NETWORK COMPLEXITY

by

Günter Hotz and Manfred Stadel

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Introduction : The use of computers in executing algorithms always leads to the question of how "expensive" these algorithms are. This can mean, for example, the amount of computing time or storage space required by a given algorithm. Such questions are handled in complexity theory; their practical significance is apparent. Upon closer examination these questions are seen to be quite complicated, and everyday problems prove to be extremely difficult to solve.

The first attempt at developing a complexity theory for general computable functions began with the axiomatic approach of Blum*. In the form of the famous speed-up theorems this approach led, however, to disappointing results. Another approach by Schnorr** with respect to optimal Gödel numberings was also taken up by Hartmanis***, but appears not to have been further handled. In the direction of a general theory the most far reaching results have been presented by Strassen in his development of the degree-bound. Although this is sharp for several interesting special cases, in general, it underestimates the complexity of polynomials quite significantly.

For several reasons the complexity of Boolean networks has received special attention. Good lower bounds for these networks would also lead to good lower bounds for polynomials. The results of Fischer and Schnorr**** show that beyond this, such results could also yield information about the complexity of general computable functions. Unfortunately, until now all of the efforts applied to the complexity of Boolean networks have led to only modest results. For example, see the report by Patterson [9], and also the papers by Paul [13] and Schnorr [15], which are quite complicated considering their results.

The goal of this paper is to examine complexity measures over an axiomatic basis. These measures include the complexity measures induced by

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* Blum,M.:" A Machine Independent Theory of the Complexity of Recursive Functions", J.ACM 14,2 (1967), 322-336.

** Schnorr,C.P.: "Optimal Enumerations and Optimal Gödel Numberings",M.Syst.Th.8(1974)

*** Hartmanis,J.+ Baker,T.P.: "On Simple Gödel Numberings and Translations" in Automata, Languages and Programming, 2nd Coll. Univ. of Saarbrücken, 1974, LNCS 14.

**** Fischer,M.J.: Lectures on Network Complexity, pres.at the Univ. of Frankfurt,1974

Schnorr,C.P.: "The Network Complexity and the Turing Complexity of Finite Functions", Acta Informatica 7, (1976), 95-1o7.

cost functions like the size complexity, depth complexity, and breadth complexity studied in [16] and [9]. Also included are the applications of entropy [12] [18] and the degree-bound [17].

We develop this theory on the basis of categories with an added monoid multiplication. It will be examined under which conditions the cost-function-induced complexity measures can be approximated by general complexity measures. Methods will be developed for constructing complexity measures, and the conditions under which the entropy can be used for the definition of complexity measures will be given. Finally we will briefly mention complexity measures for monotone functions developed over monotone elements.

This paper has its basis in the extended abstract [5]. The formal, more complete, and extended version found in the current paper is primarily the work of the second author.

§1. The Mathematical Representation of Switching Circuits.

In this paper the size complexity, depth complexity, and other complexity measures of Boolean functions represented by switching circuits will be examined. First of all we need a good mathematical representation of switching circuits.

The first idea in this direction is to represent a switching circuit by a digraph in which nodes are labeled with elementary switching elements and the edges represent wires of the circuit. This kind of representation has an important disadvantage: We are not able to distinguish between the different inputs of a switching element. If we consider only bases with commutative switching elements † this is not a handicap. But we do not want to restrict ourselves to commutative bases. Therefore, we must devise a method for distinguishing among the different input wires of the switching-elements.

To do this we may write for each switching element (node in the graph) a line as follows

$$n: \langle a; n_1, n_2, \ldots, n_k \rangle$$

where a is an elementary switching element (label of this node) with indegree k and $n_1, \ldots, n_k$ are line numbers less than n (the current line number) with the following meaning: The i-th input of a is the output of the switching element coded in line n_i.

For the inputs of the whole switching circuit we must write extra lines:

$$\begin{array}{ll} 1: & x_1 \\ 2: & x_2 \\ \ldots & \\ l: & x_l \end{array}$$

where $x_1, \ldots, x_l$ represent the input variables. Also for the two constants "true" and "false" we need extra lines:

$$\begin{array}{ll} l+1: & \text{true} \\ l+2: & \text{false.} \end{array}$$

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† An elementary switching element is called commutative iff a permutation of the input wires never changes the interpretation of the circuit as a Boolean function. For example $\wedge$, $\vee$, $\neg$ are commutative switching elements.

Since not all of the wires are outputs of the whole switching circuit, we must give a selecting function o, such that the i-th output is loaded with the output of the switching element coded in line o(i). Note also that this model only allows for elementary switching elements with only one output wire. †

Another more algebraic description of switching circuits is given in [4]. There the concept of so called X-categories is introduced. In [3] the X-categories are called "strict monoidial categories" and in [1] they are called "Kronecker categories".

1.1 Definition.

An X-category X *is a 3-tupel* $(X, \times, \varepsilon)$ *such that*

(i) X *is a category,*

(ii) $\times: X \times X \to X$ *is a covariant bifunctor (we write* $f \times g$ *instead of* $\times(f,g)$ *for morphisms* $f, g \in \mathrm{Mor}(X)$ *and, analogously* $u \times v$ *for objects* $u, v \in \mathrm{Ob}(X)$*),*

(iii) $\varepsilon \in \mathrm{Ob}(X)$ *is a special element, and*

(iv) $(\mathrm{Mor}(X), \times, 1_\varepsilon)$ *is a monoid with* $\times$ *as operation and* $1_\varepsilon: \varepsilon \to \varepsilon$*, the identity on* ε*, as unit element.*

Property (ii) implies: If $f: u' \to u$, $f': u'' \to u'$, $g: v' \to v$, $g': v'' \to v'$ are morphisms in X, then $f \times g: u' \times v' \to u \times v$ and $f' \times g': u'' \times v'' \to u' \times v'$, and we have the following equation:

$$(f \circ f') \times (g \circ g') = (f \times g) \circ (f' \times g').$$

Furthermore claim (ii) together with claim (iv) implies that $(\mathrm{Ob}(X), \times, \varepsilon)$ is also a monoid. Because of (iv) every X-category is a small category (a monoid is always a set).

1.2 Definition.

Let X, Y *be two X-categories. A functor* $\Phi: X \to Y$ *is called X-functor iff* X *regarded as a mapping* $\mathrm{Mor}(X) \to \mathrm{Mor}(Y)$ *is a monoid homomorphism.*

----

† For example a switching element consisting of two parallel negations has more than one output wires:

1.3 Definition.

(i) Let X *be an* X*-category and* $A \subset \mathrm{Mor}(X)$. A *is called a free generating system for* X *iff the following universal property holds:*
If Y *is an arbitrary* X*-category,* $\Phi_1\colon \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ *a monoid homomorphism, and* $\Phi_2\colon A \to \mathrm{Mor}(Y)$ *a mapping such that* $\Phi_2(a)\colon \Phi_1(u) \to \Phi_1(v)$ *for* $a\colon u \to v \in A$ *then there exists a uniquely determined* X*-functor* $\Phi\colon X \to Y$ *which is an extension of* Φ_1 *and* Φ_2.

(ii) A X*-category is called free iff it posses a free generating system.*

1.4 Theorem and Notation.

Given any free monoid O *and an arbitrary set* A *together with two mappings* $S, T\colon A \to O$ *there exist a free* X*-category* $F(A, O)$ *which is unique up to functorial isomorphism with free generating system* A *and* O *as monoid of objects such that every* $a \in A$ *becomes a morphism* $a\colon S(a) \to T(a)$ *in* $F(A, O)$.

For a PROOF see, for example, [1] or [7]. ■

The following example shows us how a switching circuit may be represented by a free X-category.

1.5 Example.

Let B be a basis generating all of the Boolean functions $\{\text{true}, \text{false}\}^n \to \{\text{true}, \text{false}\}^m$ for example, $B = \{\wedge, \vee, \neg\}$. In a switching circuit there are also some other elementary switching elements such as

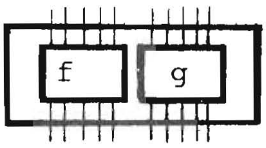
(i)	crossing of two wires	$\times$	denoted by c,
(ii)	branching of a single wire (diagonalization)	⅄	denoted by d,
(iii)	Boolean constant wires	true $\top$ false $\top$	denoted by true and false,
(iv)	truncation of a wire	$\perp$	denoted by t.

As monoid O of objects we will use $\mathbb{N}_0$ with addition as operation. If f is a switching circuit we will interpret f as a morphism

$f\colon u \to v$ with $u, v \in \mathbb{N}_0$,[†] where u denotes the number of inputs and v the number of outputs of f. The free generating system A consists of the elements of B and c, d, true, false, t where the mappings S, T are defined as follows:

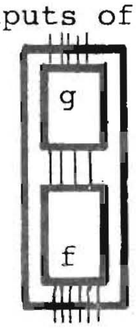
$$
\begin{array}{ll}
S(\wedge) = S(\vee) = 2, & T(\wedge) = T(\vee) = 1, \\
S(\neg) = 1, & T(\neg) = 1, \\
S(d) = 1, & T(d) = 2, \\
S(c) = 2, & T(c) = 2, \\
S(\mathrm{true}) = S(\mathrm{false}) = 0, & T(\mathrm{true}) = T(\mathrm{false}) = 1, \\
S(t) = 1, & T(t) = 0.
\end{array}
$$

The operations $\times$ and $\circ$ of $F(A, O)$ are interpreted as follows: If f and g are two circuits $f \times g$ means the circuit built by drawing f on the left of g without connecting any wires.



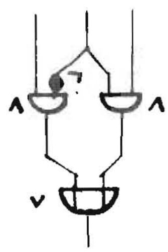
$f \times g$

If f has u inputs and g has u outputs then $f \circ g$ is defined and means the circuit built by connecting the inputs of f pairwise with the outputs of g without changing their order.



$f \circ g$

As example consider the following switching circuit:



In $F(A, O)$ this circuit is written as

$$\vee \circ (\wedge \times \wedge) \circ (1_1 \times \neg \times 1_2) \circ (1_1 \times d \times 1_1).$$

From now on we denote the so constructed free X-category by $G(B)$.

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† $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

1.6 EXAMPLES (c.f. [4], [2]).

Let M be an arbitrary set. Denote by C_M the following X-category:

$$Ob(C_M) := \mathbb{N}_0 \text{ together with } + \text{ as monoid operation,}$$

$$Mor(C_M) := \{f\colon n \to m : f \text{ is a function from } M^n \text{ to } M^m\}.$$

Thereby we identify M^o with the set $\{e\}$ consisting of a single element e. Identifying $M^o \times M^n$ and $M^n \times M^o$ with M^n we may define $f \times g$ for f, g $\in Mor(C_M)$ in the usual way. Then it is easily proved that $(Mor(C_M), \times)$ is a monoid and that C_M is indeed a (not free) X-category. As a special example we have the X-category $\mathcal{B} := C_{\{true,\ false\}}$ of Boolean functions. If G is defined as in 1.5 we have an X-functor $I\colon G(\{\wedge, \vee, \neg\}) \to \mathcal{B}$:

$$\begin{aligned}
I(\wedge)(x,y) &= x \wedge y,\\
I(\vee)(x,y) &= x \vee y,\\
I(\neg)(x) &= \neg x,\\
I(true)(e) &= true,\\
I(false)(e) &= false,\\
I(t)(x) &= e,\\
I(c)(x,y) &= (y,x),\\
I(d)(x) &= (x,x), \qquad \text{where } x, y \in M = \{true, false\}.
\end{aligned}$$

I is called the interpretation of $G(\{\wedge, \vee, \neg\})$ in $\mathcal{B}$. If f $\in$ $Mor(G(\{\wedge, \vee, \neg\}))$ is a switching circuit then I(f) is the Boolean function represented by f.

As another example let R be an arbitrary ring and denote by P_R the subcategory of C_R consisting of all polynomial † functions $R^n \to R^m$ $(n, m \in \mathbb{N}_0)$. Then P_R is also an X-category. An interpretation $I\colon G(\{+, -, *\}) \to P_R$ is defined in the obvious way.

There is a difference between the representation of a switching circuit by a free X-category and the method discussed at the beginning of this section. The following two switching circuits have the same representation as digraphs with ordered inputs but are different morphisms in $G(\{\wedge, \vee, \neg\})$:

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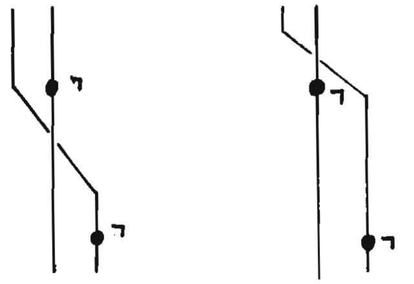
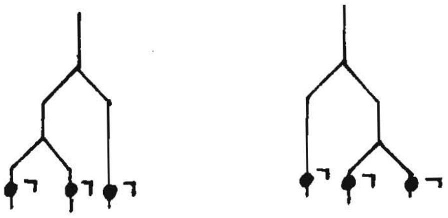
† A function $f\colon R^n \to R^m$ is called polynomial iff there exist m polynomials $p_1,\dots,p_m \in R[X_1,\dots,X_n]$ such that $f(x_1,\dots,x_n) = (p_1(x_1,\dots,x_n), \dots, p_m(x_1,\dots,x_n))$ for all $(x_1,\dots,x_n) \in R^n$.



