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# Osculating cones to Brill–Noether loci for line and vector bundles on curves and relative canonical resolutions of curves

## Dissertation

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## Abstract

The main subject of this thesis is the local geometry of the Brill–Noether loci $W_d(C)$ for a smooth curve C of genus $g$. Studying the local structure of the singular locus $W^1_d(C)$ of $W_d(C)$, we give local Torelli-type theorems for curves of genus $g \geq 4$ and $\frac{g+2}{2} \leq d \leq g-1$.

As an application, we also study the local structure of Brill–Noether loci for vector bundles on curves, especially of the generalised theta divisor. It results in a new proof of the generic injectivity of the theta map for general curves of high genus.

The second minor subject of this thesis is the study of the structure of the so-called relative canonical resolution of curves on scrolls swept out by elements in $W^1_d(C)$. We give a necessary and sufficient numerical condition for the balancedness of the bundle of quadrics for Brill–Noether general curves.



## Zusammenfassung

Das Hauptthema der vorliegenden Arbeit ist die Untersuchung der lokalen Geometrie des Brill–Noether Ortes $W_d(C)$ von glatten Kurven C vom Geschlecht $g$. Das Studieren der lokalen Strukturen des singulären Ortes $W^1_d(C)$ von $W_d(C)$ ermöglicht es uns, lokale Torelli Sätze für Kurven vom Geschlecht $g \geq 4$ und $\frac{g+2}{2} \leq d \leq g-1$ zu beweisen.

Als eine Anwendung davon, untersuchen wir ebenfalls die lokalen Strukturen der Brill–Noether Orte von Vektorbündeln auf Kurven, insbesondere des verallgemeinerten Theta Divisors. Wir geben einen neuen Beweis der generischen Injektivität der Theta Abbildung für allgemeine Kurven von großem Geschlecht.

Des Weiteren beschäftigen wir uns mit dem Studium der auftretenden Formen der sogenannten relativen kanonischen Auflösung von Kurven auf Regelvarietäten, welche von Elementen in $W^1_d(C)$ aufgespannt werden. Wir geben ein notwendiges und hinreichendes numerisches Kriterium an, das entscheidet, ob das Quadrikenbündel für Brill–Noether allgemeine Kurven balanciert ist.

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# Contents

# List of Figures

# Chapter 1

# Introduction and outline of the thesis

## 1.1 Brill–Noether theory on curves

### 1.1.1 Brill–Noether theory for line bundles

This brief introduction of the variety of special divisors on a curve follows [ACGH85].

Let C be a smooth curve of genus $g$ and let $\mathrm{Pic}^d(\mathrm{C})$ be the Picard variety consisting of isomorphism classes of line bundles of degree $d$ (or equivalently, of divisors of degree $d$ modulo linear equivalence). The subvariety $\mathrm{W}^r_d(\mathrm{C}) \subset \mathrm{Pic}^d(\mathrm{C})$ parametrising complete linear series of degree $d$ and projective dimension at least $r$, in symbols

$$\mathrm{W}^r_d(\mathrm{C}) = \left\{\mathrm{L} \in \mathrm{Pic}(\mathrm{C}) \mid \deg(\mathrm{L}) = d,\ h^0(\mathrm{C},\mathrm{L}) \geq r+1\right\},$$

is called the *Brill–Noether locus.* One can equip this locus with a scheme structure. We will recall the proof that the Brill–Noether locus is a determinantal variety in a more general setting in Section 1.1.2 (see also [ACGH85, Chapter IV, §3]). Furthermore, its expected dimension is then

$$\rho(g,d,r) = g - (r+1)(g-d+r),$$

the so-called *Brill–Noether number.* Studying first-order deformations of points in $\mathrm{W}^r_d(\mathrm{C})$ (we will present the details of this approach in Section 1.1.2), one gets the following basic theorem about the local geometry of the Brill–Noether locus.

**Theorem 1.1.1** ([ACGH85, Chapter IV, Proposition 4.2]). *Let* C *be a smooth curve of genus* $g$.

(a) *Let* $L \in W^r_d(C) \setminus W^{r+1}_d(C)$. *The tangent space to* $W^r_d(C)$ *at* L *is*

$$T_L(W^r_d(C)) = (\operatorname{Im} \mu)^{\perp}$$

*where*

$$\mu : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \longrightarrow H^0(C, \omega_C)$$

*is the cup-product map or* Petri map. *In particular,* $W^r_d(C)$ *is smooth of dimension* $\rho(g, d, r)$ *at* L *if and only if the Petri map* $\mu$ *is injective.*

(b) *Let* $L \in W^{r+1}_d(C)$. *Then*

$$T_L(W^r_d(C)) = T_L(\mathrm{Pic}^d(C)).$$

*In particular, if* $W^r_d(C)$ *has expected dimension* $\rho(g, d, r)$ *and* $r > d - g$ *(equivalently,* $\rho(g, d, r) < g$*), then* L *is a singular point of* $W^r_d(C)$.

**Basic results of Brill–Noether theory for line bundles.** Fundamental results concerning the existence of the Brill–Noether locus, its smoothness and further properties were proven by several authors. We will recall these major steps in Brill–Noether theory for line bundles. The first natural question whether $W^r_d(C)$ is nonempty if $\rho(g, d, r) \geq 0$ was independently answered by Kempf and Kleiman–Laksov.

**Theorem 1.1.2** (Existence theorem, [Kem71] and [KL72], [KL74]). *Let* C *be a smooth curve of genus* $g$. *Let* $d, r$ *be integers such that* $d \geq 1$ *and* $r \geq 0$. *If*

$$\rho(g, d, r) = g - (r + 1)(g - d + r) \geq 0,$$

*then* $W^r_d(C)$ *is nonempty. If furthermore* $r \geq d - g$, *every component of* $W^r_d(C)$ *has dimension at least equal to* $\rho(g, d, r)$.

Fulton and Lazarsfeld were able to prove the following Lefschetz (or Bertini) type of result.

**Theorem 1.1.3** (Connectedness theorem, [FL81]). *Let* C *be a smooth curve of genus* $g$. *Let* $d, r$ *be integers such that* $d \geq 1$ *and* $r \geq 0$. *If*

$$\rho(g,d,r) = g - (r+1)(g-d+r) \geq 1,$$

*then* $W^r_d(C)$ *is connected.*

A first rigorous proof of the Dimension theorem, which was already stated by Brill and Noether, was given by Griffiths and Harris.

**Theorem 1.1.4** (Dimension theorem, [GH80]). *Let* C *be a general curve of genus* $g$. *Let* $d, r$ *be integers such that* $d \geq 1$ *and* $r \geq 0$. *If*

$$\rho(g,d,r) = g - (r+1)(g-d+r) < 0,$$

*then* $W^r_d(C)$ *is empty. If* $\rho(g,d,r) \geq 0$, $W^r_d(C)$ *is reduced and of pure dimension* $\rho(g,d,r)$.

The injectivity of the cup-product map $\mu$ was first studied by Petri in a long forgotten paper [Pet25] which is the reason why we also call this map *Petri map*. Gieseker proved the injectivity of the Petri map for all line bundles over a general curve. Afterwards simpler proofs were given by Eisenbud–Harris [EH83] and Lazarsfeld [Laz86].

**Theorem 1.1.5** (Smoothness theorem/ Singularity theorem, [Gie82]). *Let* C *be a general curve of genus* $g$. *Let* D *be an effective divisor on* C. *Then the Petri map*

$$\mu : H^0(C, \mathcal{O}_C(D)) \otimes H^0(C, \omega_C \otimes \mathcal{O}_C(-D)) \longrightarrow H^0(C, \omega_C)$$

*is injective. In particular, for integers* $d, r$ *such that* $d \geq 1$, $r \geq 0$ *and* $r > d - g$,

$$\mathrm{Sing}(W^r_d(C)) = W^{r+1}_d(C).$$

We call a curve satisfying the Dimension and Smoothness theorem a *Brill–Noether general* or *Petri general* curve. A line bundle $L \in W^r_d(C)$ is called *Petri general* if the associated Petri map is injective.

**Tangent cone to** $W_d(C)$. In the following section, we will present Kempf's Singularity theorem, which describes the tangent cone to $W_d(C)$. We use the standard notation $W_d(C) = W^0_d(C)$. For an intrinsic definition of the tangent cone, a natural generalisation of the Zariski tangent space, see [ACGH85, Chapter II, §1]. For $W_d(C) \subset \mathrm{Pic}^d(C)$, the (projectivisation of the) tangent cone to $W_d(C)$ at a point L, denoted by $\mathcal{T}_L(W_d(C))$, is a natural subvariety of the canonical space $\mathbb{P}(H^0(C,\omega_C)^*)$, which is identified with the projectivisation of the tangent space $H^1(C,\mathcal{O}_C)$ to $\mathrm{Pic}^d(C)$ by Serre duality. Furthermore, it is closely related to the Petri map associated to L.

**Theorem 1.1.6** (Riemann–Kempf's Singularity theorem, [Kem73]). *Let* $L \in W^r_d(C) \subseteq W_d(C)$ *be a Petri general line bundle for* $d < g$. *Then,* $\mathcal{T}_L(W_d(C))$ *is a Cohen–Macaulay, reduced, normal subariety of* $H^0(C,\omega_C)^*$ *of degree* $\binom{h^1(C,L)}{h^0(C,L)-1}$ *whose ideal is generated by the maximal minors of the matrix associated to the Petri map*

$$\mu : H^0(C,L) \otimes H^0(C,\omega_C \otimes L^{-1}) \longrightarrow H^0(C,\omega_C).$$

*Its projectivisation is*

$$\mathbb{P}\mathcal{T}_L(W_d(C)) = \bigcup_{D\in|L|} \overline{D} \subset \mathbb{P}^{g-1}.$$

**Remark 1.1.7.** (a) For $d = g-1$, one recovers Riemann's Singularity theorem under the identification $W_{g-1}(C) = \Theta$ (see also [ACGH85, Chapter I, §3,4,5 and Chapter VI, §1]).

(b) For $r = 1$, we see that the projectivised tangent cone coincides with the scroll, which is swept out by the pencil $g^1_d = |L|$.

(c) A generalisation of Kempf's result to arbitrary Brill–Noether loci $W^r_d(C)$ can be found in [ACGH85, Chapter VI, Theorem 2.1].

The main object of this thesis will be the osculating cone to $W_d(C)$. It is a subvariety of the tangent cone, and in particular, a "finer" variety than the tangent cone to study the local geometry of $W_d(C)$. We will define this object in Section 2.3.2.

**The Torelli theorem** We would like to close the introduction to Brill–Noether theory for line bundles with the classical Torelli theorem originally proven in [Tor13].

The Picard variety $\mathrm{Pic}^0(C)$ (which we identify with the Jacobian of a curve) carries the structure of a complex torus. Using the exponential sequence $0 \to \mathbb{Z} \to \mathscr{O}_C \overset{\exp}{\to} \mathscr{O}_C^* \to 1$, one gets an isomorphism $\mathrm{Pic}^0(C) \cong H^1(C, \mathscr{O}_C)/H_1(C, \mathbb{Z})$. Furthermore, the complex torus admits a principal polarisation given by the theta divisor $\Theta$. The pair $(\mathrm{Pic}^0(C), \Theta)$ is a so-called principally polarised abelian variety (see [ACGH85, Chapter I, §3, 4]).

**Theorem 1.1.8** (Torelli). *Let* $C$ *and* $C'$ *be two smooth complex curves. If* $(\mathrm{Pic}^0(C), \Theta)$ *and* $(\mathrm{Pic}^0(C'), \Theta')$ *are isomorphic as principally polarised abelian varieties, then the curves* $C$ *and* $C'$ *are isomorphic.*

Many proofs of this theorem were given since Torelli's original proof in 1913 (see [ACGH85, Bibliographical Notes, Chapter VI]). We only mention the work of Kempf and Schreyer in [KS88] and Ciliberto and Sernesi in [CS92] and [CS95]. Using techniques and ideas of these papers, the main focus of our thesis (see Section 2 and 3) is to prove similar statements for other Brill–Noether loci $W_d(C)$.

### 1.1.2 Brill–Noether theory for vector bundles

We will recall the construction of the moduli space of stable bundles and the Brill–Noether locus as well as give an overview of the known results. We will follow Popa [Pop13] and Grzegorczyk–Teixidor i Bigas [GTiB09]. General references for the theory of moduli spaces are [LP97], [Ses82] and [HL10].

**The moduli space of stable vector bundles.** Let $C$ be a curve of genus $g$ over an algebraically closed field $\mathbb{K}$. In order to define a nice moduli space of vector bundles over $C$, we will need to consider the restriction to semistable vector bundles up to so-called S-equivalence.

Let $E \to C$ be a vector bundle of rank $r$. Its *degree* $d$ is the degree of the *determinant* $\det E := \Lambda^r E$. The *slope* of $E$ is the quotient $\mu(E) = \frac{d}{r} \in \mathbb{Q}$. We call a vector bundle *stable* (respectively, *semistable*) if

$$\mu(F) < \mu(E) \quad (\text{resp., } \mu(F) \leq \mu(E))$$

for all proper subbundles $F \subset E$ [1]. Now let $E \to C$ be a semistable vector bundle of slope $\mu$. Then there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_p = E$$

where the $E_i$ are (semistable) of slope $\mu$ and each quotient $E_i/E_{i-1}$ is a stable vector bundle. Note that any two such filtrations have the same length and, upon reordering, isomorphic stable factors (see [LP97, p.76] or [HL10, Prop. 1.5.2]). Then we define

$$gr(E) = \bigoplus_{i=1}^{p} E_i/E_{i-1},$$

the so-called *graded object* associated to E. Two semistable vector bundles are called S*-equivalent* if their associated graded objects are isomorphic. Note that S-equivalence is a slightly weaker condition than isomorphism.

**Theorem 1.1.9** ([LP97, §4-8] or [Ses82, §1]). *Fix integers $r$ and $d$ with $r \geq 1$, and a line bundle* $L \in \mathrm{Pic}^d(C)$.

*(a) There exist coarse moduli spaces $U_C(r,d)$ and $SU_C(r,L)$ for* S*-equivalence classes of semistable rank $r$ vector bundles over* C *of degree $d$ and determinant* L*, respectively.*

*(b) $U_C(r,d)$ and $SU_C(r,L)$ are projective, irreducible varieties of dimension $r^2(g-1)+1$ and $(r^2-1)(g-1)$, respectively.*

*(c) The stable loci $U'_C(r,d)$ and $SU'_C(r,L)$ are dense in $U_C(r,d)$ and $SU_C(r,L)$. The singular locus of $U_C(r,d)$ consists of the strictly semistable points except when $r = 2, g = 2$ and $d$ is even; in this case it is smooth.*

*(d) $U_C(r,d)$ and $SU_C(r,L)$ are fine moduli spaces if and only if $\gcd(r,d) = 1$, equivalently, if every bundle in $U_C(r,d)$ is actually stable.*

[1]Note that we get an equivalent definition if we replace subbundles with arbitrary coherent subsheaves (see [LP97, p. 73]).

**The Brill–Noether locus.** In the following, we only consider the moduli space $U'_{\mathrm{C}}(r,d)$ of stable vector bundles of rank $r$ and degree $d$. Our main object of consideration is the *generalised Brill–Noether locus*, defined set-theoretically by

$$\mathrm{B}^k_{r,d} \;=\; \left\{\mathrm{E} \in U'_{\mathrm{C}}(r,d) : h^0(\mathrm{C},\mathrm{E}) \geq k\right\}.$$

Note that if $r = 1$, the Brill–Noether locus $\mathrm{B}^k_{1,d}$ is classically denoted $\mathrm{W}^{k-1}_d(\mathrm{C})$ (see also Section 1.1.1). We recall the construction of $\mathrm{B}^k_{r,d}$ as a scheme following Grzegorczyk–Teixidor i Bigas [GTiB09, §2] (see also [ACGH85, Chapter IV, §3] for the line bundle case).

We assume that $\gcd(r,d) = 1$, then $U_{\mathrm{C}}(r,d) = U'_{\mathrm{C}}(r,d)$ and there exists a Poincaré bundle $\mathscr{E}$ on $\mathrm{C} \times U_{\mathrm{C}}(r,d)$ by Theorem 1.1.9 (d). Let $p_1 : \mathrm{C} \times U_{\mathrm{C}}(r,d) \to \mathrm{C}$ and $p_2 : \mathrm{C} \times U_{\mathrm{C}}(r,d) \to U_{\mathrm{C}}(r,d)$ be the two projections. We consider a divisor D on C of large degree $\deg \mathrm{D} \geq 2g - 1 - \frac{d}{r}$ and the short exact sequence

$$0 \to \mathscr{O}_{\mathrm{C}} \to \mathscr{O}_{\mathrm{C}}(\mathrm{D}) \to \mathscr{O}_{\mathrm{D}}(\mathrm{D}) \to 0.$$

We take the pullback via $p_1$ to $\mathrm{C} \times U_{\mathrm{C}}(r,d)$, tensor with $\mathscr{E}$ and push forward to $U_{\mathrm{C}}(r,d)$. This results in the sequence

$$0 \to p_{2*}(\mathscr{E}) \to p_{2*}(\mathscr{E} \otimes p_1^*(\mathscr{O}_{\mathrm{C}}(\mathrm{D}))) \xrightarrow{\gamma} p_{2*}(\mathscr{E} \otimes p_1^*(\mathscr{O}_{\mathrm{D}}(\mathrm{D}))) \to \ldots.$$

Now $p_{2*}(\mathscr{E} \otimes p_1^*(\mathscr{O}_{\mathrm{D}}(\mathrm{D})))$ is a vector bundle of rank $r \cdot \deg \mathrm{D}$ whose fiber over $\mathrm{E} \in U_{\mathrm{C}}(r,d)$ is exactly $\mathrm{H}^0(\mathrm{C}, \mathrm{E} \otimes \mathscr{O}_{\mathrm{D}}(\mathrm{D}))$. We will compute $h^0(\mathrm{C}, \mathrm{E}(\mathrm{D}))$. The degree of the Serre dual $\mathrm{K}_{\mathrm{C}} \otimes \mathrm{E}^*(-\mathrm{D})$ is $-r$ and since E is stable, $h^0(\mathrm{K}_{\mathrm{C}} \otimes \mathrm{E}^*(-\mathrm{D})) = 0$. Hence, $h^0(\mathrm{C}, \mathrm{E}(\mathrm{D})) = d + r \deg \mathrm{D} + r(1-g) =: \alpha$ is independent of the bundle $\mathrm{E} \in U_{\mathrm{C}}(r,d)$ and thus, $p_{2*}(\mathscr{E} \otimes p_1^*(\mathscr{O}_{\mathrm{C}}(\mathrm{D})))$ is a vector bundle of rank $\alpha$. Finally, we define the *generalised Brill–Noether locus* $\mathrm{B}^k_{r,d}$ as the locus where the map

$$p_{2*}(\mathscr{E} \otimes p_1^*(\mathscr{O}_{\mathrm{C}}(\mathrm{D}))) \xrightarrow{\gamma} p_{2*}(\mathscr{E} \otimes p_1^*(\mathscr{O}_{\mathrm{D}}(\mathrm{D})))$$

has rank at most $\alpha - k$ (equivalently, the fiber of $p_{2*}(\mathscr{E})$ at a point $\mathrm{E} \in \mathrm{B}^k_{r,d}$, which is $\mathrm{H}^0(\mathrm{C},\mathrm{E})$, has dimension at least $k$).

By the theory of determinantal varieties (see [ACGH85, Chapter II and Chapter IV, §3]), the generalised Brill–Noether locus is of dimension at least

$$\dim U_{\mathrm{C}}(r,d) - k \cdot (r \deg \mathrm{D} - (\alpha - k)) = \rho^k_{r,d}$$

at any point, with expected equality. Furthermore, if $B^{k+1}_{r,d} \neq U_C(r,d)$, then $B^{k+1}_{r,d}$ is contained in the singular locus, again with expected equality.

The expected dimension of $B^k_{r,d}$ is called the *Brill–Noether number*

$$\rho^k_{r,d} = r^2(g-1)+1-k(k-d+r(g-1)).$$

If the degree $d$ and the rank $r$ are not coprime, there does not exist a Poincaré bundle on $C \times U'_C(r,d)$. Though one can find an étale affine neighbourhood of a point $E \in U'_C(r,d)$ with $h^0(C,E) = k \geq 1$ where a Poincaré bundle exists and do the construction there.

The main difference between classical Brill–Noether theory (for line bundles) and Brill–Noether theory for vector bundles is that the natural generalisation of most of the basic results (Existence theorem, Dimension theorem and so on) are false in general. An overview of the state of the art is given in [GTiB09].

**The tangent spaces to $U'_C(r,d)$ and $B^k_{r,d}$.** We will now describe the tangent spaces to $U'_C(r,d)$ and $B^k_{r,d}$ (see [GTiB09, §2, p.3] or [ACGH85, Chapter IV, §4]). Let $E \in U'_C(r,d)$ be a stable vector bundle. The tangent space to $U'_C(r,d)$ at E is identified with $H^1(C,\mathrm{End}(E)) = H^1(C, E^* \otimes E)$ in the following standard procedure. A tangent vector to $U'_C(r,d)$ at E is a map from $\mathrm{Spec}(\mathbb{K}[\varepsilon]/\varepsilon^2)$ to $U'_C(r,d)$ whose image is supported at E (see [Har77, II, Exercise 2.8]). This is equivalent to a vector bundle $\mathscr{E}_\varepsilon$ on $C \times \mathrm{Spec}(\mathbb{K}[\varepsilon]/\varepsilon^2)$ extending E. We get an exact sequence

$$0 \to E \to \mathscr{E}_\varepsilon \to E \to 0$$

and the class of this extension is the corresponding element in $H^1(C,\mathrm{End}(E))$. We describe the bundle $\mathscr{E}_\varepsilon$ in terms of an element $\varphi \in H^1(C,\mathrm{End}(E))$. We choose a open covering $\{U_i\}$ of C, and denote $U_{ij} = U_i \cap U_j$ the intersection. Let $\{g_{ij}\}$ be the transition functions of the bundle E. We then represent the class $\varphi$ as a coboundary $(\varphi_{ij})$ where $\varphi_{ij} \in H^0(U_{ij},\mathrm{End}(E))$. The vector bundle is now given as the trivial extension of E on $U_i \times \mathrm{Spec}(\mathbb{K}[\varepsilon]/\varepsilon^2)$, that is $E_{U_i} \oplus \varepsilon E_{U_i}$, with transition functions

$$\begin{pmatrix} g_{ij} & \varphi_{ij} \\ 0 & g_{ij} \end{pmatrix}$$

on $U_{ij}$.

In order to get the tangent space to $B^k_{r,d}$, we have to analyse when a section of E extends to a section of $\mathscr{E}_\varepsilon$. We assume that a global section $s \in H^0(C, E)$ lifts to a section $s_\varepsilon$ of $\mathscr{E}_\varepsilon$. This means that there exist local sections $s'_i \in H^0(U_i, E_{U_i})$ such that $(s'_i, s|_{U_i})$ define $s_\varepsilon$. The local sections have to satisfy the gluing data

$$\begin{pmatrix} g_{ij} & \varphi_{ij} \\ 0 & g_{ij} \end{pmatrix} \cdot \begin{pmatrix} s'_i \\ s|_{U_i} \end{pmatrix} = \begin{pmatrix} s'_j \\ s|_{U_j} \end{pmatrix}.$$

The second row $(g_{ij}(s|_{U_i}) = s|_{U_j})$ is always satisfied since $s$ is a global section of E. The first condition can be written as

$$\varphi_{ij}(s) = s'_j - g_{ij}(s'_i),$$

equivalently, $\varphi_{ij}$ is a cocycle in $H^1(C, E)$. Finally, a section $s$ lifts to a section of $\mathscr{E}_\varepsilon$ if and only if

$$(\varphi_{ij}) \in \operatorname{Ker}\left(H^1(C, \operatorname{End}(E)) \to H^1(C, E), \nu_{ij} \mapsto \nu_{ij}(s)\right).$$

Note that the map in cohomology is induced by $s^* : \operatorname{End}(E) \to \operatorname{Hom}(\mathscr{O}_C, E), \nu \mapsto \nu \circ s$ for a section $s : \mathscr{O}_C \to E$.

If we dualise this map, we can reformulate the above fact as follows. For a vector bundle $E \in B^k_{r,d}$, the multiplication map of global sections

$$\mu : H^0(C, E) \otimes H^0(C, K_C \otimes E^*) \to H^0(C, E \otimes E^* \otimes K_C))$$

is called the ***Petri map***. The tangent space to $B^k_{r,d}$ at E is then identified with the orthogonal complement of the image of $\mu$.

Since E is stable we get $h^0(C, \operatorname{End}(E)) = 1$ and hence $h^1(C, \operatorname{End}(E)) = \dim U'_C(r, d)$. By the Riemann-Roch theorem, the Brill–Noether number is

$$\rho^k_{r,d} \; = \; h^1(C, \operatorname{End}(E)) - h^0(C, E) \cdot h^1(C, E)$$

and we have shown the well known fact that $B^k_{r,d}$ is smooth of dimension $\rho^k_{r,d}$ at E if and only if the Petri map $\mu$ is injective.

**The generalised theta divisor and the theta map.** Following [Pop13, §3, §5.2 and §7.1], we will define the generalised theta divisor and introduce the corresponding theta map.

For a line bundle $\mathrm{L} \in \mathrm{Pic}^{g-1-d}(\mathrm{C})$ there exists a theta divisor

$$\Theta_{\mathrm{L}} = \left\{ \mathrm{M} \in \mathrm{Pic}^{d}(\mathrm{C}) \mid h^0(\mathrm{C}, \mathrm{M} \otimes \mathrm{L}) \neq 0 \right\}$$

where $\Theta_{\mathrm{L}}$ is a translate of the usual theta divisor $\Theta$ on $\mathrm{Pic}^0(\mathrm{C}) = \mathrm{Jac}(\mathrm{C})$. A numerical condition to obtain a divisor is $\chi(\mathrm{M} \otimes \mathrm{L}) = 0$. We explain how to generalise this fact to vector bundles.

We fix a semistable vector bundle $\mathrm{E} \in U_{\mathrm{C}}(r,d)$. We need some notation. Let $h = \gcd(r,d)$, $r_0 = \frac{r}{h}$ and $d_0 = \frac{d}{h}$. Let F be a vector bundle of rank $\mathrm{rkF} = k \cdot r_0$ and degree $\deg \mathrm{F} = k \cdot (r_0(g-1) - d_0)$. Then, the Euler characteristic of the tensor product vanishes, that is, $\chi(\mathrm{E} \otimes \mathrm{F}) = 0$. Note that the following statements are independent of the choice of E in the S-equivalence class.

**Proposition 1.1.10** ([Pop13, p. 12]). *Let* F *be as above. If there exists an* $\mathrm{E} \in U_{\mathrm{C}}(r,d)$ *such that* $\mathrm{H}^0(\mathrm{C}, \mathrm{E} \otimes \mathrm{F}) = 0$*, then*

$$\Theta_{\mathrm{F}} := \left\{ \mathrm{E} \in U_{\mathrm{C}}(r,d) \mid h^0(\mathrm{C}, \mathrm{E} \otimes \mathrm{F}) \neq 0 \right\}$$

*is a divisor with a natural scheme structure. The same is true on* $SU_{\mathrm{C}}(r, \mathrm{L})$.

We call $\Theta_{\mathrm{F}}$ the *generalised theta divisor* on $U_{\mathrm{C}}(r,d)$. Note that the proof is similar to the construction of the generalised Brill–Noether locus and F as in Proposition 1.1.10 has to be semistable (see [Pop13, Exe. 2.8]). Furthermore, for a general choice of F, the generalised theta divisor exists.

Note that the generalised theta divisor is a special case of the so-called *twisted Brill–Noether locus*, set-theoretically defined as

$$\mathrm{B}^k_{r,d}(\mathrm{V}) \;=\; \left\{ \mathrm{E} \in U'_{\mathrm{C}}(r,d) : h^0(\mathrm{C}, \mathrm{V} \otimes \mathrm{E}) \geq k \right\}$$

where $\mathrm{V} \in U'_{\mathrm{C}}(r', d')$ (see also [TiB14] and Section 4.2).

In order to define the theta map we need one important fact about the moduli space of vector bundles due to Drézet and Narasimhan. We only state the result in the case that we will need it.

**Theorem 1.1.11** ([DN89]). *For any* $F \in U_C(k \cdot r_0, k \cdot (r_0(g-1) - d_0))$ *such that* $\Theta_F$ *is a divisor, the line bundle* $\mathscr{O}(\Theta_F)$ *on* $SU_C(r, L)$ *does not depend on the choice of* F. *The Picard group of* $SU_C(r, L)$ *is isomorphic to* $\mathbb{Z}$*, generated by an ample line bundle* $\mathscr{L}$ *(the so-called* determinant line bundle*) and* $\Theta_F \in |\mathscr{L}^k|$.

For any $k \geq 1$, we get a rational map

$$SU_C(r, L) \overset{\varphi_{|\mathscr{L}^k|}}{\dashrightarrow} \mathbb{P}(H^0(SU_C(r, L), \mathscr{L}^k)^*).$$

We restrict to the case $L = \mathscr{O}_C$ and $k = 1$, then the map $\varphi_{|\mathscr{L}|}$ is equivalent to the map taking $V \to \Theta_V \in |r\Theta|$ by Strange duality (see [Pop13, §5.2]). For simplicity, we write $SU_C(r)$ instead of $SU_C(r, \mathscr{O}_C)$. Verlinde's formula states that $H^0(SU_C(r), \mathscr{L})$ is $r^g$ dimensional (see [Pop13, Example 5.1]). Finally, the map

$$SU_C(r) \dashrightarrow \mathbb{P}^{r^g - 1} = |r\Theta|, \ V \mapsto \Theta_V$$

is called the *theta map*. Known results related to the injectivity of the theta map are summarised in Section 4.1 (see also [Bea06] or [BV12]).

## 1.2 Curves and scrolls

We give a short summary of canonical curves on rational normal scroll as presented in [Sch86]. We also follow [Har81]. Some of the facts which we will recall in this section are also included in Section 5.2.

**Rational normal scroll.** Let $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$ be a rank $d$ vector bundle on $\mathbb{P}^1$ for integers $e_1 \geq e_2 \geq \cdots \geq e_d \geq 0$ and let

$$\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$$

be the corresponding $\mathbb{P}^{d-1}$-bundle. Assume that $f = \sum_{i=1}^d e_i \geq 2$. We consider the image of $\mathbb{P}(\mathscr{E})$ under the map given by the tautological bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$:

$$j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}^r = \mathbb{P}(H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)))$$

where $r = f+d-1$. The image X is called a *rational normal scroll* of type $S(e_1,\dots,e_d)$. By [Har81, Section 3] (see also [EH87, Section 1]), the variety X is nondegenerate, irreducible of minimal degree

$$\deg X = f = r-d+1 = \operatorname{codim} X + 1.$$

Furthermore, X is smooth if and only if all $e_i > 0$ and then, the map $j : \mathbb{P}(\mathscr{E}) \to X$ is an isomorphism. If X is singular, then $j$ is a resolution of singularities. The singularities of X are rational, that is

$$j_*\mathscr{O}_{\mathbb{P}(\mathscr{E})} = \mathscr{O}_X \text{ and } R^i j_*\mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0 \text{ for } i > 0.$$

Hence for most cohomological considerations, we may replace X by $\mathbb{P}(\mathscr{E})$. In order to describe the Picard group of $\mathbb{P}(\mathscr{E})$, we denote $H = [j^*\mathscr{O}_X(1)]$ the hyperplane class and $R = [\pi^*\mathscr{O}_{\mathbb{P}^1}(1)]$ the ruling. Then the Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by H and R with intersection products

$$H^d = f,\ H^{d-1}\cdot R = 1 \text{ and } R^2 = 0.$$

Following [Sch86, (1.3)], there is an explicit identification

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)) \cong H^0(\mathbb{P}^1, (S_a\mathscr{E})(b)) \text{ for } a \geq 0,$$

where $S_a\mathscr{E}$ is the $a^{th}$ symmetric power of the vector bundle $\mathscr{E}$. Indeed, let $\mathbb{K}[s,t]$ be the coordinate ring of $\mathbb{P}^1$ and let $\varphi_i \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(H-e_iR))$ for $i = 1,\dots,d$ be the basic sections of the $i^{th}$ summand of $\mathscr{E}$. Then, a section $\psi \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR))$ can be identified with a homogeneous polynomial

$$\psi = \sum_\alpha P_\alpha(s,t)\varphi_1^{\alpha_1}\dots\varphi_d^{\alpha_d}$$

of degree $a = \alpha_1+\dots+\alpha_d$ in the $\varphi_i$ and with coefficients $P_\alpha \in \mathbb{K}[s,t]$ of degree $\deg P_\alpha = \alpha_1 e_1+\dots+\alpha_d e_d + b$. Note that this gives a description of the coordinate ring

$$R_{\mathbb{P}(\mathscr{E})} = \bigoplus_{a,b\in\mathbb{Z}} H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR))$$

of $\mathbb{P}(\mathscr{E})$ as the Cox ring $\mathbb{K}[s,t,\varphi_1,\dots,\varphi_d]$ equipped with bigrading $\deg s = \deg t = (1,0)$ and $\deg \varphi_i = (e_1-e_i, 1)$ (see also Example 1.2.1).

