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Abstract

We consider variational problems of splitting-type, i.e. the density $F : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ has an additive decomposition into two functions f and g. Assuming power growth conditions with exponents p and q for these functions, Bildhauer and Fuchs [BF2,3] show partial regularity in the general vector case and full regularity for $n = 2$ in the superquadratic situation. If the functions f and g depend on the modulus, i.e. $f(\cdot) = a(|\cdot|)$ and $g(\cdot) = b(|\cdot|)$, we generalize the statements for splitting-type variational integrals with power growth conditions to the case of N-functions a and b.

1 Introduction

In this paper we discuss regularity results for minimizers $u : \Omega \to \mathbb{R}^N$ of variational integrals

$$I[u, \Omega] := \int_\Omega F(\nabla u)\, dx, \tag{1.1}$$

where Ω denotes an open set in $\mathbb{R}^n$ and where $F : \mathbb{R}^{nN} \to [0, \infty)$ satisfies an anisotropic growth condition, i.e.

$$C_1|Z|^p - c_1 \leq F(Z) \leq C_2|Z|^q + c_2, \qquad Z \in \mathbb{R}^{nN}$$

with constants $C_1, C_2 > 0$, $c_1, c_2 \geq 0$ and exponents $1 < p \leq q < \infty$. The study of such problems was pushed by Marcellini (see [Ma1] and [Ma2]) and today it is a well known fact that there is no hope for regularity of minimizers if p and q are too far apart (compare [Gi] and [Ho] for counter examples). Under mild smoothness conditions on F (the case of (p, q)-elliptic integrands) the best known statement is the bound

$$q < p + 2$$

for regularity proved by Bildhauer and Fuchs [BF1]. To get better results additional assumptions are necessary. Therefore we consider decomposable integrands, i.e. we have

$$F(Z) = f(\widetilde{Z}) + g(Z_n)$$

for $Z = (Z_1, ..., Z_n)$ with $Z_i \in \mathbb{R}^N$ and $\widetilde{Z} = (Z_1, ..., Z_{n-1})$. Bildhauer and Fuchs assume power growth conditions for the C^2-functions f and g:

$$\begin{aligned} \lambda(1 + |\widetilde{Z}|^2)^{\frac{p-2}{2}}|\widetilde{X}|^2 \leq\ & D^2 f(\widetilde{Z})(\widetilde{X}, \widetilde{X}) && \leq \Lambda(1 + |\widetilde{Z}|^2)^{\frac{p-2}{2}}|\widetilde{X}|^2, \\ \lambda\left(1 + |Z_n|^2\right)^{\frac{q-2}{2}}|X_n|^2 \leq\ & D^2 g(Z_n)(X_n, X_n) && \leq \Lambda\left(1 + |Z_n|^2\right)^{\frac{q-2}{2}}|X_n|^2 \end{aligned} \tag{1.2}$$

with $2 \leq p \leq q$ and for $N > 1$ the structure condition

$$f(Z_1) = \widehat{f}(|Z_1|, ..., |Z_1||) \text{ and } g(Z_n) = \widehat{g}(|Z_n|),$$

with to functions $\widehat{f}$ and $\widehat{g}$ which are strictly monotonic. This is for using the maximum principle of [DLM]. In [BF2] and [BF3], the following results are proved:

- full $C^{1,\alpha}$-regularity for $n = 2$ and
- partial $C^{1,\alpha}$-regularity in general vector case, if

$$q \leq p + 2 \quad \text{and} \quad q \leq \frac{pn}{n-2}. \tag{1.3}$$

Now we suppose modulus dependence for f and g, i.e.

$$f(\widetilde{Z}) = a(|\widetilde{Z}|) \text{ and } g(Z_n) = b(|Z_n|),$$

with N-functions a and b (see [Ad]). The aim of this paper is to improve the results for splitting-type variational integrals with power growth conditions to the case of N-functions a and b. In [BF4] the authors show higher integrability theorems in this topic which are the basis for the regularity theory. In the following lines we state the conditions for a and b which are necessary for our approach. These are natural generalizations of the power growth situation.
We have for $h = a$ or $h = b$

$$\begin{gathered} h \text{ is strictly monotonic and convex with} \\ \lim_{t \to 0} \frac{h(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{h(t)}{t} = \infty. \end{gathered} \tag{A1}$$

Furthermore we assume the existence of positive numbers $\widehat{\epsilon}$ and $\widehat{h}$ such that we have for all $t \geq 0$

$$\widehat{\epsilon} \frac{h'(t)}{t} \leq h''(t) \leq \widehat{h} \frac{h'(t)}{t}. \tag{A2}$$

Since we need superquadratic growth we suppose

$$\frac{h'(t)}{t} \geq h_0 > 0 \text{ for all } t \geq 0 \tag{A3}$$

A discussion of property (A2), as well as examples for functions which satisfy the conditions above, can be found in [BF4]. Obviously we can demand, weaker than $a \leq b$ from [BF4],

$$a(t) \leq cb(t) \text{ for large } t \tag{A4}$$

for a $c > 0$ with the same results. Hereafter we analyze minimizers of

$$\mathcal{J}[w] := \int_\Omega \Big[a(|\widetilde{\nabla} w|) + b(|\partial_n w|) \Big] \, dx \tag{1.4}$$

under the assumptions (A1)-(A4).

Remark 1.1 • *By the postulated convexity and strictly monotonicity of h, we have $h(t) > 0$ and $h'(t) > 0$ for all $t > 0$, since $h(0) = h'(0) = 0$.*

- *By [BF4] we can infer from (A1) and the r.h.s. of (A2) the inequality*

$$h(2t) \le \mu h(t) \tag{1.5}$$

for all $t \ge 0$ with $\mu = 2^{\widehat{h}+1}$. Furthermore with $q := 1 + \widehat{h}$ one sees for all $t \ge 1$ (using (A2))

$$h(t) \le ct^q, \quad h'(t) \le ct^{q-1} \quad \text{and} \quad h''(t) \le ct^{q-2} \tag{1.6}$$

with a constant $c > 0$.

- *From monotonicity and Δ_2-conditions, we obtain*

$$h'(t)t \le \mu h(t) \text{ for all } t \ge 0. \tag{1.7}$$

- *For $H(Z) := h(|Z|)$, $Z \in \mathbb{R}^k$, we get*

$$\lambda \frac{h'(|Z|)}{|Z|} |Y|^2 \le D^2 H(Z)(Y,Y) \le \Lambda \frac{h'(|Z|)}{|Z|} |Y|^2 \tag{1.8}$$

for all $Y, Z \in \mathbb{R}^k$ with constants $\lambda, \Lambda > 0$. Thereby we can follow by (A3) and (1.6) the growth condition

$$c_1 |Y|^2 \le D^2 F(Z)(Y,Y) \le c_2 \left(1 + |Z|^2\right)^{\frac{q-2}{2}} |Y|^2 \tag{1.9}$$

for all $Y, Z \in \mathbb{R}^{nN}$ with constants $c_i > 0$.