Another pair of such circuits is:



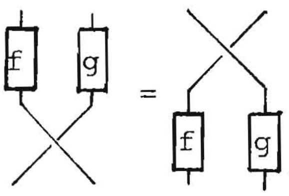
Because of this problem we will define special sorts of X-categories and use them for representing of switching circuits. First, Hotz has introduced such special X-categories, called the free D-categories (c.f. [4]).

1.7. DEFINITION (c.f. [1]).

Let X *be an X-category. We call* X

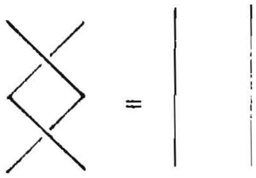
(i) *a symmetric X-category (S-category for short) iff for any two objects* u, v ∈ Ob(X) *there exist a morphism* $c_{u,v}: u\times v \to v\times u \in \mathrm{Mor}(X)$ *(called a crossing morphism) such that the following axioms hold:*

(S1) If $f: u \to r$ *and* $g: v \to s$ *are morphisms in* Mor(X) *then* $c_{r,s} \circ (f\times g) = (g\times f)\circ c_{u,v}$.



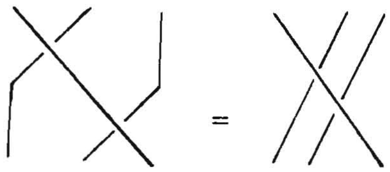
$$\begin{tikzcd} u\times v \arrow[r, "f\times g"] \arrow[d, "c_{u,v}"'] \arrow[dr, phantom, "///"] & r\times s \arrow[d, "c_{r,s}"] \\ v\times u \arrow[r, "g\times f"] & s\times r \end{tikzcd}$$

(S2) $c_{v,u} \circ c_{u,v} = 1_{u\times v}$ *for all* u, v ∈ Ob(X).



$$\begin{tikzcd} u\times v \arrow[r, "c_{u,v}"] \arrow[dr, "1_{u\times v}"'] & v\times u \arrow[d, "c_{v,u}"] \arrow[dl, phantom, "///"] \\ & u\times v \end{tikzcd}$$

(S3) $(1_v \times c_{u,w}) \circ (c_{u,v} \times 1_w) = c_{u,v\times w}$ *for all* u, v, w $\in$ Ob(X).



$$\begin{tikzcd} u\times v\times w \arrow[r, "c_{u,v}\times 1_w"] \arrow[dr, "c_{u,v\times w}"'] & v\times u\times w \arrow[d, "1_v\times c_{u,w}"] \arrow[dl, phantom, "///"] \\ & v\times w\times u \end{tikzcd}$$

(ii) *an* X*-category with finite direct products (D-category for short) iff for every* u $\in$ Ob(X) *there are morphisms* t_u: u $\to$ ε *(called truncations) such that for any two objects* v, w $\in$ Ob(X) *the following diagram is a direct product diagram in* X.

$$\begin{tikzcd} & v\times w \arrow[dl, "1_v\times t_w"'] \arrow[dr, "t_v\times 1_w"] & \\ v & & w \end{tikzcd}$$

If X *is a D-category we introduce the following notations:*

(a) Let u $\in$ Ob(X) *and consider the following diagram in* X*:*

$$\begin{tikzcd} & u \arrow[dl, "1_u"'] \arrow[d, dashed, "d_u"] \arrow[dr, "1_u"] & \\ u & u\times u \arrow[l, "1_u\times t_u"] \arrow[r, "t_u\times 1_u"'] & u \end{tikzcd}$$

There is a uniquely determined morphism d_u:u $\to$ u×u *(called diagonalization) which makes the diagram commutative:*

(D2) $1_u = (1_u \times t_u) \circ d_u = (t_u \times 1_u) \circ d_u$.

(b) Let u, v $\in$ Ob(X) *and consider the following diagram in* X*:*

$$\begin{tikzcd} & u\times v \arrow[dl, "1_u\times t_v"'] \arrow[d, dashed, "c_{u,v}"] \arrow[dr, "t_u\times 1_v"] & \\ u & v\times u \arrow[l, "t_v\times 1_u"] \arrow[r, "1_v\times t_u"'] & v \end{tikzcd}$$

There exists a uniquely determined isomorphism $c_{u,v}$: u×v $\to$ v×u *(called crossing) which makes the diagram commutative:*

(D3) $1_u \times t_v = (t_v \times 1_u) \circ c_{u,v}$ *and* $t_u \times 1_v = (1_v \times t_u) \circ c_{u,v}$.

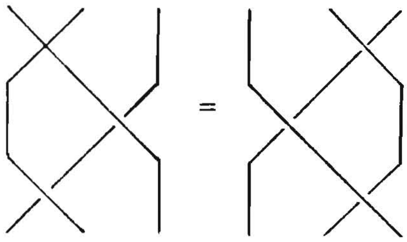
1.8 Proposition.

Let X *be an* S*-category. Then the following hold:*

(S0) $c_{u,\varepsilon} = c_{\varepsilon,u} = 1_u$ *for all* $u \in \mathrm{Ob}(X)$.

(S3') (dual to S3) $(c_{u,w}\times 1_v)\circ(1_u\times c_{v,w}) = c_{u\times v,w}$ *for all* $u,v,w \in \mathrm{Ob}(X)$.

(S4) $(c_{v,w}\times 1_u)\circ(1_v\times c_{u,w})\circ(c_{u,v}\times 1_w) = (1_w\times c_{u,v})\circ(c_{u,w}\times 1_v)\circ(1_u\times c_{v,w})$
$u,v,w \in \mathrm{Ob}(X)$.



$$\begin{tikzcd}
u\times v\times w \arrow[r, "c_{u,v}\times 1_w"] \arrow[d, "1_u\times c_{v,w}"'] & v\times u\times w \arrow[r, "1_v\times c_{u,w}"] \arrow[d, phantom, "///"] & v\times w\times u \arrow[d, "c_{v,w}\times 1_u"] \\
u\times w\times v \arrow[r, "c_{u,w}\times 1_v"'] & w\times u\times v \arrow[r, "1_w\times c_{u,v}"'] & w\times v\times u
\end{tikzcd}$$

Proof. (c.f. [1]).

(S0) $c_{u,\varepsilon} = c_{u,\varepsilon\times\varepsilon} \overset{(S3)}{=} (1_\varepsilon\times c_{u,\varepsilon})\circ(c_{u,\varepsilon}\times 1_\varepsilon) = c_{u,\varepsilon}\circ c_{u,\varepsilon}$

$\Rightarrow 1_u \overset{(S2)}{=} c_{u,\varepsilon}\circ c_{\varepsilon,u} = c_{u,\varepsilon}\circ c_{u,\varepsilon}\circ c_{\varepsilon,u} \overset{(S2)}{=} c_{u,\varepsilon}$

$\Rightarrow 1_u = 1_u\circ c_{\varepsilon,u} = c_{\varepsilon,u}$

(S3') $(c_{u,w}\times 1_v)\circ(1_u\times c_{v,w}) \overset{(S2)}{=} (c_{u,w}\times 1_v)\circ(1_u\times c_{v,w})\circ c_{w,u\times v}\circ c_{u\times v,w}$

$\overset{(S3)}{=} (c_{u,w}\times 1_v)\circ(1_u\times c_{v,w})\circ(1_u\times c_{w,v})\circ(c_{w,u}\times 1_v)\circ c_{u\times v,w}$

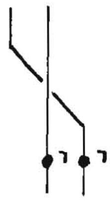
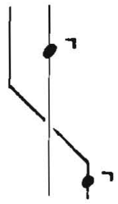
$\overset{(S2)}{=} (c_{u,w}\times 1_v)\circ(c_{w,u}\times 1_v)\circ c_{u\times v,w} \overset{(S2)}{=} c_{u\times v,w}$

(S4) $(c_{v,w}\times 1_u)\circ(1_v\times c_{u,w})\circ(c_{u,v}\times 1_w) \overset{(S3)}{=} (c_{v,w}\times 1_u)\circ c_{u,v\times w}$

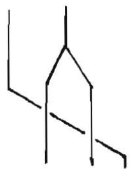
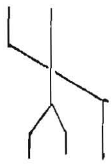
$\overset{(S1)}{=} c_{u,w\times v}\circ(1_u\times c_{v,w})$

$\overset{(S3)}{=} (1_w\times c_{u,v})\circ(c_{u,w}\times 1_v)\circ(1_u\times c_{v,w})$ ∎

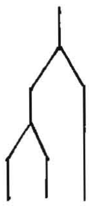
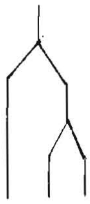
If we represent switching circuits by free S-categories which are defined in a way analogous to free X-categories (as will be shown later) then the two circuits

 and 

becomes the same morphism in the S-category (use axiom S1). Also the following two circuits have the same description in these S-category (axiom S1):

 and 

But the two circuits

 and 

are still different as morphisms in a free S-category. Therefore we will often use free D-categories for describing switching circuits.

1.9 PROPOSITION.

Let X *be a D-category. Then* X *is an S-category and the following hold:*

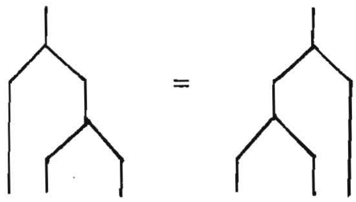
(D0) $t_\varepsilon = 1_\varepsilon$ *and* $t_{u \times v} = t_u \times t_v$ *for all* $u, v \in \mathrm{Ob}(X)$. *Further if* $f\colon u \to \varepsilon \in \mathrm{Mor}(X)$ *then* $f = t_u$. *This means that* t_u *is the only morphism* $u \to \varepsilon$ *in* $\mathrm{Mor}(X)$.

(D1) *Let* $f: u \to v\times w$, $g: u \to v\times w$ *be two morphisms in* $\mathrm{Mor}(X)$. *If*

$$(1_v\times t_w)\circ f = (1_v\times t_w)\circ g \quad \textit{and} \quad (t_v\times 1_w)\circ f = (t_v\times 1_w)\circ g$$

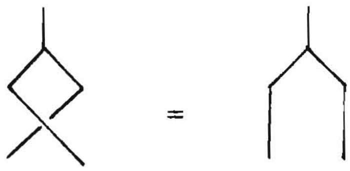
then $f = g$.

(D4) $(1_u\times d_u)\circ d_u = (d_u\times 1_u)\circ d_u$ *for all* $u \in \mathrm{Ob}(X)$.



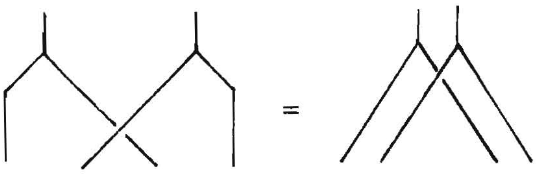
$$\begin{tikzcd} u \arrow[r, "d_u"] \arrow[d, "d_u"'] \arrow[dr, phantom, "///"] & u\times u \arrow[d, "d_u\times 1_u"] \\ u\times u \arrow[r, "1_u\times d_u"'] & u\times u\times u \end{tikzcd}$$

(D5) $c_{u,u}\circ d_u = d_u$ *for all* $u \in \mathrm{Ob}(X)$.



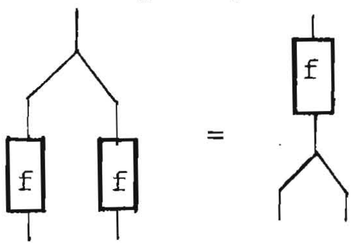
$$\begin{tikzcd} u \arrow[r, "d_u"] \arrow[dr, "d_u"'] & u\times u \arrow[d, "c_{u,u}"] \\ & u\times u \end{tikzcd}$$

(D6) $(1_u\times c_{u,v}\times 1_v)\circ(d_u\times d_v) = d_{u\times v}$ *for all* $u, v \in \mathrm{Ob}(X)$.



$$\begin{tikzcd} u\times v \arrow[r, "d_u\times d_v"] \arrow[dr, "d_{u\times v}"'] & u\times u\times v\times v \arrow[d, "1_u\times c_{u,v}\times 1_v"] \\ & u\times v\times u\times v \end{tikzcd}$$

(D7) $(f\times f)\circ d_u = d_v\circ f$ *for all morphisms* $f: u \to v \in \mathrm{Mor}(X)$.



$$\begin{tikzcd} u \arrow[r, "d_u"] \arrow[d, "f"'] \arrow[dr, phantom, "///"] & u\times u \arrow[d, "f\times f"] \\ v \arrow[r, "d_v"'] & v\times v \end{tikzcd}$$

PROOF.

