Universität des Saarlandes

Fachrichtung für Mathematik

The enumeration of real tropical curves

Dissertation zur Erlangung des Grades des
Doktors der Naturwissenschaften (Dr. rer. nat.)
der Naturwissenschaftlich-Technischen Fakultäten
der Universität des Saarlandes

vorgelegt von

Franziska Schroeter

Saarbrücken, den 7. Februar 2013

Datum des Kolloqiums: 12. Juni 2013
Dekan der Fakultät: Prof. Dr. M. D. Groves
Mitglieder des Prüfungsausschusses: Prof. Dr. J. Eschmeier (Vorsitz), Prof. Dr. H. Markwig, Prof. Dr. F.-O. Schreyer, Prof. Dr. E. Shustin, Dr. M. Weber

Zusammenfassung

Die enumerative tropische Geometrie erlaubt es, schwierige Probleme aus der algebraischen Geometrie mit kombinatorischen Methoden zu lösen. Möglich ist dies durch den degenerativen Prozess der *Tropikalisierung*, der z.B. algebraische Kurven in metrische Graphen mit speziellen Eigenschaften überführt. Bereits bekannte Resultate aus der komplexen Geometrie, wie die Invarianz von enumerativen Zahlen, die *Kontsevich*-Formel zum Zählen von rationalen Kurven in der Ebene oder die *Caporaso-Harris*-Formel lassen sich so mit weniger Aufwand gewinnen. Die enumerative reelle Geometrie hingegen hat sich lange dem komplexen Ansatz widersetzt. Hier konnte die tropische Herangehensweise ihre Stärken ausspielen und z.B. rekursive Formeln für Invarianten von reellen rationalen Kurven durch generische Punkte in der Ebene, d.h. speziellen *Welschinger*-Zahlen, hervorbringen. Im Fall von nur reellen Punkten konnten die Invarianz der Zahlen und rekursive Formeln bereits tropisch bewiesen werden. Ein wesentlicher Beitrag dieser Arbeit ist die Behandlung des Falls von beliebigen Punkten. Mithilfe von neuen, bislang rein tropischen Invarianten, den *Broccoli*-Zahlen, können wir die Invarianz der entsprechenden Welschinger-Zahlen zeigen und Formeln zu ihrer Berechnung angeben. Ferner zeigt dieser Ansatz die Möglichkeit auf, Invarianten für reelle Kurven vom Geschlecht g zu definieren. Außerdem konnten Resultate zur Charakterisierung von Punkten in spezieller Lage in tropisch-enumerativen Problemen gewonnen werden.

Abstract

Enumerative tropical geometry allows to solve technical problems from enumerative algebraic geometry using combinatorial methods. This is possible due to the degenerative process of tropicaliziation, which e.g. transforms algebraic curves into metric graphs with specific properties. Well known results from complex algebraic geometry, such as the invariance of enumerative numbers, the *Kontsevich* formula to count rational curves in the plane or the *Caporaso-Harris* formula are easier to obtain. Enumerative real geometry, however, has resisted for a long time to the complex approach. Here, the tropical approach can show its advantage by producing recursive formulas for invariants of real rational curves through generic points in the plane, which are special *Welschinger* numbers. In the case of only real points the invariance of the numbers and recursive formulas have already been proven by purely tropical means. A major contribution of the present work is the treatment of the case of arbitrary points. Via the introduction of new, at the moment purely tropical invariants, which we call *broccoli* numbers, we can prove the invariance of the corresponding Welschinger numbers and also formulas to compute them. Furthermore, this approach indicates the possibility to define invariants of real curves of any genus. Moreover, results characterizing points in special position in tropical enumerative problems have been obtained.

Contents

What is this thesis about?

Some tropical history

Tropical geometry is as much varicolored as the grains of sand of a beach are. Based on the idea of replacing objects from algebraic geometry by combinatorial objects, that we call *tropicalization*, the subject evolves now in many directions including commutative algebra, symplectic geometry, optimization, low-dimensional topology, knot theory and physics. The key feature of tropicalization is that important information about the objects are conserved. Hence, one could think of the tropicalization of algebraic objects as focusing on the important information about the problem considered.

In this work, we will focus on the interplay of algebraic geometry and tropical geometry, more precisely on enumerative aspects. In the beginning of the tropical era, Grisha Mikhalkin surprised the mathematical community with the tropical computation of certain Gromov-Witten numbers $N(d,g)$ [Mik05]. $N(d,g)$ is the number of plane complex projective curves of degree d and genus g passing through a given number of points in the plane, whose configuration ω is in some sense generic. It turns out that $N(d,g)$ does not depend on the actual configuration ω, it is an *invariant* number. To determine these numbers in terms of recursive formulas one had to make use of the high-tech but efficient tools of symplectic or algebraic intersection theory. Instead, Mikhalkin studied carefully the degeneration of complex curves to tropical objects that we call *tropical curves*. He showed that one can count tropical curves with multiplicities instead of complex curves and this gives the same number $N(d,g)$. That is known to be the *Correspondence Theorem*. By this theorem also the tropical $N(d,g)$ are invariant. Over the years, tropical moduli spaces and intersection theory have been developped [AR10], [Mik07], [GKM09]. For instance, tropical moduli spaces of rational curves carry the structure of a weighted fan which satisfies the so called *balancing condition*, see definition 2.8. This let to a proof of the invariance of the tropical $N(d,g)$ without use of algebraic geometry [GM07b]. Even more, this idea can be transfered to prove the invariance of tropical relative enumerative numbers, that cannot be obtained by the Correspondence Theorem.

Mikhalkin came also up with a purely combinatorial algorithm, the *lattice path algorithm*, to compute the numbers $N(d,g)$. For big numbers d,g this is still a long-running computation and therefore, it would be nice to have recursive formulas for these numbers. Maxim Kontsevich [KM94] found one for rational curves; a formula that works for any degree d and genus g has been discovered by Lucia Caporaso and Joe Harris [CH98]. These formulas could been reproven by tropical means in a much simplier way by Andreas Gathmann and Hannah Markwig using Mikhalkin's numbers, once the invariance of the relative tropical numbers has been shown [GM08], [GM07a].

One can also start to study properties of related objects in this tropical algebraic geometry like tropical structures in general, irreducibility of varieties, divisors, etc. It appears that not all properties from algebraic geometry translate one to one into tropical geometry. For instance, there is no unique decomposition into irreducible components of a tropical variety, see remark 3.4, and the set of all configurations of $n = 3d-1$ generic points for the enumerative problem of counting rational plane curves of degree d is a tropical subvariety of codimension 1 in $(\mathbb{R}^2)^n$, although in algebraic geometry it is only a subvariety of $(\mathbb{P}^2)^n$, see chapter 3. This is a first contribution of this thesis.

However, tropical geometry is not only a mirror of known facts in algebraic geometry but can even enlighten the situation in algebraic geometry. One of the main examples where tropical geometry does not only simplify proofs and provides means to compute things in an easier way is real enumerative geometry. The situation in real algebraic geometry is not as easy as the situation in complex algebraic geometry. It is due to the fact that real curves may be deformed into non-real curves, in particular real singularities may be deformed into complex singularities and vice versa. This implies that it is harder to prove the independence of the point configuration when we want to count plane real curves. It turns out that e.g. the number of real rational cubics through 8 real generic points is not invariant, see example 1.14. The invariance has been regained thanks to Jean-Yves Welschinger [Wel03] by the introduction of the sign of a rational curve that passes through a given point configuration, which depends on the number of a certain type of real singularities that the plane real rational curve can have. Notice that we consider here the case $g = 0$. By an approach from symplectic geometry he proved for a real rational symplectic manifold X of dimension 4 that the (signed) number of real rational J-holomorphic curves in a given homology class on X through a generic point configuration does not depend on the actual configuration, it is a *Welschinger invariant*. Unfortunately, his proof gives no hint how to compute these numbers! Looking at this in a toric-algebraic sense the statement says for a real toric unnodal Del Pezzo surface X that the number of real rational nodal curves in a certain linear system on X, corresponding to the degree of the curve, counted in a signed manner and passing through a real generic point configuration, does not depend on the choice of this configuration. We restrict ourselves here to toric surfaces as this is a condition to translate the setting into tropical geometry. Namely in this case, one can consider the underlying lattice polytope P of X to which we can associate certain tropical rational curves. Depending on the weight of each such curve it can be used to count complex rational curves, or real rational curves through real points. I.e. the difference between the real and the complex count in tropical geometry is made by the choice of weights, see definition 4.11. This weight equals the number of complex or real curves which tropicalize to the tropical curves, respectively, by the Correspondence Theorem, that also holds in a similar way for Welschinger numbers. Namely in this situation, one can show that the weighted tropical count of these rational tropical curves of given degree equals a certain Welschinger invariant and is therefore invariant itself. It is even possible to show the tropical invariance without having reference to this real Correspondence Theorem. Namely, as in the tropical proof of the invariance of the numbers $N(d, g)$, it can be proven by a local study of the corresponding tropical moduli space as the movement of a point in a non-generic configuration in order to obtain a generic configuration can be translated into the transition of one cell of codimension 1 into other cells of codimension 0 in the moduli space [IKS09]. The verification of invariance then turns out to be the check of a certain balancing condition in the moduli space. Again, this allows to define and to prove the invariance of relative tropical Welschinger numbers, and hence to obtain a recursive formula that computes tropical Welschinger numbers [IKS09]. Using the real Correspondence Theorem, this also gives a recursive formula for the Welschinger invariants from real algebraic geometry which are therefore now computable.

Until now we have only considered real rational curves passing through real points. But a real curve may also have pairs of complex conjugate points that may be refound in the generic point configuration through which the curves have to pass. In this case, the invariance of the corresponding Welschinger numbers has also been shown by Welschinger [Wel05a] and there exists also a Correspondence Theorem to relate these invariants to certain tropical Welschinger numbers. So these tropical Welschinger numbers are also invariant. The latter count in a weighted manner a new type of tropical curves whose introduction is necessary as these curves should be the tropicalization of the corresponding real curves. However, one cannot prove their invariance using a local moduli space argument as before. The reason is that the structure of the moduli space is such that cells containing curves passing through a given point configuration do not lie necessarily next to each other and hence a local proof fails.

The birth of broccoli curves

This is the initial situation in which we started to work. A new type of tropical curves, *broccoli curves* of genus 0, was built such that a local proof in their moduli space is possible again, see definition 5.21. Note that they have so far no interpretation as a certain class of real curves as this is the case for curves contributing to Welschinger numbers. The main part of this thesis, chapter 5, covers their discovery and their relation to tropical Welschinger invariants. Given a generic configuration ω of $r+s$ points in $\mathbb{R}^2$ and a collection $\Delta = (v_1, \ldots, v_n)$ of vectors in $\mathbb{Z}^2 \setminus \{0\}$ satisfying $r + 2s = n - 1$, then the *broccoli number* $N^B_{(r,s)}(\Delta, \omega)$ is roughly speaking the number of broccoli curves (counted with multiplicities) having Newton polygon P_Δ and passing through ω, see definition 5.25.

Theorem 1 - see theorem 5.26: The broccoli number $N^B_{(r,s)}(\Delta, \omega)$ does not depend on the concrete point configuration ω as long as the latter is generic.

At a first glance they have nothing to do with Welschinger invariants. But ω can be seen as the tropicalization of a configuration ω' of r real and s pairs of complex conjugate points in a real toric unnodal Del Pezzo surface X. For such a surface X we could find an algorithm which tells us how to deform a broccoli curve in order to obtain a curve or several curves contributing to the corresponding Welschinger number. This yields:

Theorem 2 - see corollary 5.60: Let X be real toric unnodal Del Pezzo surface and ω' a generic point configuration containing r real and s pairs of complex conjugate points in X and given a tautological linear system on X corresponding to Δ above. Then the broccoli invariant $N^B_{(r,s)}(\Delta, \omega)$ equals the corresponding tropical Welschinger invariant $N^W_{(r,s)}(\Delta, \omega)$, where ω is the tropicalization of ω'.

The curves that appear while we deform broccoli curves into curves contributing to Welschinger numbers constitute a new type of curves, namely *bridge curves*, see definition 5.46. Broccoli curves and curves contributing to Welschinger numbers are respectively a subclass of bridge curves, see lemma 5.52. The ensemble of bridge curves between all broccoli curves through a given configuration ω and all the curves contributing to the corresponding Welschinger numbers constitute the so called *bridge*, see remark 5.56. The equality of the numbers is due to the fact that the bridge can be seen as a weighted graph and we have invariance along this graph, see theorem 5.58. One other feature of this algorithm is that there is a direction on the bridge that informs us in which sense one has to deform a bridge curve to obtain a broccoli curve or a curve contributing to a Welschinger number.
Using Theorem 2 we could prove a recursive formula for Welschinger invariants counting curves passing through real and pairs of complex conjugate points. This formula is actually a recursive formula for broccoli invariants:

Theorem 3 - see theorem 5.75: By a Caporaso-Harris type formula the broccoli invariants $N^B_{(r,s)}(\Delta, \omega)$ can be computed recursively.

Bridge curves have the property that their weight is the product of the weights that we can associate to each of the vertices of the curve. In particular, this let us hope for a generalization of rational broccoli curves to genus $g > 0$ and therefore of Welschinger invariants to genus $g > 0$. This is work in progress. So far, the authors of [IKS09] have only defined tropical Welschinger invariants for $g > 0$ in the case of purely real points in the configuration ω. At least in some cases I could prove the invariance of broccoli numbers and this approach has chances to be generalized.

Theorem 4 - see theorems 6.2, 6.12: In certain cases the local invariance of broccoli numbers for broccoli curves of genus 1 holds.

One major open question is the intepretation of broccoli curves in real algebraic geometry, i.e. the translation from tropical geometry into algebraic geometry this time. Unfortunately, this direction is much harder as there are several options of interpretation. Namely, as we said in the beginning, tropical geometry concentrates on the important information about the object and situation you consider. The reconstruction of the thereby lost information is difficult. Nevertheless, exactly this fascinating interplay between tropical and algebraic geometry makes it worth to study this junction.

Organization of the material

Chapter 1 summarizes facts from enumerative algebraic geometry, in particular the kind of real algebraic geometry necessary to understand the range of the tropical results exposed in this thesis. We recommend this chapter to the not so experienced reader in algebraic geometry. It is written in a survey style and does not focus on details. The second chapter (2) is of a more profound nature and treats basic and known facts about tropical enumerative geometry. It covers tropical structures, different approaches to moduli spaces and Correspondence Theorems. Chapter 3 presents joint work with Andreas Gathmann that studies tropical configurations of points in special position for an enumerative problem. We discuss two notions of these configurations and describe them explictly in terms of push-forwards of divisors in tropical moduli spaces. In chapter 4 we aim to explain the relation between real algebraic enumerative geometry and real tropical enumerative geometry. The key words here are *patchworking*, real Correspondence Theorems and tropical Welschinger invariants. The main part of this thesis, i.e. the joint work with Andreas Gathmann and Hannah Markwig about rational broccoli curves is exposed in chapter 5. Finally, chapter 6 gives an idea of my work in progress about broccoli curves of genus 1.

Credits/Published work

- Chapter 3 and definitions 2.4, 2.8, 2.9, 2.10, 2.11, 2.14, 2.23, 2.24, 2.26, 2.27, 2.30, 2.31 and 2.32 of chapter 2 are excerpt (with minor modifications) of the paper [GS12] published in the Electronic Journal of Combinatorics, written together with Andreas Gathmann.
- Chapter 5 is part (with minor modifications) of the preprint [GMS13] jointly obtained with Andreas Gathmann and Hannah Markwig, accepted for publication in the Advances in Mathematics.
- Chapter 6 is my own work in progress.

Danksagung/Acknowledgment

Eine Promotion beinhaltet nicht nur das Verfassen einer längeren wissenschaftlichen Arbeit, sondern auch eine längere Lebensphase, die sehr reich an Erfahrungen ist und die ich nicht missen möchte. In ihr wird man z.B. Teil einer Gemeinschaft von Spezialisten, die sich mehr oder weniger mit der gleichen Mathematik beschäftigt, und gewöhnt sich auf Tagungen an das Reden über Mathematik in den undenkbarsten Situationen, in denen sich Arbeits- und Freizeit vermengen. Dieser ständige Austausch, der natürlich nicht ganz spaßfrei ist, und das Zusammenarbeiten mit Menschen, die teilweise am anderen Ende der Welt zu Hause sind, macht Mathematik zu einem faszinierenden und immer wieder motivierenden Erlebnis. Dafür sei all den tropischen Leuten herzlich gedankt.
Für die Mühe, diesen Prozess des Hereinfindens in den Kreis zu beschleunigen, möchte ich meiner Betreuerin *Hannah Markwig* ganz besonders danken. Immer wieder ermutigte Hannah

mich, an Konferenzen, Workshops, etc. teilzunehmen, wodurch ich schnell mit den wichtigsten Forschungstrends vertraut wurde. Das Ermutigen beschränkt sich allerdings nicht darauf; sie unterstützte mich bei (fast) allen Ideen sehr energisch. Hannah sieht man die Begeisterung für Mathematik tagtäglich an – was mir gerade in nicht ganz so rosigen Zeiten der Promotion sehr geholfen hat. Ihre offene, teilhaben lassende und wohlwollende Art hat mir gezeigt, dass man nicht alles so verbissen sehen muss (und trotzdem zufrieden sein kann), und mir einen Einblick in das akademische Leben an der Universität gewährt. Danken möchte ich ihr auch für die Zeit – unendlich viel Zeit, die sie immer bereit war zu opfern, egal wie voll ihr Terminkalender war. Ich hatte während der Promotion nie das Gefühl, im Stich gelassen zu werden. Ich wünsche jedem Doktoranden einen Betreuer wie Hannah.
Andreas Gathmann danke ich für gemeinsames Forschung, Unterstützung und Lernen lassen während der Promotionszeit.
Meinen Wegbegleitern während der Promotionszeit möchte ich ebenfalls Dank aussprechen. *Anders Jensen* für die gemeinsame Schoko-Osterhasen-Sammlung im Büro in Göttingen, Denglisch-Konversationen und seinen einzigartigen Humor. *Arne Buchholz* für seinen immer währenden Sonnenschein, geteiltes Doktorandenleid, leckeren Kuchen und Korrekturlesen. *Timo de Wolff* für lebhaften Ideenaustausch, gemeinsame Kaffeeexzesse und seinen unerschütterlichen Glauben an unser Amöben-Forschungsprojekt. *Johannes Rau* für das Klären meiner vielen Fragen beim Aufschreiben und genaues Korrekturlesen. *Simon Hampe* dafür, dass er mich in der Zeit kurz vor Abgabe im Büro ertragen hat.

I would also like to thank *Eugenii Shustin* for the pleasant stay at the Tel Aviv University last autumn and for being the external referee for this dissertation.

1 An introduction to real enumerative geometry

Counting objects seems to be – at a first glance – something childlike: small kids conceive numbers by counting. Asking the same type of questions in the context of geometry, the issue becomes harder. Which types of objects do we want to count and where do they live? Which conditions on the objects do we have to impose to ensure an answer that is useful? Often these questions are asked for objects originating in algebraic geometry and which are defined (only) over the field of complex numbers $\mathbb{C}$. For non-mathematicians this is confusing as the real numbers $\mathbb{R}$ are more natural and easier than complex numbers!

On the following few pages we try to resolve this supposed contradiction by presenting obstructions we encounter. Meanwhile we intend to enlighten the reader about the historical background of this exciting and still active field of mathematics. Our focus will be on the real situation – to prepare the following chapters.

1.1 Classical enumerative geometry

Main references are [Bru08, Kle87, KV07]. The Greek Apollonius of Perga (ca. 262 BC – ca. 190 BC) was interested in the following question: "Given a configuration of three disjoint circles in the (Euclidean) plane, how many circles are tangent to these three circles?" This can be considered as one of the earliest enumerative problems explictly stated. Important to mention here is that Apollonius clearly asked for a *particular class of objects*, circles in this case, and he did not ask for circles satisfying no conditions, nor for circles tangent to whatever configuration of three cycles. The setting of the problem is chosen such that he expects only a *finite number* of circles fulfilling his conditions. Even more: he assumed that there is only one answer – *one number* independent of the particular configuration of the three circles. The right answer is 2^3 as he figured out. Actually, the idea behind his proof is that solutions come in pairs; for each pair p_n of solutions n of the three given circles lie inside or outside the solution circle, $n \in \{0, 1, 2, 3\}$. In the following figure the three black circles are given and pairs of solutions appear in yellow, magenta, blue and grey.

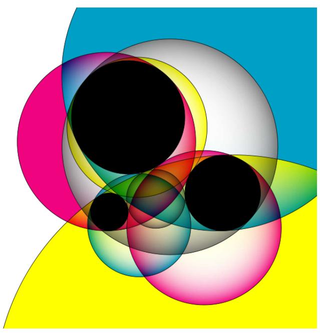


Figure 1.1: Circles of Apollonius. [Wik]

We want to call such a number an *invariant number* or shortly *invariant* if it does not depend on the conditions imposed. Observe that if we do not ask the three circles given to be disjoint, the number of solutions can be in the worst case zero! Let us summarize:

(Well-posed) Enumerative problem: Choose a class of objects and conditions such that there is only a finite number of objects satisfying these conditions. This number must not depend on the actual configuration of the conditions.

Using the first achievements of algebraic geometry like the notions of degree d, (geometric) genus g, the complex projective plane $\mathbb{P}^2$ and Bézout's theorem (proof in 1764, [Kle87]) Jakob Steiner determined in 1848 the number of plane rational cubics through 8 points in general position to be 12 [Ste48]. *General or generic position* means roughly speaking that the points impose independent conditions, i.e. no three points lie on a line, no 6 points lie on a conic, etc. to ensure that the answer to the problem is finite. To be accurate, the set of all possible configurations of n points in the plane $\mathbb{P}^2$, which are in general position for a given enumerative problem is an open, dense subset of the space of conditions $(\mathbb{P}^2)^n$. We are interested in $N(d,g)$, which is the number of irreducible plane curves of degree d and genus g passing through $n = 3d - 1 + g$ points in general position. The choice of n makes sure that the number $N(d,g)$ is finite or in other words that the parameter space of curves fulfilling the conditions is 0-dimensional. To see this, remember that for a given curve $C \subset \mathbb{P}^2$ of degree d, the linear system $|C|$ on $\mathbb{P}^2$ is a projective space of dimension $\frac{d(d+3)}{2}$ (Riemann-Roch or number of coefficients defining C). $|C|$ can be seen as an appropriate parameter space for our curves [Cap98]. Consider the subvariety $\overline{V_g(d)} \subset |C|$, which is the closure of all irreducible curves of genus g and degree d, and which is called *Severi variety* [Sev21]. It was conjectured by Severi and shown by Joe Harris that $\overline{V_g(d)}$ is irreducible [Har86]. Note that there are reducible curves in $\overline{V_g(d)}$. A general point in $V_g(d)$ is an irreducible curve having $\delta = \frac{(d-1)(d-2)}{2} - g$ nodes by the genus formula for irreducible curves. Curves having worse singularities lie in lower-dimensional strata. $\overline{V_g(d)}$ has dimension $r_g(d) = -(K_{\mathbb{P}^2} \cdot C) + g - 1 = \dim |C| - \delta$ where $K_{\mathbb{P}^2}$ is the canonical divisor of $\mathbb{P}^2$ and where we use again the genus formula for irreducible curves. Hence:

$$r_g(d) = \frac{d(d+3)}{2} - \delta = \frac{d(d+3)}{2} - \frac{(d-1)(d-2)}{2} + g = 3d - 1 + g.$$

Roughly speaking, we see that each node imposes one condition on the space of curves. Hence the number of point conditions (each point lowers the dimension by one) that we need in order to obtain a 0-dimensional space is $r_g(d)$. The degree $\deg(\overline{V_g(d)})$ equals $N(d,g)$ as long as the $r_g(d)$ points are in general position, hence we get an invariant for generic points.
The numbers $N(d,g)$ are nowadays called *Gromov-Witten invariants* (GW invariants) of the plane due to their study in quantum cohomology by Mikhail Gromov and Edward Witten in the late 20th century [Wit91]. Note that the concept of Severi varieties can be generalized: one can also construct varieties such that plane reducible curves of given degree and genus are in its open part. The numbers that we get thereby, called *Severi degrees*, are in general different from the Gromov-Witten invariants. The invariant $N(1,0) = 1$ of lines through two generic points and the invariant $N(2,0) = 1$ of conics through five generic points are known for a long time, but their origin is unclear. Chasles' student Georg Zeuthen determined the number $N(4,0) = 640$ in 1873 [Zeu73]. The next number, $N(5,0) = 87304$ was only computed more than 100 years later by Israel Vainsencher [Vai95] using recursive formulas. In meantime the study of the relation of intersection theory and enumerative geometry was pioneered by Hermann Schubert ([Sch79]) and its rigorous study encouraged by David Hilberts 15th problem [Hil01] on the ICM in Paris (1900). The modern intersection theory was not completed before the late 20th century.
Let us finish this section with an example that we will need later.

Example 1.1 (**A classical way to compute** $N(0,3) = 12$)
See [KV07, example 3.1.4] or [Cap98, section 4]. This example traces back to Schubert. Consider the Severi variety $\overline{V_0(3)} \subset \mathbb{P}^9$ of dimension 8. Its degree can be computed as the intersection number N of $L \cap \overline{V_0(3)}$ in $\mathbb{P}^9$ by Bézout for a general line $L \cong \mathbb{P}^1$ in $\mathbb{P}^9$. L can be

described as pencil of plane cubics $\{t_1C_1+t_2C_2\}_{[t_1:t_2]\in\mathbb{P}^1}$ where $C_1,\ C_2$ are smooth irreducible cubics intersecting in a configuration $\omega=(p_1,\ldots,p_9)$ of $3^2=9$ points (by Bézout), such that 8 of them are in general position. As C_1 and C_2 pass through ω, every cubic in the pencil passes through them. Hence by genericity of the points there are at worst nodal curves in the pencil. Deforming C_1 and C_2 infinitesimally we see that any two curves in the pencil intersect transversally in each p_i, i.e. these 9 points are nonsingular points for all cubics in the pencil. Blowing up $\mathbb{P}^2$ at each p_i of ω we obtain a surface S and the blow-down morphism $\pi: S\to\mathbb{P}^2$. The pencil as linear system Γ on $\mathbb{P}^2$ has base points $p_1,\ldots,p_9$. Its moving part gives rise to the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1,\ [x:y:z]\mapsto[f_1(x,y,z):f_2(x,y,z)]$, where the homogeneous polynomial f_i defines C_i such that they are w.l.o.g. $\mathbb{C}$-linear independent in the generating space $L(\Gamma)$. π then extends to a morphism $t: S\to\mathbb{P}^1$. As the blow-up is isomorphic except at its exceptional divisor $\pi^{-1}(\omega)$, but the curves in the pencil are nonsingular at ω, the fiber of t over $[t_1:t_2]$ is isomorphic to the cubic $t_2C_1-t_1C_2\subset\mathbb{P}^2$. We require that the cubics pass through $3d-1+g=8$ points, so there is only a finite number of nodal curves in the pencil passing through ω. The number N can be computed as the cardinality of the set F_{sing} of fibers $[t_1:t_2]$ over which the cubic in S is isomorphic to a nodal curve $\subset\mathbb{P}^2$. We do this by determing the Euler characteristic $\chi(S)$ in two different ways using a formula for the Euler characteristic for holomorphic maps [GH78, p. 509]. First, when we blow up a point p_i from ω, it is replaced by $\mathbb{P}^1$ in S, hence applying the formula to π yields:

$$\chi(S)=\chi(\mathbb{P}^2\setminus\omega)+9\chi(\mathbb{P}^1)=3-9+18=12. \tag{1.1}$$

The formula for the map t translates to:

$$\chi(S)=\chi(t^{-1}(\mathbb{P}^1\setminus F_{\mathsf{sing}}))+\chi(t^{-1}(F_{\mathsf{sing}}))=\chi(\mathbb{P}^1\setminus F_{\mathsf{sing}})\cdot\chi(\text{smooth cubic})+N\cdot\chi(\text{nodal cubic}). \tag{1.2}$$

As the Euler characteristic of a smooth cubic is 0 and that one of a nodal cubic is 1, we obtain by comparison of the two equations $N=12$.

1.2 Modern approach to enumerative geometry

The numbers $N(d,g)$ can be also computed as appropriate intersection products on moduli spaces. This idea goes back to 1994 when Maxim Kontsevich discovered strong relations between string theory and enumerative geometry [KM94]. This input from physics pushed a lot on the study of enumerative questions. Nowadays, the theory of Gromov-Witten invariants is an own field of mathematics where scientists from algebraic geometry, symplectic geometry and mathematical physics fruitfully collaborate. We will discuss this subject only shortly as we are more interested in its tropical version, that we are going to consider in section 2.3.

Definition 1.2 (Moduli spaces of stable maps $\overline{M}_{g,n}(X,\beta)$)
See [FP96]. Fix integers $g,n\geq 0$. Furthermore, let X be a smooth projective variety and $\beta\in H_2(X,\mathbb{Z})$ a homology class.
An *n-marked stable map of genus g* is a collection $(C,x_1,\ldots,x_n,f)$ of a projective nodal connected curve C of arithmetic genus g, smooth points $x_i\in C$ which are pairwise disjoint, and a morphism $f:C\to X$ satisfying $f_*([C])=\beta$ such that every irreducible component isomorphic to $\mathbb{P}^1$ which is mapped to a point in X contains at least 3 *special points*, i.e. markings or intersection points with other components, and every irreducible component of (geometric) genus 1 being mapped to a point contains at least one special point (*stability*). The second part of the stability condition is only necessary to exclude the case $g=1,n=0$, and $\beta=0$. Two stable n-marked maps $(C,x_1,\ldots,x_n,f)$ and $(C',x_1',\ldots,x_n',f')$ are *isomorphic* if there exists a scheme isomorphism $\varphi:C\to C'$ such that $\varphi(x_i)=x_i'$ and $f'\circ\varphi=f$. $\overline{M}_{g,n}(X,\beta)$ is the set of isomorphism classes of n-marked stable maps.

Remark 1.3 (**Properties of $\overline{M}_{g,n}(X,\beta)$**)
Let g, n, X, β be as above.

- The description as set of $\overline{M}_{g,n}(X,\beta)$ seems to be simple, but the proof of the following assertions is long and technical. $\overline{M}_{g,n}(X,\beta)$ exists as (compact) Deligne-Mumford stack/ projective coarse moduli space of expected dimension $-K_X \cdot \beta + (\dim X - 3)(1 - g) + n$ [FP96, theorem 1]. Note that in general this expected dimension does not coincide with the actual dimension. Compactness is necessary to build an intersection theory on $\overline{M}_{g,n}(X,\beta)$. For $g = 0$ and $n \geq 3$ $\overline{M}_{0,n}(X,\beta)$ is a smooth projective variety and a fine moduli space [FP96, theorem 2].

- If $X = \{pt\}$, and hence $\beta = 0$, we have $\overline{M}_{g,n}(\{pt\}, 0) = \overline{M}_{g,n}$, the moduli space of n-marked curves of (arithmetic) genus g. When $X \cong \mathbb{P}^r$ is a projective space, we simplify the notation by writing $\overline{M}_{g,n}(\mathbb{P}^r, d)$ instead of $\overline{M}_{g,n}(\mathbb{P}^r, d[\textit{line}])$.

- The open part $M_{g,n}(X,\beta)$ of $\overline{M}_{g,n}(X,\beta)$ consists of smooth n-marked stable maps. The boundary $\overline{M}_{g,n}(X,\beta) \setminus M_{g,n}(X,\beta)$ is a divisor with normal crossings [FP96, theorem 3].

Observe that the proofs of theorem 1-3 in [FP96] are for the special case of $g = 0$ and a convex variety X, as e.g. $\mathbb{P}^2$. The generalization to non-convex varieties needs virtual fundamental classes and is highly non-trivial (see below). For $g = 0$ and a convex variety the fundamental class coincides with the virtual class [HKK+03, chapter 26.1.].

Remark 1.4 (**The moduli space $\overline{M}_{g,n}$**)
A related, somewhat simpler object is the moduli space of n-marked stable curves of genus g. This space is made to classify abstract projective curves and will be needed in section 2.2. An *n-marked stable curve of genus g* is a collection $(C, x_1, \ldots, x_n)$ of a projective nodal connected curve C of arithmetic genus g, smooth points $x_i \in C$ which are pairwise disjoint, such that every rational irreducible component of C contains at least 3 *special points*, i.e. markings or intersection points with other components, and every irreducible component of (geometric) genus 1 contains at least one special point. Two stable n-marked curves $(C, x_1, \ldots, x_n)$ and $(C', x'_1, \ldots, x'_n)$ are *isomorphic* if there exists a scheme isomorphism $\varphi : C \to C'$ such that $\varphi(x_i) = x'_i$. $\overline{M}_{g,n}$ is the set of isomorphism classes of n-marked stable curves. Hence, $\overline{M}_{g,n}$ can be seen as $\overline{M}_{g,n}(X,\beta)$ but forgetting the information of the variety X and the map f. Its properties are similar to the ones of remark 1.3.

Definition 1.5 (**Evaluation maps, (algebraic) Gromov-Witten numbers**)
We define the i-th evaluation morphism $\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \to X$ as $(C, x_1, \ldots, x_n, f) \mapsto f(x_i)$. On the cohomology ring $H^*(\overline{M}_{g,n}(X,\beta), \mathbb{Z})$ one can define an intersection product $\cap$. Choose cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{Z})$, e.g. classes of points, which can be considered as conditions that the curves we want to count have to satisfy, such that the following intersection product, the (algebraic) Gromov-Witten invariant w.r.t. $g, n, X, \beta, \gamma_1, \ldots, \gamma_n$, is 0-dimensional:

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cap \ldots \cap \mathrm{ev}_n^*(\gamma_n),$$

where $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$ denotes the *virtual class of $\overline{M}_{g,n}(X,\beta)$* as in [BM96]. Its introduction is necessary as for most varieties X the actual dimension of $\overline{M}_{g,n}(X,\beta)$ is bigger than the expected one. The crucial point then is to interpret this intersection product correctly. A priori, it does not give the number of curves in X satisfying the conditions $\gamma_1, \ldots, \gamma_n$. The reason is that we have to choose a compactification $\overline{M}_{g,n}(X,\beta)$ of $M_{g,n}(X,\beta)$ in order to do intersection theory. But components of $\overline{M}_{g,n}$, often in the boundary $\overline{M}_{g,n}(X,\beta) \setminus M_{g,n}(X,\beta)$, can have too high dimension and may give contributions that we do not want. A general strategy to find enumerative numbers is as follows:

- find a suitable moduli space M (in general open) for your enumerative problem that parametrizes the objects that you want to count
- compactify this space in order to be able to apply intersection theory
- each condition of your problem corresponds to a subspace of your moduli space, e.g. a point condition for curves on a surface, cuts out a hypersurface in the moduli space
- the intersection of these subspaces corresponds to curves satisfying the conditions
- analyse this intersection product and correct by removing contributions that come from components of excessive dimension.

For instance, when M is the space of conics $\subset \mathbb{P}^2$ passing through 5 generic points in the plane, each conic is parametrized by its coefficients $(a_0 : \ldots : a_5) \in \mathbb{P}^5$. Its natural compactification is $\mathbb{P}^5$. The 5 points correspond to the intersection of 5 independent hyperplanes of degree 1. The intersection of the hyperplanes is zero-dimensional as the points are generic, so there is an unique solution. The boundary $\mathbb{P}^5 \setminus M$ contains only reducible conics, i.e. pairs of lines and double lines which cannot pass through 5 generic points, so this solution lies in the open dense part M. Note that the boundary is necessary and has an interpretation as set of degenerations of curves in M.
More generally, for moduli spaces $M = M_{g,n}(X, \beta)$ where X is a homogeneous space like $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ ([FP96, lemma 14]) or the blow-up of $\mathbb{P}^2$ in up to 3 points ([GP98, theorem 4.1]) this intersection number equals the corresponding GW invariant, and therefore the contributions at the boundary are not relevant.

So, for $n = 3d - 1 + g$, $X = \mathbb{P}^2$ and choosing the γ_i as cohomology classes of points in $\mathbb{P}^2$ in a generic point configuration we obtain:

$$N(d, g) = \int_{[\overline{M}_{g,n}(\mathbb{P}^2, d)]^{\text{vir}}} \text{ev}_1^*(\gamma_1) \cap \ldots \cap \text{ev}_n^*(\gamma_n).$$

Remark 1.6 (**Symplectic Gromov-Witten numbers**)
It is also naturally possible to define Gromov-Witten numbers via symplectic geometry [Rua96], [RT95]. Nevertheless, it has been proven that the algebraic and symplectic numbers coincide for any complex projective manifold [Sie99].

Definition 1.7 (**Psi-classes** ψ_i)
The Psi-class ψ_i is the (first) Chern class of the i-th cotangent line bundle to a given stable curve/map in $\overline{M}_{g,n}/\overline{M}_{g,n}(X, \beta)$. Considering a Psi-class as a condition that plane n-marked curves can satisfy, this condition can be interpreted as tangency condition at the marked points in most cases.

Remark 1.8 (**Characteristic numbers**)
Another type of conditions for an enumerative problem are tangency conditions. The condition for a plane curve of being tangent to a line in $\mathbb{P}^2$ is a condition of codimension 1, i.e. is a divisor in $\overline{M}_{g,3d-1+g}(\mathbb{P}^2, d)$. When we ask for the number of plane curves of degree d and genus g passing through a points and being tangent to b lines such that $3d - 1 + g = a + b$, this number is a *characteristic number*. Some, but not all of these characteristic numbers can be computed as appropriate intersection products, e.g. [KV07, section 3.6.2].

The moduli space language is useful for producing recursive formulas for the numbers $N(d, g)$ which makes it easier to compute these numbers.

Remark 1.9 (**Computing Gromov-Witten invariants**)
A formula for the numbers $N(d, 0)$ was found by Maxim Kontsevich [KM94]. He considers the forgetful map $\pi : \overline{M}_{0,n}(\mathbb{P}^2, d) \to \overline{M}_{0,4}$. As $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$, all points in $\overline{M}_{0,4}$ are linear equivalent as divisors. After choosing two points he pulls them back along π, intersects

them appropriately and compares their so obtained 0-dimensional intersection products. In this way, he finds linear relations between boundary components of $\overline{M}_{0,n}(\mathbb{P}^2, d)$.
Lucia Caporaso and Joe Harris gave in [CH98] a general approach to compute the numbers $N(d,g)$. The main idea here is to specialize the markings x_i sucessively to lie on a line. When we do so, the number of curves satisfying the conditions remains the same, but it can happen that the curves degenerate and split off components. Studying systematically how points can specialize and counting the degenerated curves instead yields a recursive formula.
Note that these formulas and their proof require a large machinery of involved tools of algebraic geometry like stacks, etc. Fortunately, the proofs become low-tech in the framework of tropical geometry [GM08, GM07a].

In [CH98] Caporaso and Harris have to make use of generalized Gromov-Witten invariants for their formula, for which they consider curves satisfying point and tancency conditions to a given line. Note that these numbers are equal in some cases to characteristic numbers, see 1.8.

Definition 1.10 (**Relative Gromov-Witten numbers**)
[GM07a, definition 2.2]. A *(finite) sequence* is a collection $\alpha = (\alpha_1, \alpha_2, ...)$ of natural numbers almost all of which are zero. Let $d \geq 0$ and g be integers, and let α and β be two sequences with $I\alpha + I\beta = d$, where $I\alpha = 1\alpha_1 + 2\alpha_2 + 3\alpha + \ldots$. Choose a fixed line $L \subset \mathbb{P}^2$. The so-called *relative Gromov-Witten number* $N^{\alpha,\beta}(d,g)$ (of the plane) is defined as the number of stable maps $(C, x_1, \ldots, x_n, f)$ of genus g with $X = \mathbb{P}^2$ and with homology class d[line] such that $f(C)$

- intersects L in α_i fixed general points of L with intersection multiplicity i each for all $i \geq 1$,
- intersects L in β_i more arbitrary points of L with intersection multiplicity i each for all $i \geq 1$, and
- passes in addition through $2d+g+|\beta|-1$ more general points in $\mathbb{P}^2$, where $|\beta| = \sum_i \beta_i$.

In the case where α is the zero sequence and $\beta = (d, 0, \ldots)$ this number is the usual Gromov-Witten number $N(d,g)$. These numbers are also invariant as proven in [CH98] using generalized Severi varieties. The formula given there is a recursive formula involving relative Gromov-Witten numbers, in particular the numbers $N(d,g)$. The formula of [CH98] also works for other surfaces than $\mathbb{P}^2$.

1.3 Real enumerative geometry

Definition 1.11 (**Real curve, real structure**)
Let C be a (-n irreducible, but not necessary smooth) complex projective curve, $\varphi : C \hookrightarrow \mathbb{P}^N$ a closed embedding for some N and its complex locus (w.r.t. φ) $C_{\mathbb{C}} = \varphi(C)$. The map $\tau_C : C_{\mathbb{C}} \to \mathbb{P}^N,\ [p_0 : \ldots : p_N] \mapsto [\overline{p_0} : \ldots : \overline{p_N}]$ is induced by complex conjugation on $\mathbb{C}$ (and depends on φ). An *antiholomorphic map* $\sigma : C_{\mathbb{C}} \to C_{\mathbb{C}}$ is a map such that $\tau_C \circ \sigma$ is holomorphic. σ is called *antiholomorphic involution* if in addition it holds $\sigma^2 = \mathrm{id}_{C_{\mathbb{C}}}$. In this case, σ is said to be a *real structure on* C and the pair (C, σ) is called a *real (projective) curve*. The *real part/locus* of C (w.r.t. φ and σ) is defined as the set of fixed points of C under σ and is written as $\mathbb{R}C$. If $\sigma = \tau_C$, then $\mathbb{R}C$ is contained in the real projective space $\mathbb{P}^N_{\mathbb{R}}$ and $C_{\mathbb{C}}$ is defined by a real homogeneous polynomial. In general we have for each real curve (C, σ) a closed embedding of C into $\mathbb{P}^N$ such that the real structure is τ_C w.r.t. that embedding [Har06, Exercise II 4.7 (a)]. Therefore, a real curve can always be described by a real homogeneous polynomial. $C_{\mathbb{C}}$ can carry several real structures. Two real structures σ, σ' on C are *equivalent* if there exists a $\mathbb{C}$-automorphism g of $C_{\mathbb{C}}$ such that $\sigma = g^{-1}\sigma' g$.
Of course, these definitions can be generalized to projective varieties.

Example 1.12 (**Real structures**)
On $\mathbb{P}^1$ we have two non-equivalent real structures: the *standard real structure* $\tau_{\mathbb{P}^1}$ and the structure given by $[z_1 : z_2] \mapsto [-\overline{z_1} : -\overline{z_2}]$. In the second case, we have $\mathbb{RP}^1 = \emptyset$. In general, $\mathbb{P}^n$ has two (non-equivalent) real structures if n is odd and one real structure, i.e. $\tau_{\mathbb{P}^n}$, if n is even [DIK00, remark 6.11.9.].

By abuse of notation, a *real plane curve* $C \subset \mathbb{P}^2$ is in the following given by a homogeneous polynomial $f \in \mathbb{R}[x, y, z]$.
The *degree of a real plane curve* is the degree the defining polynomial f and its *(geometric) genus* is the one of the corresponding complex plane curve.

Remark 1.13 (**Generic points for a real plane curve**)
If we want to play the enumerative game for real plane curves of degree d and genus g, we first observe that we have two choices of point types in the point conditions of our enumerative problem, namely real points and pairs of complex conjugate points ($f(z) = 0 \Rightarrow f(\overline{z}) = 0$). The space of real curves of degree d has (real) dimension $\frac{d(d+3)}{3}$ (argumentation as for complex curves, see 1.1). Each real point imposes one condition and each pair of complex conjugate points two. Hence the number r of real points and the number s of pairs of complex conjugate points should satisfy $r + 2s = 3d - 1 + g$ to obtain a 0-dimensional space of real curves of degree d and genus g satisfying the point conditions. It turns out when we count such curves through $3d - 1 + g$ generic points, the count depends on the position of the points and not only on the numbers r and s! A first example of this fact was given in 2000 by Degtyarev and Kharlamov:

Example 1.14 (**The number of real rational cubics through** 8 **real points**)
See [DK00, proposition 4.7.3] or [Sot11, theorem 9.4]. Here, the argument is similar to example 1.1 and that is why we stick to the same notation. Instead of 9 points in $\mathbb{P}^2$ we consider 9 real points p_i such that 8 of them are generic and blow up $\mathbb{P}^2_{\mathbb{R}}$ in these nine points. The surface that we obtain is contained in some $\mathbb{P}^M_{\mathbb{R}}$. When we compute the Euler characteristic of S via π we have to replace each p_i by $\mathbb{P}^1_{\mathbb{R}}$ in S. Hence we get:

$$\chi(S) = \chi(\mathbb{P}^2_{\mathbb{R}} \setminus \omega) + 9\chi(\mathbb{P}^1_{\mathbb{R}}) = 1 - 9 + 0 = -8. \tag{1.3}$$

Making use of the map t to compute $\chi(S)$ we should figure out which types of real curves lie in S over the regular and the singular fiber F_{sing}. Smooth real cubics have one or two topological components homeomorphic to $\mathbb{P}^1_{\mathbb{R}}$ and hence their Euler characteristic is zero. For F_{sing} we will have a closer look to the real locus $\mathbb{R}C$ of a cubic C lying in the complex pencil and over the (complex) fiber F_{sing}. There are three types of nodes:

- a node that can be seen in $\mathbb{R}C$ as crossing of two complex lines in $\mathbb{P}^2$ intersecting in a real point. This is called an *isolated node* and denoted by A_1^+ by Arnold's classification of simple singularities [AGZV88].
- it looks like the union of two real lines (*real node*), written as A_1^-.
- the last option is the empty set in $\mathbb{R}C$. This is the case when we have a *pair of complex conjugate isolated nodes* and occurs only for reducible cubics.

The following pictures are in the affine chart $\mathbb{R}^2$ of $\mathbb{P}^2_{\mathbb{R}}$. The nodes there can be described locally by the equations indicated.

$x^2 + y^2 = 0$ isolated node | $x^2 - y^2 = 0$ real node | conjugate nodes

We denote by i the number of curves with an isolated node and by n the number of curves with a real node. A curve with an isolated node has Euler characteristic 1 while the one of a cubic with a real node is -1 (union of two circles). By the result of example 1.1 we have $i+n \leq 12$. Proceeding as in this example yields:

$$\chi(S) = \chi(t^{-1}(\mathbb{P}^1_{\mathbb{R}} \setminus F_{\text{sing}})) + \chi(t^{-1}(F_{\text{sing}})) = \chi(\mathbb{P}^1_{\mathbb{R}} \setminus F_{\text{sing}}) \cdot \chi(\text{smooth cubic}) \\ + i \cdot \chi(\text{cubic with sol. node}) + n \cdot \chi(\text{cubic with real node}) = i - n. \tag{1.4}$$

By the computation of $\chi(S)$ via π, we have $i - n = -8$. Of course it holds $i, n \geq 0$. The solutions of this system of (in-)equalties then are $(i,n) \in \{(0,8),(1,9),(2,10)\}$.
Hence the number of real rational cubics through 8 generic real points can be $8, 10$ or 12. All these cases are realizable, for instance if we choose a pencil generated by two cubics with an isolated node each, then we have 12 real curves over F_{sing}.

What is going on here? The reason for this varying number is that the different types of nodes can be deformed to others when the generic points are moved, e.g. an isolated node can easily become a pair of complex conjugate nodes. The solution of this problem is the study of deformations of real curves and the conclusion that each curve should be counted with a sign to obtain an invariant. This is the work of Yves-Jean Welschinger in the framework of symplectic geometry. He considers in [Wel03, Wel05a] real rational symplectic manifolds of dimension 4 and counts real rational J-holomorphic curves in a given homology class. In the case of $\mathbb{P}^2$ or more general in the case of del Pezzo surfaces (for a definition see 1.19) the count of these J-holomorphic curves gives the same as the count of real rational curves, whose underlying complex curves are irreducible. This is due to the fact that the complex structure is generically symplectic for these surfaces [Wel03, corollary 2.3], [Rua93, theorem 7.1], which also makes sure that the number of curves is finite, [Wel03, theorem 1.11]. This celebrated result has to be considered as a milestone in real enumerative geometry which has ever since attracted much attention.

Definition 1.15 (**Welschinger invariants for $\mathbb{P}^2$** [Wel03, Wel05a])
Let $C \subset \mathbb{P}^2$ be a real nodal curve of degree d whose underlying complex curve is irreducible. The *mass of C* is defined as $m(C) := \#$ of isolated nodes in C. Fix a point configuration ω of r real and s pairs of complex conjugate points such that $\#\omega = 3d-1 = r+2s$ and the points in ω are in general position. Then the *Welschinger number w.r.t. d, g and ω* is

$$W(d,g,\omega,r,s) = \sum_{C \text{ through } \omega} (-1)^{m(C)}.$$

$(-1)^{m(C)}$ is called the *Welschinger sign of C*.

Theorem 1.16 (**Invariance of $W(d,0,\omega,r,s)$ for $\mathbb{P}^2$** [Wel03, Wel05a])
The number $W(d,0,\omega,r,s)$ does not depend on the actual configuration ω as long as the points in ω are generic.

We will therefore write $W(d,r,s)$ for the *Welschinger invariant*.

Remark 1.17 (**Natural bounds of Welschinger invariants**)
Let $\mathbb{R}C(d,0,\omega,r,s)$ be the number of real rational nodal curves of degree d whose underlying complex curves are irreducible and pass through a generic point configuration ω with $r+2s = 3d-1$ points. Then we have:

$$|W(d,r,s)| \leq \#\mathbb{R}C(d,0,\omega,r,s) \leq N(d,0).$$

Sketch of proof of thm 1.16. We stick to [IKS03a]. Consider configurations ω of r real points and s pairs of complex conjugate points in general position such that $r+2s = 3d-1 = m$. Order the points in each ω as $(p_1,\ldots,p_r,p_{r+1},\overline{p_{r+1}},\ldots,p_{r+s},\overline{p_{r+s}}) \in (\mathbb{P}^2)^m$. Define a real

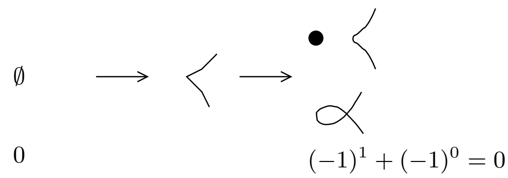
structure c on $(\mathbb{P}^2)^m$ by setting $c(x_1,\dots,x_r,x_{r+1},\dots,x_{r+2s}) = (\overline{x_1},\dots,\overline{x_r},\overline{x_{r+2}},\overline{x_{r+1}},\dots,\overline{x_{r+2s}},\overline{x_{r+2s-1}})$. This structure c is made such that the real part $\mathbb{R}(\mathbb{P}^2)^m$ of $(\mathbb{P}^2)^m$ w.r.t. c consists of all configurations ω as above. Observe that every generic configuration $\omega \in \mathbb{R}(\mathbb{P}^2)^m$ is also generic in $(\mathbb{P}^2)^m$. The idea of proof is that we choose a real point (possible if $r > 0$) or a pair of complex conjugate points (if $s > 0$) that we move continuously in $\mathbb{R}(\mathbb{P}^2)^m$ (while keeping the other points in a generic configuration fixed) in such a way that the configuration remains generic and show that the Welschinger number keeps the same along this move.
To make this more precise, we start with the move of a real point, say p_r of a generic ω. Consider the configuration ω' that we get when omitting p_r in ω. Let $\mathbb{R}\overline{V(\omega')}$ be the real part of the closure of the smooth projective variety of complex irreducible rational nodal curves of degree d passing through ω' w.r.t. the standard real structure. $\mathbb{R}\overline{V(\omega')}$ then contains real curves of degree d through ω'. The linear system $\Gamma(p_r)$ on $\mathbb{R}\overline{V(\omega')}$ is the subset of curves passing through ω. When we move p_r in $\mathbb{P}^2_{\mathbb{R}}$ also $\Gamma(p_r)$ changes. Define a projection $\pi : \mathbb{R}V(\omega') \to \mathbb{P}^2_{\mathbb{R}}$ by $\pi^{-1}(p) = \Gamma(p)$. It can be shown that $W(d,0,\omega,r,s)$ may only change (as long as ω remains generic)

a) in points of $\Gamma(p_r) \cap \mathbb{R}\overline{V(\omega')} \setminus \mathbb{R}V(\omega')$,

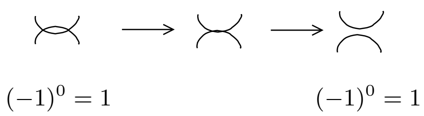
b) when p_r crosses a node (not in ω') of a curve in $\mathbb{R}V(\omega')$.

For a) let us check case by case the invariance of $W(d,0,\omega,r,s)$. A curve $C \in \mathbb{R}\overline{V(\omega')}\setminus\mathbb{R}V(\omega')$ can have:

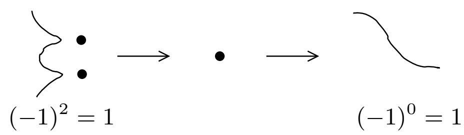
- nodes and one cusp (singularity of type A_2) at a point p_0. For a neighborhood $U(C)$ of C in $\mathbb{R}\overline{V(\omega')}$ the intersection $U(C) \cap \pi^{-1}(p)$ is either the empty curve or consists of two points, namely a curve C_1 having the cusp replaced by an isolated node and a curve C_2 with a real node instead of the cusp. The claim then follows with $m(C_1) = 1$ and $m(C_2) = 0$.



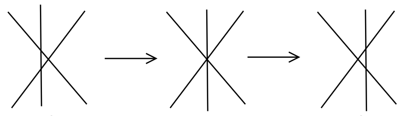
- nodes and only one tacnode with 2 branches (A_3^-) at a point p_0. Then $U(C)\cap\pi^{-1}(p)$ is a point corresponding either to a curve where the tacnode is replaced by two real nodes or a curve where the tacnode is replaced by a part without singularities:



- nodes and only one isolated tacnode (A_3^+) at p_0. In this case $U(C) \cap \pi^{-1}(p)$ is a point corresponding either to a curve where the tacnode is replaced by two isolated nodes or a curve where the tacnode is replaced by a part without singularities.

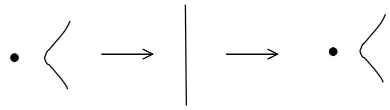


- $p_0 \in C \in \mathbb{R}\overline{V(\omega')} \setminus \mathbb{R}V(\omega')$ can be also of type D_4^-, a singularity consisting of three branches (again, all the other singularities of C are nodes). Then $U(C) \cap \pi^{-1}(p)$ is a point which corresponds to a curve with three real nodes instead of the D_4^- singularity.



$(-1)^0 = 1$ $\qquad$ $(-1)^0 = 1$

- the singularity in p_0 may be of type D_4^+, i.e. may consist of a line. Here $U(C) \cap \pi^{-1}(p)$ is a point which corresponds to a curve with one isolated node instead of the D_4^+ singularity.



$(-1)^1 = -1$ $\qquad$ $(-1)^1 = -1$

- there are two more types appearing: a reduced reducible curve consisting of two rational nodal curves intersecting transversely and not at nodes, or a rational nodal irreducible curve where one node is at a point of ω'. Also in these cases $W(d, 0, \omega, r, s)$ remains invariant.

Concerning b): if the node at p_r of a curve $C \in \mathbb{R}V(\omega')$ is isolated, then $U(C) \cap \pi^{-1}(p) \setminus \{C\}$ is empty. On the other hand, when the node at p_r is real, $U(C) \cap \pi^{-1}(p)$ contains always two points when p_r leaves the position of the node. If we move p_r, say on a line, the two curves appearing before crossing and after crossing the node are real regular homotopy equivalent and hence $W(d, 0, \omega, r, s)$ also does not change.

When we move a pair of complex conjugate points, the cases we have to consider are those of a) and the argument is similar to that one above. □

Unfortunately, this result is only true for $g = 0$.

Theorem 1.18 (**Non-Invariance of** $W(d, 1, \omega, r, s)$ **for** $\mathbb{P}^2$**, [IKS03b, theorem 3.1]**)
Let $d \geq 4$. The number $W(d, 1, \omega, r, s)$ depends on the actual configuration ω and does not yield an invariant.

Proof. Comparing with the proof of theorem 1.16 the problem occurs when considering case b). When we move p_r away from the real node of a real elliptic curve $C \in \mathbb{R}V(\omega')$ on e.g. a line, then we get on one side the empty set and on the other two points which correspond to real nodal elliptic curves with the same number of isolated nodes. □

As already noted, the concept of Welschinger numbers can be generalized to other surfaces. Welschinger numbers are originally defined as signed numbers of real rational J-holomorphic curves, so the properties of the surface should be such that this count gives the same as the count of real rational projective curves. As J-holomorphic curves are in some sense limits of complex projective curves, some curves can split off when forming the limit. This is the case when the surface contains $(-n)$-curves for $n \geq 2$.

Definition 1.19 (($-n$)**-curves, unnodal real Del Pezzo surface, toric unnodal Del Pezzo surface**)
Let S be a smooth projective irreducible surface over $\mathbb{C}$.
For $n \in \mathbb{N}_{>0}$ a $(-n)$*-curve (on S) or exceptional curve* is an irreducible smooth rational curve with self-intersection number $-n$ [DIK00, 6.1.4.]. A (-1)-curve can be obtained by blowing-up S in a point p. Blowing-up $\mathrm{Bl}_p(S)$ in a point of the exceptional curve yields an (-2)-curve, etc. In the other direction, the Castelnuovo-Grauert criterion [DIK00, 6.2.1] says that every (-1)-curve in a surface can be blown-down to a nonsingular surface.
S is called *unnodal Del Pezzo surface* if its anticanonical divisor $-K_S$ is ample. The *degree of an unnodal Del Pezzo surface* is $(-K_S)^2$. If S is an unnodal Del Pezzo of degree d, then it is one of the following surfaces [DIK00, 6.12.1.]:

- $\mathbb{P}^2$ for $d = 9$,
- $\mathbb{P}^1 \times \mathbb{P}^1$ for $d = 8$,
- for $d \in \{1, \dots, 7\}$ the blow-up $\mathrm{Bl}_{p_1,\dots,p_k}(\mathbb{P}^2)$, where $k = 9 - d$ and the k points are distinct and in general position.

