Gorenstein Modules of Finite Length

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Abstract

Wir betrachten graduierte Moduln endlicher Länge über dem gewichteten Polynomring $R = k[x_1, \ldots, x_n]$, k ein beliebiger Körper, die eine streng selbstduale Auflösung haben. Wir entwickeln eine Konstruktionsmethode für diese Gorenstein Moduln mit Hilfe symmetrischer Matrizen in dividierten Potenzen. Unser Hauptresultat ist die folgende Äquivalenz: Sei n eine ungerade natürliche Zahl. Ein graduierter R-Modul endlicher Länge besitzt eine selbstduale minimale freie Aufösung mit symmetrischer beziehungsweise schiefsymmetrischer mittlerer Matrix genau dann wenn er durch eine symmetrische beziehungsweise schiefsymmetrische Matrix in dividierten Potenzen definiert werden kann. Diese Korrespondenz hängt von der Parität von $\frac{n-1}{2}$ ab. Wir entwickeln eine Reihe von Anwendungen, zum Beispiel einen Beweis einer Vermutung von Eisenbud und Schreyer: Sei R nun der trivial gewichtete Polynomring. Der Monoid der Betti Diagramme von freien Auflösungen graduierter Cohen-Macaulay Moduln über R hängt von der Charakteristik des Grundkörpers k ab.

Abstract

We study graded modules of finite length over the weighted polynomial ring $R = k[x_1, \dots, x_n]$, k any field, having a certain strongly selfdual resolution. We give a construction method of these Gorenstein modules via symmetric matrices in divided powers. Our main result is the following equivalence: Let n be an odd integer. A graded R-module of finite length has a selfdual minimal free resolution with a symmetric respectively skew symmetric middle matrix if and only if it can be defined by a symmetric respectively skew symmetric matrix in divided powers. The correspondence depends on the parity of $\frac{n-1}{2}$.

We give applications, such as a proof of a conjecture of Eisenbud and Schreyer: Let R be trivially weighted. The monoid of Betti tables of free resolutions of graded Cohen-Macaulay modules over R depends on the characteristic of the base field k.

Contents

Introduction

0.1 Algebraic Geometry, Commutative Algebra and Computer Algebra

In his famous book Hartshorne defines algebraic geometry to be the study of the solutions of systems of polynomial equations in an affine or projective n-space ([Har77, I.8]). But "the qualitative study of systems of polynomial equations is the chief subject of commutative algebra as well", as Eisenbud formulates in "The Geometry of Syzygies" ([Eis05, Preface]). This shows the link between geometry and algebra. However usual ideal computations arising from geometric questions can not be tackled by hand anymore. The concept of Gröbner Bases and the algorithm of Buchberger became the major instrument of computational algebraic geometry, as Decker and Schreyer point out in "Varieties, Gröbner Bases and Algebraic Curves" ([DS07, Chapter 2]). It allows the efficient use of computers for solving geometric problems.
The emphasis, all theory and results of this thesis are in commutative algebra. However there are obvious geometric motivations: For a surface X of general type over $\mathbb{C}$ we know that Proj of its canonical ring $R(X)$ is the canonical model. The Godeaux and Campedelli cases, $p_g = q = 0$ and $K_X^2 = 1, 2$, are the first cases in the geography of surfaces of general type. "It is somewhat embarrassing that we are still far from having a complete treatment of them", says Reid in [Rei93, 1.1]. Let us focus onto the Campedelli case. Skew symmetric free resolutions of $R(X)$ over the symmetric algebra in its 2- and three of its 3-sections would correspond to $2-3$-canonical embeddings of X into the weighted projective space $\mathbb{P}(2,2,2,3,3,3)$. Modulo the 3-sections y_0, y_1, y_2 such a complex leads to a module of finite length with a skew symmetric resolution. Hence it would be interesting to understand how to construct the module $R(X)/(y_0, y_1, y_2)$ over a polynomial ring in three variables. Similar situations can be imagined easily whenever locally Gorenstein schemes appear.