The axioms (S2) and (S3) follow directly from the universal property of direct product diagrams. For example to prove (S2) consider the diagram

$$\begin{tikzcd}
& u\times v \arrow[dl, bend right, "1_u\times t_v"'] \arrow[dr, bend left, "t_u\times 1_v"] \arrow[d, shift right, "1_{u\times v}"'] \arrow[d, shift left, "c_{v,u}\circ c_{u,v}"] & \\
u & u\times v \arrow[l, "1_u\times t_v"] \arrow[r, "t_u\times 1_v"'] & v
\end{tikzcd}$$

in $\mathfrak{X}$ which is commutative for both morphisms $1_{u\times v}$, $c_{v,u}\circ c_{u,v}$: $u\times v \to u\times v$; this implies that they must be identical.

(S1) will be proved after (DO).

(DO) Consider the following diagram in X (recall $t_\varepsilon = t_\varepsilon\times 1_\varepsilon = 1_\varepsilon\times t_\varepsilon$, and $\varepsilon = \varepsilon\times\varepsilon$):

$$\begin{tikzcd}
& \varepsilon \arrow[dl, bend right, "1_\varepsilon"'] \arrow[dr, bend left, "t_\varepsilon"] \arrow[d, dashed, "f"] & \\
\varepsilon & \varepsilon \arrow[l, "t_\varepsilon"] \arrow[r, "t_\varepsilon"'] & \varepsilon
\end{tikzcd}$$

There must exist a (unique) morphism $f\colon \varepsilon \to \varepsilon$ such that $1_\varepsilon = t_\varepsilon\circ f = t_\varepsilon$.

Next we will prove that $f\colon u \to \varepsilon \in \mathrm{Mor}(X)$ implies $f = t_u$. Then $t_{u\times v} = t_u\times t_v$ is clear. Consider the diagram (recall $1_u = 1_u\times t_\varepsilon$, $t_u = t_u\times 1_\varepsilon$, and $u\times\varepsilon = u$):

$$\begin{tikzcd}
 & & u \arrow[dll, "1_u"'] \arrow[d, dashed, "g"] \arrow[drr, "f"] & & \\
 & & u\times\varepsilon \arrow[dll, "1_u"] \arrow[drr, "t_u"'] & & \\
u & & & & \varepsilon
\end{tikzcd}$$

There must exist a unique morphism $g\colon u \to u\times\varepsilon = u$ such that $1_u = 1_u \circ g = g$ and $f = t_u \circ g$. It follows $f = t_u$.

(S1): Let $f\colon u \to u'$, $g\colon v \to v'$ be two morphisms in $\mathrm{Mor}(X)$. Then $(1_{u'}\times t_{v'})\circ(f\times g) = f\times t_v$ and $(t_{u'}\times 1_{v'})\circ(f\times g) = t_u\times g$. This is true for $t_{v'}\circ g\colon v \to \varepsilon \in \mathrm{Mor}(X)$ and therefore by (DO) $t_{v'}\circ g = t_v$ and analogously $t_{u'}\circ f = t_u$. Now we see that $f\times g\colon u\times v \to u'\times v'$ is the unique morphism making the following diagram commutative:

$$\begin{tikzcd}
 & u\times v \arrow[dl, "1_u\times t_v"'] \arrow[dd, dashed, "f\times g"] \arrow[dr, "t_u\times 1_v"] & \\
u \arrow[dd, "f"'] & & v \arrow[dd, "g"] \\
 & u'\times v' \arrow[dl, "1_{u'}\times t_{v'}"] \arrow[dr, "t_{u'}\times 1_{v'}"'] & \\
u' & & v'
\end{tikzcd}$$

Now (S1) is easily proved by combining this diagram with the diagram defining $c_{v,u}$.

(D1) and (D4) to (D5) may similarly be proved using the universal properties of direct product diagrams and the definitions of d_u (D2) and $c_{u,v}$ (D3). ■

1.10 DEFINITION.

(i) *By* **K** *we will denote the category of all X-categories as objects together with all X-functors as morphisms.*†

(ii) *By* **S** *we will denote the category of all S-categories as objects together with all X-functors which preserve the crossing morphisms. If* $\Phi\colon X \to Y$ *is an X-functor and* X, Y *are S-categories then* Φ *preserves the crossing morphisms iff* $\Phi(c_{u,v}) = c_{\Phi(u),\Phi(v)}$ *for all* $u, v \in \mathrm{Ob}(X)$. *We will call such functors as S-functors.*

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† Recall that X-categories are small categories.

(iii) *By* $\mathbf{D}$ *we will denote the full subcategory of* $\mathbf{K}$ *consisting of all D-categories as objects.*

(iv) *Let* O *be a fixed monoid (with* $\times$ *as operation). Then we will denote by* $\mathbf{K}(O)$, $\mathbf{S}(O)$, $\mathbf{D}(O)$ *the subcategories of* $\mathbf{K}$, $\mathbf{S}$, $\mathbf{D}$ *resp. consisting of X-categories* X *with* $\mathrm{Ob}(X) = O$ *as objects and such X-functors which are the identity mapping* $\mathrm{id}\colon O \to O$ *on the objects.*

1.11 Remark.

A functor $\Phi\colon X \to Y \in \mathrm{Mor}(\mathbf{D})$ *preserves crossings, truncations, diagonalizations, and finite direct products. So* $\mathbf{D}$ *is also a full subcategory of* S.

Proof.

Because (DO) is satisfied in Y we have $\Phi(t_u) = t_{\Phi(u)}$ for all $u \in \mathrm{Ob}(X)$. Therefore Φ preserves the direct product diagrams

$$\begin{tikzcd} & u\times v \arrow[dl, "1_u\times t_v"'] \arrow[dr, "t_u\times 1_v"] & \\ u & & v \end{tikzcd}$$

Since another direct product $w \in \mathrm{Ob}(X)$ of u and v is isomorphic to $u\times v$ it can easily be proved that Φ preserves all finite direct products. From the universal properties of direct products it then follows that Φ also preserves the crossings, and diagonalizations. ∎

1.12 Definition.

(i) *Let* X *be an S-category (D-category) and* $A \subset \mathrm{Mor}(X)$. A *is called a free S-generating system (free D-generating system) for* X *iff the following universal property holds:*

If $\mathcal{Y}$ is an arbitrary S-category (D-category), Φ_1: $Ob(\mathcal{X}) \to Ob(\mathcal{Y})$ a monoid homomorphism, and Φ_2: $A \to Mor(\mathcal{Y})$ a mapping such that $\Phi_2(a)$: $\Phi_1(u) \to \Phi_1(v)$ if $a: u \to v \in A$, then there exists a uniquely defined S-functor (X-functor) Φ: $\mathcal{X} \to \mathcal{Y}$ which is an extension of Φ_1 and Φ_2.

(ii) *An S-category (D-category) is called free iff it possess a free S-generating system (D-generating system).*

1.13 THEOREM AND NOTATION.

(i) *Given any free monoid $\mathcal{O}$ and an arbitrary set A together with two mappings S, T: $A \to \mathcal{O}$ there exists a - up to an isomorphism in $\mathcal{S}$ - uniquely determined free S-category $F_S(A, \mathcal{O})$ with free S-generating system A and $\mathcal{O}$ as monoid of objects such that every $a \in A$ becomes a morphism $a: S(a) \to T(a) \in Mor(F_S(A, \mathcal{O}))$.*

(ii) *Given any free monoid $\mathcal{O}$ and an arbitrary set A together with two mappings S, T: $A \to \mathcal{O}$ such that $\varepsilon \notin T(A)$ there exists a - up to an isomorphism in $\mathcal{D}$ - uniquely determined free D-category $F_D(A, \mathcal{O})$ with free D-generating system A and $\mathcal{O}$ as monoid of objects such that every $a \in A$ becomes a morphism $a: S(a) \to T(a) \in Mor(F_D(A, \mathcal{O}))$.*

SKETCH OF A PROOF (c.f. [1], [2], [4]).

(i) Let $\mathcal{O} = \Sigma^*$ with an alphabet Σ. Define

$$A' := A \mathbin{\dot{\cup}} \{c_{s,t}: s,t \in \Sigma\}^{\dagger}$$

and extend S,T to S, T: $A' \to \mathcal{O}$ by $S(c_{s,t}) := s\times t$ and $T(c_{s,t}) := t\times s$. Let $\mathcal{X}$ be the free X-category $\mathcal{X} := F(A', \mathcal{O})$ (c.f. 2.4). Then $c_{s,t}$ becomes a morphism $c_{s,t}: s\times t \to t\times s \in Mor(\mathcal{X})$, $s,t \in \Sigma$. Define $c_{u,\varepsilon} := c_{\varepsilon,u} := 1_u$ for all $u \in \mathcal{O}$. Further define $c_{s,u}$ inductively using (S3) and then $c_{v,u}$ using (S3') for u, v $\in \mathcal{O}$. Now let R be the congruence

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† $\dot{\cup}$ denotes the disjoint union.

relation in $\mathrm{Mor}(\mathcal{X})$ generated by the axioms (S1), (S2), and (S3). Let $\Psi : \mathcal{X} \to \mathcal{X}/R$ be the canonical functor and define $F_S(A, \mathcal{O}) := \mathcal{X}/R$. It is clear that $\mathcal{X}/R$ is an S-category.

(ii) Define

$$A' := A \,\dot{\cup}\, \{t_s : s \in \Sigma\}$$
$$\dot{\cup}\, \{d_s : s \in \Sigma\}$$
$$\dot{\cup}\, \{c_{s,t} : s,t \in \Sigma\}$$

and extend S, T to S, T: $A' \to \mathcal{O}$ by $S(t_s) = S(d_s) := s$, $S(c_{s,t}) := s\times t$, $T(t_s) := \varepsilon$, $T(d_s) := s\times s$, and $T(c_{s,t}) := t\times s$. Let $\mathcal{X}$ be the free X-category $\mathcal{X} := F(A', \mathcal{O})$. Define $c_{u,v}$, d_u, and t_u for arbitrary $u, v \in \mathcal{O}$ in an analogous way as in (i) using (S3), (S3'), (DO), and (D6). Let R be the congruence relation on $\mathrm{Mor}(\mathcal{X})$ generated by the axioms (S1), (S2), (S3), and by (DO), ..., (D7). Let $\Psi : \mathcal{X} \to \mathcal{X}/R$ be the canonical functor and define $F_D(A, \mathcal{O}) := \mathcal{X}/R$. Then $\mathcal{X}/R$ is a D-category. For two given morphisms $f: w \to u$ and $g: w \to v$ in $\mathrm{Mor}(\mathcal{X}/R)$ $h := (f\times g)\circ d_w$ makes the following diagram commutative ((DO) and (D2)), and by (D1) must be unique.

$$\begin{tikzcd}
 & w \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
u & u\times v \arrow[l, "1_u\times t_v"] \arrow[r, "t_u\times 1_v"'] & v
\end{tikzcd}$$

In the both cases (i) and (ii) now we have to prove that A may be considered as a subset of $\mathcal{X}/R$ and that A then is a free S-generating system (D-generating system) for $\mathcal{X}/R$. We sketch the proof for (ii), a proof for (i) is similar.

Let $\mathcal{Y}$ be an arbitrary D-category, $\Phi_1 : \mathcal{O} \to \mathrm{Ob}(\mathcal{Y})$ a monoid homomorphism, and $\Phi_2 : A \to \mathrm{Mor}(\mathcal{Y})$ a mapping such that $\Phi_2(a) : \Phi_1(S(a)) \to \Phi_1(T(a)) \in \mathrm{Mor}(\mathcal{Y})$ for $a \in A$. Φ_2 may be extended to a mapping Φ_2': A' $\in$ Mor($\mathcal{Y}$) in an obvious way such that the images of $c_{s,t}$, d_s, and t_s are the crossings, diagonalizations, and truncations in $\mathcal{Y}$. Since A' is a free generating system for $\mathcal{X}$ there exists a unique X-functor Φ': $\mathcal{X} \to \mathcal{Y}$ extending Φ_1 and Φ_2'. Since all the relations in R also holds in $\mathcal{Y}$ we have an X-functor Φ: $\mathcal{X}/R \twoheadrightarrow \mathcal{Y}$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{X} \arrow[d, twoheadrightarrow, "\Psi"'] \arrow[dr, "\Phi'"] & \\
\mathcal{X}/R \arrow[r, "\Phi"'] \arrow[ur, phantom, "///"] & \mathcal{Y}
\end{tikzcd}$$

Clearly Φ is an extension of Φ_1 and Φ_2.