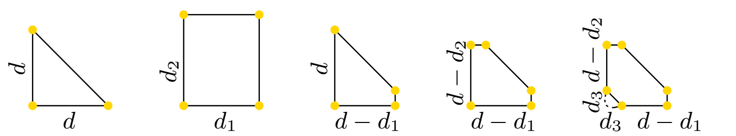
In particular, they are rational surfaces. It can be shown that the previous definition of unnodal Del Pezzo surface is equivalent to the fact that S does not contain $(-n)$-curves for $n \geq 2$ or equivalently for every effective divisor D on S it holds $D \cdot K_S < 0$ [DIK00, 6.12.2.]. It is also equivalent to say that S is 2-dimensional Fano variety.
An *unnodal real Del Pezzo surface* is a unnodal Del Pezzo surface equipped with a real structure. Up to equivalence $\mathbb{P}^1 \times \mathbb{P}^1$ has 4 real structures [DIK00, 6.11.7.], and $\mathrm{Bl}_{p_1,\dots,p_k}(\mathbb{P}^2)$ supports in general also several real structures. Note that their real part is not always connected, i.e. this can happen for $d \in \{1, \dots, 4\}$ and a bad choice of real structure, see [DIK00, 17.3].
If S is an unnodal Del Pezzo surface and the equivariant compactification of a 2-dimensional torus $T_{\mathbb{Z}^2} = (\mathbb{C}^*)^2$, then it is called *toric*. All unnodal Del Pezzo surfaces are toric for $d \geq 6$ [CLS11, example 8.3.7. and 10.5.8.], i.e. there are in total 5 toric unnodal Del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up in up to 3 points.

Remark 1.20 (**Toric surfaces**)
(Projective smooth) toric surfaces S have a nice combinatorial counterpart, namely (convex) lattice polygons. They can be used to study and express properties of the toric variety in terms of combinatorics as for instance intersection numbers of torus-invariant divisors. Let $M = \{m_1, \dots, m_s\} \subset \mathbb{Z}^2$ denote the set of lattice points of a lattice polygon P in $\mathbb{R}^2$. Then P gives rise to a toric variety S_P which is the Zariski closure $\overline{\Phi(M)}$ of the image of M under the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{P}^{s-1}$, $t \mapsto [t^{m_1} : \ldots : t^{m_s}]$. For instance, denoting the 2-dimensional standard simplex Δ_2, Φ is the d-th Veronese embedding for $P = d\Delta_2$ of $S_P = \mathbb{P}^2$. Furthermore, each facet F of P comes with a unique supporting hyperplane $H_F = \{m \in \mathbb{R}^2 | \langle m, u_F \rangle = -a_F\}$ where $u_F \in \mathbb{Z}^2$ is the inward pointing facet normal of F and unique $a_F \in \mathbb{Z}$. We then can associate to S_P and each facet F the hyperplane section $\Phi(H_F) \cap S_P$, which is a prime divisor D_F in S_P. Then $D_P = \sum_{F \text{ in } P} a_F D_F$ is a divisor on S_P.

Example 1.21
[CLS11, example 10.5.9]. Let (e_1, e_2) be the standard basis of $\mathbb{R}^2$. $P_1 = \Delta_2 = \mathrm{Conv}(0, e_1, e_2)$ yields the toric variety $S_{P_1} \cong \mathbb{P}^2$ embedded in $\mathbb{P}^2$ together with hyperplane sections $\Phi(H_F) \cap S_{P_1}$ which are lines and linearly equivalent to the divisor of a line ℓ of $\mathbb{P}^2$. For $P_2 = 2\Delta_2 = \mathrm{Conv}(0, 2e_1, 2e_2)$ the toric variety S_{P_2} is also $\mathbb{P}^2$, but this time embedded into $\mathbb{P}^5$. The hyperplane section $\Phi(H_F) \cap S_{P_2} \subset \mathbb{P}^2$ is a curve of degree 2 in $\mathbb{P}^2$ for each facet F, which as divisor is linearly equivalent to 2ℓ .

Remark 1.22 (**Tautological linear system**)
[IMS09, section 2.2.4]. For toric surfaces, Φ is always an embedding. This means that the linear system $|D_P|$ on S_P generated by the monomials t^i, $i \in P \cap \mathbb{Z}^2$, is very ample. It is called a *tautological linear system* on S_P. The divisor class of each hyperplane section in example 1.21 is such a tautological linear system on S_{P_i}.

The lattice polygons P of the toric unnodal Del Pezzo surfaces are depicted below:



$\mathbb{P}^2$ $\quad$ $\mathbb{P}^1 \times \mathbb{P}^1$ $\quad$ $\mathrm{Bl}_{p_1}(\mathbb{P}^2)$ $\quad$ $\mathrm{Bl}_{p_1,p_2}(\mathbb{P}^2)$ $\quad$ $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$

The explanation of the numbers d_i follows.

Even if it is enough to restrict from the point of view of Welschinger's proof to real unnodal Del Pezzo surfaces we will consider here only those which are also toric. The reason is that the Welschinger numbers of real toric unnodal Del Pezzo surfaces can be computed tropically.

Definition 1.23 (**Welschinger numbers for real toric Del Pezzo surfaces**)
Let S be $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown-up in up to 3 (generic) real points equipped with the standard real structure and let $|D|$ be the tautological linear system on S as in remark 1.22 w.r.t. to fixed toric coordinates. For $S = \mathbb{P}^2$ and D of degree d the polygon P is a triangle with lattice edge lengths d. If $S = \mathbb{P}^1 \times \mathbb{P}^1$ and D is of bi-degree (d_1, d_2), then P is the polygon with vertices $(0,0), (d_1, 0), (d_1, d_2), (0, d_2)$. Finally, if $S = \mathrm{Bl}_{p_1,\dots,p_k}(\mathbb{P}^2)$ and D is linear equivalent to $d\ell - \sum_{i=1}^k d_i E_i$ where ℓ denotes a line and E_i is the exceptional divisor of the blow-up at p_i, then P is the polygon as depicted above. Define $r(P)$ as the number of integer points on the boundary of P minus 1 and $\delta(P)$ as the number of interior lattice points of P. Then $r(P) = -K_S \cdot D - 1$ and $\delta(P)$ is the arithmetic genus of curves in the linear system $|D|$.
Let $0 \leq g \leq \delta(P)$ and ω be a generic point configuration of $-K_S \cdot D - 1 = r + 2s$ points lying in one connected component of $\mathbb{R}S$. Let C be a real nodal curve in S of genus g, lying in $|D|$, whose underlying complex curve is irreducible and which passes through ω. The number of isolated nodes in a such a curve C is denoted by $m(C)$ and by $(-1)^{m(C)}$ its *Welschinger sign*. The (weighted) number of all these curves is

$$W_S(g, \omega, r, s) = \sum_{C \text{ through } \omega} (-1)^{m(C)},$$

the *Welschinger number* w.r.t. S, g, ω, r and s.

Theorem 1.24 (**Invariance of $W_S(0, \omega, r, s)$ for real toric Del Pezzo surfaces**)
[Wel03, Wel05a]. The number $W_S(0, \omega, r, s)$ does not depend on the actual configuration ω as long as the points in ω remain generic.

Therefore, we will write $W_S(r, s)$ for them. In the case of $S = \mathbb{P}^2$ and when we consider curves of degree d we just write $W(d, r, s)$ as before.

Welschinger numbers for more general surfaces will be discussed in section 4.3.

Remark 1.25 (**Welschinger invariants as intersection products**)
A natural question is if it is possible to write Welschinger numbers as intersection numbers like the Gromov-Witten invariants $N(d, g)$. The first obstruction is to find the "right" moduli space for real curves or more precisely real stable maps. One can consider real isomorphisms of real curves, complex isomorphisms of real curves or the real locus of $\overline{M}_{0,n}(X, \beta)$ w.r.t. an suitable real structure, which gives different spaces [Sep89], [Sil92]. The latter approach was developed by Jake Solomon in his thesis [Sol06]. Based on the Teichmüller space description of $\mathbb{R}\overline{M}_{0,n}(X, \beta)$ due to Welschinger [Wel05b, section 1.1], which implies that $\mathbb{R}\overline{M}_{0,n}(X, \beta)$ is a real algebraic variety, he characterized Welschinger numbers als intersections [Sol06, theorem 1.8] on moduli spaces of J-holomorphic maps mapping to symplectic manifolds with boundary.

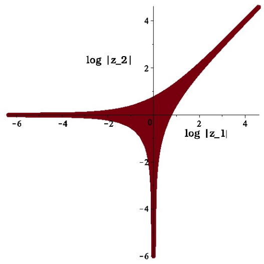
Remark 1.26 (**Computational aspects of Welschinger invariants**)
Welschinger himself computed the numbers $W(4, 1, 5) = 0$, $W(5, 0, 7) = 64$, $W(6, 1, 8) = 1024$, $W(7, 10, 0) = -14336$, $W(8, 1, 11) = -280576$ in [Wel07, corollaire 3.12]. Unfortunately, these are just computations of some special numbers, i.e. he doesn't give a general approach. More global results have been obtained in tropical geometry. Computations using lattice path algorithms were obtained in [Mik05], [IKS03b] and using floor diagrams in [BM08] for only real point configurations. For configurations containing also complex conjugate points the papers [Shu06b] and [ABLdM11] are helpful.

2 The state of the art: tropical curves and their moduli

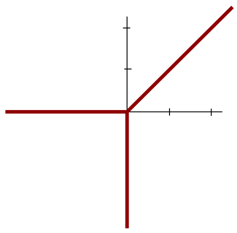
Tropical geometry is not that recent anymore: tracing back to late 80s [Sim88] it has grown rapidly since 2003 [RGST05, Mik05]. This is mainly due to the fact that people from different fields such as combinatorics, algebraic geometry and (computer) algebra worked hand in hand on problems. Since the topic is too large to expose it entirely we will have to restrict ourselves to certain aspects. For this text, it seems to be sufficient to present tropical curves from a graph-theoretical point of view keeping in mind that we are interested in counting algebraic curves and stressing this connection. We will also address their moduli spaces and what they are useful for.

2.1 Amoebas and a correspondence theorem

Consider a plane (complex) projective curve V in the torus $(\mathbb{C}^*)^2$. Applying to V the map $\text{Log}_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2$, $(z_1, z_2) \mapsto (-\frac{\log|z_1|}{\log t}, -\frac{\log|z_2|}{\log t})$ for $t = 1/\exp(1)$ and where $\log$ is the natural logarithm yields the so called *amoeba $\mathcal{A}(V)$ of V* [GKZ94, Section 6.1]. For instance, the following picture shows the amoeba of the variety $V(f)$ defined by $f(z_1, z_2) = z_1 + z_2 - 1$, i.e. we restrict $\text{Log} := \text{Log}_{1/\exp(1)}$ to $\{(z_1, -z_1 + 1)|z_1 \in \mathbb{C}\} \cap (\mathbb{C}^*)^2$. Note that the amoeba



only depends on the absolute value of z_1 and $-z_1 + 1$ here, e.g. when we replace f by $g(z_1, z_2) = -z_1 + z_2 - 1$ the amoeba is the same, $\mathcal{A}(V(f)) = \mathcal{A}(V(g))$. $\mathcal{A}(V(f))$ is closed in $\mathbb{R}^2$ since $V(f) \subset (\mathbb{C}^*)^2$ is closed and Log is a continuous, proper map. Amoebas have many interesting properties [GKZ94, The02, Mik04a], reflecting some properties of the curve. But still, they are hard to handle because of their analytic nature. For $t \to 0$ however, the amoeba gets thinner and converges to a piece-wise linear object [IMS09, theorem 1.4], which we call a *tropical line* and which is a special case of a *tropical curve*.



We now describe these tropical curves without using limits. Consider the field $K = \mathbb{C}\{\{t\}\}$ of *Puiseux series over* $\mathbb{C}$, whose non-trivial elements are formal power series of the form $p(t) = p_1 t^{a_1} + p_2 t^{a_2} + \ldots$ with $p_i \in \mathbb{C}^*$, $a_i \in \mathbb{Q}$ which have a common denominator and $a_1 < a_2 < \ldots$. It is the algebraic closure of the field $\mathbb{C}((t))$ of Laurent series over $\mathbb{C}$ [Rib99, p.

186] and has a natural non-archimedean valuation $v : K \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ given by $v(p) = a_1$ and $v(0) = \infty$. Note that v is not surjective. Let $f = \sum_i c_i \underline{x}^i \in K[x_1^{\pm}, x_2^{\pm}]$ be a non-trivial *Laurent* polynomial in 2 variables x_1, x_2 with coefficients c_i in K, and where $i = (i_1, i_2)$ is a multi-index. Define furthermore the *tropicalization of* f, denoted trop(f), which is the function $\mathbb{R}^2 \to \mathbb{R}$, $y = (y_1, y_2) \mapsto \max_i\{-v(c_i) + i \cdot y\}$. With this setting we can define two related objects

a) trop$(V(f)) = \{y \in \mathbb{R}^2 |$ trop$(f)(y)$ is achieved at least twice$\}$, and

b) the topological closure of the set $\{(-v(z_1), -v(z_2)) | \ (z_1, z_2) \in V(f)\}$ in $\mathbb{R}^2$ w.r.t. the Euclidean norm.

Kapranov's theorem published in [EKL06, theorem 2.1.1] states that the sets in a) and b) coincide. This implies that the set b) only depends on the valuation of the coefficients of f! More generally, the theorem holds for every non-archimedean valuation val on K. When val is surjective, we can leave out the "closure" in b). The set in b) can be seen as generalized amoeba, called *non-archimedean amoeba*. Indeed, given a non-archimedean valuation val on K one can consider the induced norm on K given by $|p| = \exp(-val(p))$ and $|0| = 0$. Then Log$((z_1, z_2)) = (\log|z_1|, \log|z_2|) = (-val(z_1), -val(z_2))$ for $(z_1, z_2) \in (K^*)^2$. Observe that for $val = v$ the non-archimedean amoeba equals the tropical curve of f as defined in the beginning, when allowing also zero sets in $(K^*)^2$ of Laurent polynomials. This follows from the fact that $\frac{\log|p(t)|}{\log t}$ equals $v(p(t))$ for small t and $p(t) \in K$ [Gat06, section 1.2]. There is also a third characterization of a tropical curve via initial forms, which is not relevant in this thesis and will be omitted here (see, for example, [MS11, definition 3.1.1]). The equivalence of the three characterizations for an irreducible variety $V(I)$ in $(K^*)^n$ is called *Fundamental Theorem* of tropical geometry [SS04, theorem 2.1]. Note that some people prefer to phrase tropicalizations with min instead of max, for instance in [MS11].

Example 2.1
For $f = z_1 + z_2 - 1$ the tropicalization of f is trop$(f)(y_1, y_2) = \max\{y_1, y_2, 0\}$. The maximum is achieved at least twice for $y_1 = y_2 \geq 0$, $0 = y_1 \geq y_2$ and $0 = y_2 \geq y_1$. The non-archimedean amoeba w.r.t. v consists of the closure of points which can be described as follows [MS11, example 3.1.4] using the inequality $v(a+b) \geq \min\{v(a), v(b)\}$ characterizing non-archimedean valuations:

$$(-v(z_1), -v(-z_1+1)) = \begin{cases} (-v(z_1), 0), & \text{if } v(z_1) > 0 \\ (-v(z_1), -v(z_1)), & \text{if } v(z_1) < 0 \\ (0, -a), & \text{if } v(z_1) = 0 \text{ and } z_1 = 1 + \alpha t^a + \tilde{z}_1 \\ & \ \text{with } v(\tilde{z}_1) > a > 0, \\ (0, 0), & \text{otherwise.} \end{cases}$$

Hence, the sets of a) and b) agree here. So this tropical line can be seen as the union of three half rays meeting in $(0, 0)$ with direction vectors $(-1, 0)$, $(1, 1)$ and $(0, -1)$. Note that these three vectors satisfy the *zero tension condition* or *balancing condition*, namely that their sum equals the zero vector in $\mathbb{R}^2$. This property is true for all tropical varieties [Spe05, theorem 2.5.1] and will be a feature of our combinatorial definition of tropical varieties 2.8, in particular parametrized tropical curves 2.23.

Given a plane (complex) projective curve V of degree d and of genus g in the torus $(\mathbb{C}^*)^2$ passing through a configuration ω of $3d-1+g$ points in general position in $(\mathbb{C}^*)^2$ one might be interested in tropicalizing this picture, i.e. one can consider trop$(V(f))$ and $(-v(p_1), -v(p_2)) \in \mathbb{R}^2$ for $p = (p_1, p_2) \in \omega$. Then one is tempted to ask questions like what is the number of tropical curves being the tropicalization of plane (complex) projective curves of degree d

and of genus g passing through the tropicalization of ω? Answering this question with the actual notion of tropical curve is very hard as we have always to pass through the process of tropicalization. The next section is devoted to the presentation of a more combinatorial definition of a tropical curve, which will be called parametrized tropical curve. However, the answer to the question above will given already here and is due to Grisha Mikhalkin [Mik05, theorem 1 in section 7.1].

Theorem 2.2 (**Mikhalkin's Correspondence Theorem (Version 1)**)
Given d and g and the tropicalization of a configuration ω of $3d-1+g$ points in $(\mathbb{C}^*)^2$ in general position, then the number of tropical curves trop(V) being the tropicalization of plane (complex) projective curves V of degree d and of genus g passing through the tropicalization of ω, counted each with a certain multiplicity mult(trop$(V)) \in \mathbb{N}_{>0}$, equals $N(d,g)$ and does not depend on ω. The multiplicity mult(trop(V)) will be defined in 2.35.

Note that – from the amoeba point of view from the beginning – it seems to be plausible that several classical curves are mapped to the same tropical curve.

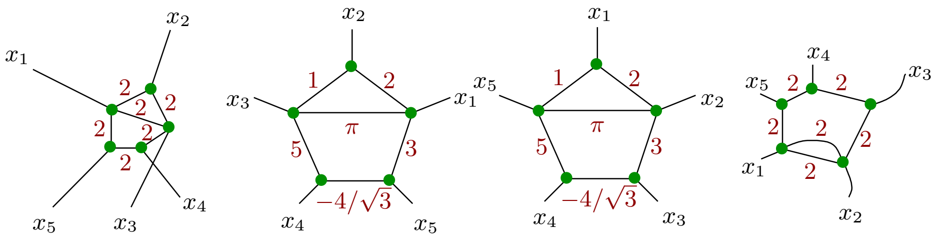
2.2 Abstract tropical curves and their moduli spaces

The notion of n-marked abstract tropical curve of given genus g, as well as that one of the corresponding moduli space $\mathcal{M}_{g,n}$, is due to Grisha Mikhalkin and has been studied in more detail by Andreas Gathmann, Michael Kerber and Hannah Markwig in [GM07b, GKM09, Mar06]. The dual bijection between cells in the classical moduli space $\overline{M}_{g,n}$ of n-marked stable curves of genus g and $\mathcal{M}_{g,n}$ is known since a while, but only proven explictly in [Cap11, theorem 4.7]. Lucia Caparaso uses there a modified notion of abstract tropical curve introduced in [BMV11], which is more useful for compactifying $\mathcal{M}_{g,n}$ nicely. But let us explain the necessary notions step by step. First, we give the definitions as in [GKM09], which differ slightly from those of Mikhalkin.

Definition 2.3 (**n-marked abstract tropical curve of genus g**)
A *metric graph* Γ is a graph with a function l on the set of its edges, taking values in $\mathbb{R}_{>0}\cup\{\infty\}$, assigning to each edge E its *length* $l(E)$. If $l(E)$ is finite, then E is called a *bounded edge*, otherwise an *unbounded edge*. Furthermore, a vertex V in Γ is called m-valent, if there are m edges being adjacent to V. We define an *n-marked (abstract) tropical curve* to be a tuple $(\Gamma; x_1, \ldots, x_n)$, where Γ is a connected metric graph with first Betti number g, i.e. the number of independent cycles in the graph, all of whose vertices are least 3-valent, and where $x_1, \ldots, x_n$ is a labeling of its unbounded edges, which we require to be ends of the graph. In the following, we will sometimes write *marked edge* x_i or *marked end* x_i for the unbounded edge with label x_i.

Definition 2.4 (**Combinatorial type of an abstract curve**)
The *combinatorial type* of an n-marked abstract tropical curve of genus g is just the curve, but forgetting the information about the edge lengths.

Example 2.5
In the following sequence of 5-marked abstract tropical curves of genus 2 vertices are depicted by green dots, i.e. the first graph has a crossing of edges which is not a vertex (this is because we have drawn the abstract graph in the plane). The first two curves have the same combinatorial type while the third and the fourth curve have a different one, respectively. The lengths of the bounded edges are pictured in red.



Remark 2.6
These curves are the tropical analogues of n-marked stable curves over $\mathbb{C}$ as defined in remark 1.4. In particular, this is motivated by the fact that the underlying graph Γ is not embedded and the stability condition is reflected by the condition that vertices are at least 3-valent. This correspondence will be studied more carefully in 2.21.

To describe tropical varieties combinatorially, we need the following basic definitions.

Definition 2.7 (**Polyhedral complex, fan**)
Pick some $r \geq 0$ and a lattice $\Lambda \cong \mathbb{Z}^r$; set $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

a) A *(closed) polyhedron* $\sigma \subset V$ *in* V is a set $\sigma = \{x \in V |\ f_1(x) = s_1, \ldots, f_n(x) = s_n, f_{n+1}(x) \geq s_{n+1}, \ldots, f_N(x) \geq s_N\}$ for some $N \in \mathbb{N}$, linear forms $f_i \in \Lambda^\vee$, and numbers $s_i \in \mathbb{R}$. If in addition $s_i = 0\ \forall i$, then σ is called a *(closed) cone in* V. $\tau \subset \sigma$ is called a *face of* σ, if τ can be described by replacing some (or none) inequalities in the characterization of σ by equalities. Alternatively, a cone can be written as $\sigma = \{\lambda_1 u_1 + \ldots + \lambda_n u_n |\ \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}\}$ with $u_i \in \Lambda$ and we then say that σ is *spanned by* $u_1, \ldots, u_n$.

b) The vector subspace of V spanned by a polyhedron moved to the origin/cone σ in V will be denoted V_σ and its lattice by $\Lambda_\sigma = \Lambda \cap V_\sigma$.

c) The *relative interior* $\overset{\circ}{\sigma}$ of a polyhedron/cone σ in V is the interior of σ in V_σ.

d) An *(abstract) polyhedral complex* is a topological space X, which is a finite disjoint union $\sqcup \varphi_i(\overset{\circ}{\sigma}_i)$ of images of injective maps $\varphi_i : \sigma_i \to X$ of the relative interior of convex polyhedra σ_i in V_i satisfying

   - the intersection $\varphi_i(\sigma_i) \cap \varphi_j(\sigma_j)$ is a closed set of X for all i, j,
   - the transition maps $\varphi_i^{-1} \circ \varphi_j$ are affine linear for $i \neq j$ where they are defined.

   The polyhedra σ_i are called the *cells* of X.

e) A *fan* in V is a finite set X of cones σ_i in V with the following properties

   - each face of a cone in X is in X and
   - the intersection $\sigma_i \cap \sigma_j$ of two cones σ_i and σ_j is a face of σ_i and σ_j.

   The cones in X are also called the *cells of* X.

Definition 2.8 (**Pure dimension of a polyhedral complex/fan, simplicial and tropical fans**)
A polyhedral complex or fan X is of *pure dimension* N if the dimension dim V_σ of each maximal cone σ (w.r.t. inclusion) is N.
Let now X be a fan in V. A cone σ is called *simplicial*, if it is spanned by linear independent vectors u_i. We say that X is *simplicial*, if all cones of X are simplicial.
We assume now that X is of pure dimension N. For $0 \leq k \leq N$ we denote the collection

of its k-dimensional cones by $X^{(k)}$. We call X a *tropical fan* if it is equipped with a weight function $w : X^{(N)} \to \mathbb{N}$ such that the *balancing condition*

$$\sum_{\sigma > \tau} w(\sigma)\, v_{\sigma/\tau} = 0 \quad \in V/V_\tau$$

holds for all $(N-1)$-dimensional cones τ, where $v_{\sigma/\tau}$ is the primitive outer normal vector of σ relative to τ. The union of all cones of X will be written as $|X| \subset V$.

Definition 2.9 (**Morphism of polyhedral complexes, morphism of fans**)
Let the notation be as above.

a) A *morphism* $f : X \to Y$ *of polyhedral complexes* X *and* Y is a continuous map f such that for each cell σ_i the image $f(\sigma_i)$ is contained in only one cell of Y , and $f|_{\sigma_i}$ is an affine linear map.

b) A *morphism* $f : X \to Y$ *of fans* X *and* Y is a $\mathbb{Z}$-linear map, i.e. a map $\tilde{f} : |X| \to |Y|$ induced by a $\mathbb{Z}$-linear map between the underlying lattices Λ_X and Λ_Y.

Moduli spaces of abstract tropical curves of genus 0 are well-known and easy to characterize, which is not the case for higher genus.

Definition 2.10 (**Moduli space $\mathcal{M}_{0,n}$ of rational n-marked abstract tropical curves**)
An *isomorphism* of rational n-marked tropical curves $(\Gamma; x_1, \ldots, x_n)$ and $(\tilde{\Gamma}; \tilde{x}_1, \ldots, \tilde{x}_n)$ is a homeomorphism $\Gamma \to \tilde{\Gamma}$ sending x_i to $\tilde{x}_i$ and mapping edges of Γ bijectively onto edges of $\tilde{\Gamma}$ by affine maps of slope ± 1. As in [Mik07] and [GKM09], we will denote by $\mathcal{M}_{0,n}$ the parameter space of all rational n-marked tropical curves modulo isomorphisms.
The moduli space $\mathcal{M}_{0,n}$ can be given the structure of a simplicial tropical fan (see remark 2.11 for the fan structure) of dimension $n-3$ in a quotient space of $\mathbb{R}^{\binom{n}{2}}$ ([GKM09, theorem 3.7]); in fact, it can be described as the tropical Grassmannian $\mathcal{G}(2,n)$ modulo its lineality space (see [SS04, theorem 3.4] or [GKM09, remark 3.9]) and is denoted there $\mathcal{G}''(2,n)$. More precisely, for each n-marked tropical curve $(\Gamma; x_1, \ldots, x_n)$ and $1 \le i < j \le n$ let $\mathrm{dist}_\Gamma(x_i, x_j)$ be the distance between the unbounded edges x_i and x_j in Γ. We thus get a map

$$\begin{aligned} \tilde{v} : \qquad\qquad \mathcal{M}_{0,n} &\to \mathbb{R}^{\binom{n}{2}} \\ (\Gamma; x_1, \ldots, x_n) &\mapsto (\mathrm{dist}_\Gamma(x_i, x_j))_{i<j} \end{aligned}$$

where we choose the lexicographic ordering of the pairs (i,j) for the coordinates in $\mathbb{R}^{\binom{n}{2}}$. We will call $\tilde{v}(\Gamma; x_1, \ldots, x_n)$ the *distance vector* of $(\Gamma; x_1, \ldots, x_n)$.
The following vectors in $\mathbb{R}^{\binom{n}{2}}$ will be of particular importance: let $I \subset \{1, \ldots, n\}$ be any subset, and denote by $\tilde{v}(I) \in \mathbb{R}^{\binom{n}{2}}$ the vector whose (i,j)-coordinate is equal to 1 if I contains exactly one of the numbers i and j, and 0 otherwise. Note that $\tilde{v}(I^c) = \tilde{v}(I)$, where I^c denotes the complement of I in $\{1, \ldots, n\}$.
The vectors $d_i := \tilde{v}(\{i\})$ for $i = 1, \ldots, n$ form a basis of the so-called lineality space mentioned above; i.e. by taking the quotient by this subspace we obtain a map

$$\begin{aligned} v : \qquad\qquad \mathcal{M}_{0,n} &\to \mathbb{R}^{\binom{n}{2}} / \langle d_1, \ldots, d_n \rangle \\ (\Gamma; x_1, \ldots, x_n) &\mapsto \overline{(\mathrm{dist}_\Gamma(x_i, x_j))_{i<j}} \end{aligned}$$

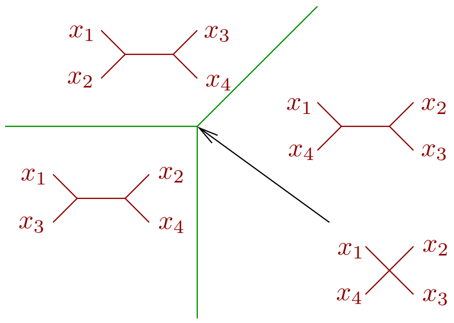
that embeds $\mathcal{M}_{0,n}$ as a tropical fan in $\mathbb{R}^{\binom{n}{2}} / \langle d_1, \ldots, d_n \rangle$ [GKM09, theorem 3.7]. For this structure of a tropical fan the weights of all top-dimensional cones are chosen to be 1, and the underlying lattice Λ is taken to be the one generated by the classes $v(I)$ of the vectors $\tilde{v}(I)$ for all $I \subset \{1, \ldots, n\}$ modulo the lineality space. In the remainder, we will always view $\mathcal{M}_{0,n}$ as a tropical fan with this embedding. We mod out the lineality space in order to make the distance $\mathrm{dist}_\Gamma(x_i, x_j)$ independent of start/end point on the unbounded edges x_i and x_j.

Remark 2.11 (**Fan structure of $\mathcal{M}_{0,n}$**)
Each cone of the tropical fan $\mathcal{M}_{0,n}$ w.r.t. the coarsest fan structure corresponds to curves of the same combinatorial type, i.e. curves which differ just by the lengths of their bounded edges. These cones are open convex polyhedra of dimension

$$n-3-\sum_{\text{vertices } V \text{ in } \Gamma} (\text{val } V - 3),$$

where $\text{val } V$ is the valence of the vertex V of any curve Γ of the given type. The one-dimensional rays of $\mathcal{M}_{0,n}$ are generated by the vectors $v(I)$ of definition 2.10 for all $I \subset \{1,\dots,n\}$ with $2 \leq |I| \leq n-2$; by construction these are just the distance vectors of curves $(\Gamma; x_1,\dots,x_n)$ in $\mathcal{M}_{0,n}$ having exactly one bounded edge of length 1, with markings in I on one and I^c on the other side. Hence there is only a finite number of combinatorial types of curves in $\mathcal{M}_{0,n}$. Note that the fan structure of $\mathcal{M}_{0,n}$ as in definition 2.10 coincides with the fan structure of the quotient $\mathcal{G}''(2,n)$, see [SS04, theorem 4.2 & proof of theorem 3.4].

Example 2.12
The space $\mathcal{M}_{0,4}$ can be embedded into $\mathbb{R}^6/\langle d_1,d_2,d_3,d_4\rangle \cong \mathbb{R}^2$. It has one top-dimensional cell corresponding to curves with a 4-valent vertex and three cells of codimension 1 associated to combinatorial types of curves with one bounded edge. Drawing its picture in $\mathbb{R}^2$ it can be seen as the tropical line of the beginning of this section (example 2.1) satisfying the balancing condition $\begin{pmatrix}0\\-1\end{pmatrix}+\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.



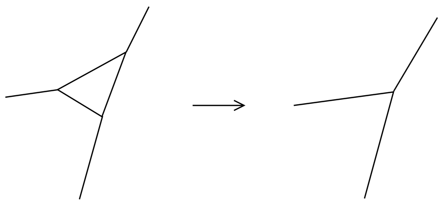
Remark 2.13 (**Grassmannian description of $\overline{M}_{0,n}$ and tropical compactification**)
Kapranov [Kap93] described $\overline{M}_{0,n}$ as Chow quotient of the classical Grassmannian $G(2,n)$ by a certain torus in $\mathbb{P}^{n-1}$. Tevelev [Tev07, theorem 5.5] has shown that it is the *tropical compactification* of $\mathcal{G}''(2,n)$, i.e. the closure of $M_{0,n}$ in the smooth toric variety associated to $\mathcal{G}''(2,n)$, which is compact. Note that this compactification seems to depend on the fan structure. Fortunately, in the case of $M_{0,n}$, there is only one coarsest fan structure [HKT09, theorems 1.10 & 1.11], hence the compactification is unique.
$\mathcal{M}_{0,n}$ is not compact, but can be compactified. Mikhalkin proposed a compactification of $\mathcal{M}_{0,n}$, where all edges are allowed to be of infinite length [Mik06, proposition 5.14]. This coincides with the tropicalization of the tropical compactification of $M_{0,n}$ defined over K [Mey11, theorem 6.6].

Remark 2.14 (**Intersection theory on $\mathcal{M}_{0,n}$**)
For many purposes $\mathcal{M}_{0,n}$ plays the tropical role of the classical moduli space of rational n-marked stable curves — even if it is not compact. Compactness is not necessary for tropical intersection theory on $\mathcal{M}_{0,n}$ so that intersection products have an enumerative significance [AR10, Rau08]. This is a particularly nice feature of the tropical world.

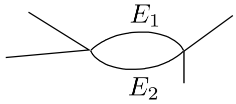
Remark 2.15 (**$\mathcal{M}_{0,n}$ as parameter space**)
So far, this description of $\mathcal{M}_{0,n}$ is only focused on $\mathcal{M}_{0,n}$ as a set. A more categorial characterization in the sense of representable moduli functors and universal families as in [Knu83]

was initiated by [Rau08, proposition 2.19], and achieved by [FH11, theorem 5.6 and corollary 5.7].

Remark 2.16 (**Problems with higher genus**)
Consider a 3-valent, n-marked abstract tropical curve of genus g. This curve has $N = n-3+g$ bounded edges of a certain length in $\mathbb{R}_{>0}$. So we can identify this curve with a point in $\mathbb{R}^N_{>0}$. If we aim to construct a moduli space for curves of genus $g > 0$ with a similar structure as for $g = 0$, we have to allow the length of one (or several) bounded edge to go to zero in order to obtain the polyhedron $\mathbb{R}^N_{\geq 0}$. But doing so, we obtain a curve with a possibly lower genus $g' < g$, as depicted below. This means that each closed polyhedron of shape $\mathbb{R}^N_{\geq 0}$ parametrizes curves of genus $\leq g$.



Also, we want to consider curves up to isomorphisms. Unfortunately, curves of genus $g > 0$ can have non-trivial automorphisms. For instance, in the picture below we have two edges E_1 and E_2 that are not distinguishable. The case $l(E_1) \neq l(E_2)$ is then problematic. One gets rid of the problem if one folds the polyhedron $\mathbb{R}^2_{\geq 0}$, whose points are of the form $(l(E_1), l(E_2))$, along the line $l(E_1) = l(E_2)$. This works fine for one polyhedron but gluing these fold polyhedra is not easy - one obtains an orbifold, see also 2.29.



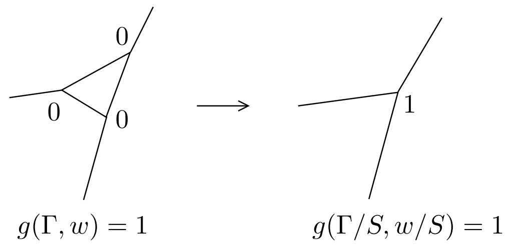
So let us now turn to a generalized concept of an abstract tropical curve due to Brannetti, Melo and Viviani [BMV11] which allows us to get rid of the problem of changing genus of curves in the moduli space.

Definition 2.17 (**n-marked abstract tropical curve of genus g (version 2)**)
Let $(\Gamma; x_1, \ldots, x_n; w)$ be a tuple consisting of a connected metric graph Γ with a weight function $w : V(\Gamma) \to \mathbb{N}$ on the set of vertices such that every vertex of weight 0 has valence at least 3, every vertex of weight 1 has valence at least 1 and where $x_1, \ldots, x_n$ is a labeling of its unbounded edges. This condition can be satisfied only if $2g - 2 + n > 0$. We define its *genus* to be $g(\Gamma, w) = g(\Gamma) + \sum_{V \in V(\Gamma)} w(V)$.
Note that this definition is equivalent to the one given in [Cap11, Cap12] by [Cap12, proposition 2.32].

Definition 2.18 (**Combinatorial type of an abstract curve (version 2)**)
[Cap12, definition 3.1.2]. The combinatorial type of an n-marked abstract tropical curve as above is the data of the curve but forgetting the information about the edge lengths.

Remark 2.19
When w is the zero function, hence Γ has at least 3-valent vertices, we retrieve the abstract tropical curve from definition 2.3.
Important is the process of *weighted contraction*, which transforms tropical curves into tropical curves of the same genus $g(\Gamma, w)$ and the same markings [Cap11, remark 3.1.3]. Let therefore $(\Gamma; x_1, \ldots, x_n; w)$ be a n-marked tropical curve and $S \subset E(\Gamma)$ a subset of its bounded edges. The weighted contraction of S consists in skrinking the edges of S to a point each such that we obtain a tropical curve $(\Gamma/S; x_1, \ldots, x_n; w/S)$, the associated map $\sigma : \Gamma \to \Gamma/S$ and the

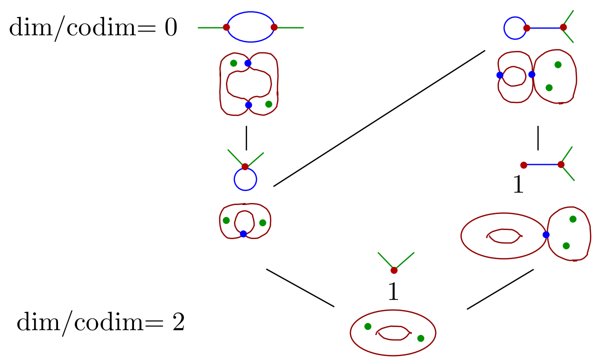
surjective map of vertices $\sigma_V : V(\Gamma) \to V(\Gamma/S)$. The weight function w/S is as follows. Given $\overline{v} \in V(\Gamma/S)$ we define $w/S(\overline{v}) = b_1(\sigma^{-1}(\overline{v})) + \sum_{v \in \sigma_V^{-1}(\overline{v})} w(v)$, where b_1 is the first Betti number.
Revising the first example from remark 2.16 in this light, the genus remains constant if we choose S to be the set of the three edges in the loop.



Remark 2.20 (**Moduli space of n-marked abstract tropical curve of genus g**)
$(\Gamma; x_1, \dots, x_n; w)$ and $(\tilde{\Gamma}; \tilde{x}_1, \dots, \tilde{x}_n; \tilde{w})$ are *isomorphic* if there there are bijections of the vertices, bounded edges of Γ and $\tilde{\Gamma}$ respectively, preserving lengths of bounded edges, weights of vertices, vertices adjacent to edges and mapping x_i to $\tilde{x}_i$ [Cap12, definition 3.1.2(7)]. This definition generalizes the one given in 2.10.
The set of isomorphism classes of n-marked abstract tropical curves of genus g is called its moduli space and is denoted $\mathcal{M}_{g,n}^{\text{trop}}$. It is a connected, Hausdorff topological space of pure dimension $3g-3+n$ [Cap12, remark 3.20, theorem 3.21(4), remark 3.23], but not a manifold [Cap12, example 3.24]. The set of curves with zero function w is open and dense in this space [Cap12, theorem 3.21(3)]. It can be compactified by letting go each bounded edge to infinity [Cap12, theorem 3.30].

Remark 2.21 (**Comparison with the classical moduli spaces**)
Given an n-marked stable curve $(C, x_1, \dots, x_n)$ of genus g, it is known (e.g. [FP96]) that we can associate to it its *dual graph*. It has a vertex for each irreducible component of C connected by bounded edges, one for each node of C lying on the components corresponding to the connected vertices. Furthermore, there is an unbounded edge for each marking x_i adjacent to the vertex corresponding to the component which contains x_i. This graph becomes the combinatorial type of an n-marked tropical curve (Γ, w), when one associates to each vertex the (geometric) genus of the corresponding irreducible component. For a given dual graph (Γ, w) we can consider the set of isomorphism classes $M_{g,n}^{\text{alg}}(\Gamma, w)$ of n-marked stable curves of genus g having (Γ, w) as dual graph, which is a subset of $\overline{M}_{g,n}$. Similarly, one can define the set of isomorphism classes $\mathcal{M}_{g,n}^{\text{trop}}(\Gamma, w)$ of tropical curves of that combinatorial type supported on (Γ, w), i.e. where only the length function on bounded edges (taking values in $\mathbb{R}_{\geq 0}$) is a degree of freedom. This is a subset of $\mathcal{M}_{g,n}^{\text{trop}}$. Then by [Cap12, theorem 4.7] there is a bijection $\{M_{g,n}^{\text{alg}}(\Gamma, w)$ with combinatorial type $(\Gamma, w)\} \to \{\mathcal{M}_{g,n}^{\text{trop}}(\Gamma, w)\}$, $M_{g,n}^{\text{alg}}(\Gamma, w) \mapsto \mathcal{M}_{g,n}^{\text{trop}}(\Gamma, w)$ with $\dim M_{g,n}^{\text{alg}}(\Gamma, w) = \operatorname{codim} \mathcal{M}^{\text{trop}}(\Gamma, w) = 3g - 3 + n - |E(\Gamma)|$, where $E(\Gamma)$ is the set of bounded edges of Γ as in 2.19 and dim/codim should be read in the respective context.

Example 2.22 ($g = 1$ **and** $n = 2$)
The bijection of cells in $\mathcal{M}_{1,2}^{\text{trop}}$ and the cells in $\overline{M}_{1,2}$ looks as follows.



In this figure, curves are represented by the corresponding Riemann surfaces. There is one cell of dimension 2 in $\overline{M}_{1,2}$, which equals the open part $M_{1,2}$. The boundary consists of two cells of dimension 1 and 0, respectively. An irreducible component of a curve is represented in red, a marking in green and a node in blue. Furthermore, the lines between different cells show the structure of the moduli space, i.e. one goes from a cell of lower dimension to one cell of higher dimension by resolving nodes, or in the context of dual graphs, by skrinking bounded edges to zero length.

2.3 Parametrized tropical curves and their moduli spaces

The tropical curves and their moduli spaces in the previous section are not useful for enumerative purpose as these curves are not embedded into some space. Indeed, we have seen in the classical context that we can write Gromov-Witten invariants as intersection product on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^2, d)$, i.e. the moduli space of stable curves embedded into $\mathbb{P}^2$. This works similarly in the tropical context. To study this in more details we first start with some definitions. If the weight function w in 2.17 is the zero function (what is always the case for $g = 0$), more details about the following can be found in [GKM09, section 4].

Definition 2.23 (**n-marked (labeled) parametrized tropical curve of degree Δ and genus g in $\mathbb{R}^2$**)
We denote a tuple $(v_1, \dots, v_m) \in (\mathbb{Z}^2 \setminus \{0\})^m$ by Δ.
A *n-marked (labeled) parametrized tropical curve of degree Δ and genus g in $\mathbb{R}^2$* is a pair (C, h) consisting of an $(n+m)$-marked abstract tropical curve $C = (\Gamma; x_1, \dots, x_{n+m})$ of genus g in the sense of 2.17 and a continuous map $h : \Gamma \to \mathbb{R}^2$ such that the following holds.

- The map h is integer affine linear on each edge E of Γ, i.e. of the form $h(t) = a + ut$ for some $a \in \mathbb{R}^2$ and $u \in \mathbb{Z}^2$. If we start parametrizing E at the vertex $V \in \partial E$ we call u the *direction* $u(E, V)$ of E with respect to V.
- At each vertex V the *balancing condition* $\sum_{E:V \in \partial E} u(E, V) = 0$ holds.
- The direction of x_i is 0 for all $i = 1, \dots, n$ (*contracted ends*).
- The direction of x_i is v_{i-n} for all $i = n+1, \dots, n+m$ (*non-contracted ends*).

Let V be a 3-valent vertex which is not adjacent to a contracted end. Let u_1, u_2, u_3 be the direction vectors of its adjacent edges. Then we define the *multiplicity* mult(V) of V to be $|\det(u_1|u_2)| = |\det(u_1|u_3)| = |\det(u_2|u_3)|$.
In our pictures we will usually only draw the image curve $h(\Gamma)$ together with the points $h(x_1), \dots, h(x_n)$. This image then has m (labeled) unbounded edges whose directions are contained in Δ. We can think of the *degree* Δ as the tropical equivalent of the homology class of an algebraic stable map. Note that the stability condition translates to the requirement that Γ has at least 3-valent vertices for $g = 0$.

In the following we will sometimes abbreviate "n-marked (labeled) parametrized tropical curve of degree Δ and genus g in $\mathbb{R}^2$" by "parametrized tropical curve".

Definition 2.24 (**Combinatorial type of a parametrized tropical curve**)
The *combinatorial type* of a parametrized tropical curve consists of the combinatorial type of the underlying abstract tropical curve and the directions of all edges in the parametrized curve.

Remark 2.25 (**Tropical degree** d)
Let (e_1, e_2) be the standard basis of $\mathbb{R}^2$ and $e_0 := -e_1 - e_2$.
If $\Delta = (\underbrace{-e_0, \ldots, -e_0}_{d \text{ times}}, \ldots, \underbrace{-e_2, \ldots, -e_2}_{d \text{ times}})$ we say that the parametrized tropical curve is of degree d and write instead of Δ simply d. The reason is that in this case the convex hull of the vectors in Δ is the lattice polygon $P = d\Delta_2$ corresponding to the toric surface $S_P = \mathbb{P}^2$ with associated divisor D of degree d as in remark 1.20.

Definition 2.26 (**Moduli space** $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$)
Two parametrized tropical curves (C, h) and $(\tilde{C}, \tilde{h})$ are *isomorphic* if there is an isomorphism φ between the underlying abstract tropical curves C and $\tilde{C}$ satisfying $\tilde{h} \circ \varphi = h$. The set of n-marked (labeled) parametrized tropical curves of given degree Δ and genus g in $\mathbb{R}^2$ modulo isomorphisms is denoted by $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$.

Remark 2.27 (**Moduli space for** $g = 0$)
Let us first consider $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. Observe that the directions of the bounded edges in a parametrized tropical curve are not fixed by Δ. But when fixing a combinatorial type (see remark 2.11) in $\mathcal{M}_{0,n+m}$ for $m = |\Delta|$ there is a unique choice for the directions of the bounded edges for a parametrized tropical curve of degree Δ in $\mathbb{R}^2$ such that the underlying graph Γ is of this combinatorial type. Hence there is a bijection between combinatorial types of the moduli spaces $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ and combinatorial types of $\mathcal{M}_{0,n+m}$ [GKM09, lemma 4.6]. Note that the number of combinatorial types in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is finite, too, for this reason.
Coordinates in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ are given by the position of a *root vertex* in $\mathbb{R}^2$ of the parametrized tropical curve and the lengths of the bounded edges (see [GM08, proposition 2.11]). Hence, given a parametrized tropical curve in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ we can send it to the image under the map v of its underlying graph Γ (see definition 2.10). So we get an isomorphism of polyhedral complexes [GKM09, proposition 4.7]

$$\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta) \overset{\cong}{\to} v(\mathcal{M}_{0,n+m}) \times \mathbb{R}^2.$$

In particular, we can consider $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ as a tropical fan of dimension $(n + m - 3) + 2$.

Remark 2.28 (**Moduli space for** $g > 0$)
The problems/obstructions described in 2.16 have to be overcome. In addition, we have to take into account that the moduli space should be constructed such that it contains the correct enumerative information, i.e. that the number of parametrized tropical curves of genus g and degree d passing through the right number of points in general position is finite and invariant, see 2.35. This means that we have to take out cells of too high dimension [Mar06, definition 4.26]. This *relevant subset* $\widetilde{\mathcal{M}}_{g,n}(\mathbb{R}^2, \Delta)$ [Mar06, definition 4.36] *of* $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ is a polyhedral complex of (pure) dimension $2n$, extending the argument of [Mar06, lemma 4.56]. To simplify the notation we will write in the following $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ instead of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{R}^2, \Delta)$.

Remark 2.29 ($\mathcal{M}_{1,n}(\mathbb{R}^2, \Delta)$ **is not a tropical fan**)
$\mathcal{M}_{1,n}(\mathbb{R}^2, \Delta)$ can not be made into a tropical fan so easily. This has been studied by Matthias Herold [Her09]. Note that cells of codimension-0 of his moduli space coincide with codimension-0 cells of our moduli space since we have $w = 0$ there. Problems occur only in cells of codimension > 0. Considering only curves of genus 1, he divides out the automorphism group of each polyhedron and glues the so obtained objects together. The result is an orbifold.

Furthermore, a balancing condition as for tropical fans holds at most polyhedra of codimension 1, but not at those where the genus disappears. Therefore, it is in general hard to construct a tropical structure for $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ if $g > 0$.

Definition 2.30 (**Evaluation map**)
For all $i = 1, \ldots, n$ define the *i-th evaluation map* by $\text{ev}_i : \mathcal{M}_{g,n}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2,\ (C, h) \mapsto h(x_i)$. These maps are well-defined and are morphisms of fans for $g = 0$ [GKM09, proposition 4.8] using remark 2.27 and morphisms of polyhedral complexes in general [Mar06, lemma 4.45]. The *evaluation map* $\text{ev} := \prod_{i=1}^{n} \text{ev}_i : \mathcal{M}_{g,n}(\mathbb{R}^2, \Delta) \to \mathbb{R}^{2n}$ is a morphism of fans for $g = 0$ and of polyhedral complexes in general, extending [Mar06, lemma 4.56].

If we want to count parametrized tropical curves such that we obtain an invariant number, we have to say, similarly to the classical situation, what we mean by points in general position. Unfortunately, there are several (canonical) definitions, but the definition depends on what we want to do with it. Version (v2) below is due to Grisha Mikhalkin.

Definition 2.31 (**Points in tropical general position**)
Fix $n > 0$ and a degree Δ such that $n = |\Delta| + g - 1$ (i.e. so that the source and target of ev have the same dimension and we expect a finite number of curves of degree Δ through n given points). A collection $\omega = (P_1, \ldots, P_n)$ of n points in $\mathbb{R}^2$ is said to be

- *in special position (v1)* if $\text{ev}^{-1}(\omega)$ is infinite;
- *in special position (v2)* if $\text{ev}^{-1}(\omega)$ is infinite or intersects polyhedra/cones of $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ of codimension > 0.

Otherwise we say that ω is *in general position* (for (v1) or (v2)). As ev is linear on each polyhedron/cone of $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$, note that $\text{ev}^{-1}(\omega)$ being infinite is equivalent to say that the map ev is not injective on (at least) one polyhedron/cone of $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ that intersects $\text{ev}^{-1}(\omega)$.

Remark 2.32
Version (v2) is typically used for enumerative purposes while version (v1) is closer to the classical definition of points in general position. In particular, parametrized tropical curves passing through a generic configuration of points (v2) do not have 4-valent vertices, and also none of the points P_i is at a vertex of $h(\Gamma)$. More details about this can be found for $g = 0$ in the next chapter 3 which is based on part 2 of the joint work with Andreas Gathmann [GS12, section 3].

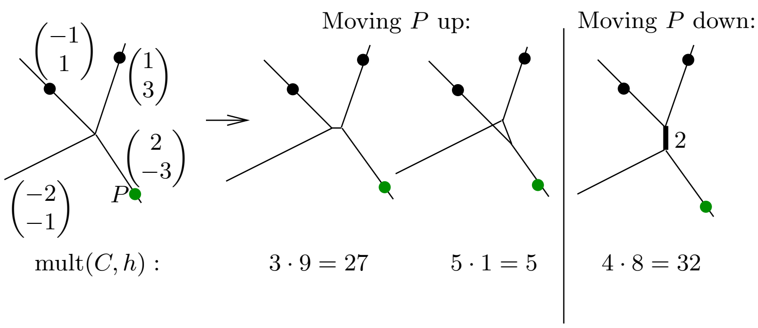
Convention 2.33
We will stick in the following to version (v2) as this definition agrees with [Mik05, definition 4.7].

Definition 2.34 (**Weight of an edge**)
Let (C, h) be an n-marked parametrized tropical curve and E a (non-contracted) edge of $h(\Gamma) \subset \mathbb{R}^2$ with direction vector $u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \in \mathbb{R}^2$. We define the *weight of E* to be the greatest common divisor (gcd) of u_x and u_y. We denote it $w(E)$. If $w(E) = 1$ we call E a *primitive* edge.
In the remainder we will draw edges of even weight as bold lines while edges of odd weight stay thin.

Definition 2.35 (**Multiplicity of a parametrized tropical curve, tropical number** $N^{\text{trop}}(\Delta, g)$)
Let (C, h) be a 3-valent parametrized tropical curve, i.e. the underlying graph Γ has only 3-valent vertices. We define the *multiplicity mult (C, h) of (C, h)* as the product $\prod_{V \text{ in } \Gamma} \text{mult}(V)$, where $\text{mult}(V)$ is the multiplicity of the vertex V as in 2.23, [Mik05, definition 4.15]. Given g, Δ and point configuration ω of $n = |\Delta| + g - 1$ points in general position in $\mathbb{R}^2$, we are interested in the number of parametrized tropical curve of genus g and degree Δ passing through

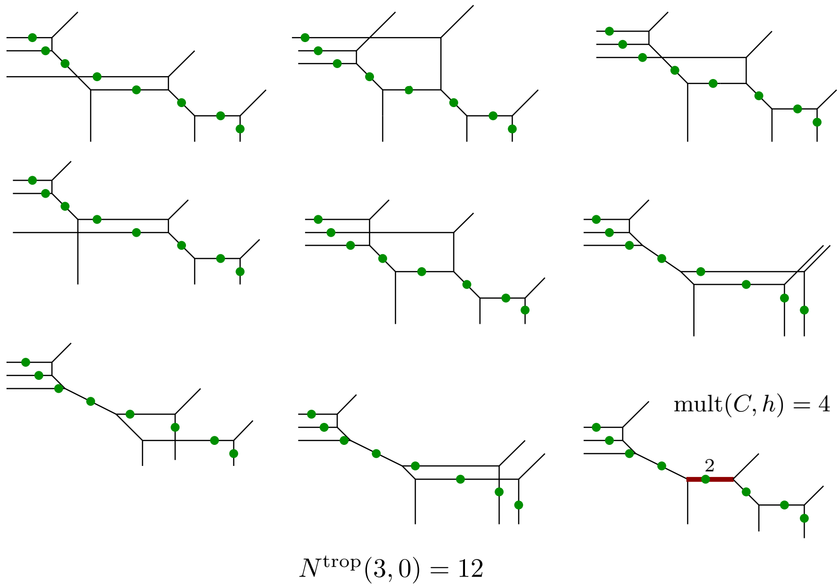
ω, that is $\sum_{C \in \mathrm{ev}^{-1}(\omega)} \mathrm{mult}(C,h)$. This number is finite by the definition of ω. A priori, it depends on ω. However, it turns out that this number does not depend on the choice of ω, see [GM07b, theorem 4.8] for a purely tropical proof. We denote it therefore by $N^{\mathrm{trop}}(\Delta, g)$. For $\Delta = d$ and if the ends are all primitive, this statement also follows from Mikhalkin's Correspondence Theorem 2.2.

Remark 2.36 (**Sketch of proof of [GM07b, theorem 4.8] or [Mar06, theorem 4.53]**)
The proof uses a connection between cells in $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ and $\mathbb{R}^{|\Delta|+g-1}$ given by the evaluation map. Namely, in order to prove the invariance of $N^{\mathrm{trop}}(\Delta, g)$, i.e. that the function $\omega \in \mathbb{R}^{|\Delta|+g-1} \to N^{\mathrm{trop}}(\Delta, g)$ is *globally* constant, the authors first note that the function is *locally* constant on the open subset of $\mathbb{R}^{|\Delta|+g-1}$ of points in general position. This is true as the multiplicity of a curve just depends on its combinatorial type and the latter is the same in each top-dimensional cell of $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ and furthermore by definition, only curves in codimension 0 of $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$ pass through points in general position. As points in special position lie in cells of codimension > 0 in $\mathbb{R}^{|\Delta|+g-1}$ but every pair of top-dimensional cells can be connected through some codimension-1 cells, it remains to prove that the function does not jump when considering curves lying in the boundary of top-dimensional cells in $\mathcal{M}_{g,n}(\mathbb{R}^2, \Delta)$. Here is an example. Consider the following 3-marked parametrized tropical curve of degree $\Delta = ((-1,1),(-2,-1),(2,-3),(1,3))$, which lies in codimension 1 of $\mathcal{M}_{0,3}(\mathbb{R}^2, \Delta)$. This means that the 3 points are in special position. (If the reader prefers, it can be regarded as a part of a curve of degree d.) When we move the point P, for instance, a little bit up or down, we obtain curves lying in codimension 0 of $\mathcal{M}_{0,3}(\mathbb{R}^2, \Delta)$ and passing through 3 points in general position. Moving the point P up there are two curves counting with multiplicity 27 and 5, respectively. When we move the point P downwards we see one curve having an edge of weight 2 as defined in 2.34 and counting with multiplicity $32 (= 5 + 27)$, so we have local invariance here. Note that each of these three curves lies in the open part of distinct top-dimensional cells having all the codimension-1 cell corresponding to the curve on left hand side as boundary.



Let $v_1, \ldots, v_4$ be the vectors in Δ (in this order). Then the general case follows from the equation $\det(v_1, v_2)\det(v_3, v_4) + \det(v_1, v_3)\det(v_2, v_4) + \det(v_1, v_4)\det(v_2, v_3) = 0$ and an analysis which types appear for which movement of P.

Example 2.37 (**The number of rational tropical cubics through** 8 **points**)
Here is a picture how the 8-marked parametrized tropical curves of genus 0 through 8 points in generic position may look like. Each curve has multiplicity 1 beside the last one, which has multiplicity 4. The total number of curves counted with multiplicities is hence 12 and agrees with the classical number as in 1.1. Observe that each curve, beside the last one, has a crossing of 2 perpendicular edges. This reflects that they have 1 node each. However for the last curve, this node can be seen as a point being located exactly in the middle of the edge of weight 2. A more precise study of tropical singularities is contained in [MMS11].



Remark 2.38 ($N^{\text{trop}}(\Delta,g)$ **as intersection number**)
There is an intersection theory for $N^{\text{trop}}(\Delta,g)$ developped in [Rau08], based on [AR10]. Among other results, the author shows that $N^{\text{trop}}(\Delta,0)$ can be written as intersection product on $\mathcal{M}_{0,n}(\mathbb{R}^2,\Delta)$ [Rau08, 3.9 & 3.10]. The extended study can be found in [MR09].

Remark 2.39 (**Multiplicities coincide**)
The Correspondence Theorem 2.2 proves for $\Delta = d$ and only primitive ends that the multiplicity mult(trop(V)) there coincides with the combinatorial multiplicity mult(C,h) [Mik05, theorem 1]. This means that the number of n-marked plane projective curves of genus g and degree d tropicalizing to $h(\Gamma)$ of a given parametrized tropical curve (C,h) can be computed with help of the embedded graph $h(\Gamma)$!

For the general Correspondence Theorem stated as follows we need an even more restricted version of points in general position than version (v2).

Definition 2.40 (**Simple** n**-marked parametrized tropical curve**)
[Mik05, definition 4.2]. An n-marked parametrized tropical curve (C,h) is called *simple* if

- Γ is 3-valent,
- there are at most two elements in $h^{-1}(y)$ for any $y\in\mathbb{R}^2$,
- $a,b\in\Gamma$ with $a\neq b$ and $h(a)=h(b)$ are not vertices of Γ.