A last property of rational normal scrolls which we would like to mention is that they are determinantal varieties. We choose a basis

$$x_{ij} = s^j t^{e_i - j} \varphi_i \text{ with } j = 0, \ldots, e_i \text{ and } i = 1, \ldots, d$$

of $\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H})) \cong \mathrm{H}^0(\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(1))$. Then, the ideal of $\mathrm{X} \subset \mathbb{P}^r$ is given by the $2 \times 2$ minors of the matrix

$$\Phi = \begin{pmatrix} x_{10} & x_{11} & \ldots & x_{1e_1-1} & x_{20} & \ldots & x_{2e_2-1} & \ldots & x_{de_d-1} \\ x_{11} & x_{12} & \ldots & x_{1e_1} & x_{21} & \ldots & x_{2e_2} & \ldots & x_{de_d} \end{pmatrix}$$

Note that the matrix $\Phi$ consists of $d$ catalecticant matrices of size $2 \times e_1, \ldots, 2 \times e_d$ and hence, the variable $x_{i,0} = \varphi_i$ does not occur in the matrix $\Phi$ if $e_i = 0$. The matrix $\Phi$ coincides with the multiplication map

$$\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{R})) \otimes \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H}-\mathrm{R})) \to \mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\mathrm{H})).$$

*Example* 1.2.1 (Scroll of type $\mathrm{S}(1, \ldots, 1, 0, \ldots, 0)$). Let $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus f} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(d-f)}$ be a rank $d$ vector bundle on $\mathbb{P}^1$ for $f \geq 2$. Then, the rational normal scroll $\mathrm{X} \subset \mathbb{P}^r$ associated to $\mathbb{P}(\mathscr{E})$ is the vanishing locus of the maximal minors of a generic $2 \times f$ matrix

$$\Phi = \begin{pmatrix} x_1 & x_2 & \ldots & x_f \\ x_{f+1} & x_{f+2} & \ldots & x_{2f} \end{pmatrix},$$

where $x_1, \ldots, x_{2f}$ are suitable independent linear forms on $\mathbb{P}^r$. The scroll X is singular along the $(d-f-1)$-dimensional linear space, whose ideal is generated by the entries of the matrix $\Phi$. The coordinate ring $\mathrm{R}_{\mathbb{P}(\mathscr{E})}$ is identified with the Cox ring $\mathbb{K}[s, t, u_1, \ldots, u_f, w_1, \ldots, w_{d-f}]$ where $\deg s = \deg t = (1,0)$, $\deg u_i = (0,1)$ and $\deg w_j = (1,1)$. The map $j : \mathbb{P}(\mathscr{E}) \to \mathrm{X}$ is given by the homogeneous forms of bidegree $(1,1)$.

**Scrolls and pencils.** Let $\mathrm{C} \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus $g$. Let further

$$\{\mathrm{D}_\lambda\}_{\lambda \in \mathbb{P}^1} \subset |\mathrm{D}|$$

be a pencil of divisors on C with $h^1(\mathrm{C}, \mathscr{O}_\mathrm{C}(\mathrm{D})) = f \geq 2$ and let $\mathrm{G} \subset \mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(\mathrm{D}))$ be the 2-dimensional subspace defining the pencil. By [EH87, Theorem 2], the variety

$$\mathrm{X} = \bigcup_{\lambda \in \mathbb{P}^1} \overline{\mathrm{D}_\lambda},$$

which is swept out by the linear span of the divisors $D_\lambda$, is a rational normal scroll of degree $f$. Furthermore, its ideal is generated by the $2\times 2$ minors of the matrix

$$\Phi : G \otimes H^0(C, \omega_C \otimes \mathscr{O}_C(-D)) \to H^0(C, \omega_C).$$

Conversely, let X be a rational normal scroll of degree $f$ containing the canonical embedded curve C. Then the ruling R on C cuts out a pencil of divisors $\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$ on C such that $h^0(C, \omega_C \otimes \mathscr{O}_C(-D)) = f$.

The type $S(e_1, \dots, e_d)$ of the scroll X can be determined as follows (see also [Sch86, (2.4)]). We consider the decomposition $D_\lambda = F + E_\lambda$, $\lambda \in \mathbb{P}^1$ in its fixed part and moving part and the following partition of $g$

$$\begin{aligned}
d_0 &= h^0(c, \omega_C) - h^0(C, \omega_C \otimes \mathscr{O}_C(-D)) \\
d_1 &= h^0(C, \omega_C \otimes \mathscr{O}_C(-D)) - h^0(C, \omega_C \otimes \mathscr{O}_C(-F-2E)) \\
&\vdots \\
d_i &= h^0(C, \omega_C \otimes \mathscr{O}_C(-F-iE)) - h^0(C, \omega_C \otimes \mathscr{O}_C(-F-(i+1)E)) \\
&\vdots
\end{aligned}$$

By [Har81], the type of the scroll X of dimension $d_0$ is given by the following partition

$$e_i = \sharp\{j | d_j \geq i\} - 1.$$

Finally, we will determine the type of the scroll swept out by a pencil $g^1_d = |L|$ for $L \in W^1_d(C) \backslash W^2_d(C)$ on a Brill–Noether general curve C. Let $L \in W^1_d(C)$ be a base point free complete line bundle of degree $d \leq g-1$ with $h^0(C, L) = 2$ such that $\rho(g, d, 1) \geq 0$. Since C is Brill–Noether general (or Petri general), the Petri map

$$\mu : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \to H^0(C, \omega_C)$$

is injective or equivalently, the kernel $\operatorname{Ker}\mu = H^0(C, \omega_C \otimes L^{-2}) = 0$. Recall that the $2\times 2$ minors generate the ideal of the rational normal scroll $X = \bigcup_{D\in|L|} \overline{D}$, and hence the scroll coincides with the projectivised tangent cone $\mathbb{P}\mathscr{T}_L(W^0_d(C))$ to $W^0_d(C)$ at L. Since the pencil |L| is base point free, we have to compute the differences

$$d_i = h^0(C, \omega_C \otimes L^{-i}) - h^0(C, \omega_C \otimes L^{-(i+1)}) \text{ for } i \geq 0.$$

But all cohomology groups $H^0(C,\omega_C \otimes L^{-i}) \subset \operatorname{Ker}\mu = 0$ vanish for $i \geq 2$ and we get

$$d_0 = g-(g-d+1) = d-1,\ d_1 = g-d+1 \text{ and } d_i = 0 \text{ for } i \geq 2.$$

It follow that the rational normal scroll X is of type $S(\underbrace{1,\dots,1}_{g-d+1},\underbrace{0,\dots,0}_{\rho=d_0-d_1})$ (see also Example 1.2.1).

## 1.3 Overview of the thesis

The goal of this thesis will be to study the geometry of the Brill–Noether locus $W^0_d(C)$ and the generalised theta divisor. Using methods of Kempf and Schreyer [Kem86], [KS88] and Ciliberto and Sernesi [CS95], we study the local geometry of $W^0_d(C)$ around general singular points in Chapter 2 and Chapter 3 and give local Torelli-type theorems. This completes the attempt to recover a general curve C of genus $g$ from its Brill–Noether locus $W^0_d(C)$ around a singular point in $W^1_d(C)$ for every possible values $g \geq 4$, $d \leq g-1$ and $\rho(g,d,1) \geq 0$.

We will give a short overview what was known before and illustrate this in the following table:

| | | | | |
|---|---|---|---|---|
| ⋮ | KS88/CS95 | | | |
| 3 | KS88/CS95 | | | |
| 2 | KS88/CS95 | Baj10 | | |
| 1 | KS88/CS95 | CS00 | CS00 | CS00 |
| 0 | Kem86 | | | |
| $\rho$ \ $g-d$ | 1 | 2 | 3 | ... |

Table 1.1: Local Torelli-type theorems for $W^0_d(C)$ for $g \geq 4$, $d \leq g-1$ and $\rho(g,d,1) \geq 0$. In order to visualise all possible cases more compactly, we use a $\rho(g,d,1) \times (g-d)$-diagram.

In [Kem86], Kempf showed that the canonically embedded curve C of genus 4 (over an algebraically closed field of characteristic at least 3) coincides with the osculating cone of order 3 to the theta divisor $\theta$ at a general double point, that is, a point of $W^1_3(C)$ such that the corresponding pencil is not semi-canonical (see also [ACGH85, Corollary on p. 232] for the result over $\mathbb{C}$). This result was extended by Kempf and Schreyer in [KS88] to general curves of genus $\geq 5$ (and $d = g-1$). In [CS92] and [CS95], Ciliberto and Sernesi also gave a Torelli theorem for curve of genus $\geq 5$ and $d = g-1$ studying the focal scheme. They were able to recover all nonhyperelliptic curves from the local geometry of the theta divisor around a double point. In [CS00], they extended their methods to $W^1_d(C)$ for curves of odd genus $g \geq 5$ such that $\rho(g,d,1) = 1$. Generalising their results, a local Torelli-type theorem in the case $g = 8$ and $\rho(8,6,1) = 2$ was obtained by Bajravani in [Baj10].

In Chapter 2, we will prove a local Torelli-type theorem for Brill–Noether general curves of even genus with $\rho(g,d,1) = 0$ over an algebraically closed field of characteristic at least 3. We will describe the osculating cone of order 3 to $W^0_d(C)$ at such a singular point. In Chapter 3, we will extend the results of [CS95], [CS00] and [Baj10] to general curve over the complex numbers of genus $g \geq 5$ such that $d \leq g-1$ and $\rho(g,d,1) \geq 1$.

Studying the tangent cone to generalised theta divisors, we were able to give a partial answer to Beauville's speculation about the injectivity of the theta map. Beauville said that the most optimistic statement of the behaviour of the theta map should be the following.

**Speculation** ([Bea06, Speculation 6.1]). *For $g \geq 3$, the theta map is generically injective if the curve is not hyperelliptic, and generically two-to-one onto its image if the curve is hyperelliptic.*

In Chapter 4, we will prove the generic injectivity of the theta map for bundles of rank $r$ and Brill–Noether general curves, where the genus $g$ grows quadratically in $r$. In [BV12], Brivio and Verra showed the generic injectivity for general curves of genus $g \geq \binom{3r}{r} - 2r - 1$, which is an exponential growth. Our approach is to show the existence of a singular point of high multiplicity of the generalised theta divisor $\Theta_V$ to a Brill–Noether general curve C, where V is a general vector bundle of rank $r$ on C. A key observation is that a determinantal representation of the tangent cone at this singular point encodes naturally the bundle V.

The last Chapter 5 is devoted to study the relative canonical resolution of a canonically embedded curve on a rational normal scroll. For a canonically embedded curve C of genus $g$ and a complete base point free pencil $g^1_k = |L|$ induced by $L \in W^1_k(C)$, let X be the rational normal scroll swept out by the pencil and let $\mathbb{P}(\mathscr{E})$ be the associated $\mathbb{P}^1$-bundle to X (see Section 1.2). The relative canonical resolution is the resolution of $C \subset \mathbb{P}(\mathscr{E})$, that is, the resolution of $\mathscr{O}_C$ as an $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module (see [Sch86] for the structure of the resolution). The syzygy modules appearing in the relative canonical resolution can be considered as bundles over $\mathbb{P}^1$ and hence, carrying a certain splitting type. We will give a sharp bound (depending on $g$ and $k$) for the balancedness of the first syzygy bundle (or bundle of quadrics) in the relative canonical resolution when the curve is Brill–Noether general. Finally, we report on an ongoing work with Christian Bopp which explains the unbalancedness of the second syzygy bundle in the relative canonical resolution of a curve of genus 9 on a scroll swept out by a $g^1_6$.

**Changes after publication**: Chapter 2, 4 and 5 have been published. In the following table, we list our post-publication changes.

- Chapter 2: We included graphics in Example 2.2.4 and 2.4.2 to illustrate the geometry. We have written a new section: Evidences for our conjecture (see Conjecture 2.1.3). We apply techniques of Section 2.3 to curves of any genus $g$ and Brill–Noether loci $W^1_d(C)$ of arbitrary dimension $\rho(g,d,1) \geq 1$.
- Chapter 5: We have written a new section: An unbalanced second syzygy bundle and K3 surfaces. We state a conjecture of an ongoing work with Christian Bopp.

# Chapter 2

# The osculating cone to special Brill–Noether loci

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**Abstract.** We describe the osculating cone to Brill–Noether loci $W^0_d(C)$ at smooth isolated points of $W^1_d(C)$ for a general canonically embedded curve C of even genus $g = 2(d-1)$. In particular, we show that the canonical curve C is a component of the osculating cone. The proof is based on techniques introduced by George Kempf.

## 2.1 Introduction

Brill–Noether theory is the study of the geometry of Brill–Noether loci $W^r_d(C)$ for a curve C, that is, schemes whose closed points consist of the set

$$\{L \in \mathrm{Pic}(C) \mid \deg L = d \text{ and } h^0(C, L) \geq r+1\} \subset \mathrm{Pic}^d(C)$$

of linear series of degree $d$ and dimension at least $r+1$. For a general curve, the dimension of $W^r_d(C)$ is equal to the Brill–Noether Number $\rho(g,d,r) = g-(r+1)(g-d+r)$ by the Brill–Noether Theorem (see [GH80]). In this paper, we study the local structure of $W^0_d(C)$ near points of $W^1_d(C)$.

[1] Some changes have been made post-publication. These are listed at the end of Section 1.3.

The Brill–Noether locus $W_d^0(C) \subset \mathrm{Pic}^d(C)$ is given locally around a point $L \in W_d^r(C)$ with $h^0(C,L) = r+1$ by the maximal minors of the matrix $(f_{ij})$ of regular functions vanishing at L arising from

$$R\pi_*\mathscr{L} : 0 \longrightarrow \mathscr{O}_{\mathrm{Pic}^d(C),L}^{h^0(C,L)} \xrightarrow{(f_{ij})} \mathscr{O}_{\mathrm{Pic}^d(C),L}^{h^1(C,L)} \longrightarrow 0,$$

the only non-trivial part of the direct image complex of the Poincaré bundle $\mathscr{L}$ on $C \times \mathrm{Spec}(\mathscr{O}_{\mathrm{Pic}^d(C),L})$ (see [Kem83]). Using flat coordinates on the universal covering of $\mathrm{Pic}^d(C) \cong H^1(C,\mathscr{O}_C)/H^1(C,\mathbb{Z})$, we may expand

$$f_{ij} \;=\; l_{ij} \;+\; q_{ij} \;+\; \textit{higher order terms},$$

where $l_{ij}$ and $q_{ij}$ are linear and quadratic forms on the tangent space $H^1(C,\mathscr{O}_C)$ of Pic(C) at L, respectively. A first local approximation of $W_d^0(C)$ is the tangent cone $\mathscr{T}_L(W_d^0(C))$ whose ideal is generated by the maximal minors of $(l_{ij})$ by Riemann-Kempf's Singularity Theorem (see [Kem73]). Recall that the analytic type of $W_d^0(C)$ at L is completely determined by the tangent cone. As its subvariety, we will study the osculating cone of order 3 to $W_d^0(C)$ at the point L, denoted by $\mathcal{OC}_3(W_d^0(C),L)$, a better approximation than the tangent cone in the given embedding into $H^1(C,\mathscr{O}_C)$.

In [Kem86], Kempf showed for a canonically embedded curve of genus 4, the osculating cone of order 3 to $W_3^0(C)$ at a point $L \in W_3^1(C)$ coincides with the curve. Using Kempf's cohomology obstruction theory, Kempf and Schreyer ([KS88]) studied the osculating cone to $W_{g-1}^0(C)$ for curve of genus $g \geq 5$ and proved a local Torelli Theorem (see Conjecture 2.1.3). Schreyer conjectured that the geometry of the osculating cone to other Brill–Noether loci $(d \neq g-1)$ is rich enough to recover the curve C. Using methods developed in [Kem86] and [KS88], we will give a positive answer for smooth pencils $L \in W_d^1(C)$ with $h^0(C,L) = 2$ and $\dim(W_d^1(C)) = \rho(g,d,1) = 0$.

To explain our main theorem, we introduce some notation. Let C be a smooth canonically embedded curve of genus $g = 2(d-1) \geq 4$ over an algebraically closed field $\Bbbk$ of characteristic different from 2. For a general curve, the Brill–Noether locus $W_d^1(C)$ is then zero-dimensional, and for every $L \in W_d^1(C)$, the multiplication map

$$\mu_L : H^0(C,L) \otimes H^0(C,\omega_C \otimes L^{-1}) \xrightarrow{\cong} H^0(C,\omega_C) \tag{2.1}$$

is an isomorphism by [Gie82]. Hence, $W_d^1(C)$ consists of $\frac{(2d-2)!}{d!\cdot(d-1)!}$ reduced isolated points. Let L be a point of $W_d^1(C)$ such that $\mu_L$ is an isomorphism.

By our choice of L, the projectivisation of the tangent cone has a simple description which is important for our considerations. Recall that the projectivisation of the tangent cone and the osculating cone live naturally in the canonical space $\mathbb{P}^{g-1} := \mathbb{P}(H^0(C,\omega_C)^*)$. By Riemann-Kempf's Singularity Theorem, $\mathbb{P}\mathcal{T}_L(W^0_d(C))$ is geometrically the scroll swept out by $g^1_d = |L|$. It is the union of planes $\overline{D} = \mathbb{P}^{d-2}$ spanned by the points of the divisor $D \in |L|$. Furthermore, the isomorphism of the multiplication map yields that the scroll is smooth and hence, coincides with the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{d-2} \subset \mathbb{P}^{g-1}$. We get the following diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{OC}_3(W^0_d(C),L)) & \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^{d-2} & \hookrightarrow & \mathbb{P}^{g-1} \\ & \searrow & \downarrow \pi & & \\ & & \mathbb{P}^1 & & \end{array}$$

where $\pi$ is the projection to the first component. A point in $\mathbb{P}^1$ corresponds to a divisor $D \in |L|$ and the fiber over D is $\pi^{-1}(D) = \overline{D}$. Our main theorem is a characterisation of the intersection of the osculating cone and a fiber $\overline{D}$.

**Theorem 2.1.1.** *Let* C *be a smooth canonically embedded curve of even genus* $g = 2(d-1) \geq 4$ *and let* $L \in W^1_d(C)$ *such that the multiplication map* $\mu_L$ *is an isomorphism. If* $\mathrm{char}(\mathbb{k}) = 0$, *then the fiber* $\overline{D}$ *of the projection* $\pi$ *intersects the osculating cone* $\mathbb{P}(\mathcal{OC}_3(W^0_d(C),L))$ *in the union of all intersections* $\overline{D_1} \cap \overline{D_2}$ *for each decomposition* $D = D_1 + D_2$ *into nonzero effective divisors. If* $\mathrm{char}(\mathbb{k}) > 0$, *then the above is true if* $\pi|_C : C \to \mathbb{P}^1$ *is tamely ramified. In particular, the osculating cone consists of* $2^{d-1} - 1$ *points in a general fiber.*

In [May13], the second author showed that all intersection points are contained in the osculating cone for a general fiber. For $g = 4$, this generalises the main theorem of [Kem86].

An immediate consequence of Theorem 2.1.1 is a Torelli-type Theorem for osculating cones to $W^0_d(C)$ at isolated singularities.

**Corollary 2.1.2.** *The general canonical curve* C *of genus* $g = 2(d-1) \geq 4$ *is an irreducible component of the osculating cone* $\mathbb{P}(\mathcal{OC}_3(W^0_d(C),L))$ *of order* 3 *to* $W^0_d(C)$ *at an arbitrary point* $L \in W^1_d(C)$.

In the case of $W^1_d(C)$ of positive dimension, we have the following local Torelli-type conjecture for Brill–Noether loci.

**Conjecture 2.1.3.** *Let* $C$ *be a general canonically embedded curve of genus* $g$ *and let* $L \in W^1_d(C)$ *be a general point where* $\dim(W^1_d(C)) \geq 1$. *Let* $V = \mathrm{Sing}(\mathbb{P}\mathcal{T}_L(W^0_d(C)))$. *The projection* $\pi_V : C \to C' \subseteq \mathbb{P}^1 \times \mathbb{P}^{h^1(C,L)-1}$ *is birational to the image of* $C$. *Consider*

$$\begin{array}{ccccccc}
\mathbb{P}(\widetilde{\mathcal{OC}_3(W^0_d(C),L)}) & \hookrightarrow & \mathbb{P}\widetilde{\mathcal{T}_L(W^0_d(C))} & \hookrightarrow & \widetilde{\mathbb{P}^{g-1}} & \longrightarrow & \mathbb{P}^{g-1} \\
 & \searrow^{\alpha} & \downarrow & & \downarrow{\scriptstyle \pi_V} & & \\
 & & \mathbb{P}^1 \times \mathbb{P}^{h^1(C,L)-1} & \longrightarrow & \mathbb{P}^{2\cdot h^1(C,L)-1} & &
\end{array}$$

*where* $\sim$ *denotes the strict transform after blowing up* $V$ *and the vertical maps are induced by the projection from* $V$. *Then,*

*(a) away from points of* $C'$ *the fibers of* $\alpha$ *are smooth or empty and*

*(b) for a smooth point* $p'$ *of* $C'$ *the corresponding point* $p$ *of* $\widetilde{C}$ *is the only singular point of the fiber of* $\alpha$ *over* $p'$.

The origin of the conjecture is the work [KS88] of Kempf and Schreyer. The authors proved the conjecture in the case $d = g-1$, where $W^0_{g-1}(C)$ is isomorphic to the theta divisor, thus implying a Torelli Theorem for general curves. In Section 2.5, we give some evidences for our conjecture and we will also prove the conjecture in three cases ($(g,d) = (7,5),(8,6),(9,7)$). Using the theory of foci, similar results are shown in [CS95], [CS00] and Chapter 3.

Our local Torelli-type Theorem recovers the original curve as a component of the osculating cone. Moreover, we can identify different components of the osculating cone. Let $C$ and $L$ be as in Theorem 2.1.1. Let further $\overline{D}$ be a general fiber of $\pi$ spanned by the divisor $D \in |L|$ on $C$ of degree $d$. Then, the osculating cone consists of $2^{d-1}-1$ points in the fiber $\overline{D}$, since there are $2^{d-1}-1$ decompositions of $D$ into nonzero effective divisors and each intersection is zero-dimensional. For $i \leq \lfloor \frac{d}{2} \rfloor$, there are $\binom{d}{i}$ points of the osculating cone in the fiber $\overline{D}$ if $i < \frac{d}{2}$ and $\frac{1}{2}\binom{d}{i}$ points if $i = \frac{d}{2}$, arising from decompositions $D = D_1 + D_2$ where $\deg(D_1) = i$ or $\deg(D_1) = d - i$. For every $i$, we get a curve $C_i$ as the closure of the union of these points. The decomposition of the curve $C_i$ depends on the monodromy group of the covering $\pi|_C : C \to \mathbb{P}^1$. A curve $C_i$ is irreducible if the monodromy group acts transitively on $C_i$. Recall that the monodromy group is the full symmetric group $S_d$ for a general curve, thus the osculating cone contains a union of $\lfloor \frac{d}{2} \rfloor$ irreducible curves. An easy example (see [Dal85, Proposition 4.1 (b)]) where an additional component of the osculating cone decomposes is the following. For a tetragonal curve of genus 6 with monodromy group $\mathbb{Z}_4$, the trigonal curve $C_2$, with the above notation, decomposes in a rational and a hyperelliptic curve.

It can also happen that there are further even higher-dimensional components of the osculating cone contained in special fibers of $\pi$.

The paper is organized as follows. Section 2.2 provides basic lemmata which we need later on and motivates our main theorem. In Section 2.3, we recall Kempf's cohomology obstruction theory. Using this theory, we will give a proof of Theorem 2.1.1 in Section 2.4.

## 2.2 Preliminaries and motivation

Throughout this paper, we fix the following notation: Let C be a smooth canonically embedded curve of even genus $g = 2(d-1) \geq 4$ over an algebraically closed field of characteristic $\neq 2$ and let $L \in W^1_d(C)$ be an isolated smooth point of the Brill–Noether locus.

We refer to [ACGH85] for basic results of Brill–Noether theory. We deduce two simple lemmata from our assumptions on the pair $(C, L)$.

**Lemma 2.2.1.** *The linear system* $|L|$ *is a base point free pencil with* $H^1(C, L^2) = 0$ *and surjective multiplication map* $\mu_L$.

*Proof.* Since L is a smooth point and $g = 2(d-1)$, the multiplication map $\mu_L : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \to H^0(C, \omega_C)$ is an isomorphism between vector spaces of the same dimension. Furthermore, $|L|$ is a base point free pencil since $H^0(C, \omega_C)$ has no base points. By the base point free pencil trick, we deduce the vanishing $H^1(C, L^2)^* = H^0(C, \omega_C \otimes L^{-2}) = \ker(\mu) = 0$ from the exact sequence

$$0 \longrightarrow \bigwedge^2 H^0(C, L) \otimes \omega_C \otimes L^{-2} \longrightarrow H^0(C, L) \otimes \omega_C \otimes L^{-1} \longrightarrow \omega_C \longrightarrow 0.$$

□

**Lemma 2.2.2.** *For any point* $p \in \mathrm{C}$*, we have*

$$\mathrm{H}^0(\mathrm{C}, \mathrm{L}^2(-p)) = \mathrm{H}^0(\mathrm{C}, \mathrm{L}).$$

*Proof.* Let $f_0 \in \mathrm{H}^0(\mathrm{C}, \mathrm{L})$ be a section vanishing at $p$. The section $f_0$ is unique since $|\mathrm{L}|$ is base point free. Let D be the divisor of zeros of $f_0$. We get an isomorphism $\mathrm{L} \cong \mathscr{O}_\mathrm{C}(\mathrm{D})$, where the section $f_0$ corresponds to $1 \in \mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(\mathrm{D}))$.

We compute the vector space $\mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D} - p))$. Since $h^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(\mathrm{D})) = 2$ and $|\mathrm{D}|$ is base point free, we find a rational function $h \in \mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(\mathrm{D}))$ whose divisor of poles is exactly D. The Riemann-Roch theorem states

$$h^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D})) - h^1(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D})) = 2\left(\frac{g}{2} + 1\right) + 1 - g = 3,$$

and since $h^1(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D})) = 0$ by Lemma 2.2.1, a basis of $\mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D}))$ is given by $(1, h, h^2)$. We conclude that

$$\mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D} - p)) = \mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(\mathrm{D}))$$

since $h^2 \notin \mathrm{H}^0(\mathrm{C}, \mathscr{O}_\mathrm{C}(2\mathrm{D} - p))$. □

As already pointed out, the tangent cone coincides with the scroll swept out by $g^1_d = |\mathrm{L}|$. In this setting, a further important object is the multiplication or cup-product map.

**Remark 2.2.3.** Let $(l_{ij})$ be the linear part of the $(d-1) \times 2$ matrix $(f_{ij})$ as in the introduction (see also Corollary 2.3.2). The matrix $(l_{ij})$ is closely related to the cup-product map $\mu_\mathrm{L}$. Indeed, by [Kem83, Lemma 10.3 and 10.6], the matrix $(l_{ij})$ describes the induced cup-product action

$$\cup : \mathrm{H}^1(\mathrm{C}, \mathscr{O}_\mathrm{C}) \to \mathrm{Hom}(\mathrm{H}^0(\mathrm{C}, \mathrm{L}), \mathrm{H}^1(\mathrm{C}, \mathrm{L})), \ b \mapsto \cup b.$$

Thus, the tangent cone corresponds to cohomology classes $b \in \mathrm{H}^1(\mathrm{C}, \mathscr{O}_\mathrm{C})$ such that the map $\cup b$ has rank $\leq 1$. By Lemma 2.2.1, the multiplication map is surjective and the homomorphism $\cup b$ always has rank $\geq 1$ for $b \neq 0$. We see that the map $\cup b$ has rank equal to 1 for points in the tangent cone.

Now, we will describe the osculating cone as a degeneracy locus of a map of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^{d-2}$. This yields the expected number of points in the intersection of the osculating cone and a fiber over $\mathbb{P}^1$ and motivates our main theorem.

Let $\Bbbk[t_0, t_1]$ and $\Bbbk[u_0, \ldots, u_{d-2}]$ be the coordinate rings of $\mathbb{P}^1$ and $\mathbb{P}^{d-2}$, respectively. Using these coordinates and the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{d-2} \subset \mathbb{P}^{g-1}$, the linear matrix $(l_{ij})$ can be expressed as the matrix $(t_i \cdot u_j)$ in Cox coordinates $\Bbbk[t_0, t_1] \otimes \Bbbk[u_0, \ldots, u_{d-2}]$. Let $(q_{ij})$ be the quadratic part of the expansion of $(f_{ij})$ in homogeneous forms on $\mathbb{P}^{g-1}$. Then, the ideal of the osculating cone is generated by homogeneous elements of bidegree $(3,3)$ in the ideal

$$
\begin{aligned}
&\mathrm{I}_{2\times 2}\left(\left(l_{ij} + q_{ij}\right)\right) \\
=&\mathrm{I}_{2\times 2}\left(\begin{pmatrix} t_0 u_0 + q_{00} & \ldots & t_0 u_{d-2} + q_{0(d-2)} \\ t_1 u_0 + q_{00} & \ldots & t_1 u_{d-2} + q_{1(d-2)} \end{pmatrix}^{\mathrm{T}}\right) \\
=&\mathrm{I}_{2\times 2}\Bigg(\underbrace{\begin{pmatrix} u_0 & \ldots & u_{d-2} \\ t_0 q_{10} - t_1 q_{00} & \ldots & t_0 q_{1(d-2)} - t_1 q_{0(d-2)} \end{pmatrix}^{\mathrm{T}}}_{=:\mathrm{A}_{(t_0,t_1)}}\Bigg)
\end{aligned}
$$

where $q_{ij}$ is of bidegree $(2,2)$ in $\Bbbk[t_0, t_1] \otimes \Bbbk[u_0, \ldots, u_{d-2}]$ (see Definition 2.3.6). In other words, the matrix $\mathrm{A}_{(t_0,t_1)}$ induces a map between vector bundles such that the osculating cone is the degeneracy locus of $\mathrm{A}_{(t_0,t_1)}$.

We assume that the degeneracy locus has expected codimension $d-2$ in $\mathbb{P}^1 \times \mathbb{P}^{d-2}$, that is, the osculating cone is a curve. We may determine the expected number of points in a general fiber over $\mathbb{P}^1$ with the help of a Chern class computation. Indeed, for a fixed point $(\lambda, \mu) \in \mathbb{P}^1$, the osculating cone is given by the finite set

$$
\{p \in \mathbb{P}^{d-2} \mid \mathrm{rk}\left(\mathrm{A}_{(\lambda,\mu)}(p) = \begin{pmatrix} u_0(p) & \ldots & u_{d-2}(p) \\ q_0(p) & \ldots & q_{d-2}(p) \end{pmatrix}^{\mathrm{T}}\right) < 2\}
$$

of points in $\mathbb{P}^{d-2}$ where $q_i \in \Bbbk[u_0, \ldots, u_{d-2}]$ is the polynomial $t_0 q_{1i} - t_1 q_{0i}$ evaluated in $(t_0, t_1) = (\lambda, \mu)$ for $i = 0, \ldots, d-2$. We compute its degree:

We define the vector bundle $\mathscr{F}$ as the cokernel of the first column of $\mathrm{A}_{(\lambda,\mu)}$. Note that $u_0, \ldots, u_{d-2}$ do not have a common zero in $\mathbb{P}^{d-2}$. Then, the vanishing locus of the section $s(2) : \mathscr{O}_{\mathbb{P}^{d-2}} \longrightarrow \mathscr{F}(2)$ induced by the second column of $\mathrm{A}_{(\lambda,\mu)}$ is the osculating cone in the fiber and its degree is the Chern class $c_{d-2}(\mathscr{F}(2))$. The total Chern class of $\mathscr{F}$ is given by

$$c(\mathscr{F}) = \frac{c(\mathscr{O}^{d-1}_{\mathbb{P}^{d-2}})}{c(\mathscr{O}_{\mathbb{P}^{d-2}}(-1))} = \frac{1}{1-t}.$$

We can transform the total Chern class to $c(\mathscr{F}) = 1+t+\cdots+t^{d-2} \mod t^{d-1}$. By the splitting principle, we find constants $\lambda_i$ such that

$$c(\mathscr{F}) = \prod_{i=1}^{d-2}(1+\lambda_i t) = 1+t+\cdots+t^{d-2}.$$

The Chern class $c_{d-2}(\mathscr{F}(2))$ is given by the coefficient of the $t^{d-2}$-term of $c(\mathscr{F}(2)) = \prod_{i=1}^{d-2}(1+(2+\lambda_i)t)$, that is, $2^{d-1}-1$. If the osculating cone intersects $\overline{\mathrm{D}}$ in a zero dimensional scheme, then the degree of $\overline{\mathrm{D}} \cap \mathrm{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ is $2^{d-1}-1$, which is consistent with the assertion of the main theorem.

