_n,s_o> = \sum_{\substack{k=1\\ y_o,y_1}}^{n-1} \alpha^{s_o}_{y_o,y_1} \sum_{\substack{y=1\\ u,v}}^{k} \alpha^{u}_{z_o,v} <z_1\ldots z_j,v><uz_{j+1}\ldots z_k,y_o> <z_{k+1}\ldots z_n,y_1>.$$

$$+ \sum_{y_o,y_1} \alpha^{s_o}_{y_o,y_1} <z_o,y_o><z_1\ldots z_n,y_1>$$

$$= \sum_{\substack{j=1\\ u,v}}^{n-1} \alpha^{u}_{z_o,v} <z_1\ldots z_j,v> \sum_{\substack{k=j\\ y_o,y_1}}^{n-1} \alpha^{s_o}_{y_o,y_1} <uz_{j+1}\ldots z_k,y_o> <z_{k+1}\ldots z_n,y_1>$$

$$+ \sum_{v} \alpha^{s_o}_{z_o,y_1} <z_1\ldots z_n,v>$$

$$= \sum_{\substack{j=1\\ u,v}}^{n} \alpha^{u}_{z_o,v} <z_1\ldots z_j,v><uz_{j+1}\ldots z_n,s_o>.$$

Therefore our claim is too true for k = n. This proofs together with (*) our lemma.

<u>LEMMA 9 :</u> Using the notation of Lemma 8 it holds

$$<z_1\ldots z_n,S_r> = <\varphi(z_1\ldots z_n), \overline{[S_1:S_r]}>.$$

Proof: From Lemma 8 it follows

$$<S_1 z_1 \ldots z_n, S_r> \; = \; <\varphi(z_1 \ldots z_n), \overline{[S_1 : S_r]}>.$$

By definition of $\mathcal{A}_R(G)$ we get

$$<S_1 z_1 \ldots z_n, S_r> \; = \; <z_1 \ldots z_n, S_r>$$

and from this directly our lemma.

If we now concatenate the homomorphisms η and φ in this sequence we get a homomorphism $h = \varphi \circ \eta$ from T^* into $R<X^{(*)}>$. This leads us to a representation theorem for c.f. languages, which is nearly the theorem of Shamir ([Sh] see too [N_i]). Shamir uses instead of $X^{(*)}$ the half group $H(X)$, that means he does not make use of the relations $x \cdot \overline{y} = 0$ for $x \neq y$.

THEOREM 6 (Shamir): To each c.f. language $L \subset T^*$ there exists a monoidhomomorphism $h: T \longrightarrow R<Z^{(*)}>$ and an additiv residual class $(\$)$ such that $L = h^{-1}((\$))$ holds.

Proof: The proof follows from lemma 9 and theorem 5 by choosing $\$ = \overline{[S_1 : S_r]}$.

Each polycyclic monoid $Z^{(*)}$ can be embedded by a monomorphism into $\{x_1, x_2\}^{(*)}$. This embedding even can be done such that $\overline{[S_1 : S_r]}$ in all cases will be mapped onto the same element $a_o \in \{x_1, x_2\}^{(*)}$. We extend this embedding to a ring homomorphism form $R<Z^{(*)}>$ into $R<\{x_1, x_2\}^{(*)}>$ and put it behind h. The resulting homomorphism let be $\overline{h}$. Then the following holds.

Corollary to THEOREM 6: For each c.f. language $L \subset T^*$ there exists a homomorphism

$$\overline{h} : T \longrightarrow R<\{x_1, x_2\}^{(*)}>$$

such that $\quad L = \overline{h}^{-1}((a_o))$

holds.

In this form this theorem was first given in [Ho.1], where it was derived from the theorem of Chomsky-Schützenberger as an algebraic version of the theorem of Greibach about a hardest language under homomorphic reduction [Gr.]. This language one gets from the representation given above by forming the c.f. language of the expressions consisting of products of polynomials of $R<\{x_1,x_2\}^*>$. The theorem of Greibach and the representation above have been found independently from the theorem of Shamir. A long time one has not payed attention to the theorem of Shamir outside of the French School, because its complexity theoretic aspects had not been seen. As in [Ho.4] has been shown one can similar representations construct for r.e., c.s., d.c.s. and other classes of languages. It seems to be possible to construct for each complexity class given by a time bound T(n) a language which is hardest in the categorie of homomorphic reductions.

We show that it is as easy as in the case of the theorem of Shamir to proof the theorem of Chomsky-Schützenberger from our theorem 5 and lemma 9. For this reason we change a little bit the definition of h, but such that lemma 9 remains applicable.

We define a homomorphism $g : T^* \longrightarrow R<(\overline{Z}\cup\overline{T})^*>$ by setting for $t\in T$

$$g(t) = \sum_{z\in X} \alpha_t^z \sum \alpha_{y,v}^u \overline{[y:x]}t\overline{t}[u:x][z:v].$$

We notice that the difference of g and h consists in two things: The codomain is different and between $\overline{[y:x]}$ and $[u:x][z:v]$ there the product $t\overline{t}$ has been inserted.