In general vector case we need the assumptions (c and ϑ positive constants)

$$\begin{aligned} b(t) &\le ct^\omega a(t) \quad \text{for an } \omega \in (0,2], \\ b(t) &\le ct^{\overline{\omega}} a(t^{\overline{\omega}}) \text{ for an } \overline{\omega} < 2, \text{ if } \omega = 2, \end{aligned} \tag{A5}$$

$$\frac{h'(t)}{t} \le h''(t) \qquad \text{for } t \ge 0, \text{ if } \omega < 1, \tag{A6}$$

$$a(t) \ge \vartheta t^{\frac{\omega}{2}(n-2)} \text{ for large } t. \tag{A7}$$

Definition 1.2 *We call a function* $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ *a local minimizer of (1.1), if we have for all* $\Omega' \Subset \Omega$

- $\int_{\Omega'} F(\nabla u)\, dx < \infty$ *and*
- $\int_{\Omega'} F(\nabla u)\, dx \leq \int_{\Omega'} F(\nabla v)\, dx$ *for all* $v \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$, $\mathrm{spt}(u - v) \Subset \Omega$.

Now we state our new result:

THEOREM 1.1 *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ *be a domain and assume (A1)-(A3).*

(a) Suppose (A4)-(A7) and let $u \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$ *be a local minimizer of (1.4). Then there is an open subset* $\Omega_0 \subset \Omega$ *with* $\mathcal{L}^n(\Omega - \Omega_0) = 0$ *and* $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ *for all* $\alpha < 1$.

(b) Let $n = 2$. *A local minimizer* $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ *of (1.4) belongs to the space* $C^{1,\alpha}(\Omega, \mathbb{R}^N)$ *for all* $\alpha < 1$.

(c) Let $N = 1$, *assume (A5), part 1, as well as*

$$a(t) \leq c t^2 b(t) \text{ for big } t. \tag{A2.12}$$

Then a local minimizer $u \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega)$ *of (1.4) belongs to the space* $C^{1,\alpha}(\Omega)$ *for all* $\alpha < 1$.

Remark 1.3

- *A comparison with [BF3] shows, that Theorem 1.1 a) is the generalization of the results concerning "splitting-type" variational integrals with power growth conditions to the case of* N*-functions. The first condition from (A5) corresponds with* $q \leq p + 2$ *which is assumed in [BF3] (Thm. 1.2) under (H1). Furthermore the* ω *from (A5), part 1, is* $q - p$, *wherefore we get in case* $a(t) = t^p$ *and* $b(t) = t^q$ *the equivalence of (A7) and* $q \leq pn/(n-2)$. *The last inequality is desired in (H2) from [BF3]. In the proof of the Caccioppoli-type inequality which is necessary for their blow up arguments, Bildhauer and Fuchs need* $q < 2p+2$. *This is exactly* $b(t) \leq c t^{\overline{\omega}} a(t^{\overline{\omega}})$ *for an* $\overline{\omega} < 2$ *in the sense of* N*-functions.*
- *In our proof we need the monotonicity of* $a(t)t^{\omega-2}$. *Furthermore we have to suppose the inequality* $a'(t)t^2 \leq c a(t) t^\omega$. *The second one is only guaranteed for* $\omega \geq 1$. *Therefore in case* $\omega < 1$ *we must have (A6). This is equivalent to the monotonicity of* $h'(t)/t$ *and allows to overcome the mentioned difficulties above. In the power growth situation with exponent* $p \geq 2$ *this is surely satisfied.*

- *Since we have superquadratic growth of the function a, in (A7) the condition*

$$\frac{\omega}{2}(n-2) \le 2$$

should be satisfied. For $n \in \{3,4\}$ we can always choose $\omega = 2$. The case $\omega < 1$ (and thereby the necessary monotonicity of $a'(t)/t$ and $b'(t)/t$) is only interesting for $n \ge 7$.

Remark 1.4 • *In [BF5] Bildhauer and Fuchs consider in 2D the same problem under the assumption*

$$\frac{h'(t)}{t} \le h''(t) \le \widehat{h}\left(1+t^2\right)^{\frac{\omega}{2}} \frac{h'(t)}{t} \tag{1.10}$$

and get regularity for $\omega < 2$. Obviously the second inequality is much weaker than in (A2). However the first inequality from (1.10) is for $\widehat{\epsilon} < 1$ clearly more restrictive than the first inequality from (A2). This shows the example

$$h(t) := \int_0^t \left[s \ln\left(1 + \frac{1}{1+s}\right) + s \right] ds$$

which satisfies (A1)-(A3) but not the l.h.s. of (1.10) for an $\widehat{\epsilon} < 1$.

Remark 1.5 • *Bildhauer and Fuchs use (A5), part 1, in scalar case and in general vector case the weaker condition $b(t) \le t^2 a(t^2)$ to show higher integrability of the solution, which is the basis of our proof. For $N = 1$ (A5) is a strong restriction if we compare this with the power growth situation (compare [BFZ], Thm. 1.1, there is no assumption between p and q if $p \ge 2$). However we have no condition between our growth exponents (see (1.9)), such that we can get an arbitrary range of anisotropy. This is the first statement in general vector case without assuming modulus dependence of the density F. Fuchs [Fu1], [Fu2] considers in this context densities with modulus dependence and conditions for the density, similar to those we have stated for f and g. In the papers of Fuchs it is possible to choose $q \ge 2$ arbitrary, too.*

- *Since we have the structure $F(Q) = a(|\widetilde{Q}|) + b(|Q_n|)$ for $Q \in \mathbb{R}^{nN}$ with monotonic functions a and b, minimizers of (1.1) satisfy a maximum principle (see [DLM]). Therefore the assumption $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ is no restriction, if we minimize w.r.t. bounded boundary data. For $n = 2$ it is possible to remove this assumption.*

2 Caccioppoli-type inequality

For a fixed $\widetilde{q} > q$ let

$$\delta := \delta(\epsilon) := \frac{1}{1 + \epsilon^{-1} + \|(\nabla u)_\epsilon\|^{2\widetilde{q}}_{L^{\widetilde{q}}(B)}}$$
$$\text{and } F_\delta(Z) := \delta\left(1 + |Z|^2\right)^{\frac{\widetilde{q}}{2}} + F(Z)$$

for $Z \in \mathbb{R}^{nN}$. For $B := B_{R_0}(x_0) \Subset \Omega$ we define u_δ as the unique minimizer of

$$I_\delta\left[w, B\right] := \int_B F_\delta(\nabla w)dx \tag{2.1}$$

in $(u)_\epsilon + W_0^{1,\widetilde{q}}(B, \mathbb{R}^N)$, where $(u)_\epsilon$ is the mollification of u. For u_δ we quote the following result (see [BF2], Lemma 1, for further references):

Lemma 2.1 • *We have as* $\epsilon \to 0$*:* $u_\delta \rightharpoonup u$ *in* $W^{1,p}(B, \mathbb{R}^N)$,

$$\delta \int_B \left(1 + |\nabla u_\delta|^2\right)^{\frac{\widetilde{q}}{2}} dx \to 0 \quad \textit{and} \quad \int_B F(\nabla u_\delta)dx \to \int_B F(\nabla u)dx.$$

- $\sup_{\delta > 0} \|u_\delta\|_{L^\infty(B)} < \infty$.
- $\nabla u_\delta \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$.