The commutative algebra problem we concentrate on is more general: We want to build modules of finite length over weighted polynomial rings having minimal free resolutions with lots of symmetry properties. The Theorem of Buchsbaum and Eisenbud ([BE77, Theorem 2.1] and A.23) states that every Gorenstein ideal of depth $= 3$ has a skew symmetric resolution. It is easy to build ideals with skew symmetric resolutions in a

polynomial ring with three variables: Every such ideal occurs as the annihilator of a homogeneous form in divided powers. Here the polynomial ring acts on the divided power algebra. Macaulay already proved this, calling such an ideal "principal system" (see [Mac16]).
We consider finite length modules over weighted polynomial rings with arbitrary many variables. They are realized as quotients of annihilators of matrices in divided powers. The question is: What are sufficient conditions for these matrices to gain a module with a symmetric respectively skew symmetric resolution in the case of an odd number of variables? Additionally: Are these conditions necessary for such a Gorenstein module of finite length?
The main result of this thesis is the answer to both of the two questions above. The theory of matrices in divided powers is developed in Chapter Two, and the answers are proved there as well. We compute many examples using the computer algebra systems [Singular] and [Macaulay2].

There is another remarkable motivation to study the special case of Gorenstein modules of finite length coming from Artinian Gorenstein factor rings of polynomial rings $R = k[x_1, \dots, x_n]$ over a field k. This time it is of pure algebraic nature: In their recent paper Eisenbud and Schreyer ([ES08]) prove a strengthened form of the Boji-Söderberg conjectures ([BS06]). For example they give an algorithm which expresses every Betti table of a finitely generated graded Cohen-Macaulay R-module as a positive rational linear combination of the Betti tables of Cohen-Macaulay R-modules with pure resolutions. Such an expression is unique in a certain sense. That is the Betti tables of Cohen-Macaulay R-modules lie inside a rational cone, whose extremal rays are Betti tables of modules with pure resolutions. Eisenbud and Schreyer also prove that for any given homological degree sequence of lenght $\leq n$ there is a corresponding Betti table of a pure minimal free resolution of a Cohen-Macaulay module. It is clear that the Betti tables for a given degree sequence are unique up to rational multiples. But it is not clear which of these multiples come from actual resolutions, respectively which points of the above mentioned cone lie inside the monoid of resolutions. Eisenbud and Schreyer conjecture that the monoid depends on the characteristic of the ground field k. We can apply our theory and symmetric resolution constructions to prove this conjecture by focusing on a certain example. This application can be found in Section 2.4.

0.2 The Main Result

The main result is developed in Chapter Two. The major aspects can be summarized briefly as follows:
Let $R = k[x_1, \dots, x_n]$ be the weighted polynomial ring over an arbitrary field k with $\deg x_l = d_l > 0$. Let M be a graded R-module of finite length. For us M is said to be *Gorenstein* if for some $s \in \mathbb{Z}$ there is a graded isomorphism $\tau : M \to \mathrm{Hom}_k(M, k)(-s)$

such that $\tau = \mathrm{Hom}_k(\tau, k)(-s)$. A slightly more general definition for Gorenstein modules of finite length is presented in 2.2.4.
Another way to put it, which is especially useful from the computational point of view, is the following:

Theorem 0.1. *M is Gorenstein if and only if there is a symmetric matrix P in divided powers, such that we have*

$$M \cong M(P).$$

Here $M(P) = \bigoplus_{j=1}^{p} R(b_j)/\mathrm{Ann}_R(P)$ is the quotient of the annihilator of P. The exact definition of $\mathrm{Ann}_R(P)$ is given in 2.1.2. The proof of Theorem 0.1 follows from Theorem 2.2.1 and Theorem 2.2.5.