Let be $\overline{g}$ the prolongation of g to a homomorphism from T^* into $R<(\overline{Z}\cup\overline{T})^{(*)}>$, by applying the canonical mapping from $R<(\overline{Z}\cup\overline{T})^*>$ into $R<(\overline{Z}\cup\overline{T})^{(*)}>$ behind g.

Obviously then it holds

Corollary to LEMMA 9:

$$<\bar{g}(w),\overline{[S_l:S_r]}> = <h(w),\overline{[S_l:S_r]}>.$$

We define now a regular set over $\bar{Z}\cup\bar{T}$.

$$REG = [S_l:S_r]\cdot\{v \mid \exists(w\in T)<g(w),v> \neq 0\}^*.$$

Let be $D(\bar{Z}\cup\bar{T})$ the Dyck-language over $\bar{Z}\cup\bar{T}$ and $\sigma:(\bar{Z}\cup\bar{T})^* \longrightarrow T^*$ the monoidhomomorphism with

$$\sigma(z) = \varepsilon \quad \text{for } z\in\bar{Z},$$
$$\sigma(\bar{t}) = \varepsilon \quad \text{for } t\in T,$$
$$\sigma(t) = t \quad \text{for } t\in T,$$

then it holds because of lemma 9 the

THEOREM 7 (Chomsky-Schützenberger):

$$L(G) = \sigma(REG\cap D(\bar{Z}\cup\bar{T})).$$

In conclusion of this section we construct a grammar for L(G) in Greibach normal form.
We define

$$\tilde{P} = \{[y:x] \longrightarrow t[z:v][u:x] \mid \alpha_t^z \cdot \alpha_{y,v}^u \neq 0\}$$

and

$$\tilde{G} = (Z,T,\tilde{P},[S_l:S_r]).$$

Obviously $\tilde{G}$ is in Greibach normal form. It holds

THEOREM 8: It is $L(G) = L(\tilde{G})$ and more precisely it holds

$$\mathrm{mult}_G(w) = \mathrm{mult}_{\tilde{G}}(w) \text{ for } w \in T^*.$$

The size of $|G|$ and $|\tilde{G}|$ relate as follows

$$|\tilde{G}| \leq 32 \cdot |P_N| \cdot |P_T| \cdot |X|,$$

where $P = P_N \cup P_T$, P_N the set of non-terminal and P_T the set of terminal productions.

<u>Proof:</u> We define a homomorphisme $h_1 : T \longrightarrow R\langle \bar{Z}^* \rangle$ by setting for $t \in T^*$

$$h_1(t) = \sum_{z \in X} \alpha_t^z \sum_{\substack{y,u,v \\ x, x \neq y}} \alpha_{y,v}^u \overline{[y:x]}[u:x][Z:v].$$

We use the canonical mapping too

$$\mu : R\langle \bar{Z}^* \rangle \longrightarrow R\langle \bar{Z}^{(*)} \rangle.$$

We write

$$h_1(w) = \sum_{m \in \bar{Z}} \alpha_m \cdot m \text{ and } \alpha_m = \langle h_1(w), m \rangle.$$

We have in our case α_z^t, $\alpha_{y,v}^u \in \{0,1\}$, because we start with δ to be a grammar.
Because h_1 is into $R\langle \bar{Z}^* \rangle$ we have too $\alpha_m \in \{0,1\}$ for $m \in \bar{Z}^*$.

We put

$$w_1(w) = \{m \in \bar{Z}^* \mid \alpha_m \neq 0,\ \mu(m) = \overline{[S_1 : S_r]}\},$$

$$w_2(w) = \bar{Z}^{(*)} - W_1(w).$$

Then we can write

$$h_1(w) = \sum_{m\in W_1(w)} \alpha_m \cdot m + \sum_{m\in W_2(m)} \alpha_m \cdot m$$

and

$$< \mu \circ h_1(w), \overline{[S_1:S_r]}> = \sum_{m\in W_1(w)} \alpha_m.$$

Because of

$$\text{mult}_G(w) = <\eta(w), S_r> = <\varphi\circ\eta(w), \overline{[S_1:S_r]}>$$

it follows

$$\text{mult}_G(w) = \sum_{m\in W_1(w)} \alpha_m.$$

Now we assign to each $m\in W_1(w)$ uniquely a derivation over $\tilde{\tilde{P}}$.
For this reason we generalize W_1 such that instead of $[S_1:S_r]$ any element of Z may be taken.
Therefore let be for $w\in T^*$ and $z\in Z$

$$W_1(w,\bar{z}) = \{m\in \bar{Z}^* \mid <h_1(w), m> = 1, \mu(m) = \bar{z}\}.$$

We construct a bijectiv mapping from $W_1(w,\bar{z})$ on

$\text{Mor}_{\tilde{\tilde{\mathcal{F}}}}(z,w)$, where $\tilde{\tilde{\mathcal{F}}}$ belongs to $\tilde{\tilde{G}}$.