We need the following Caccioppoli-type inequality, which is standard to prove:

Lemma 2.2 *For* $\eta \in C_0^\infty(B)$*, arbitrary* $\gamma \in \{1, ..., n\}$ *and* $Q \in \mathbb{R}^{nN}$ *we have*

$$\int_B \eta^2 D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)dx$$
$$\leq c \int_B D^2 F_\delta(\nabla u_\delta)([\partial_\gamma u_\delta - Q_\gamma] \otimes \nabla \eta, [\partial_\gamma u_\delta - Q_\gamma] \otimes \nabla \eta)dx$$

for a constant $c > 0$ *independent of* δ.

Now we assume (A1)-(A4) and

$$b(t) \leq ct^2 a(t^2) \text{ for large } t \tag{2.2}$$

From the proof of [BF4], Thm. 2.1, we quote

Lemma 2.3 *Under the assumptions (A1)-(A4) and (2.2) we have*

- *The functions* $a(|\widetilde{\nabla} u|)|\widetilde{\nabla} u|^2$ *and* $b(|\partial_n u|)|\partial_n u|^2$ *belong to the space* $L^1_{loc}(\Omega)$*;*
- u *belongs to the space* $W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$*.*

Additionally to the assumptions of Lemma 2.3 we have to ask for

$$b(t) \le ct^{\overline{\omega}} a(t^{\overline{\omega}}) \text{ for big } t \text{ and an } \overline{\omega} \in [1,2) \tag{2.3}$$

to prove the crucial Caccioppoli-type inequality.

Lemma 2.4 *Caccioppoli-type inequality: Under the assumptions (A1)-(A4) and (2.3) the functions*

$$\widetilde{\psi} := \int_0^{|\widetilde{\nabla} u|} \sqrt{\frac{a'(t)}{t}}\, dt, \quad \psi^{(n)} := \int_0^{|\partial_n u|} \sqrt{\frac{b'(t)}{t}}\, dt$$

satisfy for all $\eta \in C_0^\infty(B)$ *and* $P \in \mathbb{R}^{nN}$ *the inequality (sum over* γ*)*

$$\begin{aligned} &\int_B \eta^2 \left[|\nabla \widetilde{\psi}|^2 + |\nabla \psi^{(n)}|^2 \right] dx \\ \le c &\int_B D^2 F(\nabla u)([\partial_\gamma u - P_\gamma] \otimes \nabla \eta, [\partial_\gamma u - P_\gamma] \otimes \nabla \eta)\, dx. \end{aligned}$$

Proof: In an analogues way we define $\widetilde{\psi}_\delta$ and $\psi_\delta^{(n)}$ and get by Lemma 2.2 and (1.8)

$$\begin{aligned} &\int_B \eta^2 \left[|\nabla \widetilde{\psi}_\delta|^2 + |\nabla \psi_\delta^{(n)}|^2 \right] dx \\ \le c &\int_B D^2 F_\delta(\nabla u_\delta)([\partial_\gamma u_\delta - P_\gamma] \otimes \nabla \eta, [\partial_\gamma u_\delta - P_\gamma] \otimes \nabla \eta)\, dx. \end{aligned}$$

Since the r.h.s. is uniform bounded (compare [BF4], section 3), we can deduce the uniform boundedness of $\nabla \widetilde{\psi}_\delta$ and $\nabla \psi_\delta^{(n)}$ in $L^2_{loc}(B, \mathbb{R}^n)$. We obtain for a $\theta \in [0,1]$

$$|\widetilde{\psi}_\delta| \le |\widetilde{\nabla} u| \sqrt{\frac{a'(\theta |\widetilde{\nabla} u_\delta|)}{\theta |\widetilde{\nabla} u_\delta|}}.$$

If we distinguish the cases $\theta \le 1/|\widetilde{\nabla} u|$ and $\theta \ge 1/|\widetilde{\nabla} u|$, we can receive in the first situation (note $a(t) \ge ct^2$ for all $t \ge 0$ by (A3))

$$|\widetilde{\psi}_\delta| \le \sup_{t \in [0,1]} \sqrt{\frac{a'(t)}{t}} |\widetilde{\nabla} u_\delta| \le c |\widetilde{\nabla} u_\delta| \le c \sqrt{a(|\widetilde{\nabla} u_\delta|)}.$$

By (1.7) and the Δ_2-condition of a' we see if $\theta \geq 1/|\widetilde{\nabla} u|$,

$$|\widetilde{\psi}_\delta| \leq c\sqrt{|\widetilde{\nabla} u_\delta| a(|\widetilde{\nabla} u_\delta|)}.$$

After a similar argumentation for $\psi_\delta^{(n)}$ we can infer from Lemma 2.3 the uniform boundedness of $\widetilde{\psi}_\delta$ and $\psi_\delta^{(n)}$ in $L^2_{loc}(B)$ and thereby in $W^{1,2}_{loc}(B)$. Using lemma 2.3 again we get

$$\widetilde{\psi}_\delta \to \widetilde{\psi} \text{ and } \psi_\delta^{(n)} \to \psi^{(n)} \text{ in } W^{1,2}_{loc}(B)$$

From lower semicontinuity we deduce

$$\begin{aligned} \int_B \eta^2 \left[|\nabla \widetilde{\psi}|^2 + |\nabla \psi^{(n)}|^2\right] dx &\leq \liminf_{\delta\to 0} \int_B \eta^2 \left[|\nabla \widetilde{\psi}_\delta|^2 + |\nabla \psi_\delta^{(n)}|^2\right] dx \\ \leq c \liminf_{\delta \to 0} \int_B D^2 F_\delta(\nabla u_\delta)([\partial_\gamma u_\delta - P_\gamma] &\otimes \nabla \eta, [\partial_\gamma u_\delta - P_\gamma] \otimes \nabla\eta)\, dx. \quad (2.4) \end{aligned}$$

Now we have to show, that we can change limes and integral in the r.h.s. of (2.4).
By Egorov's theorem we can choose a subset S of B with $\mathcal{L}^n(B - S) \leq \kappa$ and $\nabla u_\delta \to \nabla u$ uniformly on S for an arbitrary $\kappa > 0$. For the integral over S we have the desired convergence. Now we show, that the residual "rest" becomes arbitrary small (therefore we need (2.3)). Consider the difference of

$$\begin{aligned} \mathcal{I}_\delta(B) &:= \int_B D^2 F_\delta(\nabla u_\delta)([\partial_\gamma u_\delta - P_\gamma] \otimes \nabla \eta, [\partial_\gamma u_\delta - P_\gamma] \otimes \nabla\eta)\, dx \\ \text{and } \mathcal{I}(B) &:= \int_B D^2 F(\nabla u)([\partial_\gamma u - P_\gamma] \otimes \nabla \eta, [\partial_\gamma u - P_\gamma] \otimes \nabla\eta)\, dx. \end{aligned}$$

It is enough to show

$$\limsup_{\delta\to 0} \mathcal{I}_\delta(B - S) + \mathcal{I}(B - S) \leq c\kappa^\mu \quad (2.5)$$

for an $\mu > 0$ to ends up the proof. We get by (1.8) (the following integrals are taken over $(B - S) \cap \operatorname{spt}(\eta)$)

$$\begin{aligned} \mathcal{I}_\delta(B - S) \leq c(\eta) \Bigg\{ &\int_{B-S} \frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|} |\nabla u_\delta - P|^2\, dx + \int_{B-S} \frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|} |\nabla u_\delta - P|^2\, dx \\ &+ \delta \int_{B-S} \Gamma_\delta^{\frac{\tilde{q}-2}{2}} |\nabla u_\delta - P|^2\, dx \Bigg\}. \end{aligned}$$