Now we will prove that $\Psi|_A$: A $\to$ Mor($\mathcal{X}/R$) is injective. Let M be a set such that card(M) > card(A) and let $\mathcal{C}_M$ be as in example 1.6. It is easily proved that $\mathcal{C}_M$ is a D-category. Then we may define

$$\Phi_1 : \mathcal{O} \to \mathrm{Ob}(\mathcal{C}_M) = \mathbb{N}_0 \text{ by } \Phi_1(u) := |u|^{\dagger} \text{for all } u \in \mathcal{O}$$

and

$$\Phi_2 : A \to \mathrm{Mor}(\mathcal{C}_M) \text{ such that } \Phi_2(a) : \Phi_1(S(a)) \to \Phi_1(T(a)).$$

Since card(M) > card(A), and $\Phi_1(T(a)) \geq 1$, we are able to define Φ_2 in such a way that it is injective. Then $\Phi'|_A$ is injective too and therefore $\Psi|_A$ must be injective. ■

For describing a switching circuit as a morphism in a free D-category it is not essential how the circuit paths are factored. However because of (D7) the following two circuits also have the same description.

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† $|u|$ denotes the number of letters in $u \in \Sigma^*$.

But if we are not interested in the outdegree of switching elements this is not a handicap. Our earlier method of describing switching circuits by a digraph together with a linear order on the inputs of the nodes (switching elements) is a special representation of the morphisms in $F_D(A, \mathbb{N}_0)$. We only had to consider

$$n: \langle a;\, n_1,\, \dots,\, n_k\rangle \qquad (a \in A)$$

as an abbreviation of the morphism

$$\ell_n := (1_{n-1} \times a) \circ f_{n+k-2,n_k} \circ f_{n+k-3,n_k-1} \circ \dots \circ f_{n-1,n_1},$$

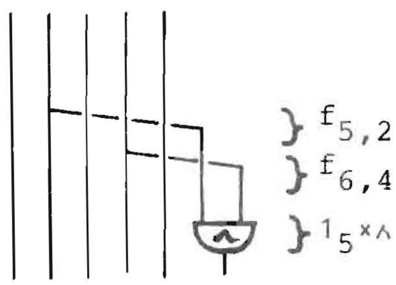
where

$$f_{\nu,\mu} := (1_\mu \times c_{1,\nu-\mu}) \circ (1_\nu \times d_1 \times 1_{\nu-\mu})$$

is the morphism which connects the μ-th wire of ν parallel wires with a new wire to the right of them. For example if

$$6: \langle\wedge;\ 2,\ 4\rangle$$

is a line then ℓ_6 is the morphism



The first l lines representing the input variables will be omitted, and the two lines for the constants true and false are

$$\ell_{l+1} := 1_l \times \mathrm{true},$$

$$\ell_{l+2} := 1_{l+1} \times \mathrm{false}.$$

Then the switching circuit represented by m such lines is the morphism

$$o \circ \ell_m \circ \dots \circ \ell_{l+1}$$

where o is the output selecting function written as a morphism in $F_D(A, \mathbb{N}_0)$.

There are several other representations and normal forms of the morphisms of an X-category, S-category, or D-category. For them we reffer the reader to [1]. As an example every morphism f in a D-category may be factored as f = f'∘f" where f" is constructed using only diagonalizations as switching elements and in f' no diagonalizations occur. then f' may be considered as the usual formula notation describing f.

The following definition generalizes the concept of Boolean and arithmetic functions in the sense of Strassen's Ω-algebras [16].

1.14 Definition.

Let M *be a set. A function* $\omega: M^n \to M$ *is called an n-ary operation on* M $(n \in \mathbb{N})$*. Let* Ω *be a set of operations on* M *together with a mapping* $S: \Omega \to \mathbb{N}$ *such that* $\omega \in \Omega$ *is an* $S(\omega)$*-ary operation on* M*. Now we define* $A := M \cup \Omega$ *and extend* S *to a mapping* $S: A \to \mathbb{N}_0$ *by setting* $S(x) = 0$ *for all* $x \in M$*. Further we define another mapping* $T: A \to \mathbb{N}_0$ *such that* $T(a) = 1$ *for all* $a \in A$*. Using theorem 1.13 we get the free D-category* $F_D(A, \mathbb{N}_0)$. *Let* C_M *be defined as in example 1.6 and define an interpretation* $I: F_D(A, \mathbb{N}_0) \to C_M$ *as follows:*

I is the identity mapping on the objects and

$$I(\omega)(x_1,\dots,x_{S(\omega)}) = \omega(x_1,\dots,x_{S(\omega)}) \quad \textit{for all } \omega \in \Omega,\ x_1,\dots,x_{S(\omega)} \in M,$$

$$I(x)(e) = x \quad \textit{for all } x \in M.$$

This means that the elements of M *are interpreted as constant functions* $\{e\} \to M$*. Now we denote by* $P_{M,\Omega}$ *the image of the functor* I*. Clearly* $P_{M,\Omega}$ *is a D-category.*

1.15 Examples.

(i) If $M = \{\text{true}, \text{false}\}$ and $\Omega = \{\wedge, \vee, \neg\}$ where $\wedge$, $\vee$, and $\neg$ are the wellknown Boolean operations on M then $P_{M,\Omega} = B$ (c.f. example 1.5).

(ii) If M = {true, false} and $\Omega = \{\wedge, \vee\}$ then $\mathcal{P}_{M,\Omega} = \mathcal{B}^m$, the D-category of all monotone Boolean functions.

(iii) If M = R is a ring and $\Omega = \{+, -, *\}$ is the set of the ring operations, then $\mathcal{P}_{M,\Omega} = \mathcal{P}_R$ is the D-category of all polynomial functions over R.

§2. Complexity Measures on X-categories.

In this section S denotes a fixed additive commutative ordered monoid with unit element 0 satisfying the following condition:

(P) $s \geq 0$ for all $s \in S$.

In this case we call S a positive monoid. For example S may be the additive semigroup $\mathbb{N}_0$ together with the natural order.

2.1. Definition.

Let X be an X-category. A complexity measure on X with values in S is a function $c\colon \mathrm{Mor}(X) \to S$ *satisfying the following axioms:*

(C1) $c(1_u) = 0$ *for all objects* $u \in \mathrm{Ob}(X)$.

(C2) $c(f \circ g) \leq c(f) + c(g)$ *for all* $f, g \in \mathrm{Mor}(X)$ *such that* $f \circ g$ *is defined.*

(C3) $c(f \times g) \leq c(f) + c(g)$ *for all* $f, g \in \mathrm{Mor}(X)$.

In the future we will often denote a complexity measure c by $|\ldots|$.

If c satisfies the stronger condition

(C2') $c(f \circ g) \leq \mathrm{Max}(c(f), c(g))$ *if* $f \circ g$ *is defined* $(f, g \in \mathrm{Mor}(X))$

instead of (C2), we call c a breadth measure on X, and if c satisfies

(C3') $c(f \times g) \leq \mathrm{Max}(c(f), c(g))$ $(f, g \in \mathrm{Mor}(X))$

instead of (C3), we call c a depth measure on X.

If X is an S-category (D-category) we call a complexity measure c on X an S-complexity measure (D-complexity measure) iff $c(c_{u,v}) = 0$ $(c(t_u) = c(d_u) = c(c_{u,v}) = 0)$ *for all* $u, v \in \mathrm{Ob}(X)$.

2.2 Examples.

Let $X = F(A, O)$ $(X = F_S(A, O))$ be a free X-category (free S-category) and $L\colon A \to S$ an arbitrary function. We may interpret S as an X-category where $\circ$ and $\times$ both mean the addition + in S and $\circ$ is always defined. The monoid $\mathrm{Ob}(S)$ consists of a single element ε. Therefore, S is also an S-category with $c_{\varepsilon,\varepsilon} := 1_\varepsilon$. Extending the function L to an X-functor (S-functor)

$$L: \mathcal{X} \to S$$

gives us a complexity measure L on $\mathcal{X}$ with values in S for which the following equations hold:

(i) $L(1_u) = 0 \quad (L(c_{u,v}) = 0$ in addition$) \qquad (u,v \in Ob(\mathcal{X}))$.

(ii) $L(f \circ g) = L(f) + L(g)$ if $f \circ g$ is defined $\qquad (f,g \in Mor(\mathcal{X}))$.

(iii) $L(f \times g) = L(f) + L(g) \qquad (f,g \in Mor(\mathcal{X}))$.

Such a complexity measure on free X-categories (free S-categories) is called a cost function. Since $L(t_v \circ f) = L(t_u) = 0$ $(f: u \to v \in Mor(\mathcal{X}))$ in a free D-category there exist only trivial cost functions (that means $L(f) = 0$ for all $f \in Mor(\mathcal{X})$).

If $\mathcal{X}$ denotes the free X-category $\mathcal{G}(A)$ of switching circuits over the basis A and $L: A \to S$ is the cost of a single switching element (in A), then for a circuit $f \in Mor(\mathcal{X})$ $L(f)$ denotes the cost of this circuit. The cost of a single switching element may be the price we must pay for it or the energy this element uses on the average for working correctly. In both cases S may be the positive monoid $\mathbb{R}_0^+$ of the nonnegative real numbers. But we also may combine the two cost functions as $L: A \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that the first component of $L(a)$ is the price and the second component is the energy consumption. In this case the semigroup S is not a submonoid of $\mathbb{R}_0^+$. The order on $S = \mathbb{R}_0^+ \times \mathbb{R}_0^+$ is the lexicographical order. This is analogous to valuations of rank > 1 in algebra.

Another important example is given in [5]: Let M be a finite set and $\mathcal{X}$ a sub-X-category of $\mathcal{C}_M$ such that $Mor(\mathcal{X})$ consists only of isomorphisms (bijections) on $\mathcal{C}_M$. Let $\Pi = (\pi_u)_{u \in \mathbb{N}_0}$ be a sequence of partitions such that

$$M^u = \bigcup_{\alpha \in \pi_u} \alpha \quad \text{where } \alpha \cap \beta = \emptyset \text{ for different } \alpha, \beta \in \pi_u.$$

Define an entropy function $H : Mor(\mathcal{X}) \to \mathbb{R}_0^+$ by

$$H_\Pi(f) := - \sum_\alpha \frac{card(\alpha)}{card(M^u)} \sum_\beta \frac{card(f(\alpha) \cap \beta)}{card(\alpha)} \log \frac{card(f(\alpha) \cap \beta)}{card(\alpha)}$$

for $f: u \to v \in Mor(\mathcal{X})$ where α runs over π_u and β runs over π_v.

Then in [5] is proved that $H\colon \mathrm{Mor}(\mathcal{X}) \to \mathbb{R}_0^+$ defined by

$$H(f) = \sup\{H_\Pi(f) : \Pi \text{ sequence of partitions as above}\}$$

is a complexity measure. This complexity measure is used to get lower bounds on the complexity of permutations (matrix transposition for example) and merging networks (c.f. [13] and [18]).

In algebra we define valuations on rings. Here, one of the axioms is $|x \cdot y| = |x| \cdot |y|$ where an equallity instead of an inequallity is postulated. Therefore in the theory of valuations on a ring or on a field we have a lot of results characterizing the set of possible valuations. But in our case we have the following difficulties:

2.3 REMARKS.

Let $\mathcal{X}$ be an X-category and $c\colon \mathrm{Mor}(\mathcal{X}) \to S$ a complexity measure on $\mathcal{X}$. If $\varphi\colon S \to S$ denotes an arbitrary monotone sublinear function† *with $\varphi(0) = 0$, then $\varphi \circ c\colon \mathrm{Mor}(\mathcal{X}) \to S$ is also a complexity measure on $\mathcal{X}$.*

PROOF: easy calculation. ∎

2.4 EXAMPLE.

If S is the positive semigroup $\mathbb{R}_0^+$, then $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined as $\varphi(x) = \log(1 + x)$ is monotone and convex and satisfies $\varphi(0) = 0$. If in general $\varphi, \psi\colon S \to S$ are monotone and convex with $\varphi(0) = \psi(0) = 0$ then $\varphi \circ \psi\colon S \to S$ is also such a function. If we have one complexity measure $c\colon \mathrm{Mor}(\mathcal{X}) \to S$ we are able to construct a lot of other complexity measures, such as $\varphi \circ c$, $\varphi \circ \varphi \circ c$, $\varphi \circ \varphi \circ \varphi \circ c$, ... which are all not very different from c.

Therefore we will later define "strict complexity measures". But before considering the set of all complexity measures on a fixed X-category $\mathcal{X}$ we will prove some theorems about complexity measures.

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† A function $\psi\colon \mathbb{R} \to \mathbb{R}$ is called sublinear iff $\psi(x + y) \leq \psi(x) + \psi(y)$.

2.5 Theorem and Notation.

Let Φ: $X \to Y$ be an X-functor defined over X-categories X and Y.

(i) If c: Mor(Y) $\to$ S *is a complexity measure on Y, then* Φ^{-1}(c): Mor(X) $\to$ S *defined by*

$$\Phi^{-1}(c)(f) := c(\Phi(f)) \qquad (f \in \mathrm{Mor}(X))$$

is a complexity measure on X.