For these Gorenstein modules the following theorem holds over any characteristic:

Theorem 0.2 (Selfdual Resolution)**.** *Let $n \geq 3$ be an odd integer, and let $m = \frac{n-1}{2}$. Assume that M is Gorenstein with $\tau = \mathrm{Hom}_k(\tau, k)(-s)$, and let $(\quad)^\vee = \mathrm{Hom}_R(\quad, R(-\sum_{l=1}^{n} d_l - s))$. Then there is a graded free resolution of M of the form*

$$0 \leftarrow M \leftarrow F_0 \stackrel{\phi_1}{\leftarrow} F_1 \leftarrow \ldots \leftarrow F_m \stackrel{\phi_{m+1}}{\leftarrow} (F_m)^\vee \leftarrow \ldots \leftarrow (F_1)^\vee \stackrel{\phi_1^\vee}{\leftarrow} (F_0)^\vee \leftarrow 0,$$

such that ϕ_{m+1} is skew if m is odd and symmetric if m is even.

The proof is given in Theorem 2.3.5. Theorem 0.2 is of special interest in the context of the applications around the non-existence of certain pure resolutions.

In Corollary 2.3.7 we give a symmetric minimization process such that the symmetry of the resolution is kept. To do so we need $\mathrm{char}\, k \neq 2$.
In Section 2.5 an equivalence is given: A symmetric resolution implies already the Gorenstein property.
These two facts might be summarized as follows:

Theorem 0.3 (Selfdual Minimal Resolution)**.** *Let $\mathrm{char}\, k \neq 2$. Let s be the top degree of M and $(\quad)^\vee = \mathrm{Hom}_R(\quad, R(-\sum_{l=1}^{n} d_l - s))$. Let $n \geq 3$ be an odd integer, and set $m := \frac{n-1}{2}$. Then M is Gorenstein if and only if its minimal graded free resolution is of the form*

$$0 \leftarrow M \leftarrow F_0 \stackrel{\psi_1}{\leftarrow} F_1 \leftarrow \ldots \leftarrow F_m \stackrel{\psi_{m+1}}{\leftarrow} (F_m)^\vee \leftarrow \ldots \leftarrow (F_1)^\vee \stackrel{\psi_1^\vee}{\leftarrow} (F_0)^\vee \leftarrow 0,$$

and satisfies the following condition: ψ_{m+1} is skew if m is odd and symmetric if m is even.

In Theorem 2.5.1 the Gorensteiness of M is shown to be true. The proof is derived from the more general Theorem 2.5.9, which is formulated in the language of isomorphisms of functors.

0.3 Synopsis of the Content

This work is composed of two chapters. One is devoted to well-known concepts and general background. It should establish a sound basis on the one hand, and it should provide a fixation of the notation we use on the other hand. For obvious reasons this stands at the beginning. However throughout the whole thesis standard notations from books like Eisenbud ([Eis94]) are used. As long as we apply basic definitions from these books we avoid recalling them if they are not central in our construction. Moreover at the very end a glossary of nonstandard notations and an index can be found.
The second chapter contains the new algebraic theory on Gorenstein modules including our main results and applications. The additionally needed knowledge — like some facts on divided powers, Gorenstein rings and local cohomology — is put into Appendix A.

At this point, let us give a little more systematic overview.
In the first chapter an introduction is given to graded modules over graded local rings and minimal free resolutions in the Noetherian case. We specify afterwards on the weighted polynomial ring. The first section also contains some background like the computation of the Koszul complex. It fixes the notation throughout this work. Our main reference are the books of Eisenbud ([Eis94]) and Bruns and Herzog ([BH93]). Finally the first section is concluded with a proof of the Hilbert's Syzygy Theorem in the case of the weighted polynomial ring.
In the second section we give two construction methods of free resolutions of modules of finite length over the weighted polynomial ring. One of them was originally given by Nielsen in [Nie81]. We recall and prove in our setting the case for modules of finite length. For later use it is necessary to define the differentials explicitly. Moreover we consider polynomial rings with non trivial grading. We point out that both resolution constructions are the same: They are canonically isomorphic. However the two of them play their own role in our construction.

