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Large-scale Quantitative Anonymity Assessment of Tor Path Selection Algorithms Against Structural Attacks

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Abstract

In this paper, we present a rigorous methodology for quantifying the anonymity provided by Tor against a variety of structural attacks, i.e., adversaries that compromise Tor nodes and thereby perform eavesdropping attacks to deanonymize Tor users. First, we provide an algorithmic approach for computing the *anonymity impact* of such structural attacks against Tor. The algorithm is parametric in the considered path selection algorithm and is, hence, capable of reasoning about variants of Tor and alternative path selection algorithms as well. Second, we present formalizations of various instantiations of structural attacks against Tor and show that the computed anonymity impact of each of these adversaries indeed constitutes a worst-case anonymity bound for the cryptographic realization of Tor. Third, we use our methodology to conduct a *rigorous, large-scale evaluation* of Tor's anonymity which establishes worst-case anonymity bounds against various structural attacks for Tor and for alternative path selection algorithms such as DistribuTor, SelekTOR, and LASTor. This yields the first rigorous anonymity comparison between different path selection algorithms. As part of our analysis, we quantify the anonymity impact of a path selection transition phase, i.e., a small number of users decides to run an alternative algorithm while the vast majority still uses the original one. The source code of our implementation is publicly available.

1 Introduction

The Internet has grown from a small network to an omnipresent backbone of our society that manages and enables commercial, social, and political activities world-wide. The indisputable benefits of this transformation are, however, accompanied by novel privacy threats: User activities are constantly tracked and profiled, and the collected information is used for targeted advertising by industry and for dragnet surveillance at the planetary scale by almost omnipotent governmental agencies. In fact, new revelations about governmental observations and large-scale user profiling by various companies make it into the news with distressing regularity.

As a result, public interest in anonymous communication systems has vastly increased, and millions of users have started to use anonymizing proxies and VPNs to anonymously browse the web. In particular the Tor network [14, 5] has received tremendous attention in this respect, both as an end-user solution, currently serving more than 1.5 million people from all over the world, and as a building block for further anonymizing systems such as the privacy-preserving operating system Tails [3]. In Tor, a user connects to a sequence of three proxies (out of a set of currently more than six thousand volunteer proxies, called *nodes*), and thereby forms a so-called *Tor circuit*. The anonymity provided by this construction inherently depends on a user's trust in these nodes and on the likelihood of selecting honest or compromised nodes in the circuit generation phase.

Assessing this degree of anonymity for different trust assumptions has spawned a multitude of research on analyzing the impact that any compromised Tor node can have on the anonymity of a user. However, most existing works provide no rigorous bounds on the provided anonymity. They are instead restricted to empirical analyses and simulations that strive to measure the anonymity impact of malicious Tor nodes; or they only consider coarse-grained, all-or-nothing attacks that would result in immediate deanonymization. The few recent approaches that aim at rigorously quantifying the anonymity of Tor against compromised nodes are restricted in scope in that they only consider simplistic adversaries and in that they are specific to individual variants of Tor's still evolving node selection algorithm. There is a lack of generic, comprehensive framework that allows for assessing anonymity against a wide selection of structural attacks (i.e., compromising Tor nodes and thereby performing eavesdropping attacks) and for comparing these variants with each other as well as with recently proposed, alternative path selection algorithms such as DistribuTor [10], SelekTOR [21], and LASTor [7].

1.1 Our Contribution

In this paper, we present a rigorous methodology for quantifying the anonymity impact of compromised Tor nodes for any variant of Tor's path selection algorithm and alternative path selection algorithms. Our contribution is twofold: we present an algorithmic approach for computing the worst-case anonymity impact of adversaries that compromise Tor nodes, and we evaluate the anonymity impact of such adversaries for different path selection algorithms.

Computing the Anonymity Impact. Algorithmically quantifying the anonymity impact of adversaries that compromise Tor nodes in a sound manner constitutes a challenging task. We strive to go beyond the prevalently considered all-or-nothing anonymity assessments, which only consider attacks in which the adversary immediately observes both ends of a communication and, hence, achieves an immediate deanonymization. A more careful investigation shows that additional conclusions that reduce anonymity can be drawn when compromising *any* node, and these conclusions are no less influential. Tor's node selection strategies can depend on properties of the sender (e.g., by using a specific algorithm) and of the recipient (imposing requirements, such as the supported ports of the connection) of a communication. Hence, we first carefully model which *observations* any subset of Tor nodes can make. After that, we show how to compute the *anonymity impact* of such observations for the commonly considered three anonymity notions: sender anonymity, recipient anonymity, and relationship anonymity. We model arbitrary structural adversaries, in the sense of compromising nodes in order to mount eavesdropping attacks, using the novel concept of *budget adversaries*. Budget adversaries have a certain budget B and a cost function f that assigns a cost to every Tor node. They can compromise an arbitrary subset of Tor nodes as long as the aggregated node cost does not exceed the budget. We show that budget adversaries can be instantiated in various ways to model different structural attacks against Tor, ranging from k-collusion adversaries that compromise a certain number of nodes to adversaries that compromise nodes based on geographic locations and adversaries that compromise nodes subject to monetary constraints. Next, we show how to compute the worst-case anonymity impact of a budget adversary based on the anonymity impact of the observations of all individual nodes. We then prove that this computed anonymity impact for every budget adversary indeed constitutes a worst-case anonymity bound for an idealized version of Tor in the AnoA framework [8] – a recent framework for proving quantitative bounds for anonymous communication protocols; moreover, these bounds are tight for adversaries that observe exactly one Tor node. Finally, we show that our bounds also hold for the cryptographic realization of Tor, up to a negligible factor.

Large-scale Evaluation of Tor's Anonymity. We demonstrate the applicability of our methodology to large-scale analyses by performing the to date largest *rigorous anonymity evaluation* of the Tor network. Based on recent Tor Metrics data [5], we compute anonymity bounds for Tor's standard path selection algorithm and several variants thereof, including LASTor [7], SelekTOR [21], DistribuTor [10], and the uniform routing strategy against a broad variety of structural adversaries. These include including k-collusion adversaries (compromising a certain number of Tor nodes), bandwidth-adversaries (compromising Tor nodes of a certain total bandwidth), predicate adversaries (compromising all nodes based on a predicate check, such as their geographic location or the Tor version they are running), and monetary adversaries that pertain to economic considerations (compromising selected nodes based on a given price function). Our evaluation yields the first rigorous, quantitative anonymity comparison between different path selection algorithms. Moreover, we explicitly cover the impact of a path selection transition phase, i.e., a small number of users already uses an alternative path selection algorithm, while the majority still relies on Tor's standard path selection algorithm. Our evaluation shows that such pioneers are highly vulnerable, even against adversaries that compromise only few Tor nodes. Moreover, we explicitly evaluate the advantage of adversaries that mount so-called guard detection attacks. We consider this to be of particular interest since Tor recently implemented a novel guard-selecting strategy [13] that restricts users to a single guard over a 9-month period. Our results in particular show that for a set of guard nodes accounting for approximately 10% of the entry bandwidth, an adversary that solely inspects the recipient without compromising any nodes can already distinguish between pairs of senders with a 2.12% advantage, in extreme cases up to 7.65%. The source code of our implementation is available [2].

1.2 Related Work

The Tor literature is rich on proposals for new path selection algorithms. Some propose to increase the anonymity of the users [7, 15, 10] by reducing the attack vectors of an adversary that controls part of the Internet infrastructure or part of the Tor network. Others propose to increase the performance (i.e., expected latency and throughput) of the Tor network [25]. Several existing works analyze or measure the anonymity of users within the Tor network. We categorize these into works that strive for rigorous worst-case guarantees and works that empirically determine anonymity.

In the category of rigorous worst-case guarantees, [17, 8, 16] analyze Tor based on an idealized functionality and probabilistic methods. All these works assume that the path selection algorithm chooses nodes uniformly at random (which Tor does not) and, hence, do not provide rigorous guarantees. Moreover, these formalizations ignore subtle, yet potentially influential, differences in adversarial observations whenever different senders or recipients impact the probabilities of the selected Tor circuits, e.g., because they use specific parameters for Tor or even alternative path selection algorithms. Closely related to this paper, Backes et al. [10] formally analyze Tor's path selection algorithm and provide an anonymity monitor, which takes into account real-life parameters such as the number of Tor nodes and their entrusted weight within the Tor consensus. Their formalization, however, is limited to Tor's path selection algorithm and DistribuTor (their own closely related alternative path selection algorithm) and to simplistic k-collusion adversaries. Moreover, the anonymity guarantees they provide significantly overestimate the adversary's impact on Tor, as they use imprecise heuristics for calculating the anonymity impact of malicious Tor nodes on the overall guarantee. In contrast, we precisely characterize the anonymity impact of observations and only slightly (and explicitly) over-approximate our worst-case guarantees for budget-adversaries in order to improve the performance of the computation. Furthermore, our methodology for calculating guarantees directly applies to all variants of Tor's path selection algorithm and to all alternative path selections.

In the category of empirical analyses without rigorous anonymity guarantees, Johnson et al. [19] present a simulation of the Tor network, based on a probabilistic (bandwidth-based) adversary that compromises a certain percentage of Tor's bandwidth. Murdoch and Watson [20] present an analysis of proposed path selection algorithms against (bandwidth-based) adversaries that can inject malicious nodes into the Tor network, subject to a specific adversarial budget. Their work inspired the formalization of our budget adversary, with the difference that our adversary compromises existing nodes instead of adding new nodes. Other works strive to analyze Tor against network-level adversaries, which we consider a highly interesting, yet orthogonal, problem. In this area, Jaggard et al. [18] propose a path selection adaptation based on network trust to reduce the impact of network adversaries. Wacek et al. [24] analyze the impact of path selection algorithms on anonymity and performance by simulating a significant fraction of the Tor network, and then they analyze the anonymity of various path selection algorithms against (AS-level) network adversaries. The amount of analyses based on simulations and measurements further underlines the importance of a rigorous approach for quantitatively assessing the anonymity of Tor's path selection algorithm and comparing it against alternative variants.

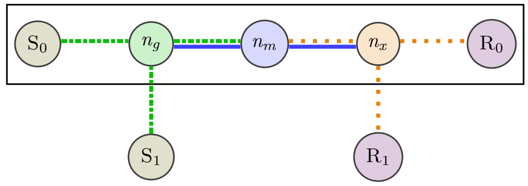

Figure 1: Upper part: A Tor circuit consisting of a sender S_0, a guard node n_g, a middle node n_m, an exit node n_x, and a recipient R_0. Colored lines for each node depict the observations this node can make. Lower part: Further observations for an alternative sender S_1 and for an alternative recipient R_1 that will be used to define different anonymity notions.

2 Observations and their Anonymity Impact

In this section, we show how to compute bounds on the anonymity provided by Tor in the presence of an adversary that can observe the communication at certain Tor nodes and potentially, at the sender or the recipient of a communication. To begin with, we characterize the possible circuit observations that such an adversary can make in Tor, introduce the anonymity notions considered in this paper, and show how to quantitatively assess the impact of circuit observations for each of these notions (Section 2.1). We then define several adversary classes that reflect different structural compromisations (Section 2.2) and show how to compute anonymity bounds for Tor against such adversaries (Section 2.3). We later instantiate these adversary classes with conceivable real-life adversaries, thereby obtaining concrete anonymity bounds for various adversarial settings.

In this section, we concentrate on the algorithmic aspects of providing anonymity bounds and use the terms *anonymity notion* and *adversary's advantage*, which are key for the overall anonymity assessment, only in an informal way. We give formal semantics to these terms in Section 3 and then rigorously prove the correctness of our computed bounds based on this semantics in Section 4.

Notation. We denote the set of Tor nodes as $\mathcal{N}$, the set of senders as $\mathcal{S}$ and the set of recipients as $\mathcal{R}$. We assume that $\mathcal{N}$, $\mathcal{S}$, and $\mathcal{R}$ are pairwise disjoint. We assume two distinguished symbols $\perp$ and $\_$ such that $\{\perp, \_\} \cap (\mathcal{N} \cup \mathcal{S} \cup \mathcal{R}) = \emptyset$. $\perp$ will later denote that no observation was made at a certain point, and $\_$ that any arbitrary observation (including $\perp$) has been made at that point.

For a probabilistic algorithm A, we write $x \leftarrow A$ to denote that x is probabilistically output by A. We write $\Pr[C; \Omega]$ to denote that event C holds true in probability space Ω. We write $[\Omega]$ to denote the carrier set of Ω, i.e., the set of all elements with non-zero probability.

We define $\phi(Y, Z)$ as $Y - Z$ if $Y > Z$, and 0 otherwise.

2.1 Defining Observations and their Anonymity Impact

Compromising any nodes involved in a Tor circuit enables the adversary to draw certain conclusions about the sender and/or the recipient of the circuit, thereby reducing their degree of anonymity.

Some of these conclusions are straightforward and result in immediate deanonymization: if the guard node is compromised, the sender is trivially deanonymized as the origin of the communication; similarly, compromising the exit node unveils the recipient as the destination of the communication. Existing papers are typically limited to such all-or-nothing observations. However, additional conclusions can be drawn when compromising *any* node, and these conclusions are no less influential. For instance, if the adversary observes or knows that the sender communicates over a specific port, all exit nodes can be excluded that do not support this port choice, and hence, any communication that involves excluded exit nodes cannot originate from that sender. Moreover, excluding exit nodes influences the probability of which nodes are being selected as guard or middle nodes in this circuit by Tor's path selection algorithm. (The selection takes so-called family relationships and further constraints into account.) Technically, this means that the a-priori probability distribution over circuits induced by Tor's path selection algorithm is now replaced by an a-posteriori distribution that is conditioned on the observations of the adversary. This enables the adversary to draw further conclusions and to thereby reduce anonymity.

Observations. Now we define which observations an adversary is able to make if certain nodes are considered compromised. We use the distinguished symbol, denoted $\perp$, to capture that an observation at a certain position in the Tor circuit has not been made. Further, we define the overall impact on anonymity if a given set of nodes is considered compromised.

Definition 1 (Observations and Circuits)**.** *We define the set of* circuits *as* $\mathcal{C} := \mathcal{S} \times \mathcal{N}^3 \times \mathcal{R}$ *and the set of* observations *as* $\mathit{Obs} := (\mathcal{S} \cup \{\perp\}) \times (\mathcal{N} \cup \{\perp\})^3 \times (\mathcal{R} \cup \{\perp\})$ *for the distinguished symbol* $\perp$*. Moreover, we define the set of all observations in which at least one position is observed as* $\mathit{Obs}_{\not\perp} = \mathit{Obs} \setminus \{(\perp, \perp, \perp, \perp, \perp)\}$*.*

For a set $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$, we now define the observations $\mathcal{O}[N](c) \in \mathit{Obs}$ made by N within a considered Tor circuit c. Intuitively, whenever a node $n \in N \cap \mathcal{N}$ is part of the circuit c, then this node as well as its successor and predecessor can be observed, see Figure 1. If $n \in N \cap (\mathcal{S} \cup \mathcal{R})$ is part of the circuit c, then sender and guard node (if $n \in \mathcal{S}$) or exit node and recipient (if $n \in \mathcal{R}$) can be obseved.

Definition 2 (Circuit Observations)**.** *For* $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$*, the* circuit observation *of* N *is a function* $\mathcal{O}[N] : \mathcal{C} \to \mathit{Obs}$ *defined as follows. For* $c = (\mathrm{S}, n_g, n_m, n_x, \mathrm{R}) \in \mathcal{C}$*, we have* $\mathcal{O}[N](c) := (n_1, \ldots n_5)$ *with*

- $n_1 := \mathrm{S}$ *if* $\{\mathrm{S}, n_g\} \cap N \neq \emptyset$*; otherwise* $n_1 := \perp$*.*
- $n_2 := n_g$ *if* $\{\mathrm{S}, n_g, n_m\} \cap N \neq \emptyset$*; otherwise* $n_2 := \perp$*.*
- $n_3 := n_m$ *if* $\{n_g, n_m, n_x\} \cap N \neq \emptyset$*; otherwise* $n_3 := \perp$*.*
- $n_4 := n_x$ *if* $\{n_m, n_x, \mathrm{R}\} \cap N \neq \emptyset$*; otherwise* $n_4 := \perp$*.*
- $n_5 := \mathrm{R}$ *if* $\{n_x, \mathrm{R}\} \cap N \neq \emptyset$*; otherwise* $n_5 := \perp$*.*

We call N *the* observation points *of* $\mathcal{O}$*.*

We furthermore define path selection algorithms.

Definition 3 (Path Selection Algorithms)**.** *A probabilistic algorithm* $\mathsf{ps} \colon \mathcal{S} \times \mathcal{R} \to \mathcal{S} \times \mathcal{N}^3 \times \mathcal{R}$ *is a* path selection algorithm *if the following holds: for all* $\mathrm{S} \in \mathcal{S}$ *and* $\mathrm{R} \in \mathcal{R}$*, we have that if* $(\mathrm{S}', n_1, n_2, n_3, \mathrm{R}') \in [\mathsf{ps}(\mathrm{S}, \mathrm{R})]$ *then* $\mathrm{S} = \mathrm{S}'$ *and* $\mathrm{R} = \mathrm{R}'$*, and* n_1, n_2, n_3 *are pairwise different.*

Anonymity Notions. We consider three common notions of communication anonymity α in this paper: *sender anonymity* ($\alpha = \alpha_{\mathrm{SA}}$, i.e., determine who is sending a message), *recipient anonymity* ($\alpha = \alpha_{\mathrm{RA}}$, i.e., determine to whom a message is being sent), and *relationship anonymity* ($\alpha = \alpha_{\mathrm{REL}}$, i.e., determine a correlation between sender and recipient). Each of these notions is defined as the (in-)ability of an adversary to distinguish two *scenarios* that differ in their involved senders and recipients. This follows the established concept of indistinguishability-based definitions in cryptography (e.g., IND-CCA secure encryption): one of these two scenarios is selected at random, a Tor circuit is created for this scenario, and the adversary is then allowed to make observations for this circuit depending on the set of compromised nodes. The adversary knows the set-up of both scenarios, makes its observations and then has to decide which scenario it currently observes. The reduction of anonymity is then defined as the *adversary's advantage*, i.e., as the probability of correctly distinguishing both scenarios.

Each of these three notions requires its own two scenarios to define the adversary's advantage with respect to this notion. This is illustrated in Figure 1: for sender anonymity, an additional sender S_1 is considered, i.e., the two scenarios differ in the sender, but share the same recipient R_0. In addition to its observations from compromised nodes, the adversary is allowed to compromise the recipient R_0 and should be able to distinguish if the communication originates at S_0 or at S_1. Similarly, an additional recipient R_1 is considered for recipient anonymity, i.e., the two scenarios differ in the recipient, but share the same sender S_0; the

$$\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(N) := \sum_{o \in Obs_{\not\perp}} \phi\Big(\Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)\right], \Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_0)\right] \Big);$$

$$\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(N) := \sum_{o \in Obs_{\not\perp}} \phi\Big(\Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)\right], \Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_1)\right] \Big);$$

$$\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(N) := \sum_{o \in Obs_{\not\perp}} \phi\Big(\big(\Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)\right] + \Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_1)\right] \big)/2,$$
$$\big(\Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_1)\right] + \Pr\left[o = \mathcal{O}[N](c), c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_0)\right] \big)/2 \Big).$$

Figure 2: Definition of $\mathbf{Impact}^{\mathsf{obs}}_X(N)$ to define observation impact (Definition 4)

adversary additionally compromises[1] the sender S_0 and tries to tell R_0 and R_1 apart. Capturing the absence of correlations to define relationship anonymity is more involved. We consider both an additional sender S_1 and an additional recipient R_1: The first relationship anonymity scenario considers the two cases that S_0 communicates with R_0 and that S_1 communicates with R_1; the second scenario considers the communication from S_0 to R_1 and from S_1 to R_0. After the scenario has been selected, one of the two described cases for this scenario is chosen uniformly at random; then a Tor circuit is created for this case and the adversary can make its observations for this circuit.

Anonymity Impact of Observations. Any circuit observation contributes information that helps an adversary to distinguish the two scenarios of the considered anonymity notion. For formally defining this *observation impact* for a given path selection algorithm ps, consider two senders $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$ and two recipients $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$, and let $\mathrm{S}_a \in \{\mathrm{S}_0, \mathrm{S}_1\}$ create a circuit to communicate with $\mathrm{R}_b \in \{\mathrm{R}_0, \mathrm{R}_1\}$. Then, the observation impact $\mathbf{Impact}^{\mathsf{obs}}_X(N)$ for the path selection algorithm ps, the anonymity notion α_X, the considered senders $\mathrm{S}_0, \mathrm{S}_1$ and recipients $\mathrm{R}_0, \mathrm{R}_1$, and a set of observation points $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$, is defined as the aggregated difference of all circuit observation probabilities for the respective scenarios of the considered anonymity notion.[2]

Definition 4 (Observation Impact). *Let $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$ denote two senders, let $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$ denote two recipients, let $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ denote a set of observation points, let ps denote a path selection algorithm, and let α_X for $X \in \{\mathrm{SA}, \mathrm{RA}, \mathrm{REL}\}$ be an anonymity notion. Then, $\mathbf{Impact}^{\mathsf{obs}}_X(N)$, as defined in Figure 2, denotes the* observation impact *of N for α_X and $\mathsf{ps}, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1$.*

For singletons $N = \{n\}$, we write $\mathbf{Impact}^{\mathsf{obs}}_X(n)$ instead of $\mathbf{Impact}^{\mathsf{obs}}_X(\{n\})$.

The empty observation. Although we exclude the empty observation $(\perp, \perp, \perp, \perp, \perp)$ from our computations, we will not exclude it from the guarantee itself. An adversary with some background knowledge that knows about the existence of a Tor circuit, but that makes no observation at all can still learn that none of its compromised Tor nodes were used in the circuit, which, roughly speaking might be more likely for one of the scenarios. We will discuss this in more detail after presenting Definition 10.

2.2 Defining Structural Compromisation

In this section, we define different classes of structural adversaries that statically compromise a certain subset of Tor nodes. For conveniently reasoning about different such adversaries in a unified manner, we define the

[1] Note that we only consider eavesdropping adversaries. In our methodology *compromising* any sender, Tor node or recipient means that the adversary sees all incoming and outgoing traffic. In particular, it cannot copy, modify or relay messages to other destinations and in case of the sender it also cannot learn the sender's secret keys.

[2] For the sake of readability, we did not explicitly include $\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1$ and ps as additional parameters of $\mathbf{Impact}^{\mathsf{obs}}_X(N)$ but consider them clear from the context, similarly in the upcoming definitions. Where necessary to clarify $\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1$ we write $\mathbf{Impact}^{\mathsf{obs}}_{X,\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_0,\mathrm{R}_1}(N)$ for this and upcoming definitions.

concept of a *budget adversary*.

Definition 5 (Budget Adversary)**.** *Given a* cost function $f\colon \mathcal{N} \to \mathbb{N} \cup \{\infty\}$ *and a* budget $B \in \mathbb{N}$*, an adversary is called a* budget adversary $\mathcal{A}_f^B$ *if it can compromise arbitrary sets of Tor nodes* $N \subseteq \mathcal{N}$*, as long as* $\sum_{n \in N} f(n) \leq B$.

We provide several instantiations for budget adversaries.

Definition 6 (k-collusion Adversary)**.** *A* k-collusion adversary *is a budget adversary* $\mathcal{A}^k_{f_{\mathsf{KofN}}}$ *that compromises up to k nodes of its choice, i.e.,* $f_{\mathsf{KofN}}(n) := 1$ *for* $n \in \mathcal{N}$.