We take $m\in W_1(w,\bar{z})$ and $w = t_o \cdot w'$ and we assume

$$<h_1(t_o), \bar{z}ab> = 1, \quad a,b\in Z.$$

Because of $\mu(m) = \bar{z}$ there exists a decomposition $w' = w_2 \cdot w_3$ such that

$$\mu(h_1(w_2)) = \bar{b} \text{ and } \mu(h_1(w_3)) = \bar{a}.$$

With that we get with $|P| := \text{card } P$

$$|\tilde{G}| \leq 2|\tilde{P}_T|+4|\tilde{P}_N| \leq 4|\tilde{P}|.$$

Now it is

$$|\tilde{P}| = \sum_{\substack{t\in T \\ z\in \overline{X}}} \alpha_t^z \sum_{u,y,v,x} \alpha_{y,v}^u \leq \left(\sum_{\substack{t\in T \\ z\in \overline{X}}} \alpha_t^z\right)\left(\sum_{u,y,v\in \overline{X}} \alpha_{y,v}^u\right) \cdot |\overline{X}|,$$

this means

$$|\tilde{P}| \leq |\overline{P}_t| \cdot |\overline{P}_N| \cdot |\overline{X}|,$$

where $\overline{P}$ is from $\overline{G}$. From this follows

$$|\tilde{P}| \leq 8|P_T| \circ |P_N| \cdot |X|$$

and

$$|\tilde{G}| \leq 32 \cdot |P_T| \cdot |P_N| \cdot |X|,$$

what has to be proofed.

Remark: From this theorem it follows immediately

$$|\tilde{G}| \leq \frac{16}{3} |G|^2 \cdot |X| < \frac{8}{3} |G|^3.$$

For large production systems, this means

$$|P_T| = O(|T| \cdot |X|), \; P_N = O(|X|^3)$$

it holds for $|T| < |X|$ and $\varepsilon > 0$

$$|\tilde{G}| \leq O(|T| \cdot |X|^5) \leq O(|G|^{2+\varepsilon}).$$

5. Syntactical congruences

In this section we transfer the syntactical congruences on our algebra $\mathcal{A}_R(G)$ and we study how this congruences relate under our representation $\varphi : \mathcal{A}_R(G) \to R\langle Z^{(*)}\rangle$. In this connection the following lemma plays a central role.

LEMMA 10: For $w \in \overline{X}^*$ let exist an $u \in \overline{Z}^*$ such that $\langle \varphi(w), [z_o:x_o]u\rangle = \alpha \neq 0$. Then there exists $w' \in X_r^*$ such that $\langle z_o ww', x_o\rangle = \alpha$.

Proof: The proof is by induction on $n = |u|$.
The case $n = 0$ follows from lemma 8.
Now let the being the lemma proofed for all u' with $|u'| \leq n$.

$$U = U_1[y:x],\ [y:x] \neq 0,1,\ |u_1| = n.$$

Then there exist $v_1, v_2, \ldots, v_m \in X_r$ such that

$$\langle yv_1v_2\ldots v_m, x\rangle = \beta \neq 0.$$

By lemma 8 we get

$$\langle \varphi(v_1\ldots v_m),\ \overline{[y:x]}\rangle = \beta.$$

Therefore it is

$$\langle \varphi(wv_1\ldots v_m),\ \overline{[z_o:x_o]u[y:x]}\rangle \geq \alpha \cdot \beta > 0.$$

So we have

$$\langle \varphi(wv_1\ldots v_m),\ \overline{[z_o:x_o]u_1}\rangle > 0.$$

From this the claim of the lemma follows inductively.

For $L \subset T^*$ we define as usually

$$u =_r v(L) \iff \underset{w}{\forall} (uw \in L \iff vw \in L).$$

$=_r (L)$ is the <u>syntactic</u> <u>right</u> <u>congruence</u>.

For an easy formulation of the following results we extend our alphabet Z by one new element $\dashv$. But we call the new alphabet again Z. And we use the appivation $\$ = \dashv \cdot [S_l:S_r]$

The idea is to annulate words, which have not the form $\overline{[S_l:S_r]}\cdot Z^*$ and which are in $\varphi(\mathcal{A}(G))$ by multiplying it from the left with \$. <u>Remember</u> $\$ \cdot \overline{z} = 0$ for $z \in Z$ and $z \neq [S_l:S_r]$ and $\$ [S_l:S_r] \cdot \overline{z} = 0$ for all $z \in Z$.

<u>THEOREM 9:</u> $w =_r 0(L) \iff \$h(w) = 0$
Here is h the homomorphism of theorem 6.