As mentioned above Chapter Two contains our main results in commutative algebra, however computer algebra systems can be used to calculate examples. We have implemented algorithms whenever possible. Roughly a correspondence between symmetric matrices in divided powers and Gorenstein modules of finite length is given. But again let us have a more systematic approach.
The first section explains the general connection between graded modules of finite length over a weighted polynomial ring over any field and matrices in divided powers. All relevant definitions concerning the divided power algebra and systems of divided powers are recalled in Appendix A.2. We tie in with the contraction action of a weighted polynomial ring R on the divided power algebra $\mathcal{D}$ at the beginning of Section 2.1. Defining what we understand by the annihilator of a divided power matrix P, we set up the central module construction $M(P)$ as a cokernel. We show that $M(P^t)$ with the transposed matrix is the k-vectorspace dual of $M(P)$. After calculating several examples we introduce the procedure `dualModule` which gives an algorithm to compute $M(P)$ for any P. A proof of

the fact that any graded module M of finite length over R is isomorphic to some $M(P)$ is given, we call such Ps the "associated matrices in divided powers". Finally it is examined how unique such a P is.

In Section 2.2 Gorenstein modules of finite length over R are defined. We prove that any such is given by a symmetric P. We compute concrete examples by hand, and show that also the opposite is true: Given a symmetric P in divided powers then $M(P)$ is Gorenstein. After defining the term it is show that even weakly Gorenstein modules have a symmetric Hilbert function. But Gorenstein modules provide more: The multiplication matrices of the generators with respect to any homogeneous form are symmetric. Towards the end we give an example for a weakly Gorenstein, but not Gorenstein module of finite length.

Section 2.3 contains the proof of the first part of our main Theorems 0.2, respectively Theorem 0.3: Any finite length Gorenstein module over R provides a symmetric respectively skew symmetric resolution as described above in Theorem 0.3. But the section is started by two lemmas concerning the selfduality properties of the Koszul complex. They and the introduced notions are essential for our construction. We use the Nielsen resolution construction from Chapter 1 to finally present the proof for the symmetry of the resolution in our case. An extensive example is calculated by hand and others using the computer algebra system [Macaulay2].

The following section is on applications of the symmetric resolution theorem of Section 2.3. On the one hand they concern Artinian Gorenstein ideals in trivially weighted polynomial rings R with n variables and Hilbert function $(1, n, n, 1)$. Assume at first $n \equiv 3 \bmod 4$. We prove that $\beta_{m,m+2}$ is even ($m := \frac{n-1}{2}$). This is true in any characteristic. Another application concerns the characteristic 2 case. Let now $n = 2^\ell - 3$ for $\ell \geq 3$. In that case we can show that $\beta_{m,m+2}$ is odd. This has a meaning for Green's conjecture in characteristic 2 and the dependence of the monoid of resolutions of finitely generated graded Cohen-Macaulay R-modules on the characteristic. All this is explained extensively within the section. Moreover let n be any integer again. We give examples for degree sequences of length $c = 2^\ell - 1 \leq n$ such that there is no Cohen-Macaulay factor ring of R of codim $= c$ with a pure resolution having this degree sequence.

The last section provides the other direction of our main Theorem 0.3. Besides this some isomorphisms of the functors $\mathrm{Ext}^n(\quad) := \mathrm{Ext}^n_R(\quad, \omega_R)$ and $(\quad)^* := \mathrm{Hom}_k(\quad, k)$ on the category of graded modules of finite length $\underline{\mathrm{grMFL}}$ over R are discussed. We derive a commutative diagram connecting them. At the beginning some technical lemmas are given and the equivalences of functors are defined. To do so we use both resolution constructions of Nielsen. Moreover we define functor isomorphisms from id to $\mathrm{Ext}^n(\mathrm{Ext}^n(\quad))$, and from id to $\mathrm{Ext}^n((\quad)^*)$. The central technical theorem of the section is a commutative diagram combining the functors $(\quad)^*$, $\mathrm{Ext}^n(\quad)$ and $\mathrm{Ext}^n((\quad)^{**})$. There is a sign appearing here depending on the number of variables, coming basically from the nature of the Koszul complex. It is needed for Theorem 0.3. An example is computed by hand, and the section is concluded with the proof of the second part of our main result, as mentioned above.