(ii) Let S *be such, that for every subset* M $\subset$ S inf(M) *exists respective to the order on* S. *If Φ considered as a mapping* Ob(X) $\to$ Ob(Y) *is bijective and* c: Mor(X) $\to$ S *is a complexity measure on X, then* Φ(c): Mor(Y) $\to$ S *defined by*

$$\Phi(c)(f) := \inf\{c(f'): \Phi(f') = f\} \qquad (f \in \mathrm{Mor}(Y))$$

is a complexity measure on Y. If inf($\emptyset$) = ∞ *does not exist in* S *but Φ is also surjective on the morphisms, then* Φ(c) *can be defined in the same way.*

Proof.

(i) obvious.

(ii) (C1): Since Φ is surjective on the objects, for every u $\in$ Ob(Y) there is a u' $\in$ Ob(X) with $\Phi(1_{u'}) = 1_u$. Therefore $\Phi(c)(1_u) = 0$.

(C2):
$$\begin{aligned} \Phi(c)(f \circ g) &= \inf\{c(h): \Phi(h) = f \circ g\} \\ &\le \inf\{c(f' \circ g'): \Phi(f') = f \text{ and } \Phi(g') = g\} \\ &\qquad \text{since } \Phi \text{ is injective on the objects} \\ &\le \inf\{c(f'): \Phi(f') = f\} \\ &\qquad + \inf\{c(g'): \Phi(g') = g\} \\ &= \Phi(c)(f) + \Phi(c)(g). \end{aligned}$$

(C3): similar to (C2) without using the assumption, that Φ is injective on the objects. ∎

The following example shows us that the supposition that Φ is bijective on the objects is necessary.

2.6 EXAMPLE.

Consider the following two free X-categories:

$\mathcal{X} = F(\{f, g_1, g_2\}, \{r, s_1, s_2, t\}^*)$, $\quad \mathcal{Y} = F(\{f', g'\}, \{u, v, w\}^*)$.

$$\begin{tikzcd} & r \arrow[d, "f"] \\ t & s_1 \arrow[l, "g_1"'] \\ & s_2 \arrow[ul, "g_2"] \end{tikzcd} \qquad\qquad \begin{tikzcd} u \arrow[d, "f'"] \\ v \arrow[d, "g'"] \\ w \end{tikzcd}$$

Let $\Phi: \mathcal{X} \to \mathcal{Y}$ be the following X-functor:

$$\Phi(r) := u, \qquad \Phi(f) := f',$$
$$\Phi(s_1) := \Phi(s_2) := v, \qquad \Phi(g_1) := \Phi(g_2) := g',$$
$$\Phi(t) := w.$$

If $c: \mathrm{Mor}(\mathcal{X}) \to S := \mathbb{R}_0^+ \cup \{\infty\}$ is defined by

$$c(f) := c(g_1) := 0, \qquad c(g_2) := 1$$

(c.f. 2.2) then we have

$$\Phi(c)(g' \circ f') = \inf\{c(h): \Phi(h) = g' \circ f'\} = 1$$

since $\Phi^{-1}(h) = \{g_1 \circ f\}$ but

$$\Phi(c)(g') = \Phi(c)(f') = 0,$$

and so $\Phi(c)$ cannot be a complexity measure on $\mathcal{Y}$.

The above problem arose because Φ was not injective on $\mathrm{Ob}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{Y})$. It is also clear that if Φ is not surjective on the objects $\Phi(c)$ cannot satisfy the axiom (C1).

Because of theorem 2.5 we are able to define the size complexity of Boolean functions. Let $I: G(\{\wedge, \vee, \neg\}) \to \mathcal{B}$ the interpretion defined in §1. On $G(\{\wedge, \vee, \neg\})$ we are able to define a cost function L with values in $\mathbb{N}_0$ such that $L(d_u) = L(c_{u,v}) = L(t_u) = 0$ $(u, v \in \mathbb{N}_0)$ and $L(\wedge) = L(\vee) = L(\neg) = 1$ (c.f. 2.2.). Then $I(L)$ is the size complexity on $\mathcal{B}$.

Throughout the rest of this section we will consider only X-categories in $\mathbf{K}(\mathcal{O})$ for a fixed monoid $\mathcal{O}$. Further we will assume $S = \mathbb{R}_0^+ \cup \{\infty\}$, which we may also consider as a semiring. Then we define:

2.7 Definition.

We denote by $\mathbf{C}$ *the following category:*

$$\mathrm{Ob}(\mathbf{C}) := \{ (X, c) : X \in \mathrm{Ob}(\mathbf{K}(\mathcal{O})) \text{ and } c \text{ is a complexity measure on } X \text{ with values in } \mathbb{R}_0^+ \cup \{\infty\}\}$$

$$\mathrm{Mor}(\mathbf{C}) := \{ \Phi : (X, c) \to (X', c') : \Phi : X \to X' \in \mathrm{Mor}(\mathbf{K}(\mathcal{O})) \text{ and } \exists\, \lambda \in \mathbb{R}^+ \text{ such that } c'(\Phi(f)) \le \lambda \cdot c(f) \text{ for all } f \in \mathrm{Mor}(X)\}.^{\dagger}$$

If $\mathrm{id}: X \to X$ *is the identity functor* ($X \in \mathbf{K}(\mathcal{O})$) *and* c_1, c_2 *are two complexity measures on* X*, then we will write*

$$c_1 \succcurlyeq c_2 \quad \textit{iff} \quad \mathrm{id}: (X, c_1) \to (X, c_2) \in \mathrm{Mor}(X).$$

Further

$$c_1 \approx c_2 \quad \textit{iff} \quad c_1 \succcurlyeq c_2 \textit{ and } c_1 \preccurlyeq c_2.$$

It is easily checked that $\mathbf{C}$ is indeed a category. Theorem 2.5 shows us that the following are morphisms in $\mathbf{C}$.

2.8 Proposition.

Let $\Phi: X \to Y$ *be a morphism in the category* $\mathbf{K}(\mathcal{O})$.

(i) If c *is a complexity measure on* Y*, then*

$$\Phi : (X, \Phi^{-1}(c)) \to (Y, c) \in \mathrm{Mor}(\mathbf{C}).$$

(ii) If c *is a complexity measure on* X*, then*

$$\Phi : (X, c) \to (Y, \Phi(c)) \in \mathrm{Mor}(\mathbf{C}).$$

Proof.

Take $\lambda = 1$. ■

---

† $\mathbb{R}^+ := \mathbb{R}_0^+ \setminus \{0\} = \{x \in \mathbb{R} : x > 0\}$

2.9 PROPOSITION.

Let $\Phi\colon X \to Y$ be a morphism in the category $\mathbf{K}(O)$.

(i) If c_1, c_2 are complexity measures on Y and $c_1 \succcurlyeq c_2$, then we have $\Phi^{-1}(c_1) \succcurlyeq \Phi^{-1}(c_2)$ on X.

(ii) If c_1, c_2 are complexity measures on X and $c_1 \succcurlyeq c_2$, then we have $\Phi(c_1) \succcurlyeq \Phi(c_2)$ on Y.

$$\begin{tikzcd}
(X,\ c_1) \arrow[r, "id"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\succcurlyeq"] & (X,\ c_2) \arrow[d, "\Phi"] \\
(Y,\ \Phi(c_1)) \arrow[r, dashed, "id"'] & (Y,\ \Phi(c_2))
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,\ \Phi^{-1}(c_1)) \arrow[r, dashed, "id"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\succcurlyeq"] & (X,\ \Phi^{-1}(c_2)) \arrow[d, "\Phi"] \\
(Y,\ c_1) \arrow[r] & (Y,\ c_2)
\end{tikzcd}$$

PROOF.

(i) $\Phi^{-1}(c_2)(f) = c_1\Phi(f) \le \lambda\cdot c_1(\Phi(f)) = \lambda\cdot\Phi^{-1}(c_1)(f)$.

(ii) $\Phi(c_2)(f) = \inf\{c_2(f')\colon \Phi(f') = f\}$
$\qquad \le \inf\{\lambda\cdot c_1(f')\colon \Phi(f') = f\} = \lambda\cdot\Phi(c_1)(f)$. ∎

In the following we will prove some simulation theorems using the following definition:

2.10 DEFINITON.

Let c be a complexity measure on an X-category X with values in $\mathbb{R}_0^+ \cup \{\infty\}$. Then c is called nondegenerate iff for all $f \in \mathrm{Mor}(X)$ $c(f) \neq \infty$.

2.11 PROPOSITION.

Let $X = F(A,\ O)$ be a free X-category with finite generating system A, c_1 a cost function on X, and c_2 an arbitrary complexity measure on X. Assume that c_1, c_2 both are nondegenerate. If $c_1(a) = 0 \Rightarrow c_2(a) = 0\ \forall\, a \in A$ *then* $c_1 \succcurlyeq c_2$.

PROOF.

Let $\lambda := \mathrm{Max}\,\{\frac{c_2(a)}{c_1(a)}\colon\ a \in A$ such that $c_1(a) \neq 0\}$. This maximum exists since A is finite and we have $0 < \lambda < \infty$ since c_1, c_2 both are nondegenerate. Let $f \in \mathrm{Mor}(X)$. We will prove by induction on the length of a sequential representation (c.f. [7]) of f, that $c_2(f) \le \lambda\cdot c_1(f)$.

(i) If $f = 1_u$ for some object $u \in \mathcal{O}$, then $c_2(f) = 0 = c_1(f)$.

(ii) If $f = (1_u \times a \times 1_v) \circ g$ with $u, v \in \mathcal{O}$, $a \in A$ and $g \in \mathrm{Mor}(\mathcal{X})$, then we have

$$\begin{aligned} c_2(f) &\le c_2(a) + c_2(g) \\ &\le \lambda \cdot c_1(a) + \lambda \cdot c_1(g) && \text{by definition of } \lambda \text{ and induction hypothesis} \\ &= \lambda \cdot (c_1(a) + c_1(g)) \\ &= \lambda \cdot c_1((1_u \times a \times 1_v) \circ g) = \lambda \cdot c_1(f) && \text{since } c_1 \text{ is a cost function on } \mathcal{X}. \quad \blacksquare \end{aligned}$$

2.12 First Simulation Theorem.

Let $\mathcal{X}$ be a finitely generated free X-category and c_1, c_2 two non degenerate cost functions on $\mathcal{X}$ with $c_1(f) = 0 \Leftrightarrow c_2(f) = 0$ for all $f \in \mathrm{Mor}(\mathcal{X})$. Then we have $c_1 \approx c_2$.

Proof.

Since every cost function is a complexity measure, we have from 3.11 that $c_1 \succcurlyeq c_2$ and $c_2 \succcurlyeq c_1$. Therefore $c_1 \approx c_2$. $\blacksquare$

2.13 Theorem (Maximality of the size complexity) (c.f. [5]).

Let $\Phi\colon \mathcal{X} \to \mathcal{Y} \in \mathrm{Mor}(\mathbf{K}(\mathcal{O}))$ where $\mathcal{X}$ is a finitely generated free X-category and Φ is surjective on the morphisms. Further, let c_1 be a non degenerate cost function on $\mathcal{X}$ and c_2 be an arbitrary non degenerate complexity measure on $\mathcal{Y}$ such that $c_1(f) = 0 \Rightarrow c_2(\Phi(f)) = 0$ for all $f \in \mathrm{Mor}(\mathcal{X})$. Then $c_2 \preccurlyeq \Phi(c_1)$.

$$\begin{tikzcd} (\mathcal{X}, c_1) \arrow[rr, "\Phi"] \arrow[dr, "\Phi"'] & & (\mathcal{Y}, c_2) \\ & (\mathcal{Y}, \Phi(c_1)) \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

In the case $\Phi = I$, $\mathcal{X} = G(\{\wedge, \vee, \neg\})$, and $\mathcal{Y} = \mathcal{B}$ this means: Every non degenerate complexity measure c on $\mathcal{B}$ with $c(d_u) = c(t_u) = c(c_{u,v}) = 0$ for all $u, v \in \mathrm{Ob}(\mathcal{B}) = \mathbb{N}_0$ is (without a constant factor λ) a lower bound for the size complexity in

$\mathcal{B}$. Further, two non degenerate size complexity measures c_1 and c_2 on $\mathcal{B}$ are equivalent ($c_1 \approx c_2$) iff $c_1(f) = 0 \Leftrightarrow c_2(f) = 0$ for all $f \in \mathrm{Mor}(\mathcal{X})$.

PROOF.

$\Phi^{-1}(c_2)$ is a non degenerate complexity measure on $\mathcal{X}$ with $c_1(f) = 0 \Rightarrow \Phi^{-1}(c_2)(f) = c_2(\Phi(f)) = 0$ for all $f \in \mathrm{Mor}(\mathcal{X})$. Then 2.11 implies that $\Phi^{-1}(c_2) \preccurlyeq c_1$ and because of 2.9 we get $\Phi(\Phi^{-1}(c_2)) \preccurlyeq \Phi(c_1)$. Now we will prove that under the supposition that Φ, regarded as a mapping Φ: $\mathrm{Mor}(\mathcal{X}) \to \mathrm{Mor}(\mathcal{Y})$ is surjective, we have $c_2 = \Phi(\Phi^{-1}(c_2))$. Let $f \in \mathrm{Mor}(\mathcal{Y})$. Then we have

$$\begin{aligned}\Phi(\Phi^{-1}(c_2))(f) &= \inf\{\Phi^{-1}(c_2)(f'):\ \Phi(f') = f\}\\ &= \inf\{c_2(\Phi(f')):\ \Phi(f') = f\}\\ &= c_2(f)\end{aligned}$$

since there exists an $f' \in \mathrm{Mor}(\mathcal{X})$ with $\Phi(f') = f$. ■

2.14 OPEN PROBLEM.

Find complexity measures which can easily be computed, but still give good lower bounds on the size complexity.