Definition 7 (Predicate Adversary)**.** *A* predicate adversary *is a budget adversary* $\mathcal{A}^1_{f_P}$ *that compromises all nodes that fulfill a given predicate P, i.e.,* $f_P(n) := 0$ *if* $P(n) = \mathsf{true}$*, and* $f_P(n) := \infty$ *otherwise.*

Examples of predicate adversaries include *geographic adversaries* that compromise all nodes within a certain country or a collaboration of countries, *Tor-Version adversaries* that can exploit vulnerabilities of specific versions of the Tor software and can compromise all nodes that run this version, and *subnet adversaries* that compromise all Tor nodes within a specific IP-subnet.

Definition 8 (Bandwidth Adversary)**.** *A resource-constrained bandwidth adversary, or* bandwidth adversary *for short, is a budget adversary* $\mathcal{A}^B_{f_{\mathtt{BW}}}$ *that compromises an arbitrary set of Tor nodes with at most an overall bandwidth of B, i.e., for* $n \in \mathcal{N}$*, we have* $f_{\mathtt{BW}}(n) := n.\mathtt{BW}$ *for* $n \in \mathcal{N}$*, where* $n.\mathtt{BW}$ *denotes the bandwidth of node n.*

This adversary model allows us to provide anonymity bounds in the presence of adversaries that manage to observe a certain percentage of all traffic within the Tor network, e.g., by adding fake nodes or by assuming control over existing nodes.

Definition 9 (Monetary Adversary)**.** *A* monetary adversary *is a budget adversary* $\mathcal{A}^B_{f_\$}$ *that compromises Tor nodes with a monthly monetary maintenance and renting cost of at most B. For a set of providers* $\mathcal{P}$ *and a function* $\mathsf{price}\colon \mathcal{P} \times \mathbb{N} \to \mathbb{N}$ *that assigns a price for each provider and offered bandwidth, we have* $f_\$(n) := \mathsf{price}(n.\mathtt{provider}, n.\mathtt{BW})$ *for* $n \in \mathcal{N}$*, where* $n.\mathtt{provider}$ *and* $n.\mathtt{BW}$ *denotes the provider and the bandwidth of node n, respectively.*

Monetary adversaries reflect adversaries with a limited budget for the operational cost of running Tor nodes.

2.3 Anonymity Impact of a Budget Adversary

We now combine our formalization of observations and of their anonymity impact from Section 2.1 with our definition of a budget adversary. Thereby, we compute a bound on the anonymity impact of a budget adversary on Tor's path selection algorithm for each of the three considered anonymity notions.

A straight forward method for computing a bound on the impact of a budget adversary is to enumerate all possible sets of nodes that this adversary could compromise, to compute the observational impact of every one of these sets, and to output the maximal such impact. To the best of our knowledge this in practice requires a huge computational effort, as for most budget adversaries, the intricate nature of the observational impact would require us to compute the impact of each such set sequentially. In this paper we instead derive a methodology to soundly approximate the anonymity impact of each individual Tor node. After calculating these impacts once, we can then apply existing integer linear maximization techniques to give a bound on the impact of the maximal set of nodes.

We distinguish two kinds of impact in the following that contribute to the overall anonymity impact of a Tor node: *direct anonymity impact* and *indirect anonymity impact*. Direct impact considers those information gained from observing compromised and honest nodes, senders and recipients. Indirect impact is more subtle: it captures what the adversary can additionally learn from the *absence* of observations, i.e., which compromised Tor nodes were *not* used in the circuit, and from which it can hence draw corresponding conclusions. In the following, we elaborate on both cases separately first.

Direct Anonymity Impact. The direct anonymity impact of a Tor node represents the observation impact (Definition 4) of this node in all circuits in which it is present, i.e., of all observations that contain the node. For sender anonymity and recipient anonymity, we consider only one node at a time, whereas for relationship anonymity, we also consider observations made by pairs of nodes. In the following, we explain this strategy for each of the anonymity notions.

For the direct anonymity impact on sender anonymity, a compromised guard node is sufficient. Consequently, each direct impact made by a compromised guard node in combination with any other node is also already made by this guard node alone. Each observation made by a compromised middle node in combination with the compromised recipient equals the observation made by a compromised middle node and a compromised exit node in the same circuit. Thus, for the direct impact it suffices to consider each compromised node n individually by calculating $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(n)$. For the direct anonymity impact on recipient anonymity, the reasoning is analogous to the reasoning for sender anonymity, where for every node n we now calculate $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n)$. For relationship anonymity, the observations made by all three nodes in a circuit equals the observation made by the guard node and the exit node. Thus, it suffices to consider all observations in which one or two nodes are compromised. We divide the set of all observations into the set of observations made by individual nodes and the observations made by two nodes. For the first set of observations, analogously to sender anonymity and recipient anonymity, we consider each compromised n individually by calculating $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(n)$. For additionally considering the second set of observations, we define two helper functions $\mathbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}$ and $\mathbf{Impact}^{\mathsf{combined}}_{\mathrm{REL}}$, see Figure 3. The intuition behind these functions is as follows:

- $\mathbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(\{n, n'\})$ defines the direct impact that a pair of nodes n and n' have on relationship anonymity. Formally, this is defined as the observation impact of $\{n, n'\}$ over all observations $o \in Obs_2$, i.e., all observations that are made by two or more compromised nodes in a circuit.

- $\mathbf{Impact}^{\mathsf{combined}}_{\mathrm{REL}}(n)$ soundly combines the observational impact $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(n)$ of n individually with a worst-case approximation over the direct impact of every pair of n with other compromised nodes n'.

Indirect Anonymity Impact. The indirect anonymity impact of a Tor node represents the observation impact of this node on all circuits in which it is not present. The observation that a compromised node is not present in a circuit can significantly modify the anonymity impact of an observation, as illustrated by the following example.

Example 1 (Indirect Impact)**.** *Consider the following set of circuits:* $X_{\mathrm{S}_0,n_g,n_m} := \{c = (\mathrm{S}, {n_g}', {n_m}', {n_x}', \mathrm{R}) \mid {n_g}' = n_g \wedge {n_m}' = n_m \wedge \mathrm{S} = \mathrm{S}_0\}$ *for a guard node* n_g *and a middle node* n_m*. On each circuit* $c \in X_{\mathrm{S},n_g,n_m}$*, an adversary that only compromises the guard makes the observation* $o := \mathcal{O}[\{n_g\}](c) = (\mathrm{S}_0, n_g, n_m, \perp, \perp)$ *independently of the exit node and the recipient. By adding an exit node* n_x *to the set of compromised nodes, the probability for the observation* o *changes. Whenever* n_x *is the exit node of the circuit, instead of* o *the adversary will make the observation* $(\mathrm{S}_0, n_g, n_m, n_x, \mathrm{R})$*. If the compromised exit node is more likely to be used for one specific recipient* R*, then not observing the exit node in this scenario leaks information about the recipient.*

We consequently capture the impact of unobserved compromised nodes by adding the impact of their absence to any given observation made by another compromised node. To formally define the indirect impact $\mathbf{Impact}^{\mathsf{ind}}_{X}$ for anonymity notion α_X, we define several helper functions for computing $\mathbf{Impact}^{\mathsf{ind}}_{X}$, see Figure 4. The intuition behind these functions is as follows:

- $\mathbf{Impact}^{(ab,cd)}_{\mathsf{indirect}}(n_g, n_x)$ defines the mutual indirect impact that a compromised guard node n_g has on the observations of a compromised exit node n_x, and vice versa.

- $\mathbf{Impact}_{\mathsf{Rec1}}$ and $\mathbf{Impact}_{\mathsf{Rec2}}$ describe the impact of compromised nodes on observations made by a compromised recipient (as used for sender anonymity): The first notion $\mathbf{Impact}_{\mathsf{Rec1}}$ considers the im-

$$\textbf{Impact}^{\mathsf{combined}}_{\mathrm{REL}}(n, \mathcal{A}^B_f) := \textbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(n) + \max_{K \overset{B-f(n),f}{\subseteq} \mathcal{N}} \left(\sum_{n' \in K} \textbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(\{n, n'\}) \right)$$

$$\textbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(N) := \sum_{o \in Obs_2} \phi\Big(\big(\Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)] + \Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_1)] \big)/2,$$
$$\big(\Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_1)] + \Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_0)] \big)/2 \Big)$$

where the set of observations that are made by two or more compromised nodes in a circuit is defined as $Obs_2 = \{\mathrm{S}_0, \mathrm{S}_1\} \times \mathcal{N}^3 \times \{\perp\} \cup \{\perp\} \times \mathcal{N}^3 \times \{\mathrm{R}_0, \mathrm{R}_1\} \cup \{\mathrm{S}_0, \mathrm{S}_1\} \times \mathcal{N}^3 \times \{\mathrm{R}_0, \mathrm{R}_1\}$

Figure 3: Helper functions for the direct impact of nodes, as used in Figure 6.

pact that any individual (compromised) node may have on the (lack of) observations of a compromised recipient; the second one $\textbf{Impact}_{\mathsf{Rec2}}$ bounds the maximal error in calculating the first one.

- $\textbf{Impact}_{\mathsf{Sen1}}$ and $\textbf{Impact}_{\mathsf{Sen2}}$ analogously describe the impact of compromised nodes on observations made by a compromised sender (as used for recipient anonymity).

Based on these helper functions, the indirect impact $\textbf{Impact}^{\mathsf{ind}}_X$ is defined in Figure 5. We explain this definition for all three cases of X:

Indirect anonymity impact for sender anonymity. $\textbf{Impact}^{\mathsf{ind}}_{\mathrm{SA}}$ is computed by combining the indirect impact of compromised guard nodes on observations made by exit nodes ($\textbf{Impact}^{(10,00)}_{\mathsf{indirect}}$) with the impact of compromised guard nodes and middle nodes on the observations made by the recipient alone ($\textbf{Impact}_{\mathsf{Rec1}}$ and $\textbf{Impact}_{\mathsf{Rec2}}$). Whenever a compromised node is more likely to be used as a guard node (or middle node) by the sender S_1, then *not* observing this node is is more likely when the sender is S_0.

Indirect anonymity impact for recipient anonymity. $\textbf{Impact}^{\mathsf{ind}}_{\mathrm{RA}}$ is computed by combining the indirect impact of compromised exit nodes on observations made by guard nodes ($\textbf{Impact}^{(01,00)}_{\mathsf{indirect}}$) with the impact of compromised middle nodes and exit nodes on the observations made by the sender alone ($\textbf{Impact}_{\mathsf{Sen1}}$ and $\textbf{Impact}_{\mathsf{Sen2}}$). Whenever a compromised node is more likely to be used as an exit node (or middle node) for contacting the recipient R_1, then *not* observing this node is is more likely when the recipient is R_0.

Indirect anonymity impact for relationship anonymity. $\textbf{Impact}^{\mathsf{ind}}_{\mathrm{REL}}$ is computed by combining all possible ways in which compromised guard nodes can impact observations of compromised exit nodes and vice versa. The observations of compromised guard nodes contain the sender of a communication and thus the adversary only needs to find out the recipient, making these cases comparable to the case of recipient anonymity; the observations of compromised exit nodes contain the recipient of a communication and thus the adversary only needs to find out the sender of a communication, making these cases comparable to sender anonymity. Since for relationship anonymity the set of observation points N neither contains any of the senders or recipients, the indirect impacts of senders and recipients do not apply here.

Defining the Anonymity Impact. We finally give the overall definition of *anonymity impact*; we will explain it for the individual anonymity notions in detail after the definition.

Definition 10 (Anonymity Impact). *Let* $\mathrm{S}_0, \mathrm{S}_1$ *be two senders, let* $\mathrm{R}_0, \mathrm{R}_1$ *be two recipients, let* ps *be a path selection algorithm, let* α_X *for* $X \in \{\mathrm{SA}, \mathrm{RA}, \mathrm{REL}\}$ *be an anonymity notion, and let* $\mathcal{A}^B_f$ *be a budget adversary. Then,* $\textbf{Impact}_X(\mathcal{A}^B_f)$*, as defined in Figure 6, defines the* anonymity impact *of* $\mathcal{A}^B_f$ *for* α_X *and* $\mathsf{ps}, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1$.

The computation of the anonymity impact $\textbf{Impact}_X(\mathcal{A}^B_f)$ depends on the considered anonymity notion α_X as follows.

$$\mathbf{Impact}^{(ab)(cd)}_{\mathsf{indirect}}(n_g, n_x) := \sum_{n_m \in \mathcal{N}} \phi(P^{ab}_{n_g,n_m,n_x}, P^{cd}_{n_g,n_m,n_x})$$
$$\mathbf{Impact}_{\mathsf{Rec1}}(n) := \sum_{n_x \in \mathcal{N}} \phi(\sum_{n' \in \mathcal{N}} (P^{10}_{n,n',n_x} + P^{10}_{n',n,n_x}), \sum_{n' \in \mathcal{N}} (P^{00}_{n,n',n_x} + P^{00}_{n',n,n_x}))$$
$$\mathbf{Impact}_{\mathsf{Rec2}}(n, n') := \sum_{n_x \in \mathcal{N}} \phi(P^{00}_{n,n',n_x} + P^{00}_{n',n,n_x}, P^{10}_{n,n',n_x} + P^{10}_{n',n,n_x})$$
$$\mathbf{Impact}_{\mathsf{Sen1}}(n) := \sum_{n_g \in \mathcal{N}} \phi(\sum_{n' \in \mathcal{N}} (P^{01}_{n_g,n,n'} + P^{01}_{n_g,n',n}), \sum_{n' \in \mathcal{N}} (P^{00}_{n_g,n,n'} + P^{00}_{n_g,n',n}))$$
$$\mathbf{Impact}_{\mathsf{Sen2}}(n, n') := \sum_{n_g \in \mathcal{N}} \phi(P^{00}_{n_g,n,n'} + P^{00}_{n_g,n',n}, P^{01}_{n_g,n,n'} + P^{01}_{n_g,n',n})$$

where $$N \overset{B,f}{\subseteq} \mathcal{N} := N \subseteq \mathcal{N} s.t. \sum_{n \in N} f(n) \leq B$$
$$P^{ab}_{n_g,n_m,n_x} := \Pr\left[(\mathrm{S}_a, n_g, n_m, n_x, \mathrm{R}_b) = c; c \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)\right]$$

Figure 4: Helper functions for the indirect impact of nodes, as used in Figure 5.

Sender anonymity. The impact of any budget adversary $\mathcal{A}^B_f$ on sender anonymity constitutes at most the aggregated observation impact of the optimal set of compromised nodes together with the observation impact $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(\mathrm{R}_0)$ of the compromised recipient R_0 (which is assumed for sender anonymity) and the indirect impact on sender anonymity, c.f., Equations (1) in Figures 5 and 6. For sender anonymity we do not need to consider the empty observation $o = (\perp, \perp, \perp, \perp, \perp)$, as we can soundly assume that the recipient is always compromised, which always leads to a non-empty observation.

Recipient anonymity. Analogously, the impact of any budget adversary $\mathcal{A}^B_f$ on recipient anonymity is, at most, the aggregated observation impact of the optimal set of compromised nodes together with the observation impact $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0)$ of the compromised sender S_0 (which is assumed for recipient anonymity) and the indirect impact on recipient anonymity, c.f., Equations (2) in Figures 5 and 6. For recipient anonymity we do not need to consider the empty observation $o = (\perp, \perp, \perp, \perp, \perp)$, as we can soundly assume that the sender is always compromised, which always leads to a non-empty observation.

Relationship anonymity. Analogously, the impact on any budget adversary $\mathcal{A}^B_f$ on relationship anonymity is, at most, the aggregated observation impact of the optimal set of compromised nodes and the indirect impact on relationship anonymity, c.f., Equations (3) in Figures 5 and 6. In contrast to sender anonymity and recipient anonymity, we need to consider a combination of nodes for the direct impact of relationship anonymity, see $\mathsf{combined}_{\mathrm{REL}}$ in Figure 4.

For relationship anonymity we technically assume that the empty observation $o = (\perp, \perp, \perp, \perp, \perp)$ has an anonymity impact of zero, while practically capturing the empty observation by calculating the anonymity impact in both directions, i.e., not only computing a guarantee for the scenarios with $b = 0$ (S_0 communicates with R_0 or S_1 communicates with R_1) against the scenarios with $b = 1$ (S_0 communicates with R_1 or S_1 communicates with R_0), but also *vice versa*.

Precision of our Calculation. For all circuit observations made by sets of compromised nodes, individual nodes, pairs of nodes, and sender or recipient, our calculation of $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(N)$, $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(N)$ and $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(N)$ precisely captures the anonymity impact for the respective notion. However, when we ag-

$$
\begin{aligned}
&(1)\ \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{SA}}(n,\mathcal{A}^B_f) := \mathbf{Impact}_{\mathsf{Rec1}}(n) + \max_{N \overset{B-f(n),f}{\subseteq} \mathcal{N}} \sum_{n'\in N} \Big(\mathbf{Impact}^{(10)(00)}_{\mathsf{indirect}}(n,n') + \mathbf{Impact}_{\mathsf{Rec2}}(n,n')\Big)\\
&(2)\ \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{RA}}(n,\mathcal{A}^B_f) := \mathbf{Impact}_{\mathsf{Sen1}}(n) + \max_{N \overset{B-f(n),f}{\subseteq} \mathcal{N}} \sum_{n'\in N} \Big(\mathbf{Impact}^{(01)(00)}_{\mathsf{indirect}}(n',n) + \mathbf{Impact}_{\mathsf{Sen2}}(n,n')\Big)\\
&(3)\ \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{REL},\mathrm{S}_a,\mathrm{S}_c,\mathrm{R}_b,\mathrm{R}_d}(n,\mathcal{A}^B_f) := \max_{K \overset{B-f(n),f}{\subseteq} \mathcal{N}\setminus\{n\}} \bigg(\frac{1}{2}\Big(\mathbf{Impact}^{(ad)(ab)}_{\mathsf{indirect}}(n,n') + \mathbf{Impact}^{(cb)(cd)}_{\mathsf{indirect}}(n,n')\\
&\qquad\qquad + \mathbf{Impact}^{(ad)(cd)}_{\mathsf{indirect}}(n',n) + \mathbf{Impact}^{(cb)(ab)}_{\mathsf{indirect}}(n',n)\Big)\bigg)
\end{aligned}
$$

Figure 5: Definition of indirect impact (for Definition 10)

$$
\begin{aligned}
&(1)\ \mathbf{Impact}_{\mathrm{SA}}(\mathcal{A}^B_f) := \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(\mathrm{R}_0) + \max_{N \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(n) + \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{SA}}(n,\mathcal{A}^B_f)\Big)\\
&(2)\ \mathbf{Impact}_{\mathrm{RA}}(\mathcal{A}^B_f) := \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0) + \max_{N \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{RA}}(n,\mathcal{A}^B_f)\Big)\\
&(3)\ \mathbf{Impact}_{\mathrm{REL}}(\mathcal{A}^B_f) := \max\big(\delta_{\mathrm{REL}}(\mathcal{A}^B_f,0,1), \delta_{\mathrm{REL}}(\mathcal{A}^B_f,1,0)\big)\\
&\text{where } \delta_{\mathrm{REL}}(\mathcal{A}^B_f,b,d) := \max_{N \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N} \Big(\mathbf{Impact}^{\mathsf{combined}}_{\mathrm{REL},\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_b,\mathrm{R}_d}(n,\mathcal{A}^B_f) + \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{REL},\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_b,\mathrm{R}_d}(n,\mathcal{A}^B_f)\Big)
\end{aligned}
$$

Figure 6: Definition of $\mathbf{Impact}_X(\mathcal{A}^B_f)$, in order to define anonymity impact (Definition 10)

gregate the impact of individual nodes in $\mathbf{Impact}_X$ in order to derive our overall bounds for the anonymity impact of a budget adversary, we might count observations made for the same circuit more than once, and therefore over-approximate the impact of the individual observations. Moreover, our bound on the (indirect) impact of nodes soundly overestimates the impact. We decided to accept this slight over-approximation for reasons of performance and scalability, as it allows us to compute bounds for budget adversaries based on each node individually; otherwise, we would have to combine all possible observations of all subsets of the set of nodes that fall within the budget. Furthermore, we implicitly assume that an adversary can mount traffic correlation attacks with perfect accuracy, i.e., whenever it observes traffic at two different points in the Tor network, we assume that the adversary can determine if this traffic belongs to the same Tor circuit. This assumption is motivated by the high accuracy achieved by recent work on traffic correlation attacks [23, 19, 20]; yet, it still constitutes an over-approximation.

3 Theoretical Underpinning

We now provide a rigorous semantics for the concepts that we informally used in the previous section, such as anonymity notions and the adversary's advantage. To this end, we cast all required formalizations in the AnoA framework [8], a framework for computing quantitative bounds for anonymous communication systems. By means of this embedding into AnoA, we show that $\mathbf{Impact}_X(\mathcal{A}^B_f)$, as defined in Definition 10, computes a bound for the notion of adversary's advantage in the AnoA framework for budget adversaries $\mathcal{A}^B_f$. In addition, we show the secure compositionality of budget adversaries in AnoA, which might be of independent interest.

3.1 Game-based Anonymity in AnoA

Anonymity Notions. The formalization of the three anonymity notions in AnoA closely follows the informal description that we gave in Section 2.1 as a challenge-response game, in which the adversary has to distinguish two scenarios. Formally, an *anonymity notion* is a function α that receives as inputs two senders: S_0 and S_1, two recipients: R_0 and R_1, and a so-called challenge bit b. It then selects one sender and one recipient, based on the challenge bit and the considered anonymity notion. For relationship anonymity, this selection is probabilistic.

Sender anonymity α_{SA}. The sender anonymity function α_{SA} selects the sender according to the challenge bit and always considers the same recipient R_0:

$$\alpha_{\mathrm{SA}}(\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, b) := (\mathrm{S}_b, \mathrm{R}_0).$$

Recipient anonymity α_{RA}. The recipient anonymity function α_{RA} selects the recipient according to the challenge bit and always considers the same sender S_0:

$$\alpha_{\mathrm{RA}}(\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, b) := (\mathrm{S}_0, \mathrm{R}_b).$$

Relationship anonymity α_{REL}. The relationship anonymity function α_{REL} selects one of the four possible sender-recipient combinations as follows: if $b = 0$, the function randomly selects one of the two pairs $(\mathrm{S}_0, \mathrm{R}_0)$ or $(\mathrm{S}_1, \mathrm{R}_1)$; if $b = 1$, it randomly selects between $(\mathrm{S}_0, \mathrm{R}_1)$ and $(\mathrm{S}_1, \mathrm{R}_0)$. In short, we obtain

$$\alpha_{\mathrm{RA}}(\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, b) := (\mathrm{S}_{b'}, \mathrm{R}_{b \oplus b'}); b' \leftarrow_R \{0, 1\}.$$

Game-based Anonymity Definition. The definition of the AnoA challenger is the final building block for the definition of the adversary's advantage in AnoA as a challenge-response game. In the AnoA framework, the challenger receives as input an anonymity notion α, a bound on the permitted challenge-messages (see below), two senders, two recipients, a path selection algorithm and the challenge bit. It then simulates the Tor protocol for the sender-recipient scenario selected by α. The adversary interacts with the challenger in order to determine which scenario is being simulated. The adversary knows all inputs to the challenger up to an uncertainty of one bit (the challenge bit b). We now describe the challenger in detail.