The last integral vanishes for $\delta \to 0$ by Lemma 2.1. For the first integral J_1^δ we obtain (note (1.7))

$$J_1^\delta \le c \int_{B-S} \frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|}\, dx + c\int_{B-S} a(|\widetilde{\nabla} u_\delta|)\, dx + c\int_{B-S} \frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|} |\partial_n u_\delta|^2\, dx$$
$$:= c\left[J_{11}^\delta + J_{12}^\delta + J_{13}^\delta\right].$$

Thereby (1.7) delivers

$$J_{11}^\delta = \int_{B-S\cap[|\widetilde{\nabla} u_\delta|\le 1]} ...dx + \int_{B-S\cap[|\widetilde{\nabla} u_\delta|>1]} ...dx \le c\kappa^3 + cJ_{12}^\delta.$$

If we choose $\alpha := 2/q + 1$ (see (1.6 for the definition of q), we see

$$a(t)^\alpha \le ct^2 a(t) \text{ for } t \ge 1,$$

and so for a suitable $\mu > 0$ by Lemma 2.3, part 1,

$$J_{12}^\delta \le c\kappa^3 + c||a(|\widetilde{\nabla} u_\delta|)||_\alpha ||1||_{\alpha'} \le c\kappa^\mu. \tag{2.6}$$

For consideration of J_{13}^δ we define for an $\omega^* \in (1,2)$

$$\mathcal{N}_b(t) := b(\sqrt{t})t^{\frac{\omega^*}{2}}.$$

Using $b(t) = \int_0^t b'(s)\, ds \le tb'(t)$ which follows from monotonicity of b' we see that $\mathcal{N}_b$ is a N-function with Δ_2-condition. By Young's inequality for N-functions we obtain

$$J_{13}^\delta \le \int_{B-S} \mathcal{N}_b(|\partial_n u_\delta|^2)\, dx + \int_{B-S} \mathcal{N}_b^* \left(\frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|}\right) dx.$$

The first integral is equal to

$$\int_{B-S} b(|\partial_n u_\delta|)|\partial_n u_\delta|^{\omega^*}\, dx.$$

If we choose $\beta^* := (q+2)/(q+\omega^*) > 1$, we get

$$\left(b(t)t^{\omega^*}\right)^{\beta^*} \le ct^2 b(t) \text{ for } t \ge 1.$$

As in (2.6) we obtain (decrease μ if necessary)

$$\int_{B-S} \mathcal{N}_b(|\partial_n u_\delta|^2)\, dx \le c\kappa^3 + c||b(|\partial_n u_\delta|)|\partial_n u_\delta|^{\omega^*}||_{\beta^*}||1||_{\beta^{*\prime}} \le c\kappa^\mu. \tag{2.7}$$

For the remaining integral in the decomposition of J_{13}^δ we need the estimation ($\widehat{b}(s) := b(\sqrt{s})s^{\frac{\omega^*-2}{2}}$)

$$\mathcal{N}_b^*(t) = \sup_{s\geq 0}[st - \mathcal{N}_b(s)] = \sup_{s\leq \widehat{b}^{-1}(t)} [t - b(\sqrt{s})s^{\frac{\omega^*-2}{2}}]s \leq \widehat{b}^{-1}(t)t.$$

It follows

$$\overline{J}_{13}^\delta := \int_{B-S} \mathcal{N}_b^* \left(\frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|} \right) dx \leq \int_{B-S} \widehat{b}^{-1} \left(\frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|} \right) \frac{a'(|\widetilde{\nabla} u_\delta|)}{|\widetilde{\nabla} u_\delta|}\, dx.$$

For an estimation of this term we have to check

$$\widehat{b}^{-1} \left(\frac{a'(t)}{t} \right) \leq t^{2+\omega^*} \text{ for large } t. \tag{2.8}$$

(2.8) is equivalent to

$$a'(t)t \leq b\big(t^{\frac{2+\omega^*}{2}}\big)t^{\frac{\omega^* 2}{2}},$$

which we can deduce from (A4) and (1.7). So we have (note (2.7))

$$\overline{J}_{13}^\delta \leq c\kappa^3 + c||a(|\widetilde{\nabla} u_\delta|)|\widetilde{\nabla} u_\delta|^{\omega^*}||_{\beta^*}||1||_{\beta^{*\prime}} \leq c\kappa^\mu.$$

All together we have showed

$$\limsup_{\delta\to 0} J_1^\delta \leq c\kappa^\mu.$$

Investigation of J_2^δ happens analogues, the only critical term is

$$\int_{B-S} \frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|} |\widetilde{\nabla} u_\delta|^2\, dx.$$

Now we define the N-function

$$\mathcal{N}_a(t) := a(\sqrt{t})t^{\frac{\omega^*}{2}}.$$

In contrast to the situation before we have to choose $\omega^* \in (\max\left\{1, \sqrt{2\overline{\omega}}, 2\overline{\omega} - 2\right\}; 2)$ (for $\omega \in [1, 2)$ the choice $\overline{\omega} = \omega$ is possible), which guarantees by (2.3)

$$\widehat{a}^{-1} \left(\frac{b'(t)}{t} \right) \leq t^{2+\omega^*} \text{ for } t \text{ big enough}$$
$$\text{with } \widehat{a}(s) := a(\sqrt{s})s^{\frac{\omega^*-2}{2}}.$$

All together we see

$$\limsup_{\delta\to 0} J_2^\delta \leq c\kappa^\mu \text{ and thereby } \limsup_{\delta\to 0} \mathcal{I}_\delta(B - S) \leq c\kappa^\mu.$$

Similarly we can show $\mathcal{I}(B - S) \leq c\kappa^\mu$. □

3 Blow-up

Motivated by [Fu2] we define the excess function as

$$E(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2\,dy + \fint_{B_r(x)} \overline{a}(|\nabla u - (\nabla u)_{x,r}|)\,dy$$

for $\overline{a}(t) := a(t)t^{\omega}$. With the results concerning higher integrability from [BF4] we can show that E is well defined. We only have to study the second integral: by monotonicity and convexity of $\overline{a}$ we deduce from Jensen's inequality the estimate

$$c \fint_{B_r(x)} \overline{a}(|\nabla u|)\,dy + c\overline{a}(|(\nabla u)_{x,r}|) \le c \fint_{B_r(x)} \overline{a}(|\nabla u|)\,dy.$$

We separate this integral into the two sets $[|\widetilde{\nabla} u| \ge |\partial_n u|]$ and $[|\widetilde{\nabla} u| \le |\partial_n u|]$ an get by Δ_2-condition of $\overline{a}$ the bound (note $a(t) \le cb(t)$)

$$c \fint_{B_r(x)} a(|\widetilde{\nabla} u|)|\widetilde{\nabla} u|^{\omega}\,dy + c \fint_{B_r(x)} b(|\partial_n u|)|\partial_n u|^{\omega}\,dy,$$

which is finite by Lemma 2.3. From Lebesgue's theorem we follow $\mathcal{L}^n(\Omega - \Omega_0) = 0$, whereby

$$\Omega_0 := \left\{ x \in \Omega : \limsup_{r\to 0} |\nabla u| < \infty \text{ und } \liminf_{r\to 0} E(x,r) = 0 \right\}.$$