Most of the well known examples of such complexity measures which are easily computed are depth measures and therefore not good lower bounds for the size complexity. Another example which yields nonlinear lower bounds is the entropy function (c.f. 2.2).

2.15 THEOREM.

Let $\mathcal{X}$ *be an* $\mathcal{X}$*-category and* r: $\mathrm{Mor}(\mathcal{X}) \to \mathbb{R}_0^+ \cup \{\infty\}$ *an arbitrary function satisfying* $r(1_u) = 0$ *for all* $u \in \mathrm{Ob}(\mathcal{X})$.

(i) c_r: $\mathrm{Mor}(\mathcal{X}) \to \mathbb{R}_0^+ \cup \{\infty\}$ *defined by*

$$c_r(f) := \sup\{r((1_u \times f \times 1_v) \circ h) - r(h):\ u, v \in \mathrm{Ob}(\mathcal{X}),\ h \in \mathrm{Mor}(\mathcal{X}) \text{ such that } (1_u \times f \times 1_v) \circ h \text{ is defined}\}$$

is a complexity measure on $\mathcal{X}$ *satisfying* $c_r(f) \geq r(f)$ *for all* $f \in \mathrm{Mor}(\mathcal{X})$.

(ii) ${}_r c\colon \mathrm{Mor}(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ *defined by*

$$ {}_r c(f) := \sup\{r(h\circ(1_u \times f \times 1_v)) - r(h)\colon u,\ v \in \mathrm{Ob}(X),\ h \in \mathrm{Mor}(X) \text{ such that } h\circ(1_u\times f\times 1_v) \text{ is defined}\} $$

is a complexity measure on X *satisfying* ${}_r c(f) \geq r(f)$ *for all* $f \in \mathrm{Mor}(X)$.

PROOF.

We will only prove (i), a proof of (ii) is similar.

(C1): $$c_r(1_w) = \sup_{u,v,h}\{r((1_u\times 1_w \times 1_v)\circ h) - r(h)\} = \sup_h \{r(h) - r(h)\} = 0.$$

(C2): $$\begin{aligned} c_r(f\circ g) &= \sup_{u,v,h}\{r((1_u\times(f\circ g)\times 1_v)\circ h) - r(h)\} \\ &\leq \sup_{u,v,h}\{r((1_u\times f\times 1_v)\circ((1_u\times g\times 1_v)\circ h)) - r((1_u\times g\times 1_v)\circ h)\} \\ &\quad + \sup_{u,v,h}\{r((1_u\times g\times 1_v)\circ h) - r(h)\} \\ &\leq c_r(f) + c_r(g). \end{aligned}$$

(C3): Let $f\colon w_1 \to w_1'$, $g\colon w_2 \to w_2' \in \mathrm{Mor}(X)$.

$$\begin{aligned} c_r(f\times g) &= \sup_{u,v,h}\{r((1_u\times f\times g\times 1_v)\circ h) - r(h)\} \\ &\leq \sup_{u,v,h}\{r((1_u\times f\times 1_{w_2'\times v})\circ((1_{u\times w_1}\times g\times 1_v)\circ h) \\ &\qquad - r((1_{u\times w_1}\times g\times 1_v)\circ h)\} \\ &\quad + \sup_{u,v,h}\{r((1_{u\times w_1}\times g\times 1_v)\circ h) - r(h)\} \\ &\leq c_r(f) + c_r(g). \end{aligned}$$

Let $f\colon w \to w' \in \mathrm{Mor}(X)$. Taking $u = v = \varepsilon$ and $h = 1_w$ we get $c_r(f) \geq r(f)$. ∎

2.16 REMARKS

(i) *If* $r\colon \mathrm{Mor}(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ *is already a complexity measure on* X *then* $c_r = {}_rc = r$.

(ii) *Let* A *be a finite generating system for* X. *If* $|\ldots|$ *denotes the size complexity on* X *and* $r\colon \mathrm{Mor}(X) \to \mathbb{R}_0^+ \cup \{\infty\}$ *satisfies*

(a) $r(1_u) = 0$ *for all* $u \in \mathrm{Ob}(X)$,

(b) $r(1_u \times f \times 1_v) \le r(f)$ *for all* $u, v \in \mathrm{Ob}(X)$, $f \in \mathrm{Mor}(X)$,

(c) $r(a \circ h) \le |a| + r(h)$ *for all* $a \in A$, $h \in \mathrm{Mor}(X)$

then we have $c_r \preccurlyeq |\ldots|$.

PROOF.

(i) $r(1_u \times f \times 1_v) \circ h) - r(h) \le r(f)$ since r is a complexity measure. It follows that $r(f) \le c_r(f) \le r(f)$ and similar $r = {}_rc$.

(ii) Let $m := \max\{|a|\colon a \in A\}$. If $f \in \mathrm{Mor}(X)$ has a representation of lenght l respective to A then $c_r(f) \le l \cdot m$. This can easily be proved by induction on l using the suppositions (b) and (c). It follows that c_r is non degenerate and therefore (by theorem 2.13) $c_r \preccurlyeq |\ldots|$. ∎

The second remark in 2.16 is often used to get lower bounds for the size complexity. Examples are Strassen's degree bound [17] or Paul's 2.5 n lower bound [13].

The following is analogous to a theorem of Strassen (c.f. [16]).

2.17 SECOND SIMULATION THEOREM (c.f. [5]).

Consider the following diagram in $\mathbf{C}$:

$$\begin{array}{ccc} (X, c) & \xrightarrow{\quad X \quad} & (X', c') \\ \Phi \big\downarrow & & \big\downarrow \Phi' \\ (Y, \Phi(c)) & & (Y', \Phi'(c')) \end{array}$$

and assume that there exists an X-functor $\Psi: \mathcal{Y} \to \mathcal{Y}'$ such that the following diagram in $\mathbf{K}(\mathcal{O})$ is commutative:

$$\begin{tikzcd} X \arrow[r, "\chi"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "///"] & X' \arrow[d, "\Phi'"] \\ \mathcal{Y} \arrow[r, "\Psi"'] & \mathcal{Y}' \end{tikzcd}$$

Then Ψ is a morphism in C too; that is $\Psi^{-1}(\Phi'(c')) \preccurlyeq \Phi(c)$.

PROOF.

$$\begin{aligned} \Phi'(c')(\Psi(f)) &= \inf\{c'(f'):\ \Phi'(f') = \Psi(f)\} \\ &\le \inf\{c'(\chi(g)):\ (\Phi' \circ \chi)(g) = \Psi(f)\} \\ &\qquad\qquad (\text{since } B \subset A \text{ implies } \inf(A) \le \inf(B)) \\ &= \inf\{c'(\chi(g)):\ (\Psi \circ \Phi)(g) = \Psi(f)\} \\ &\le \inf\{\lambda \cdot c(g):\ \Psi(\Phi(g)) = \Psi(f)\} \\ &\qquad\qquad (\text{since } \chi \in \mathrm{Mor}(\mathbf{C})) \\ &\le \lambda \cdot \inf\{c(g):\ \Phi(g) = f\} \\ &\qquad\qquad (\text{since } B \subset A \text{ implies } \inf(A) \le \inf(B)) \\ &= \lambda \cdot \Phi(c)(f). \end{aligned}$$

2.18 EXAMPLE.

Let $\psi: R \to R'$ be a ring homomorphism (R, R' rings). Then we may define the X-categories $\mathcal{P}_R$ and $\mathcal{P}_{R'}$ as in example 1.6. We denote the natural interpretations of $G(\{+, -, *\})$ in $\mathcal{P}_R$ ($\mathcal{P}_{R'}$ resp.) by I (I' resp.). The ring homomorphism ψ induces an X-functor $\Psi: \mathcal{P}_R \to \mathcal{P}_{R'}$ such that the following diagram in $\mathbf{K}(\mathbb{N}_0)$ is commutative:

$$\begin{tikzcd} G(\{+,-,*\}) \arrow[r, "id"] \arrow[d, "I"'] \arrow[dr, phantom, "///"] & G(\{+,-,*\}) \arrow[d, "I'"] \\ \mathcal{P}_R \arrow[r] & \mathcal{P}_{R'} \end{tikzcd}$$

Let L be a cost function on $G(\{+, -, *\})$ with $L(+)$, $L(-)$, $L(*)$ $\neq 0$ and $L(d_1) = L(c_{1,1}) = L(t_1) = 0$. Then $I(L)$ ($I(L')$ resp.) is the size complexity on P_R ($P_{R'}$, resp.) or, in other words, the computational complexity relative to the ring R (R' resp.). Now the second simulation theorem tells us that $\Phi^{-1}(I'(L)) \preccurlyeq I(L)$. If for example $R = \mathbb{Z}$ and $R' = \mathbb{Z}/n\mathbb{Z}$ then computing in $\mathbb{Z}$ is not easier than computing in $\mathbb{Z}/n\mathbb{Z}$. In a similar way we get n^3 as a lower bound for matrix multiplication over $\mathbb{Z}$ using only the monotone operations + and $\cdot$ from the same lower bound for monotone Boolean matrix multiplication ([8],[10]). For further examples see [5].

§3. The $\mathbb{R}^+$-Module of all the Complexity Measures on a Fixed X-Category.

In this section O will be a fixed finitely generated free monoid and all of the complexity measures considered here will have values in the semiring $\mathbb{R}_0^+$.

3.1 Proposition and Notation (c.f. [5]).

Let $X \in \mathrm{Ob}(\mathbf{K}(O))$ be an X-category. We denote by

$$\mathbf{C}(X) := \{c: \mathrm{Mor}(X) \to \mathbb{R}_0^+ : c \text{ is a complexity measure on } X\}$$

the set of all complexity measures on X. Let $c_1, c_2 \in \mathbf{K}(O)$ and $\lambda \in \mathbb{R}_0^+$. Then the following holds:

(i) $c_1 + c_2 \in \mathbf{C}(X)$ *where* $(c_1 + c_2)(f) := c_1(f) + c_2(f)$ *for all* $f \in \mathrm{Mor}(X)$.

(ii) $\lambda \cdot c_1 \in \mathbf{C}(X)$ *where* $(\lambda \cdot c_1)(f) := \lambda \cdot c_1(f)$ *for all* $f \in \mathrm{Mor}(X)$.

Thus $\mathbf{C}(X)$ is a module over the semiring $\mathbb{R}_0^+$.

Proof: elementary calculations. ∎

Denoting by $\mathbf{M}$ the category of all $\mathbb{R}^+$-modules and linear mappings we get the following theorem:

3.2 Theorem.

$C: \mathbf{K}(O) \to \mathbf{M}$ *defined by*

$$\mathbf{C}(X) := \{c: \mathrm{Mor}(X) \to \mathbb{R}^+ : c \text{ is a complexity measure on } X\}$$

$$\text{for } X \in \mathrm{Ob}(K(O)),$$

$$\mathbf{C}(\Phi)(c) := \Phi^{-1}(c) \text{ for } \Phi: Y \to X \in \mathrm{Mor}(K(O)) \text{ and } c \in \mathbf{C}(X)$$

is a contravariant functor from the category of all X-categories to the category of all $\mathbb{R}_0^+$-modules.

PROOF: elementary calculations. ■

The following remark was made by one of our students (Thiet Dung Huynh) during a seminar lecture:

Let $\Phi: X \to Y \in \mathrm{Mor}(\mathbf{K}(O))$. If Φ is surjective then $\mathbf{C}(\Phi)$ is injective. If $c_1, c_2 \in \mathbf{C}(Y)$ such that $\mathbf{C}(\Phi)(c_1) = \mathbf{C}(\Phi)(c_2)$ then we have

$$\begin{aligned} & \Phi^{-1}(c_1)(f) = \Phi^{-1}(c_2)(f) && \text{for all } f \in \mathrm{Mor}(X) \\ \Rightarrow\ & c_1(\Phi(f)) = c_2(\Phi(f)) && \text{for all } f \in \mathrm{Mor}(X) \\ \Rightarrow\ & c_1(g) = c_2(g) && \text{for all } g \in \mathrm{Mor}(Y) \text{ since } \Phi \\ & && \text{is surjective} \\ \Rightarrow\ & c_1 = c_2. \end{aligned}$$

If now Y is an arbitrary (finitely generated) X-category then we have a surjective X-functor

$$\Phi: X \to Y$$

where X is a free (finitely generated) X-category.