There are two appendices which can be found as the end of this work. The first appendix consists of two sections: One is on local cohomology and on homological algebra. Besides others the famous Theorem of Auslander and Buchsbaum and the Theorem of Buchsbaum and Eisenbud are recalled. Moreover we recall the definition of a canonical module and a Gorenstein ring. The second one is about divided powers.
The final Appendix contains general procedures around graded modules of finite length (written in [Macaulay2]), presented mainly within Chapter Two.

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Chapter 1

General Background

At the beginning we recall some well known definitions and facts. In this first Chapter we lay the algebraic background for our resolution constructions given in Chapter Two. We recapitulate some known theorems, but also give them in the notation we use throughout this whole work. Moreover we reformulate and reprove a more special setting of a resolution construction originally given by Nielsen in [Nie81].
In fact the way we will see these Nieslen resolutions gives a new insight into the topic of graded modules of finite length, which was not the central object of study of Nielsen. He derives the complex as the total complex of a double complex. Moreover he constructs it over an arbitrary scheme, while we concentrate on Spec(k) for a field k.
Also a more general construction is described by Aramova and Herzog in [AH95]. They consider arbitrary modules over local rings and compare the minimal free resolutions with a similar construction as the Nielsen complex. Finally they get an isomorphism on the level of spectral sequences associated to both constructions in a certain way.

The first section of this chapter states some very basic facts which will be used frequently. It makes sense to set up these things because we have to fix a clear notation throughout the thesis. We start by concentrating on graded rings and modules. We continue defining and constructing minimal graded free resolutions for finitely generated modules over Noetherian graded local rings. We fix the notion of Koszul complexes and finish the section by a proof of Hilbert's Syzygy Theorem over weighted polynomial rings. The second section is devoted to the Nielsen resolution construction. We define it directly without focusing on double complexes. We give two versions for the case of graded modules of finite length over a weighted polynomial ring. Both of which are canonically isomorphic to each other. But each single one will come into value in Chapter Two. Our treatment is new in the sense that the construction in [Nie81] is not focused onto the finite length case, and does not come up with the second version.

1.1 General Background on Resolutions

We start by recapitulating some basic definitions. At first we mainly follow the treatment of the book of Bruns and Herzog ([BH93]). In this thesis we use the word ring to denote a commutative ring with 1.

Definition 1.1.1 (Graded module)**.** A *graded ring* is a ring R with a decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ as $\mathbb{Z}$-modules such that $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$. It is called a *positively graded ring* if $R = \bigoplus_{i\geq 0} R_i$.
A *graded R-module* is an R-module M with a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ as $\mathbb{Z}$-module such that $R_iM_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. We call M_i the *i-th graded part of M*.
An element $r \in R$ is called *homogeneous of degree i* if $r \in R_i$ for some $i \in \mathbb{Z}$. We define this term analogously for elements of M.

Graded R-submodules of R are called *graded ideals*.
Let $d \in \mathbb{Z}$. The *d-th twist of M*, written $M(d)$, is the same as the graded module M, but we define the i-th graded part of $M(d)$ to be: $M(d)_i := M_{d+i}$.

Definition 1.1.2 (Homogeneous morphism)**.** Let R be a graded ring, and let M and N be R-graded modules. An R-module homomorphism $\phi : M \to N$ is called *homogeneous* if $\phi(M_i) \subset N_i$ for all $i \in \mathbb{Z}$.

We are nearly ready to define minimal graded free resolutions. To do so we need another notion:

Definition 1.1.3 (Graded local ring)**.** Let R be a graded ring. Let $\mathfrak{m} \subset R$ be a graded maximal ideal of R, such that every non trivial graded ideal of R is contained in $\mathfrak{m}$. Then $(R, \mathfrak{m})$ is called a *graded local ring*.