<u>Proof:</u> We assume $\$ \cdot h(w) \neq 0$. Applying lemma 10 we find w' such that $\langle S_l \eta(ww'), S_r \rangle \neq 0$, and by lemma 9 we have $ww' \in L$. Therefore it holds $w \neq_r 0(L)$.
On the other hand does there exist w' for w such that $w \cdot w' \in L$, then by lemma 9 is $\langle S_l \eta(ww'), S_r \rangle \neq 0$ and therefore $\$ \cdot h(w) \neq 0$ too. This proofs our theorem.

This theorem describes a procedure to decide $w =_r 0(L)$ for L being a c.f. language.

Now we transfer the right congruence to $\mathcal{A}_R(G)$ by defining for $p, p' \in \mathcal{A}_R(G)$

$$p =_r p'(L) \iff \underset{q \in \mathcal{A}_R(G)}{\forall} (\langle p \cdot q, S_r \rangle = 0 \iff \langle p' \cdot q, S_r \rangle = 0).$$

In a symmetrical way one defines the <u>left</u> <u>congruence</u> $=_l (L)$.

One easily sees, that this definitions for R=$\mathbb{B}$ or R=$\mathbb{N}$ define congruence relations, but this is not true for R=$\mathbb{Z}$ or R beeing a field. The same holds for the following definition of the syntactical equivalence modulo L:

$$p = p'(L) \iff \underset{q',q\in\mathcal{A}_R(G)}{\forall} (\langle q\cdot p\cdot q',S_r\rangle=0 \iff \langle q\cdot p'\cdot q',S_r\rangle=0).$$

The quotient of $\mathcal{A}_R(G)$ by the syntactical congruence yealds the <u>syntactical</u> <u>algebra</u> $\mathcal{A}_R(G)/(L)$.
Because the syntactical monoid even for c.f. languages is hard to be computed, this holds for $\mathcal{A}_R(G)/(L)$ too. Therefore it is of interest to look for algebras between $\mathcal{A}_R(G)$ and $\mathcal{A}_R(G)/(L)$.

We put

$$\mathcal{M}_r(L) = \{p\in\mathcal{A}_R(G) \mid p =_r 0(L)\}$$

and

$$\mathcal{M}(L) = \{p\in\mathcal{A}_R(G) \mid p = 0(L)\}.$$

Obviously it holds

<u>LEMMA 11:</u> $\mathcal{M}_r(L)$ is a right ideal.
$\mathcal{M}(L)$ is a two sided ideal.

Immediately on has the

<u>Corollary to Theorem 9:</u> The word problem $w \in \mathcal{M}_r(L)$ will be decided by $\$ \cdot \varphi(w)$ for R = $\mathbb{N}$ or R = $\mathbb{B}$.

Now $\varphi^{-1}(0)$ is a two sided ideal of $\mathcal{A}(G)$ and it is $\varphi^{-1}(0)\subset\mathcal{M}_r$. Therefore one may ask if $\varphi^{-1}(0)$ has an interesting syntactical property. Obviously it holds $\varphi^{-1}(0)\subset\mathcal{M}(L)$ too.

One may ask if it is possible to prolongate φ to a homomorphism $\psi : \mathcal{A}(G) \longrightarrow R\langle Y^{(*)}\rangle$ with a suitable Y, such that $\psi^{-1}(0) = \mathcal{M}(L)$ holds. Because of lemma 2 one can not do this by a homomorphism from $R\langle Z^{(*)}\rangle$ into $R\langle Y^{(*)}\rangle$. But it could be that such a prolongation from $\varphi(\mathcal{A}(G))$ into a suitable $R<Y^{(*)}>$ exists, because $1\notin\varphi(\mathcal{A}(G))$.

Presumably such a homomorphism does not exist, because each semigroup homomorphism from $Z^{(*)}$ in $R<Y^{(*)}>$, which is induced by transformations $[y:x] \longrightarrow \Sigma q[y:x]q'$ maps the elements $[y:x]\cdot[z:v]$ for $v \neq x$ into O.
Therefore it remains an

<u>Open question:</u> Do there exist non trivial representations of $\mathcal{A}_R(G)/\mathfrak{M}(L)$ in $R<Y^*>$?

Answering this question is of practical interest too, because a section u of a program of a language L is syntactically incorrect if u = O(L). By means of evaluation of $\$\cdot\varphi(w)$ we are able to find the shortest syntactically incorrect prefix of a program u L. The representation of $\mathcal{A}_R(G)/\mathfrak{M}(L)$ we are looking for would do the same for the shortest syntactically incorrect sections of a program.
One could object that the evaluation of our ring homomorphisms is not trivial. This is indeed so, if we wish to do this in a most efficient way. But there are several other important problems that are reducible on this problem.
We take the opportunity and point out some further problems which seem to be important.
The syntactical congruence of a language L(G) does not reflect the structure of G very strongly. It is as with the week equivalence of two languages L(G) = L(G') does not say much about relations between G and G'. One of the most important applycations of language theory is to describe the syntax of programming or natural languages. The semantic of this languages depends strongly an the grammars G, which generate the syntax. Therefore it seems to me that the grammars deserve more interest as the languages. Languages are only one under different properties of grammar. If the grammars G and G' describe the syntax of two programming languages and if