The AnoA challenger. The challenger $\textsc{Ch}$ is defined in Figure 7. As described above, it expects as inputs the anonymity notion α, a bound γ on the permitted challenge-messages, two senders S_0, S_1, two recipients R_0, R_1, the path selection algorithm ps and the challenge bit b. The challenger initially waits for a set $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ of compromised Tor nodes, senders and recipients. The challenger first removes illegitimate compromisation requests for senders and recipients: $\mathrm{S}_0, \mathrm{S}_1$ is removed from N for sender anonymity, $\mathrm{R}_0, \mathrm{R}_1$ is removed from N for recipient anonymity, and both $\mathrm{S}_0, \mathrm{S}_1$ and $\mathrm{R}_0, \mathrm{R}_1$ are removed from N for relationship anonymity, which reflects the respective scenarios. Then, it accepts two types of messages from the adversary: *challenge-messages*, denoted as $(\mathsf{challenge}, m)$ in Figure 7, that trigger that a challenge message is sent, and *input-messages*, denoted as $(\mathsf{input}, \mathrm{S}, m, \mathrm{R})$ in Figure 7, that send additional messages m between senders S and recipients R:

- *Challenge-messages:* Upon receiving a message $(\mathsf{challenge}, m)$, the challenger increases Ψ and only proceeds if Ψ is still less than or equal to γ. It then computes the anonymity notion α on $(\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1)$ and the challenge bit b and obtains a sender-recipient pair $(\mathrm{S}^*, \mathrm{R}^*) \in \{\mathrm{S}_0, \mathrm{S}_1\} \times \{\mathrm{R}_0, \mathrm{R}_1\}$. The challenger then simulates the Tor protocol by creating a new Tor circuit (n_g, n_m, n_x) from sender S^* to recipient R^*, and then sends the message m from S^* to R^* using Tor. We abbreviate this using the subroutine **SimulateTor**$(\mathrm{S}^*, m, \mathrm{R}^*)$ in Figure 7. Whenever a node n involved in the constructed circuit is considered compromised, i.e., $n \in N$, then the adversary is given the transcript of this communication, i.e., the messages n sent and received in this circuit.
- *Input-messages:* Upon receiving a message $(\mathsf{input}, \mathrm{S}, m, \mathrm{R})$, the challenger calls the subroutine **SimulateTor**$(\mathrm{S}, m, \mathrm{R})$, as described above. Input-messages, hence, capture additional information the adversary may have about the communication contents in the Tor network, e.g., by starting Tor communications on its own.

We now define the *reduction of anonymity* for α as the adversary's advantage in this game.

Definition 11 (Reduction of Anonymity; Advantage)**.** *Let α be an anonymity notion, $\gamma \in \mathbb{N}$, $\mathrm{S}_0, \mathrm{S}_1$ two senders, $\mathrm{R}_0, \mathrm{R}_1$ two recipients, and ps a path selection algorithm. Then, the* adversary's advantage *of an adversary $\mathcal{A}$ for these parameters is at most δ, with $0 \leq \delta \leq 1$, if for all sufficiently large $\eta \in \mathbb{N}$, we have*

$$\begin{aligned} &\Pr\left[0 = \langle \mathcal{A}(1^\eta) || \textsc{Ch}(\alpha, \gamma, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}, 0)\rangle\right] \\ \leq\ &\Pr\left[0 = \langle \mathcal{A}(1^\eta) || \textsc{Ch}(\alpha, \gamma, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}, 1)\rangle\right] + \delta. \end{aligned}$$

We say that Tor exhibits a reduction of anonymity *of at most δ under γ challenges (formally: Tor is (δ, γ)-$\mathsf{IND\text{-}ANO}$) for these parameters $\alpha, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}$ and a class A of adversaries if the adversary's advantage of all probabilistic polynomial-time adversaries $\mathcal{A} \in A$ is at most δ.*

This definition captures an eavesdropping adversary that compromises a fixed set of nodes before it starts observing the network. In particular, the adversary cannot adaptively decide which nodes to compromise.[3]

Relation to Entropy-based Anonymity Notions. Our indistinguishability-based notion reasons about the cryptographic implementation of Tor. For such cryptographic systems with their computational security guarantees, entropy-based notions, including notions that define the *effective size* of an anonymity set [22, 12], are not directly applicable. A relation between entropy-based notions and cryptographic notions might be possible along the lines of [11] that establishes a tight correspondence between the information-theoretic capacity of channels, their abstract description and finally their cryptographic instantiations. We plan to investigate this approach in the context of more comprehensive systems such as Tor in future work.

3.2 Budget Adversaries in AnoA

We now cast the notion of a budget adversary in AnoA. Intuitively, an adversary is a budget adversary if the set of compromised nodes that it sends to the challenger conforms to its budget restrictions.[4]

Definition 12 (AnoA Budget Adversary)**.** *Consider a cost function $f\colon \mathcal{N} \to \mathbb{N} \cup \{\infty\}$ and a budget $B \in \mathbb{N}$. Then a probabilistic polynomial-time adversary $\mathcal{A}_f^B$ is an* AnoA budget adversary *for f and B, if for all possible outputs N of $\mathcal{A}_f^B$ to the AnoA challenger in its first message, we have that $\sum_{n \in M} f(n) \leq B$ for $M := N \setminus (\mathcal{S} \cup \mathcal{R})$. We let A_f^B denote the class of all budget adversaries for f and B.*

All instantiations of budget adversaries defined in Section 2.2 can be cast in AnoA in an analogous manner.

As a result that we consider to be of independent interest, we show that anonymity guarantees for *individual* challenges against an AnoA budget adversary entail anonymity guarantees for an *arbitrary* (fixed) amount of challenges (for different, but related parameters). Formally, AnoA budget adversaries A_f^B are *composable* for every budget B and every cost function f.

Theorem 1 (Composition)**.** *If Tor is $(\delta, 1)$-$\mathsf{IND\text{-}ANO}$ for an anonymity notion α, two senders $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$, two recipients $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$, a path selection algorithm ps, and the class of budget adversaries A_f^B, then, Tor is also $(\gamma \cdot \delta, \gamma)$-$\mathsf{IND\text{-}ANO}$ for α, $\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}$ and A_f^B, for every $\gamma \in \mathbb{N}$.*

Proof. To show the Theorem, we leverage the composition theorem of [10]. To this end, we use the original formulation of AnoA and consider budget adversaries as wrapper machines $A_f^B(\cdot)$ that internally run an

[3]The corresponding definition in the AnoA paper [8] additionally considers a multiplicative advantage e^ε. We have set this to 1 in this paper, such that δ directly corresponds to the reduction of anonymity. Moreover, AnoA considers arbitrary probabilistic, polynomial-time Turing machines and, for technical reasons, subsequently restricts them with wrapper machines (so-called adversary classes). For the sake of presentation in our specific setting, we did not introduce the lengthy description of adversary classes, but instead restricted the adversary in the core definition and adjusted the challenger accordingly.

[4]An alternative, yet slightly more technical way of defining budget adversaries in AnoA is by using the concept of adversary classes provided by AnoA: in this case, a budget adversary would be a wrapper machine that internally runs an arbitrary adversary as a black box, but ensures that only compromisation requests N are being forwarded to the AnoA challenger that satisfy $\sum_{n \in M} f(n) \leq B$ for $M := N \setminus (\mathcal{S} \cup \mathcal{R})$. We opted for the technically simpler definition in this paper.

ANOA **Challenger** $\textsc{Ch}(\alpha, \gamma, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}, b)$

Initial message (compromisation setting)

Receive $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ as input.
if $\alpha = \alpha_{\mathrm{SA}}$, let $N := N \setminus \{\mathrm{S}_0, \mathrm{S}_1\}$.
if $\alpha = \alpha_{\mathrm{RA}}$, let $N := N \setminus \{\mathrm{R}_0, \mathrm{R}_1\}$.
if $\alpha = \alpha_{\mathrm{REL}}$, let $N := N \setminus \{\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1\}$.
$\Psi := 0$.

Upon message $(\mathsf{input}, \mathrm{S}, m, \mathrm{R})$

Run SimulateTor$(\mathrm{S}, m, \mathrm{R})$.

Upon message $(\mathsf{challenge}, m)$

$\Psi := \Psi + 1$.
if $\Psi \leq \gamma$ **then**
Compute $(\mathrm{S}^*, \mathrm{R}^*) \leftarrow \alpha(\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, b)$
Run $\mathsf{SimulateTor}(\mathrm{S}^*, m, \mathrm{R}^*)$
else abort the game.

Subroutine SimulateTor$(\mathrm{S}, m, \mathrm{R})$

Simulate the Tor protocol:
$c = (\mathrm{S}, n_g, n_m, n_x, \mathrm{R}) \leftarrow \mathsf{ps}(\mathrm{S}, \mathrm{R})$
S sends m to R via the circuit c.
for each $n \in \{\mathrm{S}, n_g, n_m, n_x, \mathrm{R}\} \cap N$ **do**
Output the transcript of n in c.

Figure 7: Definition of the AnoA Challenger

arbitrary PPT adversary $\mathcal{A}$ and ensure that the compromisation requests of $\mathcal{A}$ satisfies the constraints of a budget adversary for f and B. The composition requires that for every function f, every $B \in \mathbb{N}$ and for every anonymity function α, the adversary class A_f^B is composable as in Definition 3 from [10], i.e., that it satisfies the three properties *reliability* (the class does not start challenges on its own), *alpha-renaming* (the challenge counter holds no semantical meaning) and *simulatability* (the challenges are not handled structurally different than input messages).

Reliability: By construction, $A_f^B(\mathcal{A})$ sends messages $(\mathsf{challenge}, m)$ if and only if it receives a message $(\mathsf{challenge}, m)$ from $\mathcal{A}$. Thus, $A_f^B(\mathcal{A})$ is reliable.

Alpha-renaming: As the functional behavior of $A_f^B(\cdot)$ is completely agnostic to the challenge counter Ψ, $A_f^B(\mathcal{A})$ trivially satisfies alpha-renaming.

Simulatability: As, by construction, $A_f^B(\mathcal{A})$ only forwards challenge and input messages, we construct the following "trivial simulator" $\mathcal{S}^{\alpha}_{\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_0,\mathrm{R}_1}$ for any anonymity notion α and any two pairs of senders $\mathrm{S}_0, \mathrm{S}_1$ and recipients $\mathrm{R}_0, \mathrm{R}_1$. For a string $\vec{z} = [(z_1, b_1), \ldots, (z_n, b_n)] \in \{0,1\}^{2n}$, $\mathcal{S}^{\alpha}_{\vec{z}}$ behaves as follows. If $z_i = \mathsf{sim}$, it replaces all messages $(\mathsf{challenge}, m)$ by $(\mathsf{input}, \mathrm{S}^*, m, \mathrm{R}^*)$, where $(\mathrm{S}^*, \mathrm{R}^*) \leftarrow \alpha(\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, b_i)$. For every $\vec{z} \in \{0,1\}^{2n}$, the simulator $\mathcal{S}^{\alpha}_{\vec{z}}$ satisfies the conditions from Definition 3 in [10].

Since $A_f^B(\mathcal{A})$ satisfies all three necessary conditions, it is composable. $\square$

Remark: The proof for Theorem 1 also holds for the original, more complex description of the AnoA challenger from [8] in which the adversary chooses the senders and recipients for the challenges. In this case, the simulator computes α on the senders and recipients chosen by $\mathcal{A}$ instead of on the given fixed senders and recipients. Moreover, the proof is oblivious to the definition of the anonymity function α and also applies to the session definitions in [10].

3.3 Correctness of $\mathbf{Impact}_X$ bounds

We now show that $\mathbf{Impact}_X(\mathcal{A}_f^B)$, as defined in Definition 10, closely corresponds to the notion of adversary's advantage in the AnoA framework for budget adversaries $\mathcal{A}_f^B$, thereby establishing the output of $\mathbf{Impact}_X$ as accurate bounds for Tor against such adversaries.

We first show that our calculation of observation impact exactly corresponds to the *optimal advantage* of any adversary that makes those observations, provided that the adversary exactly compromises all nodes in a fixed set N, that it only sends a single challenge-message, and that we consider an idealization of cryptography. We call an advantage of δ *optimal* for a class of adversaries A if the adversary's advantage for sufficiently large η is at most δ for all probabilistic polynomial-time adversaries $\mathcal{A} \in A$ and if there exists an adversary $\mathcal{A} \in A$ that achieves this advantage, i.e., the less-or-equal in Definition 11 is replaced by equality for $\mathcal{A}$ for sufficiently large η.

For senders $\mathrm{S}_0, \mathrm{S}_1$, recipients $\mathrm{R}_0, \mathrm{R}_1$, and $X \in \{SA, RA, REL\}$, let $A_{\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_0,\mathrm{R}_1,X}$ be the class of probabilistic, polynomial-time adversaries that compromise $N \subseteq \mathcal{N} \cup \{\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1\}$ s.t. they are allowed by the anonymity notion, and that only send one challenge-message to $\textsc{Ch}$. More precisely, $A_{\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_0,\mathrm{R}_1,X}$ sends a set $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ to $\textsc{Ch}$ as the compromised nodes, senders and recipients, and we have that $N \cap \{\mathrm{S}_0, \mathrm{S}_1\} = \emptyset$ if $X \in \{\mathrm{SA}, \mathrm{REL}\}$ and that $N \cap \{\mathrm{R}_0, \mathrm{R}_1\} = \emptyset$ if $X \in \{\mathrm{RA}, \mathrm{REL}\}$.

To define the idealization of cryptography, we define an idealized AnoA challenger $\textsc{Ch}^*$. $\textsc{Ch}^*$ is defined exactly as $\textsc{Ch}$, with the only difference that in the subroutine **SimulateTor**, where $\textsc{Ch}$ sends the transcript of messages sent by and received by nodes and recipients $\mathcal{A}$, $\textsc{Ch}^*$ only sends the compromised nodes n of a circuit together with their predecessor and successor to $\mathcal{A}$. This models that the adversary cannot gain information about the content of encrypted messages, but that it can still determine at which point(s) in the challenge circuit it makes an observation, and then derive predecessor and successor. The only exception is that if the observation is made at the exit node or at the recipient, then the adversary would be able to see the message. However, since the adversary is allowed to choose the message in the interaction with the challenger anyway, observing it does not reveal additional information.

Lemma 1. *For every anonymity notion α_X with $X \in \{SA, RA, REL\}$, all senders $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$, all recipients $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$, and every path selection algorithm ps, we have that the optimal advantage for $A_{\mathrm{S}_0,\mathrm{S}_1,\mathrm{R}_0,\mathrm{R}_1,X}$ is equal to $\mathbf{Impact}_X^{\mathsf{obs}}(N)$.*

Proof. Let $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$ be two senders, $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$ two recipients, ps a path selection algorithm, and let $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ denote the output to $\textsc{Ch}^*$. We can ignore all senders and recipients from $(\mathcal{S} \cup \mathcal{R}) \setminus \{\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1\}$, i.e., senders or recipients that are not contained in the challenge circuit, as the observations made by these are independent of the bit b of the challenger.

Hence assume that $N \subseteq \mathcal{N} \cup \{\mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1\}$. Since $\textsc{Ch}^*$ only notifies the adversary if there is communication at entities from N, any adversary in A hence only makes observations $o \in \mathit{Obs}$, i.e., $o \in \{\mathrm{S}_0, \mathrm{S}_1, \perp\} \times (\mathcal{N} \cup \{\perp\})^3 \times \{\mathrm{R}_0, \mathrm{R}_1, \perp\}$. Since $N \cap \{\mathrm{S}_0, \mathrm{S}_1\} = \emptyset$ if $X \in \{\mathrm{SA}, \mathrm{REL}\}$ and $N \cap \{\mathrm{R}_0, \mathrm{R}_1\} = \emptyset$ if $X \in \{\mathrm{RA}, \mathrm{REL}\}$ (these cases are immediately removed by the challenger), we have that the adversary makes observations at precisely the observation points in N.

We only prove the lemma for sender anonymity; the adaptation to recipient anonymity and relationship anonymity is straightforward. We first divide the set of all observations Obs into two subsets, depending on their probability and the challenge bit. We set $\mathit{Obs}_0 := \{o \in \mathit{Obs} \mid \Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)] > \Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_0)]\}$, and $\mathit{Obs}_1 := \mathit{Obs} \setminus \mathit{Obs}_0$. An adversary $\mathcal{A}$ hence maximizes its advantage by outputting 0 if and only if it makes an observation $o \in \mathit{Obs}_0$. For $i \in \{0, 1\}$, we obtain

$$\begin{aligned} & \Pr[0 \leftarrow \langle \mathcal{A}(1^\eta) || \textsc{Ch}(\alpha, 1, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}, i) \rangle] \\ = & \sum_{o \in \mathit{Obs}_i} \Pr[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_i, \mathrm{R}_0)] \end{aligned}$$

We are left to show that the advantage of this adversary $\mathcal{A}$ is equal to $\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(N)$:

$$\begin{aligned}
& \Pr\left[0 \leftarrow \langle \mathcal{A} || \mathrm{C{\scriptstyle H}}(\alpha, 1, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}, 0)\rangle\right] \\
- \; & \Pr\left[0 \leftarrow \langle \mathcal{A} || \mathrm{C{\scriptstyle H}}(\alpha, 1, \mathrm{S}_0, \mathrm{S}_1, \mathrm{R}_0, \mathrm{R}_1, \mathsf{ps}, 1)\rangle\right] \\
= \; & \sum_{o \in Obs_0} \Pr\left[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)\right] \\
- \; & \sum_{o \in Obs_1} \Pr\left[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_0)\right] \\
= \; & \sum_{o \in Obs} \phi\Big(\Pr\left[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)\right], \\
& \qquad \Pr\left[o = \mathcal{O}[N](c); c \leftarrow \mathsf{ps}(\mathrm{S}_1, \mathrm{R}_0)\right] \Big) \\
= \; & \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(N).
\end{aligned}$$

□

Using Lemma 1 we can now show our main theorem.

Theorem 2 (Soundness)**.** *For every anonymity notion α_X with $X \in \{\mathrm{SA}, \mathrm{RA}, \mathrm{REL}\}$, all senders $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$, all recipients $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$, every path selection algorithm* ps*, and every budget B and every cost function f, Tor is $(\delta, 1)$-*IND-ANO *for the class of budget adversaries A^B_f, where $\delta = \mathbf{Impact}_X(A^B_f)$, as calculated in Section 2.2, up to a negligible additive factor.*

Since the proof of this theorem is involved, we dedicate the next section to it.

4 Proof of Soundness

We start with an overall proof outline. After that, we define the *visible elements* and the *core* of an observation, i.e., the nodes that are visible to the adversary and nodes that lead to the observation, and we describe the probability to make an observation in terms of the core and all non-core nodes (Section 4.2). Using these general concepts, we show the theorem separately for the three anonymity notions (Sections 4.3 to 4.5). Finally, we show that approximating the impacts via local maximization of the impacts is a sound over-approximation (Section 4.6).

4.1 Overall Proof Outline

The proof of Theorem 2 follows the intuition of the proofs in [10], but requires less approximation for the individual impacts of the nodes. By Theorem 1 and Lemma 22 from the full version of the AnoA framework [9], it suffices to show that the ideal functionality of Tor, $\mathcal{F}_{\mathrm{OR}}{}'$, is $(\delta, 1)$-IND-ANO for $\delta = \mathbf{Impact}_X(A^B_f)$. Once we showed this, we immediately obtain that Tor is $(\delta, 1)$-IND-ANO for $\delta = \mathbf{Impact}_X(A^B_f)$ plus a negligible additive factor, since Tor constitutes a UC-secure realization of $\mathcal{F}_{\mathrm{OR}}{}'$. Roughly, $\mathcal{F}_{\mathrm{OR}}{}'$ does not send actual onion encryptions, but only provides handles over the network and thereby eliminates cryptographic objects from the construction. For compromised nodes, $\mathcal{F}_{\mathrm{OR}}{}'$ reveals which of these handles belong together.

By Lemma 1 we know that any observation made by any set of Tor nodes (in combination with senders and recipients) N impacts anonymity by exactly $\mathbf{Impact}^{\mathsf{obs}}_X(N)$ for the anonymity notion α_X under consideration. Intuitively, the proof divides the set of all observations into distinct subsets of observations, depending on where compromised nodes are located in a circuit. Then, for every such set, we compare the impact of each observation if a set of Tor nodes (and the compromised recipient/sender) is compromised with the sum of the impacts of all compromised Tor nodes (and the compromised sender/recipient) on their own. Since we sum over all these Tor entities, for the majority of observations the impact of the sum is larger than the impact of the combined set. However, the lack of observation of certain (compromised) nodes can increase the impact that other compromised nodes have on other compromised Tor entities. In the remainder of this section, we will formally substantiate these claims.

A Simple Lemma. For our proofs we will need the following simple lemma.

Lemma 2 (Properties of ϕ)**.** *The function ϕ has the following properties:*

(i) For all $a, b, c, d \geq 0$, we have $\phi(a+b, c+d) \leq \phi(a, c) + \phi(b, d)$.

(ii) For all $a, b, c, d \geq 0$, we have $\phi(a-b, c-d) \leq \phi(a, c) + \phi(d, b)$.

Proof. We show each property via a case distinction over the two cases of the conditional within ϕ.

(i) Let a, b, c, d. We distinguish two cases:

- **Case 1:** $(a+b) \leq (c+d)$. Then $\phi(a+b, c+d) = 0 \leq 0 + 0 \leq \phi(a, c) + \phi(b, d)$.
- **Case 2:** $(a+b) > (c+d)$. Then

$$\phi(a+b, c+d) = a + b - (c+d)$$
$$= a - c + b - d \leq \phi(a, c) + \phi(b, d),$$

since by definition $\phi(X, Y) \geq X - Y$.

(ii) Let $a, b, c, d \geq 0$. We distinguish two cases:

- **Case 1:** $(a-b) \leq (c-d)$. Then $\phi(a+b, c+d) = 0 \leq 0 + 0 \leq \phi(a, c) + \phi(d, b)$.
- **Case 2:** $(a-b) > c - d$. Then

$$\phi(a-b, c-d) = a - b - (c-d) = a - c + d - b$$
$$\leq \phi(a, c) + \phi(b, d),$$

since by definition $\phi(X, Y) \geq X - Y$.

$\square$

Additional Notation for the Proof. In what follows, we write $P^{ab}[C]$ as a shortcut for $\Pr[c \in C; c \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)]$. Moreover, we write $P^{ab,cd}[C]$ instead of $\frac{1}{2}\left(P^{ab}[C] + P^{cd}[C]\right)$ for the probabilities we encounter when analyzing relationship anonymity. For an observation $o = (n_1, n_2, n_3, n_4, n_5)$ we refer to the element n_i by writing $o.n_i$. Recall that for any three Tor nodes n_g, n_m, n_x we write $P^{ab}_{n_g,n_m,n_x}$ for $\Pr[c = (\mathrm{S}_a, n_g, n_m, n_x, \mathrm{R}_b); \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)]$. .

4.2 Visible Nodes, Observation Core and Blank Observations

We continue by defining *natural* observations and by then classifying them. For every (natural) observation we define which elements are visible to the adversary. We will see that for every natural observation there is one or more sets of *core* entities that have to be compromised in order to make the respective observation. Along with these core entities we define for every observation which roles other Tor nodes have to play in order to not invalidate the observation. To this end, we define *blanked* observations, i.e., observations that only describe the type of the natural observation.