Next, we compute degree and genus of the osculating cone. As mentioned above, the osculating cone is the degeneracy locus of the matrix $\mathrm{A}_{(t_0,t_1)}$ and we assume that the degeneracy locus has expected codimension $d-2$. Let $\widetilde{\mathscr{F}}$ be the pullback of $\mathscr{F}$ by the second projection $\mathbb{P}^1 \times \mathbb{P}^{d-2} \to \mathbb{P}^{d-2}$. Then, the vanishing locus of $s(3,2) : \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^{d-2}} \longrightarrow \widetilde{\mathscr{F}}(3,2)$ induced by the second column of $\mathrm{A}_{(t_0,t_1)}$ is the osculating cone. We compute as above the Chern class $c_{d-2}(\widetilde{\mathscr{F}}(3,2))$ and get the degree of the osculating cone as the total degree on $\mathbb{P}^1 \times \mathbb{P}^{d-2}$,

$$\begin{aligned}\deg(\mathrm{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})) =& 2^{d-1}-1+3(\sum_{i=1}^{d-2} i\cdot 2^{i-1})\\ =& (3(d-1)-4)2^{d-2}+2.\end{aligned}$$

If the osculating cone has expected codimension $d-2$, the Eagon-Northcott complex resolves the osculating cone $\mathrm{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L}) \subset \mathbb{P}^1 \times \mathbb{P}^{d-2}$. We can express the Hilbert polynomial $\mathrm{H}_{\mathrm{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})}(x,y)$ of the osculating cone in terms of Betti numbers and twists appearing in the resolution. Using computer algebra software, we can compute the genus of the osculating cone. The genus is given as

$$\begin{aligned} g(\mathrm{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})) =& 1 - \mathrm{H}_{\mathrm{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})}(0,0) \\ =& 1-(1+\sum_{i=1}^{d-2}(-1)^i\sum_{j=1}^{i}\binom{d-1}{i+1}(1-3j)\binom{d-3-i-j}{d-2}) \\ =& (3(d-1)(d-2)-4)2^{d-3}+2. \end{aligned}$$

*Example* 2.2.4. Let C be a canonically embedded general curve of genus 6 and let $\mathrm{L} \in \mathrm{W}^1_4(\mathrm{C})$ be a smooth point. We denote by $\mathrm{S} = \mathbb{k}[t_0,t_1] \otimes \mathbb{k}[u_0,u_1,u_2]$ the coordinate ring of the Segre product $\mathbb{P}^1 \times \mathbb{P}^2$. As explained above, we express the ideal $\mathrm{I}(\mathrm{OC}_3(\mathrm{W}^0_4(\mathrm{C}),\mathrm{L}))$ of the osculating cone in $\mathbb{P}^1 \times \mathbb{P}^2$ as $2\times 2$ minors of the matrix

$$\mathrm{A}_{(t_0,t_1)} = \begin{pmatrix} u_0 & u_1 & u_2 \\ p_0 & p_1 & p_2 \end{pmatrix}^{\mathrm{T}},$$

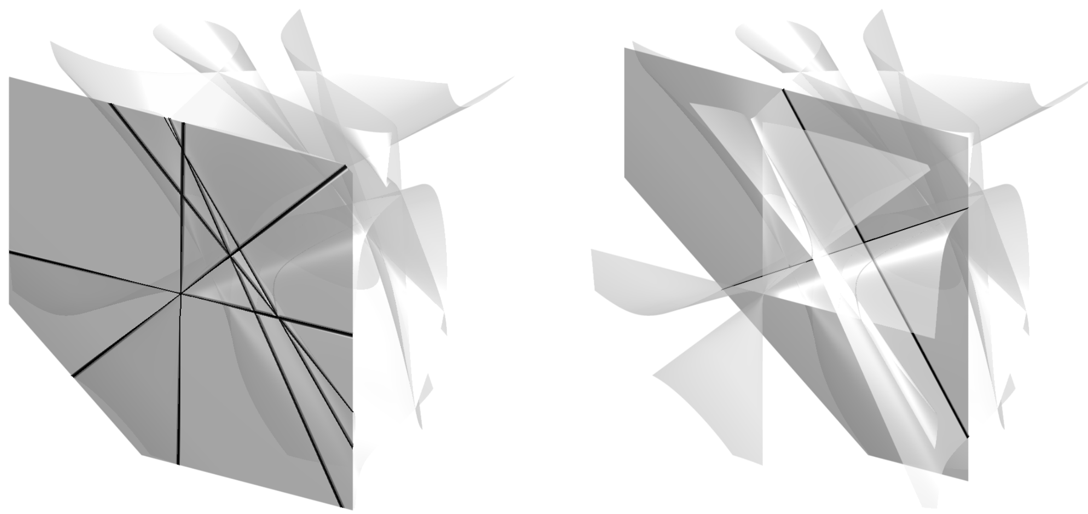
where $p_i \in \mathrm{S}_{(3,2)}$ are polynomials of bidegree $(3,2)$ for $i = 0,1,2$.

For a general curve, the map $\varphi_{|\mathrm{L}|} : \mathrm{C} \to \mathbb{P}^1$ has only simple ramification points and by Theorem 2.1.1, the osculating cone has expected codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^2$. By Hilbert-Burch Theorem, the minimal free resolution of $\mathrm{I}(\mathrm{OC}_3(\mathrm{W}^0_4(\mathrm{C}),\mathrm{L}))$ is

$$0 \longrightarrow \mathrm{S}(-3,-4) \oplus \mathrm{S}(-6,-5) \xrightarrow{\mathrm{A}_{(t_0,t_1)}} \mathrm{S}^3(-3,-3) \longrightarrow \mathrm{I}(\mathrm{OC}_3(\mathrm{W}^0_4(\mathrm{C}),\mathrm{L})) \longrightarrow 0.$$

Thus, the osculating cone is a reducible curve of degree 22 and genus 30.

As mentioned at the end of Section 2.1, we get two components of the osculating cone, that is $\mathrm{C}_1 = \mathrm{C}$ and a trigonal curve $\mathrm{C}_2$. In a general fiber of the tangent cone $\mathbb{P}^1 \times \mathbb{P}^2$, the three points of $\mathrm{C}_2$ are the intersection points of the different connection lines between the four points of C. We will illustrate this fact. We draw a real picture of the surface consisting of the connection lines in an affine chart $\mathbb{A}^1 \times \mathbb{A}^2$ of the tangent cone. We highlight two fibers.



(a) C has four real points in this fiber. We can see all six connection lines and the three further intersection point of $C_2$.

(b) C has two real points and one pair of complex conjugates. Hence, we see two real connection lines and one real intersection point of $C_2$.

Figure 2.1: Surface consisting of the connection lines in an affine chart $\mathbb{A}^1 \times \mathbb{A}^2$.

We conclude that there is a ramification point of C between the two highlighted fibers. An animation of this example can be found on the author's webpage[2]. For more detailed analysis of the osculating cone see the end of Section 2.4.

## 2.3 Kempf's cohomological obstruction theory

First, we recall variation of cohomology to provide local equations of $W^0_d(C)$ at the point L. Then, we introduce flat coordinates on an arbitrary algebraic group as in [Kem86] in order to give a precise definition of the osculating cone. Finally, we study infinitesimal deformations of the line bundle L which are related to flat curves. This results in an explicit criterion for a point $b \in H^1(C, \mathscr{O}_C)$ to lie in the osculating cone.

[2] `http://www.math.uni-sb.de/ag-schreyer/index.php/people/researchers/74-michael-hahn`

### 2.3.1 Variation of cohomology

Let $S = \mathrm{Spec}(A)$ be an affine neighbourhood of $L \in \mathrm{Pic}^d(C)$, and let $\mathscr{L}$ be the restriction of the Poincaré line bundle over $C \times \mathrm{Pic}^d(C)$ to $C \times S$. The idea of the variation of cohomology is to find an approximating homomorphism which computes simultaneously the upper-semicontinuous functions on S

$$s \mapsto h^i(C \times \{s\}, \mathscr{L} \otimes_A k(s)), \ i = 0, 1,$$

where $k(s) = A/\mathfrak{m}_s$ and $\mathfrak{m}_s$ is the maximal ideal of $s \in S$.

**Theorem 2.3.1** ([Gro63] Theorem 6.10.5 or [Kem83] Theorem 7.3). *Let* $\mathscr{M}$ *be a family of invertible sheaves on* C *parametrised by an affine scheme* $S = \mathrm{Spec}(A)$. *There exist two flat* A*-modules* F *and* G *of finite type and an* A*-homomorphism* $\alpha : F \to G$ *such that for all* A*-modules* M*, there are isomorphisms*

$$H^0(C \times S, \mathscr{M} \otimes_A M) \cong \ker(\alpha \otimes_A id_M), \quad H^1(C \times S, \mathscr{M} \otimes_A M) \cong \mathrm{coker}(\alpha \otimes_A id_M).$$

If we shrink S to a smaller neighbourhood, we may assume that F and G are free A-modules of finite type by Nakayama's Lemma. Furthermore, we may assume that the approximating homomorphism is minimal, that is, $\alpha \otimes_A k(L)$ is the zero homomorphism by [Kem83, Lemma 10.2]. Applying Theorem 2.3.1 to $\mathscr{M} = \mathscr{L}$ leads to the following corollary.

**Corollary 2.3.2.** *The local equations of* $W^0_d(C)|_S$ *at the point* L *are given by the maximal minors of a* $(d-1) \times 2$ *matrix* $(f_{ij})$ *of regular functions on* S *which vanish at* L.

*Proof.* Since $h^0(C, L) = 2$ and $\chi(\mathscr{L} \otimes k(s)) = \chi(L) = 2 - (d-1)$, $\forall s \in S$, the A-modules F and G are free of rank 2 and $d-1$, respectively. □

### 2.3.2 Flat coordinates and osculating cones

We recall the definition of the flat structure on an algebraic group and of the osculating cone according to [Kem86, Section 1 & 2].

In the analytic setting, a Lie group G and its tangent space $\mathfrak{g}$ are related by the exponential mapping, which is an analytic diffeomorphism between an open neighbourhood of the identity of G and an open neighbourhood of the origin of $\mathfrak{g}$ (see for instance

[Hel01, Chapter II, §1]). Roughly speaking, the flat structure on an algebraic group is a truncated exponential mapping. We explain this in more details.

For an algebraic group X of finite type over an algebraically closed field, let $X_{n,x}$ be the $n$-th ***infinitesimal neighbourhood*** of $x \in X$ which is given by the $(n+1)$-st power of the ideal of $x$. It is sufficient to define the flat structure on X at the identity $e$ since we can translate this to any point of X. We denote T the tangent space to X at $e$. Recall that the identity is the fixed point of the $m$-power operation which sends $g$ to $g^m$ for all $m \in \mathbb{Z}$.

**Definition 2.3.3.** The *flat structure* of $n$-th order on X at $e$ is given by an equivariant isomorphism $\iota : T_{n,0} \to X_{n,e}$ so that the multiplication by $m$ on $T_{n,0}$ coincides with the action of the $m$-power operation on $X_{n,e}$ for all integers $m$. We call a smooth subvariety $Y \subset X$ passing through ***e flat to the n-th order*** if $\iota^{-1}(Y_{n,e})$ has the form $S_{n,0}$ where S is a linear subspace of T.

**Remark 2.3.4.**

(a) If $\mathrm{char}(\Bbbk) = 0$, there is a unique equivariant flat structure of order $n$ for all $n \in \mathbb{N}$. In the limit, these flat structures give an analytic isomorphism between a neighbourhood of the 0 in T and a neighbourhood of $e$ in X, the exponential mapping.

(b) If $\mathrm{char}(\Bbbk) = p$, there exists an equivariant flat structure on X if $n$ is strictly less than $p$.

(c) Let $E \subset X$ be a smooth curve passing through $x$. Then E is flat for the flat structure of order $n$ if and only if the $m$-power operation maps $E_{n,x}$ to $E_{n,x}$ for all integers $m$.

In Subsection 2.3.3, we will give an explicit description of the flat structure on the Picard variety Pic(C).

By abuse of notation, we call $\iota$ the equivariant isomorphism defining the flat structure at an arbitrary point $x \in X$. We use the isomorphism $\iota$ to expand regular functions on X at $x$. More precisely, let $f_1, \ldots, f_m$ be regular functions on X at $x$ such that their pullback under $\iota$ forms a basis $\{x_1, \ldots, x_m\}$ for the linear functions on $T_{n,0}$.

**Definition 2.3.5.** Let $g$ be a regular function on X at $x$. An *expansion of* $g$ *in flat coordinates* is the expansion $\iota^*(g) = g_k + g_{k+1} + \cdots$ where $g_j$ is a homogeneous polynomial in the variables $x_i$ on T. A *component* of $f$ is such a homogeneous polynomial in the expansion.

Our main object of interest is the following.

**Definition 2.3.6.** Let X be an algebraic group with a flat structure and let $Y \subset X$ be a subvariety passing through $x \in X$. The *osculating cone of order* $r$ to Y at a point $x$, denoted by $\mathrm{OC}_r(Y, x) \subset T_x(X)$, is the closed scheme defined by the ideal generated by the forms

$$\{f_k | \ f_k \text{ is a component for an element } f \in I(Y), \forall k \leq r\}.$$

To get back to the Brill–Noether locus $W^0_d(C)$, we end this section with an example.

*Example* 2.3.7. For the Brill–Noether locus $W^0_d(C) \subset \mathrm{Pic}^d(C)$, the tangent cone at $L \in W^1_d(C)$ coincides with the osculating cone $\mathrm{OC}_2(W^0_d(C), L)$ of order 2. Indeed, by Corollary 2.3.2, the Brill–Noether locus $W^0_d(C)$ is given locally by the maximal minors of a $(d-1) \times 2$ matrix of regular functions vanishing at L. Thus, the ideal of the tangent cone is generated by all quadratic components in the flat expansion of regular functions in the ideal of $W^0_d(C)$.

### 2.3.3 Infinitesimal deformations of global sections

We follow [Kem83, Section 3] in order to represent points of the canonical space $H^0(C, \omega_C)^*$ by principal parts of rational functions in $\mathscr{O}_C$. Then, we introduce flat coordinates on Pic(C) and show the connection of an infinitesimal deformation of L and flat curves in Pic(C) as in [Kem86, Section 2]. Everything leads to a criterion for points in the canonical space to lie in the osculating cone.

Let $\mathscr{M}$ be an arbitrary line bundle on C and let $\mathrm{Rat}(\mathscr{M})$ be the space of all rational sections of $\mathscr{M}$. For a point $p \in C$, the space of principal parts of $\mathscr{M}$ at $p$ is the quotient

$$\mathrm{Prin}_p(\mathscr{M}) = \mathrm{Rat}(\mathscr{M}) / \mathrm{Rat}_p(\mathscr{M}),$$

where $\mathrm{Rat}_p(\mathscr{M})$ is the space of rational sections of $\mathscr{M}$ which are regular at $p$. Since a rational section of $\mathscr{M}$ has only finitely many poles, we get a mapping

$$\begin{aligned}\mathfrak{p} : \mathrm{Rat}(\mathscr{M}) &\longrightarrow \mathrm{Prin}(\mathscr{M}) := \bigoplus_{p\in C} \mathrm{Prin}_p(\mathscr{M}) \\ s &\longmapsto (s \text{ modulo } \mathrm{Rat}_p(\mathscr{M}))_{p\in C}\end{aligned}$$

and the following lemma holds.

**Lemma 2.3.8**. *[Kem83, Lemma 3.3]*
*The kernel and cokernel of* $\mathfrak{p}$ *are isomorphic to* $H^0(C,\mathscr{M})$ *and* $H^1(C,\mathscr{M})$, *respectively.*

In particular, an element $b \in H^1(C,\mathscr{O}_C)$ is represented by a collection $\beta = (\beta_p)_{p\in C}$ of rational functions, where $\beta_p$ is regular at $p$ except for finitely many $p$.

We turn to infinitesimal deformations of our line bundle $L \in W^1_d(C)$ which are determined by elements in $H^1(C,\mathscr{O}_C)$. Furthermore, we will give an explicit description of the flat structure on Pic(C).

Let $X_i$ be the infinitesimal scheme $\mathrm{Spec}(A_i)$ supported on one point $x_0$, where $A_i$ is the Artinian ring $\Bbbk[\varepsilon]/\varepsilon^{i+1}$ for $i \geq 1$. We consider the sheaf homomorphism $\mathscr{O}_C \longrightarrow \mathscr{O}^*_{C\times X_i}$ given by the truncated exponential mapping

$$s \longmapsto 1 + s\varepsilon + \frac{s^2\varepsilon^2}{2} + \cdots + \frac{s^i\varepsilon^i}{i!} =: \exp_i(s\varepsilon),$$

which is the identity on $C \times \{x_0\}$. This homomorphism induces a map between cohomology groups

$$H^1(C,\mathscr{O}_C) \longrightarrow H^1(C\times X_i, \mathscr{O}^*_{C\times X_i}),$$

where the image of a cohomology class $b \in H^1(C,\mathscr{O}_C)$ determines a line bundle on $C\times X_i$, denoted by $L_i(b)$, and whose restriction to $C \times \{x_0\}$ is the structure sheaf $\mathscr{O}_C$. For $i = 1$, this is the usual identification between $H^1(C,\mathscr{O}_C)$ and the tangent space to Pic(C) at $\mathscr{O}_C$. Furthermore, there exists a unique morphism

$$\exp_i(b) : X_i \to \mathrm{Pic}(C)$$

such that $\exp_i(b)(x_0) = \mathscr{O}_C$ and $L_i(b)$ is the pullback of the Poincaré line bundle under the morphism $id_C \times \exp_i(b)$ by the universal property of the Poincaré line bundle. The image of $\exp_i(b)$ in Pic(C) is a flat curve. Indeed, for a general $i$, we have $\exp_i(b)^n = \exp_i(nb)$ for all integers $n$. By Remark 2.3.4 (c), the image of $\exp_i(b)$ is a flat curve through $\mathcal{O}_C$ for $b \neq 0$ in the canonical flat structure on the algebraic group Pic(C).

After translation of the flat structure to the point L, we get flat curves

$$\exp_i(b) : X_i \longrightarrow \mathrm{Pic(C)}$$

with $\exp_i(b)(x_0) = L$ for $b \in H^1(C, \mathcal{O}_C)$. We will describe the infinitesimal deformation of L corresponding to such a flat curve in more detail. If $b \in H^1(C, \mathcal{O}_C)$ is represented by a collection $\beta = (\beta_p)_{p \in C}$ of rational functions, then the line bundle $L_i(b)$ is the $i$-th deformation of L whose stalk at $p$ is given by rational sections $f = f_0 + f_1\varepsilon + \cdots + f_i\varepsilon^i \in \mathrm{Rat(L)} \otimes A_i$ such that $f \exp(\beta_p \varepsilon)$ is regular at $p$. A different choice of $\beta$ gives a different but isomorphic subsheaf of $\mathrm{Rat(L)} \otimes A_i$ (see [Kem86, end of §2] for the details).

*Example* 2.3.9. If $i = 2$, then $f = f_0 + f_1\varepsilon + f_2\varepsilon^2 \in \mathrm{Rat(L)} \otimes A_2$ is a global section of $L_2(b)$ if the following three conditions are satisfied:

(a) $f_0$ is a global section of L,

(b) $f_1 + f_0\beta_p$ is regular at $p$ for all $p \in C$ and

(c) $f_2 + f_1\beta_p + f_0\beta_p^2/2$ is regular at $p$ for all $p \in C$.

Note that the conditions are independent of the representative $\beta_p$.

Furthermore, we have exact sequences

$$\begin{gathered} 0 \longrightarrow \varepsilon L \longrightarrow L_1(b) \longrightarrow L \longrightarrow 0, \\ 0 \longrightarrow \varepsilon^2 L \longrightarrow L_2(b) \longrightarrow L_1(b) \longrightarrow 0, \textit{ etc.}. \end{gathered} \tag{2.2}$$

Using the first exact sequence, we get a criterion for points in the canonical space lying in the tangent cone. We denote by $\Bbbk \cdot b$ the line spanned by a cohomology class $b$.

**Lemma 2.3.10**. *Let $0 \neq b \in H^1(C, \mathcal{O}_C)$ be a nonzero cohomology class. The point $\Bbbk \cdot b \in \mathbb{P}^{g-1}$ lies in the tangent cone if and only if there exists a global section*

$$f_0 + f_1\varepsilon \in H^0(C \times X_1, L_1(b)).$$

*In this case, $f_0 \neq 0$ is unique up to scalar.*

*Proof.* Let $0 \neq b \in H^1(C, \mathscr{O}_C)$ be a cohomology class. Applying the global section functor to the short exact sequence (2.2), we get the exact sequence

$$0 \longrightarrow H^0(C,L) \longrightarrow H^0(C \times X_1, L_1(b)) \longrightarrow H^0(C,L) \xrightarrow{\cup b} H^1(C,L)$$

where the coboundary map is given by the cup-product with $b$ by [Kem83, Lemma 10.6]. By Remark 2.2.3, $\Bbbk \cdot b$ is in the tangent cone if and only if the map $\cup b$ has rank 1. Thus, for points in the tangent cone, $H^0(C \times X_1, L_1(b))$ is three-dimensional and there exists a global section as desired. □

The following criterion provides the connection of the osculating cone of order 3 and second order deformations of L. The proof follows [KS88, Lemma 4].

**Lemma 2.3.11.** *Let $0 \neq b \in H^1(C, \mathscr{O}_C)$ be a nonzero cohomology class. The point $\Bbbk \cdot b \in \mathbb{P}^{g-1}$ lies in the osculating cone of order 3 if and only if there exists a global section*

$$f_0 + f_1\varepsilon + f_2\varepsilon^2 \in H^0(C \times X_2, L_2(b))$$

*where $f_0 \neq 0$.*

*Proof.* Applying Theorem 2.3.1 to $L_2(b)$, there exists an approximating homomorphism of $A_2$-modules

$$A_2^2 \xrightarrow{\varphi} A_2^{d-1}$$

such that $\ker(\varphi) = H^0(C \times X_2, L_2(b))$, $\operatorname{coker}(\varphi) = H^1(C \times X_2, L_2(b))$ and the pullback of $W^0_d(C)$ via the flat curve $\exp_2(b) : X_2 \longrightarrow \operatorname{Pic}(C)$ is given by the maximal minors of $\varphi$. In other words, $\varphi$ is the pullback of the matrix $(f_{ij})$ of Corollary 2.3.2. The matrix $\varphi$ is equivalent to a matrix

$$\begin{pmatrix} \varepsilon^u & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon^v & 0 & \cdots & 0 \end{pmatrix}^T$$

with $1 \leq u \leq v \leq 3$ since $\varphi \otimes k(L) = 0$.

Hence, the line $\Bbbk \cdot b$ is contained in the osculating cone $\mathrm{OC}_3(W^0_d(C), L)$ if and only if $u + v \geq 4$. Since the tangent cone is smooth and the point $b \neq 0$, the exponent $u = 1$. We conclude that the line $\Bbbk \cdot b \in \mathrm{OC}_3(W^0_d(C), L)$ if and only if $v = 3$. Since $\varepsilon^3 = 0$, there

exists a global section $f = f_0 + f_1\varepsilon + f_2\varepsilon^2 \in \mathrm{H}^0(\mathrm{C}\times\mathrm{X}_2, \mathrm{L}_2(b))$.
Restricting $\varphi$ to $\mathbb{k} \cong \mathrm{A}_0$, we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathrm{C}\times\mathrm{X}_2, \mathrm{L}_2(b)) & \longrightarrow & \mathrm{A}_2^2 & \xrightarrow{\varphi} & \mathrm{A}_2^{d-1} \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^0(\mathrm{C},\mathrm{L}) & \xrightarrow{\cong} & \mathbb{k}^2 & \xrightarrow{0} & \mathbb{k}^{d-1} \end{array}$$

and thus, $f_0$ is nonzero in $\mathrm{H}^0(\mathrm{C},\mathrm{L})$. □

Let $\mathbb{k}\cdot b$ be a point in the tangent cone and let $f_0 + f_1\varepsilon$ be the corresponding global section as in Lemma 2.3.10. The following corollary answers the question of whether a second order deformation of the global section $f_0 \in \mathrm{H}^0(\mathrm{C},\mathrm{L})$ is possible.

**Corollary 2.3.12.** *Let $0 \neq b = [\beta] \in \mathrm{H}^1(\mathrm{C},\mathscr{O}_\mathrm{C})$ be a cohomology class, such that $\mathbb{k}\cdot b$ lies in the tangent cone. Then, $\mathbb{k}\cdot b$ is contained in the osculating cone of order 3 if and only if the class $[\frac{f_0\beta^2}{2} + f_1\beta]$ is zero in $\mathrm{H}^1(\mathrm{C},\mathrm{L})/(\mathrm{H}^0(\mathrm{C},\mathrm{L})\cup[\beta])$ for the section $f_0+f_1\varepsilon \in \mathrm{H}^0(\mathrm{C}\times\mathrm{X}_1, \mathrm{L}_1(b))$ of Lemma 2.3.10.*

*Proof.* The cohomology class $[\frac{f_0\beta^2}{2} + f_1\beta]$ is zero in $\mathrm{H}^1(\mathrm{C},\mathrm{L})/(\mathrm{H}^0(\mathrm{C},\mathrm{L})\cup[\beta])$ if and only if $\frac{f_0\beta^2}{2} + f_1\beta + f_2 + \widetilde{f}\beta$ is regular for an $\widetilde{f} \in \mathrm{H}^0(\mathrm{C},\mathrm{L})$ and a rational section $f_2$. Taking Example 2.3.9 into account, this condition is satisfied if and only if $f_0 + (f_1+\widetilde{f})\varepsilon + f_2\varepsilon^2 \in \mathrm{H}^0(\mathrm{C}\times\mathrm{X}_2,\mathrm{L}_2(b))$ since $f_0 + f_1\varepsilon \in \mathrm{H}^0(\mathrm{C}\times\mathrm{X}_1,\mathrm{L}_1(b))$. Thus, it is equivalent for $b$ to be in the osculating cone of order 3 by Lemma 2.3.11. □

In the proof of Theorem 2.1.1, we will describe points in the fiber of the tangent cone over $\mathbb{P}^1$ in terms of principal parts. We need the following lemma (see [Kem86, Section 3]).

**Lemma 2.3.13.** *Let $f_0 \in \mathrm{H}^0(\mathrm{C},\mathrm{L})$ be a global section and let $\mathrm{D}$ be its divisor of zeros. The fiber of the tangent cone $\overline{\mathrm{D}}$ over the point $\mathbb{k}\cdot f_0 \in \mathbb{P}(\mathrm{H}^0(\mathrm{C},\mathrm{L}))$ is the projectivisation of the kernel of $\cup f_0 : \mathrm{H}^1(\mathrm{C},\mathscr{O}_\mathrm{C}) \to \mathrm{H}^1(\mathrm{C},\mathscr{O}_\mathrm{C}(\mathrm{D}))$, denoted by $\mathrm{K}(f_0)$. Furthermore, $\mathrm{K}(f_0)$ is generated by cohomology classes of principal parts bounded by $\mathrm{D}$, that is, elements of $\mathrm{H}^0(\mathrm{C},\mathscr{O}_\mathrm{C}(\mathrm{D})|_\mathrm{D})$ modulo the total principal part of elements in $\mathrm{H}^0(\mathrm{C},\mathscr{O}_\mathrm{C}(\mathrm{D}))$.*

*Proof.* The first statement is clear by Remark 2.2.3. For the second statement, we consider the short exact sequence $0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C(D)|_D \longrightarrow 0$ and its long exact cohomology sequence

$$0 \to \Bbbk \to H^0(C, \mathscr{O}_C(D)) \xrightarrow{\rho} H^0(C, \mathscr{O}_C(D)|_D) \to H^1(C, \mathscr{O}_C) \xrightarrow{\cup f_0} H^1(C, \mathscr{O}_C(D)).$$

Thus, $K(f_0) = \operatorname{coker}(H^0(C, \mathscr{O}_C(D)) \xrightarrow{\rho} H^0(C, \mathscr{O}_C(D)|_D))$. The map $\rho$ is the restriction of the map $\mathfrak{p}$ to the finite dimensional vector space $H^0(C, \mathscr{O}_C(D))$, that is, there is a commutative diagram

$$\begin{array}{ccc} H^0(C, \mathscr{O}_C(D)) & \xrightarrow{\rho} & H^0(C, \mathscr{O}_C(D)|_D) \\ \downarrow & & \downarrow \\ \operatorname{Rat}(\mathscr{O}_C) & \xrightarrow{\mathfrak{p}} & \operatorname{Prin}(\mathscr{O}_C). \end{array}$$

The second statement follows. □

## 2.4 Proof of the main theorem

Our proof is a modification of the proof in [Kem86, Section 4]. We fix the notation for the proof: Let

$$D = \sum_{i=1}^{n} k_i p_i$$

be an arbitrary divisor in the linear system $|L|$, where $k_i \geq 1$ and $\sum_{i=1}^{n} k_i = d$. Let $f_0 \in H^0(C, L)$ be the section whose divisor of zeros is exactly D and let $(f_0, g_0)$ be a basis of $H^0(C, L)$. Let $h := \frac{g_0}{f_0} \in H^0(C, \mathscr{O}_C(D)) \cong H^0(C, L)$. We now assume that each ramification point is tame, that is, the ramification index $k_i \geq 2$ is coprime to the characteristic of $\Bbbk$.

*Proof of Theorem 2.1.1.* We proceed as follows. We determine the condition on a point in the fiber to lie in the osculating cone and reduce this condition to a system of equations. To this end, we present the set of solutions as well as their geometry.

Let $b := [\beta] \neq 0$ be an arbitrary point in the fiber $\overline{D}$ of the tangent cone, where $\beta = (\beta_p)_{p \in C} \in \operatorname{Prin}(\mathscr{O}_C(D))$. By Lemma 2.3.13, we can choose the principal part $\beta$ such that $\beta_p$ is regular away from the support of D and the pole order at $p_i$ is bounded by $k_i$, that is, $\beta$ is spanned by elements of $H^0(C, \mathcal{O}_C(D)|_D)$.

First of all, we state the condition that the point $b$ lies in the osculating cone. Since the line $\Bbbk \cdot b$ spanned by $b$ is a point in a fiber $\overline{D}$ of the tangent cone, $f_0\beta$ is regular and $f_0 + 0\varepsilon$ is a global section of $H^0(C \times X_1, L_1(b))$ by Lemma 2.3.10. We can apply Corollary 2.3.12. The point $\Bbbk \cdot b \in \overline{D}$ is in the osculating cone $\mathfrak{OC}_3(W^0_d(C), L)$ if and only if there exist sections $f_1, f_2 \in \operatorname{Rat}(L)$ such that

$$\frac{f_0\beta^2}{2} + f_1\beta + f_2 \quad \text{is regular at } p_i. \tag{2.3}$$

Note that $f_1$ and $f_2$ are everywhere regular by Corollary 2.3.12 and Lemma 2.2.2, respectively. Thus, $f_1, f_2 \in H^0(C, L)$ and the global section $f_1 = af_0 + cg_0$ is a linear combination of $f_0$ and $g_0$. Since $f_0\beta$ is regular, condition (2.3) simplifies:

$$\frac{f_0\beta^2}{2} + cg_0\beta \quad \text{is regular at all } p_i$$

for some constant $c \in \Bbbk$.

Now, we reduce the condition to a simple system of equations. Since regularity is a local property, we study our condition at every single point $p_i$ of the support of D. To simplify notation we set $p := p_i$ and $k := k_i$.

Then, $\beta_p = \sum_{i=1}^{k} \lambda_i \beta_i$, where $\beta_i$ is the principal part of a rational function with pole of order $i$ at $p$. Hence, $\beta_1, \ldots, \beta_k$ is a basis of $H^0(C, \mathcal{O}_C(D)|_{k\cdot p})$ and $\beta_p$ is an arbitrary linear combination of this basis.

We have to choose our basis of $H^0(C, \mathcal{O}_C(D)|_{k\cdot p})$ carefully in order to get the polar behaviour in condition (2.3) at $p$ under control. More precisely, we want to have equalities $\beta_j\beta_{k+i-j} = \beta_i\beta_k$ for $i \in \{1, \ldots, k\}$ and $j \in \{i, \ldots, k\}$.
An easy local computation shows the following claim which implies our desired equalities. Here, we need that $p$ is a tame point, that is, the characteristic of $\Bbbk$ does not divide $k$.

*Claim.* There exists a basis $\{\beta_i\}_{i=1,\ldots,k}$ of $H^0(C, \mathcal{O}_C(D)|_{k\cdot p})$, that is the stalk of $\mathcal{O}_C(D)|_{k\cdot p}$ at $p$, satisfying the equations $\beta_k = h|_p$ and $\beta_i = (\beta_1)^i$ for all $i$.

*Proof.* A basis of the stalk of $\mathscr{O}_C(D)|_{k\cdot p}$ at $p$ is the images of $t^{-1}, t^{-2}, \dots, t^{-k}$ under $\mathfrak{p} : \mathrm{Rat}(\mathscr{O}_C(D)) \to \mathrm{Prin}(\mathscr{O}_C(D))$ where $t$ is a local parameter function which vanishes simply at $p$. After rescaling our local parameter, we may assume that $\beta_k := h|_p$ is of the form

$$\beta_k = (c_1 t^{-1} + c_2 t^{-2} + \dots + c_{k-1} t^{-k+1} + t^{-k})|_p \in \mathscr{O}_C(D)|_{k\cdot p}$$

for some constants $c_1, \dots, c_{k-1} \in \Bbbk$. We define $\beta_1$ to be the expansion of the $k$-th root of the rational function $F := c_1 t^{-1} + c_2 t^{-2} + \dots + c_{k-1} t^{-k+1} + t^{-k}$ up to some order. Therefore, we need the assumption that the characteristic of $\Bbbk$ does not divide $k$. To be more precise, let

$$\begin{aligned}\sqrt[k]{F} &= \sqrt[k]{c_1 t^{k-1} + \dots + c_{k-1} t + 1} \cdot \frac{1}{t} \\ &= \frac{1}{t} \cdot \left( \sum_{j=0}^{k-1} \binom{1/k}{j} (c_1 t^{k-1} + \dots + c_{k-1} t)^j + h.o.t \right)\end{aligned}$$

be the expansion of $\sqrt[k]{F}$. We define

$$\beta_1 := \frac{1}{t} \left( \sum_{j=0}^{k-1} \binom{1/k}{j} (c_1 t^{k-1} + c_2 t^{k-2} + \dots + c_{k-1} t)^j \right) \Big|_p$$

and $\beta_i := (\beta_1)^i$ for $i = 1, \dots, k-1$. Note that $\beta_i$ is the principal part of a rational function with pole of order $i$ and $\beta_1^k - \beta_k \in \mathrm{Rat}_p(\mathscr{O}_C(D))$. Hence, $\{\beta_i\}_{i=1,\dots,k}$ form a basis and $\beta_k = (\beta_1)^k$.

Recall that $f_0 \beta_k = f_0 \cdot \frac{g_0}{f_0}|_p = g_0|_p$ and $f_0 \beta_i \beta_j$ is regular for $i + j \le k$. Using our careful choice of $\beta_i$, condition (2.3) is fulfilled at the point $p$ if and only if

$$
\begin{aligned}
\frac{f_0(\beta_p)^2}{2} + cg_0\beta_p &= \frac{f_0}{2}\left(\sum_{i=1}^{k}\lambda_i\beta_i\right)^2 + cg_0\left(\sum_{i=1}^{k}\lambda_i\beta_i\right) \\
&= \frac{f_0}{2}\left(\sum_{i=1}^{k}\sum_{j=1}^{k}\lambda_i\lambda_j\beta_i\beta_j\right) + cg_0\left(\sum_{i=1}^{k}\lambda_i\beta_i\right) \\
&= \frac{f_0}{2}\left(\sum_{i=1}^{k}\sum_{j=i}^{k}\lambda_j\lambda_{k+i-j}\beta_j\beta_{k+i-j}\right) + cg_0\left(\sum_{i=1}^{k}\lambda_i\beta_i\right) \\
&= \frac{f_0}{2}\beta_k\left(\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\lambda_j\lambda_{k+i-j}\right)\beta_i\right) + cg_0\left(\sum_{i=1}^{k}\lambda_i\beta_i\right) \\
&= \frac{g_0}{2}\left(\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\lambda_j\lambda_{k+i-j} + 2c\lambda_i\right)\beta_i\right) \in \mathscr{O}_{\mathrm{C}}(\mathrm{D})|_{k\cdot p}
\end{aligned}
$$

is regular at $p$.