L(G) = L(G') then these languages as programming languages are not necessarily equal. This leads to the question to formulate structural equivalences between grammars. Different such equivalences have been defined but only one of them the "strong" equivalence is well known. This equivalences will be reflected by the existence of certain homomorphisms and products between our algebras $\mathcal{A}_R(G)$. We will come back to this problem on another place. Here we give only a definition of a finer syntactical congruence, which is identical with the normal one in the case of unambiguous grammars.

For $p,p' \in \mathcal{A}_R(G)$ we define p congruent syntactically p' modulo G:

$$p \equiv p'(G) \iff \forall_{q,q' \in \mathcal{A}_R(G)} (\langle qpq', S_r\rangle = \langle qp'q', S_r\rangle).$$

We see that the O-classes in both congruences (L) and (G) are the same.

The word-problem for the quotient algebra $\mathcal{A}(G)/(G)$ is closely related to the equivalence problem in the case of unambiguous grammars. Therefore these algebras are as one may assume, hard to be compute. It is clear that in this connection arise lot of interesting questions.

For R beeing a field we have

$$p \equiv p'(G) \iff p \equiv p' \; \mathcal{A}_R(G)/\mathfrak{M}.$$

Therefore in this case $\mathcal{A}_R(G)/(G)$ is the syntactical algebra Reutenauer [Re] associated to the formal power series belonging to the grammar G. We think it very important to study each of these cases. Restricting to $R = \mathbb{Z}$ or R to be a field makes important practical questions disappearing from the theory.

6. Unambiguous grammars, LL(k) grammars

In this section we assume always $R = \mathbb{N}$ and we write therefore only $\mathcal{A}(G)$ for $\mathcal{A}_R(G)$.

By definition it holds for unambiguous grammars

$$\langle w, S_r\rangle \leq 1 \quad \text{for } w \in T^*.$$

Because of lemma 10 this is equivalent to

$$<\$ \cdot \varphi(u), a> \leq 1 \qquad \text{for } u \in \overline{X}^* \text{ and } a \in Z^*.$$

If one goes through the proof of lemma 10 again, one sees that the following lemma is true.

LEMMA 12: Let be G an unambiguous c.f. grammar and $w \cdot w' \in L(G)$. Then there exists exactly one monom $a \in Z$ such that

$$\alpha = <S \cdot h(w), a$$
$$\alpha' = h(w), \overline{a},$$

and $\alpha = \alpha' = 1$ holds. Here is $\overline{x_1 \ldots x_k} = \overline{x_k \ldots x_1}$.

We assume in the following G to be a LL(k) grammar if not explicitly the converse will be stated. We are interested here to study $\mathcal{A}(G)$ and our representation for LL(k) grammars. As we have shown in lemma 7, it follows from f u-left prime and $Z(f) = u \cdot v$, that $v \in X_r^*$. In $\mathcal{A}(G)$ we then have $<uv, Q(f)> = 1$ if $Q(f) \in \overline{X}$. We call $v \in X_r^*$ as almost invers to u from the right if there exists $z \in \overline{X}$ such that $<uv, x> \neq 0$.

LEMMA 13: For each $u \in (\overline{X} \cup T)^*$ card $X = m$ there exist maximally $2 \cdot m^{k+2}$ elements $v \in X_r^*$, which are almost invers to u from the right side, if G is L L(k).

Proof: Let be $v \in X_r^*$ and $<uv, y> = 1$. Then we can find $f : y \longrightarrow uv$. Because of $v \in X_r^*$ f is u left prime. G is LL(k) and therefore determine $u \cdot First_k(v)$ and y the derivation tree f uniquely. Then is v uniquely determined by $u \cdot First_k(v)$ too. There exist only m^{k+1} different words of length k or shorter. Therefore there exist maximally $2 \cdot m^{k+2}$ elements which are almost invers from the right.

We define for $p \in R\, Z^{(*)}$

$$|p| = \sum_{Zu \in Z^*} <p, u>.$$

$|p|$ is the sum of the coefficients of the monoms of p which contain in the first place an invers out of Z and none else where.

LEMMA 14: For all $u \in \overline{X}^*$ it holds

$$|\varphi(u)| \leq m^{k+3}$$

Proof: Let be $|w|_\rho = w$ and $<\varphi(u), w = 0$, $w = \overline{[z:x]}w'$ and $w' \in Z^*$. By lemma 10 we find $v \in X_r^*$ such that $<zuv,x> \neq 0$. Now there exists as shown in lemma 14 not more as m^{k+1} elements $v \in X_r^*$ such that $<zuv,x> \neq 0$. There do not exists two different monoms $\overline{[z:x]}w_1'$ and $[z:x]w_2'$ which have the same v as "right inverse." From this we could conclude $<zuv,x> \geq$, which is in contradiction to the unambiguity of G. Therefore we have indeed $|\varphi(n)| \leq m^{k+3}$.