Now we state the blow up lemma:

LEMMA 3.1 *Assume (A1)-(A7) and let L be a positive number. Then for every $\tau \in (0,1)$ there is an $\epsilon = \epsilon(\tau, L) > 0$ and a constant $C_* = C_*(L)$ such that*

$$|(\nabla u)_{x,r}| \le L \text{ and } E(x,r) \le \epsilon \tag{3.1}$$

for a ball $B_r(x) \Subset \Omega$ implies the following inequality:

$$E(x,\tau r) \le C_* \tau^2 E(x,r). \tag{3.2}$$

Proof: If the statement of the lemma is false, than for every $L > 0$ there is a $\tau \in (0,1)$ and a sequence of balls $B_{r_m}(x_m) \Subset \Omega$ such that

$$|(\nabla u)_{x_m,r_m}| \le L \text{ and } E(x_m,r_m) := \lambda_m^2 \to 0 \tag{3.3}$$

$$\text{for } m \to \infty \text{ but } E(x_m, \tau r_m) > C_* \tau^2 \lambda_m^2. \tag{3.4}$$

The proof is divided into two steps.

Step 1: Scaling and limit equation
For $z \in B_1 := B_1(0)$ let

$$u_m(z) := \frac{1}{\lambda_m r_m}\Big(u(x_m + r_m z) - a_m - r_m A_m z\Big),$$
$$a_m := (u)_{x_m, r_m} \text{ and } A_m := (\nabla u)_{x_m, r_m}.$$

Thereby $(f)_{x,r}$ denotes the mean value of the function f over the ball $B_r(x)$. After scaling (3.3) reads as

$$|A_m| \le L, \ \fint_{B_1} |\nabla u_m|^2 \, dz + \lambda_m^{-2} \fint_{B_1} \overline{a}(\lambda_m |\nabla u_m|) dz = 1, \tag{3.5}$$

while from (3.4) we can follow

$$\fint_{B_\tau} |\nabla u_m - (\nabla u_m)_{0,\tau}|^2 \, dz + \lambda_m^{-2} \fint_{B_\tau} \overline{a}(\lambda_m |\nabla u_m - (\nabla u_m)_{0,\tau}|) dz > C_* \tau^2. \tag{3.6}$$

Using (3.5) we have after passing to subsequnces

$$A_m \to: A, \ \ u_m \rightharpoonup: \overline{u} \ \text{ in } \ W^{1,2}(B_1, \mathbb{R}^N), \ (\overline{u})_{0,1} = 0, \ (\nabla \overline{u})_{0,1} = 0 \tag{3.7}$$
$$\lambda_m \nabla u_m \to 0 \ \text{ in } \ L^2(B_1, \mathbb{R}^{nN}) \text{ and a.e. on } B_1. \tag{3.8}$$

Proposition 3.2 *Limit equation: The weak limit $\overline{u}$ satisfies the equation*

$$\int_{B_1} D^2F(A)\,(\nabla \overline{u}, \nabla \varphi)\; dz = 0 \textit{ for all } \varphi \in C_0^\infty(B_1, \mathbb{R}^N).$$

Proof: By minimality of u it is standard to show (compare [BF3])

$$\begin{aligned} &\int_{B_1} D^2F(A_m)(\nabla u_m, \nabla\varphi)\, dz \\ &= -\int_{B_1}\int_0^1 \left[D^2F(A_m + s\lambda_m \nabla u_m) - D^2F(A_m)\right](\nabla u_m, \nabla\varphi)\, dsdz. \end{aligned} \tag{3.9}$$

By (3.7) we get for $m \to \infty$ on the l.h.s. the desired integral from the proposition. For the r.h.s. of (3.9) we fix an $\epsilon > 0$ and get by Egorov's theorem a measurable subset A of B_1 with $\lambda_m \nabla u_m \to 0$ uniformly on A and $\mathcal{L}^n(B_1 - S) \leq \epsilon$ (therefore we need (3.8)). It follows

$$\begin{aligned} &\left| \int_S \int_0^1 \left[D^2F(A_m + s\lambda_m \nabla u_m) - D^2F(A_m) \right] (\nabla u_m, \nabla\varphi) ds dz \right| \\ &\leq \sup_{S\times[0,1]} |[...]| \, \|\nabla u_m\|_{L^2(B_1)} \|\nabla \varphi\|_{L^2(B_1)} \\ &\to 0 \text{ for } m \to \infty, \end{aligned}$$

where we use (3.7) and the continuity of D^2F. On the other hand we obtain by (3.5) and the growth of D^2F the estimate

$$\begin{aligned} T &:= \left| \int_{B_1 - S} \int_0^1 [...] \, (\nabla u_m, \nabla\varphi) \, ds dz \right| \\ &\leq c \int_{B_1 - S} |\nabla u_m||\nabla\varphi| \, dz + c \left| \int_{B - S} \int_0^1 D^2F(A_m + s\lambda_m \nabla u_m)(\nabla u_m, \nabla\varphi) \, ds dz \right| \\ &:= c \, [T_1 + T_2] \, . \end{aligned}$$

By (3.7) we receive $T_1 \leq c(\varphi)\sqrt{\epsilon}$. Splitting condition and (1.8) show

$$\begin{aligned} T_2 &\leq c \int_{B_1 - S} \int_0^1 \frac{a'(|\widetilde{A}_m + s\lambda_m \widetilde{\nabla} u_m|)}{|\widetilde{A}_m + s\lambda_m \widetilde{\nabla} u_m|} |\widetilde{\nabla} u_m||\nabla\varphi| \, ds dz \\ &\quad + c \int_{B_1 - S} \int_0^1 \frac{b'(|A_m^{(n)} + s\lambda_m \partial_n u_m|)}{|A_m^{(n)} + s\lambda_m \partial_n u_m|} |\partial_n u_m||\nabla\varphi| \, ds dz \\ &:= c \left[\widetilde{T}_2 + T_2^{(n)} \right] . \end{aligned}$$

Let

$$\begin{aligned} \widetilde{M}_1 &:= \widetilde{M}_1(s) := \left[|s\lambda_m \widetilde{\nabla} u_m| \leq K \right] , \\ \widetilde{M}_2 &:= \widetilde{M}_2(s) := \left[|s\lambda_m \widetilde{\nabla} u_m| > K \right] \end{aligned}$$

for a $K > 3L$. So one sees

$$\begin{aligned} \widetilde{T}_2 &= \int_0^1 \int_{B_1 - S} \frac{a'(|\widetilde{A}_m + s\lambda_m \widetilde{\nabla} u_m|)}{|\widetilde{A}_m + s\lambda_m \widetilde{\nabla} u_m|} |\widetilde{\nabla} u_m||\nabla\varphi| \, dz ds \\ &= \int_0^1 \int_{(B_1 - S)\cap \widetilde{M}_1} ... dz ds + \int_0^1 \int_{(B_1 - S)\cap \widetilde{M}_2} ... dz ds. \end{aligned}$$