Definition 2.41 (**Restricted general position of point configurations**)
Best presentation is [Mar06, definition 5.33]. In the setting of 2.31 a collection $\omega=(P_1,\dots,P_n)$ of n points in $\mathbb{R}^2$ is called in *restricted special position* if it is in special position (v2) and in addition ev$^{-1}(\omega)$ contains only simple curves.

Theorem 2.42 (**Mikhalkin's Correspondence Theorem (version 2)**)
[Mik05, theorem 1 in section 7.1]. Let $N(\Delta,g,\omega)$ be the number of irreducible complex algebraic curves $V\subset(\mathbb{C}^\star)^2$ of genus g defined by a polynomial $f:(\mathbb{C}^\star)^2\to\mathbb{C}$ with Newton polygon Δ passing through a configuration ω of $|(\partial\Delta\cap\mathbb{Z}^2)|+g-1$ generic points in $(\mathbb{C}^\star)^2$. Δ can then also be seen as collection of *primitive* ends in $\mathbb{R}^2$ as in 2.34. Then it holds

$$N(\Delta,g,\omega)=N^{\text{trop}}(\Delta,g).$$

Moreover, given a point configuration ω' of $|\Delta|-1+g$ points in $\mathbb{R}^2$ in restricted generic position, then there exists a point configuration ω which contains $|\Delta|-1+g$ points in $(\mathbb{C}^\star)^2$ in generic position with trop$(\omega)=\omega'$ such that for each tropical curve (C,h) passing through ω' there are exactly mult(C,h) complex curves passing through ω.

Remark 2.43
In this theorem, it is crucial that the weights of the unbounded edges in the sense of 2.34 of the considered tropical curves are all equal to 1. In fact, Mikhalkin's proof uses unparametrized tropical curves, i.e. just the data $h(\Gamma)$, and only in this case there is a one-to-one correspondence of the number of unparametrized and parametrized tropical curves, forgetting the labeling of the unmarked ends, through a given point configuration [Mar06, lemma 5.34]. This means that each unparametrized curve can uniquely (up to the labeling of the unmarked ends) be parametrized by a graph Γ' such that the (embedding) map to $\mathbb{R}^2$ identifies only finitely many points. Adding an end for each marking this gives a graph Γ together with a map $h : \Gamma \to \mathbb{R}^2$ satisfying the properties of an n-marked tropical curve.

Remark 2.44
The numbers $N(\Delta, g, \omega)$ do not depend on the choice of ω as the curves considered are curves in the toric surface S_Δ defined by the lattice polygon Δ if S_Δ is a Del Pezzo surface. Hence $N(\Delta, g, \omega)$ is an enumerative number for S_Δ in this case.

Remark 2.45 (**Tropicalization of points in classical general position**)
It follows from theorem 2.42 that points in tropical restricted general position are tropicalizations of points in classical general position.

Remark 2.46 (**Tropical relative Gromov-Witten numbers**)
In [GM07a] the authors prove the Caporaso-Harris formula [CH98] for the corresponding tropical numbers. In particular, they define *tropical relative Gromov-Witten numbers* analogous to 1.10. For a simple n-marked curve of degree d and genus g with α_i fixed and β_i non-fixed unbounded ends to the left of weight i for all i such that $n = 2d + g + \sum_i \beta_i - 1$ we define the (α, β)-*multiplicity* of (C, h) as $\text{mult}^{\alpha,\beta}(C, h) = \frac{1}{I^\alpha}\text{mult}(C, h)$, where $I^\alpha = 1^{\alpha_1} \cdot 2^{\alpha_2} \cdot \ldots$. It is clear that the α_i and β_i define a finite sequence as in 1.10, respectively. So for given $d \geq 0$, g and finite sequences α, β with $I\alpha + I\beta$ we define $N^{\alpha,\beta}_{\text{trop}}(d, g)$ as the number of (simple) n-marked tropical curves of degree d and genus g with α_i fixed and β_i non-fixed unbounded edges to the left of weight i for all i that pass through $n = 2d + g + \sum_i \beta_i$ points in general position counted with the multiplicity $\text{mult}^{\alpha,\beta}(C, h)$. It follows from [GM07b] that this number is invariant. By [GM07a, theorems 3.11 & 4.2] it equals also $N^{\alpha,\beta}(d, g)$.

3 The set of points in special position for rational n-marked plane tropical curves

In this chapter we would like to characterize points in special position as defined in 2.31 for rational n-marked tropical curves in the plane. This is part of the joint work with Andreas Gathmann and is published in [GS12]. We have seen that they play an important role for enumerative problems. Remember that in the classical situation, when the ambient space for the rational stable maps is $X = \mathbb{P}^2$, it is known that the points in special position — in the sense that there are infinitely many curves passing through them — form a subvariety of $(\mathbb{P}^2)^n$ (see [Gro66, corollaire 13.1.5] applied to the (product) evaluation map). In the tropical context a more striking result holds, namely that these points form a tropical subfan of $(\mathbb{R}^2)^n$ of codimension one, that we can describe as a push-forward of some divisor in $\mathcal{M}_{0,n}$. To be more precise, we deal here with two notions of "points in special position" which both as sets arise as a divisor pushed forward by the evaluation map. We restrict ourselves to the case where the ambient space is $\mathbb{R}^2$ as our arguments just hold there.

But let us start by recalling the necessary tropical vocabulary.

Definition 3.1 (**Tropical subfans, codimension-k skeleton**)
Pick some $r \geq 0$ and a lattice $\Lambda \cong \mathbb{Z}^r$; set $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

a) A *(tropical) subfan* Y of a (tropical) fan X in V is itself a (tropical) fan Y in V with the property that each cone of Y is contained in a cone of X. Note that the weight function of Y is not necessarily inherited by X.

b) The *codimension-k skeleton* $X_{\text{sk}}^{(N-k)}$ of a fan X in V of pure dimension N (for $0 \leq k \leq N$) consists of all cones of dimension at most $N-k$ in X. It is a pure-dimensional fan of dimension $N-k$, however with no canonical weight function associated to it.

Definition 3.2 (**Notions of intersection theory for our purpose**)
See [AR10] for more details. A *tropical (affine) k-cycle* is a weighted fan of pure dimension k in V satisfying the balancing condition but such that the weight function takes values in $\mathbb{Z}$. Hence the difference between a tropical fan of dimension k and a tropical k-cycle is just the range of the weight function. A tropical k-cycle in a tropical fan X is simply a k-cycle that is a subfan of X. More precisely, a tropical k-cycle in X is an equivalence class of weighted fans of pure dimension k, where two fans are equivalent if they have a common refinement [AR10, definition 2.8]. However, in this chapter we will not distinguish between equivalence classes and representatives of the equivalence class as the difference does not concern us here.
As before, we will denote by $|X|$ the subset of V of all cones of X (with non-zero weight). A *(Weil) divisor D* on X is a cycle in X of codimension 1.
Let X be a k-cycle. A (non-zero) *rational function on X* is a continuous function $f : |X| \to \mathbb{R}$ which is integer affine linear on each cone $\sigma \subset |X|$. We denote the linear part of the restriction of f to σ by f_σ.
Let X be a k-cycle and $f : |X| \to \mathbb{R}$ a rational function on X. The *Weil divisor $D(f)$ associated to f* is the divisor in X consisting of the codimension-1 cones $\tau \in X^{(k-1)}$ with weights

$$w(\tau) = \sum_{\sigma > \tau} w(\sigma)\, f_\sigma(v_{\sigma/\tau}) - f_\tau\Big(\sum_{\sigma>\tau} w(\sigma)\, v_{\sigma/\tau}\Big),$$

where the sum runs over all cones σ such that $\tau \subsetneq \sigma$ is a face of σ, and $v_{\sigma/\tau}$ denotes the primitive normal vector of σ relative to τ. It has been shown in [AR10, proposition 3.7] that this is indeed a tropical cycle.

Definition 3.3 (**Irreducible tropical cycles**)
A tropical cycle X in $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is said to be *(globally) irreducible* if there does not exist a tropical cycle Y of the same dimension in V such that $|Y| \subsetneq |X|$. Of course, this definition then applies to tropical fans as well.

Remark 3.4
Lemma 3.6 implies that, just as in the classical situation, the support of a tropical cycle can always be written as the union of the supports of irreducible tropical cycles. However, such a decomposition is in general not unique [GKM09, remark 2.19].

Remark 3.5
In definition 3.3 the cones of Y are just required to be contained in the union $|X|$ of the cones of X. However, the definition does not change if one requires all cones of Y to be actually cones of X, just with possibly different weights. To see this, assume there is a tropical cycle Y satisfying $|Y| \subsetneq |X|$. By passing to a common refinement with X, we can then first of all make sure that every cone of Y is contained in a cone of X. But then all cones of Y contained in the same cone of X must have the same weight due to the balancing condition, and hence can be made into a single cone.

Lemma 3.6
A tropical cycle X is irreducible if and only if "its weight function is unique up to a global multiple", i.e. if and only if for every cycle Y of the same dimension and consisting of at most the cones of X there is a rational number $\lambda \in \mathbb{Q}$ such that $w_Y(\sigma) = \lambda\, w_X(\sigma)$ for every cone σ of X.

Proof. "$\Rightarrow$": This is [GKM09, lemma 2.21].
"$\Leftarrow$": Let Y be a cycle with $|Y| \subsetneq |X|$. By remark 3.5 we can assume that each cone of Y is a cone of X, so there must be a cone σ of X with $w_Y(\sigma) = 0$. But this requires $\lambda = 0$ in our assumption, so Y would have to be the zero cycle. □

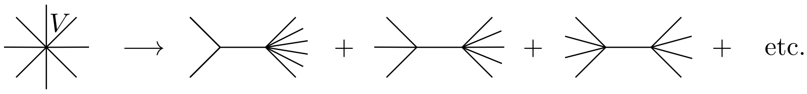
Definition 3.7 (**Tropical Psi-classes** ψ_i)
Fix $n > 2$ and $i \in \{1, \ldots, n\}$. The *Psi-class* $\psi_i \subset \mathcal{M}_{0,n}$ is the subfan of $\mathcal{M}_{0,n}$ consisting of all cones of $\mathcal{M}_{0,n}$ of curves $(\Gamma; x_1, \ldots, x_n)$ such that the marked edge x_i is adjacent to a vertex of valence at least 4 [Mik07]. Giving each top-dimensional cone the weight 1 it has the structure of a tropical subfan of $\mathcal{M}_{0,n}$ of codimension 1. Using the language of [AR10], we can rephrase this as: ψ_i is a tropical Weil divisor associated to a rational function as proven in [KM09b, proposition 3.5]. Note that a Psi-class is not defined up to rational equivalence as it is the case in classical geometry 1.7 (and should therefore better be named Psi-divisor).

Convention 3.8
When we talk in the following of "the" codimension-k skeleton of $\mathcal{M}_{0,n}$ respectively ψ_i, we mean this w.r.t. the fan structure of remarks 2.11 and 3.7.

3.1 Tropical fan description of codimension-k skeletons and Psi-classes in $\mathcal{M}_{0,n}$

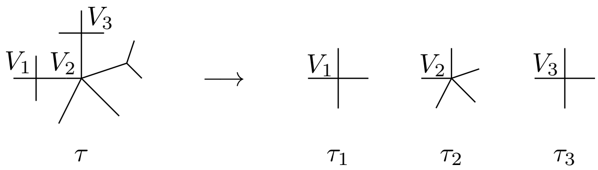
We now want to check that codimension-k skeletons of $\mathcal{M}_{0,n}$ and Psi-classes in $\mathcal{M}_{0,n}$ are in fact tropical fans, i.e. that they satisfy the balancing condition. So let us fix $k \in \{0, \ldots, n-3\}$ and a weight function $w : \mathcal{M}_{0,n}^{(n-3-k)} \to \mathbb{Z}_{>0}$ on the codimension-k skeleton of $\mathcal{M}_{0,n}$. Moreover, let $\tau \in \mathcal{M}_{0,n}^{(n-3-k-1)}$ be a cone, corresponding by remark 2.11 to a certain combinatorial type of n-marked curves. In order to verify the balancing condition for w at τ, the following toolkit will be useful.

Remark 3.9 (**About the balancing condition in** $\mathcal{M}_{0,n}$)
The cones $\sigma \in \mathcal{M}_{0,n}^{(n-3-k)}$ containing τ can be obtained by resolving one chosen vertex V of valence r at least 4 in τ in the same way as it can be resolved in $\mathcal{M}_{0,r}$ by adding one bounded edge. More precisely, this means that we replace V by two vertices joined by a bounded edge, with the r edges of V split up in every possible way onto the two new vertices such that there are at least two such edges on each side of the bounded edge. The following picture shows these types of resolutions; it can either be seen as a picture of curves in $\mathcal{M}_{0,r}$ or as a *local* picture of curves in $\mathcal{M}_{0,n}$ around V.



In order to check the balancing condition at τ it therefore suffices to split the total sum $\sum_{\sigma>\tau} w(\sigma)\, v_{\sigma/\tau}$ into parts, where each part corresponds to the resolution of one vertex V, and verify it for each part separately. This leads to the following lemma which states that the balancing condition for codimension-k cells in $\mathcal{M}_{0,n}$ can be split up into several conditions in lower-dimensional moduli spaces.

Lemma 3.10 (**Splitting the balancing condition**)
With notations as above, let $\{V_1,\dots,V_m\}$ be the set of 4- or higher valent vertices of a curve in τ, and let $r_1,\dots,r_m$ be their respective valences. Then to verify the balancing condition at τ in $\mathcal{M}_{0,n}$ it suffices to check the balancing at all 0-dimensional cones τ_i in $\mathcal{M}_{0,r_i}$ for $i \in \{1,\dots,m\}$ (corresponding to curves ("stars") having only one vertex of valence r_i).



Proof. Consider a 4- or higher valent vertex $V \in \{V_1,\dots,V_m\}$ of τ, let r be its valence, and denote by A_i for $i=1,\dots,r$ the set of marked edges behind the i-th edge of V. Thus we have $\sqcup_i A_i = \{1,\dots,n\}$. Define a linear map ϕ by

$$\phi : \mathbb{R}^{\binom{r}{2}} \to \mathbb{R}^{\binom{n}{2}}, \quad (x_{i,j})_{i<j} \mapsto (\tilde{x}_{k,l})_{k<l}$$

where

$$\tilde{x}_{k,l} = \begin{cases} x_{i,j} & \text{if } (k,l) \in A_i \times A_j \text{ or } (l,k) \in A_i \times A_j \text{ for some } 1 \le i < j \le r, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le k < l \le n$. Then by construction the distance vectors of construction 2.10 are transformed by ϕ as

$$\phi(\tilde{v}(I)) = \tilde{v}\Big(\bigcup_{i\in I} A_i\Big) \tag{1}$$

for all subsets $I \subset \{1,\dots,r\}$. In particular, for $i=1,\dots,r$ the i-th basis vector d_i of the lineality space of $\mathcal{M}_{0,r}$ is mapped to $\phi(d_i) = \tilde{v}(A_i)$.
Let $\tau_\star$ be the 0-dimensional cone in $\mathcal{M}_{0,r}$, corresponding to the star curve with one vertex. By remark 3.9 a 1-dimensional cone $\sigma_\star$ around $\tau_\star$ corresponds to a cone σ around τ in $\mathcal{M}_{0,n}$

describing the same local resolution. Let us assume that the balancing condition holds at $\tau_\star$, i.e. that

$$\sum_{\sigma_\star > \tau_\star} w(\sigma)\, \tilde{v}_{\sigma_\star/\tau_\star} = \sum_i a_i\, d_i \qquad \in \mathbb{R}^{\binom{r}{2}} \tag{2}$$

for some $a_i \in \mathbb{R}$. By remark 2.11 the normal vectors $\tilde{v}_{\sigma_\star/\tau_\star}$ in $\mathcal{M}_{0,r}$ are exactly $\tilde{v}(I)$ for the corresponding subset $I \subset \{1,\dots,r\}$ with $2 \le |I| \le r-2$, and by (1) these vectors are mapped by ϕ to the corresponding normal vectors $\tilde{v}_{\sigma/\tau}$ in $\mathcal{M}_{0,n}$. So applying ϕ to (2) we get

$$\sum_{\sigma_* > \tau_*} w(\sigma)\, \tilde{v}_{\sigma/\tau} = \sum_i a_i\, \tilde{v}(A_i) \qquad \in \mathbb{R}^{\binom{n}{2}}.$$

The vectors $\tilde{v}(A_i)$ lie in the lineality space for $|A_i| = 1$ and in V_τ otherwise, so taking the quotient by these spaces this sum reduces to zero. The claim of the lemma thus follows with the second part of remark 3.9. □

Proposition 3.11 (Codimension-k skeleton of $\mathcal{M}_{0,n}$)
Let $k \in \{0,\dots,n-3\}$. Then the codimension-k skeleton ${\mathcal{M}_{0,n}^{(n-3-k)}}_{\text{sk}}$ of $\mathcal{M}_{0,n}$ with the weight function $w : \mathcal{M}_{0,n}^{(n-3-k)} \to \{1\}$ is balanced. Hence, it is a tropical fan.

Proof. By lemma 3.10 we can reduce the proof to the local situation of a star curve with a vertex of some valence $r \in \{4,\dots,k+3\}$, i.e. to the balancing condition around the vertex $\tau_\star$ in $\mathcal{M}_{0,r}$.
Let us think of this balancing condition in terms of coordinate vectors in $\mathbb{R}^{\binom{r}{2}}$. We have to compute the sum $\tilde{v}$ of all normal vectors $\tilde{v}_{\sigma_\star/\tau_\star}$ arising from resolving the vertex of the star. Consider the first entry of this vector: here, we sum up 1 a number of times, a 1 for each type where the marked edges 1 and 2 lie on opposite sides of the bounded edge. By symmetry (i.e. no marked edge is distinguished), this sum is the same in each other entry of $\tilde{v}$. So $\tilde{v}$ is a multiple of the vector $(1,\dots,1)^\top$. Consider now the lineality space: summing up all vectors $d_1,\dots,d_r$ gives $2\cdot(1,\dots,1)^\top$. So $\tilde{v}$ is the zero vector modulo the lineality space. □

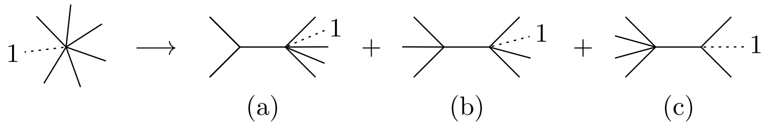
Proposition 3.12 (Codimension-k skeleton of a Psi-class ψ_i in $\mathcal{M}_{0,n}$)
Let $k \in \{0,\dots,n-4\}$ and $i \in \{1,\dots,n\}$. Then the codimension-k skeleton ${\psi_i^{(n-4-k)}}_{\text{sk}}$ of the i-th Psi-class ψ_i of $\mathcal{M}_{0,n}$ is balanced for the weight function $w : \psi_i^{(n-4-k)} \to \{1\}$. Hence, it is a tropical fan.

Proof. Again we can use lemma 3.10 to reduce the proof to the local situation of a star with a vertex of some valence $r \in \{4,\dots,k+4\}$. If the i-th unbounded edge is not adjacent to the chosen vertex then the computation is exactly the same as in the proof of proposition 3.11, so let us assume that it is adjacent to the chosen vertex. Then, as in the picture of example 3.13 below, we only have to consider resolutions of the star in which the i-th edge remains adjacent to a vertex of valence at least 4.
Think again in terms of coordinate vectors in $\mathcal{M}_{0,r}$. Again, denote the sum of all normal vectors of these resolutions by $\tilde{v}$. Then the (j,k)-coordinates of $\tilde{v}$ with $j,k \neq i$ are all the same by symmetry, and likewise for the (j,k)-coordinates where $j = i$ or $k = i$. So if we set w.l.o.g. $i = 1$ then we can write $\tilde{v}$ as

$$\tilde{v} = (\underbrace{M,\dots,M}_{r-1 \text{ times}}, N,\dots,N)^\top = M\, d_1 + \frac{N}{2}\,(-d_1 + d_2 + \dots + d_r)$$

for suitable $M, N \in \mathbb{N}$. As this is a vector in the lineality space, the balancing condition follows. □

Example 3.13 (**Codimension-2 skeleton of ψ_1 in $\mathcal{M}_{0,7}$**)
The codimension-2 skeleton of ψ_1 in $\mathcal{M}_{0,7}$ is 1-dimensional, and thus there is only one balancing condition to check, namely that around the 0-dimensional cell corresponding to the star curve with only one vertex. Allowed resolutions of this type are:



As in the proof of proposition 3.12 let $\tilde{v}$ be the sum of all normal vectors of these resolutions. Write $\tilde{v}$ as $\tilde{v} = \tilde{v}_{(a)} + \tilde{v}_{(b)} + \tilde{v}_{(c)}$, where the summands denote the parts of $\tilde{v}$ arising from resolutions of type (a), (b), (c) as in the picture above, respectively. Note that each such type corresponds to various resolutions corresponding to the choice of labeling of the marked ends. In the array below, for each resolution type the coordinates of $\tilde{v}$ are listed, where $j, k \neq 1$.

coordinate	$\tilde{v}_{(a)}$	$\tilde{v}_{(b)}$	$\tilde{v}_{(c)}$	$\tilde{v}$
$(1, j)$	$\binom{5}{4} = 5$	$\binom{5}{3} = 10$	$\binom{5}{2} = 10$	25
(j, k)	$2 \cdot \binom{4}{3} = 8$	$2 \cdot \binom{4}{2} = 12$	$2 \cdot \binom{4}{1} = 8$	28

For example, the $(1, j)$-coordinate 10 of $\tilde{v}_{(b)}$ corresponds to the 10 choices of distributing the remaining labels on the ends in type (b) if the unbounded edge j has been put at the left vertex.
So we have

$$\tilde{v}_{(a)} = 4\sum_{j=1}^{7} d_j - 3d_1, \qquad \tilde{v}_{(b)} = 6\sum_{j=1}^{7} d_j - 2d_1, \qquad \tilde{v}_{(c)} = 4\sum_{j=1}^{7} d_j + 2d_1.$$

Note that it is not just the sum $\tilde{v}$ that is zero modulo the lineality space, but also the individual vectors $\tilde{v}_{(a)}$, $\tilde{v}_{(b)}$, $\tilde{v}_{(c)}$ corresponding to the resolution types themselves. In fact, the proof of proposition 3.12 shows that the analogous statement holds for the (one-dimensional) codimension-$(n-5)$ skeleton of a Psi-class in $\mathcal{M}_{0,n}$ for all $n \geq 5$ since the symmetry argument given there also applies if we only consider a single resolution type.

Remark 3.14 (**Choice of weight function**)
Of course, the computations above depend on the chosen weight function. The moduli spaces $\mathcal{M}_{0,n}$ and the Psi-classes ψ_i, considered as tropical cycles, have all weights of their facets equal to 1, and thus it was natural in propositions 3.11 and 3.12 to also equip the k-skeletons of these cycles with the constant weight function 1. For other cycles such as for instance intersection products $\psi_1^{m_1} \cdot \ldots \cdot \psi_n^{m_n}$ of Psi-classes (where $m_1, \ldots, m_n \in \mathbb{N}_0$) this is in general no longer the case, especially when they are of codimension 2 or higher.

3.2 The tropical structure of sets of points in tropical special position

Definition 3.15 (**More intersection theory and tropical morphisms**)
We extend the definitions of definition 3.2.

a) Let X and Y be two k-cycles. After possibly adequately refining X and Y we can construct a k-cycle on $X \cup Y$, called the *sum of the cycles X and Y*, which is denoted by $X + Y$ [AR10, construction 2.13].

b) A *morphism $f : X \to Y$ of cycles X and Y* is a $\mathbb{Z}$-linear map, i.e. a map $\tilde{f} : |X| \to |Y|$ induced by a $\mathbb{Z}$-linear map between the underlying lattices Λ_X and Λ_Y.

c) Let $f : X \to Y$ be a morphism of a cycle X to an m-cycle Y, Z an n-cycle in X where $n \leq m$. Using an appropriate refinement of X we can assume that the image of each cone in X is a cone of Y. We define the *push-forward of Z along f* by $f_*(Z) = \{f(\sigma)|\ \sigma \in Z\}$. This polyhedral complex $f_*(Z)$ becomes an n-cycle in Y by giving the weights

$$w_{f_*(Z)}(\sigma') = \sum_{\substack{\sigma \in Z \\ f(\sigma)=\sigma'}} w_Z(\sigma) \cdot |\Lambda_{\sigma'}/f(\Lambda_\sigma)|$$

to the n-dimensional cones $\sigma' \in Y$ in the image of Z under f [AR10, proposition 4.6], where Λ_σ is the sublattice of Λ_X generated by σ.

d) The tropical Psi-classes of definition 3.7 can be written as the divisors associated to certain rational functions [KM09b, chapter 3]. As such, we can intersect several Psi-classes by consecutively intersecting with these rational functions. The product $\psi_1^{k_1} \cdot \cdots \cdot \psi_n^{k_n}$ is then a cycle whose support consists of all curves such that a vertex with the markings $i_1, \ldots, i_m$ has valence at least $k_{i_1} + \cdots + k_{i_m} + 3$ [KM09b, chapter 4].

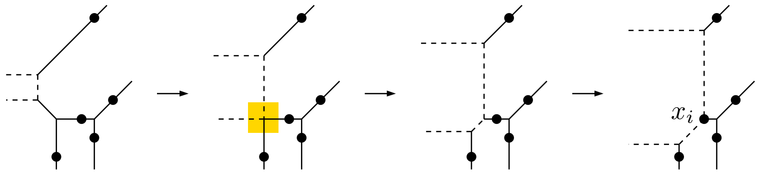
We now want to derive formulas for the locus in $\mathbb{R}^{2n}$ of points in special position, for both versions (v1) and (v2).

Remark 3.16 (**Strings**)
A parametrized tropical curve has a *string* if the underlying graph Γ contains a subgraph homeomorphic to $\mathbb{R}$ which does not intersect the closures $\overline{x_i}$ of the markings $x_1, \ldots, x_n$. By [GM08, remark 3.7] curves lying in codimension 0 of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ passing through points in special position have at least one string. Such a curve can have several strings which are not necessarily disjoint.

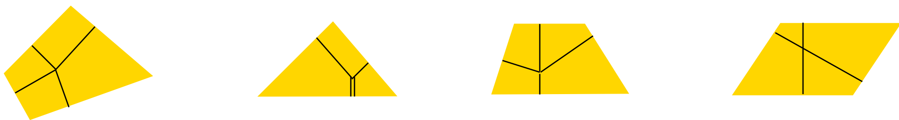
Definition 3.17 (**Free and fixed edges (in this chapter)**)
Let (C, h) with $C = (\Gamma, x_1, \ldots, x_n)$ be a parametrized tropical curve in $\mathbb{R}^2$. Let V be a vertex of C and E an adjacent edge. Then E is called a *free edge* at V if it can be connected in $\Gamma \setminus (V \cup \bigcup_{i=1}^n \overline{x_i})$ to an unmarked end. Otherwise we call E a *fixed edge* at V.

Proposition 3.18 (**Points in special position (v1)**)
Let $n = |\Delta| - 1$ and assume $n > 1$. Then the set of points in special position (v1) for curves in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ equals the support of the cycle $\text{ev}_*(\psi_1 + \cdots + \psi_n)$.

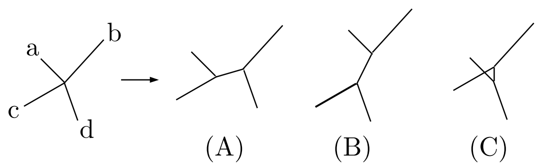
Proof. We have to show two inclusions.
In order to prove that the set of points in special position is contained in the support of $\text{ev}_*(\psi_1 + \cdots + \psi_n)$ we consider curves lying in codimension 0 of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ passing through points in special position. By remark 3.16 these curves have at least one string. The idea of the proof is that moving such a string yields a curve passing through the same point configuration, but lying in a codimension-one cone of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ and having one 4-valent vertex where one of the adjacent edges is a marking $x_1, \ldots, x_n$. (Remember that a marking x_i is an edge of the underlying abstract tropical curve that is mapped to a point in the parametrized curve, it is a contracted end. So a marking x_i adjacent to a vertex is depicted by a point on the vertex below.) This cone is often not unique as we might move the string in different directions. Moreover, the resulting curve does not necessarily lie in the boundary of the original cone: it might happen that the string first runs into a 4-valent vertex such that no marking x_i is adjacent to it. When resolving this vertex a new string appears which can be moved again. The following sequence of pictures shows the idea of these movements; there is one string which is drawn in dashed lines. In the second picture there is a 4-valent vertex without adjacent marking. The final curve has a marking at a 4-valent vertex x_i, so it lies in the support of ψ_i, which means that our set of points in special position lies in the support of $\text{ev}_*\psi_i$ and thus also of $\text{ev}_*(\psi_1 + \cdots + \psi_n)$.



To make this argument rigorous we have to give an algorithm how to move a string so that it runs into a marking. For this let us first consider a curve with a 4-valent vertex on a string without adjacent marking (as in the second picture above). The following picture shows the types of 4-valent vertices without marking, where the types distinguish which of the adjacent edges are parallel.

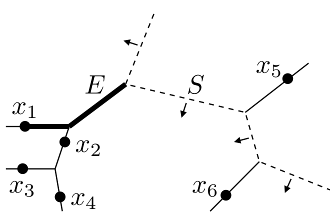


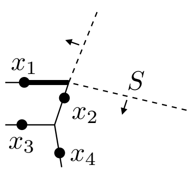
Note that at most two of the adjacent edges are fixed by the point conditions since at least two of them lie on a string. If none of the adjacent edges is fixed, the 4-valent vertex arises from the string movement in curves in codimension 0 having at least two strings joining in codimension 1 at the 4-valent vertex. In this case, it is possible to move one of the strings differently in order to obtain a 4-valent vertex with at least one adjacent fixed edge. Let us assume this in the following. Then, considering all possibilities which of the edges can be fixed in each of the above types, one can see that in each case there is a resolution of the 4-valent vertex such that at least one of the fixed adjacent edges becomes shorter. For instance, for the first 4-valent vertex from the left in the picture above the table below lists the resolution(s) (A), (B) or (C), where at least one of the fixed edges a, b, c or d gets shorter, depending on which of the adjacent edges are fixed. Note that, in the case of one fixed edge, this edge becomes shorter in each of the resolutions.



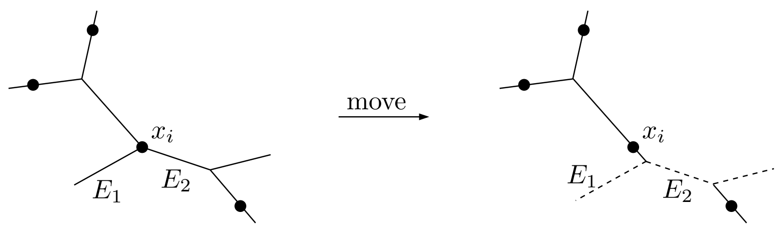
edges fixed	a, b	a, c	a, d	b, c	b, d	c, d
resolutions	(A), (C)	(B)	(A), (B)	(A), (B)	(B), (C)	(A), (C)

The algorithm now works as follows. Start with a triple (C, S, E) where C is the original curve, S a string on C, and E a fixed edge adjacent to S (such a choice is possible since there is at least one marking). Denote by $M = M(C, S, E)$ the maximum distance in C from S to a marking behind E which can be reached from S without passing other markings. The following picture on the left shows such a choice; S is again drawn with dashed lines, and M is the length of the two line segments drawn in bold.





We now claim that we can always change the curve by moving the string (as indicated by the arrows) so that M decreases — until either M becomes zero and thus the string runs into a marking (x_1 in the picture above), or the string runs into another marking elsewhere earlier (maybe x_6 in the picture above). The possibility of such a movement is obvious as long as the length of E is positive. If E shrinks to a point in the movement (as in the picture above on the right) we have a 4-valent vertex at the string with at least one fixed adjacent edge, and by our above argument we know that we can always continue to move the string so that at least one of the fixed adjacent edges becomes shorter. Choosing this edge to be E we can thus continue to decrease M (note that by changing E the set of first markings behind E is replaced by a smaller one, so this step cannot make the maximum M of their distances to S bigger). This completes the argument and yields the first inclusion of the proposition.
For the other direction, we have to show that $\text{ev}_*(\psi_1 + \cdots + \psi_n)$ contains no points in general position. As $\sum_{i=1}^n \psi_i$ is a divisor, we just have to consider curves lying in codimension one of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. So consider such a curve $C = (\Gamma; x_1, \ldots, x_n)$ in ψ_i, i.e. a curve with one 4-valent vertex V with an adjacent marking x_i and only 3-valent vertices otherwise. We have to prove that we can deform C to a codimension-0 curve that still satisfies the same point conditions.
This is obvious if C contains a string, so let us assume that this is not the case. Note that removing $\overline{x_i}$ from Γ separates Γ into 3 parts, whereas removing each of the other $n-1 = |\Delta|-2$ causes one more separation. So $\Gamma \setminus \bigcup_{j=1}^n \overline{x_j}$ consists of $|\Delta| + 1$ connected components. As none of these components can have more than one end (otherwise we would have a string) we conclude that there is precisely one bounded component with no end, whereas all other $|\Delta|$ components contain exactly one end. This means that at V (which has ψ_i and three more edges adjacent to it) at least two of the unmarked adjacent edges must be connected in $\Gamma \setminus \bigcup_{j=1}^n \overline{x_j}$ to an unbounded edge. We can then resolve V so that these two edges E_1 and E_2 remain together but separate from x_i, forming a string and thus a movement of the curve with the positions of the markings fixed.



Hence the points are by definition in special position. □

Corollary 3.19
In the same situation as above, the set of points in special position (v1) for curves in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ can be given the structure of a tropical subfan in $\mathbb{R}^{2n}$ of codimension one.

Proof. Observe that ev is a morphism of fans of dimension $2n$. The claim then follows directly from proposition 3.18 and definition 3.15 a) and c). □

Remark 3.20 (**Comparison to the classical situation**)
As stated in the introduction of this chapter, the set of n points in special position (v1) in the corresponding classical situation is a subvariety of $(\mathbb{P}^2)^n$. But in contrast to corollary 3.19 it is not necessarily a subvariety of codimension one: consider for instance conics in $\mathbb{P}^2$ through 5 points. There are infinitely many conics through these points if and only if two of them coincide or four of them lie on a line — and this forms a subvariety of codimension 2 in $(\mathbb{P}^2)^5$. The reason for the bigger dimension on the tropical side is that there are infinitely many liftings in the sense of [JMM08] to $\mathbb{P}^2$ of the points in $\mathbb{R}^2$. When tropicalizing, the algebraic curves

through each such configuration in $(\mathbb{P}^2)^n$ give rise to tropical curves passing through the given configuration of points in $(\mathbb{R}^2)^n$. As a consequence, the number of such tropical curves through the given points can be infinite although the number of algebraic curves through any lifting of them is not, i.e. the point configuration in $(\mathbb{R}^2)^n$ can be in special position (v1) although their liftings are not in special position classically.

On the other hand, returning to the case of conics, the locus of points where we find reducible curves through them (which roughly corresponds to (v2)) is the image of the locus of reducible curves in $\overline{M}_{0,5}(\mathbb{P}^2, 2)$ under the evaluation map. Here, both this locus in $\overline{M}_{0,5}(\mathbb{P}^2, 2)$ and its image in $(\mathbb{P}^2)^5$ have codimension 1, the latter being the space of all points where three of them lie on a line.

Remark 3.21 (**Generalization to curves with Psi-class conditions**)
One can generalize the result of proposition 3.18 as follows to the case of counting curves satisfying Psi-class conditions (i.e to tropical descendant Gromov-Witten invariants) as in [MR09]: fix $n > 0$, a degree Δ, and $k_1, \ldots, k_n \geq 0$ such that $n = |\Delta| - 1 - k_1 - \cdots - k_n$. If we then modify definitions 2.30 and 2.31 so that they use the moduli space $\psi_1^{k_1} \cdot \cdots \cdot \psi_n^{k_n} \cdot \mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ instead of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ we count curves through given points in $\mathbb{R}^2$ with additional Psi-class conditions, i.e. such that the valence of the vertex with attached marking x_i is (at least) $k_i + 3$ for all i [KM09b, theorem 4.1]. The proof of proposition 3.18 can then easily be adapted to show that the set of points in special position (v1) equals the support of the cycle $\text{ev}_*(\psi_1^{k_1} \cdot \cdots \cdot \psi_n^{k_n} \cdot (\psi_1 + \cdots + \psi_n))$. In fact, the first direction in the proof of the proposition remains unchanged since it is still true that curves passing through points in special position contain a string. In the second direction the curves in question will still contain exactly one bounded region in $\Gamma \setminus \bigcup_{j=1}^n \overline{x_j}$; it follows that the required movement of the curve is still possible, now resolving a (k_i+4)-valent vertex to a (k_i+3)-valent (containing the marking) and a 3-valent vertex.

Proposition 3.22 (**Points in special position (v2)**)
Fix Δ and let n equal $|\Delta| - 1$. Then the set of points in special position (v2) for curves in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ equals the support of the push-forward $\text{ev}_*(\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)_{\text{sk}}^{(2n-1)})$ of the codimension-1 skeleton of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ (note that this codimension-one skeleton is a cycle by proposition 3.11 and construction 2.26).

Proof. By definition the support of the cycle $\text{ev}_*(\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)_{\text{sk}}^{(2n-1)})$ contains only points in special position. In the other direction, if the points are in special position we can use the same argument as in the proof of proposition 3.18 to show that they lie in the push-forward of the codimension-one skeleton of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. □

Corollary 3.23
In the same situation as above, the set of points in special position (v2) for curves in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ can be given the structure of a tropical subfan in $\mathbb{R}^{2n}$ of codimension one.

Proof. Observe that ev is a morphism of tropical fans of dimension $2n$. The claim thus follows directly from proposition 3.22 and definition 3.15 c). □

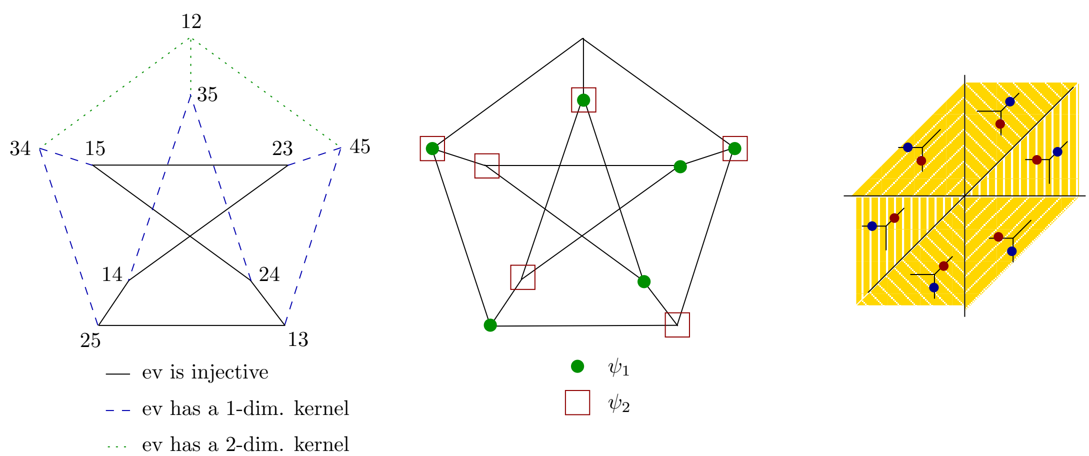
Remark 3.24 (**Reducible cycles**)
Using the characterization of proposition 3.18, the set of points in special position (v1) for curves in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ cannot be an irreducible cycle of $\mathbb{R}^{2n}$ as it is the push-forward of the reducible divisor $\psi_1 + \cdots + \psi_n$. Likewise, the set of points in special position (v2) for curves in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is not an irreducible cycle as example 3.26 shows.

Remark 3.25 (**Psi-condition**)
As in remark 3.21 we can easily generalize proposition 3.22 to the case of curves satisfying a Psi-condition ψ_i in addition to incidence conditions with points; the result is then that the set

of points in special position is the push-forward by ev of the codimension-one skeleton of ψ_i (which is a cycle by proposition 3.12). However, for more than one Psi-condition there is no similar statement as the codimension-one skeleton of a product of Psi-classes does not have a canonical choice of weights.

Example 3.26
Consider $\mathcal{M}_{0,2}(\mathbb{R}^2, \Delta)$ with $\Delta = \{e_1 + e_2, -e_1, -e_2\}$ where e_i is the i-th standard basis vector of $\mathbb{R}^2$ (i.e. curves of degree one in the notation of [GKM09, definition 4.1]). Then $\mathcal{M}_{0,2}(\mathbb{R}^2, \Delta) = \mathcal{M}_{0,5} \times \mathbb{R}^2$ by construction 2.26. The space $\mathcal{M}_{0,5}$ can be represented by the Petersen graph, depicted below twice. The *Petersen graph* is a nonplanar, connected graph with 10 vertices denoted here ij with $i, j \in \{1, \ldots, 5\}$ and $i < j$, which are connected by 15 edges such that a vertex ij is only linked to the three vertices kl with $k, l \in \{1, \ldots, 5\} \setminus \{i, j\}$. In this graph, the two-dimensional cones of $\mathcal{M}_{0,5}$ appear as edges, and the one-dimensional cones as vertices. The vertex ij corresponds to the ray of $\mathcal{M}_{0,5}$ generated by the vector $v(\{i, j\})$.
$\mathcal{M}_{0,2}(\mathbb{R}^2, \Delta)$ is particularly interesting as in this case the sets of points in special position of the two versions coincide, more precisely they even coincide as tropical fans with the weights of corollaries 3.19 and 3.23. To see this, we observe that the codimension-one skeleton of $\mathcal{M}_{0,5}$ contains the cone 12 which is not contained in $\psi_1 + \psi_2$. Furthermore, the cones 34, 35, and 45 have each weight 2 in the fan $\psi_1 + \psi_2$ but just weight 1 each in the codimension-one skeleton of $\mathcal{M}_{0,5}$. But these cones 12, 34, 35, 45 vanish when pushed forward by ev, since ev is not injective on them (in fact their images consist of the configurations of two equal points in $\mathbb{R}^2$ and thus have codimension 2 in $\mathbb{R}^2 \times \mathbb{R}^2$). The picture on the most right hand side shows the reducible fan $\text{ev}_*(\psi_1 + \psi_2) = \text{ev}_*(\mathcal{M}_{0,2}(\mathbb{R}^2, \Delta)^{(3)}_{\text{sk}})$ in $\mathbb{R}^2$. It can be obtained from the fan living in $\mathbb{R}^4$ by choosing relative coordinates, i.e. setting $h(x_1) = 0$ and considering the position of $h(x_2)$ in $\mathbb{R}^2$ relative to $h(x_1)$.

For more markings, the notions (v1) and (v2) will in general differ.



3.3 Computation of the weights of the top-dimensional cones of ev$_*(Z)$

To be able to interpret propositions 3.18 and 3.22 numerically we now want to compute the weights of the cells of codimension-1 cycles in $\mathbb{R}^{2n}$ that are of the form $\text{ev}_*(Z)$ for a cycle Z in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$. For this we first need to recall some well-known linear algebra results on elementary divisors.

Lemma 3.27 (**Theorem on elementary divisors**)
Let M be a finitely generated free module over a principal ideal domain R, and $N \subset M$ a submodule of M. Then there exists a basis $(u_1, \ldots, u_m)$ of M, a basis $(v_1, \ldots, v_n)$ of N

and $e_1, \ldots, e_n \in R \setminus \{0\}$ such that $v_i = u_i e_i$ for $i = 1, \ldots, n$ and $e_{i+1} \equiv 0 \mod e_i$ for $i = 1, \ldots, n-1$. The e_i are called the *elementary divisors of* N and are unique up to units in R.
For the quotient module it follows that $M/N \cong R^{m-n} \oplus \bigoplus_{i=1}^n R/Re_i$. Hence in the case $R = \mathbb{Z}$ the number of elements of the torsion part of M/N is $|\prod_{i=1}^n e_i|$. In the following we will denote this number by $D(M/N)$.

Proof. See for example [Lan02, theorem III.7.8]. □

In the rest of the chapter we will always use this result for the ring $R = \mathbb{Z}$. We then choose the e_i to be positive.

Lemma 3.28
In the situation of lemma 3.27 (for $R = \mathbb{Z}$) the number $D(M/N)$ is the greatest common divisor (gcd) of the $n \times n$ minors of any matrix A representing the $\mathbb{Z}$-linear map $N \hookrightarrow M$.

We therefore denote this number by $D(A)$.

Proof. See remark 3 of chapter 12.2 on page 6 of [Wae91]. From lemma 3.27 it follows that the map $N \hookrightarrow M$ can be represented by a matrix B with the elementary divisors on the diagonal and all other entries zero. As A represents the same map there exist matrices $S \in \mathrm{GL}(m, \mathbb{Z})$ and $T \in \mathrm{GL}(n, \mathbb{Z})$ such that $A = S\,B\,T$. The $n \times n$ minors of A are then integer linear functions of the $n \times n$ minors of B, and vice versa. As the gcd of the $n \times n$ minors of B equals the product $D(M/N) = \prod_{i=1}^n e_i$, this means that, up to units in $\mathbb{Z}$, the gcd of the $n \times n$ minors of A equals $D(M/N)$ as well. □

Corollary 3.29
Let $B \in \mathbb{Z}^{(n+1)\times n}$ be a matrix having a $(n-k) \times (n-k)$ part B_1 in the upper left corner, a block B_2 of dimensions $(k+1) \times k$ in the lower right corner, an arbitrary (non-quadratic) block in the upper right corner, and just zeros in the lower left corner:

$$B = \left(\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array} \right).$$

Then

$$D(B) = |\det(B_1)| \cdot D(B_2).$$

Proof. To compute the $n \times n$ minors of B, we have to erase one row of B and look at the determinants of these matrices. If we delete one of the first $n-k$ rows, the vectors in the columns of the quadratic part that remain are linearly dependent, hence these minors vanish. Deleting the j-th row with $n-k+1 \leq j \leq n+1$, we obtain a matrix that contains the block B_1 in the upper left corner, a quadratic block C_j of dimensions $k \times k$ in the lower right corner and 0 in the lower left part. Hence the determinant of such a matrix equals the product of $\det(B_1)$ and $\det(C_j)$. So by lemma 3.28 we get

$$\begin{aligned} D(B) &= \gcd\{\det(B_1) \cdot \det(C_j) : \ j = n-k+1, \ldots, n+1\} \\ &= |\det(B_1)| \cdot \gcd\{\det(C_j) : \ j = n-k+1, \ldots, n+1\} \\ &= |\det(B_1)| \cdot D(B_2), \end{aligned}$$

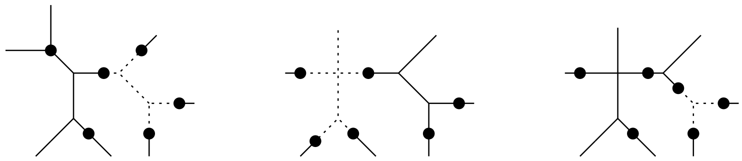
using the property $\gcd(ma, mb) = m \gcd(a, b)$ for $m \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. □

We will now apply these results to obtain formulas for the weights of push-forwards of codimension-1 cycles along the evaluation map. For this we first have to classify those cycles.

Remark 3.30 (Codimension-1 types in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$)
In the following, a connected component of $\Gamma \setminus \bigcup_{j=1}^{n} \overline{x_j}$ will be called a *region* of the curve.

Consider a cell of a codimension-1 cycle in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ on which the evaluation map is injective. It corresponds to a combinatorial type of curves having exactly one 4-valent vertex, with all other vertices being 3-valent.
If this 4-valent vertex has an adjacent marking, an argument as in the proof of the second part of proposition 3.18 shows that there is exactly one region that is bounded (by markings), whereas the others contain exactly one end. We will call this type (A); in the picture below the bounded region is drawn with dotted lines.
If the 4-valent vertex has no adjacent marking it lies in a unique region. The same argument as above then shows that

- either all regions have exactly one end (type (B) below, with the region containing the 4-valent vertex drawn with dotted lines); or
- the region with the 4-valent vertex has exactly two ends, there is one other bounded region, and all other regions have exactly one end (type (C) below, where the bounded region is drawn with dotted lines).


(A) (B) (C)

The weights of the images of those cycles under the evaluation map can be computed by the following formula.

Lemma 3.31
Let Z be a cycle of dimension $2n-1$ in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ and $\sigma' \in \text{ev}_*(Z) \subset \mathbb{R}^{2n}$ a cone of the same dimension, i.e. of codimension 1 in $\mathbb{R}^{2n}$. Then, with the convention and notations from definition 3.15 c), the weight of σ' in the cycle $\text{ev}_*(Z)$ is

$$w_{\text{ev}_*(Z)}(\sigma') = \sum_{\substack{\sigma \in Z \\ \text{ev}(\sigma)=\sigma'}} w_Z(\sigma) \cdot \text{mult}(\sigma)$$

where the multiplicity of a cone σ of dimension $2n-1$ in $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ is defined as

$$\text{mult}(\sigma) := D(\mathbb{Z}^{2n}/\text{ev}(\Lambda_\sigma)).$$

If C is any curve corresponding to a point in σ, we will write the multiplicity $\text{mult}(\sigma)$ also as $\text{mult}(C, h)$.

Proof. By definition 3.15 c) we have to sum over the numbers $w_Z(\sigma) \cdot |\Lambda_{\sigma'}/\text{ev}(\Lambda_\sigma)|$ for all $\sigma \in Z$ with $\text{ev}(\sigma) = \sigma'$. But as $\Lambda_{\sigma'}$ is by definition a saturated lattice in $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n}$, we have $\mathbb{Z}^{2n} = \Lambda_{\sigma'} \oplus \mathbb{Z}$, and thus the torsion parts of $\Lambda_{\sigma'}/\text{ev}(\Lambda_\sigma)$ and $\mathbb{Z}^{2n}/\text{ev}(\Lambda_\sigma)$ agree. □

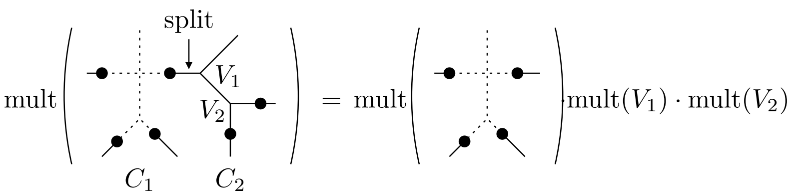
Remark 3.32
By lemma 3.27, the number $\text{mult}(\sigma) = D(\mathbb{Z}^{2n}/\text{ev}(\Lambda_\sigma))$ in lemma 3.31 can be computed as the $D(A)$ of any $(2n) \times (2n-1)$ matrix A representing the $\mathbb{Z}$-linear map $\text{ev} : \Lambda_\sigma \to \mathbb{Z}^{2n}$. By [GKM09, remark 5.2] and [GM08, example 3.3] one possibility to set up this matrix is to use the lengths of all bounded edges and the position in $\mathbb{R}^2$ of a root vertex as coordinates for Λ_σ.

We will now compute explicitly the multiplicities in lemma 3.31 for the cases of remark 3.30 and proceed in several steps.

Lemma 3.33 (**Splitting off vertex multiplicities**)
Let C be a curve in a codimension-1 cone of $\mathcal{M}_{0,n}(\mathbb{R}^2, \Delta)$ as in remark 3.30. Assume that there is an edge of C such that splitting this edge yields two parts C_1 and C_2 of C, where C_2 contains only regions with exactly one end and having only 3-valent vertices. Then

$$\text{mult}(C, h) = \text{mult}(C_1, h_1) \cdot \prod_{V \in C_2} \text{mult}(V),$$

where the product runs over all vertices in C_2 with no adjacent marking, and the multiplicity mult(V) of such a vertex V is defined as usual as the absolute value of the determinant of two of the adjacent direction vectors [GM08, definition 3.5]. The following picture shows an example.

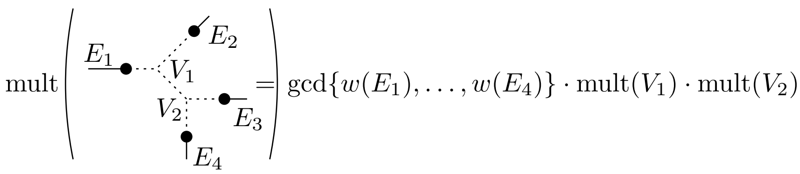


Proof. Let k be the number of unbounded ends of C_2. Then C_2 has $k-1$ markings and $2k-2$ bounded edges (including the split edge). Choosing the root vertex to be in C_1 (see remark 3.32), only the $2k-2$ coordinates of the $k-1$ markings in C_2 depend on the $2k-2$ lengths of the bounded edges in C_2. Hence the matrix for ev as in remark 3.32 has the form as in corollary 3.29, with B_1 the $(2k-2) \times (2k-2)$ block consisting of these coordinates and lengths. As the absolute value of the determinant of B_1 equals the product of the vertex multiplicities in C_2 by [GM08, proposition 3.8] and B_2 is precisely the matrix for the evaluation map on C_1, the claim follows from corollary 3.29. □

Lemma 3.34 (**Multiplicity of a bounded region**)
Let C be a 3-valent curve that has exactly one bounded region, with all other regions being single ends (such a curve occurs for instance as a part of the types (A) and (C) in remark 3.30). Then the multiplicity of C is

$$\text{mult}(C, h) = \gcd\{w(E) : \ E \text{ end in } C\} \cdot \prod_{V \in C} \text{mult}(V),$$

where the weight $w(E)$ of the edge E is the gcd of the two coordinates of the direction vector of E, and the product is taken over all vertices of C with no adjacent marking. The following picture shows an example.



Proof. We set up the matrix A for the evaluation map as in remark 3.32, with the root vertex within the bounded region. To compute the minors of A as required by lemma 3.28 we have to erase one of its rows. Note that the rows of A correspond to the coordinates of the markings in $\mathbb{R}^2$. So let us assume that we erase the row for the i-th coordinate of the marking x_j for some $i = 1, 2$ and $j = 1, \ldots, n$. Note that the length of the bounded edge E_j adjacent to x_j is needed only for the coordinates of x_j in $\mathbb{R}^2$, and so in the remaining matrix the column corresponding to E_j has at most one non-zero entry, namely for the other coordinate of x_j in

$\mathbb{R}^2$. Laplace expansion of the determinant w.r.t. the E_j column thus simply gives the product of this coordinate of x_j in $\mathbb{R}^2$ and the determinant of the evaluation matrix for the curve where the marking x_j is deleted (and thus E_j becomes an unbounded end). But this determinant just equals the product of all vertex multiplicities by [GM08, proposition 3.8].
Altogether we see that $D(A)$ is the product of all vertex multiplicities times the gcd of both coordinates of all markings in $\mathbb{R}^2$, as we have claimed. □

Corollary 3.35 (**Multiplicity of the types (A) and (C)**)
Let C be a curve in codimension 1 as in remark 3.30. If C is of type (A) with bounded region C_b then its multiplicity is

$$\text{mult}(C,h) = \gcd\{w(E) : \ E \text{ edge in } C_b \text{ with adjacent marking}\} \cdot \prod_{V \in C} \text{mult}(V).$$

If it is of type (C) with bounded region C_b then its multiplicity is

$$\text{mult}(C,h) = \gcd\{w(E) : \ E \text{ edge in } C_b \text{ with adjacent marking}\} \cdot |\det(v,v')| \cdot \prod_{V \in C} \text{mult}(V)$$

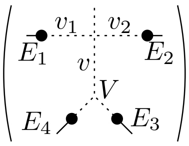
where v and v' are the directions of the two fixed adjacent edges at the 4-valent vertex (i.e. the ones that do not connect to an end within their region when coming from the 4-valent vertex). In both formulas, the product is taken over all 3-valent vertices without adjacent marking in C.

Proof. Let V be the 4-valent vertex of C. If C is of type (A) we can first use lemma 3.33 to split off all vertices behind the two unmarked edges adjacent to V that do not lead to the bounded region. This way we get the multiplicities of all split-off vertices as a factor, and are left with a curve where two of the unmarked edges adjacent to V are solitary ends (as it is already the case in the example picture in remark 3.30). Now the evaluation matrix of this curve is precisely the same as for the curve where these two ends with direction vectors v_1 and v_2 are replaced by one end with direction $v_1 + v_2$. Now in the remaining curve we can continue to split off all vertices that lie outside of the (closure of the) bounded region. This way we are left with a curve whose multiplicity has been computed in lemma 3.34. Altogether, we get the result stated in the corollary.
If C is of type (C) the procedure is very similar. We first split off all vertices behind the two free edges adjacent to V and replace the resulting two solitary ends at V by one. This makes the 4-valent vertex V into a new 3-valent one for which two adjacent direction vectors are v and v'. As above, we continue to split off all vertices that are outside of the bounded region (one of which will be the new one with multiplicity $|\det(v,v')|$), and use lemma 3.34 to obtain the result. □

Lemma 3.36 (**Multiplicity of a region with 4-valent vertex and one end**)
Let C be a curve in codimension 1 as in remark 3.30 that has a region with a 4-valent vertex and one end directly adjacent to it, all other vertices being 3-valent and all other regions being single ends. Then the multiplicity of C is

$$\text{mult}(C,h) = \gcd\{w(E) \cdot |\det(v_E^1, v_E^2)| : \ E \text{ end with marking in } C\} \cdot \prod_{V \in C} \text{mult}(V),$$

where the product is taken over all 3-valent vertices of C with no adjacent marking, and v_E^1 and v_E^2 denote the direction vectors of the two fixed edges adjacent to the 4-valent vertex that do not connect to E. The following picture shows an example.

mult 

$= \gcd\{w(E_1)\,|\det(v,v_2)|, w(E_2)\,|\det(v,v_1)|,$
$w(E_3)\,|\det(v_1,v_2)|, w(E_4)\,|\det(v_1,v_2)|\} \cdot \mathrm{mult}(V)$

Proof. The proof is very similar to that of lemma 3.34. Let V' be the 4-valent vertex of C and E its unique free end. We set up the matrix A of the evaluation map using V' as the root vertex. To compute a maximal minor of A we delete the row corresponding to the i-th coordinate of the marking x_j. Performing a Laplace expansion of the minor w.r.t. the column corresponding to the length of the bounded edge adjacent to x_j we obtain the other coordinate of x_j in $\mathbb{R}^2$ times the determinant of the evaluation matrix corresponding to the curve where the marking x_j has been deleted. In this new curve we can use the technique of lemma 3.33 to split off all vertices behind the one that lead to x_j. We can then replace the two resulting solitary ends at V' (E and the one just created by splitting off vertices) at V' by one, leading to a new 3-valent vertex with multiplicity $|\det(v_E^1, v_E^2)|$. The resulting determinant gives the product of all vertex multiplicities by [GM08, proposition 3.8]. Taking the gcd of these expressions for all rows of A yields the desired result. □

Corollary 3.37 (**Multiplicity of the type (B)**)
Let C be a curve in codimension 1 as in remark 3.30. If C is of type (B), and C' denotes the region with the 4-valent vertex, then its multiplicity is

$$\mathrm{mult}(C,h) = \gcd\{w(E)\cdot|\det(v_E^1, v_E^2)|\} \cdot \prod_{V \in C} \mathrm{mult}(V)$$

where

- the product is taken over all 3-valent vertices without adjacent marking in C;
- the gcd is taken over all edges in C' that are adjacent to a marking and lie behind one of the three fixed edges adjacent to the 4-valent vertex; and
- v_E^1 and v_E^2 denote as in lemma 3.36 the directions of the two fixed edges adjacent to the 4-valent vertex that do not connect to E.