Let $(R, \mathfrak{m})$ be a Noetherian graded local ring. Let M be a finitely generated graded $(R, \mathfrak{m})$-module. Take a homogeneous minimal system of generators $(g_1, \ldots, g_{l_0})$ of M. Let $F_0 := \bigoplus_{i=1}^{l_0} R(-\deg g_i)$ be a direct sum of twisted graded R-modules, each of them generated by e_i. Then $\phi_0 : F_0 \to M$, defined by $e_i \mapsto g_i$ is a homogeneous R-module homomorphism. $\mathscr{S}\mathrm{yz}_0(M) := \ker(\phi_0)$ is again a finitely generated graded R-module. The minimality assumption guarantees that $\mathscr{S}\mathrm{yz}_0(M) \subset \mathfrak{m}F_0$. Iterating this construction for $\mathscr{S}\mathrm{yz}_0(M)$ and the next syzygy-modules one obtains a *minimal graded free resolution.* Throughout the thesis if we work over graded local rings we mean by a *minimal resolution* always a minimal graded free resolution.
It is used to collect the terms with the same shift, that means we obtain for the minimal graded free resolution a form as

$$0 \leftarrow M \leftarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}} \leftarrow \ldots \leftarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}} \leftarrow .$$

Proposition and Definition 1.1.4 (Betti number)**.** *Let $(R, \mathfrak{m})$ be a Noetherian graded local ring, and let M be a finitely generated graded R-module. Then the numbers $\beta_{i,j}$ in a minimal graded resolution of M are uniquely determined by M.*
The numbers $\beta_{i,j}(M) = \beta_{i,j}$ are called graded Betti numbers (of M).

Proof. [BH93, Proposition 1.5.16] □

Definition 1.1.5 (Betti table)**.** Let R be any positively graded ring. Let

$$\mathbf{F} : \quad F_0 \overset{\phi_1}{\leftarrow} \ldots \overset{\phi_m}{\leftarrow} F_m \overset{\phi_{m+1}}{\leftarrow} \ldots \overset{\phi_s}{\leftarrow} F_s \leftarrow 0$$

be a complex of free R-modules $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$, such that all maps ϕ_i are homogeneous. Then the *Betti table of* $\mathbf{F}$ is the following array of numbers:

	0	1	$\ldots$	s
μ	$\beta_{0,\mu}$	$\beta_{1,\mu+1}$	$\ldots$	$\beta_{s,\mu+s}$
$\mu+1$	$\beta_{0,\mu+1}$	$\beta_{1,\mu+2}$	$\ldots$	$\beta_{s,\mu+s+1}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
ν	$\beta_{0,\nu}$	$\beta_{1,\nu+1}$	$\ldots$	$\beta_{s,\nu+s}$

If $(R, \mathfrak{m})$ is a Noetherian positively graded local ring, and M is a finitely generated graded R-module then the *Betti table of* M is defined to be a Betti table of a minimal graded resolution of M. To make it unique we start with the first non zero row and finish with the last non zero one.

From now on we focus on a more special situation. That is we consider weighted polynomial rings. This is necessary to obtain finite resolutions: For graded quotients of weighted polynomial rings this is not always the case (see for example [Eis05, 1C]).
Let us fix the following notation:

Notation 1.1.6. Let k be any field, and let $V = \langle x_1, \ldots, x_n \rangle$ be a k-vectorspace such that $\deg x_l = d_l \geq 1$. Moreover, let $R = \mathrm{Sym}(V)$ with the natural grading, that is a weighted polynomial ring.

The following lemma is a well known tool for computing Betti tables of minimal resolutions of modules over polynomial rings. We follow mainly the description in [Eis05].

Lemma 1.1.7. *Consider $k = R/\mathfrak{m}$, with $\mathfrak{m} = (x_0, \ldots, x_n)$, as graded R-module. Let*

$$\mathbf{F} : \cdots \to F_1 \to F_0 \to M$$

be a minimal free resolution of a finitely generated graded R-module M. Then $\dim_k \mathrm{Tor}_i^R(k, M)_j$, the j-th graded part of the i-th Tor*-group, is the number of the degree j elements of any minimal set of homogeneous generators of F_i.*