Definition 13 (Natural Observations; Visible Elements; Core Elements; Blanked Observations)**.** *We define the set NatObs of natural observations as $(\mathcal{S} \times \mathcal{N} \times \{\bot\}^3) \cup (\mathcal{S} \times \mathcal{N}^2 \times \{\bot\}^2) \cup (\{\bot\} \times \mathcal{N}^3 \times \{\bot\}) \cup (\{\bot\}^2 \times \mathcal{N}^2 \times \mathcal{R}) \cup (\{\bot\}^3 \times \mathcal{N} \times \mathcal{R}) \cup (\mathcal{S} \times \mathcal{N}^3 \times \{\bot\}) \cup (\{\bot\} \times \mathcal{N}^3 \times \mathcal{R}) \cup (\mathcal{S} \times \mathcal{N}^3 \times \mathcal{R})$.*

For every observation $o = (n_1, n_2, n_3, n_4, n_5) \in Obs$, we define the set of visible Tor entities *$V(o)$ as $V(o) := \{n_i \mid n_i \neq \bot, i \in \{1, 2, 3, 4, 5\}\}$.*

For every natural observation $o = (n_1, n_2, n_3, n_4, n_5) \in NatObs$, we define the core element $\mathsf{core}(o)$ *of o as follows:*

- *If* $o \in \mathcal{S} \times \mathcal{N} \times \{\bot\}^3$, *then* $\mathsf{core}(o) := \{\{n_1\}\}$.
- *If* $o \in \mathcal{S} \times \mathcal{N}^2 \times \{\bot\}^2$, *then* $\mathsf{core}(o) := \{\{n_2\}\}$.
- *If* $o \in \{\bot\} \times \mathcal{N}^3 \times \{\bot\}$, *then* $\mathsf{core}(o) := \{\{n_3\}\}$.
- *If* $o \in \{\bot\}^2 \times \mathcal{N}^2 \times \mathcal{R}$, *then* $\mathsf{core}(o) := \{\{n_4\}\}$.
- *If* $o \in \{\bot\}^3 \times \mathcal{N} \times \mathcal{R}$, *then* $\mathsf{core}(o) := \{\{n_5\}\}$.
- *If* $o \in \mathcal{S} \times \mathcal{N}^3 \times \{\bot\}$, *then* $\mathsf{core}(o) := \{\{n_1, n_3\}, \{n_2, n_3\}\}$.
- *If* $o \in \{\bot\} \times \mathcal{N}^3 \times \mathcal{R}$, *then* $\mathsf{core}(o) := \{\{n_3, n_4\}, \{n_3, n_5\}\}$.
- *If* $o \in \mathcal{S} \times \mathcal{N}^3 \times \mathcal{R}$, *then* $\mathsf{core}(o) := \{\{n_1, n_3, n_5\}, \{n_1, n_4\}, \{n_2, n_4\}, \{n_2, n_5\}\}$.

Naturally the core elements of an observation are always visible: For every natural observation $o \in \mathit{NatObs}$, *and for every* $\mathit{Core} \in \mathsf{core}(o)$, *we have* $\mathit{Core} \subseteq V(o)$.

For a natural observation $o = (n_1, n_2, n_3, n_4, n_5) \in \mathit{NatObs}$, *we define the corresponding* blanked observation $\mathsf{blank}(o)$ *as* $\mathsf{blank}(o) := (n'_1, n'_2, n'_3, n'_4, n'_5)$ *where* $n'_i := n_i$ *if* $n_i \neq \bot$ *and* $n'_i := \_$ *otherwise for a distinguished symbol* $\_$. *We extend the standard equality on observations* o *by treating* $\_$ *as a placeholder: an observation* $o = (n_1, n_2, n_3, n_4, n_5)$ *is equal to a blank observation* $o' = (n'_1, n'_2, n'_3, n'_4, n'_5)$ *if* $n_i = n'_i$ *for all* $n'_i \neq \_$.

Definition 14 (Observation Circuits)**.** *Let* $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ *and* $o \in \mathit{Obs}$ *be given. Then we define the set of* observation circuits $\mathcal{OC}(N, o)$ *of* N *and* o *as* $\mathcal{OC}(N, o) := \{c \in \mathcal{C} \mid o = \mathcal{O}[N](c)\}$. *Similarly, we define the set of* blanked observation circuits $\mathcal{OC}_{\mathsf{blank}}(N, o)$ *of* N *and* o *as* $\mathcal{OC}_{\mathsf{blank}}(N, o) := \{c \in \mathcal{C} \mid \mathsf{blank}(o) = \mathcal{O}[N](c)\}$.

We first show a basic lemma that states that an observation o by a set N can be made for a given circuit precisely if for every element *Core* of the core of o, we have that (i) the same observation is made for this circuit by *Core* and (ii) and the elements in $N \setminus \mathit{Core}$ make the corresponding blanked observation $\mathsf{blank}(o)$ for this circuit.

Lemma 3 (Classification of Core)**.** *Let* $o \in \mathit{NatObs}$ *be a natural observation, let* $\mathit{Core} \in \mathsf{core}(o)$, *and let* $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ *such that* $\mathit{Core} \subseteq N$ *and let* $c \in \mathcal{C}$ *be a circuit. Then we have*

$$\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$$

Proof. Let $o \in \mathit{NatObs}$ be a natural observation, let $\mathit{Core} \in \mathsf{core}(o)$, and let $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ such that $\mathit{Core} \subseteq N$ and let $c = (\mathrm{S}, n_g, n_m, n_x, \mathrm{R}) \in \mathcal{C}$ be a circuit. We distinguish the following cases depending on the structure of o.

- $o \in \mathcal{S} \times \mathcal{N} \times \{\bot\}^3$. Then $o = (\mathrm{S}', {n_g}', \bot, \bot, \bot)$ for some $\mathrm{S}' \in \mathcal{S}$, ${n_g}' \in \mathcal{N}$. By definition, $\mathsf{core}(o) = \big\{\{\mathrm{S}'\}\big\}$ and thus $\mathit{Core} = \{\mathrm{S}'\}$; since $\mathit{Core} \subseteq N$ we have $\mathrm{S}' \in N$. By definition of $\mathcal{O}$, we have

$$\mathcal{O}[N](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge \{n_g, n_m, n_x, \mathrm{R}_b\} \cap N = \emptyset$$

  Moreover, we have

$$\mathcal{O}[\{\mathrm{S}'\}](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}'$$

  Since $\mathsf{blank}(o) = (\_, \_, \bot, \bot, \bot)$, we have

$$\mathcal{O}[N \setminus \{\mathrm{S}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{n_g, n_m, n_x, \mathrm{R}\} \cap N = \emptyset$$

  Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \mathcal{S} \times \mathcal{N}^2 \times \{\perp\}^2$. Then $o = (\mathrm{S}', {n_g}', {n_m}', \perp, \perp)$ for some $\mathrm{S}' \in \mathcal{S}, {n_g}', {n_m}' \in \mathcal{N}$. By definition, $\mathsf{core}(o) = \{\{{n_g}'\}\}$ and thus $\mathit{Core} = \{{n_g}'\}$; since $\mathit{Core} \subseteq N$ we have ${n_g}' \in N$. By definition of $\mathcal{O}$, we have
$$\mathcal{O}[N](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge \{n_m, n_x, \mathrm{R}\} \cap (N \setminus \{{n_g}'\}) = \emptyset$$
Moreover, we have
$$\mathcal{O}[\{{n_g}'\}](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}'$$
Since $\mathsf{blank}(o) = (\_, \_, \_, \perp, \perp)$, we have
$$\mathcal{O}[N \setminus \{{n_g}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{n_m, n_x, \mathrm{R}\} \cap (N \setminus \{{n_g}'\}) = \emptyset$$
Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \{\perp\} \times \mathcal{N}^3 \times \{\perp\}$. Then $o = (\perp, {n_g}', {n_m}', {n_x}', \perp)$ for some ${n_g}', {n_m}', {n_x}' \in \mathcal{N}$. By definition, $\mathsf{core}(o) = \{\{{n_m}'\}\}$ and thus $\mathit{Core} = \{{n_m}'\}$; since $\mathit{Core} \subseteq N$ we have ${n_m}' \in N$. By definition of $\mathcal{O}$, we have
$$\mathcal{O}[N](c) = o \Leftrightarrow n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \{\mathrm{S}, n_g, n_x, \mathrm{R}\} \cap (N \setminus \{{n_m}'\}) = \emptyset$$
Moreover, we have
$$\mathcal{O}[\{{n_m}'\}](c) = o \Leftrightarrow n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}'$$
Since $\mathsf{blank}(o) = (\perp, \_, \_, \_, \perp)$, we have
$$\mathcal{O}[N \setminus \{{n_m}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{\mathrm{S}, n_g, n_x, \mathrm{R}\} \cap (N \setminus \{{n_m}'\}) = \emptyset$$
Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \{\perp\}^2 \times \mathcal{N}^2 \times \mathcal{R}$. Then $o = (\perp, \perp, {n_m}', {n_x}', \mathrm{R}')$ for some ${n_m}', {n_x}' \in \mathcal{N}, \mathrm{R}' \in \mathcal{R}$. By definition, $\mathsf{core}(o) = \{\{{n_x}'\}\}$ and thus $\mathit{Core} = \{{n_x}'\}$; since $\mathit{Core} \subseteq N$ we have ${n_x}' \in N$. By definition of $\mathcal{O}$, we have
$$\mathcal{O}[N](c) = o \Leftrightarrow n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}' \wedge \{\mathrm{S}, n_g, n_m\} \cap (N \setminus \{{n_x}'\}) = \emptyset$$
Moreover, we have
$$\mathcal{O}[\{{n_x}'\}](c) = o \Leftrightarrow n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}'$$
Since $\mathsf{blank}(o) = (\perp, \perp, \_, \_, \_)$, we have
$$\mathcal{O}[N \setminus \{{n_x}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{\mathrm{S}, n_g, n_m\} \cap (N \setminus \{{n_x}'\}) = \emptyset$$
Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \{\perp\}^3 \times \mathcal{N} \times \mathcal{R}$. Then $o = (\perp, \perp, \perp, {n_x}', \mathrm{R}')$ for some ${n_x}' \in \mathcal{N}, \mathrm{R}' \in \mathcal{R}$. By definition, $\mathsf{core}(o) = \big\{\big\{\mathrm{R}'\big\}\big\}$ and thus $\mathit{Core} = \big\{\mathrm{R}'\big\}$ and since $\mathit{Core} \subseteq N$ we have $\mathrm{R}' \in N$. By definition of $\mathcal{O}$, we have
$$\mathcal{O}[N](c) = o \Leftrightarrow n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}' \wedge \{\mathrm{S}, n_g, n_m, n_x\} \cap N = \emptyset$$
Moreover, we have
$$\mathcal{O}[\big\{\mathrm{R}'\big\}](c) = o \Leftrightarrow n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}'$$
Since $\mathsf{blank}(o) = (\perp, \perp, \perp, \_, \_)$, we have
$$\mathcal{O}[N \setminus \big\{\mathrm{R}'\big\}](c) = \mathsf{blank}(o) \Leftrightarrow \{\mathrm{S}, n_g, n_m, n_x\} \cap N = \emptyset$$
Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \mathcal{S} \times \mathcal{N}^3 \times \{\bot\}$. Then $o = (\mathrm{S}', {n_g}', {n_m}', {n_x}', \bot)$ for some $\mathrm{S}' \in \mathcal{S}, {n_g}', {n_m}', {n_x}' \in \mathcal{N}$. By definition, $\mathsf{core}(o) = \{\{{n_g}', {n_m}'\}, \{\mathrm{S}', {n_m}'\}\}$. We distinguish the following two cases, depending on *Core*.

  Case $\mathit{Core} = \{{n_g}', {n_m}'\}$: Since $\mathit{Core} \subseteq N$ we have ${n_g}', {n_m}' \in N$. By definition of $\mathcal{O}$, we have

  $$\mathcal{O}[N](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \{n_x, \mathrm{R}\} \cap (N \setminus \{{n_g}', {n_m}'\}) = \emptyset$$

  Moreover, we have

  $$\mathcal{O}[\{{n_g}', {n_m}'\}](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}'$$

  Since $\mathsf{blank}(o) = (\_, \_, \_, \_, \bot)$, we have

  $$\mathcal{O}[N \setminus \{{n_g}', {n_m}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{n_x, \mathrm{R}\} \cap (N \setminus \{{n_g}', {n_m}'\}) = \emptyset$$

  Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

  Case $\mathit{Core} = \{\mathrm{S}', {n_m}'\}$: Since $\mathit{Core} \subseteq N$ we have $\mathrm{S}', {n_m}' \in N$. By definition of $\mathcal{O}$, we have

  $$\mathcal{O}[N](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \{n_x, \mathrm{R}\} \cap (N \setminus \{\mathrm{S}, {n_m}'\}) = \emptyset$$

  Moreover, we have

  $$\mathcal{O}[\{\mathrm{S}', {n_m}'\}](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}'$$

  Since $\mathsf{blank}(o) = (\_, \_, \_, \_, \bot)$, we have

  $$\mathcal{O}[N \setminus \{\mathrm{S}', {n_m}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{n_x, \mathrm{R}\} \cap (N \setminus \{\mathrm{S}, {n_m}'\}) = \emptyset$$

  Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \{\bot\} \times \mathcal{N}^3 \times \mathcal{R}$. Then $o = (\bot, {n_g}', {n_m}', {n_x}', \mathrm{R}')$ for some ${n_g}', {n_m}', {n_x}' \in \mathcal{N}, \mathrm{R}' \in \mathcal{R}$. By definition, $\mathsf{core}(o) = \{\{{n_m}', {n_x}'\}, \{{n_m}', \mathrm{R}'\}\}$. We distinguish the following two cases, depending on *Core*.

  Case $\mathit{Core} = \{{n_m}', {n_x}'\}$: Since $\mathit{Core} \subseteq N$ we have ${n_m}', {n_x}' \in N$. By definition of $\mathcal{O}$, we have

  $$\mathcal{O}[N](c) = o \Leftrightarrow n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}' \wedge \{\mathrm{S}, n_g\} \cap (N \setminus \{{n_m}', {n_x}'\}) = \emptyset$$

  Moreover, we have

  $$\mathcal{O}[\{{n_m}', {n_x}'\}](c) = o \Leftrightarrow n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}'$$

  Since $\mathsf{blank}(o) = (\bot, \_, \_, \_, \_)$, we have

  $$\mathcal{O}[N \setminus \{{n_m}', {n_x}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{\mathrm{S}, n_g\} \cap (N \setminus \{{n_m}', {n_x}'\}) = \emptyset$$

  Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

  Case $\mathit{Core} = \{{n_m}', \mathrm{R}'\}$: Since $\mathit{Core} \subseteq N$ we have ${n_m}', \mathrm{R}' \in N$. By definition of $\mathcal{O}$, we have

  $$\mathcal{O}[N](c) = o \Leftrightarrow n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}' \wedge \{\mathrm{S}, n_g\} \cap (N \setminus \{{n_m}', \mathrm{R}\}) = \emptyset$$

  Moreover, we have

  $$\mathcal{O}[\{{n_m}', \mathrm{R}'\}](c) = o \Leftrightarrow n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}'$$

  Since $\mathsf{blank}(o) = (\bot, \_, \_, \_, \_)$, we have

  $$\mathcal{O}[N \setminus \{{n_m}', \mathrm{R}'\}](c) = \mathsf{blank}(o) \Leftrightarrow \{\mathrm{S}, n_g\} \cap (N \setminus \{{n_m}', \mathrm{R}\}) = \emptyset$$

  Thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

- $o \in \mathcal{S} \times \mathcal{N}^3 \times \mathcal{R}$. Then $o = (\mathrm{S}', {n_g}', {n_m}', {n_x}', \mathrm{R}')$ for some ${n_g}', {n_m}', {n_x}' \in \mathcal{N}, \mathrm{S}' \in \mathcal{S}, \mathrm{R}' \in \mathcal{R}$. By definition, $\mathsf{core}(o) = \left\{\left\{\mathrm{S}', {n_m}', \mathrm{R}'\right\}, \left\{\mathrm{S}', {n_x}'\right\}, \left\{{n_g}', {n_x}'\right\}, \left\{{n_g}', \mathrm{R}'\right\}\right\}$. By definition of $\mathcal{O}$ we have

$$\mathcal{O}[N](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}'$$

  Equally, for every element $\mathit{Core} \in \mathsf{core}(o)$ we have

$$\mathcal{O}[\mathit{Core}](c) = o \Leftrightarrow \mathrm{S} = \mathrm{S}' \wedge n_g = {n_g}' \wedge n_m = {n_m}' \wedge n_x = {n_x}' \wedge \mathrm{R} = \mathrm{R}'$$

  Moreover, since $\mathsf{blank}(o) = (\_, \_, \_, \_, \_)$, for every observation $o' \in \mathit{Obs}$ we have $o' = \mathsf{blank}(o)$ and thus, $\mathcal{O}[N](c) = o \Leftrightarrow \mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \mathit{Core}](c) = \mathsf{blank}(o)$.

$\square$

With this lemma in place, we can state and prove a lemma about the impact of compromised nodes on a certain observation. It states that the probability of making a certain observation by means of a set N of nodes can be computed by the probability that the core of this observation already suffices to make this observation, under the assumption that the remaining nodes (outside this core) make the corresponding blanked observation. Intuitively, this means that the core is sufficient for making the observation, and that the remaining nodes can only infer $\perp$ for those elements of the information that cannot be observed.

Lemma 4 (Impact of Compromised Nodes)**.** *Let $\mathrm{S}_a \in \mathcal{S}$ be a sender and $\mathrm{R}_b \in \mathcal{R}$ be a recipient. Let $o \in \mathit{NatObs}$ be a natural observation, $\mathit{Core} \in \mathsf{core}(o)$, and $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ such that $\mathit{Core} \subseteq N$. Then we have*

$$P^{ab}[\mathcal{OC}(N, o)] = P^{ab}[\mathcal{OC}(\mathit{Core}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \mathit{Core}, o)].$$

Proof. We have

$$\begin{aligned}
& P^{ab}[\mathcal{OC}(N, o)] = \Pr\left[c \in \mathcal{OC}(N, o); c \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)\right] = \Pr\left[\mathcal{O}[N](c) = o; c \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)\right] \\
= \; & \Pr\left[\mathcal{O}[\mathit{Core}](c) = o \wedge \mathcal{O}[N \setminus \{\mathit{Core}\}](c) = \mathsf{blank}(o); c \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)\right] \\
= \; & \Pr\left[c \in \mathcal{OC}(\mathit{Core}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \mathit{Core}, o); c \leftarrow \mathsf{ps}(\mathrm{S}_a, \mathrm{R}_b)\right] \\
= \; & P^{ab}[\mathcal{OC}(\mathit{Core}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \mathit{Core}, o)],
\end{aligned}$$

where the fourth equality follows from Lemma 3; the remaining equalities hold by definition of P^{ab} and $\mathcal{OC}$. $\square$

For any given observation, more than the core of the observation is visible to the adversary. We now analyze the impact that compromising any such visible Tor entity may have on the impact of an observation, if the entity is not already contained in the core of the observation.

We show that such entries can be of two types: (i) entities that are superfluous, i.e., compromising them does not change the impact of the observation, and (ii) entities that are destructive, i.e., compromising them makes the observation impossible and thus completely removes the impact of an observation. We combine these insights in the following lemma, where we show that ignoring all visible entities except for the core can only increase the impact of an observation.

Lemma 5 (Impact of Visible Non-core Nodes)**.** *Let $\mathrm{S}_a, \mathrm{S}_c \in \mathcal{S}$ be two senders and $\mathrm{R}_b, \mathrm{R}_d \in \mathcal{R}$ be two recipients. Let $o \in \mathit{NatObs}$ be a natural observation, $\mathit{Core} \in \mathsf{core}(o)$, and $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ such that $\mathit{Core} \subseteq N$. Then, for $N_V = N \setminus (V(o) \setminus \mathit{Core})$,*

$$\phi\left(P^{ab}[\mathcal{OC}(N, o)], P^{cd}[\mathcal{OC}(N, o)]\right) \leq \phi\left(P^{ab}[\mathcal{OC}(N_V, o)], P^{cd}[\mathcal{OC}(N_V, o)]\right)$$

Proof. Let $\mathrm{S}_a, \mathrm{S}_c \in \mathcal{S}$ be two senders and $\mathrm{R}_b, \mathrm{R}_d \in \mathcal{R}$ be two recipients. Let $o \in \mathit{NatObs}$ be a natural observation, $\mathit{Core} \in \mathsf{core}(o)$, and $N \subseteq \mathcal{N} \cup \mathcal{S} \cup \mathcal{R}$ such that $\mathit{Core} \subseteq N$, and let $N_V = N \setminus (V(o) \setminus \mathit{Core})$. By Lemma 3 we know that $\mathcal{OC}(N, o) \subseteq \mathcal{OC}(\mathit{Core}, o)$.

Let $x \in V(o) \setminus \mathit{Core}$ be any visible Tor entity of the observation that does not belong to the core. We show that we can remove x from N without reducing the impact. If $x \notin N$, $\phi\left(P^{ab}[\mathcal{OC}(N,o)], P^{cd}[\mathcal{OC}(N,o)]\right) = \phi\left(P^{ab}[\mathcal{OC}(N \setminus \{x\}, o)], P^{cd}[\mathcal{OC}(N \setminus \{x\}, o)]\right)$ trivially holds.

If $x \in N$, we distinguish two cases:

- There is a circuit $c \in \mathcal{OC}(\mathit{Core}, o)$ s.t. $c \notin \mathcal{OC}_{\mathsf{blank}}(\{x\}, o)$. By Definition 3 we know that x cannot occur in c twice with non-zero probability, so the position in which x is observed must lead to an observation that is incompatible with $\mathsf{blank}(o)$. However, by definition of $\mathcal{OC}(\mathit{Core}, o)$ and V, for every other circuit $c' \in \mathcal{OC}(\mathit{Core}, o)$, x is in the same position, which leads to an observation of the same form and thus $c' \notin \mathcal{OC}_{\mathsf{blank}}(\{x\}, o)]$ must also hold. Consequently,

$$\begin{aligned}
& \phi\left(P^{ab}[\mathcal{OC}(N,o)], P^{cd}[\mathcal{OC}(N,o)]\right) \\
= \; & \phi\left(P^{ab}[\mathcal{OC}(\mathit{Core},o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \mathit{Core}, o)], P^{cd}[\mathcal{OC}(\mathit{Core},o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \mathit{Core}, o)]\right) \\
= \; & \phi\left(P^{ab}[\emptyset], P^{cd}[\emptyset]\right) \\
= \; & \phi(0,0) \\
\leq \; & \phi\left(P^{ab}[\mathcal{OC}(N \setminus \{x\}, o)], P^{cd}[\mathcal{OC}(N \setminus \{x\}, o)]\right)
\end{aligned}$$

- Otherwise, we have $\mathcal{OC}(\mathit{Core}, o) \subseteq \mathcal{OC}_{\mathsf{blank}}(\{x\}, o)]$ and thus, $\phi\left(P^{ab}[\mathcal{OC}(N,o)], P^{cd}[\mathcal{OC}(N,o)]\right) = \phi\left(P^{ab}[\mathcal{OC}(N \setminus \{x\}, o)], P^{cd}[\mathcal{OC}(N \setminus \{x\}, o)]\right)$

Applying this reasoning for every such element $x \in V(o) \setminus \mathit{Core}$ concludes the proof. □

4.3 Proof for Sender Anonymity

We now combine the results for indirect impacts and derive our bound for sender anonymity. To this end, we first show that for any given set of compromised Tor nodes we can derive a bound according to our definitions for indirect impact.

Lemma 6 (Observations for Sender Anonymity)**.** *Let $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$ be two senders and $\mathrm{R}_0 \in \mathcal{R}$ be a recipient, and let $N = N' \cup \{\mathrm{R}_0\}$ be a set of compromised Tor entities with $N' \subseteq \mathcal{N}$ being a set of compromised Tor nodes. Then we have*

$$\begin{aligned}
& \sum_{o \in \mathit{Obs}} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
\leq \; & \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(\mathrm{R}_0) + \sum_{n \in N'} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(n) + \mathbf{Impact}_{\mathsf{Rec1}}(n) \\
& + \sum_{n' \in N' \setminus \{n\}} \Big(\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n, n') + \mathbf{Impact}_{\mathsf{Rec2}}(n, n')\Big)\Big)
\end{aligned}$$

Proof. We define the following sets of observations for sender anonymity:

- $\mathit{Obs}_{g,\mathrm{SA}} := \{(n_1, \ldots, n_5) \in \mathit{Obs} \text{ s.t. } n_1 \neq \bot\}$,
- $\mathit{Obs}_{m,\mathrm{SA}} := \{(n_1, \ldots, n_5) \in \mathit{Obs} \setminus \mathit{Obs}_{g,\mathrm{SA}} \text{ s.t. } n_2 \neq \bot\}$,
- $\mathit{Obs}_{x,\mathrm{SA}} := \{(n_1, \ldots, n_5) \in \mathit{Obs} \setminus (\mathit{Obs}_{g,\mathrm{SA}} \cup \mathit{Obs}_{m,\mathrm{SA}}) \text{ s.t. } n_3 \neq \bot\}$,
- $\mathit{Obs}_{R,\mathrm{SA}} := \mathit{Obs} \setminus (\mathit{Obs}_{g,\mathrm{SA}} \cup \mathit{Obs}_{m,\mathrm{SA}} \cup \mathit{Obs}_{x,\mathrm{SA}})$.