Proof. The proof is similar to that of corollary 3.35: first we can split off all vertices outside of C' and behind the one free end of the 4-valent vertex, and then we use lemma 3.36 for the resulting curve. □

Summarizing, we can now rephrase lemma 3.31 as follows.

Corollary 3.38
Let Z be a cycle of dimension $2n-1$ in $\mathcal{M}_{0,n}(\mathbb{R}^2,\Delta)$ and $\sigma' \in \mathrm{ev}_*(Z) \subset \mathbb{R}^{2n}$ a cone of the same dimension, i.e. of codimension 1 in $\mathbb{R}^{2n}$. Then, with the convention and notations from definition 3.15 c), the weight of σ' in the cycle $\mathrm{ev}_*(Z)$ is

$$w_{\mathrm{ev}_*(Z)}(\sigma') = \sum_{\substack{\sigma \in Z \\ \mathrm{ev}(\sigma)=\sigma'}} w_Z(\sigma)\cdot \mathrm{mult}(\sigma)$$

where the multiplicity $\mathrm{mult}(\sigma)$ of a cone σ of dimension $2n-1$ in $\mathcal{M}_{0,n}(\mathbb{R}^2,\Delta)$ is given by corollaries 3.35 and 3.37 depending on their type as in remark 3.30.
In particular, this gives an explicit formula for the cycles in $\mathbb{R}^{2n}$ of points in special position (v1) and (v2) as in propositions 3.18 and 3.22.

4 Real tropical geometry

The purpose of this section is to describe known facts about parametrized tropical curves that can be used to translate classical Welschinger theory to the tropical world.

4.1 Real tropical curves

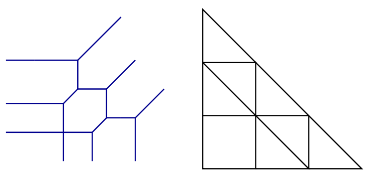
We first start with the definition of real tropical curves passing through only real points. In order to illustrate the relationship of real tropical curves to real algebraic curves we have first to understand combinatorial patchworking. Therefore, we need some vocabulary.

Remark 4.1 (**Dual/Newton subdivision of a simple parametrized tropical curve**)
[IMS09, section 2.5.1]. For an n-marked parametrized tropical curve (C,h), let P_Δ be the *Newton polygon* associated to Δ, i.e. the convex lattice polygon that we obtain when we rotate all vectors v_i in Δ by $-\pi/2$ and draw them in $\mathbb{Z}^2$ one after the other, each with lattice length equal to the weight of the corresponding edge, in a chain, starting at a lattice point of $\mathbb{Z}^2$. This chain is closed since $h(\Gamma)$ is balanced at each vertex. (C,h) defines a subdivision $P_\Delta = P_1 \cup \ldots \cup P_N$ into smaller polygons P_i which is dual to $h(\Gamma)$ in the sense that:

- components of $\mathbb{R}^2 \setminus h(\Gamma)$ are in bijection to vertices in the subdivision,
- edges of $h(\Gamma)$ are in bijection to edges of the subdivision such that an edge E of $h(\Gamma)$ is dual to an orthogonal edge of the subdivision of lattice length $w(E)$,
- vertices of $h(\Gamma)$ are in bijection to the polygons P_i such that the valence of a vertex $V \in h(\Gamma)$ equals the number of sides of the dual polygon.

Note that the multiplicity of a 3-valent vertex $V \in h(\Gamma)$ can then be computed as the lattice area of dual triangle.
A parametrized tropical curve is simple iff the dual subdivision contains only triangles and parallelograms [Mik05, lemma 4.5]. Some authors [IMS09, p. 52] call such a tropical curve also *nodal*. This is motivated by the fact that they are the tropicalization of a nodal algebraic curve [Shu06a, lemmata 3.5 & 3.6].

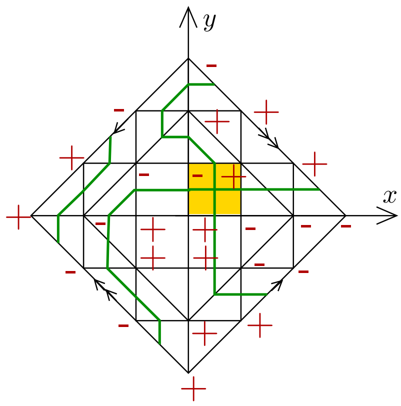
Example 4.2
The picture below gives on the left hand side the image $h(\Gamma)$ of a parametrized tropical curve of degree 3. This means that all edges are of weight 1 and there are exactly 3 of direction $(-1,0)$,$(0,-1)$ and $(1,1)$ respectively. Observe the crossing of two edges in the lower left corner. Knowing only the image $h(\Gamma)$ but not the underlying curve Γ, we cannot say if this curve is of genus 0 or 1! On the right hand side we drew the dual subdivision of this curve.



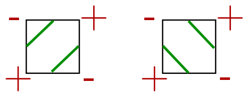
Remark 4.3 (**Combinatorial patchworking**)
[IV96], [IMS09, section 2.3.3]. Roughly speaking, *combinatorial patchworking* due to Oleg Viro is a method to construct plane real algebraic curves of degree d from weighted convex triangulations of the triangle Conv$((0,0),(d,0),(0,d))$. It is useful for instance to classify real plane curves of given degree. Also, it is a special case of Viro's more general patchworking

method as described in [IMS09, section 2.3]. In this work, we want to generalize this method to subdivisions of P_Δ containing triangles and parallelograms. We will restrict ourselves to combinatorial patchworking as the general version takes more time to explain and patchworking is not in the focus of this thesis. However, we hope that the following exposition helps to understand the flavor of patchworking and to get a feeling for why the theorems of this section are true.

Let us start with the case where $\Delta = d$, i.e. P_Δ is the triangle with corners $(0,0)$, $(d,0)$ and $(0,d)$. We attach to each vertex (i,j) in the subdivision of P_Δ a sign $\sigma_{i,j} \in \{+,-\}$. Now, we reflect this subdivision without signs at the x-axis to obtain a subdivided triangle below the x-axis. Afterward, we reflect this new triangle with corners $(0,d)$, $(d,0)$ and $(0,-d)$ at the y-axis to obtain finally a square S which is symmetric w.r.t. the x- and y-axes. We extend the signs from a point (i,j) from the original triangle to the mirror image w.r.t. an axis by preserving the sign if the (lattice) distance from (i,j) to the axis is even and changing the sign otherwise. For instance, study the example below where $d = 3$. We draw a broken line segment in S by connecting midpoints of edges whose endpoints have different signs, respectively.



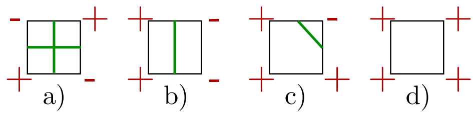
Observe that there is only a line in a triangle iff exactly two of the signs of the corners are the same and there are up to two lines in a parallelogram. The curve has one node in the parallelogram of the first quadrant. The sign distribution there is chosen such that the signs on each diagonal are the same. In principle, one could also associate one of the following broken lines to this paralleleogramm.



But this choice corresponds to a non-nodal curve. The space T we obtain if we glue the directed edge from $(0,-d)$ to $(d,0)$ to the directed edge from $(0,d)$ to $(-d,0)$ and the directed edge from $(0,d)$ to $(d,0)$ to the directed edge from $(0,-d)$ to $(-d,0)$, respectively, as indicated on the figure is homeomorphic to the real projective space $\mathbb{P}^2_{\mathbb{R}}$. In this space the broken line segment can be deformed to a connected curve of the same topology. The main result is now

Theorem 4.4 (**Patchworking theorem**)
[Vir80]. For every convex triangulation of the triangle with corners $(0,0)$, $(d,0)$ and $(0,d)$ and every choice of signs at its vertices as described above there is a nonsingular real algebraic plane projective curve of degree d and a homeomorphism $\mathbb{P}^2_{\mathbb{R}} \to T$ mapping the set of real points of this curve onto the line segment in T.

This theorem generalizes to the case where we also take account of the parallelograms [Shu06a, lemma 3.5]. Then, we get possibly nodal curves. Indeed: in total there are 4 choices of signs for a parallelogram – up to a global sign as depicted below. Case a) is in the example above. Case b) gives a node in the 4th quadrant, i.e. a square as in a) in the 4th quadrant, case c) yields no node at all and case d) correspond to a node in the third quadrant.



This theorem can be generalized to other real toric surfaces like the real unnodal Del Pezzo surfaces as in 1.19.

The next step is to explain how to construct real tropical curves.

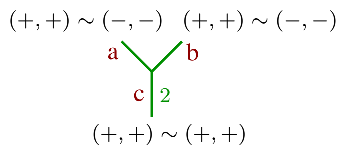
Remark 4.5 (**Construction of simple real tropical curves**)
[Mik05, definition 7.8]. Let (C, h) be a simple n-marked parametrized tropical curve such that the unbounded edges are all of weight 1 or 2. Let E be an (non-contracted) edge of direction vector u and weight w in $h(\Gamma)$. We define an equivalence relation on $(\mathbb{Z}/2\mathbb{Z})^2$ by $(x, y) \sim (x', y') :\Leftrightarrow (x, y) + u \equiv (x', y')$ mod 2. This gives us the set of equivalence classes $S_E := (\mathbb{Z}/2\mathbb{Z})^2/\sim$. The choice of an element $(x, y) \in S_E$ for an edge E is called a *phase of* E. Note that $S_E \cong (\mathbb{Z}/2\mathbb{Z})^2$ if w is even and S_E has two elements if w is odd. We assign an element $(x, y) \in S_E$ to each edge E of $h(\Gamma)$ such that the phases of three edges adjacent to a (3-valent) vertex are subject to the following conditions:

a) they are equal if the three edges are all of even weight

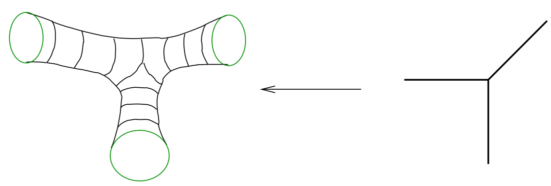
b) they have all a common representative in $(\mathbb{Z}/2\mathbb{Z})^2$ if two of the edges are odd and one is even

c) each element in $(\mathbb{Z}/2\mathbb{Z})^2$ being the representative of at least one phase is the representative of exactly two phases of edges adjacent to the vertex if all the three edges are odd.

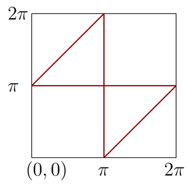
We then say that the phases are *compatible*. In the following we will identify $0 = +$ and $1 = -$.
Here is an example. The 2 means that the edge c with direction vector $(0, -2)$ is of weight 2 while the other edges a and b (of direction vectors $(-1, 1)$ and $(1, 1)$) are of weight 1. For the edges a and b we choose the equivalence class of $(+, +)$ as phase with representatives $(+, +)$ and $(-, -)$. For the edge c we also take as phase the equivalence class of $(+, +)$ with unique representative $(+, +)$. This is then a compatible choice of phases.



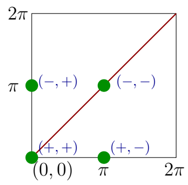
Compatible phases define a real curve as follows. Given the torus part $V = V(f) \cap (\mathbb{C}^*)^2 \subseteq \mathbb{R}^2 \times (S^1)^2$ of a plane projective curve the tropicalization trop(V) forgets the argument $\varphi = (\varphi_1, \varphi_2) \in (S^1)^2$ of a point $z = (|z_1| \cdot e^{i\varphi_1}, |z_2| \cdot e^{i\varphi_2}) \in V$, where S^1 is the unit circle. Conversely, given a parametrized tropical curve (C, h), we have to determine the argument of a point in $h(\Gamma)$ in order to lift this tropical curve to the torus part in $(\mathbb{C}^*)^2$ of a projective curve as studied in [Mik04b, section 5]. Mikhalkin associates there to each 3-valent vertex a pair of pants which all have to be glued together appropriately, i.e. we have to identify the boundary circles of the pairs of pants correctly. Each pair of pants is diffeomorphic to a sphere S^2 with three punctures, corresponding to the green boundary circles in the figure below and which can be associated to the edges of the tropical curve.



Such a pair of pants corresponds to the torus part $V(f) \cap (\mathbb{C}^*)^2$ of a projective line $f = x_1 + x_2 - x_3$ which has 3 holes associated to $x_i = 0$ each, i.e. they are associated to the three coordinates axes of $\mathbb{P}^2$. So, given a curve (C, h) we have to equip each edge with compatible data which tell us where the boundary circles of the corresponding decomposition into pairs of pants are located in $(\mathbb{C}^*)^2$. These are so called *phases* of the edges [Mik, section 18]. Writing $(S^1)^2$ as $[0, 2\pi]^2$ where we identify 0 with 2π the coordinates of the three boundary circles of the line $f = x_1 + x_2 - x_3$ are depicted below as red lines. Note that the direction vector of an edge coincides with the direction vector of the corresponding line in $[0, 2\pi]^2$.



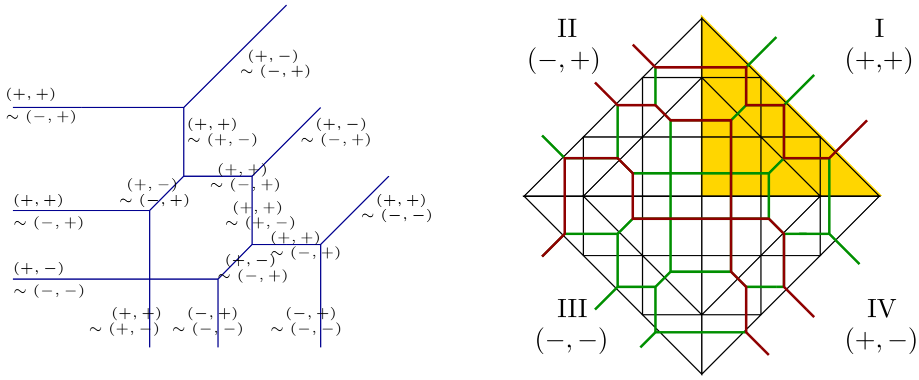
If, in addition, we want our projective curve to be real, we have to make sure that the lifted curve of $h(\Gamma)$ has non-zero intersection with $(\mathbb{R}^*)^2$ (we use the standard real structure here). Writing again $(S^1)^2$ as $[0, 2\pi]^2$ we see that the (standard) complex conjugation restricted to $(S^1)^2$ has four fix points, namely $(0, 0)$, $(\pi, 0)$, $(0, \pi)$ and (π, π) to which we can assign pairs of signs as depicted below.



So for real curves we have to make sure that the lines in $[0, 2\pi]^2$ corresponding to the position of the boundary circles pass through at least two fix points each under the complex conjugation. E.g. if the phase of an edge in our real tropical curve is $(+, +) \sim (-, -)$ then we choose the red line in the picture.
Each edge of even weight m in a tropical curve defines at least m real structures [Mik05, theorem 3 proven with help of lemma 8.24].

Remark 4.6 (**Motivation for construction 4.5**)
Let (C, h) be a simple parametrized tropical curve having only edges of odd weight and whose edges are equipped with compatible phases as in 4.5. Assume that it is of degree d. It defines a real projective curve of degree d as explained below. First, we mirror the triangle with corners $(0, 0)$, $(d, 0)$ and $(0, d)$ as in 4.3. But this time we also reflect the phases! Secondly, we assign to the first quadrant the pair $(+, +)$, to the second $(-, +)$, to the third $(-, -)$ and to the fourth $(+, -)$. Then we choose in each quadrant the edges having a phase with an representative coinciding with the pair assigned to the quadrant. This gives a broken line segment similiar to 4.3 by condition c) imposed on the phases in 4.5. Indeed, this condition makes sure that once chosen a representative of a phase of an edge there is exactly one other edge with a phase with the same representative, so a line continues at a vertex in exactly one other direction.
The following figure, taken from [Mik04c], shows a real plane projective curve of degree 3 constructed in this way. We start with a simple parametrized tropical curve (without markings) of degree 3. For edges of direction vector $(-1, 0)$ we have two choices for a phase as we have $(0, 0) + (-1, 0) \equiv (1, 0)$ mod 2 and $(1, 1) + (-1, 0) \equiv (0, 1)$ mod 2. Similarly, we have for an edge of direction $(0, -1)$ the equivalences $(0, 0) \sim (0, 1)$ and $(1, 1) \sim (1, 0)$; for an edge of direction $(1, 1)$ the equivalences $(0, 0) \sim (1, 1)$ and $(1, 0) \sim (0, 1)$. Our choice of phases is depicted on the left hand side. The tropical curve, reflects to the other quadrants, appears on

the right hand side in blue and the broken line segment is colored in red. If we identify the edges of the square as in 4.3 we obtain a connected curve which is homeomorphic to a nodal curve of degree 3 in $\mathbb{P}^2_{\mathbb{R}}$.



Definition 4.7 (**Toric Del Pezzo degrees**)
We say that a degree Δ is *toric Del Pezzo* if it consists of the primitive normal directions of edges of one of the polytopes P depicted just before definition 1.23, where each direction appears l times if l is the lattice length of the corresponding edge. We will denote the corresponding toric Del Pezzo surface S_Δ instead of S_P.

Theorem 4.8 (**Realization of real tropical curves by algebraic curves I**)
[Shu06a, proposition 6.1 b)] for irreducible curves and edges of odd weight and [Mik05, theorem 3] for the general case.
Let S_Δ be the toric surface associated to a toric Del Pezzo degree Δ with tautological linear system $|D|$, $g \in \mathbb{N}$, and $\omega = (P_1, \dots, P_r)$ a configuration of $r = -D \cdot K_{S_\Delta} - 1$ points in general position in $\mathbb{R}^2$. Let furthermore (C, h) be a simple r-marked parametrized tropical curve of degree Δ and genus g having only edges of odd weight, endowed with a compatible choice of phases on its edges, and passing through ω. Then there is generic point configuration $\omega' = (Q_1, \dots, Q_r)$ with trop$(\omega') = \omega$ and exactly one nodal real projective plane curve lying in $|D|$ and of genus g, passing through ω' and tropicalizing to $h(\Gamma)$.
For tropical curves in the setting above having at least one edge of even weight there are several real projective plane curves tropicalizing to $h(\Gamma)$.

Remark 4.9
Note that the proof of this theorem uses a more general version of patchworking than 4.3. Also, [Mik05, theorem 3] says even more than this. In fact, it is a Correspondence Theorem for real plane projective curves of degree d and genus g passing a generic configuration ω' of points. It implies that if we count real tropical curves of genus $g > 0$, this number generally depends on the point configuration ω' by theorem 1.18.

4.2 Tropical Welschinger numbers

Let us first consider the case of real points only.

Remark 4.10 (**Nodes of simple real parametrized tropical curves**)
We may wonder which types of nodes are meant as we have seen in 1.14 that there are several types of nodes for real projective plane curves. How nodes of parametrized tropical curves in general might look like was displayed in example 2.37. For parametrized real tropical curves the proof of the Correspondence Theorem [Mik05, theorem 6] gives up a partial answer. Unfortunately, the proof is by algebraic patchworking, so in context of this thesis we can only state the results. It says that if we have in the dual subdivision of the tropical curve (C, h)

a triangle with an edge of even lattice length, then there are always one or several pairs of real nodal curves having real isolated nodes of different parity each tropicalizing to (C, h). If each triangle in the subdivision has only edges of odd weight then each triangle ▲ with $l = \text{Int}(▲) \cap \mathbb{Z}^2$ integer interior points gives a multiplicative factor of l mod 2 to the parity of isolated nodes that real nodal curves tropicalizing to (C, h) have, see also [Shu06b, lemma 2.3]. This motivates the following definition which we need for the definition of tropical Welschinger numbers.

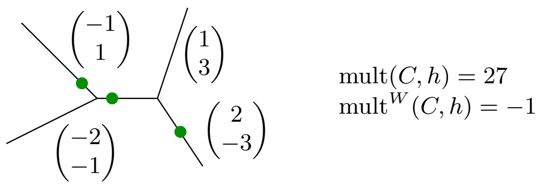
Definition 4.11 (**Welschinger multiplicity of a simple parametrized tropical curve**)
[Mik05, definition 7.19]. Let (C, h) be a marked or non-marked simple parametrized tropical curve and V a (3-valent) vertex of $h(\Gamma)$. Define its *real* multiplicity as $\text{mult}_{\mathbb{R}}(V) = (-1)^{\frac{\text{mult}(V)-1}{2}}$ if $\text{mult}(V)$ is odd and $\text{mult}_{\mathbb{R}}(V) = 0$ otherwise. The *Welschinger multiplicity or tropical Welschinger sign* $\text{mult}^W(C, h)$ is then defined as product $\prod_V \text{mult}_{\mathbb{R}}(V)$ where V ranges over all vertices in $h(\Gamma)$.

Remark 4.12
The definition of Welschinger multiplicity is equivalent to

$$\text{mult}^W(C, h) := \begin{cases} 0, & \text{if } \text{mult}(C, h) \equiv 0 \text{ mod } 4, \\ 1, & \text{if } \text{mult}(C, h) \equiv 1 \text{ mod } 4, \\ 0, & \text{if } \text{mult}(C, h) \equiv 2 \text{ mod } 4, \\ -1, & \text{if } \text{mult}(C, h) \equiv 3 \text{ mod } 4. \end{cases}$$

Note also that the Welschinger multiplicity is defined for tropical curves of any genus!

Example 4.13
The figure below shows a subgraph of $h(\Gamma)$ of a marked parametrized tropical curve. With the rule of 4.12 its Welschinger multiplicity can be easily computed as $27 \text{ mod } 4 \equiv -1$.



Example 4.14
Note that given a parametrized tropical curve having only edges of odd weight passing through points in general position, then there is only one possibility to define compatible phases on this curve. Hence, there is only one real curve tropicalizing to it by theorem 4.8 and by 4.12, this tropical curve has Welschinger multiplicity -1. Hence if $g = 1$ the real curve has only one real isolated node.

Definition 4.15 (**Tropical Welschinger numbers**)
Let S_Δ be the toric surface associated to a toric Del Pezzo degree Δ with tautological linear system $|D|$, $g \in \mathbb{N}$, and $\omega = (P_1, \ldots, P_n)$ a configuration of $n = -D \cdot K_{S_\Delta} - 1$ points in general position in $\mathbb{P}^2_{\mathbb{R}}$. Then we define the *tropical Welschinger number* $W^{\text{trop}}_\Delta(g, \omega, r, 0)$ to be the number of r-marked simple parametrized tropical curves (C, h) of genus g and lying in $|D|$ counted with multiplicity $\text{mult}^W(C, h)$ and passing through the tropicalization of ω.

Convention 4.16
If S_Δ is the projective plane $\mathbb{P}^2$ and D is of degree d we will write $W^{\text{trop}}(d, g, \omega, 3d + g - 1, 0)$ instead of $W^{\text{trop}}_\Delta(g, \omega, r, 0)$.

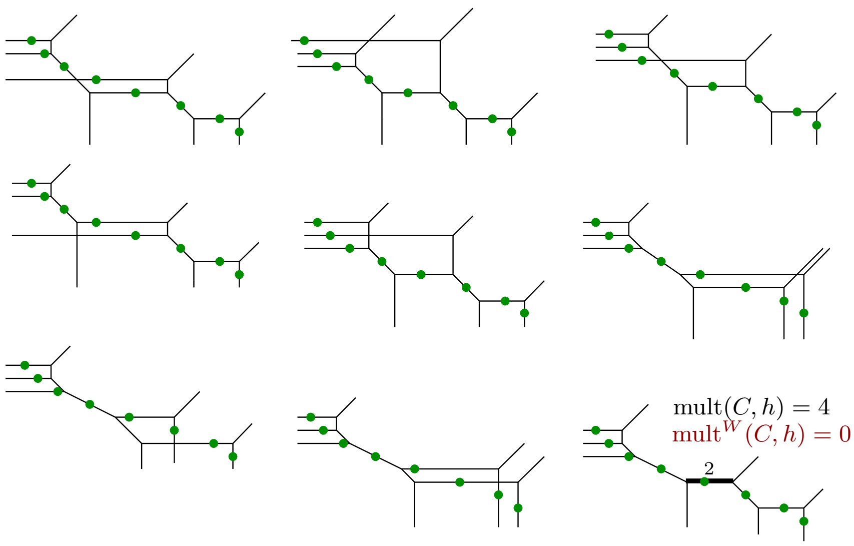
We are now ready to establish the correspondence between Welschinger and tropical Welschinger numbers for curves passing through a real point configuration.

Theorem 4.17 (**Correspondence Theorem for Welschinger numbers I**)
[Mik05, theorem 6]. In the setting above there exists for a given configuration ω' of $3d + g - 1$

generic points in $\mathbb{R}^2$ a configuration ω of $3d+g-1$ generic points in $\mathbb{P}^2_{\mathbb{R}}$ s.t. $\omega' = \text{trop}(\omega)$ and $W^{\text{trop}}(d,g,\omega',3d+g-1,0) = W(d,g,\omega,3d+g-1,0)$. That is, the tropical Welschinger number coincides with the classical Welschinger number as defined in 1.15.

Convention 4.18 (**Immediate consequences of this theorem**)
It follows from theorems 4.17 and 1.16 that for $g=0$ the number $W^{\text{trop}}(d,g,\omega,3d+g-1,0)$ does not depend on the choice of ω. We will write therefore in this case $W^{\text{trop}}(d,3d-1,0)$ instead of $W^{\text{trop}}(d,g,\omega,3d+g-1,0)$.

Example 4.19
Let us review example 2.37 in this light. All the curves there have Welschinger signs which equal the multiplicity $\text{mult}(C,h)=1$ each besides the last curve which has (by rule 4.12) Welschinger multiplicity 0 due to the edge of weight 2. In total we then have $W^{\text{trop}}(3,8,0)=8$, a number that we have seen already in 1.14.



$W^{\text{trop}}(3,8,0)=8$

Remark 4.20 (**Local proof of invariance similar to [GM07b, theorem 4.8]**)
In [IKS09] the authors prove the invariance of $W^{\text{trop}}(d,3d-1,0)$ by a local study in the moduli space as in [GM07b, theorem 4.8]. In the example of 2.36 the two curves corresponding to the movement upwards have Welschinger multiplicity -1 and 1, respectively, and the curve that we obtain when we move the point downwards has Welschinger multiplicity 0. So the general argument shows that these tropical Welschinger numbers are invariant by purely tropical means. They also proved a Caporaso-Harris type formula [IKS09, theorem 3] similar to 1.9 which enables us to compute also Welschinger numbers for high degrees recursively. For this formula they need relative Welschinger numbers in the sense of 1.10 which are invariant, too.

Remark 4.21 (**Invariance of tropical Welschinger numbers of higher genus** g)
The authors of [IKS09] also prove the invariance of tropical Welschinger numbers for higher genus g [IKS09, section 3.1] as they realized that the proof can be generalized. But this is only a tropical concept which has no classical counterpart yet, cf. theorem 1.18.

Until now we only considered real plane curves passing through a configuration of real points. To define tropical Welschinger numbers which count tropical curves passing also through pairs of complex conjugate points we have to explain which curves we are going to consider.

Convention 4.22
Let (C,h) be an n-marked parametrized tropical curve. Then we call an edge E of the underlying graph Γ *even/odd*, if the image $h(E)$ has an even/odd number of edges in the preimage.

The following definition is an adapted version of [BM08, definition 4.11] to fit with [Shu06b]. This generalizes construction 4.5 as we can also associate to edges of Shustin curves compatible phases.

Definition 4.23 (**Shustin tropical curve**)
Let (C, h) be a simple $(n = r + 2s)$-marked rational parametrized tropical curve of toric Del Pezzo degree Δ and assume that $n = |\Delta| - 1$ with $r, s \geq 0$ integers. Furthermore, let $\sigma : \Gamma \to \Gamma$ be an isometric involution such that $h = h \circ \sigma$ and there is a permutation $\tau \in S_n$ with $\sigma(x_i) = x_{\tau(i)}$ for all $1 \leq i \leq n$ and such that the following properties hold:

a) the unbounded edges of $h(\Gamma)$ are all of weight 1 or 2,

b) the images $h(x_i)$ of *real markings* $x_1, \ldots, x_r$ lie on edges of odd weight and are disjoint,

c) for the images of the *complex markings* $x_{r+1}, \ldots, x_{r+2s}$ we have x_i and $x_{\tau(i)}$ are adjacent to the same 5-valent vertex in Γ or they lie on edges E_1, E_2 such that $h(E_1)$ and $h(E_2)$ are parallel, and there are exactly s disjoint images $h(x_i)$ of complex markings,

d) the set Fix(σ) equals the set of odd edges of Γ.

Convention 4.24
In the following, we will draw images of real markings as small dots and images of complex markings as big dots. When we are in c) in the case of parallel edges E_1, E_2, then the images under h of these two edges coincide by d): it is an edge of even weight. Therefore, we will draw the image under h of these two edges as a pair of parallel edges of odd weight, each with a big dot $h(x_i)$ on them. When they are ends, we will call this pair a *double end*.

Remark 4.25
Fix(σ) is connected since we consider rational curves. It follows from the definition that images of complex markings lie on edges of even weight or on 3-valent vertices to which only edges of odd weight are adjacent, and each connected component of even edges meets the component of odd edges Fix(σ) in exactly one vertex. Also, there are $r + s$ image points $h(x_i)$.

Theorem 4.26 (**Realization of real tropical curves by algebraic curves II**)
[Shu06b, lemma 3.2]. Let Δ be a toric degree and S_Δ equipped with the standard real structure. Furthermore, let $r, s \geq 0$ be such that $r + 2s = |\Delta| - 1$ and $\omega = (P_1, \ldots, P_{r+2s})$ a complex conjugation invariant configuration of $r + 2s$ points in general position in S, where $P_1, \ldots P_r$ are conjugation invariant points each. Given an n-marked Shustin tropical curve (C, h) of degree Δ equipped with patchworking data similiar to the phases above and passing through the tropicalization of ω. Then there is at least one real projective plane rational nodal curve tropicalizing to (C, h).

The following Shustin-multiplicity was originally defined for unparametrized curves. But we will see that it can also be redefined for parametrized curves.

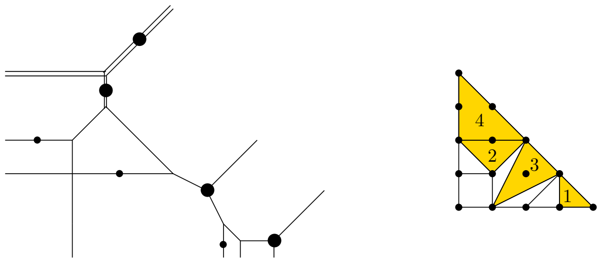
Definition 4.27 (**Shustin-multiplicity, see [Shu06b] section 2.5**)
Let (C, h) be a Shustin curve. Denote by a the number of lattice points inside triangles of this subdivision, by b the number of triangles such that all sides have even lattice length, and by c the number of triangles whose lattice area is even. Then we define the *Shustin-multiplicity* of $h(\Gamma)$ to be

$$\text{mult}_S(h(\Gamma)) := (-1)^{a+b} \cdot 2^{-c} \cdot \prod_V \text{mult}(V),$$

where the product goes over all triangles with even lattice area or dual to vertices with a complex marking.
For an unparametrized curve $h(\Gamma)$, this coincides with the definition of multiplicity in [Shu06b] section 2.5.

Remark 4.28
Note that the Shustin multiplicity does not only take values in $\{-1, 0, 1\}$ as it was the case in 4.11.

Example 4.29
The following picture shows the image $h(\Gamma)$ of a Shustin tropical curve and its dual Newton subdivision. The triangles V contributing to $\text{mult}_S(h(\Gamma))$ are shaded and labeled with their integer area; we have $\text{mult}_S(h(\Gamma)) = (-1)^{1+1} \cdot 2^{-2} \cdot 4 \cdot 2 \cdot 3 \cdot 1 = 6$.



We now want to understand how parametrized and unparametrized Shustin curves differ.

Remark 4.30 (**Labeled and unlabeled curves**)
Note that we consider parametrized curves with labeled unmarked ends, whereas the unparametrized curves in [Shu06b] come without this data. There is a $1:1$ correspondence of parametrized and unparametrized curves due to Mikhalkin as mentioned in 2.43. But we overcount each unparametrized curve by a factor that records the different ways to label the (non-fixed) unmarked ends so that we get different labeled parametrized curves. If k denotes the number of double ends then this overcounting factor is $|G(\Delta)| \cdot 2^{-k}$, where the 2^{-k} term arises because interchanging the two labels of a double end does not change the parametrized curve and $G(\Delta)$ is the subgroup of S_n that permutes the unbounded edges of the same direction.

Lemma 4.31 (**Comparison of the Shustin-multiplicity of a labeled parametrized and an unparametrized curve**)
Let (C, h) be a Shustin curve of degree Δ with image $h(\Gamma)$, and passing through points in general position. Then there are $|G(\Delta)| \cdot 2^{-k}$ labeled parametrized Shustin curves having the same image $h(\Gamma)$, where k is the number of double ends of $h(\Gamma)$.

Definition 4.32 (**Tropical Welschinger numbers II**)
Let Δ be a toric degree, $r, s \geq 0$ be such that $r + 2s = |\Delta| - 1$ and $\omega = (P_1, \ldots, P_{r+s})$ a configuration of $r + s$ points in general position in $\mathbb{R}^2$. Then the number $W_\Delta^{\text{trop}}(0, \omega, r, s)$ of Shustin curves (C, h) having r real markings and s complex markings passing through ω counted with multiplicity $\text{mult}_S^W(h(\Gamma)) = \text{mult}_S(h(\Gamma)) \cdot 2^k / |G(\Delta)|$ (using the notation of 4.30) is called *tropical Welschinger number* w.r.t. Δ, ω, r and s.

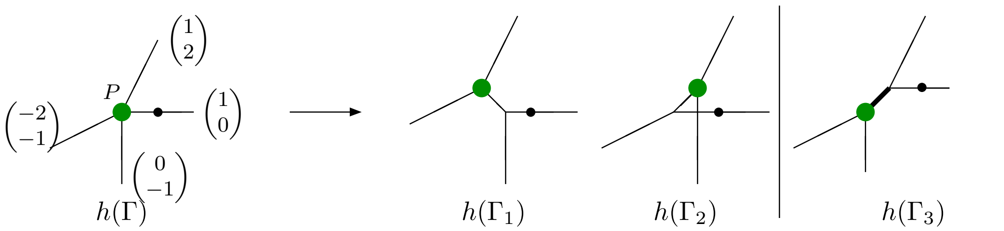
Theorem 4.33 (**Correspondence Theorem for tropical Welschinger numbers II**)
[Shu06b, theorem 3.1]. Let Δ be a toric degree and S_Δ equipped with the standard real structure. Furthermore, let $r, s \geq 0$ be such that $r + 2s = |\Delta| - 1$ and $\omega = (P_1, \ldots, P_{r+2s})$ a complex conjugation invariant configuration of $r + 2s$ points in general position in S, where $P_1, \ldots P_r$ are conjugation invariant points each and $P_{r+1}, \ldots, P_{r+2s}$ are not. Then it holds

$$W_{S_\Delta}(0, \omega, r, s) = W_\Delta^{\text{trop}}(0, \omega', r, s),$$

where ω' is the tropicalization of ω.

Remark 4.34
Again, an immediate consequence is that the tropical Welschinger numbers $W_\Delta^{\text{trop}}(0, \omega, r, s)$ do not depend on ω, as this was already proven for the algebraic Welschinger numbers, see theorem 1.24. Therefore we will write $W_\Delta^{\text{trop}}(r, s)$. Also, we will write $W^{\text{trop}}(d, r, s)$ instead of $W_\Delta^{\text{trop}}(r, s)$ if Δ corresponds to $\mathbb{P}^2$ and we consider curves of degree d.

Remark 4.35 (**A local proof of the invariance similar to 4.20 fails**)
Unfortunately, this time it is not possible to prove the invariance of the numbers $W_{\Delta}^{\text{trop}}(r,s)$ by a local study in the corresponding moduli space. Consider for instance the following image $h(\Gamma)$ of a marked curve which can be seen as a part of a Shustin curve. It is of degree $\Delta = ((-2,-1),(0,-1),(1,0),(1,2))$ and passes through a non-generic point configuration of $r = 1$ and $s = 1$ points.

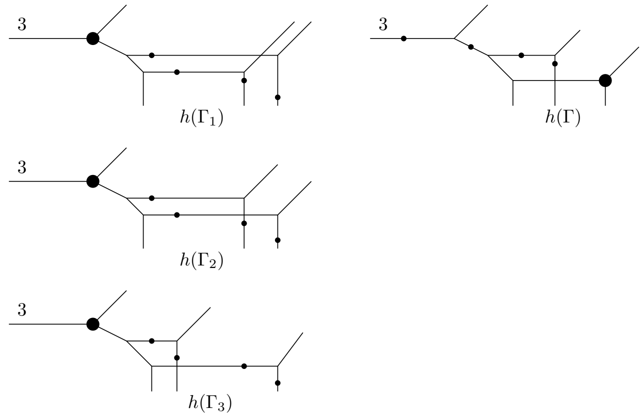


If we move the big dot P upwards we get curves with image $h(\Gamma_1)$, respectively $h(\Gamma_2)$ having multiplicities mult_S^W equal to -3, respectively 1 and passing through a generic point configuration. But when we move P downwards there is no Shustin curve at all passing through generic points! The curve with image $h(\Gamma_3)$ depicted on the right hand side is the parametrized curve that we get, but it is not a Shustin curve as the connected component of edges of even weights meets two connected components of odd weight and also the position of the complex marking is not allowed.
Curves as the latter will be studied in the next chapter 5.

Remark 4.36 (**Relative tropical Welschinger numbers**)
One can define relative tropical Welschinger numbers as it was done for Gromov-Witten numbers in 2.46. In particular, this implies that unmarked ends can be of weight > 2. In this case the Shustin-multiplicity $\text{mult}_S(C,h)$ can be defined as in 4.27. Relative tropical Welschinger numbers depend on the configuration of points as observed in [ABLdM11, section 7.2]. The following picture shows the three Shustin curves with images $h(\Gamma_1)$, $h(\Gamma_2)$, $h(\Gamma_3)$ of degree

$$\Delta = ((-3,0),(0,-1),(0,-1),(0,-1),(1,1),(1,1),(1,1))$$

passing through some given configuration ω of points. Each counts with multiplicity $\text{mult}_S^W(C,h) = 3$, so for this configuration we have in sum 9. For the configuration on the right however, there is only one Shustin curve with image $h(\Gamma)$ passing through it, and it is of multiplicity one. So in this case the total number is 1, i.e. the number depends on the choice of ω.



4.3 Properties of Welschinger invariants

In this section we want to collect general facts about Welschinger numbers. Although they concern also classical Welschinger invariants, most of the proofs concern tropical Welschinger

numbers, which are equal to the classical Welschinger numbers by the Correspondence Theorem(s). These proofs are possible due to the more accessible combinatorial nature of these invariants on the tropical side and reflect one of the advantages of tropical geometry.
The probably most natural question is:

Question 1: Given a configuaration ω of say $3d-1$ points in $\mathbb{RP}^2$ is there at least one real rational curve of degree d passing through ω?

Answer 1a: A first answer was obtained in [IKS03b, th 1.1], namely for any $d \geq 1$ there can be traced at least $d!/2$ real rational curves of degree d through any $3d-1$ generic points in $\mathbb{RP}^2$. We can skip the word generic if we also allow reducible curves. This follows directly from the inequality $W(d, 3d-1, 0) \geq d!/2$. The proof uses Mikhalkin's lattice path algorithm presented in [Mik05].

Answer 1b: [IKS04, theorem 3] generalizes this positivity result to the toric del Pezzo surfaces $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2_k$ with $k \leq 3$ with the standard real structure and D a real ample divisor on S: it holds there $W_{nD} \geq \exp(a \cdot n\log(n) + O(n))$ with $a = c_1(S) \cdot D$ and $W_{S,nD}$ is the shortcut for Welschinger invariant on S of real rational curves in $|nD|$ passing through $-K_{nD} \cdot nD - 1$ generic points and whose underlying complex curve is irreducible. This is again a tropical result.

Answer 1c: In [IKS07, theorem 1] the authors prove that the analogue result to answer 1b is true for the following surfaces equipped with a non-standard real structure: $S = \mathbb{P}^1 \times \mathbb{P}^1, (\mathbb{P}^1 \times \mathbb{P}^1)_k, (\mathbb{P}^1 \times \mathbb{P}^1)_{0,2}$ with $k \leq 2$ and $_{0,2}$ means that the surface is blown up in one pair of complex conjugate points. These surfaces are the 5 unnodal real toric Del Pezzo surfaces with a non-tautological real structures (note that $\mathbb{P}^1 \times \mathbb{P}^1)_{0,2}$ can be endowed with two different structures). But still, the tropical approach can be generalized to this situation.

Answer 1d: Generalization to non-toric Del Pezzo surfaces $S = (\mathbb{P}^1)^2_{0,1}$ and $\mathbb{P}^2_{q,s}$ with $4 \leq q + 2s \leq 5, s \leq 1$ and assuming D to be nef and big in [IKS10, theorem 7.1]. The main idea in the proof here is to blow-down exeptional divisors to obtain toric surfaces. One manages the problem of multiple fixed points which appear when blowing-down a generic point configuration in the original non-toric surface. Then one can use tropical arguments again.

Answer 1e: Even more general is [IKS12, theorem 2]. The authors prove the statement for $S = \mathbb{P}^2_{q,s}$ with $q + 2s \leq 6, s \leq 2$. The proof is does not use tropical geometry but arguments from algebraic geometry and involves a real Caporaaso-Harris type formula for S.

The second question concerns the behavior of Welschinger numbers of e.g. $\mathbb{P}^2$ when d goes to infinity compared to the corresponding Gromov-Witten numbers.

Question 2: Are $\log W_{S,nD}$ and and the logarithm of the corresponding Gromov-Witten number $N_{S,nD}$ on a surface S asymptotically equivalent when D is an ample divisor on S?

Answer 2: It holds $\frac{\log W_{,nD}}{\log N_{S,nD}} = 1$ for all the cases mentioned above. This is proven in the papers mentioned respectively.

It follows from the definition of Welschinger numbers that they equal the corresponding Gromov-Witten numbers modulo 2. Less obvious is the observation due to Grigory Mikhalkin saying that $W(d, 3d-1, 0)$ and $N(d, 0)$ are equal modulo 4 as mentioned in [Bru08, proposition 6.3].

Question 3: Does the analogue statement hold for other Del Pezzo surfaces?

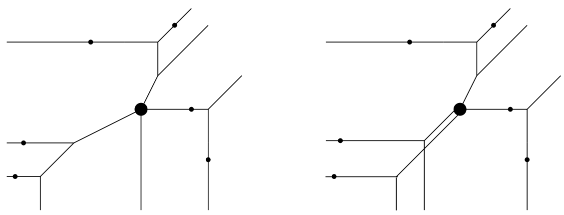
Answer 3: Yes, this works also for $\mathbb{P}^2_k$ for $1 \leq k \leq 6$ and $\mathbb{P}^1 \times \mathbb{P}^1$. See [IKS10, theorem 7.4], [IKS12, theorem 5].

5 Broccoli curves of genus 0

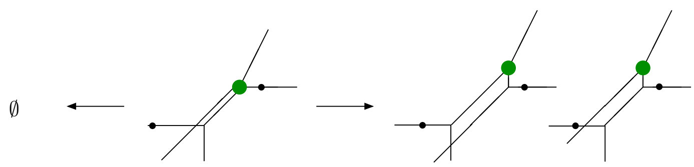
The following chapter is part of the joint paper with Andreas Gathmann and Hannah Markwig [GMS13] beside of the introduction and constitutes the main contribution of this thesis.

5.1 Motivation for broccoli curves

Let us consider the example of remark 4.35 again. As the tropical Welschinger numbers $W_{\Delta}^{\text{trop}}(r, s)$ are (globally) invariant, but not locally, this implies that there is (at least) an additional curve passing through the points of ω in special position. At the corresponding codimension-1 cell local invariance also does not hold. The differences to the invariance cancel exactly. If we consider the example as a local picture of the curve of degree 3 below, then there is a second curve passing through the points in special position such that the two differences cancel. The following picture shows these two codimension-1 curves passing through ω not in general position:

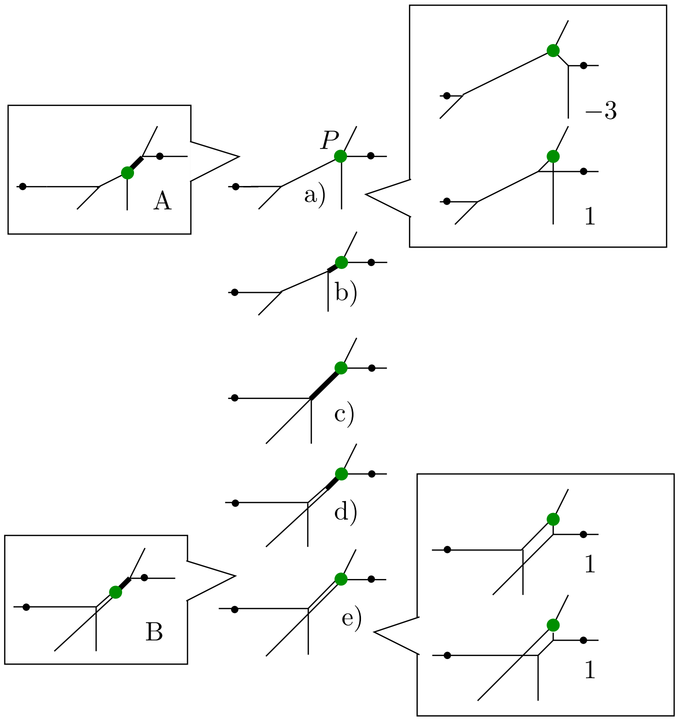


We have alraedy seen that the left picture produces a local difference of -2: locally, the difference between the numbers of curves passing through the configuration in which we move the complex point up and down is -2. The right picture now produces a local difference of $+2$:

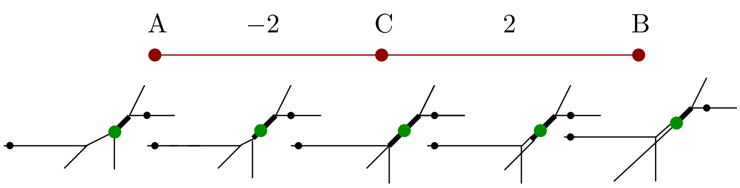


There are again two Shustin curves that have this codimension-1 curve in their boundary. They both satisfy the conditions when the complex point is moved up. Their multiplicity is 1 each. But no Shustin curve satisfies the conditions if the complex point is moved down.

So an idea would be to find out if the types above can be connected in the moduli space of Shustin curves, or more generally if we can find an algorithm that tells us how to find the other types in the moduli space. This approach is strongly related to the question how this moduli space of Shustin curves actually looks like. It is clear that Shustin curves as in definition 4.23 should lie in top-dimensional cells of this space. Assume that a Shustin curve passes through points in general position. Then we would like to see that when we move one or several of these points infinitesimally and when we consider the corresponding deformation of the curve, this deformation is also contained in the moduli space. But it is not possible to keep all deformations. Consider for instance the picture below. There, the point configuration on the left hand side is obtained from the point configuation in the middle by moving down the point P. The configuration on the right hand side comes from moving P up. Some of the curves had been considered before. When we move the point P in the (generic) point configuration on the right to the configuration on the left, we have two curves A and B passing through this generic point configuration. But both are not Shustin curves! We want our moduli space to contain Shustin curves through generic points, but we shouldn't allow top-dimensional cells with non-Shustin curves.



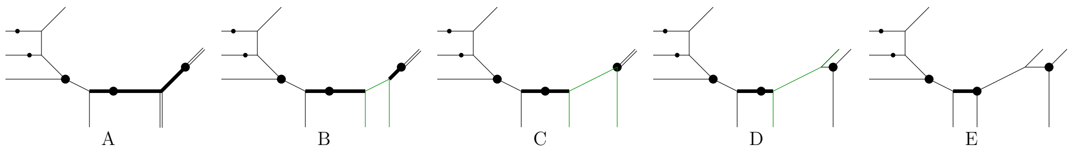
The column in the middle of the example shows a so called *bridge* connecting the Shustin curves on the right hand side. Observe that all the curves a)-e) pass through a fixed configuration of points in special position, so they lie in codimension 1 or higher of the moduli space. The curves a) and e) are Shustin curves while the curves b) - d) are not. The curve b) arises from the curve a) by pulling out an edge of weight 2 of the vertex to which P is adjacent. It looks like taking the curve A and displacing the point P to the other side of the edge of weight 2. Curve b) has a string which can be moved without changing the property that the curve passes through the points. By some algorithm one can then get from curve b) to the curve e). Hence, in total the contribution of the four Shustin curves is 0 to the Welschinger number $W^{\text{trop}}(3, 6, 1)$. Instead of doing so, one can also generalize the definition of the curves contributing such that we can assign to the curve A multiplicity -2 and to the curve B multiplicity 2. Furthermore if we can construct a bridge connecting curve A to the curve B in the moduli space of these generalized curves we can also prove the invariance, as explained below. For this example, the corresponding bridge is depicted below.



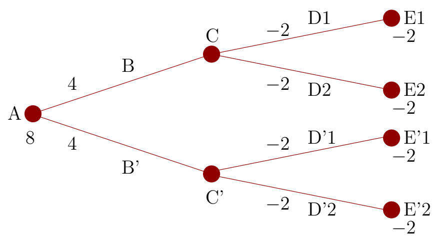
A bridge can be seen as a connected graph. Its edges are families of curves having a string. Its vertices are – in this example – either curves with a 4- or higher valent vertex in which the movement of a string of a family of curves corresponding to an edge (adjacent to this vertex) stops (as curve C) or being the curve A or B. This means that the movement along a bridge is a sequence of movements in 1-dimensional families. If we assign to the family of curves on the edge connecting A and C multiplicity -2 and to the family on the edge connecting C with B we observe that we have some sort of *balancing condition* at the vertex C: the sum of the multiplicities of the edges adjacent to this vertex equals 0. These *local balancing conditions* along a bridge ensure that the multiplicities of the curves we start with are not lost. As the multiplicities of the latter are chosen to be equal each to the sum of Shustin curves shown on the right hand side of the first bridge, this implies that the total sum of the four Shustin is equal to zero and hence does not change in example 4.35. Curves on a bridge will be called *bridge curves*; special cases of bridge curves are Shustin curves and the so called *broccoli curves* like the curves A and B.

Why should we consider this second bridge and not the first one? The main reason is that once we have defined broccoli curves, section 5.3, we can prove the invariance of *broccoli numbers* by a local argument similar to 4.20 in its moduli space. It is also possible to define *relative broccoli numbers* allowing to formulate and to prove a Caporaso-Harris type formula, see section 5.5.

Let us close this section with the observation that bridges do not only connect broccoli curves to other broccoli curves, but in general, they connect Shustin curves inside the class of Shustin curves, broccoli curves with broccoli curves, and Shustin curves with broccoli curves. Here is an example of a bridge connected the Shustin curve A to the broccoli curve E.



The strings occuring on the bridge are drawn in green. First, we pull the double end of direction vector $(0, -1)$ apart, thus introducing a string in the tropical curve. Hence the curve B is not fixed by the points but varies in a 1-dimensional family. We move the string until we hit the next vertex, which is then a 4-valent vertex, C. This is a codimension-1 cell of the bridge; the curve is fixed again by the points. To continue on the the bridge, we have to resolve the 4-valent vertex in all possible ways. Here, there is just one possible resolution D. (The precise definition explaining what resolutions are possible on a bridge can be found in section 5.4. We require that the string remains adjacent to the same even edge as before.) We move the string in D until we hit curve E. As we consider labeled curves and hence we have respectively two options to label the double end of direction $(0, -1)$ and $(1, 1)$, the bridge is actually this picture:



Let us start by oriented marked curves and their multiplicities which constitute the class of curves in which bridge curves, hence also broccoli and Shustin curves lie.

5.2 Oriented marked curves

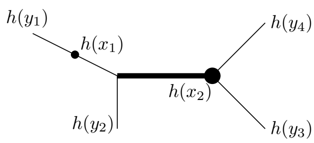
Let us redefine tropical curves and make the distinction between real and complex markings that we will later need to consider real enumerative invariants.

Definition 5.1 (**Marked curves**)
Let $r, s \in \mathbb{N}$. An (r, s)-*marked (plane tropical) curve* is a is parametrized tropical curve (C, h)

of genus 0 in $\mathbb{R}^2$ in the sense of 2.23 with $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n)$ and of degree $\Delta = (v(y_1), \ldots, v(y_n))$.
$x_1, \ldots, x_{r+s}$ is a labeling of the contracted ends, $y_1, \ldots, y_n$ a labeling of the non-contracted ends of C. We call $x_1, \ldots, x_{r+s}$ the *markings* or *marked ends*; more specifically the r ends $x_1, \ldots, x_r$ are called the *real markings*, the s ends $x_{r+1}, \ldots, x_{r+s}$ the *complex markings* of C. The other ends $y_1, \ldots, y_n$ are called the *unmarked ends*.

Convention 5.2
When drawing a marked curve (C, h) we will usually only show the image $h(\Gamma) \subset \mathbb{R}^2$, together with the image points $h(x_1), \ldots, h(x_{r+s})$ of the markings. As before, these image points will be drawn as small dots for real markings and as big dots for complex markings. The other edges will always be displayed as thin lines for odd edges and as thick lines for even edges. Unmarked contracted edges would not be visible in these pictures, but (although allowed) they will not play a special role in this chapter.

Example 5.3
Using convention 5.2, the picture shows a $(1, 1)$-marked plane curve of degree $((-2, 1), (0, -1), (1, -1), (1, 1))$. It has two 3-valent vertices and one 4-valent vertex.



The thick edge has direction $(-2, 0)$ starting at the complex marking. For clarity we have labeled all the ends in the picture, but in the future we will usually omit this as the actual labeling will not be relevant for most of our arguments.

Definition 5.4 (Combinatorial types)
Let $(C, h) \in \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ be a marked curve and let α be its *combinatorial type* in the sense of 2.24. For such a combinatorial type α we denote by $M^{\alpha}_{(r,s)}(\Delta)$ the subspace of $\mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ of all marked curves of type α.

Remark 5.5 ($\mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ as a polyhedral complex)
Our moduli space $\mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ is a polyhedral complex in the sense of 2.7, and in fact even a tropical variety (see 2.27). In this chapter we will not need its structure as a tropical variety however, but only consider $\mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ as an abstract polyhedral complex with polyhedral structure induced by the combinatorial types of the curves. The open cells of this complex are exactly the subspaces $M^{\alpha}_{(r,s)}(\Delta)$, where α runs over all combinatorial types of curves in $\mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$. The curves in such a cell (i.e. for a fixed combinatorial type) are parametrized by the position in $\mathbb{R}^2$ of a chosen root vertex and the lengths of all bounded edges (which need to be positive). Hence $M^{\alpha}_{(r,s)}(\Delta)$ can be thought of as an open polyhedron whose dimension is equal to 2 plus the number of bounded edges in the combinatorial type α. We will call this dimension the *dimension* $\dim \alpha$ of the type α.

Let us now consider enumerative questions for our curves. In addition to the usual incidence conditions we want to be able to require that some of the unmarked ends are fixed, i.e. map to a given line in $\mathbb{R}^2$. To count such curves we will now introduce the corresponding evaluation maps. Moreover, to be able to compensate for the overcounting due to the labeling of the non-fixed unmarked ends we will define the group of permutations of these ends that keep the degree fixed. First we have to generalize our definition 2.30 of the evaluation map.

Definition 5.6 (Evaluation maps and $G(\Delta, F)$)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$.

a) The *evaluation map* ev_F (with *set of fixed ends* F) on $\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$ is defined to be

$$\mathrm{ev}_F: \quad M_{(r,s)}(\Delta) \longrightarrow (\mathbb{R}^2)^{r+s} \times \prod_{i\in F} \left(\mathbb{R}^2/\langle v_i\rangle\right) \cong \mathbb{R}^{2(r+s)+|F|}$$
$$((\Gamma, x_1,\dots,x_{r+s},y_1,\dots,y_n),h) \longmapsto \big((h(x_1),\dots,h(x_{r+s})),(h(y_i):i\in F)\big).$$

In our pictures we will indicate ends that we would like to be considered fixed with a small orthogonal bar at the infinite side.

b) We denote by $G(\Delta,F)$ the subgroup of the symmetric group S_n of all permutations such that $\sigma(i)=i$ for all $i\in F$ and $v_{\sigma(i)}=v_i$ for all $i=1,\dots,n$.

For the case $F=\emptyset$ of no fixed ends we denote ev_F simply by ev as we recover definition 2.30 and $G(\Delta,F)$ by $G(\Delta)$.

Remark 5.7
As in 2.30 these evaluation maps are morphisms of polyhedral complexes in the sense that they are continuous maps that are linear on each cell $M^{\alpha}_{(r,s)}(\Delta)$ of $\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$. Note that $G(\Delta,F)$ acts on $\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$ by permuting the unmarked ends, and that ev_F is invariant under this operation. By definition, if

$$\omega = \big((P_1,\dots,P_{r+s}),(Q_i : i\in F)\big) \in (\mathbb{R}^2)^{r+s}\times\prod_{i\in F}\left(\mathbb{R}^2/\langle v_i\rangle\right)$$

then the inverse image $\mathrm{ev}_F^{-1}(\omega)$ consists of all $(r+s)$-marked curves (C,h) of degree Δ that pass through $P_i\in\mathbb{R}^2$ at the marked point x_i for all $i=1,\dots,r+s$ and map the i-th unmarked end y_i to the line $Q_i\in\mathbb{R}^2/\langle v_i\rangle$ for all $i\in F$. We call ω a *collection of conditions* for ev_F.

Of course, when counting curves we must assume that the conditions we impose are in general position so that the dimension of the space of curves satisfying them is as expected. Let us define this notion rigorously in the context of this chapter. This definition is more general than the one given in 2.31.