Proof. We tensorize the resolution to $k \otimes_R \mathbf{F}$. Then all maps in the new complex are zero, as $\mathbf{F}$ is minimal. Hence for the i-th homology of this complex we have $\mathrm{Tor}_i^R(k, M) = k \otimes_R F_i$. By the Nakayama lemma for the graded case we have that if the residue classes of elements of F_i generate $F_i/\mathfrak{m}F_i = k \otimes_R F_i$ as k-vectorspace then they generate F_i as R-module. Hence $\dim_k \mathrm{Tor}_i^R(k, M)_j$ equals the number of generators of F_i in degree j. □

The given lemma leads to a modern proof of Hilbert's Syzygy Theorem. But before we are able to give the proof, it is necessary to fix what we mean by a Koszul complex. On the other hand the Koszul complex will be an important tool for the proof of our main theorem.

Notation 1.1.8 (Koszul complex)**.** Let $R = \mathrm{Sym}(V)$ be the weighted polynomial ring as above in 1.1.6 with $V = (x_1, \ldots, x_n)$. Let W be another k-vectorspace such that $V = \mathrm{Hom}_k(W, k)$. Choose a basis $(\chi_1, \ldots, \chi_n)$ of W such that the (χ_l) are a dual to (x_l), and set $\deg \chi_l = d_l$.
Fix for the moment $x_l \in V$. We consider the assignment,

$$(w_1, \ldots, w_i) \mapsto \sum_{j=1}^{i} (-1)^{j-1} x_l(w_j) w_1 \wedge \ldots \wedge \widehat{w_j} \wedge \ldots \wedge w_i.$$

It defines an alternating map $W^i \to \bigwedge^{i-1} W$. Hence if defines by the universal property of the exterior algebra a map, the *contraction* $(x_l\neg\ ) : \bigwedge^i W \to \bigwedge^{i-1} W$.

That means on generators we have

$$(x_l \neg w_1 \wedge \ldots \wedge w_i) = \sum_{j=1}^{i} (-1)^{j-1} x_l(w_j) w_1 \wedge \ldots \wedge \widehat{w_j} \wedge \ldots \wedge w_i.$$

View $R \otimes_k \bigwedge^i W$ as graded free R-module by left multiplication, the grading given by $(R \otimes_k \bigwedge^i W)_j = \bigoplus_{j_1+j_2=j} R_{j_1} \otimes_k (\bigwedge^i W)_{j_2}$. Then we define $\delta_i : R \otimes_k \bigwedge^i W \to R \otimes_k \bigwedge^{i-1} W$ on generators by

$$r \otimes w \mapsto \sum_{l=1}^{n} r x_l \otimes (x_l \neg w).$$

A direct computation shows $\delta_{i-1} \circ \delta_i = 0$.
We call the graded complex

$$K(x): \quad 0 \to R \otimes_k \bigwedge^n W \xrightarrow{\delta_n} \ldots \to R \otimes_k \bigwedge^2 W \xrightarrow{\delta_2} R \otimes_k \bigwedge^1 W \xrightarrow{\delta_1} R \otimes_k \bigwedge^0 W \to 0$$

the *Koszul complex*. It has only homology at $H^0(K(x)) = k$ (in our case the $x_1, \ldots, x_n$ form naturally an R-sequence). For the definition and the homology statement see [BH93, Section 1.6].

Example 1.1.9. Let $R = k[x_1, x_2, x_3]$ with $\deg x_l = 1$ for all l. Let us compute the differentials with respect to the bases $(\chi_1 \wedge \chi_2 \wedge \chi_3)$, $(\chi_1 \wedge \chi_2, -\chi_1 \wedge \chi_2, \chi_2 \wedge \chi_3)$, (χ_1, χ_2, χ_3) and (1) of $\bigwedge^3 W, \bigwedge^2 W, \bigwedge^1 W$ and $\bigwedge^0 W$. Then $\delta_1 : R \otimes \bigwedge^1 W = \bigoplus_{i=1}^{3} R(-1) \to R$,

$$1 \otimes \chi_1 \mapsto x_1 \otimes x_1(\chi_1) = x_1 \otimes 1, \ \ 1 \otimes \chi_2 \mapsto x_2 \otimes 1, \ 1 \otimes \chi_3 \mapsto x_3 \otimes 1.$$