As these sets define a partitioning of Obs, we get

$$\begin{aligned}
&\sum_{o \in Obs} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \\
&= \sum_{o \in Obs_{g,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) + \sum_{o \in Obs_{m,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \\
&+ \sum_{o \in Obs_{x,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) + \sum_{o \in Obs_{R,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right)
\end{aligned}$$

We now consider each of the sets of observations individually.

- $Obs_{g,\mathrm{SA}}$. We first show that (in addition to R_0) a compromised guard node is required and sufficient for making an observation in $Obs_{g,\mathrm{SA}}$ and that additionally compromised nodes do not contribute to the impact of such observations. Let $c = (\mathrm{S}_a, n_g, n_m, n_x, \mathrm{R}_0) \in \mathcal{C}$ be any circuit with $\mathrm{S}_a \in \{\mathrm{S}_0, \mathrm{S}_1\}$ s.t. $o = \mathcal{O}[N](c) \in Obs_{g,\mathrm{SA}}$ is the observation made by the adversary for this circuit. By definition of $Obs_{g,\mathrm{SA}}$, S_a is part of the observation. Since $\{\mathrm{S}_0, \mathrm{S}_1\} \cap N = \emptyset$, we know that $n_g \in N$ is required to make the observation. Since $n_g \in N$ is also sufficient to observe the sender we get $P^{00}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)] = 0$ or $P^{10}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)] = 0$. By definition of ϕ we know that for these observations $\phi\left(P^{00}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)], P^{10}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)]\right) = P^{00}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)]$. Thus,

$$\begin{aligned}
&\sum_{o \in Obs_{g,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)], P^{10}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)]\right) \\
&= \sum_{o \in Obs_{g,\mathrm{SA}}} P^{00}[\mathcal{OC}(N \cup \{\mathrm{R}_0\}, o)] \\
&= \sum_{\substack{n_g \in N \\ n_m, n_x \in \mathcal{N}}} \Pr\left[c = (\mathrm{S}_0, n_g, n_m, n_x, \mathrm{R}_0), c \leftarrow \mathsf{ps}(\mathrm{S}_0, \mathrm{R}_0)\right] \\
&= \sum_{n_g \in N} \sum_{o \in Obs_{g,\mathrm{SA}}} \phi\big(P^{00}[\mathcal{OC}(\{n_g\}, o)], P^{10}[\mathcal{OC}(\{n_g\}, o)]\big).
\end{aligned}$$

- $Obs_{m,\mathrm{SA}}$. We first show that (in addition to R_0) a compromised middle node is required and sufficient for making an observation in $Obs_{m,\mathrm{SA}}$. We then argue that we can remove the recipient from the computation without modifying the probability and, finally, that additionally compromised nodes do not contribute to the impact of such observations. Let $c = (\mathrm{S}_a, n_g, n_m, n_x, \mathrm{R}_0) \in \mathcal{C}$ be any circuit with $\mathrm{S}_a \in \{\mathrm{S}_0, \mathrm{S}_1\}$ s.t. $o = \mathcal{O}[N](c) \in Obs_{m,\mathrm{SA}}$ is the observation made by the adversary for this circuit. We have $\mathsf{core}(o) = \{\{n_m, n_x\}, \{n_m, \mathrm{R}_0\}\}$ and set $Core = \{n_m, \mathrm{R}_0\}$. By Lemma 5 we know that

  $$\phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \leq \phi\left(P^{00}[\mathcal{OC}(N \setminus \{n_g, n_x\}, o)], P^{10}[\mathcal{OC}(N \setminus \{n_g, n_x\}, o)]\right)$$

  All nodes of the circuit are visible in the observation. Since all nodes in $N \setminus V(o)$ can thus never lead to the observation o and consequently, compromising them does not modify the probability of the observation, we know that

  $$\phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \leq \phi\left(P^{00}[\mathcal{OC}(\{n_m, \mathrm{R}_0\}, o)], P^{10}[\mathcal{OC}(\{n_m, \mathrm{R}_0\}, o)]\right)$$

  Moreover, we know that $\mathcal{O}[\{n_m, \mathrm{R}_0\}](c) = (\perp, n_g, n_m, n_x, \mathrm{R}_0)$, and $\mathcal{O}[\{n_m\}](c) = (\perp, n_g, n_m, n_x, \perp) =: o'$. Since there is only one recipient and since this holds for all respective circuits, $P^{a0}[\mathcal{OC}(\{n_m, \mathrm{R}_0\}, o)] = P^{a0}[\mathcal{OC}(\{n_m\}, o')]$, and, consequently,

  $$\sum_{o \in Obs_{m,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \leq \sum_{o \in Obs_{m,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(\{n_m\}, o)], P^{10}[\mathcal{OC}(\{n_m\}, o)]\right)$$

- $Obs_{x,\mathrm{SA}}$. All observations $o \in Obs_{x,\mathrm{SA}}$ are of the form $(\perp, \perp, n_m, n_x, \mathrm{R}_0)$ and we have $\mathsf{core}(o) = \{\{n_x\}\}$ and $V(o) = \{n_m, n_x, \mathrm{R}_0\}$. By Lemma 5 we know that

$$\begin{aligned} & \sum_{o \in Obs_{x,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \\ \leq \quad & \sum_{o \in Obs_{x,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N \setminus \{n_m, \mathrm{R}_0\}, o)], P^{10}[\mathcal{OC}(N \setminus \{n_m, \mathrm{R}_0\}, o)]\right). \end{aligned}$$

If $n_x \notin N$ then $\phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$.

Otherwise, we rewrite the probability according to Lemma 4 and get

$$\begin{aligned} & \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \\ \leq \quad & \phi\left(P^{00}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus V(o), o)], P^{10}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus V(o), o)]\right) \\ = \quad & \phi\left(\sum_{n_g \in \mathcal{N} \setminus N} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N} \setminus N} P^{1,0}_{n_g,n_m,n_x}\right) \\ = \quad & \phi\left(\sum_{n_g \in \mathcal{N}} P^{0,0}_{n_g,n_m,n_x} - \sum_{n_g \in N} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N}} P^{1,0}_{n_g,n_m,n_x} - \sum_{n_g \in N} P^{1,0}_{n_g,n_m,n_x}\right) \\ \leq \quad & \phi\left(\sum_{n_g \in \mathcal{N}} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N}} P^{1,0}_{n_g,n_m,n_x}\right) + \phi\left(\sum_{n_g \in N} P^{1,0}_{n_g,n_m,n_x}, \sum_{n_g \in N} P^{0,0}_{n_g,n_m,n_x}\right) \\ = \quad & \phi\left(P^{00}[\mathcal{OC}(\{n_x\}, o)], P^{10}[\mathcal{OC}(\{n_x\}, o)]\right) + \phi\left(\sum_{n_g \in N} P^{1,0}_{n_g,n_m,n_x}, \sum_{n_g \in N} P^{0,0}_{n_g,n_m,n_x}\right). \end{aligned}$$

Thus, if we sum over all observations $o \in Obs_{x,\mathrm{SA}}$ we get

$$\begin{aligned} & \sum_{o \in Obs_{x,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right) \\ \leq \quad & \sum_{o \in Obs_{x,\mathrm{SA}}} \Bigg(\phi\left(P^{00}[\mathcal{OC}(\{o.n_4\}, o)], P^{10}[\mathcal{OC}(\{o.n_4\}, o)]\right) \\ & \quad +\phi\left(\sum_{n_g \in N} P^{1,0}_{n_g,o.n_3,o.n_4}, \sum_{n_g \in N} P^{0,0}_{n_g,o.n_3,o.n_4}\right)\Bigg) \\ = \quad & \sum_{n_x \in N} \sum_{o \in Obs_{x,\mathrm{SA}}} \left(\phi\left(P^{00}[\mathcal{OC}(\{n_x\}, o)], P^{10}[\mathcal{OC}(\{n_x\}, o)]\right)\right) \\ & \quad + \sum_{n_x \in N} \sum_{n_m \in \mathcal{N}} \phi\left(\sum_{n_g \in N} P^{1,0}_{n_g,n_m,n_x}, \sum_{n_g \in N} P^{0,0}_{n_g,n_m,n_x}\right) \\ \leq \quad & \sum_{n_x \in N} \sum_{o \in Obs_{x,\mathrm{SA}}} \left(\phi\left(P^{00}[\mathcal{OC}(\{n_x\}, o)], P^{10}[\mathcal{OC}(\{n_x\}, o)]\right)\right) + \sum_{\substack{n_g \in N \\ n_x \in N}} \mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n_g, n_x). \end{aligned}$$

- $Obs_{R,\mathrm{SA}}$. Let $o = (\perp, \perp, \perp, n_x, \mathrm{R}_0)$ be any observation in $Obs_{R,\mathrm{SA}}$. Since $\mathsf{core}(o) = \{\{\mathrm{R}_0\}\}$ we have $Core = \{\mathrm{R}_0\}$. By Lemma 5 we know that

$$\phi\left(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\right)$$

$$\leq \quad \phi\left(P^{00}[\mathcal{OC}(N \setminus \{n_x\}, o)], P^{10}[\mathcal{OC}(N \setminus \{n_x\}, o)]\right)$$

We rewrite the probability according to Lemma 4 and yield

$$\begin{aligned}
& \phi\left(P^{00}[\mathcal{OC}(N \setminus \{n_x\}, o)], P^{10}[\mathcal{OC}(N \setminus \{n_x\}, o)]\right) \\
= \; & \phi\left(P^{00}[\mathcal{OC}(\mathit{Core}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, \mathrm{R}_0\}, o)], P^{10}[\mathcal{OC}(\mathit{Core}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, \mathrm{R}_0\}, o)]\right) \\
= \; & \phi\left(\sum_{\substack{n_g \in \mathcal{N} \setminus N \\ n_m \in \mathcal{N} \setminus N}} P^{0,0}_{n_g,n_m,n_x}, \sum_{\substack{n_g \in \mathcal{N} \setminus N \\ n_m \in \mathcal{N} \setminus N}} P^{1,0}_{n_g,n_m,n_x},\right) \\
= \; & \phi\left(\sum_{\substack{n_g \in \mathcal{N} \\ n_m \in \mathcal{N}}} P^{0,0}_{n_g,n_m,n_x} - \sum_{\substack{n \in N \\ n' \in \mathcal{N}}} \left(P^{0,0}_{n,n',n_x} + P^{0,0}_{n',n,n_x}\right) + \sum_{\substack{n_g \in N \\ n_m \in N}} P^{0,0}_{n_g,n_m,n_x},\right. \\
& \left. \sum_{\substack{n_g \in \mathcal{N} \\ n_m \in \mathcal{N}}} P^{1,0}_{n_g,n_m,n_x} - \sum_{\substack{n \in N \\ n' \in \mathcal{N}}} \left(P^{1,0}_{n,n',n_x} + P^{1,0}_{n',n,n_x}\right) + \sum_{\substack{n_g \in N \\ n_m \in N}} P^{1,0}_{n_g,n_m,n_x}\right) \\
\leq \; & \phi\left(\sum_{\substack{n_g \in \mathcal{N} \\ n_m \in \mathcal{N}}} P^{0,0}_{n_g,n_m,n_x}, \sum_{\substack{n_g \in \mathcal{N} \\ n_m \in \mathcal{N}}} P^{1,0}_{n_g,n_m,n_x}\right) \\
& + \phi\left(\sum_{\substack{n \in N \\ n' \in \mathcal{N}}} \left(P^{1,0}_{n,n',n_x} + P^{1,0}_{n',n,n_x}\right), \sum_{\substack{n \in N \\ n' \in \mathcal{N}}} \left(P^{0,0}_{n,n',n_x} + P^{0,0}_{n',n,n_x}\right)\right) \\
& + \phi\left(\sum_{\substack{n_g \in N \\ n_m \in N}} P^{0,0}_{n_g,n_m,n_x}, \sum_{\substack{n_g \in N \\ n_m \in N}} P^{1,0}_{n_g,n_m,n_x}\right) \\
\leq \; & \phi\left(P^{00}[\mathcal{OC}(\{\mathrm{R}_0\}, o)], P^{10}[\mathcal{OC}(\{\mathrm{R}_0\}, o)]\right) \\
& + \sum_{n \in N} \phi\left(\sum_{n' \in \mathcal{N}} \left(P^{1,0}_{n,n',n_x} + P^{1,0}_{n',n,n_x}\right), \sum_{n' \in \mathcal{N}} \left(P^{0,0}_{n,n',n_x} + P^{0,0}_{n',n,n_x}\right)\right) \\
& + \sum_{\substack{n_g \in N \\ n_m \in N}} \phi\left(P^{0,0}_{n_g,n_m,n_x}, P^{1,0}_{n_g,n_m,n_x}\right)
\end{aligned}$$

We combine these individual bounds for the sets of observations and yield

$$\begin{aligned}
& \sum_{o \in \mathit{Obs}} \phi\left(P^{00}[\mathcal{OC}(N, o)], P^{10}[\mathcal{OC}(N, o)]\right) \\
= \; & \sum_{o \in \mathit{Obs}_{g,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N, o)], P^{10}[\mathcal{OC}(N, o)]\right) + \sum_{o \in \mathit{Obs}_{m,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N, o)], P^{10}[\mathcal{OC}(N, o)]\right) \\
+ \; & \sum_{o \in \mathit{Obs}_{x,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N, o)], P^{10}[\mathcal{OC}(N, o)]\right) + \sum_{o \in \mathit{Obs}_{R,\mathrm{SA}}} \phi\left(P^{00}[\mathcal{OC}(N, o)], P^{10}[\mathcal{OC}(N, o)]\right) \\
\leq \; & \sum_{o \in \mathit{Obs}_{g,\mathrm{SA}}} \sum_{n_g \in N'} \phi\left(P^{00}[\mathcal{OC}(\{n_g\}, o)], P^{10}[\mathcal{OC}(\{n_g\}, o)]\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{o\in Obs_{m,\mathrm{SA}}} \sum_{n_m\in N'} \phi\left(P^{00}[\mathcal{OC}(\{n_m\},o)], P^{10}[\mathcal{OC}(\{n_m\},o)]\right)\\
&+ \sum_{o\in Obs_{x,\mathrm{SA}}} \sum_{n_x\in N'} \phi\left(P^{00}[\mathcal{OC}(\{n_x\},o)], P^{10}[\mathcal{OC}(\{n_x\},o)]\right) + \sum_{\substack{n_g\in N'\\ n_x\in N'}} \left(\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n_g,n_x)\right)\\
&+ \sum_{\substack{o\in Obs_{R,\mathrm{SA}}\\ (n_1,n_2,n_3,n_4,n_5):=o}} \Bigg(\phi\left(P^{00}[\mathcal{OC}(\{\mathrm{R}_0\},o)], P^{10}[\mathcal{OC}(\{\mathrm{R}_0\},o)]\right)\\
&\quad + \sum_{n\in N'} \phi\left(\sum_{n'\in\mathcal{N}}\left(P^{1,0}_{n,n',n_4}+P^{1,0}_{n',n,n_4}\right), \sum_{n'\in\mathcal{N}}\left(P^{0,0}_{n,n',n_4}+P^{0,0}_{n',n,n_4}\right)\right)\\
&\quad + \sum_{\substack{n_g\in N'\\ n_m\in N'}} \phi\left(P^{0,0}_{n_g,n_m,n_4}, P^{1,0}_{n_g,n_m,n_4}\right)\Bigg)\\
&\le \sum_{n\in N'}\Bigg(\sum_{o\in Obs_{\not\perp}} \phi\left(P^{00}[\mathcal{OC}(\{n\},o)], P^{10}[\mathcal{OC}(\{n\},o)]\right) + \sum_{n'\in N'} \mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n,n')\\
&+ \mathbf{Impact}_{\mathsf{Rec1}}(n) + \sum_{n'\in N'} \mathbf{Impact}_{\mathsf{Rec2}}(n,n')\Bigg) + \sum_{o\in Obs_{\not\perp}} \phi\left(P^{00}[\mathcal{OC}(\{\mathrm{R}_0\},o)], P^{10}[\mathcal{OC}(\{\mathrm{R}_0\},o)]\right)\\
&= \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(\mathrm{R}_0) + \sum_{n\in N'}\Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{SA}}(n) + \mathbf{Impact}_{\mathsf{Rec1}}(n)\\
&+ \sum_{n'\in N'\setminus\{n\}} \left(\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n,n') + \mathbf{Impact}_{\mathsf{Rec2}}(n,n')\right)\Bigg)
\end{aligned}
$$

The last equation holds because for all $n \in \mathcal{N}$, $\mathbf{Impact}^{(ab),(cd)}_{\mathsf{indirect}}(n,n) = 0$, since all probabilities of circuits are zero if the same node is used more than once in the same circuit. □

4.4 Proof for Recipient Anonymity

We now derive our bound for recipient anonymity. Both the lemma and the proof are completely analogous to the case of sender anonymity.

Lemma 7 (Observations for Recipient Anonymity)**.** *Let* $\mathrm{S}_0 \in \mathcal{S}$ *be a sender and* $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$ *be two recipients, and let* $N = N' \cup \{\mathrm{S}_0\}$ *be a set of compromised Tor entities with* $N' \subseteq \mathcal{N}$ *being a set of compromised Tor nodes. Then we have*

$$
\begin{aligned}
&\sum_{o\in Obs} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{01}[\mathcal{OC}(N,o)]\big)\\
\le\ & \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0) + \sum_{n\in N'}\Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \mathbf{Impact}_{\mathsf{Sen1}}(n)\\
+\ & \sum_{n'\in N'\setminus\{n\}}\left(\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n') + \mathbf{Impact}_{\mathsf{Sen2}}(n,n')\right)\Bigg)
\end{aligned}
$$

Proof. We define the following sets of observations for recipient anonymity:

- $Obs_{x,\mathrm{RA}} := \{(n_1,\ldots,n_5) \in Obs \text{ s.t. } n_5 \neq \perp\}$,

- $\mathit{Obs}_{m,\mathrm{RA}} := \{(n_1, \ldots, n_5) \in \mathit{Obs} \setminus \mathit{Obs}_{x,\mathrm{RA}} \text{ s.t. } n_4 \neq \bot\}$,
- $\mathit{Obs}_{g,\mathrm{RA}} := \{(n_1, \ldots, n_5) \in \mathit{Obs} \setminus (\mathit{Obs}_{x,\mathrm{RA}} \cup \mathit{Obs}_{m,\mathrm{RA}}) \text{ s.t. } n_3 \neq \bot\}$,
- $\mathit{Obs}_{S,\mathrm{RA}} := \mathit{Obs} \setminus (\mathit{Obs}_{x,\mathrm{RA}} \cup \mathit{Obs}_{m,\mathrm{RA}} \cup \mathit{Obs}_{g,\mathrm{RA}})$.

As these sets define a partitioning of Obs, we get

$$
\begin{aligned}
& \sum_{o \in \mathit{Obs}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{01}[\mathcal{OC}(N,o)]\right) \\
= & \sum_{o \in \mathit{Obs}_{x,\mathrm{RA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{01}[\mathcal{OC}(N,o)]\right) + \sum_{o \in \mathit{Obs}_{m,\mathrm{RA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{01}[\mathcal{OC}(N,o)]\right) \\
+ & \sum_{o \in \mathit{Obs}_{g,\mathrm{RA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{01}[\mathcal{OC}(N,o)]\right) + \sum_{o \in \mathit{Obs}_{S,\mathrm{RA}}} \phi\left(P^{00}[\mathcal{OC}(N,o)], P^{01}[\mathcal{OC}(N,o)]\right)
\end{aligned}
$$

The rest of the proof is completely analogous to the proof for Lemma 6. To give some intuition into the proof, we describe which parts of this proof correspond to which parts of the proof for Lemma 6.

- $\mathit{Obs}_{x,\mathrm{RA}}$ corresponds to $\mathit{Obs}_{g,\mathrm{SA}}$ in Lemma 6.
- $\mathit{Obs}_{m,\mathrm{RA}}$ corresponds to $\mathit{Obs}_{m,\mathrm{SA}}$ in Lemma 6.
- $\mathit{Obs}_{g,\mathrm{RA}}$ corresponds to $\mathit{Obs}_{x,\mathrm{SA}}$ in Lemma 6.
- $\mathit{Obs}_{S,\mathrm{RA}}$ corresponds to $\mathit{Obs}_{R,\mathrm{SA}}$ in Lemma 6.

We combine these bounds and yield

$$
\begin{aligned}
& \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(N) \\
= & \sum_{o \in \mathit{Obs}} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
= & \sum_{o \in \mathit{Obs}_{x,\mathrm{RA}}} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
+ & \sum_{o \in \mathit{Obs}_{m,\mathrm{RA}}} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
+ & \sum_{o \in \mathit{Obs}_{g,\mathrm{RA}}} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
+ & \sum_{o \in \mathit{Obs}_{S,\mathrm{RA}}} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
\leq & \sum_{o \in \mathit{Obs}_{x,\mathrm{RA}}} \sum_{n_x \in N'} \phi\left(P^{00}[\mathcal{OC}(\{n_x\},)o], P^{01}[\mathcal{OC}(\{n_x\}, o)]\right) \\
+ & \sum_{o \in \mathit{Obs}_{m,\mathrm{RA}}} \sum_{n_m \in N'} \phi\left(P^{00}[\mathcal{OC}(\{n_m\}, o)], P^{01}[\mathcal{OC}(\{n_m\}, o)]\right) \\
+ & \sum_{o \in \mathit{Obs}_{g,\mathrm{RA}}} \left(\sum_{n_g \in N'} \phi\left(P^{00}[\mathcal{OC}(\{n_g\}, o)], P^{01}[\mathcal{OC}(\{n_g\}, o)]\right) \right) + \sum_{\substack{n_g \in N' \\ n_x \in N'}} \Big(\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n_g, n_x) \Big) \\
+ & \sum_{\substack{o \in \mathit{Obs}_{S,\mathrm{RA}} \\ (n_1,n_2,n_3,n_4,n_5) := o}} \Bigg(\phi\left(P^{00}[\mathcal{OC}(\{\mathrm{S}_0\}, o)], P^{01}[\mathcal{OC}(\{\mathrm{S}_0\}, o)]\right)
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{n \in N'} \phi \left(\sum_{n' \in \mathcal{N}} \left(P^{0,1}_{n_2,n,n'} + P^{0,1}_{n_2,n',n} \right), \sum_{n' \in \mathcal{N}} \left(P^{0,0}_{n_2,n,n'} + P^{0,0}_{n_2,n',n} \right) \right) \\
& + \sum_{\substack{n_m \in N' \\ n_x \in N'}} \phi \left(P^{0,0}_{n_2,n_m,n_x}, P^{0,1}_{n_2,n_m,n_x} \right) \Bigg) \\
\leq \; & \sum_{n \in N'} \Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \sum_{n' \in N'} \mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n, n') \\
+ \; & \mathbf{Impact}_{\mathsf{Sen1}}(n) + \sum_{n' \in N'} \mathbf{Impact}_{\mathsf{Sen2}}(n, n') \Bigg) + \sum_{o \in Obs_{\not\perp}} \phi \left(P^{00}[\mathcal{OC}(\{\mathrm{S}_0\}, o)], P^{01}[\mathcal{OC}(\{\mathrm{S}_0\}, o)] \right) \\
= \; & \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0) + \sum_{n \in N'} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \mathbf{Impact}_{\mathsf{Sen1}}(n) \\
+ \; & \sum_{n' \in N' \setminus \{n\}} \Big(\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n, n') + \mathbf{Impact}_{\mathsf{Sen2}}(n, n') \Big) \Big)
\end{aligned}
$$

The last equation holds because for all $n \in \mathcal{N}$, $\mathbf{Impact}^{(ab),(cd)}_{\mathsf{indirect}}(n, n) = 0$, since all probabilities of circuits are zero if the same node is used more than once in the same circuit. □

4.5 Proof for Relationship Anonymity

We combine the results for indirect impacts and derive our bound for relationship anonymity. In contrast to sender anonymity and relationship anonymity, two aspects are different for relationship anonymity: (i) for relationship anonymity, N contains neither a sender nor a recipient from the challenge message, and (ii) for relationship anonymity we have to consider four distributions over circuits instead of two distributions over circuits. Recall that we write $P^{ab,cd}[C]$ instead of $\frac{1}{2}\left(P^{ab}[C] + P^{cd}[C]\right)$.