Definition 5.8 (**General and special position of points**)
Let $N\in\mathbb{N}$, and let $f: M\to\mathbb{R}^N$ be a morphism of polyhedral complexes (as e.g. the evaluation map ev_F of definition 5.6 a)). Then the union $\bigcup_\alpha f(M^\alpha)\subset\mathbb{R}^N$, taken over all cells M^α of M such that the polyhedron $f(M^\alpha)$ has dimension at most $N-1$, is called the locus of points *in special position* for f. Its complement is denoted the locus of points *in general position* for f.

Remark 5.9
Note that the locus of points in general position for a morphism $f: M\to\mathbb{R}^N$ is by definition the complement of finitely many polyhedra of positive codimension in $\mathbb{R}^N$. In particular, it is a dense open subset of $\mathbb{R}^N$.

Example 5.10
Let $M\subset\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$ be a polyhedral subcomplex, and let $F\subset\{1,\dots,|\Delta|\}$. Then a collection of conditions $\omega\in\mathbb{R}^{2(r+s)+|F|}$ as in remark 5.7 is in general position for $\mathrm{ev}_F: M\to\mathbb{R}^{2(r+s)+|F|}$ if and only if for each curve in M satisfying the conditions ω and every small perturbation of these conditions we can still find a curve of the same combinatorial type satisfying them.

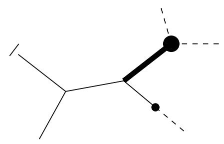
Collections of conditions in general position for the evaluation map have a special property that will be crucial for the rest of the chapter: in [GM08, remark 3.7] it was shown that every 3-valent curve $(C,h)\in\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$ through a collection of $r+s=|\Delta|-1$ points in general position for the evaluation map $\mathrm{ev}:\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)\to\mathbb{R}^{2(r+s)}$ without fixed ends has the property that each connected component of $\Gamma\backslash(x_1\cup\dots\cup x_{r+s})$ contains exactly one unmarked end. For the purposes of this chapter we need the following generalization of this statement to curves that are not necessarily 3-valent and evaluation maps that may have fixed ends.

Lemma 5.11
Let $M \subset \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ be a polyhedral subcomplex, and let ω be a collection of conditions in general position for the evaluation map $\text{ev}_F : M \to \mathbb{R}^{2(r+s)+|F|}$. Consider a curve $(C, h) \in \text{ev}_F^{-1}(\omega)$ satisfying these conditions. Then:

a) Each connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ has at least one unmarked end y_i with $i \notin F$.

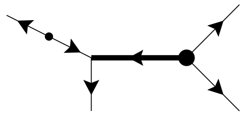
b) If the combinatorial type of C has dimension $2(r + s) + |F|$ and every vertex of C that is not adjacent to a marking is 3-valent then every connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ as in 5.11 has exactly one unmarked end y_i with $i \notin F$.

Proof. Consider a connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ and denote by Γ' its closure in Γ. We can consider Γ' as a graph, having a certain number a of unbounded fixed ends, b unbounded non-fixed ends, and c bounded ends (i.e. 1-valent vertices) at markings of C. The statement of part a) of the lemma is that $b \geq 1$, with equality holding in case b). For an example, in the picture below on the right Γ' consists of the solidly drawn lines; the curve continues in some way behind the dashed lines. Recall that fixed ends are indicated by small bars at the infinite sides. Hence in our example we have $a = 1$, $b = 1$, and $c = 2$.



By the same argument as in remark 5.5, the graph Γ' as well as the map $h|_{\Gamma'}$ is fixed by the position of a root vertex in Γ' and the lengths of all bounded edges of Γ'. But an easy combinatorial argument shows that the number of bounded edges of Γ' is equal to $a+b+2c-3-\sum_V (\operatorname{val} V - 3)$, with the sum taken over all vertices V that are not adjacent to a marking. Hence Γ' and its image $h|_{\Gamma'}$ can vary with $a+b+2c-1-\sum_V (\operatorname{val} V - 3)$ real parameters in M.
On the other hand, Γ' together with $h|_{\Gamma'}$ fixes $a+2c$ coordinates in the image of the evaluation map, namely the positions of the a fixed ends and the c markings in Γ'.
Hence $b = 0$ is impossible: then these $a + 2c$ coordinates of the evaluation map would vary with fewer than $a + 2c$ coordinates of M, meaning that the image of ev_F on the cell of C cannot be full-dimensional and thus ω cannot have been in general position. This proves a).
But in case b) $b > 1$ is impossible as well: then by assumption we have $\operatorname{val} V = 3$ for all V as above, and thus one could fix a position for the fixed ends and markings at Γ' in $\mathbb{R}^2$ and still obtain a $(b - 1)$-dimensional family for Γ' and $h|_{\Gamma'}$. As a movement in this family does not change anything away from Γ' this means that ev_F is not injective on the cell of M corresponding to C. But ev_F is surjective on this cell as ω is in general position. This is a contradiction since by assumption the source and the target of the restriction of ev_F to the cell corresponding to C have the same dimension. □

Remark 5.12
The important consequence of lemma 5.11 b) is that — whenever it is applicable — it means that there is a unique way to orient every unmarked edge of (C, h) so that it points towards the unique unmarked non-fixed end of the component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ containing the edge.



The picture shows this for the curve of example 5.3. Note that the arrow will always point inwards on fixed ends, and outwards on non-fixed ends.

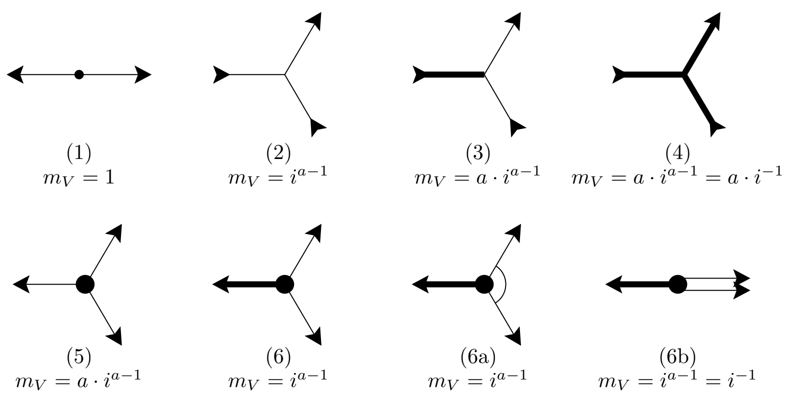
To be able to talk about this concept in the future we will now introduce the notion of oriented curves.

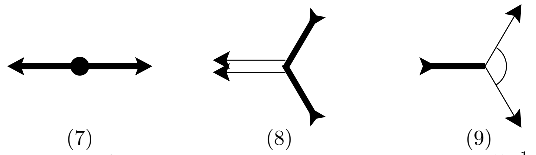
Definition 5.13 (Oriented marked curves)
An *oriented* (r, s)*-marked curve* is an (r, s)-marked curve (C, h) as in definition 5.1 in which each unmarked edge of Γ is equipped with an orientation (which we will draw as arrows in our pictures). In accordance with our above idea, the subset $F = F(C) \subset \{1, \dots, n\}$ of all i such that the unmarked end y_i is oriented inwards is called the *set of fixed ends* of C. The space of all oriented (r, s)-marked curves with a given degree Δ and set of fixed ends F will be denoted $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$; for the case $F = \emptyset$ of no fixed ends we write $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, \emptyset)$ also as $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta)$. We denote by $\mathrm{ft} : \mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F) \to \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ the obvious *forgetful map* that disregards the information of the orientations.

Remark 5.14
Obviously, our constructions and results for non-oriented curves carry over immediately to the oriented case: $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ is a polyhedral complex with cells $M^{\alpha}_{(r,s)}(\Delta, F)$ corresponding to the combinatorial types α of the oriented curves (which now include the data of the orientations of all edges). The forgetful map ft is a morphism of polyhedral complexes that is injective on each cell. There are evaluation maps on $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ as in definition 5.6 a) that are morphisms of polyhedral complexes; by abuse of notation we will write them as in the unoriented case as ev_F.

So far we have allowed any choice of orientations on the edges of our curves in $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$. To ensure that the orientations are actually as explained in remark 5.12 we will now allow only certain types of vertices. In the rest of the chapter we will study various kinds of oriented marked curves — broccoli curves and Shustin curves in section 5.3, bridge curves in section 5.4 — that differ mainly in their allowed vertex types. The following definition gives a complete list of all vertex types that will occur anywhere in this chapter.

Definition 5.15 (Vertex types and multiplicities)
We say that a vertex V of an oriented $(r+s)$-marked curve C is of a certain type if the number, parity (even or odd), and orientation of its adjacent edges is as in the following table. In addition, two arrows pointing in the same direction (as in the types (6b) and (8)) require these odd edges to be two unmarked ends with the same direction, and an arc (as in the types (6a) and (9)) means that these two odd edges must *not* be two unmarked ends with the same direction. Hence the type (6) splits up into the two subtypes (6a) and (6b). All other types in the list are mutually exclusive.





(7) $m_V = 1$ (8) $m_V = -a$ (9) $m_V = i^{a-1}$

In addition, each vertex V of one of the above types is assigned a *multiplicity* $m_V \in \mathbb{C}$ that can also be read off from the table. Here, the number a denotes the vertex multiplicity in the sense of 2.23. For the type (8) it is the absolute value of the determinant of the two even adjacent directions.

If (C, h) consists only of vertices of the above types, we denote by $n_\beta = n_\beta(C)$ the number of vertices in C of a given type β. In addition, we then define the *multiplicity* of C to be

$$m_C := \prod_{k=1}^{n} i^{\omega(y_k)-1} \cdot \prod_V m_V,$$

where the second product is taken over all vertices V of C. Although some of the vertex multiplicities are complex numbers, the following lemma shows that the curve multiplicity m_C is always real. In fact, the complex vertex multiplicities are just a computational trick that makes the "sign factor", i.e. the power of i, the same for all the vertex types (2) to (6) (which will be the most important ones), leading to easier proofs in the rest of the chapter.

Remark 5.16 (**Pick's formula**)
Let P be a (simple) lattice polygon having lattice area A and let denote I the number of lattice points in the interior of P and by B the number of lattice points on the boundary of P. Then we have $A = 2I + B - 2$ [Pic99].

Lemma 5.17
Every oriented marked curve that has only vertices of the types in definition 5.15 has a real multiplicity.

Proof. Let V be a vertex of C, and denote by $E_1, \ldots, E_q$ the adjacent unmarked edges (so $q \in \{2, 3, 4\}$ depending on the type of the vertex). Pick's formula 5.16 implies that the complex vertex multiplicity a as in definition 5.15 satisfies $a = \omega(E_1) + \cdots + \omega(E_q) \in \mathbb{Z}/2\mathbb{Z}$. By checking all vertex types we thus see that in each case

$$m_V \in \prod_{k=1}^{q} i^{\omega(E_k)-1} \cdot \mathbb{R}.$$

Now every unmarked edge is adjacent to exactly two vertices if it is bounded, and adjacent to exactly one vertex if it is unbounded. Hence

$$m_C \in \prod_E i^{2(\omega(E)-1)} \cdot \mathbb{R} = \mathbb{R},$$

where the sum is taken over all unmarked edges. □

Example 5.18
The picture of example 5.3 and remark 5.12 shows an oriented marked curve C with $F(C) = \emptyset$. Its vertices V_1, V_2, V_3, labeled from left to right, are of the types (1), (3), and (6), respectively, so that e.g. $n_{(6)} = 1$. The vertex V_3 is also of type (6a). The multiplicities of the vertices are $m_{V_1} = 1$, $m_{V_2} = 2 \cdot i^{2-1} = 2i$, and $m_{V_3} = i^{2-1} = i$. As all unmarked ends of C have weight 1 the multiplicity of C is thus $m_C = -2$.

Let us now check that, with our list of allowed vertex types, in the situation of lemma 5.11 b) the only way to orient a given curve is as explained in remark 5.12.

Lemma 5.19 (**Uniqueness of the orientation of curves**)
Let the notations and assumptions be as in lemma 5.11 b). If there is a way to make C into an oriented curve with vertices of the types (1) to (7) and so that the orientations of the unmarked ends are as given by F, this must be the orientation that lets each unmarked edge point towards the unique unmarked and non-fixed end in the component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ containing it.

Proof. By lemma 5.11 b) there is a unique orientation on C pointing on each unmarked edge towards the unmarked and non-fixed end in the component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ containing the edge. Now assume that we have any orientation on C with vertices of types (1) to (7). Denote by Γ' the subgraph of Γ where these two orientations differ; we have to show that $\Gamma' = \emptyset$.
Note that Γ' is a bounded subgraph since the orientation on the ends is fixed by F. Moreover, Γ' cannot contain an edge adjacent to a marking since all possible vertex types (1), (5), (6), and (7) with markings require the orientation on the adjacent edges precisely as in remark 5.12. So if Γ' is non-empty it must have a 1-valent vertex somewhere that is not adjacent to a marking. This can only be a vertex of the types (2), (3), or (4), and the condition of Γ' being 1-valent means that the two orientations differ at exactly one adjacent edge. But this is impossible since both orientations have the property that they have one adjacent edge pointing outwards and two pointing inwards at this vertex. □

We will end this section by computing the dimensions of the cells of $\mathcal{M}^{\text{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$.

Lemma 5.20
Let $C \in \mathcal{M}^{\text{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ be an oriented marked curve all of whose vertices are of the types listed in definition 5.15. Let α be the combinatorial type of C. Then the cell of $M^{\text{or}}_{(r,s)}(\Delta, F)$ corresponding to α has dimension

$$\dim \alpha = |\Delta| + r + n_{(7)} - n_{(8)} - 1 = 2(r+s) + |F| + n_{(9)}.$$

Proof. By remark 5.5 it suffices to show that the number of bounded edges of C is equal to

$$\text{both} \quad |\Delta| + r + n_{(7)}(C) - n_{(8)}(C) - 3 \qquad \text{and} \quad 2(r+s) + |F| + n_{(9)}(C) - 2.$$

This is easily proven by induction on the number of vertices in C: if C has only one vertex (and thus no bounded edge) it has to be one of the types in definition 5.15, and the statement is easily checked in all of these cases. If the curve C has more than one vertex we cut it at any bounded edge into two parts C_1 and C_2, making the cut edge unbounded in both parts. If $C_i \in M_{(r_i,s_i)}(\Delta_i, F_i)$ for $i = 1, 2$, then $r = r_1 + r_2$, $s = s_1 + s_2$, $|\Delta| = |\Delta_1| + |\Delta_2| - 2$, $|F| = |F_1| + |F_2| - 1$, and $n_\beta(C) = n_\beta(C_1) + n_\beta(C_2)$ for $\beta \in \{(7), (8), (9)\}$. The number of bounded edges of C is now just the number of bounded edges in C_1 and C_2 plus 1, i.e. by induction equal to

$$\begin{aligned} &|\Delta_1| + r_1 + n_{(7)}(C_1) - n_{(8)}(C_1) - 3 + |\Delta_2| + r_2 + n_{(7)}(C_2) - n_{(8)}(C_2) - 3 + 1 \\ &\quad = |\Delta| + r + n_{(7)}(C) - n_{(8)}(C) - 3 \end{aligned}$$

as well as

$$\begin{aligned} &2(r_1+s_1) + |F_1| + n_{(9)}(C_1) - 2 + 2(r_2+s_2) + |F_2| + n_{(9)}(C_2) - 2 + 1 \\ &\quad = 2(r+s) + |F| + n_{(9)}(C) - 2. \end{aligned}$$

□

5.3 Broccoli curves and Shustin curves revised

In this section we will introduce the most important type of curves considered in this paper: the broccoli curves. We define corresponding enumerative numbers, and show that they are independent of the chosen point conditions. Broccoli curves can be defined with or without orientation. Both definitions have their advantages: the oriented one is easier to state and local at the vertices, whereas the unoriented one is easier to visualize (as one does not need to worry about orientations at all). So let us give both definitions and show that they agree for enumerative purposes.
We also define tropical curves that we call Welschinger curves. Their count (for certain choices of Δ) yields Welschinger invariants. We will parametrize even non-fixed unmarked ends of Welschinger curves as two ends of half the weight— this way we can avoid this kind of splitting on the bridges of section 5.4. We will refer to such ends, i.e. pairs of non-fixed ends of the same odd direction adjacent to the same 4-valent vertex, as double ends. In the following, we will first settle how to deal with these double ends. Then we define oriented and unoriented Welschinger curves and prove that they are equivalent. We relate unoriented Welschinger curves to Shustin curves, and discuss some invariance and non-invariance properties of tropical Welschinger numbers.

Definition 5.21 (**Broccoli curves**)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$.

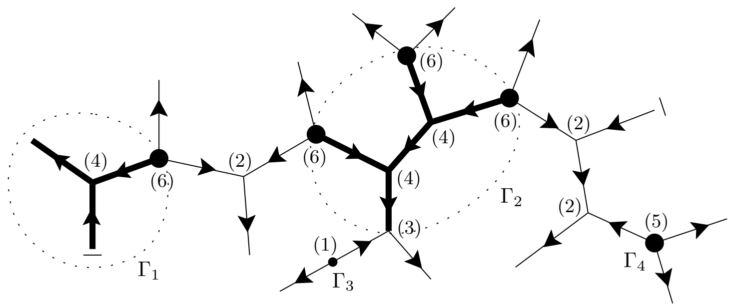
a) An oriented curve $C \in \mathcal{M}^{\text{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ all of whose vertices are of the types (1) to (6) of definition 5.15 is called an *oriented broccoli curve*.

b) Let $(C, h) \in \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$. Consider the subgraph Γ_{even} of Γ of all even edges (including the markings). The 1-valent vertices of Γ_{even} as well as the $y_i \subset \Gamma_{\text{even}}$ with $i \notin F$ are called the *stems* of Γ_{even}. We say that C is an *unoriented broccoli curve* (with set of fixed ends F) if

   (i) all complex markings are adjacent to 4-valent vertices;

   (ii) every connected component of Γ_{even} has exactly one stem.

Example 5.22
The picture below shows an oriented broccoli curve in which every allowed vertex type appears. We have labeled the vertices with their types. Note that by forgetting the orientations of the edges (and thus also disregarding the vertex types) one obtains an unoriented broccoli curve. Its subgraph Γ_{even} of even edges consists of all markings and thick edges. It has four connected components $\Gamma_1, \ldots, \Gamma_4$, and each component has exactly one stem: the non-fixed unmarked end in Γ_1, the vertex of type (3) in Γ_2, and the unique vertices in Γ_3 and Γ_4.



Of course, to count these curves we have to fix the right number of conditions to get a finite answer. This dimension condition follows e.g. for oriented broccoli curves from lemma 5.20: we must have $r + 2s + |F| = |\Delta| - 1$ since $n_{(7)} = n_{(8)} = n_{(9)} = 0$.

Proposition 5.23 (**Equivalence of oriented and unoriented broccoli curves**)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\omega \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for $\mathrm{ev}_F : \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta) \to \mathbb{R}^{2(r+s)+|F|}$ (see example 5.10).
Then the forgetful map ft of definition 5.13 gives a bijection between oriented and unoriented (r, s)-marked broccoli curves through ω with degree Δ and set of fixed ends F.

Proof. We have to prove three statements.

a) ft maps oriented to unoriented broccoli curves through ω: Let $C \in \mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ be an oriented broccoli curve. The list of allowed vertex types for C implies immediately that C then satisfies condition (i) of definition 5.21.

   To show (ii) let Γ' be a connected component of Γ_{even}. If Γ' contains no vertex of type (4) it can only be a single marking (types (1) or (5)) or a single unmarked edge with possibly attached markings (vertex types (3) together with (6), (3) with a fixed unmarked end, or (6) with a non-fixed unmarked end), and in each of these cases condition (ii) is satisfied. If there are vertices of type (4) they must form a tree in Γ', and obviously every such tree made up from type (4) vertices has exactly one outgoing end. This unique outgoing end must be a non-fixed end of C or connected to a type (3) vertex, hence in any case it leads to a stem. On the other hand, the incoming ends of the tree must be fixed ends of C or connected to a type (6) vertex, i.e. they never lead to a stem. Consequently, Γ' satisfies condition (ii).

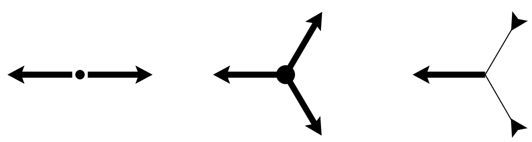
b) ft is injective on the set of curves through ω: Note that the conditions of lemma 5.11 b) are satisfied by the dimension condition of lemma 5.20 and our list of allowed vertex types. Hence lemma 5.19 implies that there is at most one possible orientation on C.

c) ft is surjective on the set of curves through ω: Let $C \in \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta)$ be an unoriented broccoli curve through ω with set of fixed ends F. Then by (i) the curve C has s 4-valent vertices at the complex markings, so by [GM08, proposition 2.11] the combinatorial type of C has dimension $|\Delta| - 1 + r - \sum_V (\mathrm{val}\, V - 3) = 2(r+s) + |F| - \sum_V (\mathrm{val}\, V - 3)$, with the sum taken over all vertices V that are not adjacent to a complex marking. But as ω is in general position this dimension cannot be less than $2(r+s) + |F|$. So we see that all vertices without adjacent complex marking are 3-valent, and that the combinatorial type of C has dimension equal to $2(r+s) + |F|$. Hence we can apply lemma 5.11 b) again to conclude that there is an orientation on C that points on each edge towards the unique non-fixed unmarked end in $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$.

   It remains to be shown that with this orientation the only vertex types occurring in C are (1) to (6). For this, note that for a vertex V

   - as we have said above, V is 4-valent if there is a complex marking at V, and 3-valent otherwise;
   - by the construction of the orientation, all edges at V are oriented outwards if there is a marking at V, and exactly one edge is oriented outwards otherwise;
   - by the balancing condition, it is impossible that exactly one edge at V is odd.

   With these restrictions, the only possible vertex types besides (1) to (6) would be the ones in the picture below.



To exclude these three cases, note that in all of them V would be contained in a connected component Γ' of Γ_{even} that contains at least one unmarked edge. So let us consider such a component, and let $W \in \Gamma' \cap (\Gamma \backslash \Gamma')$ be a vertex where Γ' meets the complement of Γ'. Then there must be an odd as well as an unmarked even edge in Γ at W, so by the balancing condition as above there are exactly two odd edges and one even unmarked edge at W. Hence W is a stem if and only if there is no marking at W. So a connection in $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ from a point in the interior of Γ' to a non-fixed unmarked end can only be via a stem — which is unique by (ii). This means that every point in the interior of Γ' must be connected in $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ to the stem. In particular, the interior of Γ' can have no further markings, which rules out the first two vertex types in the picture above. The third vertex type is impossible since this would have to be the stem and thus the connection from Γ' to the non-fixed unmarked end, which does not match with the orientation of the even edge.

□

Let us now make the obvious definition of the enumerative invariants corresponding to broccoli curves. Proposition 5.23 tells us that it does not matter whether we count oriented or unoriented broccoli curves. We choose the oriented ones here as their definition is easier. So we make the convention that from now on *a broccoli curve will always mean an oriented broccoli curve*.

Notation 5.24
We denote by $M^B_{(r,s)}(\Delta, F)$ the closure of the space of all broccoli curves in $\mathcal{M}^{\text{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$; this is obviously a polyhedral subcomplex. By lemma 5.20 it is non-empty only if the dimension condition $r+2s+|F| = |\Delta|-1$ is satisfied. Moreover, in this case it is of pure dimension $2(r+s)+|F|$, and its maximal open cells correspond exactly to the broccoli curves in $M^B_{(r,s)}(\Delta, F)$.

Definition 5.25 (Broccoli invariants)
As above, let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\omega \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for broccoli curves, i.e. for the evaluation map $\text{ev}_F : M^B_{(r,s)}(\Delta, F) \to \mathbb{R}^{2(r+s)+|F|}$. Then we define the *broccoli invariant*

$$N^B_{(r,s)}(\Delta, F, \omega) \;:=\; \frac{1}{|G(\Delta, F)|} \cdot \sum_C m_C,$$

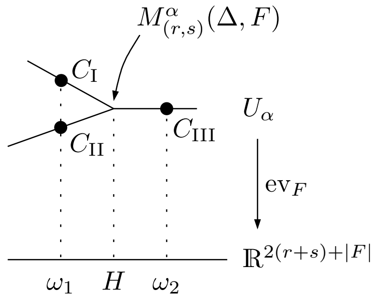
where the sum is taken over all broccoli curves C in $M^B_{(r,s)}(\Delta, F)$ of degree Δ, set of fixed ends F, and $\text{ev}(C) = \omega$. The group $G(\Delta, F)$ as in definition 2.30 b) takes care of the overcounting of curves due to relabeling the non-fixed unmarked ends. The sum is finite by the dimension statement of notation 5.24, and the multiplicity m_C is as in definition 5.15.

The main result of this section — and in fact the most important point that distinguishes our new invariants from the otherwise quite similar Welschinger invariants that we will study later in this section — is that broccoli invariants are always independent of the choice of conditions ω.

Theorem 5.26
The broccoli invariants $N^B_{(r,s)}(\Delta, F, \omega)$ are independent of the collection of conditions ω. We will thus usually write them simply as $N^B_{(r,s)}(\Delta, F)$ (or $N^B_{(r,s)}(\Delta)$ for $F = \emptyset$).

Proof. The proof follows from a local study of the moduli space $M^B_{(r,s)}(\Delta, F)$. Compared to the one for ordinary tropical curves in [GM07b] theorem 4.8 it is very similar in style and conceptually not more complicated; there are just (many) more cases to consider because we have to distinguish orientations as well as even and odd edges.

By definition, the multiplicity of a curve depends only on its combinatorial type. So it is obvious that the function $\omega \mapsto N^B_{(r,s)}(\Delta, F, \omega)$ is *locally* constant on the open subset of $\mathbb{R}^{2(r+s)+|F|}$ of conditions in general position for broccoli curves, and may jump only at the image under ev_F of the boundary of top-dimensional cells of $M^B_{(r,s)}(\Delta, F)$. This image is a union of polyhedra in $\mathbb{R}^{2(r+s)+|F|}$ of positive codimension. It suffices to show that the function $\omega \mapsto N^B_{(r,s)}(\Delta, F, \omega)$ is locally constant around a cell in this image of codimension 1 in $\mathbb{R}^{2(r+s)+|F|}$ since any two top-dimensional cells of $\mathbb{R}^{2(r+s)+|F|}$ can be connected to each other through codimension-1 cells.



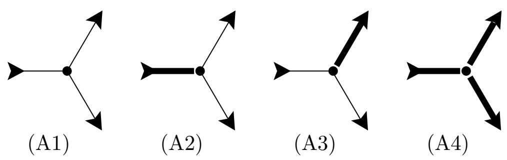
So let α be a combinatorial type in $M^B_{(r,s)}(\Delta, F)$ of dimension $2(r+s)+|F|-1$ such that ev_F is injective on $M^\alpha_{(r,s)}(\Delta, F)$ and thus maps this cell to a unique hyperplane H in $\mathbb{R}^{2(r+s)+|F|}$. As in the picture let $U_\alpha \subset M^B_{(r,s)}(\Delta, F)$ be the open subset consisting of $M^\alpha_{(r,s)}(\Delta, F)$ together with all adjacent top-dimensional cells of $M^B_{(r,s)}(\Delta, F)$. To prove the theorem we will show that for a point ω in a neighborhood of $\mathrm{ev}_F(M^\alpha_{(r,s)}(\Delta, F))$ the sum of the multiplicities of the curves in $U_\alpha \cap \mathrm{ev}_F^{-1}(\omega)$ does not depend on ω, i.e. is the same on both sides of H. In our picture this would just mean that $m_{\mathrm{I}} + m_{\mathrm{II}} = m_{\mathrm{III}}$, where $m_{\mathrm{I}}, m_{\mathrm{II}}, m_{\mathrm{III}}$ denote the multiplicities of $C_{\mathrm{I}}, C_{\mathrm{II}}, C_{\mathrm{III}}$, respectively.
Actually, we will show this in a slightly different form: to each codimension-0 type α_k in U_α we will associate a so-called *H-sign* σ_k that is 1 or -1 depending on the side of H on which $\mathrm{ev}_F(M^{\alpha_k}_{(r,s)}(\Delta, F))$ lies (it will be 0 if $\mathrm{ev}_F(M^{\alpha_k}_{(r,s)}(\Delta, F)) \subset H$). So in the picture above on the right we could take $\sigma_{\mathrm{I}} = \sigma_{\mathrm{II}} = 1$ and $\sigma_{\mathrm{III}} = -1$. We then obviously have to show that $\sum_k \sigma_k \, m_k = 0$, where the sum is taken over all top-dimensional cells adjacent to α.
To prove this, we will start by listing all codimension-1 combinatorial types α in $M^B_{(r,s)}(\Delta, F)$. They are obtained by shrinking the length of a bounded edge in a broccoli curve to zero, thereby merging two vertices into one. Depending on the merging vertex types we distinguish the following cases:

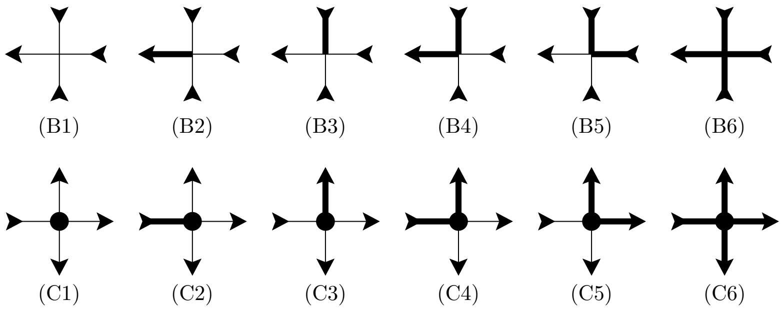
(A) a vertex (1) merging with a vertex (2)/(3), leading to a 4-valent vertex with one real marking, two outgoing edges, and one incoming edge.

(B) a vertex (2)/(3)/(4) merging with a vertex (2)/(3)/(4), leading to a 4-valent vertex with no marking, one outgoing edge, and three incoming edges.

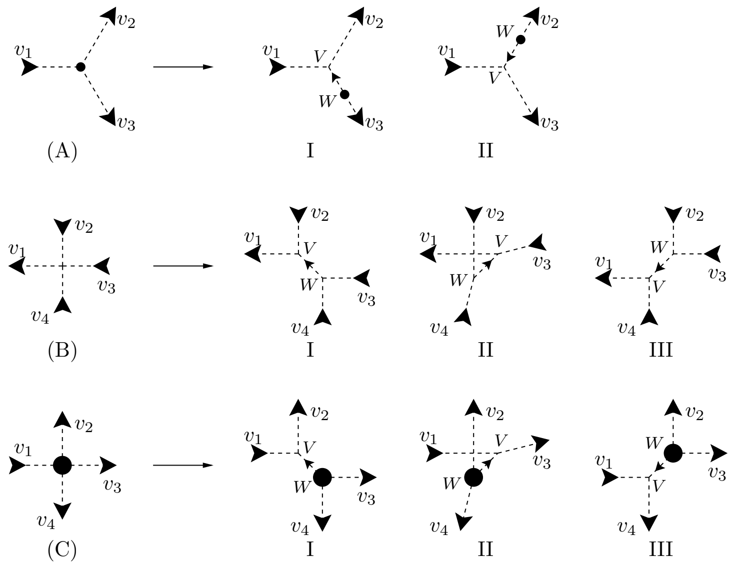
(C) a vertex (5)/(6) merging with a vertex (2)/(3)/(4), leading to a 5-valent vertex with one complex marking, three outgoing edges, and one incoming edge.

More precisely, noting that by the balancing condition it is impossible to have exactly one odd edge at a vertex, the cases (A), (B), and (C) split up into the following possibilities depending on the orientation and parity of the adjacent edges.





Next, we will list the adjacent codimension-0 types in $M^B_{(r,s)}(\Delta, F)$ (called *resolutions*) that make up U_α in the cases (A), (B), and (C). In this picture, the dashed lines can be even or odd depending on which of the subcases (A $\cdot$), (B $\cdot$), (C $\cdot$) we are in. The vectors $v_1, \dots, v_4$ will be used in the computations below; they are always meant to be oriented outwards (i.e. *not* necessarily in the direction of the orientation of the edge), so that $v_1 + v_2 + v_3 = 0$ in case (A) and $v_1 + v_2 + v_3 + v_4 = 0$ in the cases (B) and (C).



Note that the allowed vertex types for broccoli curves fix the orientation of the newly inserted bounded edge in all these resolutions; it is already indicated in the picture above. Moreover, the requirement that there cannot be exactly one odd edge at a vertex fixes the parity of the new bounded edge in all cases except (B1) and (C1). In the (B1) and (C1) cases, there are two possibilities: the four vectors $v_1, \dots, v_4$ can either be all the same in $(\mathbb{Z}_2)^2$ (in which case the new bounded edge joining V and W is even in all three types I, II, III; we call this case (B1_3) and (C1_3), respectively), or they make up two equivalence classes in $(\mathbb{Z}_2)^2$ (in which case the new bounded edge is even in exactly one of the types I, II, III; we call this case (B1_1) and (C1_1), respectively). In the (B1_1) and (C1_1) cases, we can assume by symmetry that the even bounded edge occurs in type I. So in total we now have 18 codimension-1 cases (A1), . . . , (A4), (B1_1), (B1_3), (B2),. . . (B6), (C1_1), (C1_3), (C2),. . . (C6) to consider, and in each of these cases we know the resolutions together with all parities and orientations of all edges of the curves — in particular, with the vertex types of V and W (as in the picture above). For example, in case (B6) the new bounded edge must be even in all three resolutions. Hence in all three resolutions all edges are even, and thus both vertices V and W are of type (4).

The following table lists the vertex types for V and W for all resolutions I, II, III of all codimension-1 cases. The symbol "—" means that the required vertex type is not allowed in broccoli curves and thus that a corresponding codimension-0 cell does not exist. The columns labeled m_* and μ_*/μ_* will be explained below.

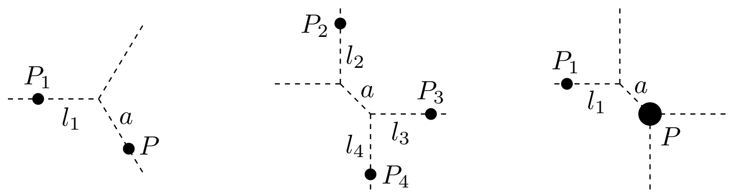
codim-1	resolution I			resolution II			
case	V	W	m_{I}	V	W	$\mu_{\mathsf{II}}/\mu_{\mathsf{I}}$	m_{II}
A1	(2)	(1)	1	(2)	(1)	-1	1
A2	(3)	(1)	(v_1, v_2)	(3)	(1)	1	(v_1, v_3)
A3	—	(1)	0	(3)	—	1	0
A4	(4)	—	0	(4)	—	1	0

codim-1	resolution I			resolution II				resolution III			
case	V	W	m_{I}	V	W	$\mu_{\mathsf{II}}/\mu_{\mathsf{I}}$	m_{II}	V	W	$\mu_{\mathsf{III}}/\mu_{\mathsf{I}}$	m_{III}
B1_1	(3)	—	0	(2)	(2)	1	1	(2)	(2)	-1	1
B1_3	(3)	—	0	(3)	—	1	0	(3)	—	1	0
B2	—	(2)	0	—	(2)	1	0	—	(2)	1	0
B3	(3)	(2)	(v_1, v_2)	(2)	(3)	1	(v_4, v_2)	(2)	(3)	-1	(v_2, v_3)
B4	(4)	—	0	—	(3)	1	0	—	(3)	1	0
B5	(3)	(3)	$(v_1, v_2)(v_3, v_4)$	(3)	(3)	1	$(v_1, v_3)(v_4, v_2)$	(3)	(4)	1	$(v_1, v_4)(v_2, v_3)$
B6	(4)	(4)	$(v_1, v_2)(v_3, v_4)$	(4)	(4)	1	$(v_1, v_3)(v_4, v_2)$	(4)	(4)	1	$(v_1, v_4)(v_2, v_3)$

codim-1	resolution I			resolution II				resolution III			
case	V	W	m_{I}	V	W	$\mu_{\mathsf{II}}/\mu_{\mathsf{I}}$	m_{II}	V	W	$\mu_{\mathsf{III}}/\mu_{\mathsf{I}}$	m_{III}
C1_1	(3)	(6)	(v_1, v_2)	(2)	(5)	1	(v_4, v_2)	(2)	(5)	-1	(v_2, v_3)
C1_3	(3)	(6)	(v_1, v_2)	(3)	(6)	1	(v_1, v_3)	(3)	(6)	1	(v_1, v_4)
C2	(3)	(5)	$(v_1, v_2)(v_3, v_4)$	(3)	(5)	1	$(v_1, v_3)(v_4, v_2)$	(3)	(5)	1	$(v_1, v_4)(v_2, v_3)$
C3	—	(5)	0	(2)	(6)	1	1	(2)	(6)	-1	1
C4	(4)	(6)	(v_1, v_2)	(3)	(6)	1	(v_1, v_3)	(3)	(6)	1	(v_1, v_4)
C5	—	(6)	0	—	(6)	1	0	(3)	—	1	0
C6	(4)	—	0	(4)	—	1	0	(4)	—	1	0

Let us now determine the H-sign of the resolutions above, i.e. figure out which of them occur on which side of H. To do this we set up the system of linear equations determining the lengths of the bounded edges of the curve in terms of the positions of the markings in $\mathbb{R}^2$. For such a given position of the markings (on the one or on the other side of H), a given resolution type is then possible if and only if the required length of the new bounded edge is positive.

More concretely, let a be the length of the newly created bounded edge, and denote by $P \in \mathbb{R}^2$ in the cases (A) and (C) the required image point for the marking. In the cases (A) and (C) the end v_1 is fixed, so say there is another marking on the v_1 end at a distance of l_1 on the graph that is required to map to a point $P_1 \in \mathbb{R}^2$. In the case (B) the ends v_2, v_3, and v_4 are fixed, so we do the same then with lengths l_2, l_3, l_4 and points $P_2, P_3, P_4 \in \mathbb{R}^2$. As an example, these notions are illustrated for the resolution I in the following picture.



(A)-I (B)-I (C)-I

The systems of linear equations that determine the relative positions of $P, P_1, \ldots, P_4$ in terms of $a, l_1, \ldots, l_4$ are then as follows (where all entries are in $\mathbb{R}^2$ and thus each row stands for two equations).

(A)-I

l_1	a	
$-v_1$	v_3	$P - P_1$

(A)-II

l_1	a	
$-v_1$	v_2	$P - P_1$

(B)-I

l_2	l_3	l_4	a	
$-v_2$	v_3	0	v_3+v_4	P_3-P_2
$-v_2$	0	v_4	v_3+v_4	P_4-P_2

(B)-II

l_2	l_3	l_4	a	
$-v_2$	v_3	0	v_1+v_3	P_3-P_2
$-v_2$	0	v_4	0	P_4-P_2

(B)-III

l_2	l_3	l_4	a	
$-v_2$	v_3	0	0	P_3-P_2
$-v_2$	0	v_4	v_1+v_4	P_4-P_2

(C)-I

l_1	a	
$-v_1$	v_3+v_4	$P-P_1$

(C)-II

l_1	a	
$-v_1$	v_2+v_4	$P-P_1$

(C)-III

l_1	a	
$-v_1$	v_2+v_3	$P-P_1$

To determine a in terms of $P, P_1, \dots, P_4$ we use Cramer's rule: if M is the (quadratic) matrix of a system of linear equations as above and M' the matrix obtained from M by replacing the a-column by the right hand side of the equation, then $a = \det M' / \det M$. But within a case (A), (B), (C) the matrix M' does not depend on the resolution I, II, III, and thus it is simply the sign of $\det M$ that tells us whether a is positive or negative, i.e. whether this resolution exists for the chosen points $P, P_1, \dots, P_4$. We can therefore take the H-sign to be the sign of $\det M$ (note that this will be 0 if and only if the relative position of $P, P_1, \dots, P_4$ is not determined uniquely by the equations and thus if and only if the codimension-0 cell maps to H). An elementary computation of the determinants shows that these H-signs are as in the following table, where (v_i, v_j) stands for the determinant of the 2×2 matrix with columns v_i, v_j (and where we have used $v_1 + v_2 + v_3 = 0$ in case (A) as well as $v_1 + v_2 + v_3 + v_4 = 0$ in the cases (B) and (C)).

	H-sign for I	H-sign for II	H-sign for III
(A)	$\operatorname{sign}(v_1, v_2)$	$\operatorname{sign}(v_1, v_3)$	
(B)	$\operatorname{sign}\big((v_1, v_2)(v_3, v_4)\big)$	$\operatorname{sign}\big((v_1, v_3)(v_4, v_2)\big)$	$\operatorname{sign}\big((v_1, v_4)(v_2, v_3)\big)$
(C)	$\operatorname{sign}(v_1, v_2)$	$\operatorname{sign}(v_1, v_3)$	$\operatorname{sign}(v_1, v_4)$

Note that these H-signs follow a special pattern: for each of the vertices V and W that is of type (2), (3), or (4) we get a factor of $\operatorname{sign}(v_i, v_j)$ in the H-sign of the resolution, where $(i, j) \in \{(1,2), (1,3), (1,4), (3,4), (4,2), (2,3)\}$ is the unique pair such that the v_i and v_j edges are adjacent to the vertex. On the other hand, by definition 5.15 the multiplicity of such a vertex is 1 in type (1), $i^{|(v_i,v_j)|-1}$ in types (2) and (6), and $|(v_i, v_j)| \cdot i^{|(v_i,v_j)|-1}$ in types (3), (4), and (5). If one replaces $|(v_i, v_j)|$ by $-|(v_i, v_j)|$ in these expressions, the vertex multiplicities remain the same for the types (1), (5) and (6), and are replaced by their negatives for the types (2), (3), and (4). It follows that the H-sign can be taken care of by replacing $a = |(v_i, v_j)|$ by (v_i, v_j) in the vertex multiplicities of definition 5.15 for V and W.
More precisely, if σ denotes the H-sign and m the multiplicity of a curve in a given resolution, then $\sigma\, m = \lambda\, \tilde{m}_V\, \tilde{m}_W$, where $\tilde{m}_V$ and $\tilde{m}_W$ are the multiplicities of the vertices V and W as in definition 5.15 with a replaced by (v_i, v_j) as above, and λ is the product of the vertex multiplicities of all other vertices. To show that the sum of these numbers over all resolutions is zero we can obviously divide by the constant λ (which is the same for the resolutions I, II, III) and only consider $\tilde{m}_V\, \tilde{m}_W$. Let us split this number as $\tilde{m}_V\, \tilde{m}_W = \mu\, m$, where μ collects all factors $i^{(v_i,v_j)-1}$ and m all factors (v_i, v_j) for V and W. The values for $m = m_{\mathsf{I}}, m_{\mathsf{II}}, m_{\mathsf{III}}$ are listed in the table of resolutions above. As for μ, note that this number is

- in case (A): $\mu_{\mathsf{I}} := i^{(v_1,v_2)-1}$ for I and $\mu_{\mathsf{II}} := i^{(v_1,v_3)-1}$ for II;
- in cases (B) and (C): $\mu_{\mathsf{I}} := i^{(v_1,v_2)+(v_3,v_4)-2}$ for I, $\mu_{\mathsf{II}} := i^{(v_1,v_3)+(v_4,v_2)-2}$ for II, and $\mu_{\mathsf{III}} := i^{(v_1,v_4)+(v_2,v_3)-2}$ for III.

To simplify these expressions we divide them by μ_{I} and get (using $v_1+v_2+v_3=0$ in (A) and $v_1+v_2+v_3+v_4=0$ in (B) and (C))

- in case (A): $\mu_{\mathsf{II}}/\mu_{\mathsf{I}} = i^{2(v_2,v_1)} = (-1)^{(v_2,v_1)}$;
- in cases (B) and (C): $\mu_{\mathsf{II}}/\mu_{\mathsf{I}} = i^{2(v_2,v_1)} = (-1)^{(v_2,v_1)}$ and $\mu_{\mathsf{III}}/\mu_{\mathsf{I}} = i^{2(v_1,v_4)} = (-1)^{(v_1,v_4)}$.

The values for these quotients are also listed in the table of resolutions above. Using these values for the quotients and $m_{\mathsf{I}}, m_{\mathsf{II}}, m_{\mathsf{III}}$, one can now easily check the required statement

$$\mu_{\mathsf{I}} \cdot m_{\mathsf{I}} + \mu_{\mathsf{II}} \cdot m_{\mathsf{II}} + \mu_{\mathsf{III}} \cdot m_{\mathsf{III}} = \mu_{\mathsf{I}} \cdot \left(m_{\mathsf{I}} + \mu_{\mathsf{II}}/\mu_{\mathsf{I}} \cdot m_{\mathsf{II}} + \mu_{\mathsf{III}}/\mu_{\mathsf{I}} \cdot m_{\mathsf{III}}\right) = 0$$

in all 18 codimension-1 cases, using the identities

- $(v_1,v_2)+(v_1,v_3)=0$ for (A),
- $(v_1,v_2)+(v_4,v_2)+(v_3,v_2)=0$, $(v_1,v_2)(v_3,v_4)+(v_1,v_3)(v_4,v_2)+(v_1,v_4)(v_2,v_3)=0$, and $(v_1,v_2)+(v_1,v_3)+(v_1,v_4)=0$ for (B) and (C),

that follow from $v_1+v_2+v_3=0$ and $v_1+v_2+v_3+v_4=0$, respectively. □

Let us now turn to Welschinger curves.

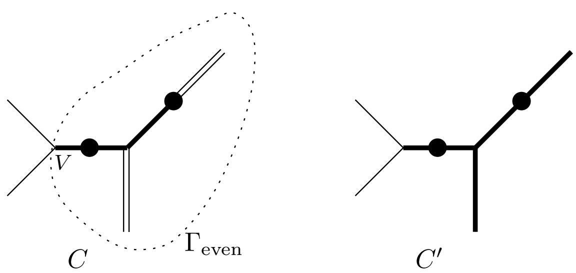
Definition 5.27 (**Double ends and end-gluing**)
Let α be a combinatorial type of $M_{(r,s)}(\Delta)$ with $\Delta=(v_1,\dots,v_n)$, and let $F \subset \{1,\dots,n\}$ be a set of fixed ends. Assume that there are exactly k pairs $i_1<j_1,\dots,i_k<j_k$ in $\{1,\dots,n\}\backslash F$ such that the unmarked ends y_{i_l} and y_{j_l} have the same odd direction and are adjacent to the same 4-valent vertex, for all $l=1,\dots,k$. We refer in the following to such a pair of ends as a *double end*. We then set

$$\Delta' = \big((v(y_i) : i \neq i_1, j_1, \dots, i_k, j_k), (2\cdot v(y_{i_1}), \dots, 2 \cdot v(y_{i_k}))\big).$$

Moreover, we define α' by gluing each pair of double ends y_{i_l} and y_{j_l} to one unmarked end of direction $2\cdot v(y_{i_l})$, and denote by $F' \subset \{1,\dots,n-k\}$ the set of entries of Δ' corresponding to the fixed ends F in Δ. There is then an associated map $M^{\alpha}_{(r,s)}(\Delta) \to M^{\alpha'}_{(r,s)}(\Delta')$ which we call the *end-gluing map*.

The analogous end-gluing map $(M^{\text{or}}_{(r,s)})^{\alpha}(\Delta,F) \to (M^{\text{or}}_{(r,s)})^{\alpha'}(\Delta',F')$ also exists for oriented curves. The map sending a combinatorial type α of $M_{(r,s)}(\Delta)$ as above to α' is injective, because if we want to produce a preimage α from α', we just have to split the last k ends of Δ', producing 4-valent vertices.

Example 5.28
The following picture shows a curve C and its image C' under the end-gluing map. Although mainly following convention 5.2, we draw double ends separately even though this is actually a feature of the graph Γ and cannot be seen in $h(\Gamma)$.



Remark 5.29
It follows from example 5.10 that if a collection of conditions $\omega \in \mathbb{R}^{2(r+s)+|F|}$ as in remark 5.7 is in general position for $\text{ev}_F : M^{\alpha}_{(r,s)}(\Delta) \to \mathbb{R}^{2(r+s)+|F|}$ then it is also in general position after end-gluing for $\text{ev}_{F'} : M^{\alpha'}_{(r,s)}(\Delta') \to \mathbb{R}^{2(r+s)+|F|}$, and vice versa. Notice also that $\dim M^{\alpha}_{(r,s)}(\Delta) = \dim M^{\alpha'}_{(r,s)}(\Delta')$: by [GM08, proposition 2.11] a combinatorial type has dimension $|\Delta|-1+r+s-\sum_V(\text{val}(V)-3)$ where the sum goes over all vertices V of Γ, and the end-gluing map decreases the number of entries of Δ by the same number as it decreases the number of 4-valent vertices. As orienting the edges does not change dimensions we conclude that the end-gluing map does not change the dimension of combinatorial types of oriented curves either.

Definition 5.30 (Γ_{even} **and roots**)
Let $(C,h) \in \mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$ and C' be the image of C under the end-gluing map of definition 5.27 and call the underlying graph Γ'. Consider the subgraph Γ'_{even} of Γ' of all even edges (including the markings), and its preimage Γ_{even}. That is, Γ_{even} consists of all even edges and all double ends of Γ. Vertices of $\Gamma_{\text{even}} \cap \overline{\Gamma\backslash\Gamma_{\text{even}}}$ as well as unmarked non-fixed *even* ends of Γ_{even} are called the *roots* of Γ_{even}.

Example 5.31
For the curve of example 5.28, the part Γ_{even} is encircled. It has one root, namely the vertex denoted by V.

Definition 5.32 (**Welschinger curves**)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2\backslash\{0\}$, and let $F \subset \{1, \ldots, n\}$.

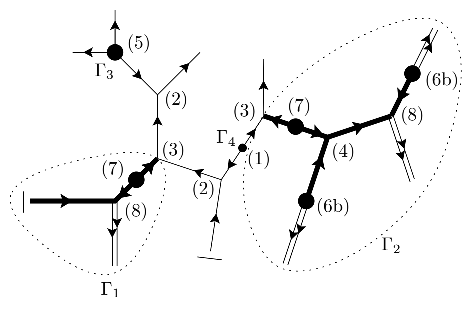
a) An oriented curve $C \in \mathcal{M}^{\text{or}}_{0,r+s}(\mathbb{R}^2,\Delta,F)$ all of whose vertices are of the types (1) to (5), (6b), (7), or (8) of definition 5.15 is called an *oriented Welschinger curve*.

b) Let $(C,h) \in \mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$, and let Γ_{even} be as in definition 5.30. We say that C is an *unoriented Welschinger curve* (with set of fixed ends F) if

   (i) complex markings are adjacent to 4-valent vertices, or non-isolated in Γ_{even};

   (ii) each connected component of Γ_{even} has a unique root.

Example 5.33
The following picture shows an oriented Welschinger curve with an even and an odd fixed end. As in example 5.28, we indicate double ends in the picture while otherwise following convention 5.2. Each vertex is labeled with its type, every allowed vertex type occurs. If we forget the orientations of the edges, we get an unoriented Welschinger curve. There are four connected components of Γ_{even}. Γ_3 consists of a complex marking and Γ_4 of a real marking. Γ_1 and Γ_2 both have one root, namely the vertex of type (3). Three complex markings are adjacent to 4-valent vertices, four are non-isolated in Γ_{even}.



As for broccoli curves, we want to show that oriented and unoriented Welschinger curves are equivalent for enumerative purposes. The following remark and lemma are needed as preparation.

Remark 5.34
Let $C \in \mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ be an oriented Welschinger curve.

a) By lemma 5.20, the curve C has $|\Delta| - |F| = r + 2s + 1 - n_{(7)} + n_{(8)}$ outward pointing ends. In particular, if $|\Delta| - 1 = r + 2s + |F|$ then $n_{(7)} = n_{(8)}$.

b) If C consists only of vertices of types (4), (6b), (7) and (8), then we have $r = 0$, $s = n_{(6b)} + n_{(7)}$, and the number of odd outward pointing ends is $2n_{(6b)} + 2n_{(8)}$. Hence in this case it follows from 5.34 that C has exactly $1 + n_{(7)} - n_{(8)}$ even outward pointing ends.

Lemma 5.35
Let $|\Delta| - 1 = r + 2s + |F|$, let $C \in \mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ be an oriented Welschinger curve, and let Γ_{even} be as in definition 5.30. Then every connected component of Γ_{even} has exactly one root.

Proof. If $\Gamma_{\mathrm{even}} = \Gamma$ then Γ has only vertices of type (4), (6b), (7), and (8). By remark 5.34 a) we have $n_{(7)} = n_{(8)}$, so from remark 5.34 b) it then follows that Γ has exactly one even outward pointing end, which is the unique root.
If $\Gamma_{\mathrm{even}} \neq \Gamma$, every connected component $\tilde{\Gamma}$ of Γ_{even} needs to be adjacent to odd edges which are not double ends. The only allowed vertex type for oriented Welschinger curves to which both even edges (resp. double edges) and odd edges (which are not double ends) are adjacent is type (3). Each vertex of type (3) yields a 1-valent vertex in Γ_{even}. Remove these 1-valent vertices from the component $\tilde{\Gamma}$, and call the resulting graph $\tilde{\Gamma}^\circ$. A vertex of type (3) leads to an outward pointing end of $\tilde{\Gamma}^\circ$. Note that $\tilde{\Gamma}^\circ$ has vertices of types (4), (6b), (7), and (8). Thus by remark 5.34 b) we have $n^{\tilde{\Gamma}^\circ}_{(8)} \leq n^{\tilde{\Gamma}^\circ}_{(7)}$, where the superscripts indicate that we refer to numbers of vertices of $\tilde{\Gamma}^\circ$. By remark 5.34 a) we have $n^C_{(7)} = n^C_{(8)}$. Since any vertex of type (7) or (8) belongs to exactly one graph $\tilde{\Gamma}^\circ$ associated to a connected component $\tilde{\Gamma}$ of Γ_{even}, and since the inequality $n^{\tilde{\Gamma}^\circ}_{(8)} \leq n^{\tilde{\Gamma}^\circ}_{(7)}$ holds for any such $\tilde{\Gamma}$, we conclude that it is an equality. Then by remark 5.34 b) every $\tilde{\Gamma}^\circ$ has exactly one even outward pointing end. It follows that every $\tilde{\Gamma}$ has exactly one root. □

With this preparation we can prove the following statement analogously to proposition 5.23.

Proposition 5.36 (Equivalence of oriented and unoriented Welschinger curves)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\omega \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for $\mathrm{ev}_F : \mathcal{M}_{0,r+s}(\mathbb{R}^2, \Delta) \to \mathbb{R}^{2(r+s)+|F|}$ (see example 5.10).
Then the forgetful map ft of definition 5.13 gives a bijection between oriented and unoriented (r, s)-marked Welschinger curves through ω with degree Δ and set of fixed ends F.

Proof. As in proposition 5.23, we have to prove three statements.

a) ft maps oriented to unoriented Welschinger curves through ω: Let $C \in \mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ be an oriented Welschinger curve. The list of allowed vertex types for C implies that C satisfies condition (i) of definition 5.32. Condition (ii) follows from lemma 5.35.

b) ft is injective on the set of curves through ω: Notice that under the end-gluing map of definition 5.27, a vertex of type (8) becomes a vertex of type (4), and type (6b) becomes (7). Thus the image C' under the end-gluing map satisfies the conditions of lemma 5.11 b) by lemma 5.20 and remark 5.29. Lemma 5.19 implies that there is at most one possible orientation on C', and it follows immediately that there is only one possible orientation on C, since double ends have to point outwards (types (6b) and (8)).

c) ft is surjective on the set of curves through ω: Let $C \in \mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$ be an unoriented Welschinger curve through ω with set of fixed ends F. Let α be the combinatorial type of C and $M^\alpha_{(r,s)}(\Delta)$ its corresponding cell in $\mathcal{M}_{0,r+s}(\mathbb{R}^2,\Delta)$. Denote by s_1 the number of isolated complex markings in Γ_{even}, and by k the number of double ends. As this means by definition 5.27 and condition (i) that there are at least s_1+k vertices of valence 4 it follows from [GM08] proposition 2.11 that the dimension of $M^\alpha_{(r,s)}(\Delta)$ is at most $|\Delta|+r+s-1-s_1-k = 2r+3s+|F|-s_1-k$. On the other hand, C passes through a collection of conditions in general position, so $\dim(M^\alpha_{(r,s)}(\Delta)) \geq 2r+2s+|F|$. It follows that

$$s - s_1 - k \geq 0. \qquad (*)$$

In fact, we want to show that we always have equality here. For this let $\tilde{\Gamma}$ be a connected component of $\Gamma_{\text{even}} \backslash \left(\overline{\Gamma\backslash\Gamma_{\text{even}}}\right)$ — i.e. we remove from Γ_{even} all attachment vertices to its complement — which is not an isolated marked end. Denote by $\tilde{\Gamma}'$ its image under the end-gluing map. Let $\tilde{s}$ be the number of complex markings belonging to $\tilde{\Gamma}$, and let $\tilde{k}$ be the number of its double ends. Then $\tilde{\Gamma}'$ contains possibly fixed even ends, the $\tilde{k}$ ends coming from the double ends, and one extra end (which is either the root itself or the edge with which it is adjacent to $\Gamma\backslash\Gamma_{\text{even}}$). If $\tilde{s} > \tilde{k}$ it follows that there is a component of $\tilde{\Gamma}'$ minus the $\tilde{s}$ complex markings which does not contain a non-fixed end, which would be a contradiction to lemma 5.11 a). Thus $\tilde{s} \leq \tilde{k}$. Summing this up over all such components $\tilde{\Gamma}$ it follows that the number $s-s_1$ of complex markings which are non-isolated in Γ_{even} satisfies $s-s_1 \leq k$. Together with $(*)$ this yields $s-s_1=k$, as desired.

Hence equality holds in all our estimates above. There are various consequences of this: first of all, we have $\dim(M^\alpha_{(r,s)}(\Delta)) = 2r+2s+|F|$, and C has exactly s vertices of valence 4, namely s_1 adjacent to complex markings which are isolated in Γ_{even}, and $s-s_1$ adjacent to double ends. All other vertices have valence 3. In particular, if the root of a connected component of Γ_{even} is not an end, it has to be at a 3-valent vertex. Also, since we have $\tilde{s}=\tilde{k}$ complex markings on the components $\tilde{\Gamma}$ above, it follows that there cannot be additional real markings on these components, since otherwise there would be a connected component of $\tilde{\Gamma}'$ without the markings again which does not contain a non-fixed end. Thus there are no real markings which are non-isolated in Γ_{even}.