Since the global section $g_0$ does not vanish at $p \in \mathrm{Supp}(\mathrm{D})$, condition (2.3) is regular at $p$ if and only if

$$\sum_{j=i}^{k}\lambda_j\lambda_{k+i-j} + 2c\lambda_i = 0$$

for all $i = 1, \ldots, k$.

At the end, we have to solve this system of equations and describe the geometry. Let $\lambda_i, c$ be a solution of the equations. In order to relate a solution to its geometry, we distinguish two cases. The second case is only relevant if the multiplicity $k \geq 2$.

**Case 1.** *If* $c \neq 0$, *then either* $\beta_p = -2c\beta_k = -2c \cdot h|_p$ *or* $\beta_p = 0$. *Geometrically, either* $\mathbb{k}\cdot[\beta_p] = \mathbb{k}\cdot[h|_p] \in \overline{k\cdot p}$ *or* $[\beta_p] = 0$.

*Proof.* Let $c$ be a nonzero constant. We consider the equation

$$\lambda_k^2 + 2c\lambda_k = 0.$$

Hence, either $\lambda_k = -2c$ or $\lambda_k = 0$. For both solutions, the equation

$$2\lambda_{k-1}\lambda_k + 2c\lambda_{k-1} = 0$$

implies that $\lambda_{k-1} = 0$. Inductively, $\lambda_i = 0$ for $1 \leq i \leq k-1$, which proves our claim.

**Case 2.** *If $c=0$, then $\beta_p = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \lambda_i \beta_i$ is an arbitrary linear combination. Geometrically, $\Bbbk \cdot [\beta_p] \in \overline{\lfloor \frac{k}{2} \rfloor \cdot p}$.*

*Proof.* If $c=0$, then the system of equation reduces to

$$\begin{cases} \lambda_k^2 = 0 & (i=k) \\ \lambda_{k-1}\lambda_k + \lambda_k\lambda_{k-1} = 0 & (i=k-1) \end{cases}$$

$$\begin{cases} \lambda_{k-2}\lambda_k + \lambda_{k-1}^2 + \lambda_k\lambda_{k-2} = 0 & (i=k-2) \\ \lambda_{k-3}\lambda_k + \lambda_{k-2}\lambda_{k-1} + \lambda_{k-1}\lambda_{k-2} + \lambda_k\lambda_{k-3} = 0 & (i=k-3) \end{cases}$$

$$\vdots$$

$$\begin{cases} \lambda_{k-2l}\lambda_k + \cdots + \lambda_{k-l}^2 + \cdots + \lambda_k\lambda_{k-2l} = 0 & (i=k-2l) \\ \lambda_{k-2l-1}\lambda_k + \cdots + \lambda_{k-l-1}\lambda_{k-l} + \lambda_{k-l}\lambda_{k-l-1} + \cdots + \lambda_k\lambda_{k-2l-1} = 0 & (i=k-2l-1) \end{cases}$$

$$\vdots$$

$$\lambda_1\lambda_k + \cdots + \lambda_{\lfloor \frac{k+1}{2} \rfloor}\lambda_{\lceil \frac{k+1}{2} \rceil} + + \cdots + \lambda_k\lambda_1 = 0 \qquad (i=1).$$

Thus, $\lambda_k = 0$ solve the first two equations. The second pair now gives $\lambda_k = \lambda_{k-1} = 0$. Inductively, we obtain $\lambda_{k-l} = 0$ for all $l$ with $k-2l \geq 1$. Whether the last equation gives a condition depends on the parity of $k$. Thus $\lambda_k = \cdots = \lambda_{\lfloor \frac{k}{2} \rfloor + 1} = 0$ and $\lambda_1, \ldots, \lambda_{\lfloor \frac{k}{2} \rfloor}$ arbitrary is the solution of the system of equations.

This completes the local study of condition (2.3). We now make use of the local description of $\beta$ which leads to a global picture. The principal part $\beta = \sum_{i=1}^n \beta_{p_i}$ is supported on D. Since the constant $c$ is the same for all local computations, either all principal parts $\beta_{p_i}$ satisfy Case 1 or all principal parts satisfy Case 2.

By Lemma 2.3.13, the total principal part of $h$ yields a relation

$$\Big[ \sum_{i=1}^{n} h|_{p_i} \Big] = 0. \qquad (*)$$

If we are in Case 1, let $\emptyset \neq \mathrm{I} \subsetneq \{1, \ldots, n\}$ be the set of indices, where $[\beta_{p_i}] = [h|_{p_i}] \neq 0$ and let $\mathrm{I}^c = \{1, \ldots, n\} \backslash \mathrm{I}$ be its complement, then the nonzero point $b = [\beta]$ lies in the osculating cone if and only if

$$\mathbb{k}\cdot b=\mathbb{k}\cdot\Big[\sum_{i\in I}h|_{p_i}\Big]\overset{(*)}{=}\mathbb{k}\cdot\Big[\sum_{i\in I^c}h|_{p_i}\Big]\in\underbrace{\overline{\sum_{i\in I}k_ip_i}}_{=:D_1}\bigcap\underbrace{\overline{\sum_{i\in I^c}k_ip_i}}_{=:D_2},$$

In Case 2, the nonzero point $b=[\beta]$ lies in the osculating cone if and only if

$$\mathbb{k}\cdot b\in\overline{\sum_{i=1}^{n}\Big\lfloor\frac{k_i}{2}\Big\rfloor p_i}=\underbrace{\overline{\sum_{i=1}^{n}\Big\lfloor\frac{k_i}{2}\Big\rfloor p_i}}_{=:D_1}\bigcap\underbrace{\overline{\sum_{i=1}^{n}\Big\lceil\frac{k_i}{2}\Big\rceil p_i}}_{=:D_2}.$$

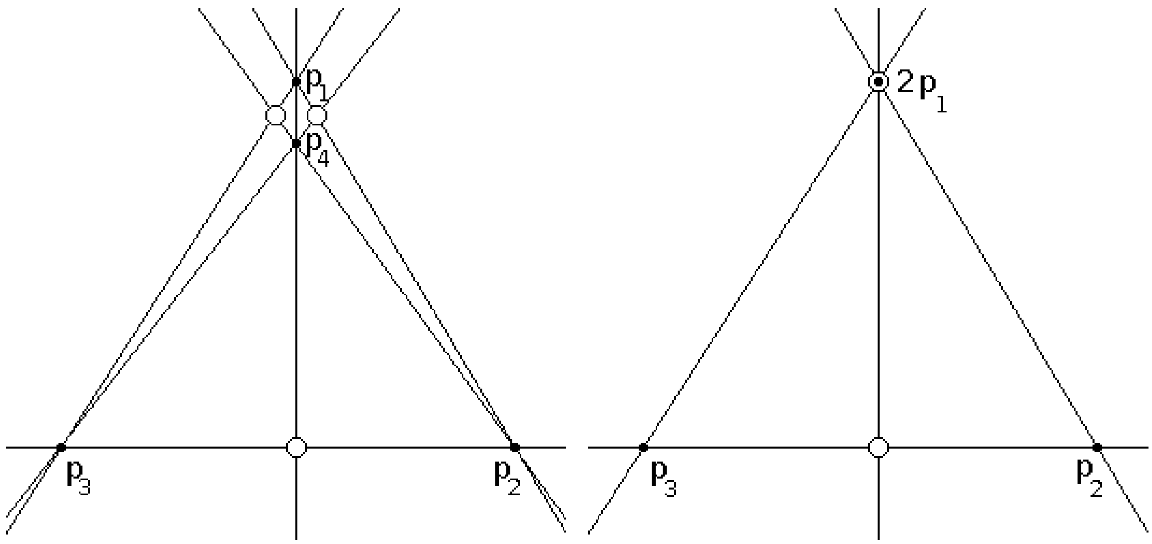
Note that $\overline{\sum_{i=1}^{n}\lfloor\frac{k_i}{2}\rfloor p_i}$ is the greatest linear span of osculating spaces to C at $p_i$ which can be expressed as an intersection $\overline{D_1}\cap\overline{D_2}$ of a nonzero effective decomposition $D=D_1+D_2$. Our theorem follows. □

**Remark 2.4.1.** The two cases, appearing in the proof, contribute differently to the osculating cone. From the first case, we always get $2^{n-1}-1$ points in the osculating cone. If the curve C has at least one ramification point in the fiber, there is a $(\sum_{i=1}^{n}\lfloor\frac{k_i}{2}\rfloor-1)$-dimensional component in the fiber by the second case. The osculating cone has a higher-dimensional component in a fiber $\overline{D}$ of the tangent cone unless $D=p_1+\cdots+p_d$ or $D=2p_1+p_2+\cdots+p_{d-1}$ or $D=3p_1+p_2+\cdots+p_{d-2}$.

*Example* 2.4.2 (Example 2.2.4 continued). In a fiber over a general point $(\lambda:\mu)\in\mathbb{P}^1$, the curve C is unramified. In a general fiber, which is a projective plane, the four distinct points of C determine three pairs of connection lines and hence, three corresponding intersection points. Thus, all intersection points, associated to the pencil $|L|$ on C, sweep out a trigonal curve $C_2$ (see also [BL04, Section 12.7]). We assume that $\pi_C:C\to\mathbb{P}^1$ has only simple branch points. By the Riemann-Hurwitz formula

$$2\cdot6-2-4\cdot(2\cdot0-2)=18,$$

the curve C has 18 simple ramification points. Applying Theorem 2.1.1 for $D=2p_1+p_2+p_3$, there are 4 points in a special fiber as mentioned above.




(a) Sketch of a fiber $\overline{p_1+p_2+p_3+p_4}$, where two points are close together. Hence, two residual points are also close together.

(b) Sketch of the fiber $\overline{2p_1+p_2+p_3}$. The point $p_1$ is also a ramification point of $C_2$. The second point of $C_2$ is the intersection of $\overline{p_2+p_3}$ and the tangent line $\overline{2p_1}$.

Figure 2.2: Two fibers of the map $\varphi_{|L|}: C \xrightarrow{4:1} \mathbb{P}^1$

By the geometry of these points, every simple ramification point of C is a simple ramification point of $C_2$, too. Hence, the genus $g(C_2)$ of $C_2$ is

$$g(C_2) = \frac{3\cdot(-2)+18}{2}+1 = 7$$

by the Riemann-Hurwitz formula. Furthermore, C and $C_2$ intersect transversally since

$$\begin{aligned}\chi(C\cap C_2) =& \chi(C)+\chi(C_2)-\chi(C\cup C_2)\\ =& \chi(C)+\chi(C_2)-\chi(\mathcal{O}_{C_3}(W^0_4(C),L)) = 1-6+1-7-(1-30) = 18.\end{aligned}$$

The osculating cone is thus the union of two transversal intersecting curves of genus 6 and genus 7.

Note that the space of the six connection lines induce an étale double cover $\widetilde{C_2}\to C_2$ of the trigonal curve $C_2$. By Recillas' theorem ([Rec74]), the Jacobian of C and the Prym variety associated to $\widetilde{C_2}\to C_2$ are isomorphic.

## 2.5 Evidences for our conjecture

In this section, we study the local structure of $\mathrm{W}^0_d(\mathrm{C})$ at a general point $\mathrm{L} \in \mathrm{W}^1_d(\mathrm{C})$ for a general canonically embedded curve C of genus $g$ and Brill–Noether number $\rho = \rho(g,d,1) \geq 1$.

Let $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ be the osculating cone of order 3 and let

$$\alpha : \mathbb{P}(\widetilde{\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})}) \to \mathbb{P}^1 \times \mathbb{P}^{h^1(\mathrm{C},\mathrm{L})-1}$$

be as in Conjecture 2.1.3, the map induced by the projection from the vertex of the tangent cone. Using techniques of Section 2.3, we show that the curve C is contained in the osculating cone $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$, a fiber of $\alpha$ is singular at any point of C and the osculating cone $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ is smooth and of dimension $\rho+1$ at a general point of C.

We use the notation as in the previous sections. Note that one can immediately generalise Lemma 2.3.10, 2.3.11, 2.3.13 and Corollary 2.3.12. Furthermore, we use the same conditions for lifting sections obtained via $\exp(([\beta]+\delta[\beta'])\varepsilon)$ over $\mathbb{k}[\varepsilon,\delta]/(\varepsilon^3,\delta^2)$ (see Section 2.3.3) for principal parts $\beta,\beta' \in \mathrm{Prin}(\mathscr{O}_\mathrm{C})$ to determine tangent vectors to $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ at some point. We use the identification of tangent vector in direction $[\beta']$ with maps from $\mathrm{Spec}(\mathbb{k}[\delta]/\delta^2)$ to $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ supported at the point spanned by $[\beta]$ (see [Har77, II, Exercise 2.8]).

**Remark 2.5.1.** We get similar lifting conditions for a section $f = f_0 + f_1\varepsilon + f_2\varepsilon^2 \in \mathrm{Rat}(\mathrm{L}) \otimes \mathbb{k}[\varepsilon,\delta]/(\varepsilon^3,\delta^2)$ as in Example 2.3.9, which are independent of the choice of the representatives of $\beta$ and $\beta'$.

In the following sections, let $n := h^1(\mathrm{C},\mathrm{L}) - 1 = g - d$ be the speciality of L.

### 2.5.1 Theoretical part

By abuse of notation, we will also use the notation $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ and $\mathscr{T}_\mathrm{L}(\mathrm{W}^0_d(\mathrm{C}))$ for the projectivisations of the osculating cone and the tangent cone, respectively.

**Remark 2.5.2.** As in Section 2.2, the osculating cone $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ sits naturally in the tangent cone $\mathscr{T}_\mathrm{L}(\mathrm{W}^0_d(\mathrm{C})) \subset \mathbb{P}^{g-1}$, which is a cone over the Segre product $\mathbb{P}^1 \times \mathbb{P}^n$.

Recall that $\mathcal{T}_{\mathrm{L}}(\mathrm{W}^0_d(\mathrm{C})) = \mathrm{X} := \bigcup_{\mathrm{D}\in|\mathrm{L}|} \overline{\mathrm{D}}$, that is the scroll swept out by the pencil $|\mathrm{L}| = g^1_d$. Furthermore, X is the image of a balanced projectivised bundle over $\mathbb{P}^1$ of the form $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n+1} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \rho}$ (see also Section 1.2 and especially, Example 1.2.1).

As a subvariety of X (or more precise, of the projectivised bundle associated to X), we can describe the determinantal structure of the osculating cone as in Section 2.2. Let $\mathrm{R}_{\mathrm{X}} = \mathbb{k}[t_0, t_1, u_0, \dots, u_n, w_1, \dots, w_\rho]$ be the bigraded coordinate ring of X, where $t_0, t_1$ are of bidegree $(1,0)$, $u_0, \dots, u_n$ are of bidegree $(0,1)$ and $w_1, \dots, w_\rho$ are of bidegree $(1,1)$ (see Example 1.2.1). Note that the embedding of the scroll is given by all homogeneous forms of bidegree $(1,1)$. The ideal of the osculating cone inside $\mathrm{R}_{\mathrm{X}}$ is generated by the $2\times 2$ minors of a $2\times(n+1)$ matrix:

$$\mathrm{I}_{2\times 2}\left(\begin{pmatrix} u_0 & \dots & u_n \\ t_0 q_{10} - t_1 q_{00} & \dots & t_0 q_{1n} - t_1 q_{0n} \end{pmatrix}\right) \tag{2.4}$$

where $q_{ij}$ are homogeneous forms of bidegree $(2,2)$ in the coordinate ring of the scroll. Restricting to a fiber $\overline{\mathrm{D}}$ of the tangent cone, the matrix is of the form

$$\begin{pmatrix} l_0 & \cdots & l_n \\ q_0 & \cdots & q_n \end{pmatrix},$$

where $l_i$ and $q_i$ are linear and quadratic forms on $\mathbb{P}^{d-2} = \overline{\mathrm{D}}$, respectively.

We start by proving the following proposition similar to the easy part of Theorem 2.1.1.

**Proposition 2.5.3.** *Let* C *be a smooth canonically embedded curve and let* L *be a smooth point of* $\mathrm{W}^1_d(\mathrm{C})$. *For a general divisor* $\mathrm{D} \in |\mathrm{L}|$, *the intersection of the linear span* $\overline{\mathrm{D}}$ *and the osculating cone* $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}), \mathrm{L})$ *contains the union of all intersection points* $\overline{\mathrm{D}_1} \cap \overline{\mathrm{D}_2}$ *for each decomposition as* $\mathrm{D} = \mathrm{D}_1 + \mathrm{D}_2$ *into nonzero effective divisors. In particular, the curve* C *is contained in the osculating cone.*

*Proof.* Let $\mathrm{D} = p_1 + \cdots + p_d$ be a general element in $|\mathrm{L}|$ consisting of $d$ distinct points and let $(f_0, g_0)$ be a basis of $\mathrm{H}^0(\mathrm{C}, \mathrm{L})$ where $\mathrm{D} = \mathrm{V}(f_0)$. We denote by $\beta_i$ the principal part of the rational function $h := \frac{g_0}{f_0}$ with simple pole at $p_i$ for $i = 1, \dots, d$. Recall that

$[\sum \beta_i] = [h] = 0$. For a decomposition $D = D_1 + D_2$, the point of intersection $\overline{D_1} \cap \overline{D_2}$ is given by

$$\left[\sum_{i \in I} \beta_i\right] = -\left[\sum_{i \in \{1,\ldots,d\} \setminus I} \beta_i\right]$$

for some nonempty subset $I \subset \{1, \ldots, d\}$. By definition, the cohomology class $[f_0(\sum_{i \in I} \beta_i)]$ vanishes in $H^1(C, L)$ and a first order lifting of $f_0$ to $f_0 + 0\varepsilon$ is possible. By our choice of principal parts (we can substitute locally the principal part corresponding to the intersection point with the rational function $h$), the cohomology class $[\frac{f_0}{2}(\sum_{i \in I} \beta_i)^2]$ is zero in $H^1(C, L)/H^0(C, L) \cup [\sum_{i \in I} \beta_i]$ since

$$\left[\frac{f_0}{2}(\sum_{i \in I} \beta_i)^2\right] = \left[\frac{f_0}{2} h(\sum_{i \in I} \beta_i)\right] = \left[\frac{g_0}{2}(\sum_{i \in I} \beta_i)\right] = \frac{g_0}{2} \cup \left[(\sum_{i \in I} \beta_i)\right].$$

□

The proof of the following lemma follows [KS88, Proposition 6].

**Lemma 2.5.4.** *Let* $p \in C$ *be a point of the curve* $C$ *and let* $V = \mathrm{Sing}(\mathcal{T}_L(W^0_d(C)))$ *be the vertex of the tangent cone. For any point* $v \in V$*, the connection line* $\overline{pv}$ *is tangent to the osculating cone* $\mathcal{OC}_3(W^0_d(C), L)$*. In particular, the fiber of* $\alpha$ *is singular at any point of the curve* $C$.

*Proof.* Let $(f_0, g_0)$ be a basis of $H^0(C, L)$, where $f_0$ vanishes at the point $p$. We define $D$ to be the zero divisor of $f_0$. Then, $h := \frac{g_0}{f_0} \in H^0(C, \mathcal{O}_C(D))$ and the point $p = \mathbb{k} \cdot [\beta_p]$ is the cohomology class of the principal part $\beta_p := h|_p$. We choose a basis $[\beta_1], \ldots, [\beta_{\rho-1}]$ of the vertex $V$ where $\beta_i$ is regular at $p$. Recall that each cohomology class in the fiber $\overline{D}$ of the tangent cone can be generated by the principal parts of $h$ without $\beta_p$.

Now let $[\beta] = [\beta_p + \sum \delta_i \beta_i]$ be a generic vector at $p$ with direction to $V$ where $\delta_i$ are indeterminates of the ring $R := \mathbb{k}[\delta_1, \ldots, \delta_{\rho-1}]/(\delta_1, \ldots, \delta_{\rho-1})^2$.

The section $f_0$ lifts to first order since $[f_0\beta] = 0$ in $H^1(C, L) \otimes R$. Geometrically, The linear span of $p$ and $V$ is clearly contained in the tangent cone. Let $f_0 + f_1\varepsilon \in H^0(C \times X_1, L_2([\beta]))$ be a lifting of $f_0$ with $f_1 = 0 + \sum \delta_i \gamma_i$. Then, the $\gamma_i$ are global regular sections, since $\beta_i$ are linear combinations of principal parts of $h$ and hence $f_0\beta$ is regular.

The obstruction to second order lifting is

$$[f_0\beta^2/2 + f_1\beta] \in H^1(C,L)\otimes R/(H^0(C,L)\otimes R)\cup[\beta].$$

We will show that the obstruction always vanishes:

$$\begin{aligned}\left[\frac{f_0\beta^2}{2} + f_1\beta\right] &= \left[\frac{f_0(\beta_p + \sum\delta_i\beta_i)^2}{2} + (\sum\delta_i\gamma_i)(\beta_p + \sum\delta_i\beta_i)\right]\\ &= \left[\frac{f_0\beta_p^2}{2} + \sum\underbrace{f_0\cdot\beta_p\cdot\beta_i\cdot\delta_i}_{=0,\text{ diff. support}} + \underbrace{\sum\beta_p\cdot\gamma_i\cdot\delta_i}_{\beta_p\cup(\sum\gamma_i\delta_i)}\right]\\ &= \left[\frac{g_0}{2}\beta_p\right]\\ &= \left[-\sum g_0\beta_i\delta_i\right] \in H^1(C,L)\otimes R/(H^0(C,L)\otimes R)\cup[\beta].\end{aligned}$$

Since the $[\beta_i]$ are a basis of the vertex, that is, elements of the kernel of $\cup$, the product $g_0\cup[\beta_i] = [g_0\beta_i] = 0$ in $H^1(C,L)$.

Geometrically, every vector at $p$ with direction $\overline{pv}$ for $v\in V$ is tangent to the osculating cone. □

**Remark 2.5.5.** We explain why the product of two principal parts with disjoint support (as in the above proof) vanishes. Let $\beta = (\beta_p) \in \bigoplus_{p\in C}\operatorname{Prin}_p(C)$ and $\omega = (\omega_p) \in \bigoplus_{p\in C}\operatorname{Prin}_p(C)$ be two principal parts where $\beta_p$ is regular for $p\notin I\subset C$ and $\omega_p$ is regular for $p\notin J\subset C$ for finite disjoint subsets I and J of C. The product of $\beta$ and $\omega$ is defined as the product of the rational functions representing the principal parts. Note that this product depends on the choice of rational functions. But since the condition for global sections of infinitesimal deformations (see Example 2.3.9 and Remark 2.5.1) is independent of the representation of the principal part, we may replace $\beta = \beta - \bigoplus_{p\in J}\beta_p$ and $\omega = \omega - \bigoplus_{p\in I}\omega_p$ and the principal part of the product is zero.

**Lemma 2.5.6.** *The osculating cone of order* 3 *to* $W^0_d(C)$ *at* L *is smooth and of dimension* $\rho+1$ *at a general point of the curve* C.

*Proof.* We follow [KS88, Proposition 5.(b)] and use the notation of the previous proof. We will find a $(n-1)$-dimensional space of tangent vectors to $\mathcal{T}_L(W^0_d(C))$ at a general point $p \in \mathrm{C}$ which are not tangent to $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$. We conclude that the dimension of the tangent space to $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ at $p$ is at most $\rho$ in the fiber $\overline{\mathrm{D}}$ of the tangent cone. Since the intersection of the osculating cone and $\overline{\mathrm{D}}$ is given by the maximal minors of a $2 \times (n+1)$ matrix, the dimension is at least $\rho$ (see Remark 2.5.2 and [BV88]). Hence, the claim follows.

We choose $n$ further points $p_1,\dots,p_n$ of C in the fiber $\overline{\mathrm{D}}$ containing $p$ such that the linear span does not intersect the vertex V. Since

$$\dim_{\mathbb{k}}(\overline{p,p_1,\dots,p_n}) + \dim \mathrm{V} = n + \rho - 1 = d - 3 < d - 2,$$

we can find such points.

The principal parts of the rational function $h \in \mathrm{H}^0(\mathrm{C},\mathcal{O}_\mathrm{C}(\mathrm{D}))$ as in the previous proof represent the points $p = \mathbb{k}\cdot[\beta_p] := \mathbb{k}\cdot[h|_p]$ and $p_i = \mathbb{k}\cdot[\beta_i] := \mathbb{k}\cdot[h|_{p_i}]$ for $i = 1,\dots,n$. Let $p' = \mathbb{k}\cdot[\beta_{p'}] := \mathbb{k}\cdot[\sum_i \lambda_i\beta_i]$ be an arbitrary point in the linear span of $p_1,\dots,p_n$. If $\mathrm{R} = \mathbb{k}[\delta]/\delta^2$, then the cohomology class $[\beta_p + \beta_{p'}\delta]$ represents a tangent vector at $p$. It is a tangent vector to $\mathcal{T}_\mathrm{L}(\mathrm{W}^0_d(\mathrm{C}))$ if and only if $[(f_0 + f_0'\delta)(\beta_p + \beta_{p'}\delta)] = 0$ in $\mathrm{H}^1(\mathrm{C},\mathrm{L}) \otimes \mathrm{R}$. Since $p$ and $p'$ are in the same fiber of the tangent cone, we can take $f_0'$ to be an arbitrary multiple of $f_0$ and the obstruction vanishes.

As a next step, we will show that the vector is not tangent to $\mathcal{OC}_3(\mathrm{W}^0_d(\mathrm{C}),\mathrm{L})$ by studying the obstruction to a second order lifting

$$[(f_0 + f_0'\delta)(\beta_p + \beta_{p'}\delta)^2/2] \in \mathrm{H}^1(\mathrm{C},\mathrm{L}) \otimes \mathrm{R}/(\mathrm{H}^0(\mathrm{C},\mathrm{L}) \otimes \mathrm{R}) \cup [\beta].$$

We have to decide whether there are sections

$$f_2 + f_2'\delta \in \mathrm{H}^0(\mathrm{C},\mathrm{L}(p + p_1 + \cdots + p_n)) \otimes \mathrm{R} \text{ and } f_1 + f_1'\delta \in \mathrm{H}^0(\mathrm{C},\mathrm{L}) \otimes \mathrm{R}$$

such that

$$(f_0 + f_0'\delta)\beta_p^2/2 - (f_1 + f_1'\delta)(\beta_p + \beta_{p'}\delta) - (f_2 + f_2'\delta)$$

is regular at $p, p_1,\dots,p_n$.

Since $d - n - 1 = \rho + 1$, $\mathrm{H}^0(\mathrm{C},\mathrm{L}(p + p_1 + \cdots + p_n)) = \mathrm{H}^0(\mathrm{C},\mathrm{L})$ by Corollary 3.3.2 and the last term is regular in any case. A general point of C is not a ramification point of $\varphi_\mathrm{L}$. Hence, $f_0\beta_p^2/2$ has a simple pol at $p$ and one needs $f_1(p) \neq 0$ to make the above expression regular at $p$. But then $f_1(p_i) \neq 0$ for $i = 1, \dots, n$ since the points are chosen in the same fiber $\overline{D}$ and L is base point free. Consequently, the term

$$(f_0'\beta_p^2/2 - f_1\beta_{p'} - f_1'\beta_p)\delta$$

has a simple pole at $p_1, \dots, p_n$. So a second order lifting is impossible and we are done. □

### 2.5.2 Computational part

We prove Conjecture 2.1.3 in three cases, namely for the pairs $(g,d) = (7,5), (8,6), (9,7)$ where $g$ is the genus of the canonically embedded curve C and $d$ is the degree of the line bundle $L \in W^1_d(C)$. In these cases, the cohomology group $H^1(C,L)$ is three dimensional. We will explain our procedure in the case $(g,d) = (8,6)$ in detail. Our main tool is the computer algebra software `Macaulay2` (see [GS]). The `Macaulay2` code is available on the following webpage:

http://www.math.uni-sb.de/ag-schreyer/index.php/people/researchers/74-michael-hahn

We will make use of the functions included in the `Macaulay2` file

`constructionOfTheOsculatingCone.m2` (see [Hof16]).

Let $C \subset \mathbb{P}^7$ be a canonically embedded curve of genus 8 and let $L \in W^1_6$ be a line bundle such that the Petri map

$$\mu : H^0(C,L) \otimes H^0(C, \omega_C \otimes L^{-1}) \to H^0(C, \omega_C)$$

is injective, that is $W^1_6(C)$ is smooth and of expected dimension 2 at L. Let

$$\varphi_{\omega_C \otimes L^{-1}} \ : \ C \longrightarrow \mathbb{P}^2 = \mathbb{P}(H^0(C, \omega_C \otimes L^{-1})^*)$$

be the birational morphism induced by the Serre dual of L. We denote $C' = \mathrm{Im}_{\varphi_{\omega_C \otimes L^{-1}}}(C)$ the birational image of C, a plane curve of degree 8. Furthermore, we assume that $C'$ has only nodes. Hence, by the genus formula, the plane curve $C'$ has $13 = \binom{7}{2} - 8$ double points. The adjoint linear system, that is, quintic curves through the 13 double points, is the canonical system on $C'$. The linear system $|L|$ on $C'$ is given by quartics through the 13 double points.

Since $h^1(C,L) = 3$, the tangent cone $\mathcal{T}_L(W^0_6(C))$ is given by the maximal minors of a $2\times 3$ matrix of linear forms on $\mathbb{P}^7$.

**Remark 2.5.7.** By Remark 2.5.2, the ideal of the osculating cone is generated by three cubic forms and three quadratic forms, which are the generators of the tangent cone. Indeed, as a subvariety of the tangent cone, the osculating cone is given by the $2\times 2$ minors of a $2\times 3$ matrix, which are homogeneous forms of bidegree $(3,3)$.
Furthermore, there are two linear relations of these cubic forms up to the ideal of the tangent cone. Note that $(u_0, u_1, u_2)$ is a syzygy of the three minors of bidegree $(3,3)$ (see (2.4) and Remark 2.5.2 for the notation) and thus, there are two linear syzygies $(t_0u_0, t_0u_1, t_0u_2)$ and $(t_1u_0, t_1u_1, t_1u_2)$ on $\mathbb{P}^7$ modulo the ideal of the tangent cone where $t_iu_j$ is identified with a linear form on $\mathbb{P}^7$.

**Remark 2.5.8.** By Lemma 2.5.4, for any point $v \in V$ of the vertex of the tangent cone, the line $\overline{pv}$ spanned by $v$ and a point $p$ of the curve $C$ is tangent to the osculating cone. We rephrase this fact in terms of polarity (see [Dol12, Chapter 1] for an introduction and overview of basic results). Let $f$ be an element of the ideal of the osculating cone. The polar form with respect to $v = (v_0, \dots, v_{g-1})$ of $f$ is $P_v(f) = \sum v_i \frac{\partial f}{\partial x_i}$. A well known result is that

$$V(f) \cap V(P_v(f)) = \left\{ x \in V(f) \mid \; v \in T_x(V(f)) \right\}.$$

We define the *polar variety* of the osculating cone with respect to the point $v$ as the intersection of all polar forms. The reformulation of Lemma 2.5.4 is that the curve $C$ is contained in the polar variety to the osculating cone w.r.t. any point $v \in V$.

**Construction 2.5.9.** We will use a reduced Macaulay2 output in order to save space. All important outputs will be presented.