LEMMA 15: Let be $u \in (\overline{X} \cup T)^*$ and $[y_o:x_o] \in Z$. If $\dashv \cdot [y_o:x_o]\varphi(u) \neq 0$, then there exists a decomposition $u = u_1 \cdot u_2$ and $w \in Z^*$ such that

$$-| \cdot [y_o:x_o]\varphi(u) = w \cdot (u_2), \quad |u_2| \leq k.$$

Proof: By lemma 10 it follows from $-| \cdot [y_o:x_o]\varphi(u) \neq 0$, that there exists $q \in X_r^*$ such that $<\varphi(u:q), \overline{[y_o:x_o]}> = 1$. Therefore we find $f: x_o \longrightarrow y_o uq$ in $\mathcal{F}$. We decompose $u = u_1 \cdot u_2$ such that $u_1 = 1$ for $|u| \leq k$ and $|u_2| = k$ in the other cases. Now let g be the uniquely determined $y_o u_1$ - left prime factor of f. G is LL(k) and therefore g is uniquely determind by x_o and $y_o u$. Therefore in $-| \cdot [y_o:x_o]\varphi(u_1)$ there exists exactly one monom w which will be not made to be o by multiplication with $\varphi(u_2)$. Therefore we have $\dashv \cdot [y_o:x_o]\varphi(u) = w \cdot \varphi(u_2)$, what the lemma claims.

From this directly follows

THEOREM 10: The word problem $w \in L(G), G \in LL(k)$ can be decided in linear space and linear time by multiplying out $\$ \cdot \varphi(u)$ sequentielly from left to right.
The method described in this theorem applied even to LR(k) languages would generally lead to exponentially growing space complexity.

The converse of our theorem 10 is not true. There exist c.f. grammars G for non deterministic languages such that their word problem can be decided by sequentially multiplying out from left to right in linear space and linear time.

Definition: We call this class of c.f. languages SMLR(N), iff

$$|\$\cdot\varphi(n)| \leq N \text{ for all } u \in T^*.$$

Obviously it holds because of lemma 15 and this remarks the

THEOREM 11:
1) The word problem for SMLR(N) can be decided in linear time and linear space.
2) $LL(k) \subset SMLR(m^{k+3})$
3) $LRSM = \bigcup_N SMLR(N)$ is closed under "$\cup$".

Open problems:

1. Is it decidable for G ∈ c.f. if G∈SMLR(N) for fixed N?
2. Is it decidable, if L(G) = L(G') for G,G'∈SMLR(N)?

This section shows that we in our theory get a pure algebraic definition of the LL(k) languages. We will show in the next section that this remains true for LR(k) languages.

THEOREM 12: $\varphi \in SMLR(N)$ is recursively undecidable.

Proof: We show that this question can be reduced on the correspondence problem of Post[Po].
Let be $(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n) \in X^* \times X^*$. The correspondence problem is the question, if there exists a sequence of natural numbers $i_1,i_2,\ldots,i_m \in \{1,\ldots,n\}$ such that

$$\alpha_{i_1} \ldots \alpha_{i_m} = \beta_{i_1} \ldots \beta_{i_m}.$$

Let S,A,B be new symbols, e. not in X. We form the polynomials

$$p_i = \overline{A}\ \alpha_i A + \overline{B}\ \beta_i B,\ p_i' = \overline{A}\alpha_i + \overline{B}\beta_i$$

for i = 1,...,n and

$$q_j = \overline{x}_j \text{ for } x_j \in X,$$
$$r = \overline{S}(S_1+S_2)$$

We ask, does there exist a product $f \in \{p_i, p_i', q_j \surd \left| \begin{matrix} i = 1,\ldots,n \\ j = 1,\ldots,n \end{matrix} \right.\}^*$

such that

$$|S(A+B)f| > 2$$

Obviously this holds iff the correspondence problem has a solution.
In the case of SMLR there are the monoms of p of length ≤ 3.
One reduces the general case onto this special case by decomposing

$$p_i = p_{1,1} \cdots p_{i,l_i}$$

where the p_{i,l_i} have degree ≤ 3.

Let be

$$q = \overline{A}a_1 \ldots a_l A + \overline{B}b_1 \ldots b_r B \text{ and } a_i, b_i \in X,$$

and

$$l \geq r.$$

$A_1, A_2, \ldots, A_{l-1}$ and $B_1, \ldots, B_{l-1}$ are new symbols. We define

$$q_1 = \overline{A}a_1 A_1 + \overline{B}b_1 B_1,$$
$$q_i = \overline{A}_{i-1} a_i A_i + \overline{B}_{i-1} b_i B_i \text{ for } i = 2, \ldots, l-1,$$
$$q_l = \overline{A}_{l-1} a_l A + \overline{B}_{l-1} b_l B.$$

Here we understand $b_{r+1} = \dots = b_l = 1$.