The combinatorial type of the image C' of C under the end-gluing map is of dimension $\dim(M^\alpha_{(r,s)}(\Delta)) = 2r+2s+|F|$ by remark 5.29. Since C has 4-valent vertices only at complex markings resp. double ends, it follows that C' has 4-valent vertices only at complex markings, and so we can apply lemma 5.11 to C' to see that there is an orientation on C' that points on each edge towards the unique non-fixed unmarked end in $\Gamma'\backslash(x_1\cup\cdots\cup x_{r+s})$. We can define an orientation on C by orienting double ends just as the end they map to under the end-gluing map.

It remains to be shown that, for this orientation of C, we only have the vertex types (1) to (5), (6b), (7) or (8). As in the proof of proposition 5.23 c), all edges adjacent to a vertex V point outwards if there is a marking at V, and exactly one points outwards otherwise. It is impossible that exactly one edge at V is odd. We have seen that V is 4-valent if it is adjacent to a double end, or to a complex marking, and 3-valent otherwise. The only vertex types compatible with all these restrictions are the types (1) to (8), and the three special ones in the picture of the proof of proposition 5.23 c). Type (6a) cannot appear since each root has to be 3-valent by the above. The left picture in the proof of proposition 5.23 c) is excluded since there are no non-isolated real markings in Γ_{even}. The middle picture is excluded since we have 4-valent vertices only at isolated complex markings or double ends. The right picture would be a root of a component $\tilde{\Gamma}$ as above. But because of the orientation there is no connection from this vertex via one of the odd edges to a non-fixed unmarked end without passing a marking. With $\tilde{k}$ non-fixed ends and $\tilde{k}$ complex markings in $\tilde{\Gamma}$ this would again lead to a connected component of Γ minus the markings with no non-fixed end, a contradiction to lemma 5.11 a).

□

Remark 5.37 (**Unoriented Welschinger curves after end-gluing**)
In addition to definition 5.32 b) we can also describe unoriented Welschinger curves after the end-gluing: fix a degree $\Delta = (v_1, \ldots, v_n)$ and $F \subset \{1, \ldots, n\}$. We then allow curves of any degree $\Delta' = \big((v(y_i) : i \neq i_1, j_1, \ldots, i_k, j_k), (2 \cdot v(y_{i_1}), \ldots, 2 \cdot v(y_{i_k}))\big)$ for some $i_1 < j_1, \ldots, i_k < j_k$ in $\{1, \ldots, n\} \setminus F$ such that the unmarked ends y_{i_l} and y_{j_l} have the same odd direction. For a curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_{n-k}, h) \in M_{(r,s)}(\Delta')$, we define Γ_{even} as in definition 5.21 as the subgraph of all even edges. We then require that complex markings are adjacent to 4-valent vertices, or non-isolated in Γ_{even}; and that each connected component of Γ_{even} has a unique root.

Now we define enumerative numbers of Welschinger curves. As for broccoli curves, we work with oriented Welschinger curves from now on, keeping in mind that it does not matter whether we count oriented or unoriented Welschinger curves by proposition 5.36.

Notation 5.38
Let $r + 2s + |F| = |\Delta| - 1$, and denote by $M^W_{(r,s)}(\Delta, F)$ the closure of the space of all Welschinger curves in $\mathcal{M}^{\text{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$. This is obviously a polyhedral subcomplex. By lemma 5.20 it is of pure dimension $2(r + s) + |F|$, and its maximal open cells correspond exactly to the Welschinger curves in $M^W_{(r,s)}(\Delta, F)$. For $F = \emptyset$ we write $M^W_{(r,s)}(\Delta, F)$ also as $M^W_{(r,s)}(\Delta)$.

Definition 5.39 (**New Welschinger numbers**)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\omega \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for Welschinger curves, i.e. for the evaluation map $\text{ev}_F : M^W_{(r,s)}(\Delta, F) \to \mathbb{R}^{2(r+s)+|F|}$. Then we define the *new Welschinger number*

$$N^W_{(r,s)}(\Delta, F, \omega) \;:=\; \frac{1}{|G(\Delta, F)|} \cdot \sum_C m_C,$$

where the sum is taken over all Welschinger curves C in with degree Δ, set of fixed ends F, and $\text{ev}(C) = \omega$. As in the case of broccoli invariants, the group $G(\Delta, F)$ compensates for the overcounting of curves due to relabeling the non-fixed unmarked ends (see remark 4.30). The sum is finite by the dimension statement of notation 5.38, and the multiplicity m_C is as in definition 5.15. For $F = \emptyset$ we abbreviate the numbers as $N^W_{(r,s)}(\Delta, \omega)$.

In contrast to the broccoli invariants of definition 5.25 we will see in remark 5.43 that these Welschinger numbers will in general depend on the choice of conditions ω.

Example 5.40 (**New Welschinger numbers for degrees with non-fixed even ends**)
In two special cases when the degree $\Delta = (v_1, \ldots, v_n)$ contains one or several non-fixed even ends we can actually compute the new Welschinger numbers directly:

a) Assume that Δ contains more than one non-fixed even end. Consider a Welschinger curve (C, h) contributing to the number $N^W_{(r,s)}(\Delta, F, \omega)$. Every even non-fixed end belongs to a connected component of Γ_{even} and is a root. Since every connected component has a unique root by definition 5.32 b) (ii) it follows that such a component cannot meet the remaining part $\overline{\Gamma \setminus \Gamma_{\text{even}}}$. But as the curve is connected this means that Γ_{even} can have only one connected component and thus only one root. This is a contradiction, showing that there is no Welschinger curve with more than one non-fixed even end, and thus that in this case

$$N^W_{(r,s)}(\Delta, F, \omega) = 0.$$

b) Assume now that Δ contains exactly one non-fixed end of weight 2, of direction v_1, and only non-fixed edges of weight 1 otherwise.

Assume that $N^W_{(r,s)}(\Delta,\omega) \neq 0$. By the same argument as in a) each curve contributing to $N^W_{(r,s)}(\Delta,\omega)$ is totally even (containing one even and $\frac{|\Delta|-1}{2}$ double ends). Hence $|\Delta|$ must be odd and must contain each vector v_i $(i \neq 1)$ twice. Without restriction we can assume that $v_i = v_{i+\frac{|\Delta|-1}{2}}$ for $1 < i \leq \frac{|\Delta|-1}{2} + 1$. Furthermore, it then follows that $r = 0$ and $s = \frac{|\Delta|-1}{2}$.

In other words, each curve contributing to $N^W_{(0,s)}(\Delta,\omega)$ contains only vertices of type (4), (6b), (7), and (8). We can thus interpret the number $N^W_{(0,s)}(\Delta,\omega)$ as a "double complex enumerative number" in the following sense: let $\Delta' = (\frac{1}{2}v_1, v_2, \ldots, v_{\frac{|\Delta|-1}{2}+1})$ and denote by $N^{\text{trop}}_s(\Delta', 0, \omega)$ the number of (3-valent) tropical curves (without labeled ends) passing through ω similar to 2.35, i.e. each curve is counted with its usual complex multiplicity as in 2.35. If we forget the labels of the non-marked ends, the set of curves contributing to $N^W_{(0,s)}(\Delta,\omega)$ is then obviously in bijection to the set of curves contributing to $N^{\text{trop}}_s(\Delta', 0, \omega)$ by multiplying each direction vector (after end-gluing) with $\frac{1}{2}$. However, $N^W_{(0,s)}(\Delta,\omega)$ is not quite equal to $N^{\text{trop}}_s(\Delta', 0, \omega)$ since the multiplicities of the curves are slightly different:

- If the vector $\frac{1}{2}v_1$ occurs d times in Δ' then there are d choices in the count of $N^W_{(0,s)}(\Delta,\omega)$ which of the ends of the "double complex curve" is the weight-2 end of the Welschinger curve.
- As we count Welschinger curves with labeled ends to get the number $N^W_{(0,s)}(\Delta,\omega)$, we overcount each curve without labeled ends by a factor of $|G(\Delta)| \cdot 2^{-\frac{|\Delta|-1}{2}}$ (see remark 4.30), since $\frac{|\Delta|-1}{2}$ is the number of double ends.
- Under the bijection, each vertex of type of type (4) and (8) maps to a vertex of complex multiplicity $\frac{a}{4}$. Denote by Γ' the graph after end-gluing and forgetting the marked points. This graph has $\frac{|\Delta|-1}{2} + 1$ ends and is 3-valent, thus we have $n_{(4)} + n_{(8)} = \frac{|\Delta|-1}{2} - 1$. Therefore we overcount each Welschinger curve by an additional factor of $4^{\frac{|\Delta|-1}{2}-1}$.
- In addition, we count each Welschinger curve with a sign, namely $i \cdot (-1)^{n_{(8)}} \cdot i^{-n_{(4)}-n_{(6\text{b})}}$, where the factor of i arises because of the end of weight 2 and the other factors arise because of the vertex multiplicities. The number of ends of the graph Γ' equals $n_{(6\text{b})} + n_{(7)} + 1 = \frac{|\Delta|-1}{2} + 1$, thus we have $n_{(4)} + n_{(8)} + 1 = n_{(6\text{b})} + n_{(7)}$. Since $n_{(7)} = n_{(8)}$ by 5.34, we can conclude $n_{(4)} + 1 = n_{(6\text{b})}$, thus the sign above equals $(-1)^{n_{(8)}} \cdot i^{-2n_{(4)}} = (-1)^{n_{(4)}+n_{(8)}} = (-1)^{\frac{|\Delta|-1}{2}-1}$.

Taking all these factors together, it follows that

$$\begin{aligned} N^W_{(0,s)}(\Delta,\omega) &= d \cdot (-1)^{\frac{|\Delta|-1}{2}-1} \cdot 2^{-\frac{|\Delta|-1}{2}} \cdot 4^{\frac{|\Delta|-1}{2}-1} \cdot N^{\text{trop}}_s(\Delta', 0, \omega) \\ &= d \cdot (-1)^{\frac{|\Delta|-1}{2}-1} \cdot 2^{\frac{|\Delta|-1}{2}-2} \cdot N^{\text{trop}}_s(\Delta', 0, \omega). \end{aligned}$$

In particular, in this case $N^W_{(0,s)}(\Delta,\omega)$ does not depend on the exact position of the points ω.

We will see in example 5.65 that in some cases these results hold for broccoli invariants as well.

Remark 5.41 (**Welschinger curves compared to Shustin curves 4.23**)
Notice that (unoriented) Welschinger curves af end-gluing where all unmarked ends are non-fixed and of weight 1 or 2 correspond precisely to Shustin curves considered in 4.23 (in the way described in remark 5.37). Namely, adding an end for each marking and splitting each even unmarked end into a double end then gives a graph Γ together with a map $h : \Gamma \to \mathbb{R}^2$ satisfying the conditions of definition 5.32 b) and the involution σ. It follows from definition 4.23 that each connected component has one root.

We will now show that the multiplicity 5.15, which looks at first a little different, coincides with the Shustin-multiplicity 4.27.

Lemma 5.42 (**New Welschinger numbers compared to 4.32**)
Let (C, h) be a Welschinger curve of degree Δ with no fixed ends, satisfying $w(y_i) = 1$ for all $i = 1, \ldots, n$, and passing through points in general position as in example 5.10. Then the multiplicity m_C and the Shustin multiplicity $\text{mult}_S(h(\gamma))$ of C are related by $m_C = 2^k \cdot \text{mult}_S(h(\Gamma))$, where k is the number of double ends of C.
It follows therefore from remark 5.41 and remark 4.30 that for $F = \emptyset$ and Δ consisting of primitive vectors (i.e. of directions of weight one) our Welschinger number $N^W_{(r,s)}(\Delta, \omega)$ of definition 5.39 equals the number 4.32 of Shustin curves, counted with their Shustin multiplicities as in definition 4.27.

Therefore, we will now talk about Welschinger numbers.
Proof. It follows from the list of allowed vertex types and their multiplicities that a vertex V contributes a factor of $\text{mult}(V)$ to m_C if and only if V is adjacent to a complex marking or dual to a triangle with even lattice area.

The number c of triangles with even lattice area equals $n_{(3)}+n_{(4)}+n_{(8)}$. Let $\tilde{\Gamma}$ be a connected component of Γ_{even}. We know that $\tilde{\Gamma}$ has a unique root. Since $w(y_i) = 1$ for all $i = 1, \ldots, n$, this root cannot be an end of Γ, so it has to be a vertex of type (3) in Γ, i.e. a 1-valent vertex in Γ_{even}. Remove the 1-valent vertex from $\tilde{\Gamma}$, thus producing an end, apply the end-gluing map of definition 5.27, and forget all markings (straightening the 2-valent vertices). Call the resulting graph $\tilde{\Gamma}^\circ$. This graph is 3-valent and has $1 + n^{\tilde{\Gamma}}_{(6b)} + n^{\tilde{\Gamma}}_{(8)}$ ends, and thus it has $n^{\tilde{\Gamma}}_{(6b)} + n^{\tilde{\Gamma}}_{(8)} - 1$ vertices. But this number of 3-valent vertices also equals $n^{\tilde{\Gamma}}_{(4)} + n^{\tilde{\Gamma}}_{(8)}$, and so $n^{\tilde{\Gamma}}_{(6b)} + n^{\tilde{\Gamma}}_{(8)} = n^{\tilde{\Gamma}}_{(4)} + n^{\tilde{\Gamma}}_{(8)} + 1 = n^{\tilde{\Gamma}}_{(4)} + n^{\tilde{\Gamma}}_{(8)} + n^{\tilde{\Gamma}}_{(3)}$. Since this holds for any $\tilde{\Gamma}$, it follows that $n_{(6b)} + n_{(8)} = n_{(3)} + n_{(4)} + n_{(8)}$. Thus $k = c$, where k denotes the number of double ends. The factor 2^k in the lemma thus corresponds exactly to the factor 2^{-c} in the definition 4.27 of $\text{mult}_S(h(\Gamma))$.

Hence it only remains to show that $(-1)^{a+b}$ equals the sign contribution coming from factors of i in the definition 5.15 of m_C, where a denotes the number of lattice points in the interior of triangles and b denotes the number of triangles such that all sides have even lattice length. We refer to the power of i in the vertex multiplicity m_V of definition 5.15 as the sign.

Consider a vertex V and let $A = \text{mult}(V)$. If V is of type (2) to (5), assume the three adjacent (non-marked) edges have weights w_1, w_2 and w_3. By Pick's formula 5.16, $A = 2I + B - 2$, where I denotes the number of lattice points in the interior of the triangle dual to V and B denotes the number of lattice points on the boundary. By our assumptions, $B = w_1+w_2+w_3$. If V is of type (2) or (5), then its sign is

$$i^{A-1} = (-1)^{\frac{A-1}{2}} = (-1)^{\frac{2I+w_1+w_2+w_3-2-1}{2}} = (-1)^I \cdot (-1)^{\frac{w_1-1}{2}} \cdot (-1)^{\frac{w_2-1}{2}} \cdot (-1)^{\frac{w_3-1}{2}}.$$

If V is of type (3), its sign is

$$i^{A-1} = i^{-1} \cdot i^{A} = i^{-1} \cdot (-1)^{\frac{A}{2}} = i^{-1} \cdot (-1)^{\frac{2I+w_1+w_2+w_3-2}{2}}$$
$$= i^{-1} \cdot (-1)^{I} \cdot (-1)^{\frac{w_1-1}{2}} \cdot (-1)^{\frac{w_2-1}{2}} \cdot (-1)^{\frac{w_3}{2}},$$

where we assume that w_3 is the even weight. For type (4), we get

$$i^{A-1} = i^{-1} \cdot i^{A} = i^{-1} \cdot (-1)^{\frac{A}{2}} = i^{-1} \cdot (-1) \cdot (-1)^{\frac{2I+w_1+w_2+w_3}{2}}$$
$$= i^{-1} \cdot (-1) \cdot (-1)^{I} \cdot (-1)^{\frac{w_1}{2}} \cdot (-1)^{\frac{w_2}{2}} \cdot (-1)^{\frac{w_3}{2}}.$$

We write the sign of type (6b) as $i^{-1} = i \cdot (-1) = i \cdot (-1)^{\frac{2}{2}}$, and 2 is the weight of the even adjacent edge (since the double ends are of weight 1 by assumption). The sign of (8) is

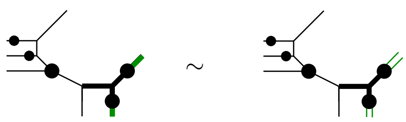
$$-1 = (-1) \cdot i^{A} = (-1) \cdot (-1)^{I} \cdot (-1)^{\frac{w_1}{2}} \cdot (-1)^{\frac{w_2}{2}},$$

where w_1 and w_2 are the weights of the two adjacent even edges. This is true since the two edges of the same direction which are adjacent to (8) are ends and thus their weight is 1 by assumption. The sign of (1) can be written as $1 = (-1)^{\frac{w_1-1}{2}} \cdot (-1)^{\frac{w_2-1}{2}}$, where $w_1 = w_2$ is the odd weight of the adjacent edges. Analogously, we can write the sign of (7) as $1 = (-1)^{\frac{w_1}{2}} \cdot (-1)^{\frac{w_2}{2}}$, where now $w_1 = w_2$ is the even weight of the adjacent edges.
Notice that the product of the factors $(-1)^{I}$ which appear for each vertex dual to a triangle is $(-1)^{a}$. Also, for each vertex of type (4) and (8) — which are the vertices dual to triangles such that all sides have even lattice length — we have a factor of (-1) which yields $(-1)^{b}$ as product. In addition, we have extra factors of i^{-1} for each vertex of type (3) and (4), and i for each vertex of type (6b). But since $n_{(4)} + n_{(3)} = n_{(6b)}$ as we have seen above, these extra factors cancel. Furthermore, we have factors of $(-1)^{\frac{w-1}{2}}$ for each edge of odd weight ending at a vertex, and $(-1)^{\frac{w}{2}}$ for each even edge. Every bounded edge ends at two vertices, so these contributions cancel. Since we require that the weights of all ends are 1, the corresponding factors for the ends are just 1. Thus all the factors $(-1)^{\frac{w-1}{2}}$ resp. $(-1)^{\frac{w}{2}}$ cancel, and it follows that the sign equals $(-1)^{a+b}$, as required. □

Notice that a toric Del Pezzo degree consists of directions of weight one, so the requirements of lemma 5.42 are satisfied.

Remark 5.43 (**Welschinger numbers are not locally invariant in the moduli space**)
It is a striking feature of the Welschinger numbers $N^{W}_{(r,s)}(\Delta, \omega)$ that, although they are invariant under ω in the cases mentioned in theorem 4.33, one cannot show this by a local study of the moduli space as in the proof of theorem 5.26 as we have seen in remark 4.35. In short, the reason for this is that the absence of the vertex type (6a) breaks the local invariance argument in the codimension-1 case (C1_1) (see the proof of theorem 5.26, in particular the table of codimension-1 cases and their resolutions).

Remark 5.44 (**Comparing broccoli curves with Welschinger curves**)
There are broccoli curves, which are also Welschinger curves when we identify Welschinger curves before and after after end-gluing 5.27.



Hence the intersection of the subset of broccoli curves with the subset of Welschinger curves is not empty in $\mathcal{M}^{\mathrm{or}}_{0,r+s}(\mathbb{R}^2, \Delta, F)$ or $M_{(r,s)}(\Delta)$.

5.4 Bridge curves and the invariance along bridges

The aim of the following section is to prove that for toric Del Pezzo degrees Δ (see definition 4.7) the Welschinger numbers $N^W_{(r,s)}(\Delta,\omega)$ coincide with the broccoli invariants $N^B_{(r,s)}(\Delta,\omega)$ (see corollary 5.60). Since broccoli invariants are independent of the chosen conditions, this result provides a tropical proof of the invariance of Welschinger numbers, without having to use the detour via the Correspondence and the Welschinger theorem. When considering degrees Δ that are not toric Del Pezzo, the equivalence of Welschinger numbers and broccoli invariants no longer holds, and consequently the Welschinger numbers may actually not be invariant.
We start with the definition of the class of bridge curves. It is a special case of the class of oriented marked curves and includes oriented broccoli and Welschinger curves. When a bridge curve is a broccoli curve having vertices of type (6a) or a Welschinger curve having vertices of type (8), this curve allows to start a so called *bridge*, that is, a 1-dimensional family of bridge curves connecting broccoli and Welschinger curves. We show the invariance of the curve multiplicities m_C along these bridges, which then leads to the equality of broccoli and Welschinger numbers mentioned above.
Throughout this section let $r,s \geq 0$, let $\Delta = (v_1,\dots,v_n)$ be a collection of vectors in $\mathbb{Z}^2 \setminus \{0\}$, and let $F \subset \{1,\dots,n\}$ such that $|\Delta| - 1 = r + 2s + |F|$. Moreover, fix conditions $\omega \in \mathbb{R}^{2(r+s)+|F|}$ in general position for $\text{ev}_F : M^{\text{or}}_{(r,s)}(\Delta,F) \to \mathbb{R}^{2(r+s)+|F|}$ as in definition 5.8 and example 5.10, and consider only curves satisfying these conditions.

Remark 5.45
Note that by lemma 5.20 an oriented curve $C \in M^{\text{or}}_{(r,s)}(\Delta,F)$ all of whose vertices are of the types (1) to (9) of definition 5.15 satisfies $n_{(7)} = n_{(8)} + n_{(9)}$ (similarly to remark 5.34 a) for Welschinger curves).

Definition 5.46 (**Bridge curves**)
Let r, s, Δ, and F be as in remark 5.45. A *bridge curve* consists of the data of:

- an oriented curve $C \in M^{\text{or}}_{(r,s)}(\Delta,F)$ all of whose vertices are of the types (1) to (9) of definition 5.15, and
- a bijection between its vertices of type (7) and those of types (8) or (9) (see remark 5.45),

such that the following conditions hold:

a) There is at most one vertex of type (9).

b) Each vertex of type (8) or (9) is connected to its corresponding vertex of type (7) (under the given bijection) starting with one of its even edges by a sequence of edges with no markings on them.

c) Consider the set M of vertices of type (6a) and (7); by abuse of notation we will sometimes also think of it as the set of all complex markings at these vertices. We split this set as $M = M_{(8)} \dot\cup M_{(9)} \dot\cup M_{(6a)}$, where

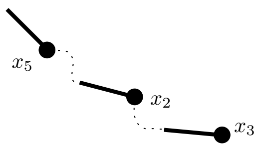
   - $M_{(8)}$ contains the vertices of type (7) corresponding to vertices of type (8) under the given bijection,
   - $M_{(9)}$ contains the vertices of type (7) corresponding to vertices of type (9) under the given bijection,
   - $M_{(6a)}$ contains the vertices of type (6a).

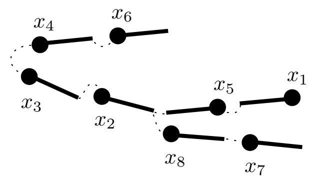
   We define a partial order on M by considering each vertex in M with one even adjacent edge — in the case of a vertex of type (7) we take the edge that does not connect this vertex to its corresponding vertex of type (8) or (9). For complex markings $x_i \neq x_j$ in

M we say $x_i < x_j$ if the unique path connecting x_i and x_j does not pass through the even edge of x_i, but does pass through the even edge of x_j. Refine this partial order to a total order by considering vertices which are minimal under the partial order and comparing the (numerical) value of their markings. Choose the numerically minimal one and repeat the procedure without the chosen vertex until all vertices are ordered. We require now that the labeling of the complex markings is chosen such that vertices in $M_{(8)}$ are smaller than vertices in $M_{(9)}$, and vertices in $M_{(9)}$ are smaller than vertices in $M_{(6a)}$.

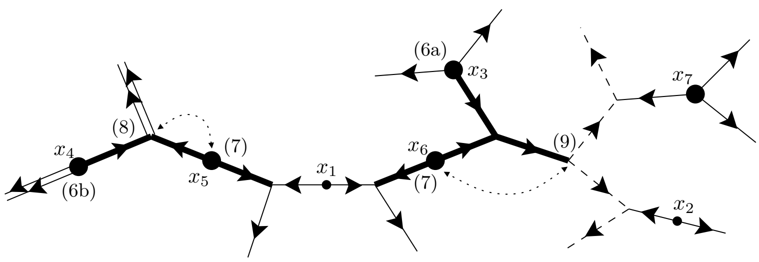
The multiplicity m_C of a bridge curve C is given as usual by definition 5.15.

Example 5.47
For an example of the partial order in definition 5.46 c) consider the picture below on the left, in which x_2, x_3, and x_5 are the complex markings of type (6a) or (7). We have $x_5 < x_2 < x_3$, where dotted lines stand for parts of the graph between the distinguished edges and vertices. In this case, the total order on M of definition 5.46 c) agrees with this partial order. In the picture on the right however we get the partially ordered sets $x_7 < x_8 < x_5 < x_1$, $x_7 < x_8 < x_2 < x_3$, $x_6 < x_4$, and the total order $x_6 < x_4 < x_7 < x_8 < x_2 < x_3 < x_5 < x_1$.





Example 5.48
An example of a bridge curve (containing a vertex of type (9)) is given in the following picture; the bijection between the vertices of type (7) and those of types (8) and (9) is indicated by the dotted arrows. We have labeled the vertices by their types only in the cases (6), (7), (8), and (9) since these are the most relevant ones for our study of bridge curves. In this example we have $M = \{x_3, x_5, x_6\}$ and $M_{(8)} = \{x_5\}$, $M_{(9)} = \{x_6\}$, $M_{(6a)} = \{x_3\}$. The partial order on M is given by $x_6 < x_3$ and the total order by $x_5 < x_6 < x_3$. The dashed edges are ordinary odd edges; they form a string as explained in definition 5.53 and remark 5.54.



Remark 5.49
From the allowed vertex types of definition 5.15 it follows that the sequence of edges of definition 5.46 b) connecting each vertex of type (7) to its corresponding vertex of type (8) or (9) just contains even edges which are then adjacent to vertices of type (4).

Remark 5.50
The choice of the total order refining the partial order in definition 5.46 c) is not important. While the definition of bridge curves depends on this choice, the result of invariance in theorem 5.58 does not.

Remark 5.51 (**Dimension of the space of bridge curves**)
These (oriented) bridge curves can be constructed with the bridge algorithm 5.62 from oriented broccoli or Welschinger curves without changing the conditions ω. In particular, bridge curves

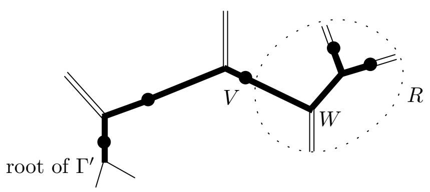
are curves passing through conditions in general position. In fact, since the number of our conditions is $2(r+s)+|F|$ it follows from lemma 5.20 that the space of bridge curves of a given combinatorial type through ω is 0-dimensional if there is no vertex of type (9) (i.e. if $M_{(9)} = \emptyset$), and 1-dimensional otherwise. If we even have $M_{(8)} = M_{(9)} = \emptyset$ or $M_{(9)} = M_{(6a)} = \emptyset$, the bridge curves specialize to the broccoli and Welschinger curves that we already know:

Lemma 5.52 (**Broccoli and Welschinger curves as bridge curves**)
For fixed r, s, Δ, F the operation of forgetting the correspondence between the vertices of type (7) and those of types (8) or (9) of definition 5.46 induces bijections between curves through ω

$$\{\text{bridge curves with } M_{(8)} = M_{(9)} = \emptyset\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{\text{oriented broccoli curves}\}$$

and $$\{\text{bridge curves with } M_{(9)} = M_{(6a)} = \emptyset\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{\text{oriented Welschinger curves}\}.$$

Proof. First of all, given a bridge curve with $M_{(8)} = M_{(9)} = \emptyset$, it follows directly $n_{(7)} = n_{(8)} = n_{(9)} = 0$. Hence the curve consists only of vertices of types (1) to (6) and is therefore a broccoli curve. In the same way, $M_{(9)} = M_{(6a)} = \emptyset$ for a bridge curve implies $n_{(9)} = 0$ and $n_{(6a)} = 0$ by definition 5.46 c). So we obtain a Welschinger curve. Hence the two maps of the lemma (from left to right) are well-defined.
Conversely, an oriented broccoli curve has only vertices of type (1) to (6). Hence $M_{(8)} = M_{(9)} = \emptyset$, and the correspondence between vertices of types (7), (8), and (9) is trivial. So the statement of the lemma about broccoli curves is obvious.
Analogously, we have $M_{(9)} = M_{(6a)} = \emptyset$ for each oriented Welschinger curve as we just allow vertices of types (1) to (5), (6b), (7), and (8). Conditions a) and c) of definition 5.46 are clear. So we have to prove the existence and uniqueness of a correspondence between the vertices of type (7) and (8) that satisfies b). To do this, we perform an induction over the number $n_{(7)}$ of vertices of type (7) in the underlying graph Γ. For $n_{(7)} = 0$ there is nothing to show. Let V be such a vertex of type (7) in a connected component Γ' of Γ_{even} such that the part of $\Gamma' \setminus \{V\}$ not containing the root of Γ' (see definitions 5.30 and 5.32 b) and the equivalence of oriented and unoriented Welschinger curves through conditions in general position in proposition 5.36) contains no other vertices of type (7). Using remark 5.34 b) for the encircled part R in the picture below, we know that it has exactly one vertex W of type (8). Now V and W are obviously connected by a sequence of even edges as required by definition 5.46 b), and moreover V is the only vertex of type (7) that W can be connected to without passing through other markings. Cut off R and replace V by a vertex of type (6b). Applying the induction hypothesis to the rest of Γ, we obtain the required existence and uniqueness of the bijection between the vertices of type (7) and (8).



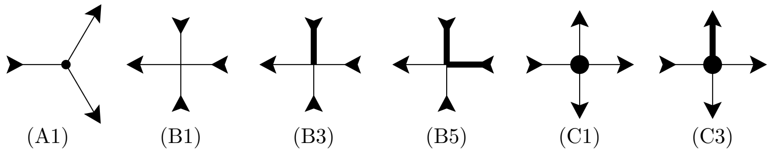
□

We will now study the 1-dimensional types of bridge curves through ω and the boundary cases to which they can degenerate.

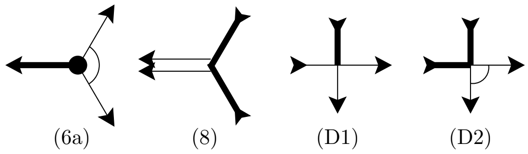
Definition 5.53 (**Strings**)
Let $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h) \in M^{\text{or}}_{(r,s)}(\Delta, F)$ be an oriented marked curve. As in remark 3.16, a *string* of C is a subgraph of Γ (after the end-gluing of definition 5.27) homeomorphic to $\mathbb{R}$ which does not intersect the closures $\overline{x_i}$ of the marked points and whose two ends are not fixed.

Remark 5.54
A bridge curve with a vertex of type (9) contains a unique string (containing this vertex) since the orientation of the two odd edges prescribes that they both lead in a unique way to a non-fixed unbounded end without passing through any markings (see example 5.48). Note that the allowed vertex types require that these paths to the non-fixed unbounded ends go only through vertices of types (2) and (3). In particular, the string then contains only odd edges. On the other hand, a curve without vertex of type (9) does not contain a string.
By remark 5.51, a bridge curve through conditions in general position that has a vertex of type (9) (and thus a string) moves in a 1-dimensional family — namely by moving this string, as already observed in [GM08, remark 3.6]. Let us now figure out what boundary cases can occur at the end of such 1-dimensional families.

Lemma 5.55 (**Codimension-**1 **cases for bridge curves**)
Let C be a bridge curve through ω with a vertex of type (9), thus having a string as in remark 5.54. This string can be moved until two vertices of C merge. The possible resulting vertices are as follows; we call them *codimension-*1 *cases for bridge curves*. As before, the arc in type (D2) means that the two odd edges must not be ends of the same direction.



two vertices of type (1) – (6) merging



one vertex of type (9) and one vertex of type (2) – (4) or (7) merging

Proof. For the terminology used in the following, we refer to the proof of theorem 5.26.

Case 1: Assume the two vertices merging are of types (1) to (6). Then V is a vertex of type (A $\cdot$), (B $\cdot$), or (C $\cdot$). The bridge curve we started with has already a vertex W of type (9). Hence, just resolutions that do not create a vertex of type (9) are allowed. As C originates from a bridge curve with a string, two of the edges adjacent to V are contained in the string; more precisely by remark 5.54 there must be one incoming and one outgoing odd edge. If we just consider vertices where not all resolutions have multiplicity 0, the only possible vertices which are left then are (A1), $(\mathrm{B1}_1)$, (B3), (B5), $(\mathrm{C1}_1)$, $(\mathrm{C1}_3)$, and (C3).

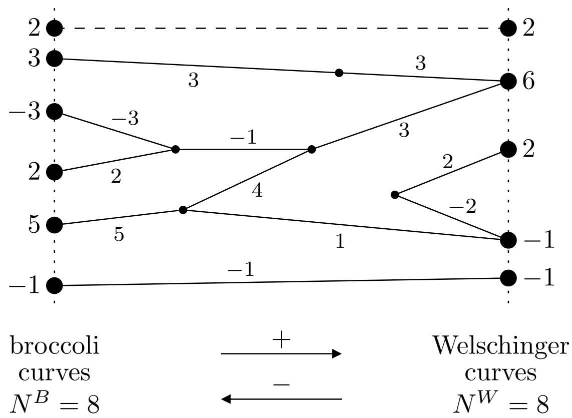
Case 2: One vertex is of type (1) to (8) and the other one of type (7) or (8). Note that the string has to pass through one of the merging vertices in order to create the codimension-1 case. So we cannot have two vertices of type (7) and/or (8) as they do not allow the existence of the string. We thus need one vertex of type (1) to (6) which has one incoming and one outgoing odd edge, i.e. a vertex of type (3) merging with a vertex of type (7). But in this case, this vertex of type (7) (which necessarily lies in $M_{(8)}$) is bigger than the type (7) vertex in $M_{(9)}$ corresponding to the type (9) vertex at which the string starts — in contradiction to part c) of the definition 5.46 of a bridge curve. And indeed, the vertex arising from merging type (3) with (7) has no other legal resolution, so such a case does not appear. Case 2 is thus impossible.

Case 3: One of the vertices is of type (9). Then the other vertex must be of type (2) to (4) or (7) as the other vertices of type (1), (5), (6), (8) do not fit together with the parity and the direction of the edges adjacent to the vertex of type (9).

- If V arises from merging a vertex of type (9) with a vertex of type (7) we obtain a bridge curve with a vertex of type (6a), but without vertex of type (9).
- Merging a vertex of type (9) with a vertex of type (3) gives a bridge curve with a vertex of type (8) or (D2), depending on whether the resulting two odd edges are ends of the same direction or not.
- If the second vertex is of type (2) or (4), we obtain a vertex of type (D1) resp. (D2).

□

Remark 5.56 (**Bridge graphs and bridges**)
We are now able to explain the idea of bridges connecting broccoli to Welschinger curves more precisely. For this let us construct a so-called *bridge graph* as follows: the edges are the 1-dimensional types of bridge curves through ω (i.e. those containing a vertex of type (9) and thus a string), and the vertices are their 0-dimensional boundary degenerations as described in lemma 5.55 (we will see in lemma 5.59 that in the toric Del Pezzo case the string movement actually ends at both sides and thus leads to two vertices for each edge in the bridge graph). Note that the bijection between vertices of type (7) and those of types (8) and (9) that we have for the 1-dimensional types can be extended to a map between vertices in the 0-dimensional boundary types. We identify two such 0-dimensional boundary types, i.e. represent them by the same vertex in the bridge graph, if they have the same underlying oriented curve and this map between vertices agrees, where we discard any mapping of a vertex to itself (which can occur if a type (7) vertex merges with a type (9) vertex to one of type (6a)).
Note that some vertices in the bridge graph correspond to bridge curves with no type (9) vertex, whereas others (corresponding to codimension-1 cases (A $\cdot$), (B $\cdot$), (C $\cdot$), (D $\cdot$)) are not bridge curves in the sense of our definition. Included are however (as we will see in theorem 5.58) all broccoli and Welschinger curves through ω, so that we can think of the bridge graph as connecting broccoli and Welschinger curves. We will call a connected component of the bridge graph a *bridge*.
The following picture shows a schematic example of a bridge graph. Its vertices corresponding to broccoli and Welschinger curves are drawn as big dots (on the left resp. right hand side of the diagram), the other ones as small dots. The dashed line indicates a curve which is both broccoli and Welschinger (i.e. has $M_{(8)} = M_{(9)} = M_{(6a)} = \emptyset$), so it does not correspond to an edge in the bridge graph. The broccoli and Welschinger curves, as well as the 1-dimensional types of bridge curves, are labeled with their multiplicities as in definition 5.15.



The idea to prove the equality of broccoli and Welschinger numbers is now that there is a *local balancing condition* on the bridge graph, i.e. that (as in the picture above) at each vertex the sum of the incoming equals the sum of the outgoing curve multiplicities when we move from the broccoli to the Welschinger side. To make this idea work, we first of all have to see that the edges of the bridge graph have a natural orientation so that it is well-defined which direction leads to the broccoli and which to the Welschinger side.

Definition 5.57 (**Direction of string movement**)
For a given bridge curve C with a vertex V of type (9) consider the even edge E adjacent to V. Changing the length of E induces the movement of the string in C. Namely, making this edge longer makes the curve "more Welschinger"; we want to call this the *positive direction* $(+)$ of the string movement. Making E shorter leads to a "more broccoli" like curve; we want to call this the *negative direction* $(-)$ of the string movement.

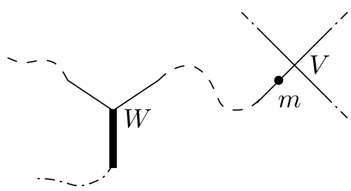
Theorem 5.58 (**Invariance along bridges**)
Let C be an oriented curve containing a vertex V of one of the codimension-1 types (A·), (B·), (C·), (6a)/(8), or (D·) as in lemma 5.55, and only vertices of types (1) to (9) otherwise. Assume as in lemma 5.55 that C arises from moving a string in a bridge curve with a vertex of type (9). Consider all bridge curves C' that resolve C and that have matching bijections between their vertices of type (7) and those of type (8) and (9). (In the language of remark 5.56 this means that C corresponds to a vertex and C' to its adjacent edges in the bridge graph.)
The curves C' all contain a string and thus we can define $\mathrm{sign}_{C'}$ as the direction of the movement of the string away from C. Then $\sum_{C'} \mathrm{sign}_{C'} \cdot m_{C'}$ equals either

a) m_C if C is a broccoli curve (i.e. we are on the left side of the bridge graph in remark 5.56);

b) $-m_C$ if C is a Welschinger curve (i.e. we are on the right side of the bridge graph);

c) 0 in all other cases.

Proof. For the terminology used in the following, we refer to the proof of theorem 5.26. We consider the resolving bridge curves C' and distinguish the types of V as in lemma 5.55.
Case 1: V is a vertex of type (A·), (B·), or (C·) (we are then in case c) of the theorem). We then compare the H-sign in the proof of theorem 5.26 with the direction of the string movement for C'. Imagine to put a marking m on the bounded edge adjacent to V that connects this vertex on the string to the vertex W of type (9). We know from 5.55 that V can be resolved into two vertices of types (1) to (6). As the two odd edges adjacent to W are contained in the string, the 1-dimensional movement of the marking m generated by resolving V is reflected by the 1-dimensional movement of the string and hence by varying the length of the even edge at W:



Thus the H-sign equals the sign defined by the direction of the string movement (up to the same sign for all resolutions). Since we proved $\sum_{C'} (H\text{-sign}) \cdot m_{C'} = 0$ in theorem 5.26 already, it only remains to be shown in each case that all resolving curves are actually bridge curves, i.e. satisfy the conditions a) to c) of definition 5.46. Condition a) is always satisfied as we do not create a vertex of type (9).
Concerning condition b) of the definition of a bridge curve, note that in the cases (B·) the connection between vertices of type (7), (8), and (9) are not modified as no vertices of type (7), (8), and (9) and no markings are involved. Hence, condition b) is satisfied in all resolutions in this case. In the resolutions of vertices of type (A·) and (C·), no vertices of type (4) are

involved, which are however necessary by remark 5.49 to connect vertices of type (7) and (8), (9). Hence, also in these cases condition b) is satisfied in all resolutions.
Looking at condition c) of definition 5.46, the cases (A $\cdot$) and (B $\cdot$) are easy to manage as no vertices of type (6a) and (7) are involved (the partition of M and the total order are not changed). For the case (C $\cdot$) we have to go into more details.

- ($\mathrm{C1}_1$) Resolution (I) has a supplementary vertex V of type (6). If the supplementary vertex is of type (6b), it is not contained in M and need not be considered, so let us assume that V is of type (6a). Then the set M contains one more element (lying in $M_{(6\mathrm{a})}$) compared to the resolutions (II) and (III). The string contains the edge v_1 and therefore, the vertex contained in $M_{(9)}$ also lies behind v_1. Hence, V is bigger than the vertex of $M_{(9)}$ under the partial order. As the total order refines the partial order condition c) is still satisfied.

- ($\mathrm{C1}_3$) All three resolutions contain one more vertex of type (6a) in $M_{(6\mathrm{a})}$ than C. But also in this case, this new vertex is bigger than the already existing vertex in $M_{(9)}$. Condition c) is thus satisfied for all three resolutions simultaneously.

- (C3) Here, there are just two resolutions with a vertex of type (6a), where each time the new bounded edge is odd. The edge v_2 is even as before, the vertex in $M_{(9)}$ lies behind v_1, so the vertex in $M_{(9)}$ and this vertex can be compared under the total order but not under the partial order. Hence, condition c) satisfied in both cases simultaneously.

In total, we can conclude that conditions b) and c) are fulfilled for all resolutions (if for any).
Case 2: V is a vertex of type (6a) or (8) (note that V is a priori not unique then since C has in general several vertices of type (6a) or (8)). We want to resolve vertices in this curve such that the resolutions are bridge curves with a vertex of type (9). The other way around we can ask ourselves which vertices in a bridge curve with vertex of type (9) can be merged in order to create C. After testing all possibilities we obtain two cases:

- (A) the vertex of type (9) can melt with a vertex of type (7) into a vertex of type (6a);

- (B) the vertex of type (9) can melt with a vertex of type (3) into a vertex of type (8), if the odd outgoing edge of the vertex of type (3) is an end and if one of the odd outgoing edges of the vertex of type (9) is also an end of the same direction.

Hence if we want to go the other way around, we can resolve

- (A) a vertex of type (6a) into a vertex of type (7) and a vertex of type (9);

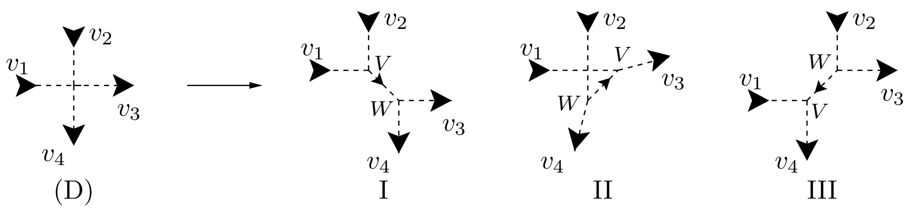
- (B) a vertex of type (8) into a vertex of type (3) and a vertex of type (9). The so newly created bounded edge can have both orientations, due to the symmetric situation at the vertex of type (8). The question is just which of the vertices will become the vertex of type (3) and which one the vertex of type (9).

For these two types of resolutions we have to check if the conditions b) and c) of the definition 5.46 of a bridge curve are satisfied.

- (A) The set M remains the same as before resolving. The connections between vertices considered in condition b) also remain the same. Before resolving the marking is at a vertex in $M_{(6\mathrm{a})}$, but after resolving it becomes a vertex in $M_{(9)}$. This is just allowed if the marking was the smallest element in $M_{(6\mathrm{a})}$, which is the case for exactly one marking if we assume $M_{(6\mathrm{a})} \neq \emptyset$. Then the partial and the total order on M also remain the same and condition c) is satisfied.

(B) The set M is conserved also in this case. Consider the marking x_i which corresponds to the vertex of type (8). In order to satisfy condition b) of the definition we have to meet the vertex of type (9) at its even edge if we start at the marking. This means that we must choose the orientation of the inserted bounded edge such that this holds. To satisfy condition c) the marking x_i has to be the biggest point in $M_{(8)}$ (assuming $M_{(8)} \neq \emptyset$). We need this since, after resolving the vertex, the marking lies in $M_{(9)}$ and not anymore in $M_{(8)}$. But note that we still have two resolutions as we have two possibilities to enumerate the two odd edges at the vertex of type (8) that we resolve.

Observe that both the multiplicity of the curve in (A) and the sum of the multiplicities of the two resolutions from (B) equal the multiplicity of C — due to the fact that the multiplicity of the vertex of type (8) resolved in (B) is the double of the multiplicity of the vertex of type (3) after the resolution. Thus, as the even edge E adjacent to the type (9) vertex becomes longer in (A) and shorter in the resolutions (B), the invariance holds if $M_{(8)} \neq \emptyset \neq M_{(6a)}$ so that both cases (A) and (B) exist. If $M_{(8)}$ is empty, the bridge curve we are looking at is a broccoli curve by lemma 5.52. We then resolve a vertex of type (6a) by making E longer. Hence $\mathrm{sign}_{C'} \cdot m_{C'}$ is plus the broccoli multiplicity. In the same way, if $M_{(6a)}$ is empty, the considered bridge curve is a Welschinger curve by lemma 5.52. As we then resolve a vertex of type (8), E becomes shorter, so $\mathrm{sign}_{C'} \cdot m_{C'}$ is minus the Welschinger multiplicity.
Case 3: V is a vertex of type (D1) or (D2) (we are then in case c) of the theorem). Remember from lemma 5.55 that V can then be resolved into a vertex of type (2) to (4) and a vertex of type (9). The vertex of type (7) corresponding to the vertex of type (9) has to lie behind one of the even edges at the 4-valent vertex by definition 5.46 b); we choose it to be behind the edge with direction v_2. The orientation and the parity of the bounded edge which appears when resolving are determined.



Observe that resolution I does not exist for the vertex of type (D1) as the 3-valent vertices that appear then are not allowed for bridge curves. The vertices appearing are listed in the table below. The last column $m_{\mathrm{I/II/III}}$ shows the absolute value of the product of the two vertex multiplicities in the resolutions I, II, and III.

codim-1	resolution I			resolution II			resolution III		
case	V	W	m_{I}	V	W	m_{II}	V	W	m_{III}
D1				(2)	(9)	1	(2)	(9)	1
D2	(4)	(9)	$\lvert(v_1, v_2)\rvert$	(3)	(9)	$\lvert(v_1, v_3)\rvert$	(3)	(9)	$\lvert(v_1, v_4)\rvert$

We have to check if conditions b) and c) of definition 5.46 are satisfied. Connections between vertices of type (7) to vertices of type (8) are not modified as no vertices of type (7), (8) and markings are involved in the resolutions. Similarly, the connection between the vertex of type (9) and the corresponding vertex of type (7) is not modified as the vertex of type (7) lies behind the edge of direction v_2. Hence, condition b) is satisfied in all resolutions or in none of them. As no markings are involved in the resolutions, the set M, the splitting of M, and the total order are also preserved. So condition c) holds in all three resolutions or in none of them.
In order to prove the local invariance we also have to compute the direction of the string movement as in definition 5.57. In resolution I we create a vertex of type (9), so the edge E of definition 5.57 becomes longer.

As in the proof of theorem 5.26 we can imagine to have for the other resolutions II and III two other markings $P_1, P_2 \in \mathbb{R}^2$ on the edges v_1, v_2 as these are fixed. Hence we have two bounded edges of lengths l_1 and l_2, in addition to the (by resolving) new inserted bounded edge of length a. The direction of the string movement as in definition 5.57 is positive if and only if l_2 becomes longer when a becomes longer. We can describe the condition that the curve has to pass through the given point conditions by the following linear systems of equations in the variables l_1, l_2, a.

II

l_1	l_2	a	
$-v_1$	v_2	$-v_1 - v_3$	$P_2 - P_1$

III

l_1	l_2	a	
$-v_1$	v_2	$-v_1 - v_4$	$P_2 - P_1$

Obviously, these systems both have a one-dimensional space of solutions. In case II the homogeneous solution vector (l_1, l_2, a) has the following entries:

$$l_1 = (v_2, -v_1 - v_3), \qquad l_2 = -(-v_1, -v_1 - v_3), \qquad a = (-v_1, v_2),$$

where as above (v_i, v_j) is the determinant of the matrix consisting of the column vectors v_i, v_j. So in order to determine the direction of the string movement we have to multiply the signs of l_2 and a, that is $\operatorname{sign}(v_1, v_3)\operatorname{sign}(v_1, v_2)$. In case III we just have to substitute the vector v_3 by v_4 and obtain therefore as sign $\operatorname{sign}(v_1, v_4)\operatorname{sign}(v_1, v_2)$. So in total the sign for the directions of the string movements are given by the following table.

	sign for I	sign for II	sign for III
(D)	1	$\operatorname{sign}((v_1, v_3)(v_1, v_2))$	$\operatorname{sign}((v_1, v_4)(v_1, v_2))$

We are now able to verify the local invariance. We will use the same identities to deal with vertex multiplicities and signs as in the proof of theorem 5.26. Mainly, we use the formulas $\operatorname{sign}(v_i, v_j) i^{|(v_i,v_j)|-1} = i^{(v_i,v_j)-1}$ if $|(v_i, v_j)|$ is odd and $i^{|(v_i,v_j)|-1} = i^{(v_i,v_j)-1}$ if $|(v_i, v_j)|$ is even.

In case (D1), we then obtain for the product of the vertex multiplicities together with the direction of the string movement in the resolutions II and III:

$$\begin{aligned}
(\mathrm{II}) &= \operatorname{sign}((v_1, v_3)(v_1, v_2)) \cdot i^{|(v_1,v_3)|-1} \cdot i^{|(v_2,v_4)|-1} = \operatorname{sign}(v_1, v_2) \cdot i^{(v_1,v_3)+(v_4,v_2)-2},\\
(\mathrm{III}) &= \operatorname{sign}((v_1, v_4)(v_1, v_2)) \cdot i^{|(v_1,v_4)|-1} \cdot i^{|(v_2,v_3)|-1} = \operatorname{sign}(v_1, v_2) \cdot i^{(v_1,v_4)+(v_2,v_3)-2}.
\end{aligned}$$

We have $\operatorname{sign}(v_1, v_2) \neq 0$ since v_1 and v_2 cannot be parallel as our curves pass through conditions in general position. Dividing equation (III) by (II) yields $i^{2(v_3,v_1)} = (-1)^{(v_3,v_1)} = -1$ as (v_3, v_1) is odd. Hence (II)+(III)$= 0$.

Similarly, for (D2) we obtain:

$$\begin{aligned}
(\mathrm{I}) &= |(v_1, v_2)| \cdot i^{|(v_1,v_2)|-1} \cdot i^{|(v_3,v_4)|-1} = \operatorname{sign}(v_1, v_2) \cdot (v_1, v_2)\, i^{(v_1,v_2)+(v_3,v_4)-2},\\
(\mathrm{II}) &= \operatorname{sign}((v_1, v_3)(v_1, v_2)) \cdot |(v_1, v_3))| \cdot i^{|(v_1,v_3)|-1} \cdot i^{|(v_2,v_4)|}\\
&= \operatorname{sign}(v_1, v_2) \cdot (v_1, v_3)\, i^{(v_1,v_3)+(v_4,v_2)-2},\\
(\mathrm{III}) &= \operatorname{sign}((v_1, v_4)(v_1, v_2)) \cdot |(v_1, v_4)| \cdot i^{|(v_1,v_4)|-1} \cdot i^{|(v_2,v_3)|-1}\\
&= \operatorname{sign}(v_1, v_2) \cdot (v_1, v_4)\, i^{(v_1,v_4)+(v_2,v_3)-2}.
\end{aligned}$$

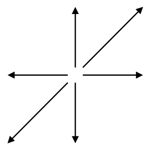
Let us divide all three terms by $\operatorname{sign}(v_1, v_2)\, i^{(v_1,v_2)+(v_3,v_4)-2}$. For (I) we then get (v_1, v_2). In term (II) we obtain $i^{2(v_2,v_1)} \cdot (v_1, v_3) = (-1)^{(v_2,v_1)} \cdot (v_1, v_3) = (v_1, v_3)$ as (v_2, v_1) is even. Finally, for (III) we get $i^{2(v_1,v_4)} \cdot (v_1, v_4) = (-1)^{(v_1,v_4)} \cdot (v_1, v_4) = (v_1, v_4)$ as (v_1, v_4) is also even. So we have (I)+(II)+(III)$= (v_1, v_2) + (v_1, v_3) + (v_1, v_4) = 0$.

Hence we have shown the invariance for all codimension-1 cases for bridge curves. □

In order to prove the equality of broccoli and Welschinger numbers with the idea of remark 5.56 we need one more final ingredient: that each edge in the bridge graph is actually bounded, i.e. that the string movement in each 1-dimensional type of bridge curves is bounded in both directions by a codimension-1 case. It is actually only this last step that requires a toric Del Pezzo degree and thus spoils the equality of broccoli and Welschinger numbers (as well as the invariance of Welschinger numbers, see example 4.36) in other cases.

Lemma 5.59 (**Boundedness of bridges**)
Assume that Δ is a toric Del Pezzo degree (see definition 4.7). Let C be a bridge curve through ω with a vertex of type (9), thus having a string as in remark 5.54. Then the movement of the string within this combinatorial type is bounded in both directions.

Proof. Assume that we have a bridge curve through ω with a string that can be moved infinitely far. By the proof of [GM08, proposition 5.1] such a string then has to consist of two edges which are both ends of the curve.



As we are dealing with bridge curves the string must then consist of the two odd edges adjacent to the vertex of type (9). From the definition of the vertex type (9) we know that the two ends cannot have the same direction. Considering definition 4.7 of toric Del Pezzo degrees we thus see that these ends have two of the directions shown in the picture on the right. But in all these cases the third direction at the vertex of type (9) would be odd (in contradiction to the definition of type (9)) or 0 (which is impossible for curves through conditions in general position). Hence the string movement cannot be unbounded.

□

Corollary 5.60 (**Welschinger numbers = broccoli invariants in the toric Del Pezzo case**)
Let $r, s \geq 0$, let $\Delta = (v_1, \dots, v_n)$ be a toric Del Pezzo degree, and let $F \subset \{1, \dots, n\}$ such that $|\Delta| - 1 = r + 2s + |F|$. Fix a configuration ω of conditions in general position. Then $N^W_{(r,s)}(\Delta, F, \omega) = N^B_{(r,s)}(\Delta, F, \omega)$.

Proof. By theorem 5.58 and definitions 5.25 and 5.39 we have

$$|G(\Delta, F)| \cdot \left(N^B_{(r,s)}(\Delta, F, \omega) - N^W_{(r,s)}(\Delta, F, \omega)\right) = \sum_{C} \sum_{C'} \operatorname{sign}_{C'} \cdot m_{C'},$$

where the sum is taken over all C as in theorem 5.58 and all resolutions C' of C (i.e. over all vertices and adjacent edges in the bridge graph of remark 5.56). Note that this in fact a finite sum since there are only finitely many types of bridge curves. Now by lemma 5.59 each 1-dimensional type C' of bridge curves occurs in this sum exactly twice with the same multiplicity, once with a positive and once with a negative sign. Hence the sum is 0, proving the corollary. □

This also gives a tropical proof of

Corollary 5.61 (**Invariance of Welschinger numbers in the toric Del Pezzo case**)
With the assumptions and notations as in corollary 5.60, the Welschinger numbers $N^W_{(r,s)}(\Delta, F, \omega)$ are independent of the conditions ω.

Proof. This follows from corollary 5.60 and theorem 5.26. □

In the remaining part of this section we want to construct bridges explicitly and give some examples. The following algorithm, which follows from the proof of theorem 5.58, shows how to construct a bridge from a given starting point.

Algorithm 5.62 (**Bridge algorithm**)
Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a toric Del Pezzo degree, and let $F \subset \{1, \ldots, n\}$ be such that $|\Delta| - 1 = r + 2s + |F|$. Fix a configuration ω of conditions in general position. Consider a bridge curve C passing through ω; we want to construct the bridge that contains C.

(1) If C is a broccoli and Welschinger curve simultaneously (hence $M_{(8)} = M_{(9)} = M_{(6a)} = \emptyset$), do nothing.

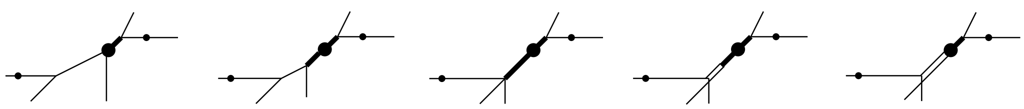
(2) Given a bridge curve C with $M_{(9)} \neq \emptyset$ (hence with a string) together with a direction for the movement of the string, move the string in the direction until we hit a codimension-1 type C' as in lemma 5.55. Go to (2) with each new resolution in the direction away from C'.

(3) If the curve is a broccoli curve, that is $M_{(8)} = M_{(9)} = \emptyset$, choose the smallest vertex in $M_{(6a)}$ under the total order defined in 5.46 c). Pull out an even edge of this vertex of type (6a) in order to create a vertex of type (7) and a vertex of type (9), thus producing a bridge curve with a string and a direction for the movement. Go to (2).

(4) If the curve is a Welschinger curve, that is $M_{(9)} = M_{(6a)} = \emptyset$, choose the vertex of type (8) corresponding to the biggest vertex in $M_{(8)}$ under the total order defined in 5.46 c). Pull apart the two odd edges in order to create a string between the two even edges and a direction for the movement. We thus transform the vertex of type (8) into a vertex of type (3) and a vertex of type (9). Go to (2).

(5) If the curve is a bridge curve with $M_{(9)} = \emptyset$, but $M_{(8)} \neq \emptyset \neq M_{(6a)}$, we can choose the biggest vertex (under the total order) in $M_{(8)}$ or the smallest in $M_{(6a)}$ in order to construct the bridge in direction "broccoli" or in direction "Welschinger". Transform the vertex as described in the two last items, respectively, thus producing a bridge curve with a string and a direction. Go to (2).

Example 5.63 (**A bridge connecting only broccoli curves**)
Following algorithm 5.62, the following picture shows a bridge connecting one broccoli curve (a) to another broccoli curve (e) (and to no Welschinger curve). In curve (c) we resolve a 4-valent vertex of type (D1). The types (b) and (d) are 1-dimensional, the other three 0-dimensional.