1. Construction of a plane octic $C'$ of genus 8 with 13 ordinary double points.

```
i1 : setRandomSeed"osculatingCone";
i2 : load"constructionOfTheOsculatingCone.m2"
-- ground field and canonical ring PP^7
i3 : kk = ZZ/10007
i4 : PP7 = kk[x_0..x_7]
-- coordinate ring of PP^2
i5 : PP2 = kk[y_0..y_2]
-- we choose randomly 13 kk-rational points in the plane
i6: thirteenPoints = intersect(apply(13, i ->
                           ideal(random(1, PP2), random(1, PP2))));
i7 : degree thirteenPoints, dim thirteenPoints

o7 = (13, 1)

-- we construct the plane model C'
i8 : C' = ideal(gens saturate(thirteenPoints^2) *
          random(source gens saturate(thirteenPoints^2), PP2^{1: -8}));
i9 : singC'=saturate ideal mingens(ideal(jacobian C')+C');
i10 : degree C', genus C', degree singC', dim singC', singC'==thirteenPoints

o10 = (8, 21, 13, 1, true)
```

2. Computation of the canonical system on $C'$, of the canonical embedding $C$ of $C'$ and of the tangent cone $X = \mathcal{T}_L(W^0_d(C))$. In our construction the tangent cone is always given by the $2 \times 2$ minors of the following matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{pmatrix}$$

where the coordinate ring of $\mathbb{P}^7$ is $\mathbb{k}[x_0, \dots, x_7]$.

```
-- note that the canonical map is the blowup of PP^2 in thirteenPoints
i11 : blowup = map(PP2, PP7, gens thirteenPoints*matrix basis(5, thirteenPoints));
-- canonical embedding of C'
i12 : C = ideal mingens preimage_blowup(C');
i13 : M = matrix{{x_0,x_1,x_2},{x_3,x_4,x_5}}
i14 : X = minors(2,M)
i15: X + C == C

o15 = true

i16 : sixPoints = C + ideal(random(kk)*M^{0}+random(kk)*M^{1});
i17 : degree sixPoints, dim sixPoints, betti gens sixPoints

                    0  1
o17 = (6, 1, total: 1 18)
                 0: 1  3
                 1: .  15
```

3. Construction of the osculating cone – we compute three cubic hypersurfaces

$$F_1 = V(f_1), F_2 = V(f_2), F_3 = V(f_3) \subset \mathbb{P}^7$$

with the following properties

   (a) $F_1, F_2, F_3$ contain the curve C, that is $f_1, f_2, f_3 \in I(C)$ (see Proposition 2.5.3),

   (b) all polars w.r.t. the vertex $V = V(x_0, \dots, x_5)$ of the tangent cone contain the curve C (see Lemma 2.5.4 and Remark 2.5.8), that is for $i = 1,2,3$ and for any point $v = (v_0, \dots, v_7) \in V$, the polar form

$$P_v(f_i) := \sum_{j=0}^{7} v_j \frac{\partial f_i}{\partial x_j} \in I(C),$$

   (c) $F_1$ contains $V(x_0, x_1, x_3, x_4)$, $F_2$ contains $V(x_0, x_2, x_3, x_5)$ and $F_3$ contains $V(x_1, x_2, x_4, x_5)$ (see Remark 2.5.2 and 2.5.7),

   (d) there are two linear relations between $f_1, f_2, f_3$ modulo the ideal of the tangent cone (see Remark 2.5.7), namely

$$x_0 f_1 + x_1 f_2 + x_2 f_3 = 0 \text{ and } x_3 f_1 + x_4 f_2 + x_5 f_3 = 0 \in \mathbb{k}[x_0, \dots, x_7]/I(\mathcal{T}_L(W^0_d(C))).$$

**Remark 2.5.10**. Conditions (b) and (c) are linear in the coefficients of a generic cubic in the ideal of the canonical curve. We use the function `genericCubic(C,L1,L2,a)` (see [Hof16]).

```
-- we construct generic cubics with the following properties
-- (a) containing C
-- (b) polars w.r.t. V(x_0..x_5) contain C
-- (c) goes through V(x_l;l\in L)
--     where L\in {{0,1,3,4},{0,2,3,5},{1,2,4,5}}
-- using the function "genericCubic(C,L1,L2,a)"
--  which computes a generic cubic in the ideal of C,
--  whose polars w.r.t. V(x_l,l\in L1) contain C and
--  goes through V(x_l;l\in L2).
i18 : time (cubic0, coeffa) = genericCubic(C, {0,1,2,3,4,5}, {0,1,3,4}, a);
```

```
     -- used 5.65669 seconds
i19 : time (cubic1, coeffb) = genericCubic(C, {0,1,2,3,4,5}, {0,2,3,5}, b);
     -- used 5.67274 seconds
i20 : time (cubic2, coeffc) = genericCubic(C, {0,1,2,3,4,5}, {1,2,4,5}, c);
     -- used 6.03041 seconds

-- (d) the generic cubics have to satisfy special linear relations:
-- x_2*cubic0-x_1*cubic1+x_0*cubic2 \in scroll and
-- x_5*cubic0-x_4*cubic1+x_3*cubic2 \in scroll
-- we find all possible generic cubics satisfying the two linear relations
-- initializing the ring R, where all computations take place
i21 : s0 = rank source vars coeffa;
i22 : s1 = rank source vars coeffb;
i23 : s2 = rank source vars coeffc;
i24 : q = rank source matrix basis(4,X);
i25 : coeffd = kk[d_1..d_q];
i26 : coeff = coeffa ** coeffb ** coeffc ** coeffd;
i27 : R = coeff[flatten entries vars PP7];

-- substituting the generic cubics into R
i28 : cubic0=sub(cubic0,R);
i29 : cubic1=sub(cubic1,R);
i30 : cubic2=sub(cubic2,R);

-- generic quartic in X
i31 : quartics = gens X*matrix basis(4,X);
i32 : quartic = (sub(quartics,R)*
          transpose sub((vars coeff)_{s0+s1+s2..s0+s1+s2+q-1},R))_(0,0);

-- equations in the coefficients of
-- cubic0, cubic1, cubic2 and quartic for condition (d):
i33 : equa1 = sub(
        (coefficients(sub(x_2,R)*cubic0-sub(x_1,R)*cubic1+sub(x_0,R)*cubic2
         -quartic))#1,coeff);
```

```
i34 : equa2 = sub(
        (coefficients(sub(x_5,R)*cubic0-sub(x_4,R)*cubic1+sub(x_3,R)*cubic2
         -quartic))#1,coeff);
-- solutions:
i35 : solution1 = sub(syz diff(vars coeff, equa1), kk);

               164        67
o35 : Matrix kk    <--- kk

i36 : solution2 = sub(syz diff(vars coeff, equa2), kk);

               164        67
o36 : Matrix kk    <--- kk

-- all common solutions:
i37 : sols = intersectionSpace(solution1^{0..s0+s1+s2-1},
                               solution2^{0..s0+s1+s2-1});

               72        67
o37 : Matrix kk   <--- kk
```

**Remark 2.5.11.** We have a 67-dimensional space of cubics but the dimension of cubics containing X is 22. Hence, there is a unique (up to X, $67-3\cdot 22=1$) solution of cubic hypersurfaces satisfying the above properties modulo the ideal of X. In particular, the osculating cone is completely determined by these properties.

```
i38 : solution = sols * random(source sols,kk^1);

-- substituting the solution in the generic cubics
i39 : helpCubic0 = apply(s0,i->(a_(i+1) => solution_(i,0)));
i40 : helpCubic1 = apply(s1,i->(b_(i+1) => solution_(s0+i,0)));
i41 : helpCubic2  =apply(s2,i->(c_(i+1) => solution_(s0+s1+i,0)));
i42 : f1 = sub(sub(cubic0,helpCubic0),PP7);
i43 : f2 = sub(sub(cubic1,helpCubic1),PP7);
i44 : f3 = sub(sub(cubic2,helpCubic2),PP7);
```

4. The ideal of the osculating osculating cone is generated by $f_1, f_2, f_3$ and generators of the ideal of the tangent cone. At the end, we show the correctness of our computation and therefore, verify Conjecture 2.1.3.

```
-- finally, the oculating cone
i45 : OscCone = (ideal(f1,f2,f3) + X)
i46 : use PP7;
i47 : OscCone + C == C

o47 = true

-- vertex of the tangent cone X
i48 : V = ideal(x_0..x_5);

-- points spanning the vertex
i49 : gensPointsSpanVertex = pointsInHyperplane(V);

-- coordinates of the spanning points
i50 : coordPointsSpanVertex = apply(gensPointsSpanVertex,p->coordinatesOfPoint(p));

-- polars w.r.t to these points
i51 : polars = apply(flatten entries gens OscCone, f->(
        ideal apply(coordPointsSpanVertex,p->(
        polarOfHypersurfaceAtPoint(f,p)))));

-- the polar variety to the osculating cone w.r.t. the vertex V:
i52 : polar = ideal polars + X;

-- we intersect the polar variety and the osculating cone
-- the intersection is the curve and the vertex V
i53 : ideal mingens saturate(polar + OscCone,V) == C

o53 = true
```

# Chapter 3

# Focal schemes to families of secant spaces to canonical curves

**Abstract**. For a general canonically embedded curve C of genus $g \geq 5$, let $d \leq g-1$ be an integer such that the Brill–Noether number $\rho(g,d,1) = g-2(g-d+1) \geq 1$. We study the family of $d$-secant $\mathbb{P}^{d-2}$'s to C induced by the smooth locus of the Brill–Noether locus $W^1_d(C)$. Using the theory of foci and a structure theorem for the rank one locus of special 1-generic matrices by Eisenbud and Harris, we prove a Torelli-type theorem for general curves by reconstructing the curve from its Brill–Noether loci $W^1_d(C)$ of dimension at least 1.

## 3.1 Introduction and motivation

For a general canonically embedded curve C of genus $g \geq 5$ over $\mathbb{C}$, we study the local structure of the Brill–Noether locus $W^1_d(C)$ for an integer $\lceil \frac{g+3}{2} \rceil \leq d \leq g-1$. Our main object of interest is the focal scheme associated to the family of $d$-secant $\mathbb{P}^{d-2}$'s to C. The focal scheme arises in a natural way as the degeneracy locus of a map of locally free sheaves associated to a family of secant spaces to a curve. In other words, the focal scheme (or the scheme of first-order foci) consists of all points where a secant intersects its infinitesimal first-order deformation.

In [CS92] and [CS95], Ciliberto and Sernesi studied the geometry of the focal scheme associated to the family of $(g-1)$-secant $\mathbb{P}^{g-3}$'s induced by the singular locus $\mathrm{W}^1_{g-1}(\mathrm{C})$ of the theta divisor, and they gave a conceptual new proof of Torelli's theorem. Using higher-order focal schemes for general canonical curves of genus $g = 2m+1$, they showed in [CS00] that the family of $(m+2)$-secants induced by $\mathrm{W}^1_{m+2}(\mathrm{C})$ also determines the curve. These are the extremal cases, that is, the degree $d$ is maximal or minimal with respect to the genus $g$ (in symbols $d = g-1$ or $d = \frac{g+3}{2}$ and $g$ odd). The article [Baj10] of Bajravani can be seen as a first extension of the previous results to another Brill–Noether locus ($g = 8$ and $d = 6 = \lceil \frac{g+3}{2} \rceil$).

Combining methods of [CS95],[CS00] and [CS10], we will give a unified proof which shows that the canonical curve is contained in the focal schemes parametrised by the smooth locus of any $\mathrm{W}^1_d(\mathrm{C})$ if $d \leq g-1$ and $\rho(g,d,1) = g - 2(g-d+1) \geq 1$. Moreover, we have the following Torelli-type theorem.

**Main theorem 3.1.1.** *A general canonically embedded curve of genus $g$ can be reconstructed from its Brill–Noether locus $\mathrm{W}^1_d(\mathrm{C})$ if $\lceil \frac{g+3}{2} \rceil \leq d \leq g-1$.*

In [PTiB92], G. Pirola and M. Teixidor i Bigas proved a generic Torelli-type theorem for $\mathrm{W}^r_d(\mathrm{C})$ if $\rho(g,d,r) \geq 2$, or $\rho(g,d,r) = 1$ and $r = 1$. Whereas they used the global geometry of the Brill–Noether locus to recover the curve, our theorem is based on the local structure around a smooth point of $\mathrm{W}^1_d(\mathrm{C}) \subset \mathrm{W}_d(\mathrm{C})$. Only first-order deformations are needed.

Our proof follows [CS00]. We show that the first-order focal map is in general 1-generic and apply a result of D. Eisenbud and J. Harris [EH92] in order to describe the rank one locus of the focal matrix. Two cases are possible. The rank one locus of the focal matrix consists either of the support of a divisor D of degree $d$ corresponding to a line bundle $\mathscr{O}_{\mathrm{C}}(\mathrm{D}) \in \mathrm{W}^1_d(\mathrm{C})$ or of a rational normal curve. Even if we are not able to decide which case should occur on a general curve (see Section 3.4 for a discussion), we finish our proof by studying focal schemes to a family of rational normal curves induced by the first-order focal map.

In Section 3.2, we recall the definition of focal schemes as well as general facts and known results about focal schemes. Section 3.3 is devoted to prove the generalisation of the main theorem of [CS00] to arbitrary positive dimensional Brill–Noether loci.

## 3.2 The theory of foci

We recall the definition as well as the construction of the family of $d$-secant $\mathbb{P}^{d-2}$'s induced by an open dense subset of $C^1_d$. Afterwards we introduce the characteristic or focal map and define the scheme of first-order foci of rank $k$ associated to the above family. We give a sligthly generalised definition of the scheme of first- and second-order foci compared to [CS95]. In Section 3.2.2, we recall the basic properties of the scheme of first-order foci. Our approach and the notation follow [CS10].

### 3.2.1 Definition of the scheme of first-order foci

*Notation* 3.2.1. Let C be a Brill–Noether general canonically embedded curve of genus $g \geq 5$, and let $d \leq g-1$ be an integer such that the Brill–Noether number $\rho := \rho(g,d,1) = g-2(g-d+1) \geq 1$. Let $C^1_d$ be a variety parametrising effective divisors of degree $d$ on C moving in a linear system of dimension at least 1 (see [ACGH85, IV, §1]). Let $\Sigma \subset W^1_d(C)$ be the smooth locus of $W^1_d(C)$. Furthermore, let $\alpha_d : C^1_d \to W^1_d(C)$ be the Abel-Jacobi map (see [ACGH85, I, §3]) and let $S = \alpha_d^{-1}(\Sigma)$. Then $\alpha : S \to \Sigma$ is a $\mathbb{P}^1$-bundle, and in particular S is smooth of pure dimension $\rho+1$. For every $s \in S$, we denote by $D_s$ the divisor of degree $d$ on C defined by $s$ and $\Lambda_s = \overline{D_s} \subset \mathbb{P}^{g-1}$ its linear span, which is a $d$-secant $\mathbb{P}^{d-2}$ to C. We get a $(\rho+1)$-dimensional family of $d$-secant $\mathbb{P}^{d-2}$'s parametrised by S:

$$\begin{array}{ccc} \underline{\Lambda} \subset S \times \mathbb{P}^{g-1} & \xrightarrow{q} & \mathbb{P}^{g-1} \\ \downarrow p & & \\ S & & \end{array}$$

We denote by $f : \underline{\Lambda} \to \mathbb{P}^{g-1}$ the induced map.

**Construction 3.2.2** (of the family $\underline{\Lambda}$). Let $\mathbf{D}_d \subset C_d \times C$ be the universal divisor of degree $d$ and let $\mathbf{D}_S \subset S \times C$ be its restriction to $S \times C$. We denote by $\pi : S \times C \to S$ the projection. We consider the short exact sequence

$$0 \to \mathcal{O}_{S\times C} \to \mathcal{O}_{S\times C}(\mathbf{D}_S) \to \mathcal{O}_{\mathbf{D}_S}(\mathbf{D}_S) \to 0.$$

By Grauert's Theorem, the higher direct image $R^1\pi_*(\mathcal{O}_{\mathbf{D}_S}(\mathbf{D}_S)) = 0$ vanishes and we get a map of locally free sheaves on S

$$R^1\pi_*(\mathcal{O}_{S\times C}) \to R^1\pi_*(\mathcal{O}_{S\times C}(\mathbf{D}_S)) \to 0$$

whose kernel is a locally free sheaf $\mathscr{F} \subset R^1\pi_*(\mathcal{O}_{S\times C}) \cong \mathcal{O}_S \otimes H^1(C, \mathcal{O}_C)$ of rank $d-1 = g-(g-d+1)$. The family $\underline{\Lambda}$ is the associated projective bundle

$$\underline{\Lambda} = \mathbb{P}(\mathscr{F}) \subset S \times \mathbb{P}^{g-1}.$$

**Remark 3.2.3.** We can also construct the family $\underline{\Lambda}$ from the Brill–Noether locus $W_d(C)$ and its singular locus $W^1_d(C)$. At a singular point $L \in W^1_d(C)\backslash W^2_d(C)$, the projectivised tangent cone to $W_d(C)$ at L in the canonical space $\mathbb{P}^{g-1}$ coincides with the scroll

$$X_L = \bigcup_{D\in|L|} \overline{D}$$

swept out by the pencil $g^1_d = |L|$. Hence, the ruling of $X_L$ is the one-dimensional family of secants induced by $|L|$. Varying the point L yields the family $\underline{\Lambda}$. See also [CS95, Theorem 1.2]. We conclude that the family $\underline{\Lambda}$ is determined by $W_d(C)$ and its singular locus $W^1_d(C)$.

We define the scheme of first-order foci of the family $\underline{\Lambda}$ following [CS10]. Let

$$\mathcal{N} := \mathcal{N}_{\underline{\Lambda}/S\times\mathbb{P}^{g-1}}$$

be the normal sheaf of $\underline{\Lambda}$ in $S \times \mathbb{P}^{g-1}$ and let

$$T(p)|_{\underline{\Lambda}} := p^*(T_S)|_{\underline{\Lambda}}$$

be the restriction of the tangent sheaf along the fibers of $p$ to $\underline{\Lambda}$. There is a map

$$\chi : T(p)|_{\underline{\Lambda}} \to \mathcal{N}$$

called the *global characteristic map* of the family $\underline{\Lambda}$, which is defined by the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{T}(p)|_{\underline{\Lambda}} & \xrightarrow{\chi} & \mathcal{N} & & \\
 & & & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{T}_{\underline{\Lambda}} & \longrightarrow & \mathrm{T}_{\mathrm{S}\times\mathbb{P}^{g-1}}|_{\underline{\Lambda}} & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle df} & & \downarrow & & & & \\
 & & f^*(\mathrm{T}_{\mathbb{P}^{g-1}}) & = & q^*(\mathrm{T}_{\mathbb{P}^{g-1}})|_{\underline{\Lambda}} & & & &
\end{array}$$

For every $s \in \mathrm{S}$ the homomorphism $\chi$ induces a homomorphism

$$\chi_s : \mathrm{T}_{\mathrm{S},s} \otimes \mathscr{O}_{\Lambda_s} \to \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}}$$

called the *characteristic map* or *first-order focal map* of the family $\underline{\Lambda}$ at a point $s$. One can also define the characteristic map of an arbitrary family (see [CS10, §1]).

**Remark 3.2.4.** (a) Fix an $s \in \mathrm{S}$. We have $\Lambda_s = \mathbb{P}(\mathrm{U})$, where $\mathrm{U} \subset \mathrm{V} = \mathrm{H}^1(\mathrm{C}, \mathscr{O}_\mathrm{C})$ is a vector subspace of dimension $d-1$. The normal bundle of $\Lambda_s$ in $\mathbb{P}^{g-1}$ is given by

$$\mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}} = \mathrm{V}/\mathrm{U} \otimes \mathscr{O}_{\Lambda_s}(1)$$

and

$$\mathrm{H}^0(\Lambda_s, \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}}) = \mathrm{Hom}(\mathrm{U}, \mathrm{V}/\mathrm{U}).$$

The characteristic map is of the form

$$\chi_s : \mathrm{T}_{\mathrm{S},s} \otimes \mathscr{O}_{\Lambda_s} \to \mathrm{V}/\mathrm{U} \otimes \mathscr{O}_{\Lambda_s}(1).$$

(b) By the universal property of $\mathbb{G}(d-2, \mathbb{P}^{g-1})$, the family $\underline{\Lambda}$ gives a morphism

$$\varphi : \mathrm{S} \to \mathbb{G}(d-2, \mathbb{P}^{g-1}).$$

The linear map induced by the characteristic map

$$\mathrm{H}^0(\chi_s) : \mathrm{T}_{\mathrm{S},s} \to \mathrm{Hom}(\mathrm{U}, \mathrm{V}/\mathrm{U})$$

is the differential $d\varphi_s$ at the point $s$ (see also [Ful84, p. 198 f]). We may assume that $\varphi$ is generically finite to its image. Hence, the differential of $\varphi$ is injective at a general point of S.

Since S and the family $\underline{\Lambda}$ are smooth, all sheaves in the above diagram are locally free and by diagram-chasing, it follows that

$$\ker(df) = \ker(\chi).$$

We get

$$\dim(f(\underline{\Lambda})) = \dim(\underline{\Lambda}) - \mathrm{rk}(\ker(\chi)).$$

We define the first- and the second-order foci (of rank $k$) of a family $\underline{\Lambda}$.

**Definition 3.2.5.**

(a) Let $V(\chi)_k$ be the closed subscheme of $\underline{\Lambda}$ defined by

$$V(\chi)_k = \{p \in \underline{\Lambda} \mid \mathrm{rk}(\chi(p)) \leq k\}.$$

Then, $V(\chi)_k$ is the *scheme of first-order foci of rank k* and the fiber of $V(\chi)_k$ over a point $s \in S$

$$(V(\chi)_k)_s = V(\chi_s)_k \subset \Lambda_s$$

is the *scheme of first-order foci of rank k at s*.

(b) If $V(\chi)_k$ induces a family of rational normal curves $\underline{\Gamma}$, that is, for a general $s \in S$ the fiber $\Gamma_s = V(\chi_s)_k$ is a rational normal curve, let $\Psi$ be the characteristic map of $\underline{\Gamma}$. We call the first-order foci of rank $k$ of the family $\underline{\Gamma}$, that is,

$$V(\Psi)_k = \{p \in \underline{\Gamma} \mid \mathrm{rk}(\Psi(p)) \leq k\},$$

the *second-order foci of rank k* of $\underline{\Lambda}$.

**Remark 3.2.6.** Our definition of scheme of first- (or second-) order foci is a slight generalisation of the definitions given before. Note that if $k = \min\{\mathrm{rk}(T(p)|_{\underline{\Lambda}}), \mathrm{rk}(\mathcal{N})\} - 1 = \min\{\mathrm{codim} S, \mathrm{codim}_{S\times\mathbb{P}^{g-1}}(\underline{\Lambda})\} - 1$, we get the classical definition (see [CS92],[CS95]) of first-order foci. Furthermore, our definition is inspired by the definition of higher-order foci of [CS10].

**Remark 3.2.7.** (a) The equality $V(\chi)_s = V(\chi_s)$ is shown in [CC93, Proposition 14].

(b) If $\chi$ has maximal rank, that is, if $\chi$ is either injective or has torsion cokernel, then $V(\chi)_k$ is a proper closed subscheme of $\underline{\Lambda}$ for $k \leq \min\{\mathrm{rk}(T(p)|_{\underline{\Lambda}}), \mathrm{rk}(\mathcal{N})\} - 1$.

(c) In Section 3.3 we study the scheme of first-order foci of rank 1 of the family $\underline{\Lambda}$.

### 3.2.2 Properties of the scheme of first-order foci

The following proposition is proven in [CS95].

**Proposition 3.2.8.** *For $s \in S$, we have*

$$D_s \subset V(\chi_s)_1.$$

*In particular, the canonical curve C is contained in the scheme of first-order foci.*

*Proof.* Let $p \in \mathrm{Supp}(D_s)$. Then there exists a codimension 1 family of divisors and hence $d$-secants containing the point $p$. Therefore, there is a codimension 1 subspace $T \subset T_{S,s}$ such that the map $\chi_s(p)|_T$ is zero. We conclude that the focal map $\chi_s$ has rank at most 1 in points of $\mathrm{Supp}(D_s)$. □

An important step in the proof of our main theorem is to show that the first-order focal map $\chi_s$ is 1-generic. The general definition of 1-genericity can be found in [Eis88]. In our case, a reformulation of the definition is the following.

**Proposition 3.2.9.** *The matrix $\chi_s$ is 1-generic if and only if for each nonzero element $v \in T_{S,s}$, the homomorphism*

$$H^0(\chi_s)(v) \in \mathrm{Hom}(U, V/U)$$

*is surjective.*

We recall what is known about the 1-genericity of the matrix $\chi_s$.

**Proposition 3.2.10** ([CS95, Theorem 2.5], [CS00, Theorem 2], [Baj10]). *Let $s \in S$ be a general point.*

*(a) If $D_s$ is a divisor of degree $g-1$ cut on C by $\Lambda_s$, then the matrix $\chi_s$ is 1-generic (equivalently, $V(\chi_s)_1$ is a rational normal curve) if and only if the pencil $|D_s|$ is base point free.*

*(b) If $\rho := \rho(g, d, 1) = 1$, then the matrix $\chi_s$ is 1-generic (equivalently, $V(\chi_s)_1$ is a rational normal curve).*

*(c) If* $g = 8$ *and* $d = 6$*, then the matrix* $\chi_s$ *is* 1*-generic.*

**Remark 3.2.11** ([Ser06, p. 253]). Another fact related to the 1-genericity of $\chi_s$ is the following: Let

$$\begin{array}{c} \underline{\Lambda}_\varepsilon \subset \mathrm{Spec}(\Bbbk[\varepsilon]) \times \mathbb{P}^{g-1} \\ \downarrow \\ \mathrm{Spec}(\Bbbk[\varepsilon]) \end{array}$$

be the first order deformation of $\Lambda_s$ defined by $H^0(\chi_s)(v)$ for a vector $v \in T_{S,s}$. Then, $H^0(\chi_s)(v)$ is surjective if and only if $q(\underline{\Lambda}_\varepsilon) \subset \mathbb{P}^{g-1}$ is not contained in a hyperplane. Furthermore, the definition of the first-order foci at a point $s \in S$ depends only on the geometry of the family $\underline{\Lambda}$ in a neighbourhood of $s$. A point in $V(\chi_s)_k$ is a point where the fiber $\Lambda_s$ intersects a codimension $k$ family of infinitesimally near ones.

## 3.3 Proof of the main theorem

The strategy of the proof is the same as in [CS00]. We assume that the canonically embedded curve C is a Brill–Noether general curve. Recall that $g$ and $d$ are chosen such that the Brill–Noether number $\rho := \rho(g, d, 1) \geq 1$. We begin by showing some standard properties of a line bundle over a Brill–Noether general curve which we will use later on. Then we prove that the matrix $\chi_s$ is 1-generic for general $s \in S$ and study the rank one locus of $\chi_s$ which will be the divisor $D_s$ or a rational normal curve. In the second case, we study the second-order focal locus. In both cases we can recover the canonical curve.

**Lemma 3.3.1.** *Let* C *be a Brill–Noether general curve and let* $L \in W^1_d(C)$ *be a smooth point. Then* $|L|$ *is base point free,* $H^1(C, L^2) = 0$ *and* $g^{\rho+2}_{2d} = |L^2|$ *maps* C *birational to its image (it is not composed with an involution).*

*Proof.* All of our claims follow directly from the generality assumption. We just mention that the map induced by $|L^2|$ can not be composed with an irrational involution. Hence, if the map is not birational, it is composed with a $g^1_{d'}$ for $d' \leq \frac{2d}{\rho+2}$ which is impossible for a Brill–Noether general curve. □

**Corollary 3.3.2.** *Let* $C$ *be a Brill–Noether general curve and let* $L \in W^1_d(C)$ *be a smooth point. For* $i \geq 1$ *and* $p_1, \dots, p_i \in \operatorname{Supp}(D)$ *for* $D \in |L|$ *general, we have*

$$h^0(C, L^2(-p_1 - \dots - p_i)) = 2d - i + 1 - g.$$

*In particular,* $H^0(C, L^2(-p_1 - \dots - p_{\rho+1})) = H^0(C, L)$ *and* $H^1(C, L^2(-p_1 - \dots - p_i)) = 0$ *for* $i = 1, \dots, \rho + 1$.

*Proof.* Since $|L^2|$ maps C birational to its image, $H^0(C, L^2(-p_1 - \dots - p_i)) = H^0(C, L^2(-p_1 - \dots - p_{i+1}))$ if the images of the two points $p_i$ and $p_{i+1}$ are the same point. This does not happen for a general choice. □

Using Lemma 3.3.1 and Corollary 3.3.2, our proof of the following lemma is identical to [CS00, Theorem 2]. We clarify and generalise their arguments.

**Lemma 3.3.3.** *With the assumptions of Lemma 3.3.1, the focal matrix* $\chi_s : T_{S,s} \otimes \mathcal{O}_{\Lambda_s} \to \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}}$ *is* 1*-generic for a sufficiently general* $s \in S$.

*Proof.* By Proposition 3.2.9, the matrix $\chi_s$ is 1-generic if and only if for each nonzero element $v \in T_{S,s}$, the homomorphism $H^0(\chi_s)(v) \in \operatorname{Hom}(U, V/U)$ is surjective.

We consider the first order deformation $\underline{\Lambda}_\varepsilon \subset \operatorname{Spec}(k[\varepsilon]) \times \mathbb{P}^{g-1}$ defined by $H^0(\chi_s)(\theta)$ for a nonzero vector $\theta \in T_{S,s}$. Note that $H^0(\chi_s)(\theta)$ is surjective if and only if the image $q(\underline{\Lambda}_\varepsilon) \subset \mathbb{P}^{g-1}$ is not contained in a hyperplane. Let $D_\varepsilon \subset \operatorname{Spec}(k[\varepsilon]) \times \mathbb{P}^{g-1}$ be the first order deformation of the divisor $D_s$ defined by $\theta \in T_{S,s}$. Then

$$q(\underline{\Lambda}_\varepsilon) \supset q(D_\varepsilon)$$

and the curvilinear scheme $q(D_\varepsilon)$ corresponding to a divisor on C satisfies

$$D_s \leq q(D_\varepsilon) \leq 2D_s.$$

We show for all possible cases that $q(D_\varepsilon)$ is not contained in a hyperplane.

<u>Case 1:</u> The vector $\theta$ is tangent to $\alpha_d^{-1}(L)$, equivalently the family $D_\varepsilon$ deforms the divisor $D_s$ in the linear pencil $|L|$. Let $\varphi_L$ be the morphism defined by the pencil. Then we get

$$q(D_\varepsilon) = \varphi_L^*(\theta),$$

where we identify $\theta$ with a curvilinear scheme of $\mathbb{P}^1$ supported at the point $s \in \mathbb{P}^1$. Since $|L|$ is base point free, we have $q(D_\varepsilon) = 2D_s$. Therefore, the curvilinear scheme $q(D_\varepsilon)$ is not contained in a hyperplane since $H^0(C, K_C - 2D_s)^* = H^1(C, 2D_s) = H^1(C, L^2) = 0$. We are done in this case.

Case 2: We assume that $\theta \in T_{S,s} \setminus \{0\}$ is not tangent to $\alpha_d^{-1}(L)$ at $s$. Let

$$q(D_\varepsilon) = p_1 + \cdots + p_k + 2(p_{k+1} + \cdots + p_d)$$

where $D_s = p_1 + \cdots + p_d$ and $k \geq 0$.

Case 2 (a): We assume $k \leq \rho$. We have

$$\begin{aligned} H^0(C, K_C - q(D_\varepsilon))^* &= H^1(C, p_1 + \cdots + p_k + 2(p_{k+1} + \cdots + p_d)) \\ &= H^1(C, 2D_s - p_1 - \cdots - p_k) \\ &= H^1(C, L^2(-p_1 - \cdots - p_k)) = 0 \end{aligned}$$

by Corollary 3.3.2. Hence, the curvilinear scheme $q(D_\varepsilon)$ is not contained in a hyperplane and $H^0(\chi_s)(\theta)$ is surjective.

Case 2 (b): We assume $k \geq \rho + 1$. In the following, we will show that this case can not occur. The vector $\theta$ is also tangent to $p_1 + \cdots + p_k + C_{d-k}$. We denote by $E_s$ the divisor $E_s = p_{k+1} + \cdots + p_d$. Then the tangent space to $p_1 + \cdots + p_k + C_{d-k}$ is given by $H^0(E_s, \mathcal{O}_{E_s}(D_s))$ which is a subspace of $H^0(D_s, \mathcal{O}_{D_s}(D_s))$. The short exact sequence

$$0 \to \mathcal{O}_C \to L \to \mathcal{O}_{D_s}(D_s) \to 0$$

induces a linear map

$$H^0(D_s, \mathcal{O}_{D_s}(D_s)) \xrightarrow{\delta} H^1(C, \mathcal{O}_C)$$

which we identify with the differential of $\alpha_d$ at $s$ (see [ACGH85, Chapter IV, §2, Lemma 2.3]). The image of $\theta \in H^0(E_s, \mathcal{O}_{E_s}(D_s))$ is therefore contained in the linear span of $E_s$. After projectivising, we get

$$[\delta(\theta)] \in \overline{E_s} = \overline{p_{k+1} + \cdots + p_d} \subset \Lambda_s \subset \mathbb{P}^{g-1}.$$

Since $\theta$ is not tangent to $\alpha_d^{-1}(L)$, the vector $\theta$ is also tangent to $W_d^1(C)$ and therefore the image point $[\delta(\theta)]$ is contained in the vertex $V = T_L(W_d^1(C))$ of $X_L$, the scroll swept out by the linear pencil $|L|$. Hence, for every sufficiently general $D \in |L|$, there is an effective divisor $E$ of degree $d-\rho-1$ such that $D = E + p_1 + \cdots + p_{\rho+1}$ and $V \cap \overline{E} \neq \emptyset$. Hence, $\dim(\overline{D_s + E}) \leq d - 2 + d - \rho - 1$ and equivalently,

$$h^0(C, D_s + E) = \deg(D_s + E) - \dim(\overline{D_s + E}) + 1 \geq 3.$$

But by Corollary, 3.3.2 $H^0(C, L^2(-p_1 - \cdots - p_{\rho+1})) = H^0(C, L)$, a contradiction. □

Note that

$$\chi_s : T_{S,s} \otimes \mathscr{O}_{\Lambda_s} \to \mathscr{N}_{\Lambda_s/\mathbb{P}^{g-1}}$$

is a map between rank $\rho+1$ and $n = h^1(C, L)$ vector bundles of linear forms in $\mathbb{P}^{d-2} = \Lambda_s$. Since $d = \rho + 1 + n$ and $\chi_s$ is 1-generic by Lemma 3.3.3, we may apply the following theorem due to Eisenbud and Harris.

**Theorem 3.3.4** ([EH92, Proposition 5.1]). *Let* $M$ *be an* $(a+1) \times (b+1)$ 1*-generic matrix of linear forms on* $\mathbb{P}^{a+b}$. *If* $D_1(M) = \{x \in \mathbb{P}^{a+b} \mid \mathrm{rk}(M(x)) \leq 1\}$ *contains a finite scheme* $\Gamma$ *of length* $\geq a+b+3$, *then* $D_1(M)$ *is the unique rational normal curve through* $\Gamma$ *and* $M$ *is equivalent to the catalecticant matrix.*

We get the following corollary.

**Corollary 3.3.5.** *For* $s \in S$ *sufficiently general, the rank one locus* $V(\chi_s)_1$ *is either* $D_s$ *or a rational normal curve through* $D_s$.

*Proof.* By Lemma 3.3.3, we may apply Theorem 3.3.4. Note that $D_s \subset V(\chi_s)_1$ (there exists a codimension 1 family in $S$ of $\Lambda_s$ containing a point of the support of $D_s$). □

**Remark 3.3.6.** (a) The scheme of first-order foci at $s \in S$ of the family $\underline{\Lambda}$ is a secant variety to $V(\chi_s)_1$.