Since $\mathbf{C}(\Phi)$ is injective we may consider $\mathbf{C}(Y)$ as a submodule of $\mathbf{C}(X)$. Therefore in studying all the complexity measures on all (finitely generated) X-categories in $\mathrm{Ob}(\mathbf{K}(O))$ it is enough to consider the complexity measures on (finitely generated) free X-categories. But on the other hand, for the same reason, it is harder to classify the complexity measures on a free X-category than on a special X-category. For example, D-complexity measures on a D-category Y have properties which are not satisfied by all complexity measures on the free X-category X generating Y, even if they have zero-values on the switching elements interpreted as crossings, truncations, and diagonalizations.

3.3 THEOREM.

Let $|\ldots|$ *be a D-complexity measure on a D-category* X *with* $\mathrm{Ob}(X) = \mathbb{N}_0$. *Then the following holds:*

(i) $|f\times g| = |g\times f|$ *for all* $f\colon u \to u',\ g\colon v \to v' \in \mathrm{Mor}(X)$.

(ii) $|f| \le |f\times g|$ *for all* $f\colon u \to u',\ g\colon v \to v' \in \mathrm{Mor}(X),\ u \neq 0$.

(iii) $|1_u\times f\times 1_v| = |f|$ *for all* $f\colon w \to w' \in \mathrm{Mor}(X),\ w \neq 0$.

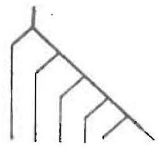
PROOF.

(i) $|f\times g| = |c_{v',u'}\circ(g\times f)\circ c_{u,v}| \le |g\times f|$ for $|c_{u,v}| = |c_{v',u'}|$ $= 0$, and vice versa.

(ii) Let $h_v\colon 1 \to v$ be defined inductively as follows ($v \in \mathbb{N}$):

$$h_1 := 1_1,$$

$$h_{v+1} := (1_1\times h_v)\circ d_1.$$



Then $(1_1\times t_v)\circ h_{1+v} = 1_1$ and therefore

$$f = (1_{u'}\times t_{v'})\circ(f\times g)\circ(t_{u-1}\times h_{1+v}), \qquad (u-1 \in \mathbb{N}_0 \text{ since } u \neq 0$$

$|f| \le |f\times g|$ by (C2).

(iii) $|f| \le |1_u\times f\times 1_v| \le |f|$ by (ii) and (C3). ∎

3.3 (iii) gives an answer to a question in [5], p. 408 f.

3.4 THEOREM AND NOTATION.

(i) *Let* $X \in \mathrm{Ob}(\mathbf{K}(0))$ *and* $\approx$ *the equivalence relation on* $\mathbf{C}(X)$ *defined in 2.7. Then* $\approx$ *is a congruence relation on* $\mathbf{C}(X)$ *considered as an* $\mathbb{R}_0^+$*-module. We define* $\widetilde{\mathbf{C}}(X) := \mathbf{C}(X)/\!\approx$.

(ii) *Let* $\Phi\colon Y \to X \in \mathrm{Mor}(\mathbf{K}(0))$ *then there exists a canonical module homomorphism* $\widetilde{\mathbf{C}}(\Phi)\colon \widetilde{\mathbf{C}}(X) \to \widetilde{\mathbf{C}}(Y)$ *such that the following diagram is commutative, and* $\widetilde{\mathbf{C}}\colon \mathbf{K}(0) \to \mathbf{M}$ *becomes a functor.*

$$\begin{tikzcd}
\mathbf{C}(X) \arrow[r, "\mathbf{C}(\Phi)"] \arrow[d] \arrow[dr, phantom, "/\!/\!/"] & \mathbf{C}(Y) \arrow[d] \\
\widetilde{\mathbf{C}}(X) \arrow[r, "\widetilde{\mathbf{C}}(\Phi)"'] & \widetilde{\mathbf{C}}(Y)
\end{tikzcd}$$

PROOF.

(i) Let $c_1, c_2 \in \mathbf{C}(X)$ with $c_1 \preccurlyeq c_2$. Then there is a $\lambda \in \mathbb{R}^+$ such that $c_1(f) \leq \lambda \cdot c\ (f)$ for all $f \in \mathrm{Mor}(X)$. If $c \in \mathbf{C}(X)$ is another complexity measure we have for all $f \in \mathrm{Mor}(X)$:

$$(c_1 + c)(f) \leq \lambda \cdot c_2(f) + c(f) \leq \lambda' \cdot (c_2(f) + c(f))$$
$$= \lambda' \cdot (c_2 + c)(f)$$

where $\lambda' := \mathrm{Max}\,\{\lambda, 1\}$. Therefore $c_1 + c \preccurlyeq c_2 + c$. If $\mu \in \mathbb{R}_0^+$ then clearly $\mu \cdot c_1 \preccurlyeq \mu \cdot c_2$. Thus it follows that $\approx$ is a congruence relation on $\mathbf{C}(X)$.

(ii) We only must show that $c_1 \approx c_2$ implies $\Phi^{-1}(c_1) \approx \Phi^{-1}(c_2)$. Let $c_1, c_2 \in \mathbf{C}(X)$ with $c_1 \preccurlyeq c_2$ that means there exists a $\lambda \in \mathbb{R}^+$ such that $c_1(f) \leq \lambda \cdot c_2(f)$ for all $f \in \mathrm{Mor}(X)$. It follows

$$\Phi^{-1}(c_1)(g) = c_1(\Phi(g)) \leq \lambda \cdot c_2(\Phi(g)) = \lambda \cdot \Phi^{-1}(c_2)(g)$$

for all $g \in \mathrm{Mor}(X)$. Therefore $\Phi^{-1}(c_1) \preccurlyeq \Phi^{-1}(c_2)$ and analogously $\Phi^{-1}(c_2) \preccurlyeq \Phi^{-1}(c_1)$. ∎

Let $X \in \mathrm{Ob}(\mathbf{K}(\mathcal{O}))$ and $c \in \mathbf{C}(X)$ such that c is not bounded above by a constant. If $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the function $\varphi(x) = \log(1 + x)$ defined in 2.4 then the complexity measures c, $\varphi \circ c$, $\varphi \circ \varphi \circ c$, ... are pairwise inequivalent. Furthermore they are algebraically indepedent over $\mathbb{R}$. If we call the maximal number of algebraically indepedent elements in an $\mathbb{R}^+$-module the dimension of that module then $\mathbf{C}(X)$ has in general infinite dimension.

As noted in example 2.4 above we will now define strict complexity measures.

§4. Strict complexity measures.

4.1 DEFINITION.

Let X be an X-category and c a complexity measure on X. c is called a strict complexity measure iff it satisfies

(CS3) $\quad c(f \times g) = c(f) + c(g)$ *for all* $f, g \in \mathrm{Mor}(X)$.

Since it seems to be very difficult to get good lower bounds on the size complexity of Boolean functions we consider monotone Boolean functions. Let B^m denote the subcategory of B having only monotone Boolean functions as morphisms. Clearly B^m is a D-category. Let

$$I: G(\{\wedge, \vee\}) \to B^m$$

be the natural interpretation. If L is the following cost function then I(L) is a strict complexity measure (for a proof see [6] or [12], for another proof see 4.7).

$$L(\wedge) := L(\vee) := 1,$$
$$L(t_1) := L(d_1) := L(c_{1,1}) := 0.$$

From example 2.4 it is clear that there exists complexity measures which are not strict. The following example shows that even the size complexity must not be a strict complexity measure. Further this example shows that there are non strict complexity measures on B^m. Therefore monotonicity is not only a property of the X-category but also of the complexity measure considered; however we do not know whether "strict" is enough to characterize "monotone" complexity measures.

4.2 EXAMPLE.

Let R be a finite ring and consider P_R. Let $I: G(\{+, -, \cdot\}) \to P_R$ be the natural interpretation and $|\ldots| = I(L): \mathrm{Mor}(P_R) \to \mathbb{R}_0^+$ where L ist the following cost function on $G(\{+, -, \cdot\})$:

$$L(+) := L(-) := L(\cdot) := 1,$$
$$L(t_1) := L(d_1) := L(c_{1,1}) = 0.$$

For an $n \times n$ matrix $A \in R^{n \times n}$ define the morphism $f_A: n \to n \in \mathrm{Mor}(\mathcal{P}_R)$ by $f_A(v) = A \cdot v$ for all column vectors $v \in R^n$. If B is another $n \times n$ matrix over R then we have

$$A \cdot B = (\underbrace{f_A \times f_A \times \ldots \times f_A}_{n \text{ times}})(B)$$

where B is considered as a sequence of n column vectors from R^n. By Strassen's matrix multiplication we know that

$$|\underbrace{f_A \times f_A \times \ldots \times f_A}_{n \text{ times}}| = O(n^{\log 7}).$$

If we take $\mathcal{B}$ instead of $\mathcal{P}_R$ we know that

$$|\underbrace{f_A \times f_A \times \ldots \times f_A}_{n \text{ times}}| = O(n^{\log 7} \log n \log\log n \log\log\log n).$$

On the other hand $f_A = f_B \Leftrightarrow A = B$ for two matrices $A, B \in R^{n \times n}$, and therefore there are at least r^{n^2} different morphisms f_A with $A \in R^{n \times n}$ where $r = \mathrm{card}(R)$. Further, a simple counter argument shows that there are at most $(3(1 + n + r)^2)^l$ morphisms $f \in \mathrm{Mor}(\mathcal{P}_R)$ such that $|f| \leq l$. Now there must exist a matrix $A \in R^{n \times n}$ such that

$$(3(1 + n + r)^2)^l \geq r^{n^2} \text{ where } l = |f_A|.$$

If $r > 1$ it follows that $l = |f_A|$ grows asymptoticaly at least as fast as $\frac{n^2}{\log n}$ and therefore

$$n|f_A| > |\underbrace{f_A \times f_A \times \ldots \times f_A}_{n \text{ times}}| \quad \text{for } n \text{ large enough.}$$

Using $\mathcal{B}$ instead of $\mathcal{P}_R$ the same holds true. Thus since $f_A \in \mathrm{Mor}(\mathcal{B}^m)$ we see that the restriction of the size complexity on $\mathcal{B}$ is not a strict complexity measure on $\mathcal{B}^m$.

Another such example is the fast Fourier transformation.

In [12] Paul has shown that for every $\varepsilon > 0$ there are morphisms $f \in \mathrm{Mor}(\mathcal{B})$ such that

$$|f \times f| \leq (1 + \varepsilon) \cdot |f|.$$

For giving a precise meaning to the proposition "the size complexity measure on $\mathcal{B}$ is not strict" we define strict equivalence classes of complexity measures:

4.3 DEFINITION.

Let X be an X-category and $\tilde{c} \in \tilde{\mathbf{C}}(X)$. We call $\tilde{c}$ strict iff there is a strict complexity measure $c \in \tilde{c}$. We use the notation "the size complexity measure on X is strict" iff there is a strict complexity measure on X which is equivalent to a non degenerate size complexity measure on X having no zero values on elementary switching elements (without truncations, crossings, diagonalizations, and constant morphisms).

4.4 PROPOSITION.

The size complexity measure on P_R for a finite ring R is not strict.

PROOF.

Assume that there is a strict complexity measure c which is equivalent to the size complexity $|\ldots|$ with $|+| = |\cdot| = |-| = 1$. Then there are $\lambda, \lambda' \in \mathbb{R}^+$ such that $c \leq \lambda \cdot |\ldots|$ and $|\ldots| \leq \lambda' \cdot c$. Consider example 4.2 and let $n \in \mathbb{N}$ such that $|\underbrace{f_A \times \ldots \times f_A}_{n \text{ times}}| \leq \frac{1}{2} \frac{n}{\lambda\lambda'} |f_A|$ for a matrix $A \in R^{n \times n}$. Then we have the following contradiction:

$$\begin{aligned} c(\underbrace{f_A \times \ldots \times f_A}_{n \text{ times}}) &\leq \lambda |\underbrace{f_A \times \ldots \times f_A}_{n \text{ times}}| \\ &\leq \frac{1}{2} \frac{n}{\lambda'} |f_A| \\ &\leq \frac{1}{2} n\, c(f_A) = \frac{1}{2} c(\underbrace{f_A \times \ldots \times f_A}_{n \text{ times}}). \quad \blacksquare \end{aligned}$$

On the other hand there are a lot of examples where the size complexity is strict.

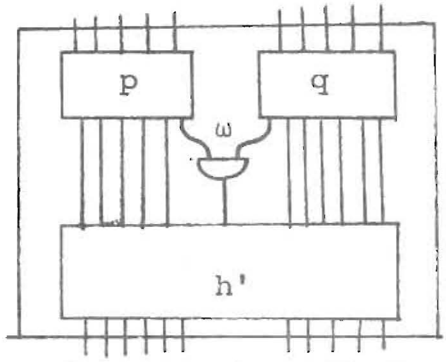
Let M be a set and Ω a set of unary and binary operations on M. In the following we will prove in some special cases that the size complexity measure on $P_{M,\Omega}$ is strict. The proofs will use the technic of considering a "first mixed switching element".