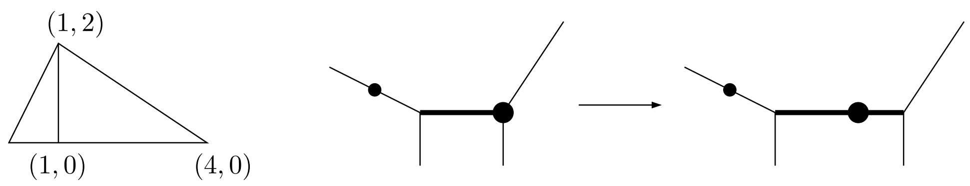


(a) (b) (c) (d) (e)

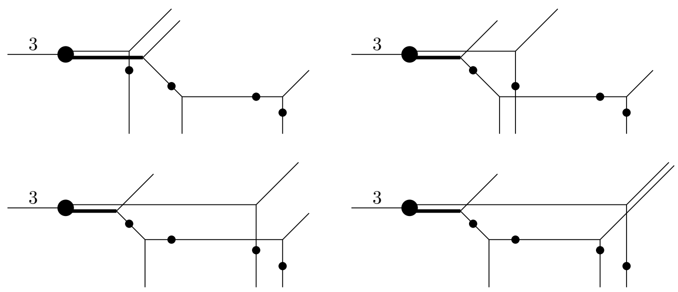
An example of a bridge connecting a broccoli curve with a Welschinger curve can be found in section 5.1.

Example 5.64 (**Two cases that are not toric Del Pezzo**)
The boundedness of bridges of lemma 5.59, and consequently the equality of broccoli and Welschinger numbers as well as the invariance of Welschinger numbers, are false in general for degrees that are not toric Del Pezzo:

a) Consider the following Newton polytope and its subdivision. It is obviously not toric Del Pezzo. A broccoli curve having this Newton subdivision is depicted on the right hand side. Starting the bridge as in algorithm 5.62 yields a string going to infinity (very right hand side), so the broccoli curve is not connected to a Welschinger curve by a bridge.



b) Recall example 4.36 where we have shown that Welschinger numbers are not invariant if we do not have a toric Del Pezzo degree. If we choose the point configuration $\mathcal{P}$ as in example 4.36, the Welschinger curves C_1, C_2, C_3 with multiplicity 3 shown there are also broccoli curves, and in addition there are 4 more broccoli curves passing through $\mathcal{P}$ as depicted below.



Each of them has multiplicity -2, so the broccoli invariant is $N^B_{(r,s)}(\Delta,\omega) = 3 \cdot 3 + 4 \cdot (-2) = 1$. In particular, it is not equal to $N^W_{(r,s)}(\Delta,\omega) = 9$. Indeed, starting a bridge at the complex marking of each of the four curves above gives a curve having a string going to infinity as in a), so the contribution of -8 to the broccoli invariant is not seen on the Welschinger side.

Example 5.65 (**Broccoli invariants for degrees with non-fixed even ends**)
By remark 5.54 the ends of a string are always unfixed and odd. In particular, this means that the proof of lemma 5.59 (and thus also of the equality of broccoli and Welschinger numbers) only requires that the *unfixed odd* ends in Δ are those occurring in a toric Del Pezzo degree. Let us review example 5.40 from this point of view.

a) If Δ has more than one non-fixed even end, and all other non-fixed ends are only those occurring in a toric Del Pezzo degree, then the result $N^W_{(r,s)}(\Delta, F, \omega) = 0$ of example 5.40 a) implies that also $N^B_{(r,s)}(\Delta, F) = 0$.

b) If Δ has one non-fixed even end, and all other ends are non-fixed and among those occurring in a toric Del Pezzo degree, then the formula for $N^W_{(r,s)}(\Delta,\omega)$ of example 5.40 b) holds in the same way for $N^B_{(r,s)}(\Delta)$.

5.5 The Caporaso-Harris formula for broccoli curves

In this section, we establish a Caporaso-Harris formula for broccoli curves of degree dual to the triangle with endpoints $(0,0)$, $(d,0)$ and $(0,d)$. This is a recursive formula computing all broccoli invariants with weight conditions on fixed and non-fixed left ends in addition to the usual point conditions. As usual for Caporaso-Harris type formulas, the idea to obtain these relations is to move one of the point conditions to the far left so that the curve splits into a left part (passing through the moved point) and a right part (passing through the remaining points). Since broccoli invariants of curves with ends of weight one (i.e. of degree d) equal Welschinger numbers $N^W_{(r,s)}(d)$ by corollary 5.60 and the latter equal Welschinger invariants

$W_{\mathbb{P}^2}(d,r,s)$ by the Correspondence theorem 4.33, our formula then computes all Welschinger invariants of the plane recursively.
It is also possible to use Welschinger curves directly to establish a similar formula. However, since the numbers of Welschinger curves of degree dual to the triangle with endpoints $(0,0)$, $(d,0)$, and $(0,d)$ and with ends of higher weight are not invariant (as we have seen in example 4.36), the arguments are then getting significantly more complicated as one always has to pick special configurations of points. This is the content of [ABLdM11]. There, the authors pick a configuration of points such that the Welschinger curves passing through these points decompose totally into floors (see proposition 5.73), and count them by means of floor diagrams. This yields a recursive formula for floor diagrams which also computes all Welschinger invariants of the plane.
Let us first enlarge the notation of definition 1.10.

Notation 5.66
Let $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_{m'})$, $\alpha^1 = (\alpha^1_1, \ldots, \alpha^1_{m_1})$, ..., $\alpha^k = (\alpha^k_1, \ldots, \alpha^k_{m_k})$ be finite sequences with $\alpha_i, \beta_i, \alpha^j_i \in \mathbb{N}$. For simplicity, we will usually consider them to be infinite sequences by setting the remaining entries to 0. We then define:

a) $\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$,

b) $\alpha \leq \beta :\Leftrightarrow \alpha_i \leq \beta_i$ for all i,

c) $\alpha < \beta :\Leftrightarrow \alpha \leq \beta$ and $\alpha \neq \beta$,

d) $\binom{n}{\alpha_1,\ldots,\alpha_m} := \frac{n!}{\alpha_1! \cdot \ldots \cdot \alpha_m! (n-\alpha_1-\ldots-\alpha_m)!}$ for $|\alpha| \leq n$,

e) $\binom{\alpha}{\alpha^1,\ldots,\alpha^k} := \prod_i \binom{\alpha_i}{\alpha^1_i,\ldots,\alpha^k_i}$.

Furthermore, we define e_k to be the sequence having only 0 as entries except a 1 in the k-th entry.

Definition 5.67 (Broccoli curves of type (α, β))
Let $d > 0$, and let α and β be two sequences satisfying $I\alpha + I\beta = d$. We define $\Delta(\alpha,\beta)$ to be the degree consisting of d times the vectors $(0,-1)$ and $(1,1)$ each, and $\alpha_i + \beta_i$ times $(-i,0)$ for all i (in any fixed order). Let $F(\alpha,\beta) \subset \{1,\ldots,|\Delta(\alpha,\beta)|\}$ be a fixed subset with $|\alpha|$ elements such that the entries of $\Delta(\alpha,\beta)$ with index in F are α_i times $(-i,0)$ for all i. If no confusion can result we will often abbreviate $\Delta(\alpha,\beta)$ as Δ and $F(\alpha,\beta)$ as F.
Broccoli curves in $M^B_{(r,s)}(\Delta, F)$ will be called *curves of type* (α,β). We speak of their unmarked ends with directions $(-i,0)$ as the *left ends*. So α_i and β_i are the numbers of fixed and non-fixed left ends of weight i, respectively.

Definition 5.68 (Relative broccoli invariants)
Let $\Delta = \Delta(\alpha,\beta)$ and $F = F(\alpha,\beta)$ be as in definition 5.67, and r,s such that the dimension condition $|\Delta| - 1 - |F| = 2d + |\beta| - 1 = r + 2s$ is satisfied. To simplify notation, we define the *relative broccoli invariant*

$$N^d(\alpha,\beta,s) := N^B_{(r,s)}(\Delta(\alpha,\beta), F(\alpha,\beta)).$$

Remark 5.69 (Unlabeled non-fixed ends)
Notice that by remark 4.30 a broccoli curve without labels on the unmarked ends yields $2^{-k} \cdot |G(\Delta,F)|$ labeled curves contributing to the broccoli invariant, where $|G(\Delta,F)|$ as in definition 2.30 b) denotes the number of ways to relabel the non-fixed unmarked ends without changing the degree, and $k = n_{(6b)} + n_{(8)}$ is the number of double ends. In contrast, in the definition 5.25 of broccoli invariants we multiply the number of broccoli curves with $\frac{1}{|G(\Delta,F)|}$. Thus a curve without labels contributes 2^{-k} to the count. Hence, when counting broccoli curves whose non-fixed unmarked ends are not labeled, we have to change the multiplicity of vertices

of type (6b) to $\frac{1}{2} \cdot i^{-1}$. In the following, we will drop the labels of the non-fixed ends and change the multiplicity accordingly. Note that for the degree Δ and F as above we have $|G(\Delta, F)| = d! \cdot d! \cdot \beta_1! \cdot \beta_2! \cdot \cdots$.

Remark 5.70
It follows from theorem 5.26 that $N^d(\alpha, \beta, s)$ is invariant, i.e. does not depend on the choice of the conditions. Note that if $\alpha = (0)$ and $\beta = (d)$ then

$$N^d((0),(d),s) = N^B_{(r,s)}(d) = N^W_{(r,s)}(d) = W_{\mathbb{P}^2}(d, 3d-2s-1, s),$$

where the second equality follows from theorem 5.58 and the last equality from theorem 4.33.

Now we describe the properties of configurations ω of points that we obtain by moving one of the point conditions (w.l.o.g. P_1) to the left. Let us show first that then curves satisfying these conditions decompose into a left and a right part.

Lemma 5.71 (**Decomposing curves into a left and right part**)
Let Δ and F be as in definition 5.67, and let $2d + |\beta| - 1 = r + 2s$. Fix a small real number $\epsilon > 0$ and a large one $N > 0$. Choose $r + s$ (real and complex) points $P_1, \dots, P_{r+s}$ and $|\alpha|$ y-coordinates for the fixed left ends in general position such that

- the y-coordinates of all P_i and the fixed ends are in the open interval $(-\epsilon, \epsilon)$,
- the x-coordinates of $P_2, \dots, P_{r+s}$ are in $(-\epsilon, \epsilon)$,
- the x-coordinate of P_1 is smaller than $-N$.

Let $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h) \in M^B_{(r,s)}(\Delta, F)$ be a broccoli curve satisfying these conditions. Then no vertex of C can have its y-coordinate below $-\epsilon$ or above ϵ. There is a rectangle $R = [a, b] \times [-\epsilon, \epsilon]$ (with $a \geq -N$, $b \leq -\epsilon$ only depending on d) such that $R \cap h(\Gamma)$ contains only horizontal edges of C.

Proof. Notice that it follows from lemma 5.11 that each connected component of C minus the marked points contains exactly one non-fixed unmarked end, a statement analogous to [GM07a, remark 2.10]. The fact that the y-coordinates of the vertices of C cannot be above ϵ or below $-\epsilon$ and the existence of the rectangle R follow analogously to the first part of the proof of [GM07a, theorem 4.3]. □

A configuration of points and y-coordinates for the fixed left ends as in lemma 5.71 can be obtained from any other by moving P_1 far to the left. So in this situation the curves decompose into a left and a right part connected by only horizontal edges in the rectangle R. A picture showing this can be found in example 5.74. In the following, we study the possibilities for the shapes of the left and right part.

Notation 5.72 (**Left and right parts**)
With notations as in lemma 5.71, cut C at each bounded edge e such that $h(e) \cap R \neq \emptyset$. Denote the component passing through P_1 by C_0 (the left part), and the union of the other connected components by $\tilde{C}$ (the right part).

Proposition 5.73 (**Possible shapes of the left and right part**)
Let C_0 and $\tilde{C}$ be the left and right part of a broccoli curve as in lemma 5.71 and notation 5.72.

a) If C_0 has no bounded edges, it looks like (A), (B), or (C) in the picture below (in which the edges are labeled with their weights). Moreover:
   - In case (A), $\tilde{C}$ is an irreducible curve of type $(\alpha + e_k, \beta - e_k)$.
   - In case (B), $\tilde{C}$ is an irreducible curve of type $(\alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2})$.

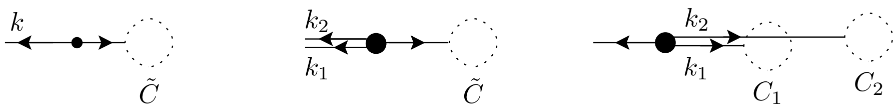
- In case (C), $\tilde{C}$ decomposes into two connected components C_1 and C_2 of types (α^1, β^1) resp. (α^2, β^2) with $I\alpha^j + I\beta^j = d_j$ for $j = 1, 2$, $d_1 + d_2 = d$, $\alpha^1 + \alpha^2 = \alpha + e_{k_1} + e_{k_2}$, and $\beta^1 + \beta^2 = \beta - e_{k_1+k_2}$. The curve C_j for $j = 1, 2$ passes through r_j real and s_j complex given points, where $2d_j + |\beta^j| - 1 = r_j + 2s_j$.

In case (A) (for real P_1) the left end is odd, in the cases (B) and (C) (for complex P_1) exactly one of the three edges adjacent to P_1 is even.

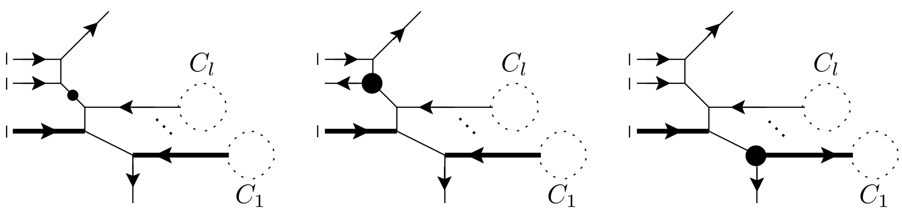
b) If C_0 has bounded edges (it is then called a *floor*), it looks like (D), (E), or (F) in the picture below, and has one end of direction $(0, -1)$ and one of direction $(1, 1)$. We call the ends of C_0 of direction $(i, 0)$ for $i > 0$ the *right ends*. Moreover:

- In case (D) (for real P_1), C_0 has only fixed left and right ends.
- In case (E) (for complex P_1), P_1 is adjacent to a left non-fixed end of C_0, and all other left and right ends of C_0 are fixed.
- In case (F) (for complex P_1), P_1 is adjacent to a right non-fixed end of C_0, and all other left and right ends of C_0 are fixed.

In any case, $\tilde{C}$ consists of some number l of connected components $C_1, \ldots, C_l$. Each C_j is a curve of some type (α^j, β^j) with $I\alpha^j + I\beta^j = d_j$ and $\sum_{j=1}^{l} d_j = d - 1$. The curve C_j for $j = 1, \ldots, l$ passes through r_j real and s_j complex given points, where $2d_j + |\beta^j| - 1 = r_j + 2s_j$. Note that (D), (E), and (F) are meant to be schematic pictures in which the thin and thick horizontal edges are just examples. The non-horizontal edges are always odd however.



(A) (B) (C)

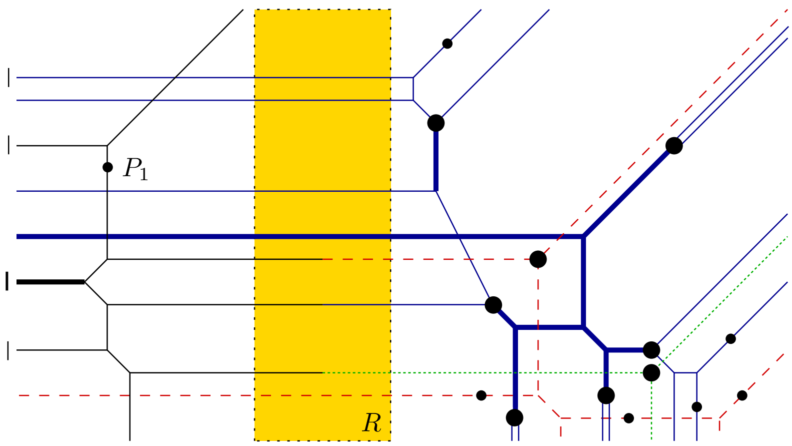


(D) (E) (F)

Proof. a) Assume C_0 contains no bounded edge and P_1 is real. Then C_0 contains exactly one vertex, of type (1). Both adjacent edges are ends of C_0. Since C is connected, one of the ends of C_0 results from cutting a bounded horizontal edge of C. Because of the balancing condition, it follows that the other end is of direction $(-k, 0)$ for some $k > 0$, which has to be odd since P_1 is of vertex type (1). Hence we are then in case (A).
Assume now that P_1 is complex. Then C_0 consists of a vertex of type (5) or (6). At least one of the adjacent edges is of direction $(k, 0)$ for some $k > 0$ since it results from cutting a horizontal bounded edge. The other adjacent edges are ends of C. It follows from the balancing condition that all three adjacent edges are horizontal, and so we have type (B) or (C). Exactly one of the adjacent edges is even (and so vertex type (5) is impossible). In (A) and (B), we just cut one edge, so it follows that $\tilde{C}$ is irreducible and of the degree as claimed above. In (C), we cut two edges, so $\tilde{C}$ consists of two connected components C_1 and C_2. Ends of C_1 and C_2 are either ends of C or the two cut edges. Denote their weights by k_1 resp.

k_2, then it follows that C_j is of a type (α^j, β^j) for $j = 1, 2$ with $\alpha^1 + \alpha^2 = \alpha + e_{k_1} + e_{k_2}$ and $\beta^1 + \beta^2 = \beta - e_{k_1+k_2}$. If $2d_j + |\beta^j| - 1 < r_j + 2s_j$ for $j = 1$ or $j = 2$, then it follows that there is a connected component of Γ minus the marked ends which does not contain a non-fixed unmarked end, a contradiction to lemma 5.11. Thus we have $2d_j + |\beta^j| - 1 \geq r_j + 2s_j$, and since $2d_1 + |\beta^1| - 1 + 2d_2 + |\beta^2| - 1 = 2d + |\beta| - 3 = r + 2(s - 1) = r_1 + 2s_1 + r_2 + 2s_2$ it follows that $2d_j + |\beta^j| - 1 = r_j + 2s_j$ for $j = 1, 2$.
b) Now assume that C_0 contains a bounded edge. By lemma 5.11, each connected component of C minus the marked points contains exactly one non-fixed unmarked end. If P_1 is real, removing the marked end x_1 satisfying $h(x_1) = P_1$ from Γ produces 2 connected components; if it is complex it produces 3 connected components. It follows that C_0 contains at most 2 non-fixed ends of C if P_1 is real, or 3 if P_1 is complex. Ends of C_0 are of direction $(k, 0)$ for some k (resulting from cutting horizontal bounded edges of C) or ends of C. If C_0 contains a bounded edge then C_0 cannot lie entirely in a horizontal line, since otherwise the length of such a bounded edge could not be fixed by our conditions. It follows by the balancing condition that C_0 must have ends of direction $(0, -1)$ and $(1, 1)$, and in fact an equal number of them. But since ends of these directions are non-fixed and we have at most 3 non-fixed ends of C in C_0, we conclude that there is exactly one end of direction $(0, -1)$ and $(1, 1)$ each. Since all other ends of C_0 are horizontal, it follows from the balancing condition that the directions of the bounded edges of C_0 are $\pm(a, 1)$ for some a. In particular, they are all odd.
If P_1 is real, C_0 cannot have more non-fixed ends of C than the two ends of direction $(0, -1)$ and $(1, 1)$. So then all left and right ends of C_0 are fixed, and we are in case (D). If P_1 is complex, there can be one non-fixed left end of C_0, which then has to be adjacent to P_1 as in case (E). Otherwise, P_1 has to be adjacent to a horizontal edge connecting C_0 with $\tilde{C}$. This is true because by the directions of the ends of C_0 and the balancing condition we can conclude that every vertex of C_0 is adjacent to an edge of direction $(k, 0)$ for some (positive or negative) k. Thus we are then in case (F).
Assume we have to cut l edges to produce C_0 and $\tilde{C}$, then $\tilde{C}$ consists of l connected components. Each connected component is a curve of some type (α^j, β^j) with $I\alpha^j + I\beta^j = d_j$. It follows from the balancing condition that $\sum_{j=1}^{l} d_j = d - 1$. The equations $2d_j + |\beta^j| - 1 = r_j + 2s_j$ follow as in part 5.73. □

Example 5.74
The picture shows an example of a curve C decomposing into a floor C_0 of type (D) on the left and a reducible curve $\tilde{C}$ on the right. C is of type $((3, 1), (3, 1))$ passing through $r = 7$ real and $s = 8$ complex points satisfying $2d + |\beta| - 1 = 20 + 4 - 1 = 23 = r + 2s$. We have chosen to move a real point to the left of the others.



The reducible curve $\tilde{C}$ consists of three connected components, C_1 (green dotted), C_2 (red dashed) and C_3 (blue solid). C_1 is a curve of type $((0), (1))$ passing through $s_1 = 1$ complex points, satisfying $2d_1 + |\beta^1| - 1 = 2 + 1 - 1 = 2 = r_1 + 2s_1$. C_2 is a curve of type $((0),(2))$ passing through $r_2 = 3$ real and $s_2 = 1$ complex points satisfying $2d_2 + |\beta^2| - 1 = 4 + 2 - 1 = 5 = r_2 + 2s_2$. C_3 is a curve of type $((1),(3,1))$ passing through $r_3 = 3$ real and $s_3 = 6$ complex points satisfying $2d_3 + |\beta^3| - 1 = 12 + 4 - 1 = 15 = r_3 + 2s_3$. We have $d_1 + d_2 + d_3 = 1 + 2 + 6 = d - 1$. All three curves are connected to C_0 via a horizontal edge of weight 1. We have $\beta = (3,1) = \beta^1 + \beta^2 + \beta^3 - 3e_1$ and $\alpha^1 + \alpha^2 + \alpha^3 = (1) < \alpha = (3,1)$.

Note that in the situation above there is always a unique possibility for C_0 once we are given the left and right ends of C_0 (together with their position for fixed ends) as well as the position of P_1. Thus, to determine $N^d(\alpha,\beta,s)$, we just have to determine the different contributions from all possibilities for $\tilde{C}$. This is the content of the following theorem. We are grateful to Inge Sandstad Skrondal, who worked in his master thesis [Skr12] on this formula, for pointing out a small inequality mistake.

Theorem 5.75 (**Caporaso-Harris formula for $N^d(\alpha,\beta,s)$**)
The following two recursive formulas hold for the invariants $N^d(\alpha,\beta,s)$, where we use the notation $r := 2d + |\beta| - 2s - 1$ (resp. $r_j := 2d_j + |\beta_j| - 2s_j - 1$ for all j) for the corresponding number of real markings in the invariant:

a) (Moving a real point to the left) If $r > 0$ then

$$N^d(\alpha,\beta,s) = \sum_{k \text{ odd}} N^d(\alpha + e_k, \beta - e_k, s) \quad \text{(A)}$$

$$+ \sum \frac{1}{l!} \binom{s}{s_1,\ldots,s_l} \binom{r-1}{r_1,\ldots,r_l} \binom{\alpha}{\alpha^1,\ldots,\alpha^l} \prod_{m \text{ even}} (-m)^{\alpha'_m} \prod_{\substack{j=1 \\ k_j \text{ even}}}^{l} k_j$$

$$\cdot \prod_{j=1}^{l} \left(\beta^j_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right) \quad \text{(D)}$$

where we set $\alpha' := \alpha - \sum_{j=1}^{l} \alpha^j$, and where the sum in (D) runs over all $l \geq 0$ and all $\alpha^j, \beta^j, k_j \geq 1, d_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ satisfying $\sum_j \alpha^j < \alpha$, $\sum_j (\beta^j - e_{k_j}) = \beta$, $\sum_j d_j = d - 1$, $\sum_j s_j = s$.

b) (Moving a complex point to the left) If $s > 0$ then

$$N^d(\alpha,\beta,s) = \sum -\tfrac{1}{2} N^d(\alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2}, s - 1) \quad \text{(B)}$$

$$+ \sum \frac{1}{2} \binom{s-1}{s_1,s_2} \binom{r}{r_1,r_2} \binom{\alpha}{\alpha^1,\alpha^2} \cdot \prod_{j=1}^{2} N^{d_j}(\alpha^j + e_{k_j}, \beta^j, s_j) \quad \text{(C)}$$

$$+ \sum \frac{1}{l!} \binom{s-1}{s_1,\ldots,s_l} \binom{r}{r_1,\ldots,r_l} \binom{\alpha}{\alpha^1,\ldots,\alpha^l} M_k \prod_{m \text{ even}} (-m)^{\alpha'_m} \prod_{\substack{j=1 \\ k_j \text{ even}}}^{l} k_j$$

$$\cdot \prod_{j=1}^{l} \left(\beta^j_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right) \quad \text{(E)}$$

$$+ \sum \frac{1}{(l-1)!} \binom{s-1}{s_1,\ldots,s_l} \binom{r}{r_1,\ldots,r_l} \binom{\alpha}{\alpha^1,\ldots,\alpha^l} \tilde{M}_{k_1} \prod_{m \text{ even}} (-m)^{\alpha'_m} \prod_{\substack{j=2 \\ k_j \text{ even}}}^{l} k_j$$

$$\cdot N^{d_1}(\alpha^1 + e_{k_1}, \beta^1, s_1) \prod_{j=2}^{l} \left(\beta^j_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right) \quad \text{(F)}$$

where as above $\alpha' := \alpha - \sum_{j=1}^{l} \alpha^j$, and where the sums run over

(B) all $k_1, k_2 \geq 1$ such that at least one of them is odd;

(C) all $\alpha^j, \beta^j, k_j \geq 1, d_j \geq 1, s_j \geq 0$ for $j \in \{1,2\}$ such that at least one of k_1, k_2 is odd, $\sum_j \alpha^j = \alpha$, $\sum_j \beta^j = \beta - e_{k_1+k_2}$, $\sum_j d_j = d$, $\sum_j s_j = s-1$.

(E) all $l \geq 0$ and all $\alpha^j, \beta^j, k \geq 1, k_j \geq 1, d_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that $\sum_j \alpha^j \leq \alpha$, $\sum_j(\beta^j - e_{k_j}) = \beta - e_k$, $\sum_j d_j = d-1$, $\sum_j s_j = s-1$.

(F) all $l \geq 1$ and all $\alpha^j, \beta^j, k_j \geq 1, d_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that $\sum_j \alpha^j < \alpha$, $\beta^1 + \sum_{j>1}(\beta^j - e_{k_j}) = \beta$, $\sum_j d_j = d-1$, $\sum_j s_j = s-1$.

Here, the numbers M_k and $\tilde{M}_k$ are defined by

$$M_k = \begin{cases} k & \text{if } k \text{ odd,} \\ -1 & \text{if } k \text{ even} \end{cases} \quad \text{and} \quad \tilde{M}_k = \begin{cases} k & \text{if } k \text{ odd,} \\ 1 & \text{if } k \text{ even.} \end{cases}$$

Of course, for both equations it is assumed that the sums are taken only over choices of variables such that all occurring sequences have only non-negative entries and all relative broccoli invariants satisfy the dimension condition.

Proof. As we have mentioned already we move one of the point conditions to the far left, so that each curve satisfying the conditions decomposes into a left part C_0 and a right part $\tilde{C}$. Since we have studied the possibilities for C_0 and $\tilde{C}$ in proposition 5.73 already it only remains to understand the different contributions to the relative broccoli invariant from each of these cases.
a) The first formula arises from moving a real point to the left, so we have the cases (A) and (D).

(A) C_0 consists of one vertex of multiplicity 1, and $\tilde{C}$ has the same ends as C, with one odd non-fixed left end replaced by a fixed one. Thus we only have to sum over all possibilities of weights of this left end.

(D) We have to sum over all possibilities for $\tilde{C}$ to split into l connected components $C_1, \ldots, C_l$, where C_j is of type (α^j, β^j) with $I\alpha^j + I\beta^j = d_j$ and passes through r_j real and s_j complex points of $P_2, \ldots, P_{r+s}$. The right ends of C_0 are the gluing points for $C_1, \ldots, C_l$. They are fixed for C_0 and thus non-fixed for $C_1, \ldots, C_l$, i.e. they belong to $\beta^1, \ldots, \beta^l$. Let k_j be the weight of the edge with which C_0 and C_j are connected. Then we have $\sum_{j=1}^l (\beta^j - e_{k_j}) = \beta$. Also, we have $\sum_{j=1}^l \alpha^j < \alpha$, and $\alpha' = \alpha - \sum_{j=1}^l \alpha^j$ is the sequence of fixed left ends adjacent to C_0. The multinomial coefficient $\binom{s}{s_1,\ldots,s_l}$ gives the number of possibilities how the s complex points of $P_2, \ldots, P_{r+s}$ can be distributed among the C_j. The second and third multinomial coefficient give the corresponding number for the real points and the fixed left ends, respectively.

It remains to take care of different multiplicity factors. First of all note that every fixed left end adjacent to C_0 (described by α') is not a fixed end of $\tilde{C}$ any more, so when counting the contribution from $\tilde{C}$ instead of C we lose a factor of i^{k-1} for every such end of weight k (remember that the weights of the ends of a curve C enter into the multiplicity m_C, see definition 5.15). Also, each such fixed end is adjacent to a vertex of C_0 whose multiplicity is $i^{k-1} \cdot k$ if k is even and i^{k-1} if k is odd. Thus, we lose a factor $i^{2k-2} = (-1)^{k-1} = 1$ if k is odd, and $k \cdot i^{2k-2} = k \cdot (-1)^{k-1} = -k$ if k is even. Therefore we have to multiply by $\prod_{m \text{ even}} (-m)^{\alpha'_m}$.

Similarly, for $j = 1, \ldots, l$ the end of weight k_j with which C_j is connected to C_0 yields a factor of i^{k_j-1} in the multiplicity of $\tilde{C}$ that we do not need for C. The vertex of C_0 adjacent to such an edge has multiplicity $k_j \cdot i^{k_j-1}$ if k_j is even, and i^{k_j-1} if k_j is odd. Thus we need to multiply by $\prod_{j:\, k_j \text{ even}}^l k_j$.

The factors $\beta^j_{k_j}$ stand for the number of possibilities with which of the $\beta^j_{k_j}$ non-fixed ends of weight k_j the component C_j is connected to C_0. The factor $\frac{1}{l!}$ takes care of the overcounting due to the labeling of the components $C_1, \dots, C_l$. As C_0 has one end of direction $(0,-1)$ and $(1,1)$ each it is clear that we must have $\sum_j d_j = d-1$.

b) In the second formula we move a complex point to the left, so we have four summands corresponding to the possibilities (B), (C), (E), and (F).

(B) We have to sum over all possibilities k_1 and k_2 for the weights of the two left ends which are adjacent to P_1. If we sum over all tuples (k_1, k_2), we overcount by a factor of 2 since these two weights are unordered. Therefore we multiply by $\frac{1}{2}$. For summands with $k_1 = k_2$, the $\frac{1}{2}$ takes care of the factor of $\frac{1}{2}$ in the multiplicity of the vertex of C_0 that we have to include when counting curves without labels at the unmarked ends (see remark 5.69). We lose factors of i^{k_1-1} and i^{k_2-1} since these two ends are not ends of $\tilde{C}$, and we lose a factor of i^{-1} for the vertex of C_0. Instead, we have a factor of $i^{k_1+k_2-1}$ for the end of $\tilde{C}$ with which it is glued to C_0. Thus, we have to multiply by -1.

(C) In this case we have to sum over all choices of the connecting weights k_1 and k_2 (which are fixed ends for C_1 and C_2), degrees d_1 and d_2, and numbers s_1 and s_2 of complex markings on each component. The symmetry factor $\frac{1}{2}$ cancels the overcounting due to the labeling of the two components. The binomial factors count the possibilities how the complex and real points and the fixed ends can be distributed among C_1 and C_2. In C_0, we have the left end contributing $i^{k_1+k_2-1}$ and a vertex contributing i^{-1}, in $\tilde{C}$ we have instead the two ends contributing i^{k_1-1} and i^{k_2-1}. So we do not need to multiply by a factor to take care of these multiplicities.

(E) The terms are essentially as in (D) above, except that in addition we have to sum over all possibilities for the weight k of the non-fixed left end adjacent to P_1. Also, this non-fixed end is not an end of any of the C_j, so the condition $\sum_j(\beta^j - e_{k_j}) = \beta$ has to be changed to $\sum_j(\beta^j - e_{k_j}) = \beta - e_k$. In addition to the factors of (D) we lose a factor of i^{k-1} for the end, and of i^{k-1} if k is even and $k \cdot i^{k-1}$ if k is odd for the vertex at P_1. So altogether we have to multiply by $i^{2k-2} = (-1)^{k-1} = -1$ if k is even and by k if k is odd.

(F) We get again a similar summand as in (E). However, here instead of summing over the possibilities for k we now have to choose one of the C_j — call it C_1 — which is adjacent to P_1. This component will then have an additional fixed end of weight k_1. So in the invariant for C_1 we have to replace α^1 by $\alpha^1 + e_{k_1}$; at the same time however we do not have to multiply this invariant by $\beta^1_{k_1}$ as C_1 is connected to C_0 by a fixed end. The fixed end of weight k_1 of C_1 contributes a factor of i^{k_1-1} to $\tilde{C}$. We lose the multiplicity of the vertex at P_1 which is i^{k_1-1} if k_1 is even and $k_1 \cdot i^{k_1-1}$ if k_1 is odd. Hence we have to multiply by $\tilde{M}_{k_1}$.

□

Of course, theorem 5.75 now gives recursive formulas for all broccoli invariants $N^d(\alpha, \beta, s)$, and thus in particular by remark 5.70 also for the Welschinger numbers $W_{\mathbb{P}^2}(d, 3d-2s-1, s)$.

5.6 Explicit computations of broccoli numbers

For small degrees d, the computations of theorem 5.75 can be driven by hand. The following table shows all invariants $N^d(\alpha, \beta, s)$ for $d \leq 3$.
The numbers in the last line respectively are those that correspond to the degree d Welschinger

invariants. They agree with ones computed with the formula in [ABLdM11]. The entries in the second last line are all 0 in accordance with example 5.40 b).

α, β	$s = 0$	$s = 1$
$(1), (0)$	1	
$(0), (1)$	1	1

α, β	$s = 0$	$s = 1$	$s = 2$
$(0), (0, 1)$	0	0	-1
$(2), (0)$	1	1	
$(0, 1), (0)$	-2	-2	
$(1), (1)$	1	1	1
$(0), (2)$	1	1	1

α, β	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$(0, 0, 1), (0)$	3	1	-1		
$(0, 1), (1)$	-12	-8	-4	0	
$(1, 1), (0)$	-8	-4	0		
$(1), (0, 1)$	0	0	0	0	
$(1), (2)$	8	6	4	2	
$(2), (1)$	8	6	4	2	
$(3), (0)$	6	4	2		
$(0), (0, 0, 1)$	3	1	-1	-3	
$(0), (1, 1)$	0	0	0	0	
$(0), (3)$	8	6	4	2	0

Notice that our computations involve all cases of theorem 5.75 except case (C). So these results verify our formula only in parts.

More test data have been computed by Inge Sandstad Skrondal in his master thesis [Skr12]. He implemented our formula in Java and got results up to degree 6. Unfortunately, his code takes a lot of time and memory to check the conditions on the sequences α^j, β^j and d_j. So the program did not computed all the numbers of degree 6. We only list the numbers starting with degree 4 as the numbers of degree ≤ 3 agree with the results of our computations. As one can see the absolute Welschinger numbers of degree 4 and 5 coincide the ones computed in [ABLdM11].
It is worth to have a look at Inge's thesis as he also found analogue formulas to theorem 5.75 for $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2_k$ with $k \leq 2$.

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
$(0),(0,0,0,1)$	0	0	0	0	8	
$(0),(0,2)$	0	0	0	0	0	
$(0),(1,0,1)$	108	44	12	-4	-20	
$(0),(2,1)$	0	0	0	0	0	16
$(0),(4)$	240	144	80	40	16	0
$(0,0,0,1),(0)$	-72	-16	8	32		
$(0,0,1),(1)$	75	33	11	1	-5	
$(0,1),(0,1)$	0	0	0	0	-16	
$(0,1),(2)$	-288	-160	-80	-32	0	
$(0,2),(0)$	120	48	8	-32		
$(1),(0,0,1)$	33	11	1	-5	-15	
$(1),(1,1)$	0	0	0	0	0	
$(1),(3)$	240	144	80	40	16	0
$(1,0,1),(0)$	33	11	1	-5		
$(1,1),(1)$	-240	-124	-56	-20	0	
$(2),(0,1)$	0	0	0	0	0	
$(2),(2)$	240	144	80	40	16	
$(2,1),(0)$	-124	-56	-20	0		
$(3),(1)$	216	126	68	34	16	
$(4),(0)$	126	68	34	16		

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
$(0),(0,0,0,0,1)$	189	33	-7	-11	21	105		
$(0),(0,1,1)$	0	0	0	0	0	-32		
$(0),(1,0,0,1)$	0	0	0	0	0	0		
$(0),(1,2)$	0	0	0	0	0	0	0	
$(0),(2,0,1)$	8208	3156	1056	252	-16	-28	192	
$(0),(3,1)$	0	0	0	0	0	0	0	
$(0),(5)$	18264	9096	4272	1872	744	248	64	64
$(0,0,0,0,1),(0)$	189	33	-7	-11	21			
$(0,0,0,1),(1)$	-5184	-1600	-352	32	128	0		
$(0,0,1),(0,1)$	0	0	0	0	0	-32		
$(0,0,1),(2)$	4320	1764	640	188	32	20		
$(0,1),(0,0,1)$	-1080	-352	-72	16	-24	-192		
$(0,1),(1,1)$	0	0	0	0	0	0		
$(0,1),(3)$	-18192	-8544	-3744	-1488	-496	-128	-128	
$(0,1,1),(0)$	-864	-264	-48	8	-64			
$(0,2),(1)$	9792	3904	1376	352	0	128		
$(1),(0,0,0,1)$	0	0	0	0	0	0		
$(1),(0,2)$	0	0	0	0	0	0		
$(1),(1,0,1)$	3888	1392	416	64	-48	-48		
$(1),(2,1)$	0	0	0	0	0	0	0	
$(1),(4)$	18264	9096	4272	1872	744	248	64	
$(1,0,0,1),(0)$	-1728	-416	-32	64	0			
$(1,0,1),(1)$	2736	1012	320	68	-16	-12		
$(1,1),(0,1)$	0	0	0	0	0	64		
$(1,1),(2)$	-16272	-7392	-3104	-1168	-368	-128		
$(1,2),(0)$	4032	1440	416	64	128			
$(2),(0,0,1)$	1152	380	96	-4	-32	-36		
$(2),(1,1)$	0	0	0	0	0	0		
$(2),(3)$	18264	9096	4272	1872	744	248	64	
$(2,0,1),(0)$	1044	336	84	0	-12			
$(2,1),(1)$	-11664	-5024	-1984	-688	-176	-64		
$(3),(0,1)$	0	0	0	0	0	-32		
$(3),(2)$	17304	8520	3952	1712	680	248		
$(3,1),(0)$	-5088	-2016	-720	-208	-64			
$(4),(1)$	13560	6472	2912	1232	488	216		
$(5),(0)$	6504	2928	1248	504	216			

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$
$(0),(0,0,0,0,0,1)$	0	0	0	0	0	0	-336		
$(0),(0,0,2)$	15714	4122	762	18	-30	-6	474		
$(0),(0,1,0,1)$	0	0	0	0	0	0	0		
$(0),(0,3)$	0	0	0	0	0	0	0	0	
$(0),(1,0,0,0,1)$	39150	8838	1278	-250	-274	102	1086		
$(0),(1,1,1)$	0	0	0	0	0	-32	-192	-2976	
$(0),(2,0,0,1)$	0	0	0	0	0	0	0	-1152	
$(0),(2,2)$	0	0	0	0	0	0	0	0	
$(0),(3,0,1)$	1215360	424992	136704	38592	8192	352			

6 Broccoli curves of genus 1

The purpose of this section is to describe the problems we encounter when we want to carry over the concept of broccoli curves of genus 0 to genus 1. This study is work in progress so we are only able to present partial results. Broccoli curves of genus 1 are a good subject to study as they could be the source of new invariants of real curves of genus 1 on the classical side once we are able translate broccoli invariants to the classical world. Note that the construction of new tropical invariants of curves of genus $g > 0$ in [IKS09] is for curves passing through only real points.
The main interplay is between the construction of the corresponding moduli space, the local invariance of broccoli numbers in this moduli space and the adding of new vertex types to definition 5.15, respectively 5.21, in order to define broccoli curves of genus 1. Indeed, heuristically one could start to take the vertex types of definition 5.21 and construct curves of degree Δ with a cycle passing through $|\Delta| - 1 + g = |\Delta| - 1 + 1 = |\Delta|$ points in generic position in the sense of 5.8 (note that this definition depends on the structure of the moduli space).
First we have to exclude cells of too high dimension in our moduli space under construction similar to 2.28 and [KM09a]. Defining the *deficiency* def(α) of a combinatorial type α as in [KM09a, definition 2.7] as

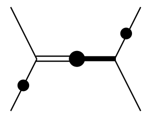
$$\text{def}(\alpha) = \begin{cases} 2, & \text{if } g = 1 \text{ and the cycle is mapped to a point in } \mathbb{R}^2 \\ 1, & \text{if } g = 1 \text{ and the cycle is mapped to a line in } \mathbb{R}^2 \\ 0, & \text{otherwise.} \end{cases}$$

The dimension of the corresponding cell in the moduli space is $|\Delta| + r - \sum_V(\text{val } V - 3) - \sum_W(\text{val } W - 4) + \text{def}(\alpha)$ similar to [KM09a, lemma 3.1], where V are vertices in $h(\Gamma)$ without a big dot and W are vertices in $h(\Gamma)$ with a big dot. We want to eliminate cells of this moduli space which have dimension bigger than $|\Delta| + r$. Uncomplicated are curves with a cycle such that the direction vectors of edges adjacent to a vertex in the cycle are not the same, i.e. they span $\mathbb{R}^2$ and def$(\alpha) = 0$. Note that in this case, as it can be seen below, we can treat such curves as rational broccoli curves as in the last chapter if we cut the curve on an edge of the cycle.

We can make sure that the number of conditions is the right one if we conserve the cut as one gluing condition, i.e. the information of the two newly created unbounded edges that have to be glued together to regain a broccoli curve of genus 1. Indeed, before we had $(|\Delta|-1)+1$ and after cutting we need $(|\Delta|-1+2)+0$ conditions where the 2 comes from two new unbounded edges. This approach is similar to [KM09a, remark 3.6]. If def$(\alpha) = 1$ oder def$(\alpha) = 2$ the situation is more complicated.

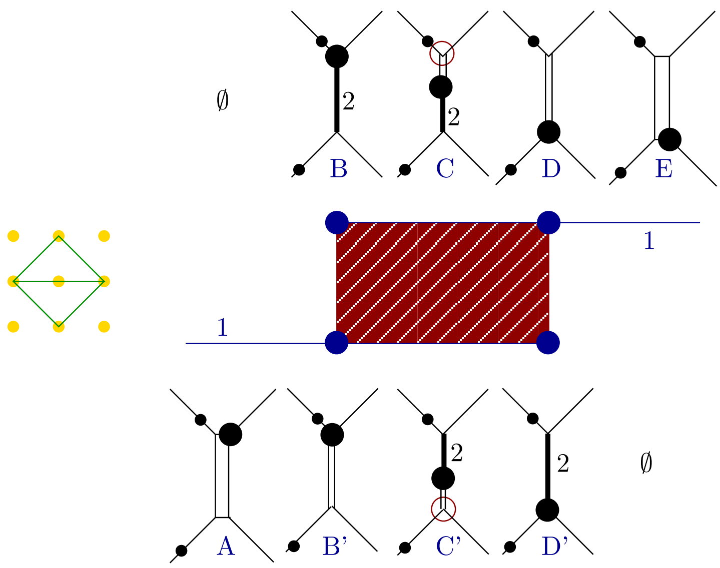
a) b)

In case a), where $\operatorname{def}(\alpha) = 1$, the dimension of the cell is 8, but the right dimension would be 7. This is reflected in the picture by the fact that the flat cycle can move on the horizontal edge when we allow only 3 real markings to fix the curve. So the dimension is too big. Note that the picture b) is dynamic: there is a curve with a moving loop. So $\operatorname{def}(\alpha) = 2$. Here, the right dimension is $|\Delta| + r = 7$. But as the loop is there the actual dimension is 9, i.e. the dimension is too big, too. This is obvious as the evaluation map can not be injective, since the loop can shrink and increase as depicted. The only type of a flat cycle that is possible is depicted below.

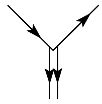


Namely, it is clear that the evaluation map is never injective when the curve has a loop. So $\operatorname{def}(\alpha) = 1$, and hence in order to be in the right dimension, we need a 4-valent vertex in the curve. But the evaluation map is only injective if the 4-valent vertex is located next to the cycle. We are supposed to need $|\Delta| = 4$ conditions and we actually need 4 conditions. The dimension is $4 + 4 - 1 - 1 = 2|\Delta|$. In the following we allow curves having a flat cycle next to a 4-valent vertex as above. Note that in the case of only real conditions which are not allowed on a vertex, flat cycles play no role as the evalution map is then not injective.

Assuming that the multiplicity of a curve of the right dimension is real we then have to prove that if we move a point in the configuration ω such that ω remains generic and deform the curve accordingly, then the multiplicity (or the sum of multiplicities if we cross codimension-1 cells of the moduli space) keeps invariant. When we do so, it may happen that a new vertex type in the deformed curves appears. In this case we have to decide how to proceed: if we want to allow this vertex type in our definition of broccoli curves of genus 1 or if we do not need this vertex type and can establish the invariance in another way. If we keep it, then we have to think about its multiplicity. Consider for instance the situation below.



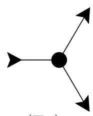
Here we have $\Delta = ((-1,-1),(1,1),(-1,1),(1,-1))$. Let us consider the first row. In the sequence B-E we move the complex marking vertically downwards. The curves B and D pass through points in special position, while the curves C, E pass through points in general position. Also, the curves C-E are of topological genus 1, while the curve B is of topological genus 0 (remember that they are all of genus 1 in the sense of definition 2.17). When we move the complex marking in the curve B upwards there is no curve that we obtain by deformation of B. Notice that E contains only vertex types of 5.21 but the curve C has a new vertex type:



So whatever multiplicity we assign to C, the first row will never be invariant! In order to obtain invariance, i.e. local invariance in the moduli space, one could consider cells in the moduli space together as indicated by the red box. This means that in the example we think of the curves C and C' as being in one cell of the moduli space even if we actually do not modify our moduli space. The second row with curves A to D' is somewhat symmetric to the first as there is e.g. no curve E'. Considering the whole picture we have invariance if we assign to the pair C & C' multiplicity 1. But this situation does not tell us what precisely is the multiplicity of C or C'. Also, we cannot yet deduce what the multiplicity of the new vertex type should be.
In some special cases we can prove local invariance which we are going to present. If we can prove local invariance in all possible cases and if we can verify that some "connectedness in codimension-1" argument holds, then we are done. Problematic is here that multiplicities of curves are likely not to be products of vertex multiplicities as in 5.15.

6.1 Invariance in a first case

Definition 6.1 (**New codimension-**1 **type (E1)**)
We define the vertex type (E1) appearing in codimension-1 in this first class of examples as depicted below. This completes the list given in the proof of theorem 5.26.

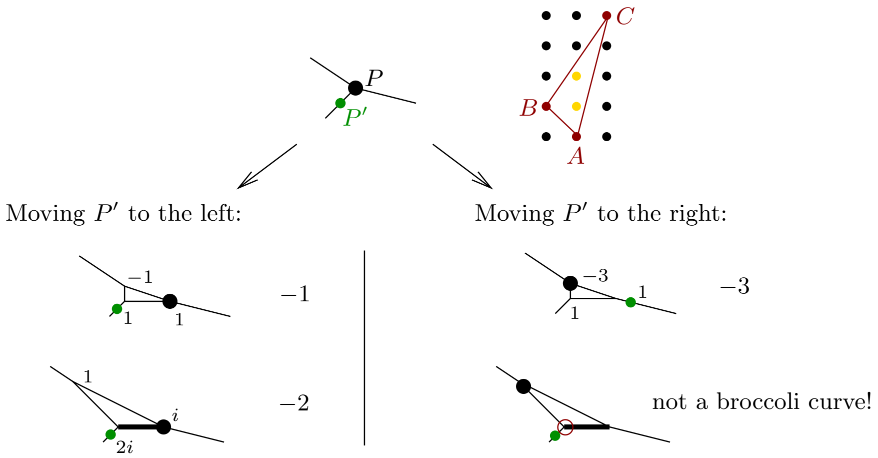


(E1)

Theorem 6.2 (**Local picture**)
Let $\triangle ABC \subset \mathbb{R}^2$ be a lattice triangle with corners A, B, and C, whose lattice points are in the set $\{A, B, C\}$ or in the interior int$(\triangle ABC)$. Assume there are n lattice points in int$(\triangle ABC)$. Consider a broccoli curve dual to $\triangle ABC$ consisting of a vertex of type (E1) as in 6.1 and a vertex of type (1) of 5.15 passing through one big dot marking P and one thin dot P' such that (P, P') is a point configuration in special position.
Then it holds for the multiplicities mult$_{C_l}$/mult$_{C_r}$ of broccoli curves C_l/C_r of genus 1 appearing when moving P' to the left/right hand side and passing through P and P':

$$\sum_{C_l} \text{mult}_{C_l} = \sum_{C_r} \text{mult}_{C_r} \qquad \text{and} \qquad \Big|\sum_{C_l} \text{mult}_{C_l}\Big| = \frac{n(n+1)}{2}.$$

Example 6.3
To make the statement of the theorem clear, consider the following example.



In this case, $\Delta = ((-3, 2), (-1, -1), (4, -1))$ and $\triangle ABC = \text{Conv}((1, 0), (0, 1), (2, 4))$, having $n = 2$ lattice points in $\text{int}(\triangle ABC)$. The two curves C_l are depicted on the left hand side. Each of them corresponds to the subdivision of $\triangle ABC$ where one of the lattice points inside $\triangle ABC$ is connected to the 3 corners of $\triangle ABC$. They are both broccoli curves in the sense of 5.21 with multiplicity -1 and -2, respectively. On the right hand side there is one curve C_r corresponding to the subdivision of $\triangle ABC$ in which the upper lattice points inside $\triangle ABC$ is connected to the 3 corners of $\triangle ABC$. This curve has multiplicity -3. The curve drawn below is not a broccoli curve as the encircled vertex does not exist in the list 5.15. We observe that the sum of multiplicities on the left and on the right hand side are the same and it holds $|-3| = \frac{3 \cdot 2}{2}$. Also, this example motivates some remarks.

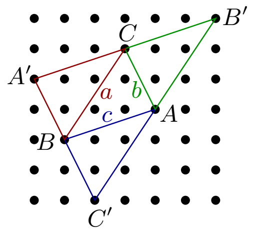
Remark 6.4

- The codimenson-1 cells in this (local) moduli space correspond to a curve with a contracted cycle.
- Curves appearing as resolutions of the cycle are not always broccoli curves.
- n equals the number of possible resolutions of the contracted cycle on the left respectively right hand side.
- No new vertex types need to be introduced for this theorem.
- An analogous statement holds when P is moved instead.
- The theorem contains actually two claims: the one of invariance while moving P and the closed formula for the sum of multiplicities on the left and on the right hand side, respectively.

We will now show the theorem giving two useful lemmata first.

Definition 6.5

Let $\triangle ABC$ be a lattice triangle as in theorem 6.2 with corners A, B, C. When we reflect the triangle $\triangle ABC$ at the side $a = \overline{BC}$ we obtain the triangle $\triangle BCA'$ with corners B, C, A' as indicated in the figure below. We proceed analogously in order to find the triangles with corners A, C, B' and A, B, C', respectively.



Denoting the *lattice area* by ℓ we define the sum

$$S_{xY'} := \sum_{Q} \ell(\triangle xQ),$$

where $x \in \{a, b, c\}$, $Y \in \{A, B, C\}$ such that the letter x is different from the letter in Y and the sum goes over all lattice points Q inside the parallelogramm $\square ABCY'$. In the same manner, denoting by odd ℓ the function which takes value 0 if the lattice area is even and as value the lattice area if the lattice area is odd, we then define the sum

$$S_{xY'}^{\text{odd}} := \sum_{Q} \text{odd } \ell(\triangle xQ),$$

where $x \in \{a, b, c\}$, $Y \in \{A, B, C\}$ such that the letter x is different from the letter in Y and the sum goes over all lattice points Q inside the parallelogramm $\square ABCY'$.

Remark 6.6
The points Q of 6.5 will not lie on an edge of any triangle defined in 6.5 by the assumption of theorem 6.2.

Lemma 6.7
In the context of theorem 6.2 we have $\ell(\triangle ABC) = 2n + 1$ and for the sums defined in 6.5 it holds:

$$S_{xY'} = n(2n + 1) \qquad \text{and} \qquad S_{xY'}^{\text{odd}} = n^2.$$

Furthermore, for any $x \in \{a, b, c\}$ the function $\ell(\triangle xQ)$ takes every value between 1 and $2n$ exactly once, where Q is a lattice point inside the parallelogramm $\square ABCY'$.

Proof. Consider lattice points Q inside a parallelogramm $\square ABCY'$ for w.l.o.g. $Y = A$ and $x = c$. First, we claim that the lattice areas of the triangles $\triangle cQ$ are pairwise different. Therefore, we choose w.l.o.g. B as origin of our affine coordinate system. Assume there are points Q_1 and Q_2 such that it holds $\ell(\triangle cQ_1) = \ell(\triangle cQ_2)$. Assume that the direction vector of the side c is (x, y) and that of the vector $\overrightarrow{BQ_i}$ is called (x_i, y_i), then $\ell(\triangle cQ_i)$ can be computed as $\left|\det\begin{pmatrix} x & x_i \\ y & y_i \end{pmatrix}\right|$. So the equality of the areas is equivalent to $|xy_1 - yx_1| = |xy_2 - yx_2|$, which is equivalent to say there is $\gamma \in \mathbb{Z} \setminus \{0\}$ with $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \gamma \begin{pmatrix} x \\ y \end{pmatrix}$. But then one of the points Q_i cannot lie inside $\square ABCA'$ as the side a is of lattice length 1. Hence, the areas cannot be the same. Next, we claim that $\ell(\triangle XYQ)$ takes all values between 1 and $2n$, where n is, as in theorem 6.2, the number of lattice points in the interior of $\triangle ABC$. So, as there are $2n$ lattice points inside $\square ABCX'$ and using the first claim, we only have to prove that the lattice area $(\triangle cQ)$ cannot be bigger than $2n$, since $\ell(\triangle ABC) = 2n + 1$ by Pick's formula 5.16. But this is clear as the lattice area of $\triangle ABC$ and $\triangle BCA'$ are both $2n+1$, and therefore the triangles $\triangle cQ$ with Q inside $\square ABCA'$ should have a smaller area than $2n + 1$. Hence,

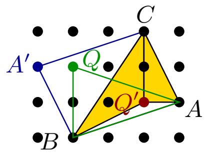
$$\begin{aligned}
S_{cA'} &= 1 + 2 + \ldots + 2n = \frac{2n(2n+1)}{2} = n(2n+1) \text{ and} \\
S_{cA'}^{\text{odd}} &= 1 + 3 + \ldots + (2n-1) = \sum_{i=1}^{\lfloor \frac{2n-1}{2} \rfloor} i + \sum_{i=1}^{\lceil \frac{2n-1}{2} \rceil} i \\
&= \frac{\lfloor \frac{2n-1}{2} \rfloor \cdot (\lfloor \frac{2n-1}{2} \rfloor + 1)}{2} + \frac{\lceil \frac{2n-1}{2} \rceil \cdot (\lceil \frac{2n-1}{2} \rceil + 1)}{2} = \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = n^2.
\end{aligned}$$

□

Lemma 6.8
Consider the parallelogramm $\square ABCA'$ as in definition 6.5. Then for each lattice point Q in the interior of the triangle $\triangle BCA'$ there is exactly one lattice point Q' in the interior of $\triangle ABC$ such that

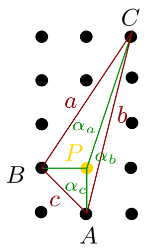
$$\ell(\triangle ABQ) = \ell(\triangle ACQ') + \ell(\triangle BCQ').$$

Denote by $f_{A'}$ the function which associates to each point Q the point Q' as above.



Proof. Let Q be a lattice point inside $\triangle BCA'$. Then the point Q' inside $\triangle ABC$ which we obtain when we reflect Q at the midpoint of the side a satisfies $\ell(\triangle cQ') + \ell(\triangle cQ) = \ell(\triangle ABC)$, because $\overrightarrow{BQ} + \overrightarrow{BQ'} = \overrightarrow{BC}$. The point Q' is the only point in $\square ABCA'$ having this property by lemma 6.7. Then, obviously we have $\ell(\triangle cQ) = \ell(\triangle ABC) - \ell(\triangle cQ') = \ell(\triangle BCQ') + \ell(\triangle ACQ')$. □

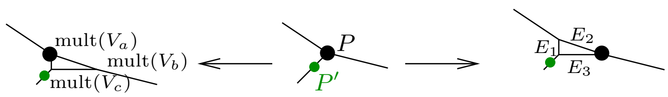
Proof of thm 6.2. Let Q be a lattice point in the interior of $\triangle ABC$. Define α_a^Q to be the lattice area of the triangle $\triangle BCQ$, analogously for α_b^Q and α_c^Q. α_i^Q can be odd (o) or even (e) for each $i \in \{a, b, c\}$. Then in total, there are four cases. Note that the case where all α_i are even is not possible since the total lattice area is $2n + 1$.



Case	α_a^Q	α_b^Q	α_c^Q
1	o	o	o
2	e	e	o
3	e	o	e
4	o	e	e

(1)

Consider now a point configuration $\omega = (P, P')$ as in theorem 6.2. Then, if we move P to the left or right hand side, there is exactly one curve passing through points in general position on each side. This curve does not contain necessarily broccoli vertices, see 5.21. Depending on the parity of the weight of the edges E_1, E_2, E_3 the multiplicity mult(V_i) in the sense of 2.23 contributes to the multiplicity m_C as in 5.15 of each of the two curves.



It is clear by remark 4.1 that mult$(V_i) = \alpha_i^Q$ for $i \in \{a, b, c\}$. So, if α_i^Q is even, then at least one of the edges adjacent to the corresponding vertex V_i is even by Pick's formula 5.16. Knowing the parity of α_i^Q for all $i \in \{a, b, c\}$ we also know which of the edges E_j are even or odd. Note that given a triangle $\triangle ABC$ as in theorem 6.2 having lattice area $2n + 1$, then the total sign of the multiplicity of each broccoli curve C passing through a generic point configuration (P, P') is $i^{\text{mult}(V_a)-1} \cdot i^{\text{mult}(V_b)-1} \cdot i^{\text{mult}(V_c)-1} = i^{2n+1-3} = i^{2n-2} \in \{-1, 1\}$. So the multiplicities m_C up to global sign are as follows depending on the case.