(b) If $d = g-1$ or $\rho = 1$, the focal matrix $\chi_s$ is a $2 \times (g-3)$ or $n \times 2$-matrix, respectively. Hence, the rank one locus is the scheme of first-order foci, which is a rational normal curve in $\Lambda_s$. We recover the cases of [CS95] and [CS00].

**Corollary 3.3.7.** *Let* $C$ *be a Brill–Noether general canonically embedded curve. If* $V(\chi_s)_1 = D_s$ *for sufficiently general* $s \in S$, *the family* $\underline{\Lambda}$ *determines the canonical curve* $C$.

**For the rest of this section, we assume that $\Gamma_s = V(\chi_s)_1$ is a rational normal curve for $s \in S$ sufficiently general.**

Let $\Sigma$ be the smooth locus of $W^1_d(C)$ and $L \in \Sigma$.

Let $U \subset \alpha_d^{-1}(L))$ be a Zariski open dense set such that $\Gamma_s = V(\chi_s)_1$ for all $s \in U$. We define the surface

$$\Gamma_L = \overline{\bigcup_{s \in U} \Gamma_s}$$

and

$$\Gamma_{\mathbb{P}^{g-1}} = \overline{\bigcup_{L \in \Sigma} F_L}.$$

Let

$$\begin{array}{ccc} \underline{\Gamma} \subset S' \times \mathbb{P}^{g-1} & \xrightarrow{q} & \mathbb{P}^{g-1} \\ \downarrow p & & \\ S' & & \end{array}$$

be the family induced by all rational normal curves, that is, for $s \in S'$, $\Gamma_s = V(\chi_s)_1$ is a rational normal curve. The family $\underline{\Gamma}$ is the rank one locus of the global characteristic map $\chi$ and the variety $\Gamma_{\mathbb{P}^{g-1}}$ is the image of the family $\underline{\Gamma}$ under the second projection $q$.

**Remark 3.3.8**. In the case $d = g-1$ ($\rho = 1$, respectively), one has a precise geometric description of the rational surface $\Gamma_L$. It is birational to $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ (a quadric cone in $\mathbb{P}^3$). We have not found a similar geometrical meaning of the surface $\Gamma_L$ in the other cases (see also Question 1).

**Lemma 3.3.9**. *The variety* $\Gamma_{\mathbb{P}^{g-1}}$ *has dimension at least* 3.

*Proof.* Note that there is a map $\Gamma_L \to \mathbb{P}^1 = \alpha_d^{-1}(L)$ such that the general fiber is a rational curve. Hence, the surface $\Gamma_L$ is rational. Assume that $\Gamma_L = \Gamma_{L'}$ for all $L' \in \Sigma$. Since the scrolls $X_{L'}$ are algebraically equivalent to each other, the rulings on them cut out a $(\rho+1)$-dimensional family of algebraically equivalent rational curves on $\Gamma_L$, the focal curves. (We can also argue that all $d$-secant to C are algebraically equivalent, thus the intersection with $\Gamma_L$ yields a $(\rho+1)$-dimensional family of algebraically equivalent focal curves.) On the desingularization of $\Gamma_L$, all of them are linear equivalent since $\Gamma_L$ is regular ($H^1(\Gamma_L, \mathcal{O}_{\Gamma_L}) = 0$). This implies that all $g^1_d$'s on C are linear equivalent, hence C has a $g^{\rho+1}_d$. A contradiction to the generality assumption on C. □

For the convenience of the reader, we recall the definition of the second-order foci of the family $\underline{\Lambda}$ (see also Definition 3.2.5). We apply the theory of foci to the family $\underline{\Gamma} \subset S' \times \mathbb{P}^{g-1}$ and get the characteristic map

$$\psi : T(p)|_{\underline{\Gamma}} \to \mathcal{N}_{\underline{\Gamma}/S' \times \mathbb{P}^{g-1}}$$

of vector bundles of rank $\rho + 1$ and $g - 2$, respectively. For $s \in S'$, we call the closed subscheme of $\Gamma_s$ defined by $\mathrm{rk}(\psi_s) \leq k$ the *scheme of second-order foci of rank $k$ at $s$* (of the family $\underline{\Lambda}$).

We will show that the scheme of second-order foci of rank 1 at $s \in S'$ of the family $\underline{\Lambda}$ is a finite scheme containing the divisor $D_s$ and compute its degree.

**Lemma 3.3.10.** *Let $\psi_s : T_{S',s} \otimes \mathcal{O}_{\Gamma_s} \to \mathcal{N}_{\Gamma_s/\mathbb{P}^{g-1}}$ be the characteristic map for general $s \in S'$. Then the rank of $\psi_s$ at a general point of $\Gamma_s$ is at least two, that is,*

$$\mathrm{rk}_{p \in \Gamma_s}(\psi_s(p)) \geq 2 \textit{ if } \dim(\Gamma_{\mathbb{P}^{g-1}}) \geq 3.$$

*Proof.* We recall the connection of the rank and the dimension of $\Gamma_{\mathbb{P}^{g-1}}$ as in [CS10, page 6]. Since $\dim(\Gamma_{\mathbb{P}^{g-1}}) = \rho + 2 - \mathrm{rk}(\ker(\psi))$, the rank of $\psi_s$ at the general point $p \in \Gamma_s$ is

$$\mathrm{rk}_{p \in \Gamma_s}(\psi_s(p)) = \dim(T(p)|_{\underline{\Gamma}}) - \mathrm{rk}(\ker(\psi)) = \rho + 1 - \mathrm{rk}(\ker(\psi)) = \dim(\Gamma_{\mathbb{P}^{g-1}}) - 1.$$

The lemma follows from Lemma 3.3.9. The above fact is also shown in [CC93, page 98]. □

We now consider for a general $s \in S'$ the rank one locus of $\psi_s$ which is a proper subset of $\Gamma_s$ by Proposition 3.3.10.

**Lemma 3.3.11.** *The degree of $V(\psi_s)_1 \subset \Gamma_s = V(\chi_s)_1$ is at most $d + \rho$.*

*Proof.* We imitate the proof of [CS00, Theorem 3]. Let $s \in S'$ be a general point and let $\Gamma_s \subset \mathbb{P}^{d-2} = \Lambda_s$ be the rank 1 locus of the map

$$\chi_s : T_{S',s} \otimes \mathcal{O}_{\Lambda_s} \to \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}}.$$

Note that the normal bundle of $\Gamma_s$ splits

$$\mathcal{N}_{\Gamma_s/\mathbb{P}^{g-1}} = (\mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}} \otimes \mathcal{O}_{\Gamma_s}) \oplus \mathcal{N}_{\Gamma_s/\Lambda_s} = \mathcal{O}_{\Gamma_s}(d-2)^{\oplus n} \oplus \mathcal{O}_{\Gamma_s}(d)^{\oplus d-3}.$$

Hence, the map $\psi_s$ is given by a matrix

$$\psi_s = \begin{pmatrix} A \\ B \end{pmatrix}$$

where A is a $n\times(\rho+1)$-matrix and B is a $(d-3)\times(\rho+1)$-matrix. The matrix A represents the map $(\chi_s : T_{S,s}\otimes\mathcal{O}_{\Lambda_s} \to \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}})|_{\Gamma_s}$ and therefore has rank 1 and is equivalent to a catalecticant matrix. Let $\{s,t\}$ be a basis of $H^0(\Gamma_s, \mathcal{O}_{\Gamma_s}(1))$. In an appropriate basis, the matrix A is of the following form

$$A = \begin{pmatrix} t^{d-2} & t^{d-3}s & \cdots & t^{d-2-\rho}s^{\rho} \\ t^{d-3}s & \ddots & & t^{d-2-\rho-1}s^{\rho+1} \\ \vdots & & & \vdots \\ t^{d-2-n+1}s^{n-1} & t^{d-2-n}s^{n} & \cdots & s^{d-2} \end{pmatrix}$$

$$= \begin{matrix} t^{n-1}\cdot \\ t^{n-1}s\cdot \\ \vdots \\ s^{n-1}\cdot \end{matrix} \begin{pmatrix} t^{\rho} & t^{\rho-1}s & \cdots & s^{\rho} \\ t^{\rho} & \ddots & & s^{\rho} \\ \vdots & & & \vdots \\ t^{\rho} & t^{\rho-1}s & \cdots & s^{\rho} \end{pmatrix}.$$

We see that the rank 1 locus of $\psi_s$ is the rank 1 locus of the following matrix

$$N = \begin{pmatrix} t^{\rho} & t^{\rho-1}s & \cdots & s^{\rho} \\ & B & & \end{pmatrix}.$$

Since $V(\psi_s)_1 \neq \Gamma_s$ by Lemma 3.3.10, we have

$$\deg(V(\psi_s)_1) = \deg(D_1(N)) \leq \min\{\text{degree of elements of } I_{2\times2}(N)\} \leq \rho + d.$$

□

**Proposition 3.3.12.** *Let* $s\in L$ *be a sufficiently general point. Then,* $V(\psi_s)_1$ *is the union of* $D_s$ *and* $\rho$ *points which are the intersection of* $\Gamma_s = V(\chi_s)_1$*, and the vertex of the scroll* $X_L$ *swept out by the pencil* $|L|$.

*Proof.* As in the proof of Proposition 3.2.8, one can show that the points in the support of $D_s$ are contained in $V(\psi_s)_1$.

Next, we show that the vertex in $\Lambda_s$ is given by a column of the matrix $\chi_s$. Again, we imitate the proof of [CS95, Proposition 4.2]. Each column of the $n \times (\rho+1)$-matrix $\chi_s$ is a section of the rank $n$ vector bundle $V/U \otimes \mathcal{O}_{\Lambda_s}(1)$ (where $U \subset V$ is the affine subspace representing $\Lambda_s$) corresponding to an infinitesimal deformation of $\Lambda_s$. Each section vanishes in a $\rho - 1 = (d-2-n)$-subspace of $\Lambda_s$ which is a $\rho$-secant of $\Gamma_s$. Since $\chi_s$ is 1-generic, we get a $(\rho+1)$-dimensional family of infinitesimal deformations of $\Lambda_s$ induced by all columns. Hence, one column corresponds to the deformation in the scroll $X_L$. The corresponding section vanishes at the vertex V. □

As in the case $V(\chi_s)_1 = D_s$, we get the following Torelli-type theorem using Remark 3.2.3.

**Corollary 3.3.13.** *A Brill–Noether general canonically embedded curve* C *is uniquely determined by the family* $\underline{\Lambda}$. *More precise, the canonical curve* C *is a component of the scheme of first- or second-order foci of the family* $\underline{\Lambda}$ *induced by the Brill–Noether locus* $W_d(C)$ *and (the smooth locus of) its singular locus* $W^1_d(C)$ *of dimension at least one.*

## 3.4 The first-order focal map

For a general curve C and a sufficiently general point $s \in S$, the rank one locus of the focal map $\chi_s$ at $s$ is either $d$ points or a rational normal curve. In the second case, the focal matrix at $s$ is catalecticant (see Corollary 3.3.5).

As mentioned above, the articles [CS95] and [CS00] of Ciliberto and Sernesi are the extremal cases ($d = g-1$ and $\rho = 1$, respectively), where the rank one locus is always a rational normal curve. We propose the following question.

**Question 1.** *When is the focal matrix* $\chi_s$ *catalecticant for a general curve* C *and a sufficiently general point* $s \in S$*?*

We conjecture that only in the extremal cases $d = g-1$ and $\rho = 1$ the rank one locus of $\chi_s$ is a rational normal curve for a general curve C and a general point $s \in S$. For the rest of this section, we explain the reason for our conjecture.

Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus $g$ and let $L \in W^1_d(C)$ be a smooth point such that the rank one locus of the focal matrix $\chi_s : T_{S,s} \otimes \mathcal{O}_{\Lambda_s} \to \mathcal{N}_{\Lambda_s/\mathbb{P}^{g-1}}$

is a rational normal curve $\Gamma_s$ in $\mathbb{P}^{d-2}$ for $s \in |L|$ sufficiently general. Let $X_L = \bigcup_{s\in|L|} \overline{D_s}$ be the scroll swept out by the pencil $|L|$. We get a rational surface

$$\Gamma_L = \overline{\bigcup_{s\in|L| \text{ gen}} \Gamma_s} \subset X_L$$

defined as in the previous section. The rational normal curve $\Gamma_s$ intersects the vertex V of $X_L$ in $\rho(g,d,1)$ points by Proposition 3.3.12. Note that the scroll $X_L$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^{h^1(C,L)-1}$ with vertex V. Hence, projection from the vertex V yields a rational surface in $\mathbb{P}^1 \times \mathbb{P}^{h^1(C,L)-1}$ whose general fiber in $\mathbb{P}^{h^1(C,L)-1}$ is again a rational normal curve. We have shown the following proposition.

**Proposition 3.4.1.** *Let* $C \subset \mathbb{P}^{g-1}$ *be a canonically embedded curve of genus* $g$ *and let* $L \in W^1_d(C)$ *be a smooth point such that the rank one locus of the focal matrix* $\chi_s$ *is a rational normal for* $s \in |L|$ *sufficiently general. Then, the image of* C *in* $\mathbb{P}^1 \times \mathbb{P}^{h^1(C,L)-1}$ *given by* $|L| \times |\omega_C \otimes L^{-1}|$ *lies on a rational surface of bidegree* $(d', h^1(C,L)-1)$ *for some* $d'$.

*Proof.* The proposition follows from the preceeding discussion. We only note that the map given by $|L| \times |\omega_C \otimes L^{-1}|$ is the same as the projection of $\mathbb{P}^{g-1}$ along the vertex V of the canonically embedded C. □

*Example* 3.4.2. We explain the above circumstance for a curve C of genus 8 with a line bundle $L \in W^1_6(C)$. The residual line bundle $\omega_C \otimes L^{-1}$ has degree 8 and $H^0(C, \omega_C \otimes L^{-1})$ is three dimensional. Let $C'$ be the image of C in $\mathbb{P}^1 \times \mathbb{P}^2$ given by $|L| \times |\omega_C \otimes L^{-1}|$. We think of $C' \to \mathbb{P}^1$ as a one-dimensional family of six points in the plane. If our assumption of Proposition 3.4.1 is true, the six points lie on a conic in every fiber over $\mathbb{P}^1$. Computing a curve of genus 8 with a $g^1_6$ in `Macaulay2` shows that these conics do not exist. Hence, our assumption of Proposition 3.4.1, that is, the rank one locus of the focal matrix $\chi_s$ is a rational normal curve for $s \in |L|$ sufficiently general, does not hold for a general curve.

If $\rho(g,d,1) = 2d - g - 2 \geq 2$ and $d < g - 1$, we do not expect the existence of such a rational surface for a curve of genus $g$ and a line bundle of degree $d$ as above. Indeed, $m$ general points in $\mathbb{P}^r$ do not lie on a rational normal curve if $m > r+3$. But the inequality $\rho(g,d,1) = 2d - g - 2 \geq 2$ implies $d > (h^1(C,L) - 1) + 3$. Using our `Macaulay2` package (see [BH15]), we could show in several examples ($(g,d) = (8,6), (9,7), (10,8), (9,6)$) that the rational surface of bidegree $(d', h^1(C,L)-1)$ of Proposition 3.4.1 does not exist. This confirms our conjectural behaviour of the first-order focal map.

# Chapter 4

# Tangent cones to generalised theta divisors and generic injectivity of the theta map

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**Abstract.** Let C be a Petri general curve of genus $g$ and E a general stable vector bundle of rank $r$ and slope $g-1$ over C with $h^0(C,E) = r+1$. For $g \geq (2r+2)(2r+1)$, we show how the bundle E can be recovered from the tangent cone to the theta divisor $\Theta_E$ at $\mathscr{O}_C$. We use this to give a constructive proof and a sharpening of Brivio and Verra's theorem that the theta map $SU_C(r) \dashrightarrow |r\Theta|$ is generically injective for large values of $g$.

## 4.1 Introduction

Let C be a nonhyperelliptic curve of genus $g$ and $L \in \mathrm{Pic}^{g-1}(C)$ a line bundle with $h^0(C,L) = 2$ corresponding to a general double point of the Riemann theta divisor $\Theta$. It is well known that the projectivised tangent cone to $\Theta$ at L is a quadric hypersurface $R_L$ of rank 4 in the canonical space $|K_C|^*$, which contains the canonically embedded curve.

Quadrics arising from tangent cones in this way have been much studied: Green [Gre84] showed that the $R_L$ span the space of all quadrics in $|K_C|^*$ containing C; and both Kempf and Schreyer [KS88] and Ciliberto and Sernesi [CS92] have used the quadrics $R_L$ in various ways to give new proofs of Torelli's theorem.

In another direction: Via the Riemann–Kempf singularity theorem [Kem73], one sees that the rulings on $R_L$ cut out the linear series $|L|$ and $|K_C L^{-1}|$ on the canonical curve. Thus the data of the tangent cone and the canonical curve allows one to reconstruct the line bundle L. In this article we study a related construction for vector bundles of higher rank.

Let $V \to C$ be a semistable vector bundle of rank $r$ and integral slope $h$. We consider the set

$$\left\{ M \in \mathrm{Pic}^{g-1-h}(C) \ : \ h^0(C, V \otimes M) \geq 1 \right\}. \tag{4.1}$$

It is by now well known that for general V, this is the support of a divisor $\Theta_V$ algebraically equivalent to a translate of $r \cdot \Theta$. If V has trivial determinant, then in fact $\Theta_V$, when it exists, belongs to $|r\Theta|$.

For general V, the projectivised tangent cone $\mathscr{T}_M(\Theta_V)$ to $\Theta_V$ at a point M of multiplicity $r+1$ is a determinantal hypersurface of degree $r+1$ in $|K_C|^*$ (see for example Casalaina Martin–Teixidor i Bigas [CMTiB11]). Our first main result (§4.3.2) is a construction which from $\mathscr{T}_M(\Theta_V)$ recovers the bundle $V \otimes M$, up to the involution $V \otimes M \mapsto K_C \otimes M^{-1} \otimes V^*$. This is valid whenever $V \otimes M$ and $K_C \otimes M^{-1} \otimes V^*$ are globally generated.

We apply this construction to give an improvement of a result of Brivio and Verra [BV12]. To describe this application, we need to recall some more objects. Write $SU_C(r)$ for the moduli space of semistable bundles of rank $r$ and trivial determinant over C. The association $V \mapsto \Theta_V$ defines a map

$$\mathscr{D} \colon SU_C(r) \dashrightarrow |r\Theta| = \mathbb{P}^{r^g - 1}, \tag{4.2}$$

called the *theta map*. Drezet and Narasimhan [DN89] showed that the line bundle associated to the theta map is the ample generator of the Picard group of $SU_C(r)$. Moreover, the indeterminacy locus of $\mathscr{D}$ consists of those bundles $V \in SU_C(r)$ for which (4.1) is the whole Picard variety. This has been much studied; see for example Pauly [Pau10], Popa [Pop99] and Raynaud [Ray82].

Brivio and Verra [BV12] showed that $\mathscr{D}$ is generically injective for a general curve of genus $g \geq \binom{3r}{r} - 2r - 1$, partially answering a conjecture of Beauville [Bea06, §6]. We apply the aforementioned construction to give the following sharpening of Brivio and Verra's result:

**Theorem 4.1.1.** *For $r \geq 2$ and $C$ a Petri general curve of genus $g \geq (2r+2)(2r+1)$, the theta map (4.2) is generically injective.*

In addition to giving the statement for several new values of $g$ when $r \geq 3$ (our lower bound for $g$ depends quadratically on $r$ rather than exponentially), our proof is constructive, based on the method mentioned above for explicitly recovering the bundle $V$ from the tangent cone to the theta divisor at a point of multiplicity $r+1$. This gives a new example, in the context of vector bundles, of the principle apparent in [KS88] and [CS92] that the geometry of a theta divisor at a sufficiently singular point can encode essentially all the information of the bundle and/or the curve.

Our method works for $r = 2$, but in this case much more is already known: Narasimhan and Ramanan [NR69] showed, for $g = 2$ and $r = 2$, that $\mathscr{D}$ is an isomorphism $SU_C(2) \xrightarrow{\sim} \mathbb{P}^3$, and van Geemen and Izadi [vGI01] generalised this statement to nonhyperelliptic curves of higher genus. Note that our proof of Theorem 4.1.1 is not valid for hyperelliptic curves (see Remark 4.4.3).

Here is a more detailed summary of the article. In §4.2, we study semistable bundles $E$ of slope $g-1$ for which the *Petri trace map*

$$\bar{\mu} : H^0(C, E) \otimes H^0(C, K_C \otimes E^*) \to H^0(C, K_C)$$

is injective. A bundle $E$ with this property will be called *Petri trace injective.* We prove that for large enough genus, the theta divisor of a generic $V \in SU_C(r)$ contains a point $M$ of multiplicity $r+1$ such that $V \otimes M$ and $K_C \otimes M^{-1} \otimes V^*$ are Petri trace injective and globally generated.

Suppose now that $E$ is a vector bundle of slope $g-1$ with $h^0(C, E) \geq 1$. If $\Theta_E$ is defined and $\mathrm{mult}_{\mathscr{O}_C}(\Theta_E) = h^0(C, E)$, then the tangent cone to $\Theta_E$ at $\mathscr{O}_C$ is a determinantal hypersurface in $|K_C|^* = \mathbb{P}^{g-1}$ containing the canonical embedding of $C$. We prove (Proposition 4.3.3 and Corollary 4.3.5) that if $C$ is a general curve of genus $g \geq (2r+2)(2r+1)$, and $E$ a globally generated Petri trace injective bundle of rank $r$ and slope

$g-1$ with $h^0(\mathrm{C},\mathrm{E}) = r+1$, then the bundle E can be reconstructed up to the involution $\mathrm{E} \mapsto \mathrm{K_C} \otimes \mathrm{E}^*$ from a certain determinantal representation of the tangent cone to $\Theta_\mathrm{E}$ at $\mathcal{O}_\mathrm{C}$. By a classical result of Frobenius (whose proof we sketch in Proposition 4.3.7), any two such representations are equivalent up to transpose. The generic injectivity of the theta map for a Petri general curve (Theorem 4.4.1) can then be deduced by combining these facts and the statement in §4.2 that the theta divisor of a general $\mathrm{V} \in SU_\mathrm{C}(r)$ contains a suitable point of multiplicity $r+1$.

We assume throughout that the ground field is $\mathbb{C}$. The reconstruction of E from its tangent cone in §4.3.2 is valid for an algebraically closed field of characteristic zero or $p > 0$ not dividing $r+1$.

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## 4.2 Singularities of theta divisors of vector bundles

### 4.2.1 Petri trace injective bundles

Let C be a projective smooth curve of genus $g \geq 2$. Let $\mathrm{V} \to \mathrm{C}$ be a stable vector bundle of rank $r \geq 2$ and integral slope $h$, and consider the locus

$$\left\{ \mathrm{M} \in \mathrm{Pic}^{g-1-h}(\mathrm{C}) : h^0(\mathrm{C}, \mathrm{V} \otimes \mathrm{M}) \geq 1 \right\}. \tag{4.3}$$

If this is not the whole of $\mathrm{Pic}^{g-1-h}(\mathrm{C})$, then it is the support of the theta divisor $\Theta_\mathrm{V}$.

The theta divisor of a vector bundle is a special case of a *twisted Brill–Noether locus*

$$\mathrm{B}^n_{1,g-1-h}(\mathrm{V}) \ := \ \left\{ \mathrm{M} \in \mathrm{Pic}^{g-1-h}(\mathrm{C}) : h^0(\mathrm{C}, \mathrm{V} \otimes \mathrm{M}) \geq n \right\}. \tag{4.4}$$

The following is central in the study of these loci (see for example Teixidor i Bigas [TiB14, §1]): For $\mathrm{E} \to \mathrm{C}$ a stable vector bundle, we consider the *Petri trace map*:

$$\bar{\mu} : \mathrm{H}^0(\mathrm{C},\mathrm{E}) \otimes \mathrm{H}^0(\mathrm{C}, \mathrm{K_C} \otimes \mathrm{E}^*) \xrightarrow{\mu} \mathrm{H}^0(\mathrm{C}, \mathrm{K_C} \otimes \mathrm{End}\,\mathrm{E}) \xrightarrow{\mathrm{tr}} \mathrm{H}^0(\mathrm{C}, \mathrm{K_C}). \tag{4.5}$$

Then for $E = V \otimes M$ and $M \in B^n_{1,g-1-h}(V) \backslash B^{n+1}_{1,g-1-h}(V)$, the Zariski tangent space to $B^n_{1,g-1-h}(V)$ at M is exactly $\mathrm{Im}\left(\bar{\mu}\right)^{\perp}$. This motivates a definition:

**Definition 4.2.1.** Suppose $E \to C$ is a vector bundle with $h^0(C,E) = n \geq 1$. If the map $\mu$ above is injective, we will say that E is *Petri injective*. If the composed map $\bar{\mu}$ is injective, we will say that E is *Petri trace injective*.

**Remark 4.2.2.**

(1) Clearly, a Petri trace injective bundle is Petri injective. For line bundles, the two notions coincide.

(2) Suppose $V \in U_C(r,d)$ where $U_C(r,d)$ is the moduli space of semistable rank $r$ vector bundles of degree $d$. If $E = V \otimes M$ is Petri trace injective for $M \in \mathrm{Pic}^e(C)$, then $B^n_{1,e}(V)$ is smooth at M and of the expected dimension

$$h^1(C, \mathcal{O}_C) - h^0(C, V \otimes M) \cdot h^1(C, V \otimes M).$$

(3) We will also need to refer to the usual generalised Brill–Noether locus

$$B^n_{r,d} \ = \ \left\{ E \in U_C(r,d) : h^0(C,E) \geq n \right\}.$$

If E is Petri injective then this is smooth and of the expected dimension

$$h^1(C, \mathrm{End}\, E) - h^0(C,E) \cdot h^1(C,E)$$

at E. See for example Grzegorczyk and Teixidor i Bigas [GTiB09, §2].

(4) Petri injectivity and Petri trace injectivity are open conditions on families of bundles $\mathcal{E} \to C \times B$ with $h^0(C, \mathcal{E}_b)$ constant. Later, we will discuss the sense in which these properties are "open" when $h^0(C, \mathcal{E}_b)$ may vary.

We will also need the notion of a Petri general curve:

**Definition 4.2.3.** A curve C is called *Petri general* if every line bundle on C is Petri injective.

By [Gie82], the locus of curves which are not Petri general is a proper subset of the moduli space $M_g$ of curves of genus $g$, the so called *Gieseker–Petri locus*. The hyperelliptic locus is contained in the Gieseker–Petri locus. Apart from this, in general not much is known about the components of the Gieseker–Petri locus and their dimensions. For an overview of known results, we refer to [TiB88], [Far05] and [BS11] and the references cited there.

**Proposition 4.2.4.** *Suppose* $V$ *is a stable bundle of rank* $r$ *and integral slope* $h$. *Suppose* $M_0 \in \mathrm{Pic}^{g-1-h}(C)$ *satisfies* $h^0(C, V \otimes M_0) \geq 1$, *and furthermore that* $V \otimes M_0$ *is Petri trace injective. Then the theta divisor* $\Theta_V \subset \mathrm{Pic}^{g-1-h}(C)$ *is defined. Furthermore, we have equality* $\mathrm{mult}_{M_0}\Theta_V = h^0(C, V \otimes M_0)$.

*Proof.* Write $E := V \otimes M_0$. It is well known that via Serre duality, $\bar{\mu}$ is dual to the cup product map

$$\cup\colon H^1(C, \mathcal{O}_C) \to \mathrm{Hom}\big(H^0(C,E), H^1(C,E)\big).$$

By hypothesis, therefore, $\cup$ is surjective. Since E has Euler characteristic zero, $h^0(C,E) = h^1(C,E)$. Hence there exists $b \in H^1(C, \mathcal{O}_C)$ such that $\cdot \cup b\colon H^0(C,E) \to H^1(C,E)$ is injective. The tangent vector $b$ induces a deformation of $M_0$ and hence of E, which does not preserve any nonzero section of E. Therefore, the locus

$$\Big\{M \in \mathrm{Pic}^{g-1-h}(C) : h^0(C, V \otimes M) \geq 1\Big\}$$

is a proper sublocus of $\mathrm{Pic}^{g-1-h}(C)$, so $\Theta_V$ is defined. Now we can apply Casalaina-Martin and Teixidor i Bigas [CMTiB11, Proposition 4.1], to obtain the desired equality $\mathrm{mult}_{M_0}\Theta_V = h^0(C, V \otimes M_0)$. □

## 4.2.2 Existence of good singular points

In this section, we study global generatedness and Petri trace injectivity of the bundles $V \otimes M$ for $M \in B^{r+1}_{1,g-1}(V)$ for general C and V. The main result of this section is:

**Theorem 4.2.5.** *Suppose* $C$ *is a Petri general curve of genus* $g \geq (2r+2)(2r+1)$ *and* $V \in SU_C(r)$ *a general bundle. Then there exists* $M \in \Theta_V$ *such that* $h^0(C, V \otimes M) = r+1$, *and both* $V \otimes M$ *and* $K_C \otimes M^{-1} \otimes V^*$ *are globally generated and Petri trace injective.*

The proof of this theorem has several ingredients. We begin by constructing a stable bundle $E_0$ with some of the properties we are interested in. Let F be a semistable bundle of rank $r-1$ and degree $(r-1)(g-1)-1$, and let N be a line bundle of degree $g$.

**Lemma 4.2.6.** *A general extension* $0 \to F \to E \to N \to 0$ *is a stable vector bundle.*

*Proof.* Any subbundle G of E fits into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & N(-D) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_1} & & \downarrow & & \downarrow{\scriptstyle \iota_2} & & \\ 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where D is an effective divisor on C. If $\iota_2 = 0$, then $\mu(G) = \mu(G_1) \leq \mu(F) < \mu(E)$. Suppose $\iota_2 \neq 0$, and write $s := \mathrm{rk}(G_1)$. If $s \neq 0$, the semistability of F implies that

$$\deg(G_1) \;\leq\; s(g-1) - \frac{s}{r-1},$$

so in fact $\deg(G_1) \leq s(g-1)-1$. As $\deg(N) = g$, we have $\deg(G) \leq (s+1)(g-1)$. Thus we need only exclude the case where $\deg(G_1) = s(g-1)-1$ and $D = 0$, so $\iota_2 = \mathrm{Id}_N$. In this case, the existence of the above diagram is equivalent to $[E] = (\iota_1)_*[G]$ for some extension G, that is, $[E] \in \mathrm{Im}\,((\iota_1)_*)$. It therefore suffices to check that

$$(\iota_1)_* : H^1(C, \mathrm{Hom}(N, G_1)) \;\to\; H^1(C, \mathrm{Hom}(N, F))$$

is not surjective. This follows from the fact, easily shown by a Riemann–Roch calculation, that $h^1(C, \mathrm{Hom}(N, F/G_1)) > 0$.

If $s = 0$, then we need to exclude the lifting of $G = N(-p)$ for all $p \in C$, that is,

$$[E] \;\notin\; \bigcup_{p \in C} \Big(\mathrm{Ker}\big(H^1(C, \mathrm{Hom}(N, F)) \;\to\; H^1(C, \mathrm{Hom}(N(-p), F))\big)\Big).$$

A dimension count shows that this locus is not dense in $H^1(C, \mathrm{Hom}(N, F))$. □

**Lemma 4.2.7.** *Suppose* $h^0(C, N) \geq h^1(C, F)$. *Then for a general extension* $0 \to F \to E \to N \to 0$, *the coboundary map is surjective.*

*Proof.* Clearly it suffices to exhibit one extension $E_0$ with the required property. We write $n := h^1(C,F)$ for brevity.

Let $0 \to F \to \tilde{F} \to \tau \to 0$ be an elementary transformation with $\deg(\tau) = n$ and such that the image of $\Gamma(C,\tau)$ generates $H^1(C,F)$. We may assume that $\tau$ is supported along $n$ general points $p_1,\dots,p_n$ of C which are not base points of $|N|$. Then $\tau_{p_i}$ is generated by an element

$$\phi_i \in \left(\frac{F(p_i)}{F}\right)_{p_i}$$

defined up to nonzero scalar multiple. We write $[\phi_i]$ for the class in $H^1(C,F)$ defined by $\phi_i$.

Now $h^0(C,N) \geq n$ and the image of C is nondegenerate in $|N|^*$. As the $p_i$ can be assumed to be general, they impose independent conditions on sections of N. We choose sections $s_1,\dots,s_n \in H^0(C,N)$ such that $s_i(p_i) \neq 0$ but $s_i(p_j) = 0$ for $j \neq i$. For $1 \leq i \leq n$, let $\eta_i$ be a local section of $N^{-1}$ such that $\eta_i(s_i(p_i)) = 1$.

Let $0 \to F \to E_0 \to N \to 0$ be the extension with class $[E_0]$ defined by the image of

$$\left(\eta_1 \otimes \phi_1,\dots,\eta_n \otimes \phi_n\right)$$

by the coboundary map $\Gamma(C, N^{-1} \otimes \tau) \to H^1(C, N^{-1} \otimes F)$. Then $s_i \cup [E_0] = [\phi_i]$ for $1 \leq i \leq n$. Hence the image of $\cdot \cup [E_0]$ spans $H^1(C,F)$. □

We now make further assumptions on F and N. If $r = 2$, then $g \geq (2r+2)(2r+1) = 30$. Hence by the Brill–Noether theory of line bundles on C, we may choose a line bundle F of degree $g-2$ with $h^0(C,F) = 2$ and $|F|$ base point free. If $r \geq 3$: Since $g \geq 3$, we have $(r-1)(g-1)-1 \geq r$. Therefore, by [BBPN15, Theorem 5.1] we may choose a semistable bundle F of rank $r-1$ and degree $(r-1)(g-1)-1$ which is generated and satisfies $h^0(C,F) = r$, so $h^1(C,F) = r+1$.

Furthermore, again by Brill-Noether theory, since $g \geq (2r+2)(2r+1)$ we may choose $N \in \mathrm{Pic}^g(C)$ such that $h^0(C,N) = 2r+2$ and $|N|$ is base point free. By Lemma 4.2.7, we may choose an $(r+1)$-dimensional subspace $\Pi \subset H^0(C,E)$ lifting from $H^0(C,N)$.