Lemma 8 (Observations for Relationship Anonymity)**.** *Let* $\mathrm{S}_0, \mathrm{S}_1$ *be two senders and* $\mathrm{R}_0, \mathrm{R}_1$ *be two recipients, and let* $N \subseteq \mathcal{N}$ *be a set of compromised Tor nodes. Under the assumption that* $P^{00,11}[\mathcal{OC}(N, (\perp, \perp, \perp, \perp, \perp))] \leq P^{01,10}[\mathcal{OC}(N, (\perp, \perp, \perp, \perp, \perp))]$

$$
\begin{aligned}
& \sum_{o \in Obs} \phi\big(P^{00,11}[\mathcal{CO}(N, o)], P^{01,10}[\mathcal{CO}(N, o)] \big) \\
\leq \; & \sum_{n \in N} \Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(\{n\}) + \sum_{n' \in N} \Big(\mathbf{Impact}^{\mathsf{obs}-2}_{\mathrm{REL}}(\{n, n'\}) \Big) \\
& + \sum_{n' \in N} \Big(\frac{1}{2} \mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n, n') + \frac{1}{2} \mathbf{Impact}^{(10),(11)}_{\mathsf{indirect}}(n, n') \\
& \qquad + \frac{1}{2} \mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n', n) + \frac{1}{2} \mathbf{Impact}^{(01),(11)}_{\mathsf{indirect}}(n', n) \Big) \Bigg)
\end{aligned}
$$

Proof. We define the following sets of observations for relationship anonymity, where each set is indexed by the positions of the compromised nodes that make the respective observations:

- $Obs_{gx,\mathrm{REL}} := \{(n_1, \ldots, n_5) \in Obs \text{ s.t. } (n_1 \neq \perp \wedge n_5 \neq \perp)\}$,
- $Obs_{gm,\mathrm{REL}} := \{(n_1, \ldots, n_5) \in Obs \setminus Obs_t \text{ s.t. } n_1 \neq \perp \wedge n_4 \neq \perp\}$,

- $Obs_{mx,\mathrm{REL}} := \{(n_1, \dots, n_5) \in Obs \setminus Obs_t \text{ s.t. } n_2 \neq \perp \wedge n_5 \neq \perp\}$,
- $Obs_{m,\mathrm{REL}} := \{(n_1, \dots, n_5) \in Obs \setminus (Obs_t \cup Obs_{gm,\mathrm{REL}} \cup Obs_{mx,\mathrm{REL}}) \text{ s.t. } n_2 \neq \perp \wedge n_4 \neq \perp\}$,
- $Obs_{g,\mathrm{REL}} := \{(n_1, \dots, n_5) \in Obs \setminus (Obs_t \cup Obs_{gm,\mathrm{REL}} \cup Obs_{mx,\mathrm{REL}} \cup Obs_{m,\mathrm{REL}}) \text{ s.t. } n_2 \neq \perp\}$,
- $Obs_{x,\mathrm{REL}} := \{(n_1, \dots, n_5) \in Obs \setminus (Obs_t \cup Obs_{gm,\mathrm{REL}} \cup Obs_{mx,\mathrm{REL}} \cup Obs_{m,\mathrm{REL}}) \text{ s.t. } n_4 \neq \perp\}$,
- $Obs_{r,\mathrm{REL}} := Obs \setminus (Obs_t \cup Obs_{gm,\mathrm{REL}} \cup Obs_{mx,\mathrm{REL}} \cup Obs_{m,\mathrm{REL}} \cup Obs_{g,\mathrm{REL}} \cup Obs_{x,\mathrm{REL}}) = \{(\perp, \perp, \perp, \perp, \perp)\}$.

We now consider each of the sets of observations individually.

- $Obs_{gx,\mathrm{REL}}$. Since neither the sender nor the recipient can be compromised for relationship anonymity, each observation in this set is of the form $o = (\mathrm{S}_a, n_g, n_m, n_x, \mathrm{R}_b)$ and we either have $n_g, n_x \in N$, or $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$. Since $\{n_g, n_x\} \in \mathsf{core}(o)$, by Lemma 5 and by the fact that the observation can only be made by the nodes n_g, n_m, n_x that belong to the circuit we get $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \leq \phi\left(P^{00,11}[\mathcal{OC}(\{n_g, n_x\}, o)], P^{01,10}[\mathcal{OC}(\{n_g, n_x\}, o)]\right)$
- $Obs_{gm,\mathrm{REL}}$. Each observation in this set is of the form $o = (\mathrm{S}_a, n_g, n_m, n_x, \perp)$ and we have $\mathsf{core}(o) = \{\{\mathrm{S}_a, n_m\}, \{n_g, n_m\}\}$. Since $\mathrm{S}_a \notin N$ for relationship anonymity, we assume $Core = \{n_g, n_m\} \subseteq N$ to make this observation, as otherwise $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$. Since $\{n_g, n_m\} \in \mathsf{core}(o)$, by Lemma 5 and by the fact that the observation can only be made by the nodes n_g, n_m, n_x that belong to the circuit we get $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \leq \phi\left(P^{00,11}[\mathcal{OC}(\{n_g, n_m\}, o)], P^{01,10}[\mathcal{OC}(\{n_g, n_m\}, o)]\right)$.
- $Obs_{mx,\mathrm{REL}}$. Analogously to the previous case, each observation in this set is of the form $o = (\perp, n_g, n_m, n_x, \mathrm{R}_b)$ and we have $\mathsf{core}(o) = \{\{n_m, n_x\}, \{n_m, \mathrm{R}_b\}\}$. Since $\mathrm{R}_b \notin N$ for relationship anonymity, we assume $Core = \{n_m, n_x\} \subseteq N$ to make this observation, as otherwise $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$. Since $\{n_m, n_x\} \in \mathsf{core}(o)$, by Lemma 5 and by the fact that the observation can only be made by the nodes n_g, n_m, n_x that belong to the circuit we get $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \leq \phi\left(P^{00,11}[\mathcal{OC}(\{n_m, n_x\}, o)], P^{01,10}[\mathcal{OC}(\{n_m, n_x\}, o)]\right)$.
- Obs_m. Each observation in this set is of the form $o = (\perp, n_g, n_m, n_x, \perp)$ and we have $\mathsf{core}(o) = \{\{n_m\}\}$. We assume $Core = \{n_m\} \subseteq N$ to make this observation, as otherwise $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$. Since $\{n_m\} \in \mathsf{core}(o)$, by Lemma 5 and by the fact that the observation can only be made by the nodes n_g, n_m, n_x that belong to the circuit we get $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \leq \phi\left(P^{00,11}[\mathcal{OC}(\{n_m\}, o)], P^{01,10}[\mathcal{OC}(\{n_m\}, o)]\right)$.
- $Obs_{g,\mathrm{REL}}$. Each observation in this set is of the form $o = (\mathrm{S}_a, n_g, n_m, \perp, \perp)$ and we have $\mathsf{core}(o) = \{\{n_g\}\}$. We assume $Core = \{n_g\} \subseteq N$ to make this observation, as otherwise $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$. Since $\{n_g\} \in \mathsf{core}(o)$, by Lemmas 4 and 5 we get

$$
\begin{aligned}
& \phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \\
\leq\ & \phi\left(P^{00,11}[\mathcal{OC}(N \setminus \{n_m\}, o)], P^{01,10}[\mathcal{OC}(N \setminus \{n_m\}, o)]\right) \\
=\ & \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_g, n_m\}, o)],\right. \\
& \left. P^{01,10}[\mathcal{OC}(\{n_g\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_g, n_m\}, o)]\right)
\end{aligned}
$$

We now distinguish two cases depending on the sender S_a:

  - Case $\mathrm{S}_a = \mathrm{S}_0$. Then

$$
\phi\left(P^{00,11}[\mathcal{OC}(\{n_g\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_g, n_m\}, o)],\right.
$$

$$
\begin{aligned}
& P^{01,10}[\mathcal{OC}(\{n_g\},o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_g, n_m\},o)]) \\
= \ & \frac{1}{2}\phi\left(\sum_{n_x \in \mathcal{N}\setminus N} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_x \in \mathcal{N}\setminus N} P^{0,1}_{n_g,n_m,n_x}\right) \\
= \ & \frac{1}{2}\phi\left(\sum_{n_x \in \mathcal{N}} P^{0,0}_{n_g,n_m,n_x} - \sum_{n_x \in N} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_x \in \mathcal{N}} P^{0,1}_{n_g,n_m,n_x} - \sum_{n_x \in N} P^{0,1}_{n_g,n_m,n_x}\right) \\
\leq \ & \frac{1}{2}\phi\left(\sum_{n_x \in \mathcal{N}} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_x \in \mathcal{N}} P^{0,1}_{n_g,n_m,n_x}\right) + \frac{1}{2}\phi\left(\sum_{n_x \in N} P^{0,1}_{n_g,n_m,n_x}, \sum_{n_x \in N} P^{0,0}_{n_g,n_m,n_x}\right) \\
\leq \ & \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\},o)], P^{01,10}[\mathcal{OC}(\{n_g\},o)]\right) + \frac{1}{2}\sum_{n_x \in N}\phi\left(P^{0,1}_{n_g,n_m,n_x}, P^{0,0}_{n_g,n_m,n_x}\right)
\end{aligned}
$$

– Case $\mathrm{S}_a = \mathrm{S}_1$. Then

$$
\begin{aligned}
& \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\},o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_g, n_m\},o)],\right. \\
& \left. P^{01,10}[\mathcal{OC}(\{n_g\},o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_g, n_m\},o)]\right) \\
= \ & \frac{1}{2}\phi\left(\sum_{n_x \in \mathcal{N}\setminus N} P^{1,1}_{n_g,n_m,n_x}, \sum_{n_x \in \mathcal{N}\setminus N} P^{1,0}_{n_g,n_m,n_x}\right) \\
= \ & \frac{1}{2}\phi\left(\sum_{n_x \in \mathcal{N}} P^{1,1}_{n_g,n_m,n_x} - \sum_{n_x \in N} P^{1,1}_{n_g,n_m,n_x}, \sum_{n_x \in \mathcal{N}} P^{1,0}_{n_g,n_m,n_x} - \sum_{n_x \in N} P^{1,0}_{n_g,n_m,n_x}\right) \\
\leq \ & \frac{1}{2}\phi\left(\sum_{n_x \in \mathcal{N}} P^{1,1}_{n_g,n_m,n_x}, \sum_{n_x \in \mathcal{N}} P^{1,0}_{n_g,n_m,n_x}\right) + \frac{1}{2}\phi\left(\sum_{n_x \in N} P^{1,0}_{n_g,n_m,n_x}, \sum_{n_x \in N} P^{1,1}_{n_g,n_m,n_x}\right) \\
\leq \ & \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\},o)], P^{01,10}[\mathcal{OC}(\{n_g\},o)]\right) + \frac{1}{2}\sum_{n_x \in N}\phi\left(P^{1,0}_{n_g,n_m,n_x}, P^{1,1}_{n_g,n_m,n_x}\right)
\end{aligned}
$$

Thus, overall we get

$$
\begin{aligned}
& \sum_{o \in Obs_{g,\mathrm{REL}}} \phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \\
\leq \ & \sum_{\substack{o \in Obs_{g,\mathrm{REL}} \\ (n_1,n_2,n_3,n_4,n_5) := o}} \left(\sum_{n_g \in N} \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\},o)], P^{01,10}[\mathcal{OC}(\{n_g\},o)]\right)\right. \\
& \left. + \frac{1}{2}\sum_{n_x \in N}\phi\left(P^{0,1}_{n_2,n_3,n_x}, P^{0,0}_{n_2,n_3,n_x}\right) + \frac{1}{2}\sum_{n_x \in N}\phi\left(P^{1,0}_{n_2,n_3,n_x}, P^{1,1}_{n_2,n_3,n_x}\right)\right) \\
\leq \ & \sum_{n_g \in N}\sum_{o \in Obs_{g,\mathrm{REL}}} \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\},o)], P^{01,10}[\mathcal{OC}(\{n_g\},o)]\right) \\
& + \frac{1}{2}\sum_{\substack{n_g \in N \\ n_m \in N \\ n_x \in N}} \phi\left(P^{0,1}_{n_g,n_m,n_x}, P^{0,0}_{n_g,n_m,n_x}\right) + \frac{1}{2}\sum_{\substack{n_g \in N \\ n_m \in N \\ n_x \in N}} \phi\left(P^{1,0}_{n_g,n_m,n_x}, P^{1,1}_{n_g,n_m,n_x}\right) \\
= \ & \sum_{n_g \in N}\sum_{o \in Obs_{g,\mathrm{REL}}} \phi\left(P^{00,11}[\mathcal{OC}(\{n_g\},o)], P^{01,10}[\mathcal{OC}(\{n_g\},o)]\right) \\
& + \frac{1}{2}\sum_{\substack{n_g \in N \\ n_x \in N}} \mathbf{Impact}^{(01)(00)}_{\mathsf{indirect}}(n_g, n_x) + \frac{1}{2}\sum_{\substack{n_g \in N \\ n_x \in N}} \mathbf{Impact}^{(10)(11)}_{\mathsf{indirect}}(n_g, n_x).
\end{aligned}
$$

- $Obs_{x,\mathrm{REL}}$. Each observation in this set is of the form $o = (\perp, \perp, n_m, n_x, \mathrm{R}_b)$ and we have $\mathsf{core}(o) = \{\{n_x\}\}$. We assume $Core = \{n_x\} \subseteq N$, as otherwise $\phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) = \phi(0,0) = 0$. Since $\{n_x\} \in \mathsf{core}(o)$, by Lemmas 4 and 5 we get

$$\begin{aligned} & \phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \\ \leq\; & \phi\left(P^{00,11}[\mathcal{OC}(N \setminus \{n_m\}, o)], P^{01,10}[\mathcal{OC}(N \setminus \{n_m\}, o)]\right) \\ =\; & \phi\left(P^{00,11}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, n_m\}, o)], P^{01,10}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, n_m\}, o)]\right) \end{aligned}$$

We now distinguish two cases depending on the recipient R_b:

  - Case $\mathrm{R}_b = \mathrm{R}_0$. Then

$$\begin{aligned} & \phi\left(P^{00,11}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, n_m\}, o)],\right. \\ & \left.P^{01,10}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, n_m\}, o)]\right) \\ =\; & \frac{1}{2}\phi\left(\sum_{n_g \in \mathcal{N} \setminus N} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N} \setminus N} P^{1,0}_{n_g,n_m,n_x}\right) \\ =\; & \frac{1}{2}\phi\left(\sum_{n_g \in \mathcal{N}} P^{0,0}_{n_g,n_m,n_x} - \sum_{n_g \in N} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N}} P^{1,0}_{n_g,n_m,n_x} - \sum_{n_g \in N} P^{1,0}_{n_g,n_m,n_x}\right) \\ \leq\; & \frac{1}{2}\phi\left(\sum_{n_g \in \mathcal{N}} P^{0,0}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N}} P^{1,0}_{n_g,n_m,n_x}\right) + \frac{1}{2}\phi\left(\sum_{n_g \in N} P^{1,0}_{n_g,n_m,n_x}, \sum_{n_g \in N} P^{0,0}_{n_g,n_m,n_x}\right) \\ \leq\; & \phi\left(P^{00,11}[\mathcal{OC}(\{n_x\}, o)], P^{01,10}[\mathcal{OC}(\{n_x\}, o)]\right) + \frac{1}{2}\sum_{n_g \in N} \phi\left(P^{1,0}_{n_g,n_m,n_x}, P^{0,0}_{n_g,n_m,n_x}\right) \end{aligned}$$

  - Case $\mathrm{R}_b = \mathrm{R}_1$. Then

$$\begin{aligned} & \phi\left(P^{00,11}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, n_m\}, o)],\right. \\ & \left.P^{01,10}[\mathcal{OC}(\{n_x\}, o) \cap \mathcal{OC}_{\mathsf{blank}}(N \setminus \{n_x, n_m\}, o)]\right) \\ =\; & \frac{1}{2}\phi\left(\sum_{n_g \in \mathcal{N} \setminus N} P^{1,1}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N} \setminus N} P^{0,1}_{n_g,n_m,n_x}\right) \\ =\; & \frac{1}{2}\phi\left(\sum_{n_g \in \mathcal{N}} P^{1,1}_{n_g,n_m,n_x} - \sum_{n_g \in N} P^{1,1}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N}} P^{0,1}_{n_g,n_m,n_x} - \sum_{n_g \in N} P^{0,1}_{n_g,n_m,n_x}\right) \\ \leq\; & \frac{1}{2}\phi\left(\sum_{n_g \in \mathcal{N}} P^{1,1}_{n_g,n_m,n_x}, \sum_{n_g \in \mathcal{N}} P^{0,1}_{n_g,n_m,n_x}\right) + \frac{1}{2}\phi\left(\sum_{n_g \in N} P^{0,1}_{n_g,n_m,n_x}, \sum_{n_g \in N} P^{1,1}_{n_g,n_m,n_x}\right) \\ \leq\; & \phi\left(P^{00,11}[\mathcal{OC}(\{n_x\}, o)], P^{01,10}[\mathcal{OC}(\{n_x\}, o)]\right) + \frac{1}{2}\sum_{n_g \in N} \phi\left(P^{0,1}_{n_g,n_m,n_x}, P^{1,1}_{n_g,n_m,n_x}\right) \end{aligned}$$

  Thus, overall we get (analogously to the case for $Obs_{g,\mathrm{REL}}$)

$$\begin{aligned} & \sum_{o \in Obs_{x,\mathrm{REL}}} \phi\left(P^{00,11}[\mathcal{OC}(N,o)], P^{01,10}[\mathcal{OC}(N,o)]\right) \\ \leq\; & \sum_{n_x \in N} \sum_{o \in Obs_{x,\mathrm{REL}}} \phi\left(P^{00,11}[\mathcal{OC}(\{n_x\}, o)], P^{01,10}[\mathcal{OC}(\{n_x\}, o)]\right) \end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{n_g\in N\\ n_x\in N}}\mathbf{Impact}^{(10)(00)}_{\mathsf{indirect}}(n_g,n_x)+\frac{1}{2}\sum_{\substack{n_g\in N\\ n_x\in N}}\mathbf{Impact}^{(01)(11)}_{\mathsf{indirect}}(n_g,n_x).$$

- $Obs_{r,\mathrm{REL}}$. This set only contains the observation $o_r = (\bot,\bot,\bot,\bot,\bot)$ and by assumption $P^{00,11}[\mathcal{OC}(N,(\bot,\bot,\bot,\bot,\bot))] \leq P^{01,10}[\mathcal{OC}(N,(\bot,\bot,\bot,\bot,\bot))]$. Thus, $\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)=0$.

As these sets define a partitioning of Obs, we get

$$\begin{aligned}
&\sum_{o\in Obs}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)\\
=&\sum_{o\in Obs_{gx,\mathrm{REL}}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)+\sum_{o\in Obs_{gm,\mathrm{REL}}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)\\
&+\sum_{o\in Obs_{mx,\mathrm{REL}}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)+\sum_{o\in Obs_{m}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)\\
&+\sum_{o\in Obs_{g}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)+\sum_{o\in Obs_{x}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)\\
&+\sum_{o\in Obs_{r}}\phi\left(P^{00,11}[\mathcal{OC}(N,o)],P^{01,10}[\mathcal{OC}(N,o)]\right)\\
\leq&\sum_{\substack{n\in N\\ n'\in N}}\left(\sum_{o\in Obs_{gx,\mathrm{REL}}\cup Obs_{gm,\mathrm{REL}}\cup Obs_{mx,\mathrm{REL}}}\phi\left(P^{00,11}[\mathcal{OC}(\{n,n'\},o)],P^{01,10}[\mathcal{OC}(\{n,n'\},o)]\right)\right)\\
&+\sum_{n\in N}\left(\sum_{o\in Obs_{m}}\phi\left(P^{00,11}[\mathcal{OC}(\{n\},o)],P^{01,10}[\mathcal{OC}(\{n\},o)]\right)\right.\\
&+\sum_{o\in Obs_{g}}\phi\left(P^{00,11}[\mathcal{OC}(\{n\},o)],P^{01,10}[\mathcal{OC}(\{n\},o)]\right)\\
&+\frac{1}{2}\sum_{n_x\in N}\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n_x)+\frac{1}{2}\sum_{n_x\in N}\mathbf{Impact}^{(10),(11)}_{\mathsf{indirect}}(n,n_x)\\
&+\sum_{o\in Obs_{x}}\phi\left(P^{00,11}[\mathcal{OC}(\{n\},o)],P^{01,10}[\mathcal{OC}(\{n\},o)]\right)\\
&\left.+\frac{1}{2}\sum_{n_g\in N}\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n_g,n)+\frac{1}{2}\sum_{n_g\in N}\mathbf{Impact}^{(01),(11)}_{\mathsf{indirect}}(n_g,n)\right)\\
\leq&\sum_{n\in N}\left(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(\{n\})+\sum_{n'\in N}\left(\mathbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(\{n,n'\})\right)\right.\\
&+\sum_{n'\in N}\left(+\frac{1}{2}\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n')+\frac{1}{2}\mathbf{Impact}^{(10),(11)}_{\mathsf{indirect}}(n,n')\right.\\
&\left.\left.+\frac{1}{2}\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n',n)+\frac{1}{2}\mathbf{Impact}^{(01),(11)}_{\mathsf{indirect}}(n',n)\right)\right)
\end{aligned}$$

$\square$

4.6 Approximating the Set of Compromised Nodes

It remains to be shown that the guarantees we compute can be approximated for every budget adversary. To this end we show that the guarantee we get by maximizing the (slightly over-approximated) impact of every node leads to a bound on the advantage of every adversary from the respective class.

Lemma 9 (Approximating N)**.** *For every anonymity notion α_X with $X \in \{\mathrm{SA}, \mathrm{RA}, \mathrm{REL}\}$, all senders $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$, all recipients $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$, every path selection algorithm ps, and every budget B and every cost function f, let $\mathcal{A}_f^B$ be any budget adversary that compromises the nodes in the set $N \overset{B,f}{\subseteq} \mathcal{N}$. Then the impact of compromising N can be bounded as follows:*

(1) $\mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(N \cup \{\mathrm{R}_0\}) \leq \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(\mathrm{R}_0) + \max\limits_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum\limits_{n \in N^\dagger} \Big(\mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(n) + \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{ind}}(n, \mathcal{A}_f^B)\Big).$

(2) $\mathbf{Impact}_{\mathrm{RA}}^{\mathsf{obs}}(N \cup \{\mathrm{S}_0\}) \leq \mathbf{Impact}_{\mathrm{RA}}^{\mathsf{obs}}(\mathrm{S}_0) + \max\limits_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum\limits_{n \in N^\dagger} \Big(\mathbf{Impact}_{\mathrm{RA}}^{\mathsf{obs}}(n) + \mathbf{Impact}_{\mathrm{RA}}^{\mathsf{ind}}(n, \mathcal{A}_f^B)\Big).$

(3) $\mathbf{Impact}_{\mathrm{REL}}^{\mathsf{obs}}(N) \leq \max\limits_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum\limits_{n \in N^\dagger} \Big(\mathbf{Impact}_{\mathrm{REL}}^{\mathsf{combined}}(n) + \mathbf{Impact}_{\mathrm{REL}}^{\mathsf{ind}}(n, \mathcal{A}_f^B)\Big)$

Proof. Let $\mathrm{S}_0, \mathrm{S}_1 \in \mathcal{S}$ be two senders and $\mathrm{R}_0, \mathrm{R}_1 \in \mathcal{R}$ be two recipients, and let $N' \overset{B,f}{\subseteq} \mathcal{N}$ be any set of compromised Tor nodes within the budget. We show each part separately. In each part we use the fact that $\mathbf{Impact}_{\mathsf{indirect}}^{(ab),(cd)}(n, n) = 0$, which holds since a node cannot occur in the same circuit twice with non-zero probability.