Case	m_C on the left	m_C on the right
1	α_a^Q	α_b^Q
2	α_b^Q	α_a^Q
3	α_c^Q	/
4	/	α_c^Q

(2)

We now intend to express the sum of multiplicities m_C going over all lattice points Q inside the triangle $\triangle ABC$ taken when P is moved to the left hand side and to the right hand side, respectively, as linear combination of the sums $S_{xY'}$ and $S_{xY'}^{\text{odd}}$. We claim that the sum on the left hand side is

$$-\frac{1}{2}S_{cA'}^{\text{odd}} + \frac{1}{2}S_{aB'}^{\text{odd}} - \frac{1}{2}S_{bC'}^{\text{odd}} + \frac{1}{2}S_{cA'}, \tag{3}$$

and that one on the right hand side

$$-\frac{1}{2}S_{cA'}^{\text{odd}} - \frac{1}{2}S_{aB'}^{\text{odd}} + \frac{1}{2}S_{bC'}^{\text{odd}} + \frac{1}{2}S_{cA'}. \tag{4}$$

By lemma 6.7 each sum does not depend on the choice of the parallelogramm $ABCX'$. Let us first consider the sum (3). We have

$$\begin{aligned}
-\frac{1}{2}S_{cA'}^{\text{odd}} &= -\frac{1}{2}\sum_{\substack{Q \text{ in } \square ABCA' \\ \text{s.t. } \ell(\triangle cQ) \text{ is odd}}} \ell(\triangle cQ) \\
&= -\frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_c^Q \text{ is odd}}} \alpha_c^Q + \sum_{\substack{Q \text{ in } \triangle BCA' \\ \text{s.t. } \ell(\triangle cQ) \text{ is odd}}} \ell(\triangle cQ)\Big) \\
&\overset{6.8}{=} -\frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_c^Q \text{ is odd}}} \alpha_c^Q + \sum_{\substack{Q \text{ in } \triangle BCA' \\ \text{s.t. } \ell(\triangle cQ) \text{ is odd}}} (\alpha_a^{f_{A'}(Q)} + \alpha_b^{f_{A'}(Q)})\Big) \\
&= -\frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_c^Q \text{ is odd}}} \alpha_c^Q + \sum_{\substack{Q' \text{ in } \triangle ABC \text{ s.t.} \\ \alpha_a^{Q'} \text{ is even} \\ \text{and } \alpha_b^{Q'} \text{ is odd}}} (\alpha_a^{Q'} + \alpha_b^{Q'}) + \sum_{\substack{Q' \text{ in } \triangle ABC \text{ s.t.} \\ \alpha_a^{Q'} \text{ is odd} \\ \text{and } \alpha_b^{Q'} \text{ is even}}} (\alpha_a^{Q'} + \alpha_b^{Q'})\Big).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}S_{aB'}^{\text{odd}} &= \frac{1}{2}\sum_{\substack{Q \text{ in } \square ABCB' \\ \text{s.t. } \ell(\triangle aQ) \text{ is odd}}} \ell(\triangle aQ) \\
&= \frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_a^Q \text{ is odd}}} \alpha_a^Q + \sum_{\substack{Q \text{ in } \triangle ACB' \\ \text{s.t. } \ell(\triangle aQ) \text{ is odd}}} \ell(\triangle aQ)\Big) \\
&\overset{6.8}{=} \frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_a^Q \text{ is odd}}} \alpha_a^Q + \sum_{\substack{Q \text{ in } \triangle ACB' \\ \text{s.t. } \ell(\triangle aQ) \text{ is odd}}} \alpha_b^{f_{B'}(Q)} + \alpha_c^{f_{B'}(Q)}\Big) \\
&= \frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_a^Q \text{ is odd}}} \alpha_a^Q + \sum_{\substack{Q' \text{ in } \triangle ABC \text{ s.t.} \\ \alpha_b^{Q'} \text{ is even} \\ \text{and } \alpha_c^{Q'} \text{ is odd}}} (\alpha_b^{Q'} + \alpha_c^{Q'}) + \sum_{\substack{Q' \text{ in } \triangle ABC \text{ s.t.} \\ \alpha_b^{Q'} \text{ is odd} \\ \text{and } \alpha_c^{Q'} \text{ is even}}} (\alpha_b^{Q'} + \alpha_c^{Q'})\Big).
\end{aligned}$$

$$
\begin{aligned}
-\frac{1}{2}S^{\text{odd}}_{bC'} &= -\frac{1}{2}\sum_{\substack{Q \text{ in } \square ABCC' \\ \text{s.t. } \ell(\triangle bQ) \text{ is odd}}} \ell(\triangle bQ) \\
&= -\frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_b^Q \text{ is odd}}} \alpha_b^Q + \sum_{\substack{Q \text{ in } \triangle ABC' \\ \text{s.t. } \ell(\triangle bQ) \text{ is odd}}} \ell(\triangle bQ)\Big) \\
&\overset{6.8}{=} -\frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_b^Q \text{ is odd}}} \alpha_b^Q + \sum_{\substack{Q \text{ in } \triangle ABC' \\ \text{s.t. } \ell(\triangle bQ) \text{ is odd}}} \alpha_a^{f_{C'}(Q)} + \alpha_c^{f_{C'}(Q)}\Big) \\
&= -\frac{1}{2}\Big(\sum_{\substack{Q \text{ in } \triangle ABC \\ \text{s.t. } \alpha_b^Q \text{ is odd}}} \alpha_b^Q + \sum_{\substack{Q' \text{ in } \triangle ABC \text{ s.t.} \\ \alpha_a^{Q'} \text{ is even} \\ \text{and } \alpha_c^{Q'} \text{ is odd}}} (\alpha_a^{Q'} + \alpha_c^{Q'}) + \sum_{\substack{Q' \text{ in } \triangle ABC \text{ s.t.} \\ \alpha_a^{Q'} \text{ is odd} \\ \text{and } \alpha_c^{Q'} \text{ is even}}} (\alpha_a^{Q'} + \alpha_c^{Q'})\Big).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}S_{cA'} &= \frac{1}{2}\sum_{Q \text{ in } \square ABCA'} \ell(\triangle cQ) \\
&= \frac{1}{2}\Big(\sum_{Q \text{ in } \triangle ABC} \alpha_c^Q + \sum_{Q \text{ in } \triangle BCA'} \ell(\triangle cQ)\Big) \\
&\overset{6.8}{=} \frac{1}{2}\Big(\sum_{Q \text{ in } \triangle ABC} \alpha_c^Q + \sum_{Q \text{ in } \triangle BCA'} (\alpha_a^{f_{A'}(Q)} + \alpha_b^{f_{A'}(Q)})\Big) \\
&= \frac{1}{2}\Big(\sum_{Q \text{ in } \triangle ABC} \alpha_c^Q + \sum_{Q' \text{ in } \triangle ABC} (\alpha_a^{Q'} + \alpha_b^{Q'})\Big).
\end{aligned}
$$

So in total we have

$$
-\frac{1}{2}S^{\text{odd}}_{cA'} + \frac{1}{2}S^{\text{odd}}_{aB'} - \frac{1}{2}S^{\text{odd}}_{bC'} + \frac{1}{2}S_{cA'} = \sum_{\substack{Q \text{ in } \triangle ABC \\ \text{in case } 1}} \alpha_a^Q + \sum_{\substack{Q \text{ in } \triangle ABC \\ \text{in case } 2}} \alpha_b^Q + \sum_{\substack{Q \text{ in } \triangle ABC \\ \text{in case } 3}} \alpha_c^Q,
$$

This is equal to the absolute value of the sum of the multiplicities m_C using the tables (1) and (2). By lemma 6.7 we can compute (3) as

$$
-\frac{1}{2}S^{\text{odd}}_{cA'} + \frac{1}{2}S^{\text{odd}}_{aB'} - \frac{1}{2}S^{\text{odd}}_{bC'} + \frac{1}{2}S_{cA'} = \frac{1}{2}(n(2n+1) - n^2) = \frac{n(n+1)}{2}.
$$

Proceeding in an analogous way for (4), we obtain

$$
\begin{aligned}
&-\frac{1}{2}S^{\text{odd}}_{cA'} - \frac{1}{2}S^{\text{odd}}_{aB'} + \frac{1}{2}S^{\text{odd}}_{bC'} + \frac{1}{2}S_{cA'} \\
&= \sum_{\substack{Q \text{ in } \triangle ABC \\ \text{in case } 2}} \alpha_a^Q + \sum_{\substack{Q \text{ in } \triangle ABC \\ \text{in case } 1}} \alpha_b^Q + \sum_{\substack{Q \text{ in } \triangle ABC \\ \text{in case } 4}} \alpha_c^Q \\
&= \frac{n(n+1)}{2}.
\end{aligned}
$$

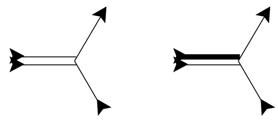
Hence, the sums of multiplicities m_C when moving P to the left hand side, respectively to the right hand side, are equal. □

6.2 Invariance for a second class of examples

Next, we treat a class of examples which generalizes the example of the introduction of this chapter. In particular, we need a new vertex type in order to prove the invariance. Let us first define the curve we deal with in this section.

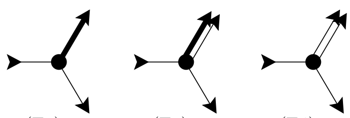
Definition 6.9 (**Oriented broccoli curve of genus** 1)
An *oriented broccoli curve of genus* 1 is a (r,s)-marked (plane tropical) curve (C,h) of genus 1 with $C=(\Gamma,x_1,\ldots,x_{r+s},y_1,\ldots,y_n)$ where

- $x_1,\ldots,x_r$ are its real markings,
- $x_{r+1},\ldots,x_{r+s}$ its complex markings,
- $y_1,\ldots,y_n$ its unmarked ends,
- which is oriented in the sense of 5.13,
- and all of whose vertices are of the type (1)-(6) of 5.15 and/or of type (10)-(11) as depicted below.

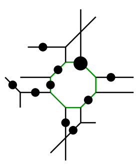


We are not going to define unoriented broccoli curves of genus 1 (neigher we will show their equivalence to oriented curves) as we did in 5.21 since definition 6.9 is work in progress and may be modified at a later state of this work.

Definition 6.10 (**New codimension-**1 **types (E2)-(E4)**)
We define vertex types (E2)-(E4) appearing in codimension-1 in this second class of examples as depicted below. This expands the list given in the proof of theorem 5.26 and of definition 6.1.



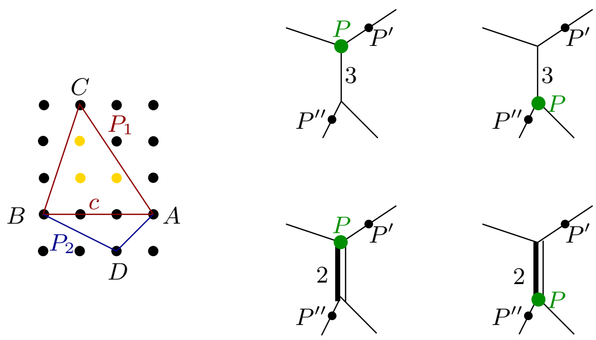
Definition 6.11 (n-**cycles**)
Let (C,h) be a parametrized tropical curve. We define an n-cycle to be a subgraph of $h(\Gamma)$ being a cycle and having n vertices.



Theorem 6.12
Let $\Delta=(v(y_1),\ldots,v(y_4))$ a tropical degree containing only primitive vectors and consider the dual subdivision (as defined in 4.1) $P_\Delta=P_1\cup P_2$ which satisfies:

- P_1 is a triangle having a horizontal side c of lattice length m and lattice height h w.r.t. c, but whose two other sides a and b are of lattice length 1. P_1 may contain lattice points in its interior.
- P_1 is glued at the side c to the triangle P_2 which has lattice height 1 w.r.t. the side c and whose other two sides a' and b' are of lattice length 1. Therefore, it does not contain any lattice points in its interior.

Consider a broccoli curve dual to P_Δ consisting of a vertex of type (E1) or (E2) as in 6.1 respectively 6.10, two vertices of type (1) and one vertex of type (2) or (3) of 5.15 passing through one big dot P and two thin dots P' and P'' such that (P, P', P'') is a point configuration in special position. Simultaneously, consider a broccoli curve dual to P_Δ consisting of a vertex of type (E3) or (E4) as in 6.10, two vertices of type (1) and one vertex of type (11) or (10) of 6.9 passing through the same configuration (P, P', P'') in special position. For both curves, there are two possiblities - P can lie on the upper vertex or on the lower vertex as depicted below which we will call *first* and *second codimension*-1 *situation* in the following.



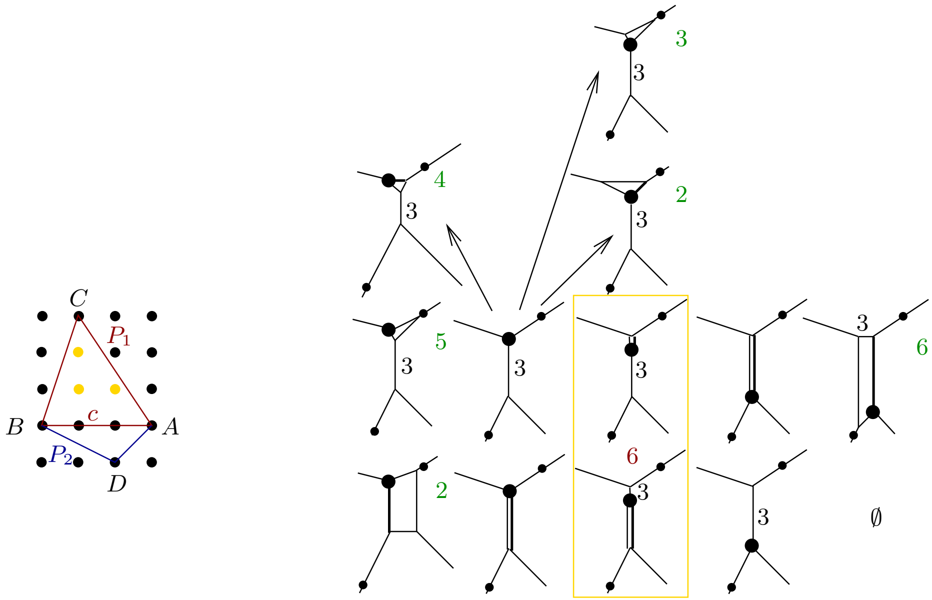
One obtains the second from the first situation by moving P vertically downwards. Then it holds for the multiplicities $\text{mult}_{C_l}/\text{mult}_{C_r}$ of broccoli curves C_l/C_r of genus 1 passing through (P, P', P'') appearing when moving P upwards from the first codimension-1 situation, respectively downwards from the second codimension-1 situation:

$$\sum_{C_l} \text{mult}_{C_l} = \sum_{C_r} \text{mult}_{C_r}.$$

That is, we can define an appropriate multiplicity for the bunch of flat cycles appearing between these codimension-1 situations as specified in 6.18, 6.19, and 6.20.

Remark 6.13 (**Choice of moved points**)
Analogous statements hold if we move P' or P'' instead.

Example 6.14
Considering the situation in the figure of theorem 6.12 we have three lattice points in the interior of P_1. Each of them gives rise to a cycle in the curve when we move P such that (P, P', P'') is generic in a similar way to what happens in theorem 6.2. As there are no lattice points inside P_2, there is no cycle contribution from P_2. Somewhat new compared to the case in theorem 6.2 are 4-cycles and flat cycles. They are possible because we consider a dual subdivision with two polygons at the same time. Curves with a 4-cycle contain only vertex types of definition 5.21 and therefore we can compute their multiplicity m_C. The multiplicity of a curve with a flat cycle has to be determined since it contains new vertices – in this example of type (11) and in the example of the introduction of this chapter of type (10). The study of examples like this one gives only information about the sum of multiplicities of curves with a flat cycle as they always appear together. In this case here the sum of the multiplicities of the two curves in the yellow box is 6 and in the example of the introduction of this chapter the sum is 1.



As we will see, this sum of multiplicities depends on the parity of the lattice lengths m and h as defined in 6.12.

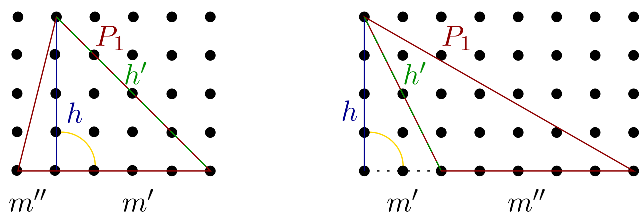
Remark 6.15 (**Cases in theorem 6.12**)
The following cases are possible for m and h in the situation of theorem 6.12.

Case 1: m and h are even,

Case 2: m and h are odd,

Case 3: m is even and h is odd.

The case where m is odd and h is even does not occur. Namely, assume that h is even and m is odd. Denote by m' and m'' the parts of m subdivided by the height h if the h lies inside $\triangle ABC$ and choose m' and m'' as depicted on the right hand side otherwise. As $m = m' + m''$ respectively $m = m''$ is odd, then w.l.o.g. m' in the figure below is even. By the theorem of Pythagoras $m'^2 + h^2 = h'^2$ the lattice length h' is also even, which is a contradiction to the requirement of theorem 6.12 that the vectors in Δ are primitive.



Lemma 6.16
Let Q be a lattice point on the side c and denote by m_1 the lattice length of $\overline{BQ}$ and by m_2 the lattice length of $\overline{QA}$. Then it holds for the lattice length l of $\overline{QC}$ in the cases of 6.15:

Case 1: if m_1 und m_2 are odd, then l is even; if m_1 and m_2 are even, then l is odd,

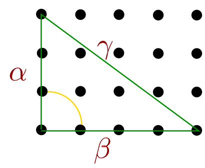
Case 2: l is always odd,

Case 3: l is always odd.

Proof. First note that we have for a right-angled lattice triangle as below

$$\gamma = \gcd(\alpha, \beta), \tag{5}$$

where α, β, γ are the lattice lengths of the corresponding edges.



Using relation (5) we can treat cases 2 and 3 simultaneously. Namely, given a point Q as in the theorem choose $R \in \mathbb{Z} \times \{0\}$ such that the lattice length of $\overline{CR}$ is h and $\triangle CQR$ is a triangle with right angle at R. Set $h = \alpha$ and $l = \gamma$. If the lattice length β of $\overline{RQ}$ is even, then gcd(α, β) is odd. If the lattice length β of $\overline{RQ}$ is odd, then gcd(α, β) is odd since the gcd of two odd numbers is always odd.
It remains to prove the statement for case 1. It is clear that the m_i have the same parity. If $R \in \mathbb{Z} \times \{0\}$ is as above, then $\triangle CRA$ and $\triangle CRB$ have both a right angle at R. Since the sides a and b are of lattice length one, the lattice lengths of $\overline{BR}$ and $\overline{AR}$ should be odd by (5). If Q is such that the m_i are of odd lattice length, then $\overline{RQ}$ is of even length. Hence, considering the triangle $\triangle CQR$ which is right at Q, we know by (5) that $\overline{QC}$ should be of even length. If, in contrast, the m_i are of even length, then $\overline{RQ}$ is of odd length and therefore by (5) $\overline{QC}$ is of odd length. □

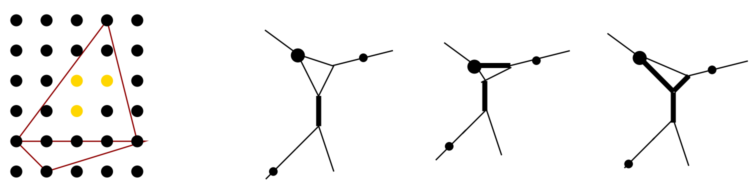
We now consider the cases one after the other. Let us start with a remark that will be useful in the following. The proof of theorem 6.12 follows then from the lemmata 6.18, 6.19 and 6.20.

Remark 6.17 (**Possible cycles**)
Disregarding flat cycles, it is clear by the definition of a dual subdivision that tropical curves dual to P_Δ having a cycle correspond to triangulations of P_Δ having an interior vertex. As we are interested in counting curves passing through generic point configurations and as the point configuration contains one big dot and two thin dots, curves with a non-flat cycle (that contribute) should have an n-cycle with $3 \leq n \leq 4$ vertices as we have only 4 ends in the situation of theorem 6.12.

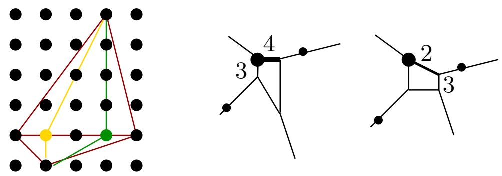
Lemma 6.18 (**Case 1**)
In case 1 of 6.15 invariance holds at the two codimension-1 situations of theorem 6.12. In particular, when we move P upwards from the first codimension-1 situation we obtain only curves with broccoli vertices from chapter 5 and the same holds for moving P downwards from the second codimension-1 situation. The codimension-0 cells we obtain when we move P downwards from the first codimension-1 situation or upwards from the second codimension-1 cell contain flat cycles with vertices as in 6.10 and they count in sum 0.

Proof.

- First we claim that if we move P upwards from the first codimension-1 situation then only 4-cycles contribute. Indeed, we have seen in section 1 that lattice points in the interior of a lattice triangle (dual to a tropical curve) should be seen as contracted 3-cycles. In the situation of theorem 6.12 there are only lattice points inside P_1. Since m is of even lattice length, curves dual to P_Δ having a 3-cycle have a vertical edge of even weight. The types occuring just differ by the number and the position of edges of even weight in the cycle, e.g.:

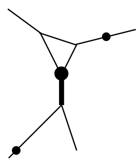


These types have in common that they are not broccoli curves. Hence, there are no curves with 3-cycles which contribute when we move P upwards. Curves with a flat cycle do not appear when we move P upwards since flat cycles need to be mapped to the edge of $h(\Gamma)$ dual to c, but as the position of P' is fixed, this edge cannot be prolonged for P moved upwards. Considering curves with 4-cycles they only contribute if at most one of the four edges in the cycle is even by an argumentation similiar to the case of 3-cycles. More precisely, they correspond to triangulations of P_Δ such that there is a lattice point Q on the side c with m_i are odd, because when the m_i are odd then the two vertical edges in a 4-cycle are odd and by lemma 6.16 the upper horizontal edge is of even weight. The lower horizontal edge must be of odd weight since P_2 has lattice height 1.



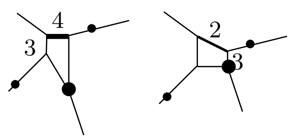
Broccoli curves with such a 4-cycle all have the same sign because the total lattice area $p = m \cdot (h+1)$ of P_Δ is constant and 4-cycles correspond to decompositions of $P_\Delta = Q_1 \cup \ldots \cup Q_4$ where the Q_i are triangles. Therefore, using 5.15 the sign of a broccoli curve dual to this decomposition is $(-1)^{q_1} \cdot (-1)^{q_2} \cdot (-1)^{q_3} \cdot (-1)^{q_4} = (-1)^{q_1+q_2+q_3+q_4-4} = (-1)^{p-4}$, where q_i is the lattice area of Q_i.

- When we move P downwards from the first codimension-1 situation then 4-cycles are not possible anymore: the moved point has to be in the vicinity of the upper vertex of the initial special configuration of points, i.e. in a 4-cycle on one of the upper vertices. But the right vertex is not possible since then the points would not be in general position and the left vertex corresponds to moving P upwards. But this time, 3-cycles are possible and count if there are only odd edges in cycle:



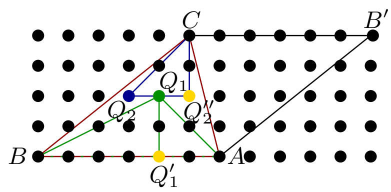
Broccoli curves with such a 3-cycle have all the same sign and this sign equals the sign of broccoli curves with a 4-cycle as above. This is true since the total lattice area p of P_Δ is constant and 3-cycles correspond to decompositions of $P_1 = Q_1 \cup \ldots \cup Q_3$ in P_Δ, where the Q_i are triangles. Therefore, using 5.15 the sign of a broccoli curve dual to P_Δ with this decomposition is $(-1)^{q_1} \cdot (-1)^{q_2} \cdot (-1)^{q_3} \cdot (-1)^{p_2} = (-1)^{q_1+q_2+q_3+q_4-4} = (-1)^{p-4}$, where q_i is the lattice area of Q_i and p_2 is the lattice area of P_2. The last type of cycles that appear here are flat cycles.

- If we move P downwards from the second codimension-1 situation, 4-cyles are possible again at a certain moment but flat cycles are not anymore as the situation is symmetric to the one above. Also, there are only 4-cycles possible which have exactly one edge of even weight and the other being odd. But as the edge of even weight is the upper horizontal edge in the cycle, curves with such a 4-cycle are not broccoli curves.



> 3-cycles are not possible since P_2 does not contain lattice points in the interior. So there are no broccoli curves which count!

We define the sum of multiplicities of curves with a flat cycle to be zero. We then claim that the invariance in each codimension-1 situation holds, i.e. we have to show that the sum of multiplicities of broccoli curves with 4-cycles appearing when we move P upwards from the first codimension-1 situation equals the sum of broccoli curves having a 3-cycle when we move P downwards from the first codimension-1 situation. This implies that the invariance also holds for the second codimension-1 situation since the sum of multiplicities of curves with a flat cycle is zero and there are no other curves contributing when we move P upwards from the second codimension-1 situation and no curves at all when we move P downwards from the second codimension-1 situation.
Let us first consider broccoli curves with a 4-cycle. So there is a lattice point Q on the side c with m_i both odd. If the m_i have different value, then there are two possible broccoli curves with a 4-cycle associated to m_1 and m_2. Namely, the one with a vertical edge of weight m_1 on the left hand side and a vertical edge of weight m_2 on the right hand side and vice versa. When we move P upwards from the first codimension-1 situation then the first curve has up to sign multiplicity $m_C = m_2 \cdot h$ and the second one $m_C = m_1 \cdot h$. If $m_1 = m_2$ there is only one broccoli curve associated having multiplicity $m_1 \cdot h = m_2 \cdot h = \frac{m \cdot h}{2}$. Hence, we are interested in the number of partitions of m with m_i odd. If $4|m$ there are $\frac{m}{4}$ pairs $(m_1, m_2) \equiv (m_2, m_1)$ contributing to the sum of broccoli curves with a 4-cycle with the lattice area of P_1 which is $m \cdot h$. The m_i satisfy $m_1 \neq m_2$. If $4 \nmid m$ then we have $\lceil \frac{m}{4} \rceil$ pairs $(m_1, m_2) \equiv (m_2, m_1)$, where one is of the form $m_1 = m_2$ contributing with $\frac{m \cdot h}{2}$ and the others contribute with $m \cdot h$.
Let us now consider broccoli curves with a 3-cycle. Lattice points Q inside P_1 only contribute if the lattice distance to each of the corners A, B and C is odd. We claim that this is the case if Q lies on a line parallel to the side c that has odd lattice distance to c. Namely, if Q lies on a line parallel to c with even distance, then we can project Q in an orthogonal way down to c and get thereby a point Q'. Then the triangles $\triangle AQ'Q$ and $\triangle BQ'Q$ are right angled at Q'. When the lattice lengths of $\overline{BQ'}$ and $\overline{AQ'}$ are even, then by property (5) also $\overline{BQ}$ and $\overline{AQ}$ are even. If this is not the case, then $\overline{QC}$ is even. Namely, if we project C to the line parallel to c passing through Q we get a point Q''. $\overline{QQ''}$ has even lattice length, since the projection of C to c always divides m into two odd numbers m' and m''. Therefore, considering the right angled triangle $\triangle CQQ''$ property (5) implies that $\overline{QC}$ has even lattice length. A similar argumentation shows when Q lies on a line parallel to c which has odd lattice distance to c it has odd lattice distance to any of the three points A, B and C. On each line parallel to c we have m lattice points inside the parallelogram $\square ABCB'$ with corners A, B, C and B'. Taking in account only odd heights, we have in total $\frac{h}{2} \cdot m$ lattice points inside $\square ABCB'$, hence in the triangle $\triangle ABC$ there are $\frac{h}{4} \cdot m$. As we are interested in the sum of multiplicities of such broccoli curves with a 3-cycle we have to multiply this quantity with m, as this is the lattice area of P_2.



Hence, for $4|m$ the contribution of the curves with a 4-cycle is

$$\frac{m}{4} \cdot (m \cdot h)$$

and equals the contribution of the curves with a 3-cycle

$$\frac{h \cdot m}{4} \cdot m.$$

For $4 \nmid m$ we have for the curves with a 4-cycle

$$\begin{aligned}(\lceil \frac{m}{4} \rceil - 1) \cdot m \cdot h + \frac{m \cdot h}{2} =& (\frac{m+2}{4} - 1) \cdot m \cdot h + \frac{m \cdot h}{2} \\ =& \frac{m-2}{4} \cdot m \cdot h + \frac{m \cdot h}{2} \\ =& (\frac{1}{2} + \frac{m-2}{4}) \cdot m \cdot h \\ =& \frac{m^2 \cdot h}{4}\end{aligned}$$

which is again equal (up to sign) to the sum of multiplicities of curves with a 3-cycle:

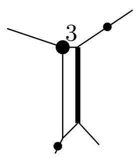
$$\frac{h \cdot m}{4} \cdot m.$$

□

Lemma 6.19 (**Case 2**)
In case 2 of 6.15 invariance holds at the two codimension-1 situations of theorem 6.12. In particular, when we move P upwards from the first codimension-1 situation we obtain only curves with broccoli vertices from chapter 5 and the same holds for moving P downwards from the second codimension-1 situation. The codimension-0 cells we obtain when we move P downwards from the first codimension-1 situation or upwards from the second codimension-1 cell contain flat cycles with vertices as in 6.10 and they count in sum $\frac{m^2-1}{4} \cdot h$ up to sign.

Proof.

- When we move P upwards from the first codimension-1 situation, then there are 4-cycles contributing as in lemma 6.18. As m is odd, we should consider this time partitions of m, where m_1 is even and m_2 is odd, and vice versa. But with the point configuration of theorem 6.12, only the case m_1 even and m_2 odd will contribute since otherwise the curves are not broccoli. For instance, in example 6.14 the following curve ($m_1 = 1$, $m_2 = 2$) appears on the left hand side of the picture.



  In addition, there are also 3-cycles this time as m is odd. Since m and h are both odd, also the lattice area $\ell(\triangle ABC) = m \cdot h$ is odd. Considering partitions of an odd number N into 3 summands N_i, one observes that only the cases, where a) all numbers N_i are odd or b) exactly two of the numbers are even, are possible. Taking these numbers N_i to be the areas of the 3 lattice triangles in a dual subdivision of $\triangle ABC$, this implies that only 3-cycles with at most one even edge in the cycle are possible. The sign of the 4- and 3-cycles contributing is the same as the total lattice area of $P_\triangle$ is constant - see the proof of the case in 6.18 for an exact argumentation.

- If we move P downwards from the first codimension-1 situation, 4-cycles are not longer possible - compare with the analogous situation in the proof of 6.18. But we have 3-cycles with up to one even edge and flat cycles. The 3-cycles have the same sign as 4- and 3-cycles appearing when moving P upwards from the first codimension-1 situation.

- When we move P downwards from the second codimension-1 situation, 4-cycles (that count) are possible again. They have the property that m_1 is odd and m_2 is even since otherwise they are not broccoli curves. But 3-cycles do not occur here as P_2 does not contain interior lattice points. Flat cycles also do not arise here, because of the position of the point P relative to the point P''.

In order to show the invariance we first figure out what the contribution of broccoli curves is that appear when we move P upwards from the first codimension-1 situation. Concerning 4-cycles we have to determine the number of partitions of m such that m_1 is even and m_2 is odd. Each such 4-cycle counts with multiplicity m_1 (up to sign). The sum of all numbers between 1 and $m-1$ is $\frac{m\cdot(m-1)}{2}$ and the sum of odd numbers between 1 and $m-1$ is $(\frac{m-1}{2})^2$. So the total contribution of 4-cycles is

$$\frac{m\cdot(m-1)}{2} - (\frac{m-1}{2})^2 = \frac{m^2-1}{4}.$$

For the contribution of 3-cycles one has to argue in a similar way as in the proof of theorem 6.2. Define $S_{xY'}$ and $S_{xY'}^{\text{odd}}$ as in 6.5. First, we see that for $S_{cA'}^{\text{odd}}$ we should consider those points in $\square ABCA'$, which lie on odd height w.r.t. the edge c. As there are m such points Q on each height h_i and the lattice area of $\triangle cQ$ is $h_i \cdot m$, we have

$$S_{cA'}^{\text{odd}} = m \cdot \sum_{\substack{1\leq h_i<h \\ \text{with } h_i \text{ odd}}} h_i \cdot m = (\frac{h-1}{2})^2 \cdot m^2.$$

Furthermore $S_{aB'}^{\text{odd}} = S_{bA'}^{\text{odd}} = (\frac{m\cdot h-1}{2})^2 - h\cdot(\frac{m-1}{2})^2$ what can be seen as follows. Similar to the proof of lemma 6.7 one can show that the values of $\ell(\triangle xQ)$ with $x=a$, respectively $x=b$, are pairwise different. As this lattice area can be computed as absolute value of the determinant of the vectors $\overrightarrow{BC}/\overrightarrow{AC}$ (with y-coordinate h) and $\overrightarrow{BQ}/\overrightarrow{CQ}$ (with y-coordinate different from h) in columns, it follows that $\ell(\triangle xQ)$ cannot be divisible by h. $\ell(\triangle xQ)$ should be smaller than $\ell(ABC) = m\cdot h$. Since there are $m\cdot(h-1)$ points Q in the interior, all the numbers between 1 and $m\cdot h-1$ which are not divisible by h appear exactly once as lattice area $\ell(\triangle xQ)$. Hence, the equality $S_{aB'}^{\text{odd}} = S_{bC'}^{\text{odd}}$ holds since exactly the same numbers occur in the sums. In general, we have $S_{cA'}^{\text{odd}} \neq S_{bA'}^{\text{odd}}$. The sum of odd numbers between 1 and $m\cdot h-1$ is $(\frac{m\cdot h-1}{2})^2$ and the sum of odd numbers between 1 and $m\cdot h-1$ divisible by h is $h\cdot(\frac{m-1}{2})^2$. Hence

$$S_{aB'}^{\text{odd}} = S_{bA'}^{\text{odd}} = (\frac{m\cdot h-1}{2})^2 - h\cdot(\frac{m-1}{2})^2.$$

The sum $S_{cA'}$ equals $1\cdot m^2 + \ldots + (h-1)\cdot m^2 = \frac{m^2\cdot(h-1)\cdot h}{2}$. Now, we claim that the contribution of the 3-cycles can be expressed as

$$\frac{1}{2}S_{cA'} - \frac{1}{2}S_{cA'}^{\text{odd}} - \frac{1}{2}S_{bA'}^{\text{odd}} + \frac{1}{2}S_{aB'}^{\text{odd}} = \frac{1}{2}S_{cA'} - \frac{1}{2}S_{cA'}^{\text{odd}}.$$

To show this, first notice that the statement of lemma 6.8 also holds here. Then, we proceed analogously to the proof of theorem 6.2. So the multiplicities m_C up to global sign are as follows depending on the case (described in the proof there):

Case	m_C on the left
1	α_a^Q
2	α_b^Q
3	α_c^Q
4	/

When we split $\frac{1}{2}S_{cA'}$, $-\frac{1}{2}S^{\text{odd}}_{cA'}$, $-\frac{1}{2}S^{\text{odd}}_{bA'}$, $+\frac{1}{2}S^{\text{odd}}_{aB'}$ as we did it the proof of theorem 6.2 we see that

$$\begin{aligned}&\frac{1}{2}S_{cA'} - \frac{1}{2}S^{\text{odd}}_{cA'} - \frac{1}{2}S^{\text{odd}}_{bA'} + \frac{1}{2}S^{\text{odd}}_{aB'}\\ &= \sum_{\substack{Q \text{ in } \triangle ABC\\ \text{in case } 1}} \alpha_a^Q + \sum_{\substack{Q \text{ in } \triangle ABC\\ \text{in case } 2}} \alpha_b^Q + \sum_{\substack{Q \text{ in } \triangle ABC\\ \text{in case } 3}} \alpha_c^Q\\ &= \frac{1}{8}\cdot(h^2-1)\cdot m^2.\end{aligned}$$

So the contribution of 4-cycles and 3-cycles is

$$\frac{m^2h^2+m^2-2}{8}.$$

Considering the contribution from 3-cycles when we move P downwards from the first codimension-1 situation, this is equal to

$$\frac{1}{2}S_{cA'} + \frac{1}{2}S^{\text{odd}}_{cA'} - \frac{1}{2}S^{\text{odd}}_{bA'} - \frac{1}{2}S^{\text{odd}}_{aB'} = \frac{1}{2}S_{cA'} + \frac{1}{2}S^{\text{odd}}_{cA'} - S^{\text{odd}}_{bA'}. \tag{6}$$

This can be shown by observing that the multiplicities m_C up to global sign are as follows depending on the case as described in the proof of theorem 6.2:

Case	m_C on the right
1	α_c^Q
2	/
3	α_a^Q
4	α_b^Q

and splitting the summands (on the left hand side) in (6) as in the proof of theorem 6.2. Then it follows:

$$\begin{aligned}&\frac{1}{2}S_{cA'} + \frac{1}{2}S^{\text{odd}}_{cA'} - \frac{1}{2}S^{\text{odd}}_{bA'} - \frac{1}{2}S^{\text{odd}}_{aB'}\\ &= \sum_{\substack{Q \text{ in } \triangle ABC\\ \text{in case } 3}} \alpha_a^Q + \sum_{\substack{Q \text{ in } \triangle ABC\\ \text{in case } 4}} \alpha_b^Q + \sum_{\substack{Q \text{ in } \triangle ABC\\ \text{in case } 1}} \alpha_c^Q\\ &= \frac{1}{8}(h^2m^2-2hm^2+2h+m^2-2).\end{aligned}$$

So we have invariance at this first codimension-1 situation if the contribution of the flat cycles is up to sign

$$\frac{m^2-1}{4}\cdot h.$$

Let us now study the second codimension-1 situation. The contribution of the 4-cycles when we move P downwards is exactly $\frac{m^2-1}{4}\cdot h$ since we are interested in partitions of m where m_1 is odd and m_2 is even and each of these curves contributes with multiplicity $m_1 \cdot h$ (compare with the computation for the 4-cycles in the first codimension-1 situation). As 4-cycles are the only broccoli curves that appear when we move P downwards and there are only flat cycles when we move P upwards (the 3-cycles only appear when we move P downwards from the first codimension-1 situation!), we have also invariance for the second codimension-1 situation when the flat cycles count (up to sign) $\frac{m^2-1}{4}\cdot h$. □

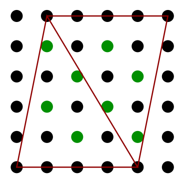
Lemma 6.20 (**Case 3**)
In case 3 of 6.15 invariance holds at the two codimension-1 situations of theorem 6.12. In particular, when we move P upwards from the first codimension-1 situation we obtain only curves with broccoli vertices from chapter 5 and the same holds for moving P downwards from the second codimension-1 situation. The codimension-0 cells we obtain when we move P downwards from the first codimension-1 situation or upwards from the second codimension-1 cell contain flat cycles with vertices as in 6.10 and they count in sum $\frac{m^2}{4}$ up to sign.

Proof.

- When we move P upwards from the first codimension-1 situation, only 4-cycles with m_1 and m_2 odd contribute, since m is even again. Namely, this implies that only odd edges are allowed inside 3-cycles and they are not broccoli curves by an analogous argument to the one given in the proof of 6.18. Also, it was proven there that 4-cycles with m_i both even do not yield broccoli curves. The number of partitions of m with m_i both odd is

$$\begin{aligned} 4|m: &\quad \frac{m}{4} \text{ pairs, which each counts } m\cdot h \text{, so in total } \frac{m^2\cdot h}{4},\\ 4\nmid m: &\quad \lceil\frac{m}{4}\rceil \text{ pairs, one is of the form } m_1=m_2 \text{ and counts } \frac{m\cdot h}{2}\\ &\quad \text{and the other count } m\cdot h \text{ each, so in total } \frac{m^2\cdot h}{4}\text{, too.}\end{aligned}$$

- If we move P downwards from the first codimension-1 situation, 3-cycles having only odd edges and flat loops are possible. We have to count suitable lattice points Q inside $\triangle ABC$, namely those having odd lattice length to each point A, B and C as we did it in the proof of 6.18. By a similar proof to the one of 6.18 in the parallelogram $\square ABCB'$ one shows that every second point (starting with the second from the left) on a line parallel to c of odd height h satisfies the conditions and every second point (starting with the first from the left) on a line parallel to c of even height h, too.



So in $\square ABCB'$, there are in total $\frac{m\cdot(h-1)}{2}$ points contributing each with multiplicity m, so the total contribution of 3-cycles is

$$\frac{m\cdot(h-1)}{4}\cdot m.$$

- Moving P downwards from the second codimension-1 situation there are 4-cycles with m_i both odd, exclusively. The number of partitions is the same as computed for the 4-cycles above, but each pair counts with multiplicity m, if $m_1 \neq m_2$ and $\frac{m}{2}$ otherwise. So in total:

$$\begin{aligned} 4|m: &\quad \frac{m^2}{4},\\ 4\nmid m: &\quad \frac{m^2}{4}\text{, too.}\end{aligned}$$

It is hence easy to see that invariance holds in both codimension-1 situations if the contribution of the flat cycles is $\frac{m^2}{4}$ up to sign. □

6.3 Outlook

The example class of the previous section can be generalized easily to the case in which the lower triangle P_2 is also allowed to contain lattice points in its interior. Namely, one has then to consider the cases 1-3 of 6.15 for both triangles P_1 (with height h_1) and P_2 (with height h_2), respectively. The multiplicity of the flat cycles is then depending on the case (the first number below is for P_1 and the second number for P_2):

Case 1 + 1: $\frac{m^2}{4} \cdot h_1 \cdot h_2$,

Case 1 + 3: $\frac{m^2 \cdot h_1}{4}(h_2 - 1)$,

Case 2 + 2: $\frac{m^2 - 1}{4}(h_1 + h_2 - 1)$,

Case 3 + 3: $\frac{m^2}{4}(h_2 - h_1 \cdot h_2 + h_1)$.

These results are consistent with the results from the last section since if we choose $h_2 = 1$ above we obtain the formulas from lemmata 6.18, 6.19 and 6.20. A little bit nasty is that we cannot (so far) write these multiplicities as products of vertex multiplicities even if we allow correction terms from the cycles. Once we constructed genus-1 broccoli curves, it is of course an interesting question to come up with meaningful real counterparts.

In a larger context, it would be nice if one can characterize the moduli space of bridge curves and the locus therein of broccoli curves of genus 0 and of Welschinger curves, respectively, a bit more.
Another, in some sense natural, question is if the bridge for a given generic point configuration, considered as a graph, is always a tree and if this is only true as long one considers curves of genus 0.
The answer to these and similar questions will help to understand the nature of bridges and broccoli curves better, which is so far only combinatorial.

Bibliography

[ABLdM11] Aubin Arroyo, Erwan Brugallé, and Lucia Lopez de Medrano, *Recursive formulas for Welschinger invariants*, Int. Math. Res. Not. (2011), no. 5, 1107 – 1134, arXiv: 0809.1541.

[AGZV88] Vladimir I. Arnold, Sabir M. Gusein-Zade, and Aleksandr N. Varchenko, *Singularities of differentiable maps*, Birkhäuser, 1988.

[AR10] Lars Allermann and Johannes Rau, *First steps in tropical intersection theory*, Math. Z. **127** (2010), no. 3, 601–617.

[BM96] Kai Behrend and Yuri Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), no. 1, 1–60, arXiv: alg-geom/9506023.

[BM08] Erwan Brugallé and Grigory Mikhalkin, *Floor decompositions of tropical curves: the planar case*, Proceedings of the 15th Gökova Geometry-Topology Conference (2008), 64–90, arXiv: 0812.3354.

[BMV11] Silvia Brannetti, Margarida Melo, and Filippo Viviani, *On the tropical Torelli map*, Adv. Math. **226** (2011), no. 3, 2546–2586.

[Bru08] Erwan Brugallé, *Géométries énumératives complexe, réelle et tropicale*, arXiv: 0808.0019, 2008.

[Cap98] Lucia Caporaso, *Counting curves on surfaces: a guide to new techniques and results*, European Congress of Mathematics, Budapest, July 22-26, 1996, vol. 1, Birkhäuser, 1998, arXiv: alg-geom/9611029, pp. 136–149.

[Cap11] ______, *Algebraic and tropical curves: comparing their moduli spaces*, to appear in the Handbook of Moduli, edited by Gavril Farkas and Ian Morrison (2011), arXiv: 1101.4821.

[Cap12] ______, *Geometry of tropical moduli spaces and linkage of graphs*, J. Comb. Theo. Ser. A **119** (2012), 579–598, arXiv: 1001.2815.

[CH98] Lucia Caporaso and Joe Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), 345–392, arXiv: alg-geom/9608025.

[CLS11] David Cox, John Little, and Hal Schenck, *Toric varieties*, Graduate Studies in Mathematics, AMS, 2011.

[DIK00] Alexander Degtyarev, Ilia Itenberg, and Viatcheslav Kharlamov, *Real Enriques surfaces*, Springer LMN 1746, 2000.

[DK00] Alexander Degtyarev and Viatcheslav Kharlamov, *Topological properties of real algebraic varieties: du côté de chez Rokhlin*, Uspekhi Mat. Nauk **55** (2000), no. 4(334), 129–212, arXiv: math/0004134.

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.

[FH11] Georges François and Simon Hampe, *Universal families of rational tropical curves*, to appear in Canad. J. Math (2011), arXiv: 1105.1674.

[FP96] William Fulton and Rahul Pandharipande, *Notes on stable maps and quantum cohomology*, Proceedings of the 1995 Santa Cruz conference, 1996, arXiv: alg-geom/9608011.

[Gat06] Andreas Gathmann, *Tropical algebraic geometry*, Jahresbericht der DMV **108** (2006), no. 1, 3–32, arXiv: math.AG/0601322.

[GH78] Phillip A. Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[GKM09] Andreas Gathmann, Michael Kerber, and Hannah Markwig, *Tropical fans and the moduli space of rational tropical curves*, Compos. Math. **145** (2009), no. 1, 173–195, arXiv: 0708.2268.

[GKZ94] Israel M. Gelfand, Mikhail Kapranov, and Andrei Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, 1994.

[GM07a] Andreas Gathmann and Hannah Markwig, *The Caporaso-Harris formula and plane relative Gromov-Witten invariants in tropical geometry*, Math. Ann. **338** (2007), 845–868, arXiv: math.AG/0504392.

[GM07b] ———, *The numbers of tropical plane curves through points in general position*, J. Reine Angew. Math. **602** (2007), 155–177, arXiv: math.AG/0504390.

[GM08] ———, *Kontsevich's formula and the WDVV equations in tropical geometry*, Adv. Math. **217** (2008), 537–560, arXiv: math.AG/0509628.

[GMS13] Andreas Gathmann, Hannah Markwig, and Franziska Schroeter, *Broccoli curves and the tropical invariance of Welschinger numbers*, Adv. Math. **240** (2013), 520–574, arXiv: 1104.3118.

[GP98] Lothar Göttsche and Rahul Pandharipande, *The quantum cohomology of blow-ups of $\mathbb{P}^2$ and enumerative geometry*, J. Differ. Geom. **48** (1998), no. 1, 61–90, arXiv: alg-geom/9611012.

[Gro66] Alexander Grothendieck, *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. de l'IHÉS, 1966.

[GS12] Andreas Gathmann and Franziska Schroeter, *Irreducible cycles and points in special position in moduli spaces for tropical curves*, Elect. J. Comb. **19** (2012), no. 4, P26, arXiv: 1104.3307.

[Har86] Joe Harris, *On the Severi problem*, Invent. Math. **84** (1986), no. 3, 445–461.

[Har06] Robert Hartshorne, *Algebraic geometry*, Springer GTM 52, 2006.

[Her09] Matthias Herold, *Tropical orbit spaces and the moduli spaces of elliptic tropical curves*, arXiv: 0911.1491, 2009.

[Hil01] David Hilbert, *Mathematische Probleme*, Arch. Math. Phys. **1** (1901), 43–63.

[HKK+03] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow, *Mirror symmetry*, Clay Mathematics Monographs, AMS, 2003.

[HKT09] Paul Hacking, Sean Keel, and Jenia Tevelev, *Stable pair, tropical, and log canonical compactifications of moduli spaces of Del Pezzo surfaces*, Invent. Math. **178** (2009), no. 1, 173–227.

[IKS03a] Ilia Itenberg, Viatcheslav Kharlamov, and Eugenii Shustin, *Appendix to "Welschinger invariant and enumeration of real rational curves"*, arXiv: math.AG/0312142, 2003.

[IKS03b] ______, *Welschinger invariant and enumeration of real rational curves*, Int. Math. Res. Not. **49** (2003), 2639 – 2653, arXiv: math/0303378.

[IKS04] ______, *Logarithmic equivalence of Welschinger and Gromov-Witten invariants*, Russ. Math. Surv. **59** (2004), no. 6, 1093–1116, arXiv: 0407188.

[IKS07] ______, *New cases of logarithmic equivalence of Welschinger and Gromov-Witten invariants*, Proc. Steklov Math. Inst. **258** (2007), 65–73, arXiv: 0612782.

[IKS09] ______, *A Caporaso-Harris type formula for Welschinger invariants of real toric Del Pezzo surfaces*, Comment. Math. Helv. **84** (2009), 87–126, arXiv: math.AG/0608549.

[IKS10] ______, *Welschinger invariants of small non-toric Del Pezzo surfaces*, to appear in J. Europ. Math. Soc. (2010), arXiv: 1002.1399.

[IKS12] ______, *Welschinger invariants of real Del Pezzo surfaces of degree* ≥ 3, arXiv: 1108.3369, 2012.

[IMS09] Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin, *Tropical Algebraic Geometry*, 2nd ed., Birkhäuser, 2009.

[IV96] Ilia Itenberg and Oleg Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Math. Intelig. **18** (1996), no. 4, 19–28.

[JMM08] Anders N. Jensen, Hannah Markwig, and Thomas Markwig, *An algorithm for lifting points in a tropical variety*, Collect. Math. **59** (2008), no. 2, 129–165.

[Kap93] Mikhail M. Kapranov, *Chow quotients of Grassmannians I*, Adv. Soviet Math. (1993), 29–110, arXiv: alg-geom/9210002.

[Kle87] Steven L. Kleiman, *Intersection theory and enumerative geometry: a decade in review*, Algebraic Geometry, Bowdoin 1985, vol. 46 of Proc. Symp. Pure Math, 1987, pp. 321–370.

[KM94] Maxim Kontsevich and Yuri I. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562, arXiv: hep-th/9402147.

[KM09a] Michael Kerber and Hannah Markwig, *Counting tropical elliptic plane curves with fixed j-invariant*, Comment. Math. Helv. **84** (2009), no. 2, 387–427, arXiv: 0608472.

[KM09b] ______, *Intersecting psi-classes on tropical* $\mathcal{M}_{0,n}$, Int. Math. Res. Not. (2009), 221–240.

[Knu83] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II*, Math. Scand. **52** (1983), 161–199.

[KV07] Joachim Kock and Israel Vainsencher, *An invitation to quantum cohomology: Kontsevich's formula for rational plane curves*, Progress in Mathematics, Birkhäuser, 2007.

[Lan02] Serge Lang, *Algebra*, Springer GTM, 2002.

[Mar06] Hannah Markwig, *The enumeration of plane tropical curves*, Ph.D. thesis, TU Kaiserslautern, 2006, https://kluedo.ub.uni-kl.de/frontdoor/index/index/docId/1756.

[Mey11] Henning Meyer, *Intersection theory on tropical toric varieties and compactifications of tropical parameter spaces*, Ph.D. thesis, TU Kaiserslautern, 2011, https://kluedo.ub.uni-kl.de/frontdoor/index/index/year/2011/docId/2323.

[Mik] Grigory Mikhalkin, *Tropical geometry - lecture notes by Eric Katz*, www.ma.utexas.edu/rtgs/geomtop/rtg/notes/Mikhalkin_Tropical_Lectures.pdf.

[Mik04a] ______, *Amoebas of algebraic varieties and tropical geometry*, arXiv: math/0403015 (2004).

[Mik04b] ______, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology **43** (2004), no. 5, 1035–1065, arXiv: math/0205011.

[Mik04c] ______, *Different faces of geometry*, ch. Tropical Degeneration and the Limits of Amoebas, Springer, 2004.

[Mik05] ______, *Enumerative tropical geometry in $\mathbb{R}^2$*, J. Amer. Math. Soc. **18** (2005), 313–377, arXiv: math.AG/0312530.

[Mik06] ______, *Tropical geometry and its applications*, Proceedings of the International Congress of Mathematicians Madrid 2006 (2006), 827–852, arXiv: math/0601041.

[Mik07] ______, *Moduli spaces of rational tropical curves*, Proceedings of the Gökova Geometry-Topology Conference GGT 2006 (2007), 39–51.

[MMS11] Hannah Markwig, Thomas Markwig, and Eugenii Shustin, *Tropical curves with a singularity in a fixed point*, Man. Math. **137** (2011), no. 3-4, 383–418, arXiv: 0909.1827.

[MR09] Hannah Markwig and Johannes Rau, *Tropical descendant Gromov-Witten invariants*, Man. Math. **129** (2009), no. 3, 293–335.

[MS11] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, book in preparation, last update 2011, http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf.

[Pic99] Georg Pick, *Geometrisches zur Zahlenlehre*, Sitzungsberichte des deutschen naturwissenschaftlich-medicinischen Vereines für Böhmen "Lotos" in Prag. (Neue Folge) **19** (1899), 311–319.

[Rau08] Johannes Rau, *Intersections on tropical moduli spaces*, arXiv: 0812.3678, 2008.

[RGST05] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald, *First steps in tropical geometry*, Idempotent mathematics and mathematical physics. Vienna, 2003 (G. L. Litvinov et al., ed.), AMS, Contemp. Math. 377, 289–317, 2005, arXiv: math/0306366.

[Rib99] Paulo Ribenboim, *The theory of classical valuations*, Springer, 1999.

[RT95] Yongbin Ruan and Gang Tian, *A mathematical theory of quantum cohomology*, J. Differ. Geom. **42** (1995), no. 2, 259–367.

[Rua93] Yongbin Ruan, *Symplectic topology and extremal rays*, Geom. Funct. Ana. **3** (1993), no. 4, 395–430.

[Rua96] ______, *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. J. **83** (1996), no. 2, 461–500.

[Sch79] Hermann Schubert, *Kalkül der abzählenden Geometrie*, Teubner, Leipzig, 1879.

[Sep89] Mika Seppälä, *Complex algebraic curves with real moduli*, J. Reine Angew. Math. **387** (1989), 209–220.

[Sev21] Francesco Severi, *Vorlesungen über algebraische Geometrie, Anhang F*, Teubner, Leipzig, 1921.

[Shu06a] Eugenii Shustin, *A tropical approach to enumerative geometry*, St. Petersburg Math. J. **17** (2006), no. 2, 343–375, very similiar to arXiv: math/0211278.

[Shu06b] ______, *A tropical calculation of the Welschinger invariants of real toric Del Pezzo surfaces*, J. Alg. Geom. **15** (2006), no. 2, 285–322, arXiv: mathAG/0406099.

[Sie99] Bernd Siebert, *Algebraic and symplectic Gromov-Witten invariants coincide*, Ann. Inst. Fourier **49** (1999), no. 6, 1743–1795.

[Sil92] Robert Silhol, *Moduli problems in real algebraic geometry*, Real Algebraic Geometry: proceedings of the conference held in Rennes, France, 1991 (M. Coste, ed.), Springer LNM 1524, 1992.

[Sim88] Imre Simon, *Recognizable sets with multiplicities in the tropical semiring*, Mathematical Foundations of Computer Science. Carlsbad, 1988 (M. P. Chytil, L. Janiga, and V. Koubek, eds.), Springer LMN 324, 1988.

[Skr12] Inge Sandstad Skrondal, *Calculations of tropical Welschinger numbers via broccoli curves*, Master's thesis, University of Oslo, 2012, https://www.duo.uio.no/bitstream/handle/123456789/10747/SkrondalsMasteroppgave.pdf.

[Sol06] Jake P. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, arXiv: math/0606429 (2006).

[Sot11] Frank Sottile, *Real solutions to equations from geometry*, University Lecture Series, AMS, 2011.

[Spe05] David E. Speyer, *Tropical geometry*, Ph.D. thesis, UC Berkeley, 2005, www-personal.umich.edu/∼speyer/thesis.pdf.

[SS04] David Speyer and Bernd Sturmfels, *The tropical Grassmannian*, Adv. in Geom. **4** (2004), 389–411.

[Ste48] Jakob Steiner, *Allgemeine Eigenschaften der algebraischen Curven*, Berichte der Akad. Wiss. zu Berlin (1848), 310–315, also published in Crelle XLVII, 1854.

[Tev07] Jenia Tevelev, *Compactifications of subvarieties of tori*, Am. J. Math. **129** (2007), no. 4, 1087–1104.

[The02] Thorsten Theobald, *Computing amoebas*, Exp. Math. **11** (2002), no. 4, 513–526.

[Vai95] Israel Vainsencher, *Enumeration of n-fold tangent hyperplanes to a surface*, J. Alg. Geom. **4** (1995), 503–526, arXiv: alg-geom/9312012.

[Vir80] Oleg Viro, *Curves of degree 7, curves of degree 8 and the Ragsdale conjecture*, Soviet. Math. **Dokl. 22** (1980), 566–570.

[Wae91] Bartel van der Waerden, *Algebra, Volume II*, Springer, 1991.

[Wel03] Jean-Yves Welschinger, *Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 4, 341–344.

[Wel05a] ______, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. **162** (2005), no. 1, 195–234.

[Wel05b] ______, *Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants.*, Duke Math. J. **127** (2005), no. 1, 89–121.

[Wel07] ______, *Optimalité, congruences et calculs d'invariants des variétés symplectiques réelles de dimension quatre*, arXiv: 0707.4317, 2007.

[Wik] Wiki, *Wikipedia*, Pictures from wikipedia are proteged by the GNU license and can be copied without permission.

[Wit91] Edward Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1** (1991), 243–310.

[Zeu73] Hieronymus Georg Zeuthen, *Almindelige egenskaber ved systemer af plane kurver*, Kongelige Danske Vidensk abernes Selskabs Skrifter - Naturvidenskabelig og Mathematisk **10** (1873), 285–393.