By (3.7) we can bound the first integral by $c(K,\varphi)\sqrt{\epsilon}$ from above and for the second one we find the bound (remember (1.7) and the Δ_2-condition of a')

$$\lambda_m c(K,\varphi)\lambda_m^{-2}\int\limits_0^1 \int\limits_{(B_1-S)\cap\widetilde{M}_2} a'(|\lambda_m\widetilde{\nabla}u_m|)\lambda_m|\widetilde{\nabla}u_m|\,dzds$$
$$\leq \lambda_m c(K,\varphi)\lambda_m^{-2}\int\limits_{B_1}\overline{a}(|\lambda_m\nabla u_m|)\,dzds,$$

which vanishes by (3.5) for $m\to\infty$. Analogous we obtain from a corresponding definition for $M_2^{(n)}$ by (A5)

$$T_2^{(n)} \leq c(K,\varphi)\sqrt{\epsilon}+\lambda_m c(K,\varphi)\lambda_m^{-2}\int\limits_0^1 \int\limits_{(B_1-S)\cap M_2^{(n)}} b(|\lambda_m\partial_n u_m|)\,dzds$$
$$\leq c(K,\varphi)\sqrt{\epsilon}+\lambda_m c(K,\varphi)\lambda_m^{-2}\int\limits_{B_1}\overline{a}(|\lambda_m\nabla u_m|)\,dzds.$$

It follows

$$\limsup_{m\to\infty} T \leq c(K,\varphi)\sqrt{\epsilon}.$$

Since $\epsilon>0$ is arbitrary, we have proved proposition 3.2. □

The equation in proposition 3.2 is an elliptic system with constant coefficients and elliptic bounds which only depend on L. Thereby we get

$$⨍_{B_\tau} |\nabla\overline{u}-(\nabla\overline{u})_{0,\tau}|^2\,dz \leq C^*\tau^2 ⨍_{B_1} |\nabla\overline{u}-(\nabla\overline{u})_{0,1}|^2\,dz \tag{3.10}$$

for an constant $C^*=C^*(L)$. Futheremore we have $(\nabla\overline{u})_{0,1}=0$ and from

$$⨍_{B_1} |\nabla u_m|^2\,dz \leq 1 \quad \text{follows} \quad ⨍_{B_1} |\nabla\overline{u}|^2\,dz \leq 1 \tag{3.11}$$

by lower semicontinuity (see (3.7)). So we can deduce from (3.10) the inequality

$$⨍_{B_\tau} |\nabla\overline{u}-(\nabla\overline{u})_{0,\tau}|^2\,dz \leq C^*\tau^2. \tag{3.12}$$

Choosing $C_* = 2C^*$ we have the contradiction to (3.6) if we can show the following strong convergences:

$$\nabla u_m \to \nabla \overline{u} \text{ in } L^2_{loc}(B_1, \mathbb{R}^{nN}) \tag{3.13}$$

$$\lambda_m^{-2} \fint_{B_r} \overline{a}(\lambda_m |\nabla u_m - (\nabla u_m)_{0,r}|) dz \to 0 \text{ for all } r < 1 \ . \tag{3.14}$$

By this two informations the l.h.s. of (3.6) converge against the l.h.s. of (3.12).

Step 2: Strong convergence of the scaled functions
Now we prove (3.13) and (3.14): From lemma 2.4 we deduce by (A3) for $Q = A_m$ after sacling (sum over γ)

$$\int_{B_1} \eta^2 \left|\nabla^2 u_m\right|^2 dx \le c \int_{B_1} D^2F(\lambda_m \nabla u_m + A_m)(\partial_\gamma u_m \otimes \nabla\eta, \partial_\gamma u_m \otimes \nabla\eta) dx$$

for $\eta \in C_0^\infty(B_1)$. Using (1.8) we receive

$$\begin{aligned}\int_{B_1} \eta^2 \left|\nabla^2 u_m\right|^2 dx \le c(\eta) &\int_{B_1} \frac{a'(|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|)}{|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|} |\nabla u_m|^2 dx \\ + c(\eta) &\int_{B_1} \frac{b'(|A_m^{(n)} + \lambda_m \partial_n u_m|)}{|A_m^{(n)} + \lambda_m \partial_n u_m|} |\nabla u_m|^2 dx \\ := c(\eta) &\left\{ \widetilde{U} + U^{(n)} \right\}.\end{aligned}$$

A partition into the sets $[|\lambda_m \widetilde{\nabla} u_m| \le K]$ and $[|\lambda_m \widetilde{\nabla} u_m| > K]$ for a $K > 3L$ shows us if $\omega \ge 1$ by (1.7)

$$\begin{aligned}\widetilde{U} &\le c(K) \int_{B_1} |\nabla u_m|^2 dx + c(K) \int_{B_1 \cap [\ldots > K]} a'(|\lambda_m \widetilde{\nabla} u_m|) |\nabla u_m|^2 dx \\ &\le c(K) \int_{B_1} |\nabla u_m|^2 dx + c(K) \lambda_m^{-2} \int_{B_1} \overline{a}(|\lambda_m \nabla u_m|) dx \\ &\le c(K),\end{aligned}$$

using (3.7) and (3.5). If $\omega < 1$ we use the monotonicity of $a'(t)/t$ and the Δ_2-condition of a' and see

$$\widetilde{U} \le c(K) \int_{B_1} |\nabla u_m|^2 dx + c\lambda_m^{-2} \int_{B_1} \overline{a}(|\lambda_m \nabla u_m|) dx$$

directly. For $U^{(n)}$ we separate

$$\begin{aligned}U^{(n)} &= \lambda_m^{-2}\int_{B_1} \frac{b'(|A_m^{(n)}+\lambda_m\partial_n u_m|)}{|A_m^{(n)}+\lambda_m\partial_n u_m|}|\lambda_m\partial_n u_m|^2\,dx\\ &+ \lambda_m^{-2}\int_{B_1} \frac{b'(|A_m^{(n)}+\lambda_m\partial_n u_m|)}{|A_m^{(n)}+\lambda_m\partial_n u_m|}|\lambda_m\widetilde{\nabla} u_m|^2\,dx\\ &=: U_1^{(n)}+U_2^{(n)}.\end{aligned}$$

A suitable partition shows (compare (1.7) and (A5))

$$\begin{aligned}U_1^{(n)} &\le c(K)\int\limits_{B_1}|\nabla u_m|^2\,dx + c(K)\lambda_m^{-2}\int\limits_{B_1\cap[...>K]} b(|\lambda_m\partial_n u_m|)\,dx\\ &\le c(K)\int\limits_{B_1}|\nabla u_m|^2\,dx + c(K)\lambda_m^{-2}\int\limits_{B_1}\overline{a}(|\lambda_m\nabla u_m|)\,dx\\ &\le c(K).\end{aligned}$$

Analogous we obtain by (1.7) and the Δ_2-condition of b

$$\begin{aligned}U_2^{(n)} &\le c(K)\int\limits_{B_1}|\nabla u_m|^2\,dx + c\lambda_m^{-2}\int\limits_{B_1\cap[...>K]}\frac{b(|A_m^{(n)}+\lambda_m\partial_n u_m|)}{|A_m^{(n)}+\lambda_m\partial_n u_m|^2}|\lambda_m\widetilde{\nabla} u_m|^2\,dx\\ &\le c(K)\int\limits_{B_1}|\nabla u_m|^2\,dx + c\lambda_m^{-2}\int\limits_{B_1\cap[...>K]}\frac{b(|\lambda_m\partial_n u_m|)}{|\lambda_m\partial_n u_m|^2}|\lambda_m\widetilde{\nabla} u_m|^2\,dx.\end{aligned}$$

For the last integral we get by (A5) the bound

$$c\lambda_m^{-2}\int\limits_{B_1\cap[...>K]} a(|\lambda_m\partial_n u_m|)|\lambda_m\partial_n u_m|^{\omega-2}|\lambda_m\widetilde{\nabla} u_m|^2\,dx.$$

For $\omega\ge 1$ $a(t)t^{\omega-2}$ is monotonic and so we find the bound

$$c\lambda_m^{-2}\int\limits_{B_1}\overline{a}(\lambda_m\nabla u_m|)\,dx\le c$$

by (3.5). If $\omega<1$ we get this estimate directly using monotonicity of $b'(t)/t$. All together we have showed uniform boundedness of u_m in $W^{2,2}_{loc}(B,\mathbb{R}^N)$ (note (3.7)) and we get $u_m \rightharpoondown \overline{u}$ in $W^{2,2}_{loc}(B,\mathbb{R}^N)$ (after passing to subsequences) and (3.13) using Kondrachov's theorem.