**Proposition 4.2.8.** *Let* F, N *and* $\Pi$ *be as above, and let* $0 \to F \to E \to N \to 0$ *be a general extension. Then the restricted Petri trace map* $\Pi \otimes H^0(C, K_C \otimes E^*) \to H^0(C, K_C)$ *is injective.*

*Proof.* Choose a basis $\sigma_1, \ldots, \sigma_{r+1}$ for $\Pi$. For each $i$, write $\widetilde{\sigma_i}$ for the image of $\sigma_i$ in $H^0(C, N)$.

By Lemma 4.2.7, there is an isomorphism $H^1(C, E) \xrightarrow{\sim} H^1(C, N)$. Hence, by Serre duality, the injection $K_C \otimes N^{-1} \hookrightarrow K_C \otimes E^*$ induces an isomorphism on global sections. Choose a basis $\tau_1, \ldots, \tau_{2r+1}$ for $H^0(C, K_C \otimes E^*)$. For each $j$, write $\widetilde{\tau_j}$ for the preimage of $\tau_j$ by the aforementioned isomorphism.

For each $i$ and $j$ we have a commutative diagram

$$\begin{array}{ccccc} E^* & \xrightarrow{{}^t\sigma_i} & \mathcal{O}_C & \xrightarrow{\tau_j} & K_C \otimes E^* \\ \uparrow & \nearrow_{\widetilde{\sigma_i}} & & \searrow^{\widetilde{\tau_j}} & \uparrow \\ N^{-1} & & & & K_C \otimes N^{-1} \end{array}$$

where the top row defines the twisted endomorphism

$$\mu\left(\sigma_i \otimes \tau_j\right) \in H^0(C, K_C \otimes \operatorname{End} E^*) = H^0(C, K_C \otimes \operatorname{End} E).$$

Clearly this has rank one. As it factorises via $K_C \otimes N^{-1}$, at a general point of C the eigenspace corresponding to the single nonzero eigenvalue is identified with the fibre of $N^{-1}$ in $E^*$. Hence the Petri trace $\bar{\mu}(\sigma_i \otimes \tau_j)$ may be identified with the restriction to $N^{-1}$. By the diagram, we may identify this restriction with

$$\mu_N\left(\widetilde{\sigma_i} \otimes \widetilde{\tau_j}\right) \in H^0(C, K_C),$$

where $\mu_N$ is the Petri map of the line bundle N. Since C is Petri, $\mu_N$ is injective. Hence the elements $\bar{\mu}(\sigma_i \otimes \tau_j) = \mu_N(\widetilde{\sigma_i} \otimes \widetilde{\tau_j})$ are independent in $H^0(C, K_C)$. This proves the statement. □

Before proceeding, we need to recall some background on coherent systems (see [BBPN08, §2] for an overview and [BGaPMnN03] for more detail): We recall that a ***coherent system of type*** $(r, d, k)$ is a pair $(W, \Pi)$ where W is a vector bundle of rank $r$ and degree $d$ over C, and $\Pi \subseteq H^0(C, W)$ is a subspace of dimension $k$. There is a stability condition for coherent systems depending on a real parameter $\alpha$, and a moduli space $G(\alpha; r, d, k)$ for equivalence classes of $\alpha$-semistable coherent systems of type $(r, d, k)$. If $k \geq r$, then by [BGaPMnN03, Proposition 4.6] there exists $\alpha_L \in \mathbb{R}$ such that $G(\alpha; r, d, k)$

is independent of $\alpha$ for $\alpha > \alpha_L$. This "terminal" moduli space is denoted $G_L$. Moreover, the locus

$$U(r,d,k) \ := \ \{(W,\Pi) \in G_L : \ W \text{ is a stable vector bundle}\}$$

is an open subset of $G_L$. For us, $d = r(g-1)$ and $k = r+1$. To ease notation, we write $U := U(r, r(g-1), r+1)$.

Let now $N_1$ be a line bundle of degree $g$ with $h^0(C, N_1) \geq r+2$. Let F be as above, and let $0 \to F \to E \to N_1 \to 0$ be a general extension.

**Lemma 4.2.9.** *For a general subspace* $\Pi \subset H^0(C,E)$ *of dimension* $r+1$, *the coherent system* $(E,\Pi)$ *defines an element of* $U$.

*Proof.* Recall the bundle $E_0$ defined in Lemma 4.2.7, which clearly is generically generated. Let us describe the subsheaf $E_0'$ generated by $H^0(C,E_0)$.

Write $p_1 + \cdots + p_{r+1} =: D$. Clearly $s \cup [E_0] = 0$ for any $s \in H^0(C, N_1(-D))$. Since the $p_i$ are general points,

$$h^0(C, N_1(-D)) \ = \ h^0(C,N_1) - (r+1) \ = \ \dim\big(\mathrm{Ker}(\cdot \cup [E_0] : H^0(C,N_1) \to H^1(C,F)\big).$$

Therefore, the image of $H^0(C,E_0)$ in $H^0(C,N_1)$ is exactly $H^0(C,N_1(-D))$. As the subbundle $F \subset E_0$ is globally generated, $E_0'$ is an extension $0 \to F \to E_0' \to N_1(-D) \to 0$. Dualising and taking global sections, we obtain

$$0 \to H^0(C, N_1^{-1}(D)) \to H^0(C, (E_0')^*) \to H^0(C, F^*) \to \cdots$$

Since both $N_1^{-1}(D)$ and $F^*$ are semistable of negative degree, $h^0(C, N_1^{-1}(D)) = h^0(C,F^*) = 0$, so $h^0(C,(E_0')^*) = 0$.

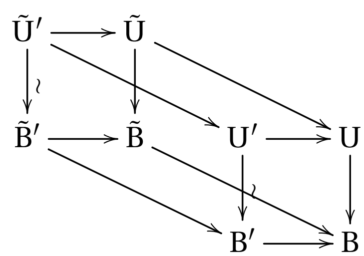
Now let $\Pi_1 \subset H^0(C,E_0)$ be any subspace of dimension $r+1$ generically generating $E_0$. Since $h^0(C,(E_0')^*) = 0$, by [BBPN08, Theorem 3.1 (3)] the coherent system $(E_0,\Pi_1)$ defines a point of $G_L$. Since generic generatedness and vanishing of $h^0(C,(E')^*)$ are open conditions on families of bundles with a fixed number of sections, the same is true for a generic $(E,\Pi)$ where E is an extension $0 \to F \to E \to N_1 \to 0$. By Lemma 4.2.6, in fact $(E,\Pi)$ belongs to U. □

**Lemma 4.2.10.** *For generic* E *represented in* U, *we have* $h^0(C,E) = h^0(C, K_C \otimes E^*) = r+1$.

*Proof.* Since C is Petri general, $B^{r+2}_{1,g}$ is irreducible in $\mathrm{Pic}^g(C)$. Thus there exists an irreducible family parametrising extensions of the form $0 \to F \to E \to N_1 \to 0$ where F is as above and $N_1$ ranges over $B^{r+2}_{1,g}$. This contains the extension $E_0$ constructed above. By Lemma 4.2.7, a general element $E_1$ of the family satisfies $h^0(C, E_1) = r+1$. By semicontinuity, the same is true for general E represented in U. □

Now by [BBPN08, Theorem 3.1 (4) and Remark 6.2], the locus U is irreducible. Write B for the component of $B^{r+1}_{r,r(g-1)}$ containing the image of U, and $B'$ for the sublocus $\{E \in B : h^0(C,E) = r+1\}$. Set $U' := U \times_B B'$; clearly $U' \cong B'$.

Let $\tilde{B} \to B$ be an étale cover such that there is a Poincaré bundle $\mathscr{E} \to \tilde{B} \times C$. In a natural way we obtain a commutative cube




where all faces are fibre product diagrams. By a standard construction, we can find a complex of bundles $\alpha\colon K^0 \to K^1$ over $\tilde{B}$ such that $\mathrm{Ker}(\alpha_b) \cong H^0(C, K_C \otimes \mathscr{E}^*_b)$ for each $b \in \tilde{B}$. Following [ACGH85, Chapter IV], we consider the Grassmann bundle $\mathrm{Gr}(r+1, K^0)$ over $\tilde{B}$ and the sublocus

$$\mathscr{G} \ := \ \{\Lambda \in \mathrm{Gr}(r+1, K^0) : \alpha|_\Lambda = 0\}.$$

Write $\mathscr{G}_1 := \mathscr{G} \times_{\tilde{B}} \tilde{U}$. The fibre of $\mathscr{G}_1$ over $(E_b, \Pi) \in \tilde{U}$ is then $\mathrm{Gr}(r+1, H^0(C, K_C \otimes E^*_b))$.

Now let $E_0$ be a bundle as constructed in Lemma 4.2.7 with $h^0(C, E_0) = 2r+2$, and let $\Pi_0$ be a generic choice of $(r+1)$-dimensional subspace of $H^0(C, E_0)$. We may assume $\tilde{U}$ is irreducible since U is. Since $\tilde{U} \to U$ is étale, by Lemma 4.2.8 in fact U is also smooth at $(E_0, \Pi_0)$ (cf. [BGaPMnN03, Proposition 3.10]). Therefore, we may choose a one-parameter family $\{(E_t, \Pi_t) : t \in T\}$ in $\tilde{U}$ such that $(E_{t_0}, \Pi_{t_0}) = (E_0, \Pi_0)$ while $(E_t, \Pi_t)$ belongs to $\tilde{U}'$ for generic $t \in T$. Since the bundles have Euler characteristic zero, for generic $t \in T$ there is exactly one choice of $\Lambda \in \mathscr{G}_1|_{(E_t, \Pi_t)}$. Thus we obtain a section $T \backslash \{0\} \to \mathscr{G}_1$. As $\dim T = 1$, this section can be extended uniquely to 0. We obtain thus a triple $(E_0, \Pi_0, \Lambda_0)$ where $\Lambda \subset H^0(C, K_C \otimes E_0^*)$ has dimension $r+1$. By Lemma 4.2.8, this triple is Petri trace injective. Hence

$$(E_t, \Pi_t, \Lambda_t) \;=\; (E, H^0(C,E), H^0(C, K_C \otimes E^*))$$

is Petri trace injective for generic $t \in T$. Thus a general bundle E represented in $\tilde{U}'$ is Petri trace injective.

Furthermore, by [BBPN08, Theorem 3.1 (4)], a general $(E, \Pi) \in U$ is globally generated (not just generically). Thus we obtain:

**Proposition 4.2.11.** *A general element* E *of the irreducible component* $B \subseteq B^{r+1}_{r,r(g-1)}$ *is Petri trace injective and globally generated with* $h^0(C,E) = r+1$.

Now we can prove the theorem:

*Proof of Theorem 4.2.5.* Consider the map $a\colon SU_C(r) \times \mathrm{Pic}^{g-1}(C) \to \mathcal{U}_C(r, r(g-1))$ given by $(V, M) \mapsto V \otimes M$. This is an étale cover of degree $r^{2g}$. We write $\overline{B}$ for the inverse image $a^{-1}(B)$. Since $a$ is étale, we have $T_{(V,M)}\overline{B} \;\cong\; T_{V\otimes M}B$ for each $(V,M) \in B$. In particular,

$$\dim \overline{B} \;=\; \dim B \;=\; \dim \mathcal{U}_C(r, r(g-1)) - (r+1)^2. \tag{4.6}$$

We write $p$ for the projection $SU_C(r) \times \mathrm{Pic}^{g-1}(C) \to SU_C(r)$, and $p_1$ for the restriction $p|_{\overline{B}}\colon \overline{B} \to SU_C(r)$.

**Claim**: $p_1$ is dominant.

To see this: For $(V,M) \in \overline{B}$, the locus $p_1^{-1}(V)$ is identified with an open subset of the twisted Brill–Noether locus

$$B^{r+1}_{1,g-1}(V) \;=\; \left\{ M \in \mathrm{Pic}^{g-1}(C) : h^0(C, V \otimes M) \geq r+1 \right\} \;\subseteq\; \mathrm{Pic}^{g-1}(C).$$

Moreover, for each such $(V,M)$, we have

$$\dim_M \left(p_1^{-1}(V)\right) \;=\; \dim\left(T_M\left(B^{r+1}_{1,g-1}(V)\right)\right) \;=\; \dim \mathrm{Im}\,(\bar{\mu})^{\perp}.$$

Since $V \otimes M$ is Petri trace injective, this dimension is $g - (r+1)^2$. By semicontinuity, a general fibre of $p_1$ has dimension at most $g - (r+1)^2$. Therefore, in view of (4.6), the image of $p_1$ has dimension at least

$$\left(\dim \mathcal{U}_C(r, r(g-1)) - (r+1)^2\right) - \left(g - (r+1)^2\right) \;=\; \dim \mathcal{U}_C(r, r(g-1)) - g \;=\; \dim SU_C(r).$$

As $SU_C(r)$ is irreducible, the claim follows.

Now we can finish the proof: Let $V \in SU_C(r)$ be general. By the claim, we can find $(V, M) \in \overline{B}$ such that $h^0(C, V \otimes M) = r + 1$ and $V \otimes M$ is globally generated and Petri trace injective. By Proposition 4.2.4, the theta divisor $\Theta_V$ exists and satisfies $\mathrm{mult}_M \Theta_V = h^0(C, V \otimes M) = r + 1$. Lastly, by considering a suitable sum of line bundles, one sees that the involution $E \mapsto K_C \otimes E^*$ preserves the component $\overline{B}$. Since a general element of $\overline{B}$ is globally generated, in general both $V \otimes M$ and $K_C \otimes M^{-1} \otimes V^*$ are globally generated. □

## 4.3 Reconstruction of bundles from tangent cones to theta divisors

### 4.3.1 Tangent cones

Let Y be a normal variety and $Z \subset Y$ a divisor. Let $p$ be a smooth point of Y which is a point of multiplicity $n \geq 1$ of Z. A local equation $f$ for Z near $p$ has the form $f_n + f_{n+1} + \cdots$, where the $f_i$ are homogeneous polynomials of degree $i$ in local coordinates centred at $p$. The projectivised tangent cone $\mathscr{T}_p(Z)$ to Z at $p$ is the hypersurface in $\mathbb{P}T_pY$ defined by the first nonzero component $f_n$ of $f$. (For a more intrinsic description, see [ACGH85, Chapter II.1].)

Now let C be a curve of genus $g \geq (r+1)^2$. Let E be a Petri trace injective bundle of rank $r$ and degree $r(g-1)$, with $h^0(C, E) = r + 1$. By Proposition 4.2.4 (with $h = g - 1$), the theta divisor $\Theta_E$ is defined and contains the origin $\mathscr{O}_C$ of $\mathrm{Pic}^0(C)$ with multiplicity $h^0(C, E) = r + 1$.

By [CMTiB11, Theorem 3.4 and Remark 3.8] (see also Kempf [Kem73]), the tangent cone $\mathscr{T}_{\mathscr{O}_C}(\Theta_E)$ to $\Theta_E$ at $\mathscr{O}_C$ is given by the determinant of an $(r+1) \times (r+1)$ matrix $\Lambda = \left(l_{ij}\right)$ of linear forms $l_{ij}$ on $H^1(C, \mathscr{O}_C)$, which is related to the multiplication map $\bar{\mu}$ as follows: In appropriate bases $(s_i)$ and $(t_j)$ of $H^0(C, E)$ and $H^0(C, K_C \otimes E^*)$ respectively, $\Lambda$ is given by

$$\left(l_{ij}\right) = \left(\bar{\mu}(s_i \otimes t_j)\right).$$

Hence, via Serre duality, $\Lambda$ coincides with the cup product map

$$\cup \colon H^0(C, E) \otimes H^1(C, \mathscr{O}_C) \to H^1(C, E).$$

Thus the matrix $\Lambda = (l_{ij})$ is a matrix of linear forms on the canonical space $|K_C|^*$.

In the following two subsections, we will show on the one hand that one can recover the bundle E from the determinantal representation of the tangent cone $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$ given by the matrix $\Lambda$. On the other hand, up to changing bases in $H^0(C,E)$ and $H^1(C,E)$ there are only two determinantal representations of the tangent cone, namely $\Lambda$ or $\Lambda^t$. Thus the tangent cone determines E up to an involution.

We will denote by $\varphi$ the canonical embedding $C \hookrightarrow |K_C|^*$.

### 4.3.2 Reconstruction of the bundle from the tangent cone

As above, let $\Lambda = (l_{ij})$ be the determinantal representation of the tangent cone given by the cup product mapping. We identify the source of $\Lambda$ with $H^0(C,E)$ and the target with $H^1(C,E)$:

$$H^0(C,E) \otimes \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{\Lambda} H^1(C,E) \otimes \mathcal{O}_{\mathbb{P}}.$$

We recall that the Serre duality isomorphism sends $b \in H^1(C,E)$ to the linear form

$$\cdot \cup b\colon H^0(C, K_C \otimes E^*) \;\to\; H^1(C,K_C) \;=\; \mathbb{C}.$$

In the following proofs, we will use principal parts in order to represent cohomology classes of certain bundles. We refer to [Kem83] or [Pau03, §3.2] for the necessary background. See also Kempf and Schreyer [KS88].

**Lemma 4.3.1.** *Suppose that* $h^0(C,E) = r+1$ *and* E *and* $K_C \otimes E^*$ *are globally generated. Then the rank of* $\Lambda|_C = \varphi^*\Lambda$ *is equal to* $r = \mathrm{rk}E$ *at all points of* C. *In particular, the canonical curve is contained in* $\mathcal{T}_{\mathcal{O}_C}(\Theta_E)$.

*Proof.* For each $p \in C$, write $\beta_p$ for a principal part with a simple pole supported at $p$. Then (see [KS88]) the cohomology class $[\beta_p]$ is identified with the image of $p$ by $\varphi$. Therefore, at $p \in C$, the pullback $\Lambda|_C$ is identified with the cup product map

$$[\beta_p] \otimes s \;\mapsto\; [\beta_p] \cup s.$$

The kernel of $[\beta_p] \cup \cdot$ contains the subspace $H^0(C, E(-p))$, which is one-dimensional since E is globally generated and $h^0(C,E) = r+1$. If $\mathrm{Ker}([\beta_p] \cup \cdot)$ has dimension greater than 1, then there is a section $s' \in H^0(C,E)$ not vanishing at $p$ such that

$$[\beta_p \cdot s'] \;=\; [\beta_p] \cup s' \;=\; 0 \;\in\; H^1(C,E).$$

By Serre duality, this means that

$$\left[\beta_p \cdot \langle s'(p), t(p)\rangle\right]$$

is zero in $H^1(C, K_C)$ for all $t \in H^0(C, K_C \otimes E^*)$. Hence the values at $p$ of all global sections of $K_C \otimes E^*$ belong to the hyperplane in $(K_C \otimes E^*)|_p$ defined by contraction with the nonzero vector $s'(p) \in E|_p$. Thus $K_C \otimes E^*$ is not globally generated, contrary to our hypothesis. □

**Remark 4.3.2**. Casalaina-Martin and Teixidor i Bigas in [CMTiB11, §6] prove more generally that if E is a general vector bundle with $h^0(C, E) > kr$, then the $k$th secant variety of the canonical image $\varphi(C)$ of C is contained in $\mathscr{T}_{\mathscr{O}_C}(\Theta_E)$.

**Proposition 4.3.3**. *Let* E *be a vector bundle with* $h^0(C, E) = r + 1$, *such that both* E *and* $K_C \otimes E^*$ *are globally generated. Then the image of* $\Lambda|_C$ *is isomorphic to* $T_C \otimes E$.

*Proof.* As $\varphi^* \mathscr{O}_{\mathbb{P}^{g-1}}(-1) \cong T_C$, the pullback $\varphi^* \Lambda = \Lambda|_C$ is a map

$$\Lambda|_C \colon T_C \otimes H^0(C, E) \to \mathscr{O}_C \otimes H^1(C, E).$$

Write $L := \det(E)$, a line bundle of degree $r(g-1)$. Then $\det(K_C \otimes E^*) = K_C^r \otimes L^{-1}$. As $K_C \otimes E^*$ is globally generated, the evaluation sequence

$$0 \to K_C^{-r} \otimes L \to \mathscr{O}_C \otimes H^0(C, K_C \otimes E^*) \to K_C \otimes E^* \to 0$$

is exact. For each $p \in C$, the image of $\left(K_C^{-r} \otimes L\right)|_p$ in $H^0(C, K_C \otimes E^*)$ is exactly $\mathbb{C} \cdot t_p$, where $t_p$ is the unique section, up to scalar, of $K_C \otimes E^*$ vanishing at $p$.

Dualising, we obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_C \otimes E & \longrightarrow & \mathscr{O}_C \otimes H^0(C, K_C \otimes E^*)^* & \xrightarrow{e} & K_C^r \otimes L^{-1} & \longrightarrow & 0 \\
 & & & \nearrow & \uparrow{\scriptstyle \wr}\ \text{Serre} & & & & \\
 & T_C \otimes H^0(E) & & \xrightarrow{\Lambda|_C} & \mathscr{O}_C \otimes H^1(C, E) & & & &
\end{array}$$

Here $e_p$ can be identified up to scalar with the map $f \mapsto f(t_p)$ where $t_p$ is as above.

Now for each $p \in C$, the image

$$[\beta_p] \cup H^0(C, E) \ \subset\ H^1(C, E) \ \cong\ H^0(C, K_C \otimes E^*)^*$$

annihilates $t_p \in H^0(C, K_C \otimes E^*)$, since the principal part $\beta_p \cdot t_p$ is everywhere regular. Therefore, $\Lambda|_C$ factorises via $\mathrm{Ker}(e) = T_C \otimes E$. Since $\mathrm{rk}(\Lambda|_C) \equiv r$ by Lemma 4.3.1, we have $\mathrm{Im}\,(\Lambda|_C) \cong T_C \otimes E$. □

**Remark 4.3.4.** A straightforward computation shows also that

$$\mathrm{Ker}(\Lambda|_C) \cong T_C \otimes L^{-1} \text{ and } \mathrm{Coker}(\Lambda|_C) \cong K_C^r \otimes L^{-1}.$$

We will also want to study the transpose $\Lambda^t$, which we will consider as a map

$$\Lambda^t : \mathscr{O}_{\mathbb{P}}(-1) \otimes H^0(C, K_C \otimes E^*) \ \to\ \mathscr{O}_{\mathbb{P}} \otimes H^1(C, K_C \otimes E^*).$$

The proof of Proposition 4.3.3 also shows:

**Corollary 4.3.5.** *Let* E *and* $\Lambda$ *be as above. Then the image of* $\Lambda^t|_C$ *is isomorphic to* $E^*$.

**Remark 4.3.6.** In order to describe the cokernel of $\Lambda|_C$, it is also enough to know in which points of C a row of $\Lambda|_C$ vanishes. Dualising the sequence

$$0 \to K_C^r \otimes L^{-1} \to \mathscr{O}_C \otimes H^0(C, K_C \otimes E^*) \to K_C \otimes E^* \to 0,$$

we see that $H^0(C, K_C \otimes E^*)^*$ is canonically identified with a subspace of $H^0(C, K_C^r \otimes L^{-1})$. Using the description of

$$T_C \otimes H^0(C, E) \xrightarrow{\Lambda|_C} \mathscr{O}_C \otimes H^1(C, E) \xrightarrow{\sim} \mathscr{O}_C \otimes H^0(C, K_C \otimes E^*)^*$$

as in the above proof, one sees that a row vanishes exactly in a divisor associated to $K_C^r \otimes L^{-1}$. Hence, the cokernel is isomorphic to $K_C^r \otimes L^{-1}$.

### 4.3.3 Uniqueness of the determinantal representation of the tangent cone

In order to show the desired uniqueness of the determinantal representation of the tangent cone, we use a classical result of Frobenius. See [Fro97] and also for a modern proof [Die49], [Wat87] and the references there. For the sake of completeness we will give a sketch of a proof following Frobenius.

**Proposition 4.3.7.** *Suppose* $r \geq 1$. *Let* A *and* B *be* $(r+1)\times(r+1)$ *matrices of independent linear forms, such that the entries of* A *are linear combinations of the entries of* B *and* $\det(\mathrm{A}) = k\cdot\det(\mathrm{B})$ *for a nonzero constant* $k \in \mathbb{C}$. *Then, there exist invertible matrices* $\mathrm{S},\mathrm{T} \in Gl(r+1,\mathbb{C})$, *unique up to scalar, such that* $\mathrm{A} = \mathrm{S}\cdot\mathrm{B}\cdot\mathrm{T}$ *or* $\mathrm{A} = \mathrm{S}\cdot\mathrm{B}^t\cdot\mathrm{T}$.

*Proof by Frobenius [Fro97, pages 1011-1013].* Note that for $r \geq 1$ only one of the above cases can occur and the matrices S and T are unique up to scalar. Indeed, let $\mathrm{A} = \mathrm{SBT} = \mathrm{S}'\mathrm{BT}'$ and set $b_{ii} = 1$ and $b_{ij} = 0$ if $i \neq j$, then $\mathrm{ST} = \mathrm{S}'\mathrm{T}'$. Set $\mathrm{U} = \mathrm{T}(\mathrm{T}'^{-1}) = \mathrm{S}(\mathrm{S}'^{-1})$, thus $\mathrm{UB} = \mathrm{BU}$. Since U commutes with every matrix, we have $\mathrm{U} = k\cdot\mathrm{E}_r$ and hence $\mathrm{S}' = k\cdot\mathrm{S}$ and $\mathrm{T}' = \frac{1}{k}\cdot\mathrm{T}$. Similar one can show that $\mathrm{B}^t$ is not equivalent to B. Note also that there is no relation between any minors of A or B.

For $l = 0,\dots,r$, let $c^l_{ij}$ be the coefficient of $b_{ll}$ in $a_{ij}$ and let $y$ be a new variable. We substitute $b_{ll}$ with $b_{ll}+y$ in A and B and get new matrices, denoted by $\left(a_{ij} + y\cdot c^l_{ij}\right)$ and $\mathrm{B}^l$, respectively. Since $\det\mathrm{B}^l$ is linear in $y$, the coefficient of $y^2$ in $\det\left(a_{ij} + y\cdot c^l_{ij}\right) = \det\mathrm{B}^l$ has to vanish. But the coefficient is the sum of products of $2\times 2$ minors of $\left(c^l_{ij}\right)$ and $(r-1)\times(r-1)$ minors of A. Since there are no relations between any minors of A, all $2\times 2$ minors of $\left(c^l_{ij}\right)$ vanish. Hence, $\left(c^l_{ij}\right)$ has rank one for any $l$ and we can write $c^l_{ij} = p^l_i q^l_j$ where $p^l$ and $q^l$ are column and row vectors, respectively.

Let $\mathrm{B}_0 = \mathrm{B}|_{\{b_{ij}=0, i\neq j\}}$ and $\mathrm{A}_0 = \mathrm{A}|_{\{b_{ij}=0, i\neq j\}}$. Then

$$\mathrm{A}_0 = \mathrm{PB}_0\mathrm{Q}$$

where $\mathrm{P} = \left(p^l_i\right)_{0\leq i,l\leq r}$ and $\mathrm{Q} = \left(q^l_j\right)_{0\leq l,j\leq r}$. Since $\det(\mathrm{A}_0) = c\cdot\det(\mathrm{B}_0) = c\cdot b_{00}\cdot\ldots\cdot b_{rr}$, we get $\det(\mathrm{P})\cdot\det(\mathrm{Q}) = c$, hence P and Q are invertible.

Let $\widetilde{\mathrm{B}} = \mathrm{P}^{-1}\mathrm{AQ}^{-1}$. By definition $\widetilde{\mathrm{B}}|_{\{b_{ij}=0,i\neq j\}} = \mathrm{B}_0$. Thus, the entries $\widetilde{b_{ij}}$ for $i \neq j$ and $v_i = \widetilde{b_{ii}} - b_{ii}$ are linear function in $b_{ij}$ for $i \neq j$. Furthermore, we have

$$\det(\widetilde{\mathrm{B}}) = \det(\mathrm{P}^{-1}\mathrm{AQ}^{-1}) = \det(\mathrm{P}^{-1}\mathrm{Q}^{-1})\cdot\det(\mathrm{A}) = \frac{1}{c}\cdot c\cdot\det(\mathrm{B}) = \det(\mathrm{B}).$$

Comparing the coefficient of $b_{11}b_{22}\cdots b_{rr}$ in $\det(\widetilde{\mathrm{B}})$ and $\det(\mathrm{B})$, we get $v_0 = 0$. Similarly, $v_i = 0$ for $0 \leq i \leq r$. Comparing the coefficients of $b_{22}\cdots b_{rr}$, we get $b_{12}b_{21} = \widetilde{b_{12}}\widetilde{b_{21}}$ and in general

$$b_{ij}b_{ji} = \widetilde{b_{ij}}\widetilde{b_{ji}},\ i \neq j.$$

Comparing the coefficients of $b_{33}\cdots b_{rr}$, we get $b_{12}b_{23}b_{31}+b_{21}b_{13}b_{32}=\widetilde{b_{12}}\widetilde{b_{23}}\widetilde{b_{31}}+\widetilde{b_{21}}\widetilde{b_{13}}\widetilde{b_{32}}$ and in general

$$b_{ij}b_{jk}b_{ki}+b_{ji}b_{ik}b_{kj}=\widetilde{b_{ij}}\widetilde{b_{jk}}\widetilde{b_{ki}}+\widetilde{b_{ji}}\widetilde{b_{ik}}\widetilde{b_{kj}},\ i\neq j\neq k\neq i.$$

A careful study of these equations shows that either

$$\widetilde{b_{ij}}=\frac{k_i}{k_j}b_{ij} \text{ and } \widetilde{\mathrm{B}}=\mathrm{KBK}^{-1} \text{ or } \widetilde{b_{ij}}=\frac{k_i}{k_j}b_{ji} \text{ and } \widetilde{\mathrm{B}}=\mathrm{KB}^t\mathrm{K}^{-1}$$

where $\mathrm{K}=\left(k_i\delta_{ij}\right)_{0\leq i,j\leq r}$. The claim follows. □

We now assume that E is a Petri trace injective bundle. Let $\Lambda=\left(l_{ij}\right)$ be a determinantal representation of the tangent cone $\mathscr{T}_{\mathscr{O}_\mathrm{C}}(\Theta_\mathrm{E})$ as above. By Petri trace injectivity, the matrix $\Lambda$ is $(r+1)$-generic (see [Eis88] for a definition), that is, there are no relations between the entries $l_{ij}$ or any subminors of $\Lambda$.

**Corollary 4.3.8.** *For a curve of genus $g\geq(r+1)^2$ and a Petri trace injective bundle* E *with $r+1$ global sections of degree $r(g-1)$, any determinantal representation of the tangent cone $\mathscr{T}_{\mathscr{O}_\mathrm{C}}(\Theta_\mathrm{E})\subset|\mathrm{K}_\mathrm{C}|^*$ is equivalent to $\Lambda$ or $\Lambda^t$.*

*Proof.* Let $\alpha$ be any determinantal representation of the tangent cone $\mathscr{T}_{\mathscr{O}_\mathrm{C}}(\Theta_\mathrm{E})$ in $|\mathrm{K}_\mathrm{C}|^*$. Then, $\alpha$ is an $(r+1)\times(r+1)$ matrix of linear entries, since the degree of the tangent cone is $r+1$. Furthermore, the entries $\alpha_{ij}$ of $\alpha$ are linear combinations of the entries $l_{ij}$ of $\Lambda$. Indeed, assume for some $k,l$ that $\alpha_{kl}$ is not a linear combination of the $l_{ij}$. Then, $\mathscr{T}_{\mathscr{O}_\mathrm{C}}(\Theta_\mathrm{E})$ would be the cone over $\mathrm{V}(\alpha_{kl})\cap\mathscr{T}_{\mathscr{O}_\mathrm{C}}(\Theta_\mathrm{E})$. Hence, the vertex of $\mathscr{T}_{\mathscr{O}_\mathrm{C}}(\Theta_\mathrm{E})$ defined by the entries $l_{ij}$ would have codimension strictly less than $(r+1)^2$; a contradiction to the independence of the $l_{ij}$. The corollary follows from Proposition 4.3.7. □

## 4.4 Injectivity of the theta map

**Theorem 4.4.1.** *Suppose $r\geq 2$. Let* C *be a Petri general curve of genus $g\geq(2r+2)(2r+1)$. Then the theta map $\mathscr{D}\colon SU_\mathrm{C}(r)\dashrightarrow|r\Theta|$ is generically injective.*

*Proof.* Let $\mathrm{V}\in SU_\mathrm{C}(r)$ be a general stable bundle. By Theorem 4.2.5, there exists $\mathrm{M}\in\Theta_\mathrm{V}$ such that $h^0(\mathrm{C},\mathrm{V}\otimes\mathrm{M})=r+1$, the bundle $\mathrm{V}\otimes\mathrm{M}=:\mathrm{E}$ is Petri trace injective, and E and $\mathrm{K}_\mathrm{C}\otimes\mathrm{E}^*$ are globally generated.

Note that tensor product by $M^{-1}$ defines an isomorphism $\Theta_V \xrightarrow{\sim} \Theta_E$ inducing an isomorphism $\mathscr{T}_M(\Theta_V) \xrightarrow{\sim} \mathscr{T}_{\mathscr{O}_C}(\Theta_E)$. In order to use the results of the previous sections, we will work with $\Theta_E$.