(1) By Lemma 6 we know that for $N = N' \cup \{\mathrm{R}_0\}$ we have

$$
\begin{aligned}
& \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(N) \\
= \; & \sum_{o \in Obs} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big) \\
\leq \; & \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(\mathrm{R}_0) + \sum_{n \in N'} \Bigg(\mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(n) + \mathbf{Impact}_{\mathsf{Rec1}}(n) \\
& + \sum_{n' \in N' \setminus \{n\}} \Big(\mathbf{Impact}_{\mathsf{indirect}}^{(10)(00)}(n, n') + \mathbf{Impact}_{\mathsf{Rec2}(n,n')}\Big)\Bigg) \\
\leq \; & \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(\mathrm{R}_0) + \max_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n \in N^\dagger} \Bigg(\mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(n) + \mathbf{Impact}_{\mathsf{Rec1}}(n) \\
& + \max_{N^* \overset{B-f(n),f}{\subseteq} \mathcal{N}} \sum_{n' \in N^*} \Big(\mathbf{Impact}_{\mathsf{indirect}}^{(10)(00)}(n, n') + \mathbf{Impact}_{\mathsf{Rec2}(n,n')}\Big)\Bigg) \\
= \; & \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(\mathrm{R}_0) + \max_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n \in N^\dagger} \Big(\mathbf{Impact}_{\mathrm{SA}}^{\mathsf{obs}}(n) + \mathbf{Impact}_{\mathrm{SA}}^{\mathsf{ind}}(n, \mathcal{A}_f^B)\Big).
\end{aligned}
$$

(2) By Lemma 7 we know that for $N = N' \cup \{\mathrm{S}_0\}$ we have

$$
\begin{aligned}
& \mathbf{Impact}_{\mathrm{RA}}^{\mathsf{obs}}(N) \\
= \; & \sum_{o \in Obs} \phi\big(P^{00}[\mathcal{OC}(N,o)], P^{10}[\mathcal{OC}(N,o)]\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0) + \sum_{n\in N'} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \mathbf{Impact}_{\mathsf{Sen1}}(n) \\
&+ \sum_{n'\in N'\setminus\{n\}} \Big(\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n') + \mathbf{Impact}_{\mathsf{Sen2}}(n,n')\Big)\Big) \\
&\leq \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0) + \max_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N^\dagger} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \mathbf{Impact}_{\mathsf{Sen1}}(n) \\
&+ \max_{N^* \overset{B-f(n),f}{\subseteq} \mathcal{N}} \sum_{n'\in N^*} \Big(\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n') + \mathbf{Impact}_{\mathsf{Sen2}}(n,n')\Big)\Big) \\
&= \mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(\mathrm{S}_0) + \max_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N^\dagger} \Big(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{RA}}(n) + \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{RA}}(n,\mathcal{A}^B_f)\Big).
\end{aligned}
$$

(3) By Lemma 8 we know that for $N = N'$ we have

$$
\begin{aligned}
&\sum_{o\in Obs} \phi\big(P^{00,11}[\mathcal{CO}(N,o)], P^{01,10}[\mathcal{CO}(N,o)]\big) \\
\leq &\sum_{n\in N} \Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(\{n\}) + \sum_{n'\in N} \Big(\mathbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(\{n,n'\})\Big) \\
&+ \sum_{n'\in N} \Big(+\frac{1}{2}\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n') + \frac{1}{2}\mathbf{Impact}^{(10),(11)}_{\mathsf{indirect}}(n,n') \\
&+\frac{1}{2}\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n',n) + \frac{1}{2}\mathbf{Impact}^{(01),(11)}_{\mathsf{indirect}}(n',n)\Big)\Bigg) \\
= &\sum_{n\in N} \Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(\{n\}) + \sum_{n'\in N\setminus n} \Big(\mathbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(\{n,n'\})\Big) \\
&+ \sum_{n'\in N\setminus n} \Big(+\frac{1}{2}\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n') + \frac{1}{2}\mathbf{Impact}^{(10),(11)}_{\mathsf{indirect}}(n,n') \\
&+\frac{1}{2}\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n',n) + \frac{1}{2}\mathbf{Impact}^{(01),(11)}_{\mathsf{indirect}}(n',n)\Big)\Bigg) \\
\leq &\max_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N^\dagger} \Bigg(\mathbf{Impact}^{\mathsf{obs}}_{\mathrm{REL}}(\{n\}) + \max_{N^* \overset{B-f(n),f}{\subseteq} \mathcal{N}} \sum_{n'\in N^*} \Big(\mathbf{Impact}^{\mathsf{obs-2}}_{\mathrm{REL}}(\{n,n'\})\Big) \\
&+ \max_{N^* \overset{B-f(n),f}{\subseteq} \mathcal{N}} \sum_{n'\in N^*} \Big(+\frac{1}{2}\mathbf{Impact}^{(01),(00)}_{\mathsf{indirect}}(n,n') + \frac{1}{2}\mathbf{Impact}^{(10),(11)}_{\mathsf{indirect}}(n,n') \\
&+\frac{1}{2}\mathbf{Impact}^{(10),(00)}_{\mathsf{indirect}}(n',n) + \frac{1}{2}\mathbf{Impact}^{(01),(11)}_{\mathsf{indirect}}(n',n)\Big)\Bigg) \\
= &\max_{N^\dagger \overset{B,f}{\subseteq} \mathcal{N}} \sum_{n\in N^\dagger} \Big(\mathbf{Impact}^{\mathsf{combined}}_{\mathrm{REL}}(n,\mathcal{A}^B_f) + \mathbf{Impact}^{\mathsf{ind}}_{\mathrm{REL}}(n,\mathcal{A}^B_f)\Big)
\end{aligned}
$$

□

Lemma 9 concludes our proof. We have shown that our anonymity guarantees indeed are sound upper bounds on the anonymity impact of a budget adversary.

5 Evaluation

In this section, we apply the computation proposed in Section 2 to recent Tor Metrics data [5] to quantify the anonymity impact of various budget adversaries. Each of these adversaries is evaluated for Tor's standard path selection algorithm (short: TorPS) and for several commonly considered variants including SelekTOR [21], DistribuTor [10], and LASTor [7]. Furthermore, we compute the anonymity impact of a path selection transition phase, i.e., a small number of pioneering users decide to use an alternative path selection algorithm, while the remaining users still run the original algorithm. We stress again that our evaluation assesses the anonymity impact of structural attacks only, without taking any potential countermeasures by the users or the Tor developers into account. Moreover, our evaluation only addresses the anonymity impact, and disregards a potentially detrimental performance impact of the respective path selection algorithm.

We structure the section as follows. We first briefly review the evaluated path selection algorithms. We then describe how we implemented the computation of $\mathbf{Impact}_X(A_f^B)$, how we selected senders and recipients for our analyses and which adversaries we considered in our evaluation. Finally, we present and discuss the corresponding results for the anonymity impact.

5.1 Evaluated Path Selection Algorithms

The computation of $\mathbf{Impact}_X$ in Section 2 relies on the probability distribution $\mathsf{ps}(\mathrm{S}_i, \mathrm{R}_j)$ over Tor circuits that is induced by the considered path selection algorithm for sender S_i and recipient R_j. For our evaluation, we concretely instantiate this distribution using the Tor network consensus data. The dependence of Tor's path selection algorithm on a multitude of parameters (e.g., individual flags and weights of Tor nodes, family relations, TCP ports required for a connection, parameters selected by senders, etc.) makes this a non-trivial task. In addition to Tor's default path selection algorithm, several variants have been proposed that strive to improve performance or anonymity under specific assumptions. All of these variants are characterized by the different probability distributions over Tor circuits they induce, and they can, hence, be evaluated by means of $\mathbf{Impact}_X$ as well.

TorPS – Tor's Standard Path Selection. We utilize MATor [10] for computing the distribution of TorPS. TorPS randomly selects nodes based on their flags in the Tor consensus (e.g., only nodes with the *guard* flag can become guard nodes, nodes with the flag *bad-exit* cannot be used as exit nodes, etc.) and, in addition, for the exit node based on whether the ports required by the user are offered by the Tor node. The path selection weights this random choice with the weight in the Tor consensus. If the sender has created at least one circuit, the guard node selected in that circuit is used as the guard node in all subsequent circuits as well. We refer to Tor's specification [6] and to MATor [10] for a more detailed description.

UniformTor. Many existing works abstract Tor's actual path selection as a uniform path selection algorithm. In this variant of Tor, all (eligible) entry, middle, and exit nodes are chosen with the same weight.

SelekTOR. SelekTOR [21] restricts the Tor client to always select an exit node from a specific country, e.g., in order to bypass geo-restrictions of websites and services. SelekTOR only differs from TorPS in that the weights of all Tor exit nodes outside the considered country are set to zero. In our evaluation, we consider a SelekTOR configuration with exit nodes in the US.

DistribuTor. DistribuTor [10] aims to mitigate the anonymity impact of Tor nodes with very large bandwidths by distributing the usage of guard nodes and exit nodes to the greatest possible extent. To this end, DistribuTor modifies the node weights so that nodes with a very large bandwidth are mostly used as middle nodes. Never used as middle nodes are those nodes with the guard or exit flag that have a low bandwidth.

LASTor. LASTor [7] groups Tor nodes together into so-called clusters based on their physical location (latitude and longitude), which it infers by their IP address via GeoIP. LASTor first selects a guard cluster, a middle cluster, and an exit cluster. In this weighted random selection, LASTor weights the clusters inverse to the distance of the path over them, where this path starts with the sender and ends with the recipient, thereby reducing the expected physical distance. After selecting clusters, LASTor selects a node from each cluster uniformly at random.

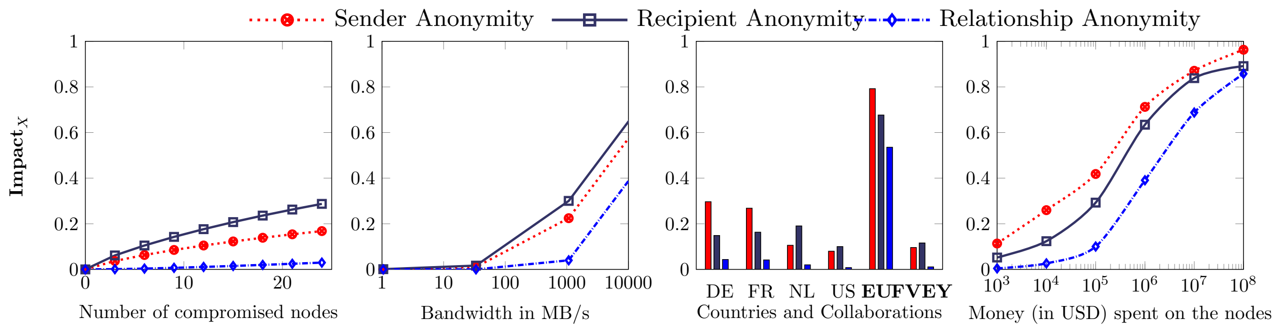

Figure 8: **Impact**$_X$ for different adversaries classes (from left to right): k-collusion, bandwidth, geographic, and monetary.

5.2 Implementation

We have implemented the computation of **Impact**$_X$ as an extension of the MATor [10] tool. MATor already takes care of computing the probability distribution over Tor circuits for a given Tor network consensus and the respective server descriptors. Hence, we added the calculations from Section 2 for any given budget adversary $A_f^B(\cdot)$, the desired anonymity notion α_X, and concrete senders $\mathrm{S}_0, \mathrm{S}_1$ and recipients $\mathrm{R}_0, \mathrm{R}_1$. To this end, we first compute the individual impacts $\mathbf{Impact}^{\mathsf{obs}}$ for all observations of individual nodes, pairs of nodes and – depending on the anonymity notion – relevant end points. Leveraging the computation from $\mathbf{Impact}_X^{\mathsf{obs}}$ to $\mathbf{Impact}_X$ requires us to solve the underlying integer maximization problems, e.g., to determine $N \subseteq \mathcal{N}$ such that $\sum_{n \in N} f(n) \leq B$ becomes maximal. While this problem is known to be NP-hard, we can solve it using a simple dynamic programming algorithm since the number of Tor nodes, and hence the size of the considered instances, is sufficiently small. The source code of our implementation is available [2].

5.3 Senders and Recipients

Recall that our computations are with respect to specific senders $\mathrm{S}_0, \mathrm{S}_1$ and recipients $\mathrm{R}_0, \mathrm{R}_1$. For the sake of evaluation, we hence consider concrete users in the following: the IP addresses from the affiliations of the PC chairs of PETS2015 and PETS2016. The first user, S_0, establishes a Tor circuit from Drexel University in Philadelphia; the second user, S_1, connects from Indiana University in Bloomington. As possible destinations, we have selected TU Darmstadt as R_0 and KU Leuven as R_1. For both destinations, we only required the HTTPS port 443 as the by far most widely used port for Tor connections (Tor is mainly used via the Tor-Browser bundle, which includes HTTPS-Everywhere).

5.4 Evaluated Adversary Classes

We consider the following six instances of budget adversaries in our analysis. We evaluate the first four instances for all considered path selection algorithms, whereas the last two instances are specific for TorPS.

***k*-collusion adversary.** We evaluate the k-collusion adversary for up to 25 compromised nodes, i.e., for a budget B ranging from 0 to 25.

Bandwidth adversary. We evaluate the bandwidth adversary for a budget B ranging from 1 MB/s to 10 GB/s.

Geographic adversary. We evaluate several adversaries that compromise all nodes in a given country or set of countries. We consider the four top countries according to offered Tor bandwidth: Germany, France, the Netherlands and the US. Moreover, we consider a collaboration of all countries of the European Union (abbreviated EU) and a collaboration of the US, New Zealand, Canada, the United Kingdom, and Australia (the so-called Five Eyes, abbreviated FVEY).

Monetary adversary. We evaluate a monetary adversary with a monthly budget B in US dollars, ranging from 10^3 to 10^8 US dollars. Recall that the cost function $f_\$$ assigns each node its monthly cost, depending on a price function $\mathsf{price}(n.\mathsf{provider}, n.\mathtt{BW})$. We instantiate price for the 8 largest providers hosting Tor nodes (Amazon, DigitalOcean, Hetzner, LeaseWeb, myLoc, Online, OVH, and STRATO), accounting for approximately $\frac{1}{3}$ of Tor bandwidth as follows. For each provider P in this list we set $\mathsf{price}(P, \mathtt{BW})$ to the cost of the cheapest server offered by this provider that has at least a bandwidth of $\mathtt{BW}$. For all remaining nodes (that are not hosted by these providers), $\mathsf{price}(\cdot)$ assigns the average consumer price per bandwidth, depending on the node's country, taken from Ookla's NetIndex [1] per country.

Vulnerable Tor Versions (TorPS only): We evaluate a predicate adversary for a critical Tor software update. The recently released update 0.2.6.10 solves many stability issues. The Tor blog recommends that every Tor node running an older version, especially an older version of 0.2.6 should update [4]. For presenting an example analysis of a Tor version predicate, we assume that there was a vulnerability in all Tor versions prior to the 0.2.6 branch and that the Vulnerable Tor adversary compromises all Tor nodes that run a Tor version between 0.2.6 and 0.2.6.10.

Guard Discovery Adversary (TorPS only): Recently, Tor has implemented a new strategy for selecting guard nodes called "One Fast Guard for Life" ([13]) that aims at improving longtime sender anonymity (and relationship anonymity to some extent). A sender selects a guard node once and uses it continuously over a period of 9 months to mitigate the danger imposed by frequently selecting fresh guard nodes and, moreover, to mitigate the danger imposed by selecting a recognizable set of guard nodes. We evaluate the anonymity impact of the first four aforementioned adversaries on the anonymity of guard nodes, effectively measuring their success to perform guard discovery attacks. Strictly speaking, this adversary does not constitute a single budget adversary, but a specific setting that is parametric in a considered budget adversary.

5.5 Results

Unless stated otherwise in a specific experiment, we used the following data in our evaluation: Our evaluation was conducted on Tor network consensus data over the course of one year (August 2014-July 2015), where we calculated the anonymity impact of the considered adversaries on four consensus data per day (at midnight, 6 a.m., noon and 6 p.m.). Additionally, we conducted extensive evaluations over the course of one year for the first four adversaries considered in Section 5.4, with the following budget choices: 10 compromised nodes for k-collusion adversary, 1 GB/s for the bandwidth adversary, Five Eyes for the geographic adversary, and 100,000 USD/mo monthly budget for the monetary adversary. The results – for sender, recipient and relationship anonymity – are depicted in Figure 13 in Appendix A, where we averaged the results per day to minimize day-time dependencies. For all other graphs we averaged all results to minimize any short-time impacts.

Remark: Please note that all of our evaluations consider the *worst-case* adversary for the respective class, i.e., we calculate and plot the maximal adversarial impact within this class to give an anonymity guarantee against this type of adversary. For example, the k-collusion adversary will compromise the k nodes with the highest impact for the considered anonymity notion and not just some subset of k nodes. Likewise, we consider the worst-case adversary for each notion separately, i.e., the adversary may compromise different nodes for each of the anonymity notion.

5.5.1 Evaluating Tor's Path Selection Algorithm

The results of our evaluation of TorPS are depicted in Figure 8.

Results – k-collusion adversary (left, Fig. 8). Our results confirm the (known) strong anonymity impact of a small number of high-bandwidth Tor nodes – a collusion of 10 Tor nodes results in a reduction of sender anonymity of 9.2%, a reduction of recipient anonymity of 15.4% and a reduction of relationship anonymity of 0.8%. The reduction of anonymity grows sub-linearly in k for sender anonymity and recipient anonymity, and more than linearly in k for relationship anonymity. A reduction of anonymity of $> 95\%$

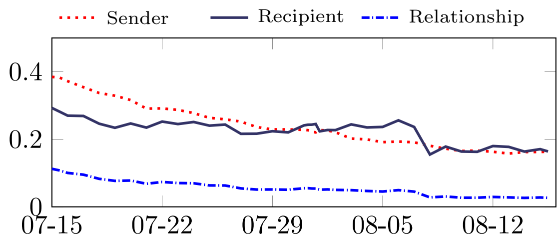

Figure 9: Adversarial **Impact** for sender, recipient and relationship anonymity against the *Vulnerable Tor Version* adversary.

amounts to 915 compromised nodes for sender anonymity, 330 compromised nodes for recipient anonymity, and 1230 compromised nodes for relationship anonymity.

Results – bandwidth adversary (2nd left, Fig. 8). The plot shows the reduction of anonymity on a logarithmic scale axis. An adversary compromising nodes with at most 1 GB/s of average bandwidth achieves a reduction of sender anonymity of 22.4%, a reduction of recipient anonymity of 30.0%, and a reduction of relationship anonymity of 3.8%. A reduction of anonymity of $> 95\%$ amounts to compromised bandwidth of 250 GB/s (for sender anonymity), 65 GB/s (for recipient anonymity), and 310 GB/s (for relationship anonymity).

Results – geographic adversary (2nd right, Fig. 8). Our results in particular show that no country on its own can successfully break relationship anonymity with a significant probability. However, their collaboration, in the case of the European Union, would be capable of deanonymize a significant amount of Tor traffic (anonymity impact of 53% for relationship anonymity), which significantly surpasses the advantage of the Five Eyes adversary (1.1%). We stress again that these results are specific to structural attacks against nodes, and do not take adversary-controlled network structure into account (such as monitoring traffic of domestic ISPs).

Results – monetary adversary (right, Fig. 8). An adversary running Tor nodes with a monthly cost of 100,000 USD at most reduces sender anonymity by 41.8%, recipient anonymity by 29.3% and relationship anonymity by 10.0%. The smaller reduction of recipient anonymity compared to sender anonymity stems from the fact that the prices of hosting guard nodes, on average, are significantly lower than the prices of hosting exit nodes. A reduction of anonymity of $> 95\%$ amounts to monthly costs of 8.75 Mio. USD (for sender anonymity), 27.5 Mio. USD (for recipient anonymity), and 40 Mio. USD (for relationship anonymity).

Vulnerable Tor Versions (Fig. 9). Our evaluation shows that on August 15th, i.e., one month after the release of the fix, the Tor version adversary still achieves a reduction of anonymity of 16.33% for sender anonymity, of 16.4% for recipient anonymity and of 2.67% for relationship anonymity. We refer to Figure 9 for a graph showing the reduction of anonymity of this adversary over the course of one month after the release of Tor version 0.2.6.10.

Anonymity Against Guard Discovery Attacks (Fig. 12). We selected a set of Tor consensus data (28 consensus data from July 23rd to July 29th 2015, taken each 6 hours) and from each of these consensus data we selected the top 25 guard nodes that share their /16 subnet with at least one exit node (which effectively affects around 45% of Tor guard nodes). Subsequently, we compared the anonymity impact of a budget adversary in distinguishing these guards, i.e., for all pairs of selected guard nodes (n_{g_0}, n_{g_1}) with $n_{g_0} \neq n_{g_1}$, we proceeded as follows: First, S_0 selects n_{g_0} as guard node and S_1 selects n_{g_1} as guard node. Second, we set the costs for compromising n_{g_0} or n_{g_1} to ∞ to disallow the adversary to compromise the respective guard nodes. Finally, we compute the sender anonymity impact.