For proving (3.14) we define

$$\widetilde{\psi}_m := \frac{1}{\lambda_m}\left\{\int_0^{|\widetilde{A}_m+\lambda_m\widetilde{\nabla}u_m|}\sqrt{\frac{a'(t)}{t}}dt - \int_0^{|\widetilde{A}_m|}\sqrt{\frac{a'(t)}{t}}dt\right\},$$
$$\psi_m^{(n)} := \frac{1}{\lambda_m}\left\{\int_0^{|A_m^{(n)}+\lambda_m\partial_n u_m|}\sqrt{\frac{b'(t)}{t}}dt - \int_0^{|A_m^{(n)}|}\sqrt{\frac{b'(t)}{t}}dt\right\}.$$

We have to show

$$\widetilde{\psi}_m \text{ and } \psi_m^{(n)} \text{ are uniformly bounded in } W^{1,2}_{loc}(B). \tag{3.15}$$

From Jensen's inequality we deduce

$$\begin{aligned}\int_{B_1}|\widetilde{\psi}_m|^2\,dz &\le \int_{B_1}\int_0^1 \frac{a'(|\widetilde{A}_m+s\lambda_m\widetilde{\nabla}u_m|)}{|\widetilde{A}_m+s\lambda_m\widetilde{\nabla}u_m|}|\widetilde{\nabla}u_m|^2\,dsdz\\ &= \int_0^1\int_{B_1}\frac{a'(|\widetilde{A}_m+s\lambda_m\widetilde{\nabla}u_m|)}{|\widetilde{A}_m+s\lambda_m\widetilde{\nabla}u_m|}|\widetilde{\nabla}u_m|^2\,dzds.\end{aligned}$$

We define for $K > 3L$

$$\begin{aligned}M_1 &:= M_1(s) := [s|\lambda_m\widetilde{\nabla}u_m| \le K],\\ M_2 &:= M_2(s) := [s|\lambda_m\widetilde{\nabla}u_m| > K]\end{aligned}$$

and separate

$$\int_{B_1}|\widetilde{\psi}_m|^2 = \int_0^1\int_{B_1\cap M_1}...dzds + \int_0^1\int_{B\cap M_2}...dzds.$$

For the first inetgral we see

$$c(K)\int_{B_1}|\widetilde{\nabla}u_m|^2\,dz \le c(K)$$

by (3.7). If $\omega \ge 1$ we can bound the second one by (remember (1.7) and the Δ_2-condition of a')

$$\begin{aligned}&c(K)\int_0^1\int_{B_1\cap M_2} a'(|\widetilde{A}_m+s\lambda_m\widetilde{\nabla}u_m|)|\widetilde{\nabla}u_m|^2dzds\\ &\le c(K)\lambda_m^{-2}\int_{B_1}\overline{a}(\lambda_m\nabla u_m|)\,dx \le c(K).\end{aligned}$$

In the other situation we use monotonicity of $a'(t)/t$. So we get $\sup_m ||\widetilde{\psi}_m||_2 < \infty$. By 2.4 we receive after scaling for B_t with $t < 1$ for a suitable η

$$\int_{B_t} |\nabla \widetilde{\psi}_m|^2 \, dz \leq \int_{B_1} \eta^2 \frac{a'(|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|)}{|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|} \left| \frac{\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m}{|\widetilde{A}_m + \lambda_m \widetilde{\nabla} u_m|} \nabla \widetilde{\nabla} u_m \right|^2 \, dz$$
$$\leq c \int_{B_1} D^2 F(\lambda_m \nabla u_m + A_m)(\partial_\gamma u_m \otimes \nabla \eta, \partial_\gamma u_m \otimes \nabla \eta) dx.$$

We have estimated the r.h.s. in the proof of (3.13) and so we get uniform boundedness of $\widetilde{\psi}_m$ in $W^{1,2}_{loc}(B)$. Examination of $\psi_m^{(n)}$ is analogous, the only critical term is

$$\mathcal{W} := \int_0^1 \int_{B \cap M_2} \frac{b'(|A_m^{(n)} + s\lambda_m \partial_n u_m|)}{|A_m^{(n)} + s\lambda_m \partial_n u_m|} |\partial_n u_m|^2 dz ds$$

with a suitable definiton for $M_2 = M_2(s)$. A first estimation shows

$$\begin{aligned}
\mathcal{W} \leq & \, c \int_0^1 \int_{B_1 \cap M_2} \frac{b(|A_m^{(n)} + s\lambda_m \partial_n u_m|)}{|A_m^{(n)} + s\lambda_m \partial_n u_m|^2} |\partial_n u_m|^2 dz ds \\
\leq & \, c\lambda_m^{-2} \int_0^1 \int_{B_1 \cap M_2} a(|A_m^{(n)} + s\lambda_m \partial_n u_m|) |A_m^{(n)} + s\lambda_m \partial_n u_m|^{\omega - 2} |\lambda_m \partial_n u_m|^2 dz ds \\
\leq & \, c\lambda_m^{-2} \int_{B_1} \overline{a}(|\lambda_m \nabla u_m|) \, dx \leq c
\end{aligned}$$

by (1.7), (A5) and Δ_2 in case $\omega \geq 1$. If $\omega < 1$ we use monotonicity of $b'(t)/t$ and $b(t) \leq t^\omega a(t)$ for big t. Hence (3.15) is proved. □

For the proof of (3.14) note

$$\lambda_m^{-2} c \fint_{B_r(x)} \overline{a}(\lambda_m |\nabla u_m - (\nabla u_m)_{0,r}|) \, dz \leq \lambda_m^{-2} c \fint_{B_r(x)} a(\lambda_m |\widetilde{\nabla} u_m|) |\lambda_m \widetilde{\nabla} u_m|^\omega \, dy$$
$$+ \lambda_m^{-2} c \fint_{B_r(x)} b(|\lambda_m \partial_n u_m|) |\lambda_m \partial_n u_m|^\omega \, dy.$$

Using (3.15) we show that the r.h.s vanishes for $m \to \infty$. Let $A_K(r) := B_r \cap [|\lambda_m \widetilde{\nabla} u_m \leq K]$ then

$$a(t) = \int_0^t \int_0^\tau a''(s) \, ds d\tau \leq c(K) t^2 \text{ for } t \leq K.$$