Now let

$$\alpha\colon \mathscr{O}_{\mathbb{P}^{g-1}}(-1)\otimes\mathbb{C}^{r+1} \to \mathscr{O}_{\mathbb{P}^{g-1}}\otimes\mathbb{C}^{r+1}$$

be a map of bundles of rank $r+1$ over $\mathbb{P}^{g-1}$ whose determinant defines the tangent cone $\mathscr{T}_{\mathscr{O}_C}(\Theta_E)$. By Corollary 4.3.8, the map $\alpha$ is equivalent either to $\Lambda$ or $\Lambda^t$, where $\Lambda$ is the representation given by the cup product mapping as defined in §4.3. Therefore, by Proposition 4.3.3 and Corollary 4.3.5, the image $E'$ of $\alpha|_C$ is isomorphic either to $T_C\otimes E = V\otimes M\otimes T_C$ or to $E^* = V^*\otimes M^{-1}$. Thus V is isomorphic either to

$$E'\otimes K_C\otimes M^{-1} \qquad \text{or to} \qquad (E')^*\otimes M^{-1}. \tag{4.7}$$

Now since in particular $g > (r+1)^2$, the open subset $\{M\in \mathrm{Pic}^{g-1}(C) : h^0(C, V\otimes M) = r+1\}\subseteq B^{r+1}_{1,g-1}(V)$ has a component of dimension $g-(r+1)^2 \geq 1$. Therefore, we may assume that $M^{2r} \not\cong K_C^r$. Thus only one of the bundles in (4.7) can have trivial determinant. Hence there is only one possibility for V.

In summary, the data of the tangent cone $\mathscr{T}_M(\Theta_V)$ and the point M, together with the property $\det(V) = \mathscr{O}_C$, determine the bundle V up to isomorphism. In particular, $\Theta_V$ determines V. □

**Remark 4.4.2.** The involution $M\mapsto K_C\otimes M^{-1}$ defines an isomorphism of varieties $\Theta_V \xrightarrow{\sim} \Theta_{V^*}$. We observe that the transposed map $\Lambda^t$ occurs naturally as the cup product map defining the tangent cone $\mathscr{T}_{K_C\otimes M^{-1}}(\Theta_{V^*})$.

**Remark 4.4.3.** If C is hyperelliptic, then the canonical map factorises via the hyperelliptic involution $\iota$. Thus the construction in §4.3.3 can never give bundles over C which are not $\iota$-invariant. We note that Beauville [Bea88] showed that in rank 2, if C is hyperelliptic then the bundles V and $\iota^*V$ have the same theta divisor.

# Chapter 5

# Resolutions of general canonical curves on rational normal scrolls

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**Abstract**. Let $C \subset \mathbb{P}^{g-1}$ be a general curve of genus $g$, and let $k$ be a positive integer such that the Brill–Noether number $\rho(g,k,1) \geq 0$ and $g > k+1$. The aim of this short note is to study the relative canonical resolution of C on a rational normal scroll swept out by a $g^1_k = |L|$ with $L \in W^1_k(C)$ general. We show that the bundle of quadrics appearing in the relative canonical resolution is unbalanced if and only if $\rho > 0$ and $(k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4} > 0$.

## 5.1 Introduction

Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus $g$ that admits a complete base point free $g^1_k$, then the $g^1_k$ sweeps out a rational normal scroll X of dimension $d = k-1$ and degree $f = g-k+1$. One can resolve the curve $C \subset \mathbb{P}(\mathscr{E})$, where $\mathbb{P}(\mathscr{E})$ is the $\mathbb{P}^{d-1}$-bundle associated to the scroll X. Schreyer showed in [Sch86] that this so-called *relative canonical resolution* is of the form

[1]Some changes have been made post-publication. These are listed at the end of Section 1.3.

$$0 \to \pi^* N_{k-2}(-k) \to \pi^* N_{k-3}(-k+2) \to \cdots \to \pi^* N_1(-2) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_C \to 0,$$

where $\pi : C \to \mathbb{P}^1$ is the map induced by the $g^1_k$ and $N_i = \bigoplus_{j=1}^{\beta_i} \mathcal{O}_{\mathbb{P}^1}(a_j^{(i)})$.

To determine the splitting type of these $N_i$ is an open problem. If C is a general canonical curve with a $g^1_k$ such that the genus $g$ is large compared to $k$, it is conjectured that the bundles $N_i$ are balanced, which means that $\max|a_j^{(i)} - a_l^{(i)}| \leq 1$. This is known to hold for $k \leq 5$ (see e.g. [DP15] or [Bop15]). Gabriel Bujokas and Anand Patel [BP15] gave further evidence to the conjecture by showing that all $N_i$ are balanced if $g = n \cdot k + 1$ for $n \geq 1$ and the bundle $N_1$ is balanced if $g \geq (k-1)(k-3)$.
The aim of this short note is to provide a range in which the first syzygy bundle $N_1$, hence the relative canonical resolution, is unbalanced for a general pair $(C, g^1_k)$ with non-negative Brill–Noether number $\rho(g,k,1)$. Our main theorem is the following.

**Main theorem 5.1.1.** *Let* $C \subset \mathbb{P}^{g-1}$ *be a general canonical curve, and let k be a positive integer such that* $\rho := \rho(g,k,1) \geq 0$ *and* $g > k+1$. *Let* $L \in W^1_k(C)$ *be a general point inducing a* $g^1_k = |L|$. *Then the bundle* $N_1$ *in the relative canonical resolution of* C *is unbalanced if and only if* $(k - \rho - \frac{7}{2})^2 - 2k + \frac{23}{4} > 0$ *and* $\rho > 0$.

After introducing the relative canonical resolution, we prove the above theorem in Section 5.3. The strategy for the proof is to study the birational image $C'$ of C under the residual mapping $|\omega_C \otimes L^{-1}|$. Quadratic generators of $C'$ correspond to special generators of $C \subset \mathbb{P}(\mathcal{E})$ whose existence forces $N_1$ to be unbalanced in the case $\rho > 0$. Under the generality assumptions on C and L, one obtains a sharp bound for which pairs $(k, \rho)$, the curve $C'$ has quadratic generators. Finally in section 5.4.1, we state a more precise conjecture about the splitting type of the bundles in the relative canonical resolution.

Our theorem and conjecture are motivated by experiments using the computer algebra software *Macaulay2* ([GS]) and the package `RelativeCanonicalResolution.m2` [BH15].

## 5.2 Relative canonical resolutions

In this section we briefly summarize the connections between pencils on canonical curves and rational normal scrolls in order to define the relative canonical resolution.

Furthermore, we give a closed formula for the degrees of the bundles $N_i$ appearing in the relative canonical resolution. Most of this section follows Schreyer's article [Sch86].

**Definition 5.2.1.** Let $e_1 \geq e_2 \geq \cdots \geq e_d \geq 0$ be integers, $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$, and let $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the corresponding $\mathbb{P}^{d-1}$-bundle.
A *rational normal scroll* $X = S(e_1, \ldots, e_d)$ of type $(e_1, \ldots, e_d)$ is the image of

$$j : \mathbb{P}(\mathscr{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r,$$

where $r = f + d - 1$ with $f = e_1 + \cdots + e_d \geq 2$.

In [Har81] it is shown that the variety X defined above is a non-degenerate $d$-dimensional variety of minimal degree $\deg X = f = r - d + 1 = \operatorname{codim} X + 1$. If $e_1, \ldots, e_d > 0$, then $j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r$ is an isomorphism. Otherwise, it is a resolution of singularities. Since $R^i j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0$, it is convenient to consider $\mathbb{P}(\mathscr{E})$ instead of X for cohomological considerations.
It is furthermore known, that the Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by the ruling $R = [\pi^* \mathscr{O}_{\mathbb{P}^1}(1)]$ and the hyperplane class $H = [j^* \mathscr{O}_{\mathbb{P}^r}(1)]$ with intersection products

$$H^d = f, \quad H^{d-1} \cdot R = 1, \quad R^2 = 0.$$

Now let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus $g$, and let further

$$g^1_k = \{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subset |D|$$

be a pencil of divisors of degree $k$. If we denote by $\overline{D_\lambda} \subset \mathbb{P}^{g-1}$ the linear span of the divisor, then

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(k-1)$-dimensional rational normal scroll of degree $f = g - k + 1$. Conversely, if X is a rational normal scroll of degree $f$ containing a canonical curve, then the ruling on X cuts out a pencil of divisors $\{D_\lambda\} \subset |D|$ such that $h^0(C, \omega_C \otimes \mathscr{O}_C(D)^{-1}) = f$.

**Theorem 5.2.2** ([Sch86], Corollary 4.4). *Let* C *be a curve with a complete base point free* $g^1_k$*, and let* $\mathbb{P}(\mathscr{E})$ *be the projective bundle associated to the scroll* X*, swept out by the* $g^1_k$.

(a) $C \subset \mathbb{P}(\mathscr{E})$ *has a resolution* $F_\bullet$ *of type*

$$0 \to \pi^* N_{k-2}(-kH) \to \pi^* N_{k-3}((-k+2)H) \to \cdots \to \pi^* N_1(-2H) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

*with* $\pi^* N_i = \sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a_j^{(i)} R)$ *and* $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

(b) *The complex* $F_\bullet$ *is self dual, that is,* $\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (f-2)R)) \cong F_\bullet$

According to [DP15], the resolution $F_\bullet$ above is called the *relative canonical resolution.*

**Remark 5.2.3.** A generalisation of Theorem 5.2.2 can be found in [CE96] for covers $\pi : X \to Y$ of degree $k$. In [CE96], the authors used the Tschirnhausen bundle $\mathscr{E}_T$ defined by

$$0 \to \mathscr{O}_Y \to \pi_*(\mathscr{O}_X) \to \mathscr{E}_T^\vee \to 0$$

to construct relative resolutions. Note that for covers of $\mathbb{P}^1$, $\mathscr{E}_T = \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^1}(2)$ and therefore, the degrees of the syzygy bundles $N_i$ in [CE96] differ slightly from the ones given in Proposition 5.2.9.

**Definition 5.2.4.** We say that a bundle of the form $\sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(nH + a_j R)$ is *balanced* if $\max_{i,j} |a_j - a_i| \leq 1$. The relative canonical resolution is called balanced if all bundles occurring in the resolution are balanced.

**Remark 5.2.5.** To determine the splitting type of the bundle $\mathscr{E}$, one can use [Sch86, (2.5)]. It follows that the $\mathbb{P}^1$-bundle $\mathscr{E}$ associated to the scroll is always balanced for a Petri-general curve C with a $g_k^1$ if $\rho(g,k,1) \geq 0$.
If C is a general $k$-gonal curve and the degree $k$ map to $\mathbb{P}^1$ is determined by a unique $g_k^1$, then it follows by [Bal89] that $\mathscr{E}$ is balanced as well.

**Remark 5.2.6.** If all $a_j^{(i)} \geq -1$, one can resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules occurring in the relative canonical resolution of C by Eagon-Northcott type complexes. An iterated mapping cone gives a possibly non-minimal resolution of the curve $C \subset \mathbb{P}^{g-1}$. In [Sch86], Schreyer used this method to classify all possible Betti tables of canonical curves up to genus 8. An implementation of this construction can be found in the *Macaulay2*-package [BH15].

We will give a lower bound on the integers $a_j^{(1)}$ appearing in the resolution $F_\bullet$.

**Proposition 5.2.7.** *Let* C *be a general canonically embedded curve of genus* g, *and let* $k \geq 4$ *be an integer such that* $\rho(g,k,1) \geq 0$ *and* $g > k+1$. *Let further* $L \in W^1_k(C)$ *be a general point inducing a complete base point free* $g^1_k$. *Then with notation as in Theorem 5.2.2, all twists* $a^{(1)}_j$ *of the bundle* $N_1$ *are non-negative.*

*Proof.* As usual, we denote by $\mathbb{P}(\mathscr{E})$ the $\mathbb{P}^1$-bundle induced by the $g^1_k$. We consider the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. Twisting of the relative canonical resolution by 2H and pushing forward to $\mathbb{P}^1$, we get an isomorphism $\pi_*(\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H)) \cong N_1 = \bigoplus_{j=1}^{\beta_1} \mathscr{O}_{\mathbb{P}^1}(a^{(1)}_j)$. Then, all twists $a^{(1)}_j$ are non-negative if and only if

$$h^1(\mathbb{P}^1, N_1(-1)) = h^1(\mathbb{P}^1, \pi_*(\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R))) = h^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) = 0.$$

We consider the long exact cohomology sequence

$$0 \to H^0(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) \to H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \to H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_C(2H-R)) \to$$
$$\to H^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) \to \dots$$

obtained from the standard short exact sequence.

The vanishing of $H^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R))$ is equivalent to the surjectivity of the map

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \longrightarrow H^0(C, \mathscr{O}_C(2H-R)).$$

From the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) & \longrightarrow & H^0(C, \mathscr{O}_C(2H-R)) \\ \uparrow & & \uparrow \eta \\ H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)) \otimes H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) & \xrightarrow{\cong} & H^0(C, \mathscr{O}_C(H)) \otimes H^0(C, \mathscr{O}_C(H-R)), \end{array}$$

we see that it suffices to show the surjectivity of $\eta$.

Note that the system $|H-R|$ on C is $\omega_C \otimes L^{-1}$. The residual line bundle $\omega_C \otimes L^{-1} \in W^{g-k}_{2g-2-k}(C)$ is general since L is general. Hence, the residual morphism induced by $|\omega_C \otimes L^{-1}|$ is birational for $g-k \geq 2$ by [GH80, Section 0.b (4)].

We may apply [AS78, Theorem 1.6] and get a surjection

$$\bigoplus_{q\geq 0} \mathrm{Sym}_q(H^0(C, \omega_C \otimes L^{-1})) \otimes H^0(C, \omega_C) \longrightarrow \bigoplus_{q\geq 0} H^0(C, \omega_C \otimes (\omega_C \otimes L^{-1})^q),$$

that is, the $\mathrm{Sym}(H^0(C, \omega_C \otimes L^{-1}))$-module $\bigoplus_{q\in\mathbb{Z}} H^0(C, \omega_C \otimes (\omega_C \otimes L^{-1})^q)$ is generated in degree 0. In particular, this implies the surjectivity of $\eta$. □

**Remark 5.2.8.** Using the projective normality of $C \subset \mathbb{P}(\mathscr{E})$, one can show that all twists $a_j^{(1)}$ of $N_1$ are greater or equal to $-1$. There exist several examples where $N_1$ has negative twists (see [Sch86]). We conjecture that all $a_j^{(i)} \geq -1$ and in general $a_j^{(i)} \geq 0$.

It is known that the degrees of the bundles $N_i$ can be computed recursively. However, we did not find a closed formula for the degrees in the literature.

**Proposition 5.2.9.** *The degree of the bundle* $N_i$ *of rank* $\beta_i = \frac{k}{i+1}(k-2-i)\binom{k-2}{i-1}$ *in the relative canonical resolution* $F_\bullet$ *is*

$$\deg(N_i) = \sum_{j=1}^{\beta_i} a_j^{(i)} = (g-k-1)(k-2-i)\binom{k-2}{i-1}.$$

*For* $i = 1, 2$ *one obtains* $\deg(N_1) = (k-3)(g-k-1)$ *and* $\deg(N_2) = (k-4)(k-2)(g-k-1)$.

*Proof.* The degrees of the bundles $N_i$ can be computed by considering the identity

$$\chi(\mathscr{O}_C(\nu)) = \sum_{i=0}^{k-2} (-1)^i \chi(F_i(\nu)). \tag{5.1}$$

If $b \geq -1$, we have

$$h^i(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH+bR)) = \begin{cases} h^i(\mathbb{P}^1, S_a(\mathscr{E})(b)), & \text{for } a \geq 0 \\ 0, & \text{for } -k < a < 0 \\ h^{k-i}(\mathbb{P}^1, S_{-a-k}(\mathscr{E})(f-2-b)), & \text{for } a \leq -k \end{cases}$$

where $f = \deg(\mathscr{E}) = g-k+1$. As in the construction of the bundles in [CE96, Proof of Step B, Theorem 2.1], one obtains that the degree of $N_i$ is independent of the splitting type of the bundle. Hence, we assume that $a_j^{(i)} \geq -1$ and therefore, we can apply the above formula to all terms in $F_\bullet$.

We compute the degree of $N_n$ by induction. The base case is straightforward. We twist the relative canonical resolution by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(n+1)$ and compute the Euler characteristic of each term. By the Riemann-Roch Theorem, $\chi(\mathscr{O}_C(n+1)) = (2n+1)g - (2n+1)$. Applying the above formula yields

$$\chi(F_i(n+1)) = \begin{cases} \binom{k-1+n}{k-2} + f\binom{k-1+n}{k-1}, & \text{for } i = 0 \\ (\deg(N_i)+\beta_i)\binom{k-2+n-i}{k-2} + \beta_i f\binom{k-2+n-i}{k-1}, & \text{for } n \geq i \geq 1 \\ 0, & \text{for } i \geq n+1 \end{cases}$$

Substituting all formulas in (5.1), we get

$$\begin{aligned}(2n+1)g-(2n+1) = &\binom{k-1+n}{k-2}+f\binom{k-1+n}{k-1}\\ &+\sum_{i=1}^{n-1}(-1)^i\left((\deg(\mathrm{N}_i)+\beta_i)\binom{k-2+n-i}{k-2}+\beta_i f\binom{k-2+n-i}{k-1}\right)\\ &+(-1)^n(\deg(\mathrm{N}_n)+\beta_n).\end{aligned}$$

Using the induction step, the alternating sums simplify to

$$\sum_{i=1}^{n-1}(-1)^i\deg(\mathrm{N}_i)\binom{k-2+n-i}{k-2} = (f-2)(2n+1-nk)+(-1)^{n+1}(f-2)(k-2-n)\binom{k-2}{n-1}$$

$$\sum_{i=1}^{n-1}(-1)^i\beta_i\binom{k-2+n-i}{k-2} = k-\binom{k-1+n}{k-2}+(-1)^{n+1}\frac{k}{n+1}(k-2-n)\binom{k-2}{n-1}$$

$$\sum_{i=1}^{n-1}(-1)^i\beta_i f\binom{k-2+n-i}{k-1} = nkf-f\binom{k-1+n}{k-1}$$

and we get the desired formula for $\deg(\mathrm{N}_n)$. □

## 5.3 The bundle of quadrics

Let $\mathrm{C}\subset\mathbb{P}^{g-1}$ be a general canonically embedded genus $g$ curve, and let $k$ be a positive integer such that the Brill–Noether number $\rho := \rho(g,k,1)$ is non-negative and $g > k+1$. Let $\mathrm{L}\in \mathrm{W}^1_k(\mathrm{C})$ general. Then, we denote by X the rational normal scroll swept out by the $g^1_k = |\mathrm{L}|$ and by $\mathbb{P}(\mathscr{E})\to \mathrm{X}$ the projective bundle associated to X. By Remark 5.2.5, the bundle $\mathscr{E}$ on $\mathbb{P}^1$ is of the form

$$\mathscr{E} = \bigoplus_{i=1}^{k-1-\rho}\mathscr{O}_{\mathbb{P}^1}(1)\oplus\bigoplus_{i=1}^{\rho}\mathscr{O}_{\mathbb{P}^1}.$$

By Theorem 5.2.2, the resolution of the ideal sheaf $\mathscr{I}_{\mathrm{C}/\mathbb{P}(\mathscr{E})}$ is of the form

$$0\longleftarrow \mathscr{I}_{\mathrm{C}/\mathbb{P}(\mathscr{E})}\longleftarrow \mathrm{Q} := \sum_{j=1}^{\beta_1}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+a_j^{(1)}\mathrm{R})\longleftarrow \ldots$$

where $\beta_1 = \frac{1}{2}k(k-3)$. We denote Q the bundle of quadrics. By Proposition 5.2.9, we know the degree of $\mathrm{N}_1 = \pi_*(\mathrm{Q})$ is precisely

$$\deg(\mathrm{N}_1) = \sum_{j=1}^{\beta_1} a_j^{(1)} = (k-3)(g-k-1).$$

By Proposition 5.2.7, all $a_i$ are non-negative. Since each summand of Q corresponds to a non-zero global section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2\mathrm{H}-a_j^{(1)}\mathrm{R})$, we get $2\cdot e_1 - a_j^{(1)} \geq 0$. Hence $a_j^{(1)} \leq 2$ for all $j$. It follows that the bundle of quadrics Q is of the following form

$$\mathrm{Q} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2\mathrm{H})^{\oplus l_0} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2\mathrm{H}+\mathrm{R})^{\oplus l_1} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2\mathrm{H}+2\mathrm{R})^{\oplus l_2}.$$

We will describe the possible generators of $\mathcal{I}_{\mathrm{C}/\mathbb{P}(\mathcal{E})}$ in $\mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2\mathrm{H}-2\mathrm{R}))$. Therefore, we consider the residual line bundle $\omega_\mathrm{C} \otimes \mathrm{L}^{-1}$ with

$$h^0(\mathrm{C}, \omega_\mathrm{C} \otimes \mathrm{L}^{-1}) = f = g-k+1 \text{ and } \deg(\omega_\mathrm{C} \otimes \mathrm{L}^{-1}) = 2g-k-2.$$

By [GH80, Section 0.b (4)], $|\omega_\mathrm{C} \otimes \mathrm{L}^{-1}|$ induces a birational map for $g > k+1$.

**Lemma 5.3.1.** *Let* $\mathrm{C}' \subset \mathbb{P}^{g-k}$ *be the birational image of* C *under the residual linear system* $|\omega_\mathrm{C} \otimes \mathrm{L}^{-1}|$. *There is a one-to-one correspondence between quadratic generators of* $\mathrm{C}' \subset \mathbb{P}^{g-k}$ *and quadratic generators of* $\mathrm{C} \subset \mathbb{P}(\mathcal{E})$ *contained in* $\mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2\mathrm{H}-2\mathrm{R}))$.

*Proof.* Since $\rho \geq 0$, the scroll X is a cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{g-k}$. Let $p : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^{g-k}$ be the projection on the second factor. An element $q$ of $\mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2\mathrm{H}-2\mathrm{R}))$ corresponds to a global section of $\mathrm{H}^0(\mathbb{P}^1, \mathrm{S}_2(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(-2))$ which does not depend on the fiber over $\mathbb{P}^1$. Hence, the image of $\mathrm{V}(q)$ under the projection yields a quadric containing $\mathrm{C}'$. Conversely, the pullback under the projection $p$ of a quadratic generator of $\mathrm{C}' \subset \mathbb{P}^{g-k}$ does not depend on the fiber and has therefore to be contained in $\mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2\mathrm{H}-2\mathrm{R}))$. □

We are now interested in a bound on $k$ and $\rho$ such that the curve $\mathrm{C}'$ lies on a quadric.

**Lemma 5.3.2.** *For a general curve* C *and a general line bundle* $\mathrm{L} \in \mathrm{W}_k^1(\mathrm{C})$, *the curve* $\mathrm{C}' \subset \mathbb{P}^{g-k}$ *lies on a quadric if and only if the pair* $(k, \rho)$ *satisfies the inequality*

$$\left(k-\rho-\frac{7}{2}\right)^2 - 2k + \frac{23}{4} > 0.$$

*Proof.* By [JP15], the map

$$\mathrm{H}^0(\mathbb{P}^{g-k}, \mathscr{O}_{\mathbb{P}^{g-k}}(2)) \to \mathrm{H}^0(\mathrm{C}', \mathscr{O}_{\mathrm{C}'}(2))$$

has maximal rank for a general curve C and a general line bundle $\omega_{\mathrm{C}} \otimes \mathrm{L}^{-1}$. Using the long exact cohomology sequence to the short exact sequence

$$0 \to \mathscr{I}_{\mathrm{C}'}(2) \to \mathscr{O}_{\mathbb{P}^{g-k}}(2) \to \mathscr{O}_{\mathrm{C}'}(2) \to 0,$$

we see that $\mathrm{C}'$ lies on a quadric if and only if

$$h^0(\mathbb{P}^{g-k}, \mathscr{O}_{\mathbb{P}^{g-k}}(2)) - h^0(\mathrm{C}', \mathscr{O}_{\mathrm{C}'}(2)) > 0.$$

We compute the Hilbert polynomial of $\mathrm{C}'$: $h_{\mathrm{C}'}(n) = (2g-k-2)n+1-g$ and get $h_{\mathrm{C}'}(2) = 3g-2k-3$. The dimension of the space of quadrics in $\mathbb{P}^{g-k}$ is $\binom{g-k+2}{2}$. Hence,

$$h^0(\mathbb{P}^{g-k}, \mathscr{O}_{\mathbb{P}^{g-k}}(2)) - h^0(\mathrm{C}', \mathscr{O}_{\mathrm{C}'}(2)) = \binom{g-k+2}{2} - 3g + 2k + 3 > 0. \tag{5.2}$$

Expressing $g$ in terms of $k$ and $\rho$, the inequality (5.2) is equivalent to

$$(k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4} > 0.$$

□

*Proof of the Main Theorem.* As mentioned above, the bundle $\mathrm{Q} = \pi^*\mathrm{N}_1$ is of the form $\mathrm{Q} = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H})^{\oplus l_0} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+\mathrm{R})^{\oplus l_1} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+2\mathrm{R})^{\oplus l_2}$ (see also Proposition 5.2.7). By Lemma 5.3.1, the bundle of quadrics is balanced if no quadratic generator of $\mathrm{C}' \subset \mathbb{P}^{g-k}$ exists. So, we are done for pairs $(k,\rho)$ with $(k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4} \le 0$.

It remains to show that the bundle of quadrics is unbalanced in the case $\rho > 0$ for pairs $(k,\rho)$ satisfying the inequality in Lemma 5.3.2.

Let $k$ and $\rho$ be non-negative integers satisfying the above inequality, and let $l_2 = h^0(\mathrm{C}', \mathscr{I}_{\mathrm{C}'}(2)) = (k-\rho-\frac{7}{2})^2 - 2k + \frac{23}{4}$ be the positive dimension of quadratic generators of the ideal of $\mathrm{C}'$. By Lemma 5.3.1, the bundle Q is now unbalanced if a summand of the type $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H})$ exists. Such a summand exists if and only if the following inequality holds

$$l_0 = \beta_1 - l_2 - l_1 = \beta_1 - l_2 - \left(\sum_{i=1}^{\beta_1} a_i - 2 \cdot l_2\right) > 0. \tag{5.3}$$

An easy calculation shows that the inequality (5.3) is equivalent to

$$l_0 = \binom{\rho+1}{2} > 0.$$

□

For pairs $(k,\rho)$ in the following marked region, the bundle Q is unbalanced.

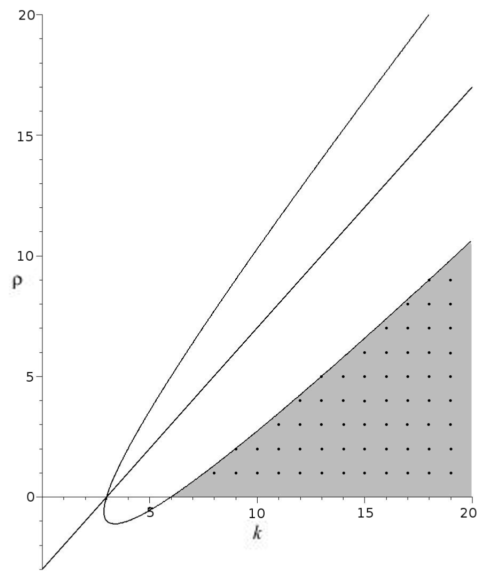


Figure 5.1: The conic: $\left(k-\rho-\frac{7}{2}\right)^2-2k+\frac{23}{4}=0$ and the line: $k-\rho-3=0 \Leftrightarrow g=k+1$.

**Remark 5.3.3.** With our presented method, the whole first linear strand of the resolution of $C' \subset \mathbb{P}^{g-k}$ lifts to the resolution of $C \subset \mathbb{P}(\mathscr{E})$. See also Example 5.4.1.

## 5.4 Outlook

### 5.4.1 Example and open problems

*Example* 5.4.1. Using [BH15], we construct a nodal curve $C \subset \mathbb{P}^{18}$ of genus 19 with a concrete realization of $L \in W^1_{11}(C)$. The ideal of the scroll X swept out by $|L|$ is given by the $2\times 2$ minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & \dots & x_{16} \\ x_1 & x_3 & \dots & x_{17} \end{pmatrix}.$$

The resolution of the birational image $C'$ of $C$ under the map $|\omega_C \otimes L^{-1}|$ has the following Betti table

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | - | - | - | - | - | - | - |
| 1 | - | 13 | 9 | - | - | - | - | - |
| 2 | - | - | 91 | 259 | 315 | 197 | 56 | 1 |
| 3 | - | - | - | - | - | - | - | 2 |

Assuming that the relative canonical resolution is as balanced as possible, the first part of the relative canonical resolution is of the following form

$$0 \leftarrow \mathscr{I}_{C/\mathbb{P}(\mathscr{E})} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R)^{\oplus 13} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 30} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H) \end{matrix} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+3R)^{\oplus 9} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 192} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 30} \end{matrix} \leftarrow \cdots$$

Using the *Macaulay2*-Package [BH15], our experiments lead to conjecture the following:

**Conjecture 5.4.2.** *(a) Let* $C \subset \mathbb{P}^{g-1}$ *be a general canonical curve, and let* $k$ *be a positive integer such that* $\rho := \rho(g,k,1) \geq 0$. *Let* $L \in W^1_k(C)$ *be a general point inducing a* $g^1_k = |L|$. *Then for bundles* $N_i = \bigoplus \mathscr{O}_{\mathbb{P}^1}(a_j^{(i)})$, $i = 2, \ldots, \lceil \frac{k-3}{2} \rceil$ *there is the following sharp bound*

$$\max_{j,l} |a_j^{(i)} - a_l^{(i)}| \leq \min\{g-k-1, i+1\}.$$

*In particular, if* $g - k = 2$, *the relative canonical resolution is balanced.*

*(b) For general pairs* $(C, g^1_k)$ *with* $\rho(g,k,1) \leq 0$, *the bundle* $N_1$ *is balanced.*

**Remark 5.4.3.** (a) In order to verify Conjecture (b), it is enough to show the existence of one curve with these properties. With the help of [BH15], we construct a $g$-nodal curve on a normalized scroll swept out by a $g^1_k$ and compute the relative canonical resolution. Then, Conjecture (b) is true for

$$(k,\rho) \in \{6,7,8,9\} \times \{-8,-7,\ldots,-1,0\} \text{ where } g = 2k - \rho - 2.$$

(b) We found several examples (e.g. $(g,k) = (17,7), (19,8), \ldots$) of $g$-nodal $k$-gonal curves where some of the higher syzygy modules $N_i$, $i \geq 2$ are unbalanced. We believe that the generic relative canonical resolution is unbalanced in these cases.

### 5.4.2 An unbalanced second syzygy bundle and K3 surfaces

Let $C \subset \mathbb{P}^8$ be a curve of genus 9. Let X be the scoll swept out by a $g^1_6$ and let $\mathbb{P}(\mathscr{E})$ be the corresponding projectivised bundle. Computing the relative canonical resolution of C on $\mathbb{P}(\mathscr{E})$ (using [BH15]) yields the following unbalanced resolution

$$\mathscr{I}_{C/\mathbb{P}(\mathscr{E})} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 6} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus 3} \end{matrix} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 2} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 12} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H)^{\oplus 2} \end{matrix} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-4H+2R)^{\oplus 3} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-4H+R)^{\oplus 6} \end{matrix} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-6H+2R) \leftarrow 0.$$

A natural question is if we have computed a general curve in the moduli space of curves of genus 9, or equivalently, if the general curve of genus 9 has an unbalanced second syzygy bundle. In an ongoing work with Christian Bopp, we will try to answer this question. In the remaining section, we will state our conjecture and give an idea of a proof without explaining the details.

We need some notation. Let $\mathscr{F}^{\mathfrak{h}}$ be the moduli space of $\mathfrak{h}$-lattice polarised K3 surfaces where $\mathfrak{h}$ is a rank 3 lattice given by the following intersection matrix with respect to the ordered basis $\{H, C, N\}$ (of effective classes)

$$\mathfrak{h} = \begin{pmatrix} 14 & 16 & 5 \\ 16 & 16 & 6 \\ 5 & 6 & 0 \end{pmatrix}$$

See [Dol96] for the construction of this moduli space. We will consider the finite cover $\mathscr{F}^{\mathfrak{h}}_8 \to \mathscr{F}^{\mathfrak{h}}$ when choosing an ample polarisation H of a K3 surface $S \in \mathscr{F}^{\mathfrak{h}}$ (see also [Bea04]). Let

$$\mathscr{P}^{\mathfrak{h}}_8 = \{(S, C') \mid S \in \mathscr{F}^{\mathfrak{h}}_8 \text{ and } C' \in |\mathscr{O}_S(C)| \text{ smooth}\}$$

be the open subset of the tautological $\mathbb{P}^9$-bundle on $\mathscr{F}^{\mathfrak{h}}_8$ and let

$$\mathscr{W}^1_{9,6} = \{(C, L) \mid C \in M_g \text{ and } L \in W^1_6(C)\}$$

be the universal Brill-Noether variety. Our conjecture is the following

**Conjecture 5.4.4.** *The morphism*

$$\phi : \mathscr{P}_8^{\mathfrak{h}} \to \mathscr{W}^1_{9,6}, \ (S, C) \mapsto (C, \mathscr{O}_S(N) \otimes \mathscr{O}_C)$$

*is dominant. In particular, on an open subset of* $\mathscr{W}^1_{9,6}$*, the fibers of* $\phi$ *are rational curves and* $\mathscr{P}_8^{\mathfrak{h}}$ *and* $\mathscr{F}_8^{\mathfrak{h}}$ *are unirational.*

The key idea of a proof should be the following observation. The submatrix

$$A : \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 6} \leftarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 2}$$

of the relative canonical resolution only involves rank 4 syzygies, that is, the entries of a generalised column of A span the fourdimensional vector space $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R))$. Furthermore, the syzygy scheme associated to a generalized column of A is a K3 surface $S \in \mathscr{F}_8^{\mathfrak{h}}$ (see [GvB07] for a definition of the syzygy scheme). We could check these facts computationally for our example. Given a K3 surface $S \in \mathscr{F}_8^{\mathfrak{h}}$ and a genus 9 curve $C \subset S$, one can show that S lies on the scroll X swept out by $|\mathscr{O}_S(N) \otimes \mathscr{O}_C|$ on C. In particular, the K3 surface forces the second syzygy bundle of the relative canonical resolution of C on X to be unbalanced. A detailed analysis of this interplay and a semicontinuity argument should yield a proof of the above conjecture.

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