We see $$q = q_1 \cdot q_2 \dots q_l.$$

By doing this decomposition for each p_i, p_i', $i = 1, \dots, n$ with sets of new variables whose intersection is pairwise empty, we get a reduction that shows that

$$|S(A+B)f| > 2$$

remains undecidable even if we restrict our question to the case degree $(p_i) \le 3$.
It remains open, if there exists G such that φ_G defines our polynomials.

7. LR(k) - Grammars

We here derive similar results as in the section before. The only difference comes in by substituting $R\langle X^{(*)}\rangle$ by $\mathcal{A}_R(G) \bmod \mathfrak{u}_r(L)$ in the characterisation of LR(k). A first information we get by the following

LEMMA 16: Let G be a LR(k) - grammar and $u \in T^*$. If $u = \tilde{u}_1 + \dots + \tilde{u}_m$ $(\mathfrak{U}_r(L))$ with $\tilde{u}_i \neq_r O(L)$, then $m \le (|\bar{X}|+1)^k$ holds.

Proof: From $\tilde{u}_i \neq_r O(L)$ it follows that there exists v such that $u \cdot v \in L(G)$. Let be $f: S_r \longrightarrow u \cdot v$ the derivation of $u \cdot v$ from S_r. Then u $\mathrm{First}_k(v)$ determine uniquely an u-leftbase g of f. Then $Q(g) = \tilde{u}_i$. This means that u $\mathrm{First}_k(v)$ uniquely determines the index i by the condition $\tilde{u}_i \cdot \mathrm{First}_k(v) \neq_r O$. Now there are maximally only $(|\bar{X}|+1)^k$ words f' of length $|v'| \le k$, which select an index i by the condition $\tilde{u}_i \cdot v' \neq_r O(L)$. Therefore $m \le (|\bar{X}|+1)^k$ as claimed by the lemma.

This lemma not yet characterizes LR(k)-grammars. But going a second time through the proof of lemma 17, we see that $\tilde{u}_1, \dots, \tilde{u}_m$ have a common prefix, which uniquely is determined by u. This we see from the decomposition $u = u_1 \cdot u_2$ such that $|u_2| = k$. Therefore it holds one direction of the

THEOREM 13: The c.f. grammar G is of type LR(k) iff for each $u \in T^*$ it holds $u = \tilde{u} \cdot p$, $\tilde{u} \in X_l^*$, $p = \tilde{u}_1 + \ldots + \tilde{u}_m$, $\tilde{u}_i \in X_l^*$ and $|\tilde{u}_i| \leq k$.

To proof this theorem completely it is sufficient to show, that the word problem $w \in L(G)$ can be decided by a deterministic pda. We will not proof this here, because it is a simple consecuence of the following theorem, which concerns a more general class of c.f. languages.
We generalize LR(k) as before LL(k) in the following

Definition: The c.f. grammar G is in the class BSLR(N) iff for all $u \in T^*$ holds:

from $u = \alpha_1 \cdot \tilde{u}_1 + \ldots + \alpha_m \cdot \tilde{u}_m (\equiv_r (L))$, $L = L(G)$, $\tilde{u}_i \in X_l^*$, $\alpha_i \cdot \in R$

$R = \mathbb{N}$ it follows $\sum_{i=1}^{m} \| \alpha_i \| \leq N$, $\| \alpha_i \| = \begin{cases} 1 & \text{if } \alpha_i \neq 0 \\ 0 & \text{if } \alpha_i = 0. \end{cases}$

The letters BS come from bounded size and LR from the use of the right congruence $=_r(L)$.

THEOREM 14: The word problem $w \in L(G)$ for G BSLR(N) can be decided sequentially in time $O(|w|)$.

Proof: We first give the idea of the proof. For each of the words u_i we have to compute $\$\varphi(\tilde{u}_i)$ to decide $\tilde{u}_i =_r O(L)$. This computation can be done sequentially because $\tilde{u}_i \in X_l^*$. But to compute $\$\varphi(u_i \cdot \eta(+))$ is more difficult, because $\tilde{u}_i \cdot z$, can produce several words in X_l^*, which are of very difficult length. This could lead to a n^2 algorithm. We overcome this difficulty by computing for each prefix v of u_i all possible results of $v \cdot z$ for $z \in X_r$ in advance. It will happen in this computations that we get the same word u_i in different ways. Therefore we have this to check, or to use a data structure, which makes this checking superfluous.