4.5 LEMMA.

Let M *and* Ω *be as above,* $A := M \cup \Omega$ *and* $I\colon F_D(A, \mathbb{N}_0) \to P_{M,\Omega}$ *the natural interpretation. Then every* $h \in \mathrm{Mor}(F_D(A, \mathbb{N}_0))$ *such that* $I(h) = f*g$ *with* $f\colon u \to u'$, $g\colon v \to v' \in \mathrm{Mor}(P_{M,\Omega})$ *may be written as*

(i) $h = p \times q$ *with* $I(p) = f$ *and* $I(q) = g$ *or as*

(ii) $h = h' \bullet (1_{w_1} \times \omega \times 1_{w_2}) \circ (p \times q)$ *with* $p\colon u \to w_1$, $q\colon v \to w_2$, $h\colon w_1 + w_2 - 1 \to u' + v' \in \mathrm{Mor}(F_D(A, \mathbb{N}_0))$ *and a binary operation* $\omega \in \Omega$, *such that* $I(t_{w_1-1} \times 1_1) \bullet p)$ *and* $I((1_1 \times t_{w_2-1}) \bullet q)$ *both are not constant functions.*



In the second case ω *is called a first mixed switching element in the circuit* h *respective to the partition* u+v *of the input wires of* h.

PROOF: Induction on the size of h using the relations S1,..., S3 and D0,...,D7 satisfied in a D-category.

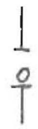
4.6 THEOREM.

Let R *be a free semiring. Then the size complexity measure on* P_R *is strict.*

Sketch of a PROOF.

Let $f\colon u \to u'$, $g\colon v \to v' \in \mathrm{Mor}(P_R)$ and let $h \in \mathrm{Mor}(F_D(R\cup\{+,\cdot\}, \mathbb{N}_0))$ a minimal switching circuit such that $I(h) = f \times g$ where $I\colon F_D(R\cup\{+,\cdot\}, \mathbb{N}_0) \to P_R$ is the natural interpretation as in Lemma 4.5 above. If $h = p \times q$ with $I(p) = f$ and $I(q) = g$ there is nothing to prove. Otherwise let ω be a first mixed switching element. Since h is minimal h' is minimal too. Further there must exist a path from ω to one of the output wires of h' (otherwise we may simplify the circuit h and eliminate ω in it using DO). Since R is free the only chance to lose the dependece of the output of ω along this path is multiplying by the constant 0. (This argument is wrong if R is a ring and "-" is a switching element!) But instead of

we may use



with lower cost. Therefore there is an output wire of h' such that $I((t_{w_1'} \times 1_1 \times t_{w_2'}) \circ h')$ depends on the output of ω (w_1', $w_2' \in \mathbb{N}_0$ such that $w_1' + 1 + w_2' = u' + v'$). This means $I((t_{w_1'} \times 1_1 \times t_{w_2'}) \circ h')$ is a polynomial $H' \in R[X_1', \ldots, X'_{w_1-1}, Z', Y'_{w_2-1}, \ldots, Y_1']$ in which there is at least one monomial containing Z' and having a nonzero coefficient. Let $I(p) = (P_1, \ldots, P_{w_1})$ with polynomials $P_i \in R[X_1, \ldots, X_u]$ and $I(q) = (Q_{w_2}, \ldots, Q_1)$ with polynomials $Q_i \in R[Y_1, \ldots, Y_v]$. Now we get the polynomial $H := I((t_{w_1'} \times 1_1 \times t_{w_2'}) \circ h) = (t_{w_1'} \times 1_1 \times t_{w_2'}) \circ (f \times g)$ by substituting P_i for X_i' $(1 \leq i \leq w_1-1)$, Q_i for Y_i' $(1 \leq i \leq w_2-1)$ and $P_{w_1} \, \omega \, Q_{w_2}$ for Z' in H'. Since P_{w_1} and Q_{w_2} both are not constant functions over R there are monomials with nonzero coefficients containing at least one X_i $(1 \leq i \leq u)$ and monomials containing at least one Y_i $(1 \leq i \leq v)$. This is a contradiction to $H = (t_{w_1'} \times 1_1 \times t_{w_2'}) \circ (f \times g)$ since the identity theorem for polynomials over a free semiring holds. ∎

4.7 THEOREM.

Let $(R, +, \cdot)$ *be an ordered semiring. Assume that* R *is positive (that means* $x \geq 0$ *for all* $x \in R$*) and that there exists a maximal element* $\infty \in R$ *(such that* $x \leq \infty$ *for all* $x \in R$*). Consider the natural interpretation* $I: F_D(\{+, \cdot, 0, \infty\}, \mathbb{N}_0) \to C_R$ *and denote its image by* P_R' *(note that here we do not allow arbitrary constants as switching elements). Then the size complexity on* P_R' *is strict.*

PROOF.

Let $f\colon u \to u'$, $g\colon v \to v' \in \mathrm{Mor}(P_R')$ and let $h \in \mathrm{Mor}(F_D(\{+,\cdot,0,\infty\}, \mathbb{N}_0))$ a minimal switching circuit such that $I(h) = f \times g$. Assume further that h has a minimal number of mixed switching elements. Let ω be a first mixed switching element in h and let p, q, and h' as in Lemma 4.5. Further let $I(p) = (P_1,\dots,P_{w_1})$ with polynomials $P_i \in R[X_1,\dots,X_u]$, $I(q) = (Q_{w_2},\dots,Q_1)$ with polynomials $Q_i \in R[Y_1,\dots,Y_v]$, and $I(h') = (H_1',\dots,H_{u'}',H_{u'+1}',\dots,H_{u'+v'}')$ with polynomials $H_i' \in R[X_1',\dots,X_{w_1-1}',Z',Y_{w_2-1}',\dots,Y_1']$. As a consequence of the assumption that only 0 and ∞ are allowed as constants, the polynomials P_{w_1} and Q_{w_2} (which are not constant by lemma 4.5) have no constant terms and therefore $P_{w_1}(0,\dots,0) = 0$, $P_{w_1}(\infty,\dots,\infty) = \infty$ and Q_{w_2} similar.

Let $\omega = \cdot$ and consider $H_i'(P_1,\dots,P_{w_1-1},P_{w_1}\cdot Q_{w_2},Q_{w_2-1},\dots,Q_1) = H_i \in R[X_1,\dots,X_u,Y_1,\dots,Y_v]$. If $1 \le i \le u'$ then $H_i = H_i\big|_{Y_1=0,\dots,Y_v=0}$ and we have

$$\begin{aligned} &H_i'(P_1,\dots,P_{w_1-1},P_{w_1}\cdot Q_{w_2},Q_{w_2},\dots,Q_1)\\ &= H_i'(P_1,\dots,P_{w_1-1},0,0,\dots,0)\\ &\le H_i'(P_1,\dots,P_{w_1-1},0,Q_{w_2-1},\dots,Q_1)\\ &\le H_i'(P_1,\dots,P_{w_1-1},P_{w_1}\cdot Q_{w_2},Q_{w_2-1},\dots,Q_1). \end{aligned}$$

These inequalities hold since R is an ordered semiring and therefore all polynomial functions $R^n \to R$ are monotone. Now it follows

$$\begin{aligned} &H_i'(P_1,\dots,P_{w_1-1},P_{w_1}\cdot Q_{w_2},Q_{w_2-1},\dots,Q_1)\\ &= H_i'(P_1,\dots,P_{w_1-1},0,Q_{w_2-1},\dots,Q_1). \end{aligned}$$

If $u' \le i \le u'+v'$ we get the same equation. This proves that the first mixed switching element ω may be replaced by the constant 0. This is a contradiction to the assumption that h has a minimal number of mixed switching elements.

If $\omega = +$ we have $H_i = H_i|_{Y_1=\infty,\ldots,Y_v=\infty}$ for $1 \leq i \leq u'$ and therefore

$$H_i'(P_1,\ldots,P_{w_1-1},P_{w_1}+Q_{w_2},Q_{w_2-1},\ldots,Q_1)$$
$$= H_i'(P_1,\ldots,P_{w_1-1},\infty,\infty,\ldots,\infty)$$
$$\geq H_i'(P_1,\ldots,P_{w_1-1},\infty,Q_{w_2-1},\ldots,Q_1)$$
$$\geq H_i'(P_1,\ldots,P_{w_1-1},P_{w_1}+Q_{w_2},Q_{w_2-1},\ldots,Q_1).$$

This leads to the equation

$$H_i'(P_1,\ldots,P_{w_1-1},P_{w_1}+Q_{w_2},Q_{w_1-1},\ldots,Q_1)$$
$$= H_i'(P_1,\ldots,P_{w_1-1},\infty,Q_{w_2-1},\ldots,Q_1)$$

which holds for all i, $1 \leq i \leq u'+v'$. We may replace ω by the constant ∞ and receive a contradiction too. ■

4.8 Korollar.

The size complexity on B^m is strict.

Proof: B^m satisfies the suppositions of theorem 4.7.

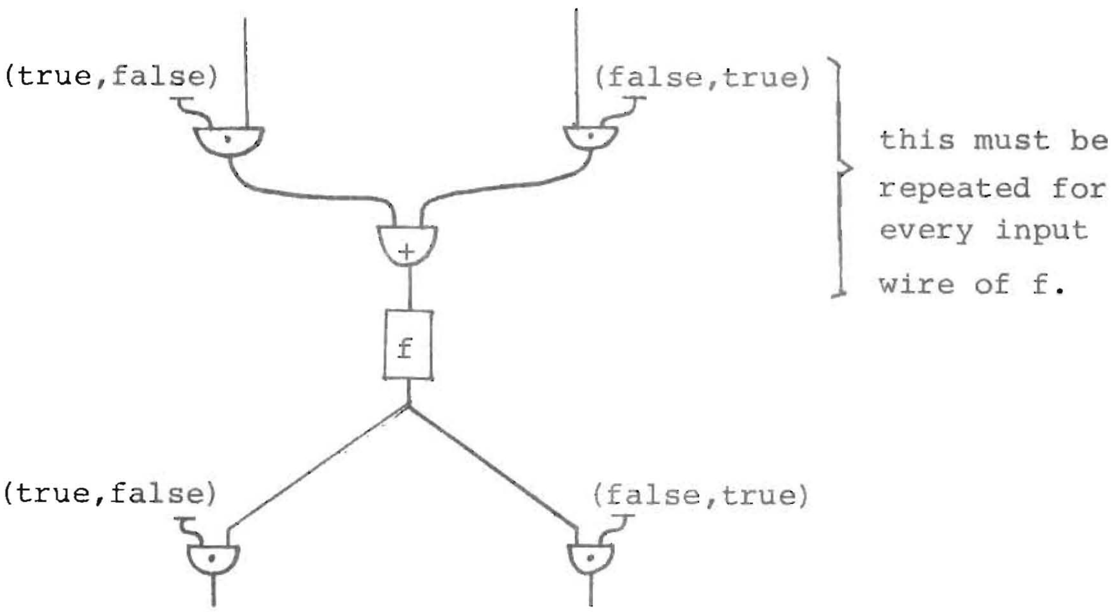
The following example shows, that allowing arbitrary constants as switching elements, the theorem 4.7 becomes wrong.

4.9 Example.

Let R = {true, false} × {true, false} the cartesian product. R becomes a semiring if we define $(x, y) + (x', y') := (x \vee x', y \vee y')$ and $(x, y)\cdot(x', y') := (x \wedge x', y \wedge y')$. Further the lexicographical order (based on "false < true") makes R to a positive ordered semiring with maximal element ∞ := (true, true).

Now let $f'\colon\{\text{true, false}\}^n \to \{\text{true, false}\} \in \mathrm{Mor}(\mathcal{B}^m)$ be a Boolean function. Clearly f' may be extended to a function $f\colon R^n \to R$ by $f((x,y)) := (f'(x), f'(y))$. Let $|\ldots|$ denote the size complexity on $\mathcal{B}^m$ and $\|\ldots\|$ the size complexity on $\mathcal{P}_R$. Then we have $|f'| = \|f\|$ (a switching circuit for f is also a switching circuit for f' and vice versa). Let f_1, f_2 $\in \mathrm{Mor}(\mathcal{P}_R)$ be the functions represented by the following two switching circuits:

Then $\|f_1\| \geq |f'| = \|f\|$ and $\|f_2\| \geq |f'| = \|f\|$. On the other hand $f_1 \times f_2$ may be computed using the following switching circuit:



Therefore $\|f_1 \times f_2\| \leq \|f\| + O(n)$. If we now take f complex enough we have shown that $\|\ldots\|$ is not a strict complexity measure.

4.10 Open Problems.

1) *Give a characterization of all strict complexity measures on a fixed X-category.*

2) *Find a definition of "monotone X-categories" as a generalization of* $\mathbb{B}^m$.

3) *Find a definition of "monotone complexity measures" on a (monotone) X-category.*

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