Even without compromised nodes, a compromised recipient reduces sender anonymity by 2.12% on average. We furthermore evaluated our four aforementioned adversaries against the selected pairs of guard nodes. Our results, depicted in Figure 12 in Appendix A, show that the relationship between the adversarial advantage in the normal case and our guard discovery strategy differs strongly for the considered

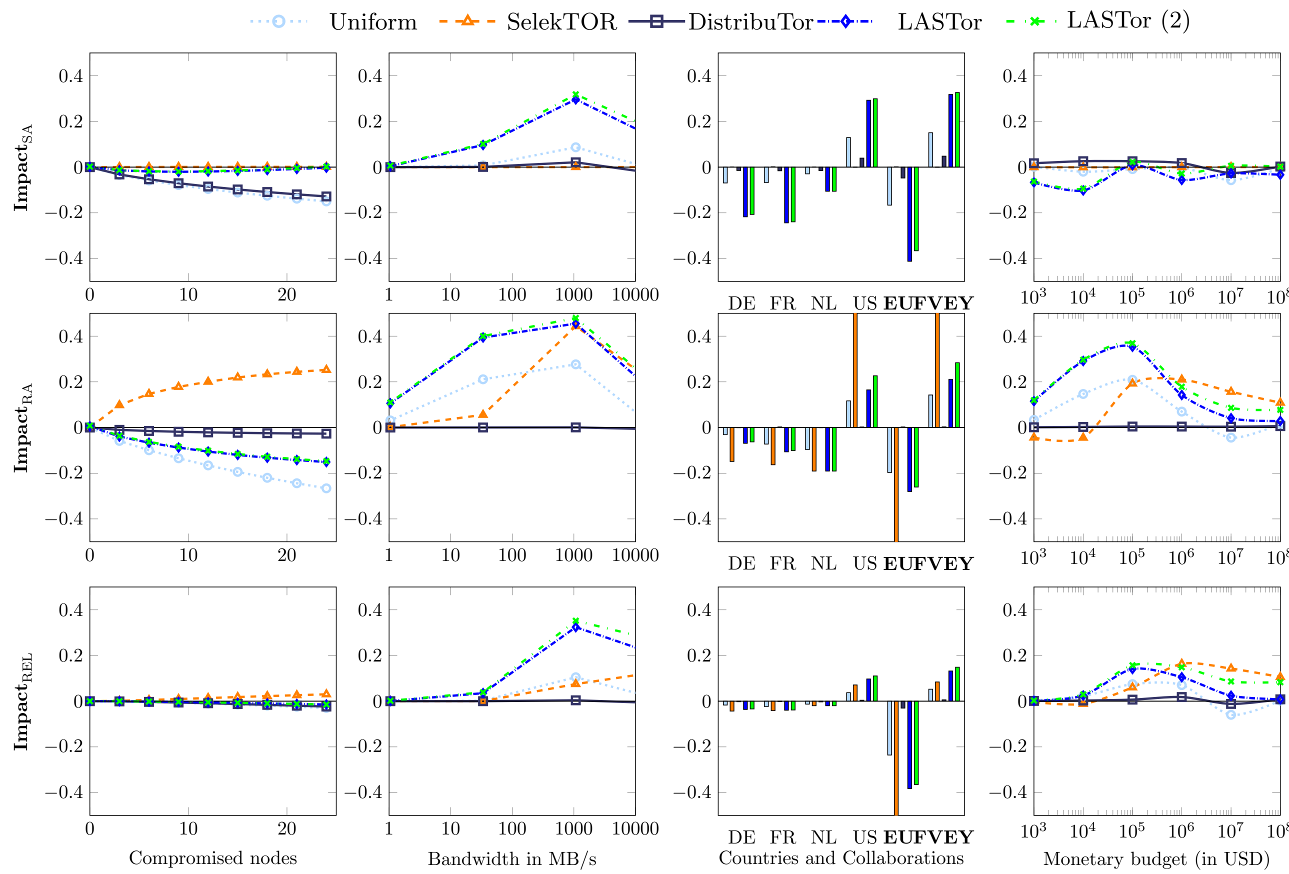

Figure 10: The difference between $\mathbf{Impact}_X$ for sender (top), recipient (middle) and relationship (bottom) anonymity of alternative path selection algorithms and TorPS. Cionsidered adversaries classes (from left to right): k-collusion, bandwidth, geographic, and monetary.

adversaries. For the k-collusion adversary for instance, with more than 8 compromised nodes, the adversary obtains a smaller anonymity impact in the guard-discovery scenario; in contrast to that, the difference in the anonymity impact of a monetary adversary becomes larger the more money it spends.

5.5.2 Alternative Path Selection Algorithms

The results of our evaluation of alternative path selection algorithms are depicted in Figure 10 – from top to bottom: sender anonymity, recipient anonymity, and relationship anonymity, showing the differences between the anonymity impacts of the alternative path selection algorithm and TorPS.

UniformTor. It is commonly believed that the uniform distribution over all nodes offers the highest degree of anonymity. Against k-collusion adversaries, this is certainly true. However, an adversary that compromises a certain amount of bandwidth can compromise a large number of low-bandwidth nodes even with a small budget. In case of a monetary adversary, 10 Mio. USD constitutes the break-even point at which the uniform path selection actually exhibits better anonymity guarantees (but at this point, anonymity has degenerated to a large extent anyway). The anomaly is caused by expensive nodes contributing a very small advantage in case of uniform path selection.

SelekTOR. As SelekTOR only restricts the choice for an exit node, sender anonymity is comparable to Tor, while relationship anonymity and in particular recipient anonymity suffers significantly, for essentially

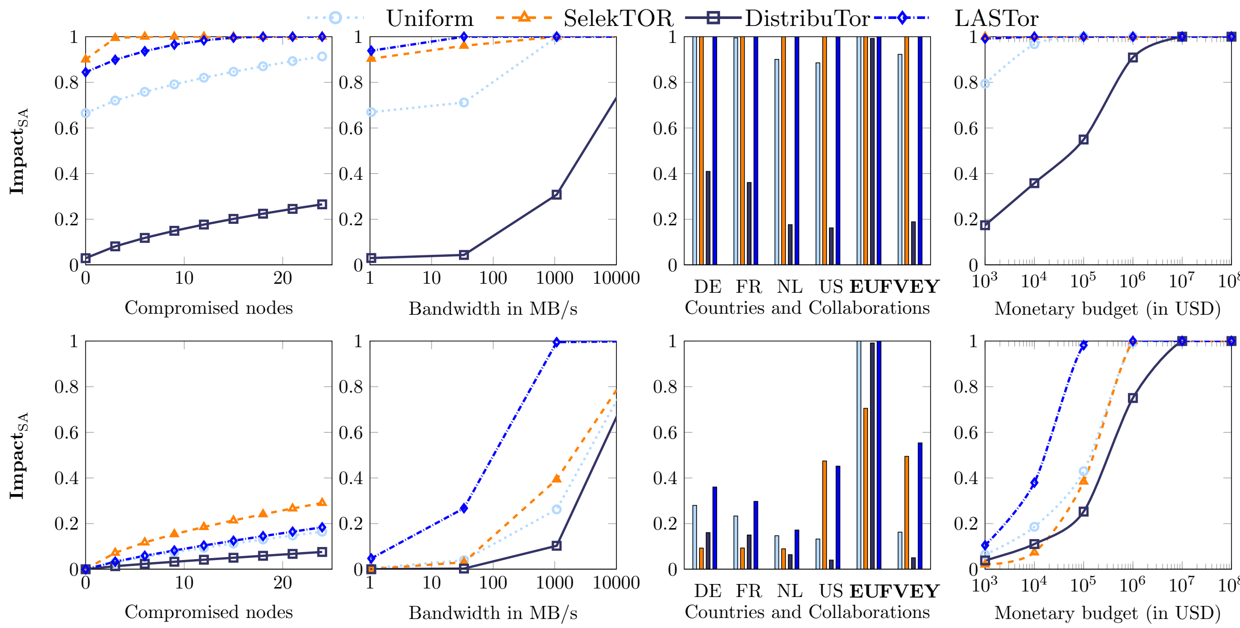

Figure 11: **Impact**$_X$ for sender and relationship anonymity *during the transition phase* for different adversarial strategies (from left to right): k-collusion, bandwidth, geographic, and monetary.

all considered adversaries. As expected, the geographic adversary that compromises all nodes within the US can break recipient anonymity with 100% probability (it always controls the exit node and can perform a traffic correlation).

DistribuTor. DistribuTor in particular modifies the selection of entry nodes by capping the possible weights of nodes at a certain point. Consequently, it achieves better sender anonymity and relationship anonymity guarantees against k-collusion adversaries (against which it has been designed), but does not exhibit a clear advantage against bandwidth adversaries. Since it uses modified weights (especially for entry nodes), its sender anonymity is, in comparison to Tor's sender anonymity, slightly less prone against European country adversaries, but more vulnerable against the US country adversary (since there are more smaller entry nodes in the US).

LASTor. Since our structural adversaries do not perform network-based attacks, an evaluation of LASTor, which is designed to counter network-based attacks, is slightly unfair. Still we gained interesting insights by our analysis. The uniform distribution of the node weights within LASTor's "geo-location buckets", leads to significantly better results against a k-collusion adversary. However, even a small amount of compromised bandwidth suffices for completely breaking anonymity, as even a small, compromised middle node can gain information about the location of both sender and recipient of a communication: entry and exit nodes have significantly different weights for different locations. Note that LASTor (as presented in [7]) additionally restricts circuits depending on whether traffic is expected to be routed through the same autonomous system twice. This additional restriction, however, only increases the advantage of a structural adversary.

To further evaluate LASTor, we also ran analyses swapping one recipient with one sender (TU Darmstadt with Indiana University). This is displayed as *LASTor (2)* in Figure 10. As expected, this more diverse selection of sender and recipient had a negative impact on the anonymity guarantees.

5.5.3 Transition Phase

In existing evaluations of alternative path selection algorithms, the major impeding anonymity factor that is typically omitted is the so-called transition phase, i.e., a small number of users is already using an alternative of Tor's path selection algorithm, whereas the vast majority still uses the standard one. Intuitively, there are not yet enough users with the alternative path selection algorithm to provide a sufficiently large anonymity set. Figure 11 depicts the adversary's advantage in such scenarios (for sender and relationship anonymity only, since these are the only two anonymity notions affected by this algorithmic transition).

Our analyses show that even adversaries that do not compromise any single Tor node have a tremendous advantage in distinguishing a user that relies on an alternative path selection algorithm from a regular Tor user: for SelekTOR, the adversary has an advantage of 92.89%, for LASTor an advantage of 85.87% and for the uniform path selection, the adversary has an advantage of 68.14%. In case of SelekTOR, this advantage arises mostly from the fact, that a normal user would choose a US exit node with only $\approx 7\%$ probability, whereas a SelekTOR user always uses such an exit node. For LASTor and the uniform path selection algorithm, circuits containing small nodes are chosen with a much higher probability in comparison to TorPS. The effect of the transition phase on DistribuTor is less drastic, but still noticeable. We attribute this result to the close similarity of DistribuTor and TorPS. However, the fact that the weights of entry nodes are heavily modified grants compromised middle nodes an advantage in distinguishing between a TorPS user and a DistribuTor user.

Mitigating the Risk of the Transition Phase. The high vulnerability of users that use a non-standard path selection algorithm indicates that a slow and voluntary transition from one algorithm to another might alienate the (few) users that migrate first and thus significantly weaken their anonymity. We think that as soon as a transition is necessary, the novel algorithm should be rolled out to all users at once, in order to shorten the transition phase. With this strategy, all users would intuitively remain in the same anonymity set.

6 Conclusion

In this paper, we have presented a rigorous methodology for quantifying the anonymity provided by Tor against a variety of structural attacks, i.e., adversaries that compromise Tor nodes and thereby perform eavesdropping attacks to deanonymize Tor users. We have made the following two tangible contributions. First, we have provided the first algorithmic approach for soundly computing the anonymity impact of such structural attacks against Tor. We have devised formalizations of various instantiations of structural attacks against Tor, and we have subsequently proven that the computed anonymity impact for each of these adversaries constitutes a worst-case anonymity bound for the cryptographic realization of Tor, up to a negligible additive factor. We have furthermore shown that our approach is sequentially composable, which we consider to be of independent interest. Second, we have demonstrated the applicability of our approach to large-scale analyses by performing the to date largest rigorous anonymity evaluation of Tor's anonymity. Concretely, we have established worst-case anonymity bounds against various structural attacks for Tor and for alternative path selection algorithms such as DistribuTor, SelekTOR, and LASTor, yielding the first rigorous, quantitative anonymity comparison between these algorithms. We have moreover quantified the anonymity impact of a path selection transition phase, showing that a small number of pioneering users who decide to run a (potentially improved) alternative path selection algorithm first while the majority of users still runs the original algorithm, face severe risks of being deanonymized.

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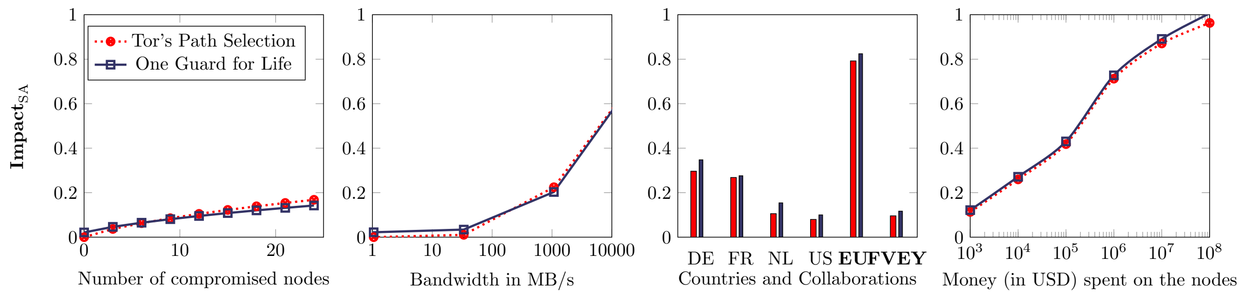

Figure 12: Adversarial **Impact** against guard discovery attack, for different strategies (from left to right): k-collusion adversary, resource-constrained bandwidth adversary, geographic adversary, and monetary adversary.

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A Appendix: Postponed Figures

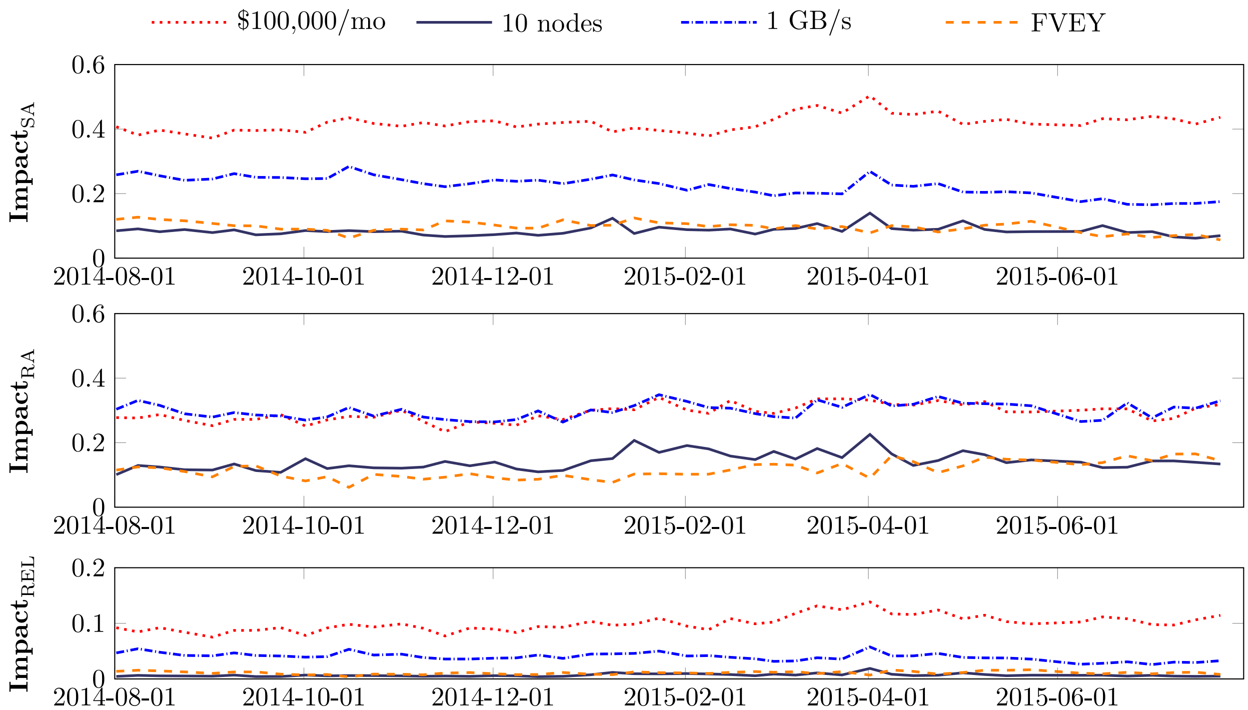

Figure 13: Changes in the adversarial **Impact** against selected adversarial strategies for Tor's path selection during the last year. Note, that the Y axis scales from 0 to 0.6 for sender and recipient anonymity plots, and from 0 to 0.3 for relationship anonymity.

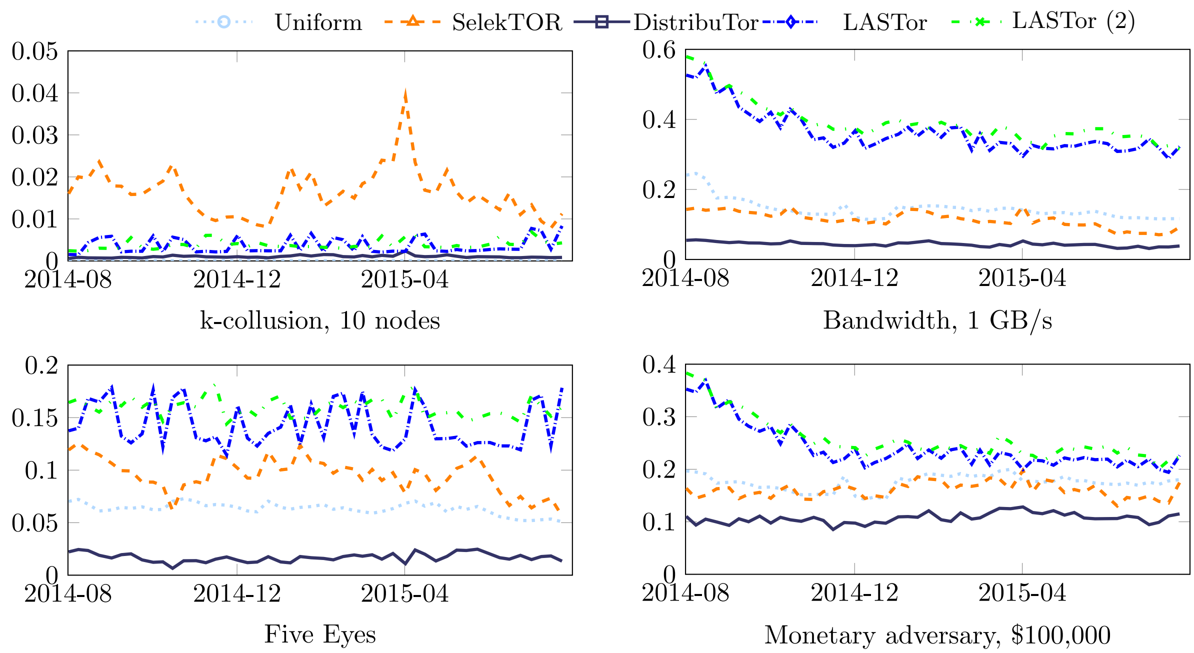

Figure 14: Changes in the adversarial **Impact** against relationship anonymity for different strategies (from top left): k-collusion adversary compromising 10 nodes, resource-constrained bandwidth adversary compromising 1 GB/s, Five Eyes countries and monetary adversary with budget of $100,000 per month, for alternative path selection algorithms. **Note**, that each plot has an individual scale on Y axis to improve readability.

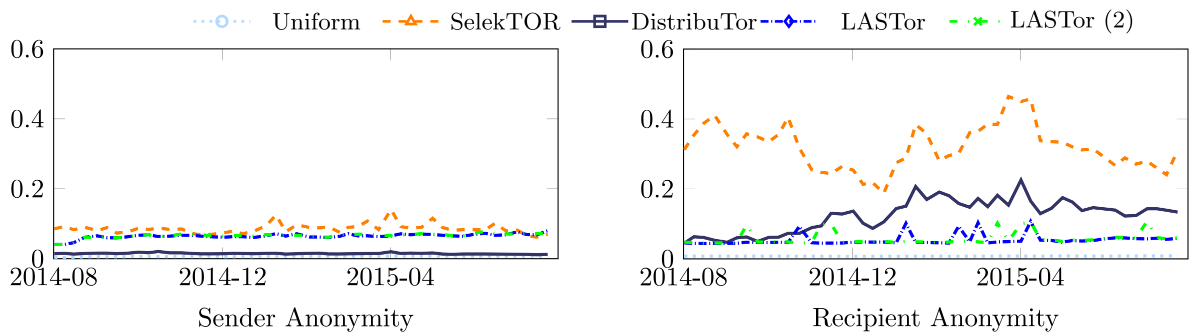

Figure 15: Changes in the adversarial **Impact** of k-collusion adversary compromising 10 nodes, in sender and recipient anonymity, for alternative path selection algorithms.

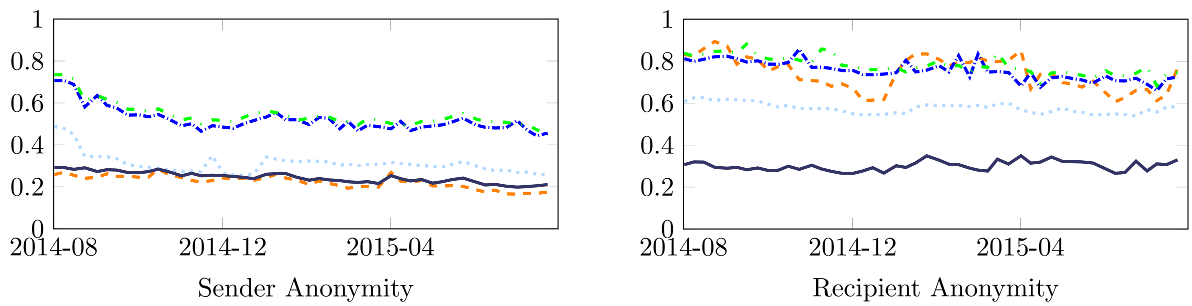

Figure 16: Changes in the adversarial **Impact** of resource-constrained bandwidth adversary compromising nodes of total bandwidth up to 1 GB/s, in sender and recipient anonymity, for alternative path selection algorithms.

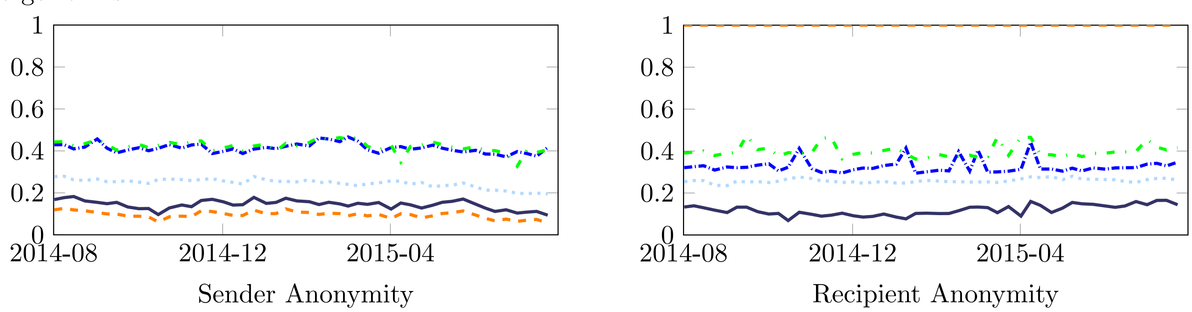

Figure 17: Changes in the adversarial **Impact** of Five Eyes countries, in sender and recipient anonymity, for alternative path selection algorithms.

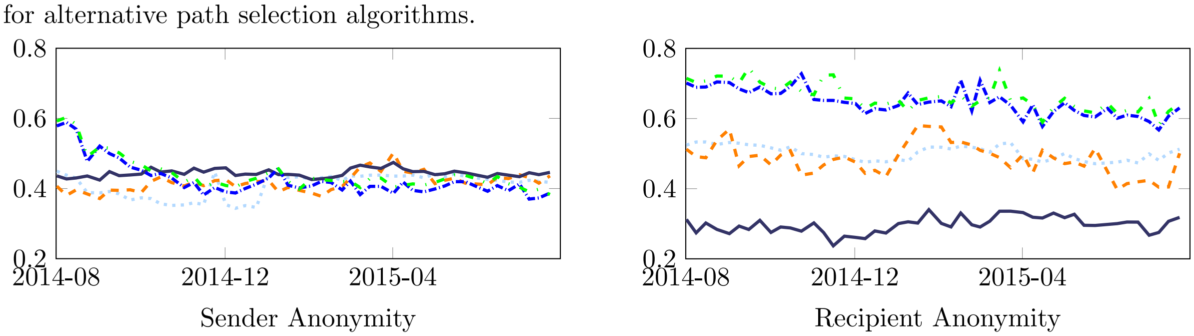

Figure 18: Changes in the adversarial **Impact** of monetary adversary with budget $100,000, in sender and recipient anonymity, for alternative path selection algorithms.