So we receive

$$\lambda_m^{-2} \int\limits_{A_K(r)} a(\lambda_m|\widetilde{\nabla} u_m|)|\lambda_m \widetilde{\nabla} u_m|^\omega \, dy$$

$$\leq c(K) \left[\lambda_m^\omega \int\limits_{A_K(r)} \left\{ |\widetilde{\nabla} u_m|^\omega + |\widetilde{\nabla} \overline{u}|^\omega \right\} |\widetilde{\nabla} u_m - \widetilde{\nabla} \overline{u}|^2 \, dy + \lambda_m^\omega \int\limits_{A_K(r)} |\widetilde{\nabla} \overline{u}|^{2+\omega} \, dy \right].$$

From uniform boundedness of $\nabla \overline{u}$ and (3.13) we deduce

$$\lambda_m^{-2} \int\limits_{A_K(r)} a(\lambda_m|\widetilde{\nabla} u_m|)|\lambda_m \widetilde{\nabla} u_m|^2 \, dy \to 0, \ \ m \to \infty. \tag{3.16}$$

On $\overline{A}_K(r) := B_r \cap [|\lambda_m \widetilde{\nabla} u_m \geq K]$ we have for $K > 3L$

$$\widetilde{\psi}_m \geq \frac{1}{\lambda_m} \int_{\frac{1}{3}|\lambda_m \widetilde{\nabla} u_m|}^{\frac{2}{3}|\lambda_m \widetilde{\nabla} u_m|} \sqrt{\frac{a'(t)}{t}} dt \geq \frac{c}{\lambda_m} \sqrt{a(|\lambda_m \widetilde{\nabla} u_m|)}$$

by monotonicity of a' and (1.7). It follows

$$\widetilde{\psi}_m^2 \geq c\lambda_m^{-2} a(|\lambda_m \widetilde{\nabla} u_m|) \text{ on } \overline{A}_K(r). \tag{3.17}$$

Thereby we get following the lines of [Fu2]

$$\lambda_m^{-2} \int_{\overline{A}_K(r)} a(\lambda_m|\widetilde{\nabla} u_m|)|\lambda_m \widetilde{\nabla} u_m|^\omega \, dy \leq c(r) \left(\int_{\overline{A}_K(r)} |\lambda_m \widetilde{\nabla} u_m|^{\frac{\omega}{2} n} dz \right)^{2/n}$$

by (3.15) and Sobolev's inequality. In the following we need (A7) to show

$$\lambda_m^{-2} \int_{\overline{A}_K(r)} a(\lambda_m|\widetilde{\nabla} u_m|)|\lambda_m \widetilde{\nabla} u_m|^\omega \, dy \leq \ c(r) \left(\int_{\overline{A}_K(r)} a(|\lambda_m \widetilde{\nabla} u_m|)^{\frac{n}{n-2}} \, dz \right)^{2/n}$$
$$\leq c(r) \left(\lambda_m^{\frac{2n}{n-2}} \int_{\overline{A}_K(r)} \widetilde{\psi}_m^{\frac{2n}{n-2}} \, dz \right)^{2/n} \leq \ c(r) \lambda_m^{\frac{4}{n-2}} \to 0, \ \ m \to \infty,$$

by (3.17) and (3.15). Together with (3.16) we have showed

$$\lim_{m \to \infty} \lambda_m^{-2} \fint\limits_{B_r(x)} a(\lambda_m|\widetilde{\nabla} u_m|)|\lambda_m \widetilde{\nabla} u_m|^\omega \, dy = 0.$$

For $\lambda_m^{-2} \int_{B_r(x)} b(|\lambda_m \partial_n u_m|)|\lambda_m \partial_n u_m|^\omega \, dy$ we can argue similarly and so we get (3.14). Note that we have (A4) and thereby

$$b(t) \geq ct^{\frac{\omega}{2}(n-2)} \text{ for large } t.$$

4 Proof of theorem 1.1 b)

Let $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$, $n = 2$ and fix $x_0 \in \Omega$. Now it is possible to find a radius $R > 0$ with $u \in L^\infty_{loc}(\partial B_R(x_0), \mathbb{R}^N)$ (compare [Bi2], section 4, for details). From the maximim principle of [DLM] we get $u \in L^\infty_{loc}(B_R(x_0), \mathbb{R}^N)$. For $0 < \epsilon \ll 1$ $(u)_\epsilon$ denotes the mollification of u with radius ϵ (see [Ad]). Now we choose $R_0 < R$ and get $\sup_{\epsilon>0} \|(u)_\epsilon\|_\infty < \infty$. We define the regularisation on $B_{R_0}(x_0)$ and get the statements of Lemma 2.1 without assuming local boundedness of the minimizer. Now we only need (A1)-(A3). The proof is a little modification from [BF5], Thm. 1.1. The only condition in [BF9] which is stronger than ours is

$$\frac{h'(t)}{t} \leq h''(t) \tag{4.1}$$

which is needed to show ($i \in \{1, 2\}$, $h = a$ or $h = b$)

$$\int_{|\partial_i u|/2}^{|\partial_i u|} \sqrt{\frac{h'(t)}{t}}\, dt \geq \frac{|\partial_i u|}{2} \sqrt{\frac{h'(|\partial_i u|/2)}{|\partial_i u|/2}} \geq h(|\partial_i u|/2)^{\frac{1}{2}} \geq ch(|\partial_i u|)^{\frac{1}{2}}.$$

This is the only part in the proof of [BF5] were (4.1) is strongly needed and can not be replaced by the first inequality of (A2) with $\widehat{\epsilon} < 1$. In our situation we get ($\vartheta \in [1/2, 1]$)

$$\int_{|\partial_1 u_\delta|/2}^{|\partial_1 u_\delta|} \sqrt{\frac{h'(t)}{t}}\, dt \geq \frac{|\partial_1 u_\delta|}{2} \sqrt{\frac{h'(\vartheta|\partial_1 u_\delta|)}{\vartheta|\partial_1 u_\delta|}} \geq ch(|\partial_1 u_\delta|)^{\frac{1}{2}}$$

by Δ_2-condition of h'. Now we can reproduce the proof of [BF5] and get the statement of Theorem 1.1 b).

5 Proof of theorem 1.1 c)

Here it is enough to ask for (A1)-(A3), too, together with $b(t) \leq ct^2 a(t)$ and $a(t) \leq ct^2 b(t)$ for large t. In [BF4] (proof of Thm. 2.3) is showed

$$\sup_{\epsilon>0} \|\nabla u_\epsilon\|_{L^t(B_\rho)} < \infty$$

for all $\rho < R$ and all $t < \infty$ (where u_ϵ denotes the Hilbert-Haar-regularisierung). Bildhauer and Fuchs assume in place of $a(t) \leq ct^2 b(t)$ the stronger condition $a(t) \leq b(t)$. If you have a look at the inequalities (5.9) and (5.10) from [BF4], you see that our assumptions are enough the iterate. By (1.9) we have an $(2, q)$-elliptic integrand. Now we can reproduce the proof of [Bi], Thm 5.22 and get ∇u in $L^{\infty}_{loc}(B, \mathbb{R}^n)$ (see [Bi], p. 66, for details). By standard theory for elliptic equations or variational problems with standard growth conditions (compare [Gi2]) we can follow the claim of Theorem 1.1 c).

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