To proof our theorem we define two new functions.
For $f \in R\langle X^{(*)}\rangle$ we define

$$\text{suffix}(f) = \{z \in Z \mid \exists_{v} \langle f, vz\rangle \neq 0\}.$$

To each $x \in X_1$ we assign a mapping $\psi(x) : 2^Z \longrightarrow 2^Z$ by definition

$$\psi(x)(z) = \{y \mid \exists_{v} \langle \varphi(x), \bar{z}vy\rangle \neq 0\},$$

and

$$\psi(x)(Z') = \bigcup_{z \in Z'} \psi(x)(z) \quad \text{for } Z' \subset Z.$$

It follows immediatly

$$\text{suffix}(\$\,\varphi(ux)) = \psi(x)(\text{suffix}(\$\,\psi(u)).$$

This property we use to compute $u \cdot t = \tilde{v}_1 + \ldots + \tilde{v}_n (\mathfrak{A}_r)$.
It holds in $\mathcal{A}_R(G)$ for $\tilde{u}_i = u'_i \cdot x_i$

$$u \cdot = \sum_{i=1}^{m} u_i \cdot \eta(t) = \sum_i \sum_{z \in \bar{X}} \tilde{u}_i \cdot z \cdot \alpha_z^t$$

$$= \sum_i \sum_{z \in X_1} \tilde{u}_i \cdot z \cdot \alpha_z^t + \sum_i \sum_{\substack{z \in X_r \\ y \in \bar{X}}} u'_i \; y \; \alpha^y_{x_i, z}$$

$$= \sum_{\substack{i \\ \psi(z)\text{suffix}(u_i) \neq \emptyset}} \sum_{z \in X_1} \tilde{u}_i \cdot z \cdot \alpha_z^t + \sum_i \sum_{\substack{z \in X_r \\ y \in \bar{X}}} u'_i \cdot y \; \alpha^y_{x_i} \cdot z$$

This relation is recursiv because the second sum is of the same character as the whole sum. The recursion could run $|u|$ steps, what would lead to a $|u|^2$ algorithm. To use this relation more efficiently, we construct a tree like data structur which represents $\tilde{u}_1+\ldots+\tilde{u}_m$ by a tree and which contains feed back edges to shorten the recursion.

Definition of the tree T(u).

T(u) is an oriented tree. The root of the tree is \$. The other vertices of the tree are $\{v \mid \exists_i v \text{ prefix of } \tilde{u}_i\}$. The set of edges is

$$\{(v,x) \mid vx \text{ prefix of } a\tilde{u}_i,\ x \in X_1\}.$$

v is the start vertex of (v,x) and vx the end vertex of (v,x). We label the vertices of T(u) by

$$\mu(v) = \$\ \text{suffix}(v).$$

From our recursive relation it follows

$$\mu(vx) = \psi(x)(\mu(v)).$$

This means that μ can sequentially be computed on the tree.

Now we introduce backward edges in T(u).

There exists a backward edge from v_1 to v_2 iff

$$v_2 \text{ is prefix of } v_1,\ v_1 \neq v_2,$$

and

$$\text{if } v_1 = v_2 \cdot v \text{ then there exists } X \in X_r \text{ and } z \in X_1 \text{ such that } \langle vx,z\rangle \neq 0.$$

We denote this edge by (v_1,v_2,x,z). v_1 is the start vertex and v_2 the end vertex and $\langle x,z\rangle$ is the 'label' of (v_1,v_2,x,z). The number of backward edges from v_2 is bounded by $|X_r| \cdot N$, otherwise we got a contradiction to the assumtion $G \in BSLR(N)$.

We have $u \in L(G)$ iff S_r is edge in $T(u)$. To proof our theorem it is therefore sufficient to show, that $T(u \cdot t)$ can be constructed in constant time from $T(u)$. To proof this we look at the vertex u_i.

a) Let $\langle \eta(t), z \rangle \neq 0$ and $z \in X_l$.
By computing $\psi(z)\mu(u_i)$ we decide if (u_i, z) is an edge in $T(u \cdot t)$. The time for this computation depends only on G, not on $|ut|$.

b) Let $\langle \eta(t), z \rangle \neq 0$ and $z \in X_r$.
We look through the backward edges from u_i, if there are some with the label $\langle x, z \rangle$. If (v_1, v_2, x, z) is a backward edge, then (v_2, x) is an edge in $T(u \cdot t)$.
We maximally have to look through

$$|X_r| \cdot |X_l| \cdot N^2$$

edges. This number again depends only on G.

c) We have to compute the new backward edges for $t(u \cdot t)$.
Let be vx a new vertex in $T(ut)$. Then for all $y \in X_r$ we compute

$$vxy = v \cdot \sum_{z \in \overline{X}} \alpha^z_{x,y} \cdot z.$$

This we can do as before under b) by using the backward edges from v. Again we need not more as $N^2 \cdot |X|^2$ steps.

d) It is not necessary to delete the edges of $T(u)$, which do not appear in $T(ut)$ explicity by keeping a list of the 'leafs' of the tree. Notice 'leaf' means here the vertices, which represent one of the u_i.

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