$\delta_2 : R \otimes \bigwedge^2 W = \oplus_{i=1}^{3} R(-2) \to \oplus_{i=1}^{3} R(-1)$:

$$1 \otimes \chi_1 \wedge \chi_2 \mapsto x_1 \otimes \chi_2 - x_2 \otimes \chi_1,$$
$$-1 \otimes \chi_1 \wedge \chi_3 \mapsto -x_1 \otimes \chi_3 + x_3 \otimes \chi_1, \ \ 1 \otimes \chi_2 \wedge \chi_3 \mapsto x_2 \otimes \chi_3 - x_3 \otimes \chi_2.$$

Finally δ_3 maps as follows:

$$1 \otimes \chi_1 \wedge \chi_2 \wedge \chi_3 \mapsto x_1 \otimes \chi_2 \wedge \chi_3 - x_2 \otimes \chi_1 \wedge \chi_3 + x_3 \otimes \chi_1 \wedge \chi_2.$$

Now we have the machinery to give a modern proof of Hilbert's Syzygy Theorem. Especially the theorem gives an upper bound on the Betti table. We follow the proof given for the usual polynomial ring in [Eis05].

Theorem 1.1.10 (Hilbert's Syzygy Theorem)**.** *Let $R = \mathrm{Sym}(V)$ as in 1.1.6. Let M be a finitely generated graded R-module. The graded Betti number $\beta_{i,j}(M)$ is the dimension* *of the homology, at the term* $(M \otimes_k \bigwedge^i W)_j = \bigoplus_{\mu=0}^{j-i}(M_\mu \otimes_k (\bigwedge^i W)_{j-\mu})$*, of the complex of vectorspaces*

$$0 \to (M \otimes_k \bigwedge^n W)_j \to \dots (M \otimes_k \bigwedge^{i+1} W)_j \to (M \otimes_k \bigwedge^{i} W)_j \to (M \otimes_k \bigwedge^{i-1} W)_j \to \dots \to (M \otimes_k \bigwedge^{0} W)_j.$$

In particular we have $\beta_{i,j}(M) \leq \sum_{\mu=0}^{j-i}((\dim_k M_\mu) \cdot (\dim_k(\bigwedge^i W)_{j-\mu}))$*, and* $\beta_{i,j}(M) = 0$ *for* $i \geq n+1$*.*

Proof. Let $\beta_{i,j} = \beta_{i,j}(M)$. By Lemma 1.1.7 we know $\beta_{i,j} = \dim_k \operatorname{Tor}_i^R(M,k)_j$. We compute these groups by the free resolution of k as R-module given by the Koszul complex. So $\operatorname{Tor}_i^R(M,k)_j$ is the degree j-part of the homology of $M \otimes_R K(x)$ at

$$M \otimes_R R \otimes_k \bigwedge^i W = M \otimes_k \bigwedge^i W.$$

The differentials of $M \otimes_R K(x)$ are homogeneous, hence the complex decomposes as direct sum of complexes of vector spaces:

$$(M \otimes_k \bigwedge^{i+1} W)_j \to (M \otimes_k \bigwedge^{i} W)_j \to (M \otimes_k \bigwedge^{i-1} W)_j.$$

This finishes the proof. □

1.2 Nielsen Resolutions and Modules of Finite Length

Let M be any module over any ring R. Then a chain $M = M_0 \supset M_1 \supset \dots \supset M_n$ of submodules of M is called a *composition series* if all M_j/M_{j+1} are nonzero simple modules. The *length of* M is the least length of a composition series of M, or ∞ if there is no finite such series. M is called a module *of finite length* if there is a finite composition series of M. The Jordan-Hölder theorem gives that the length of any finite composition series is the same.

Proposition 1.2.1. *M is of finite length if and only if M is Artinian and Noetherian.*

Proof. [Eis94, Theorem 2.13] □

From now on throughout this section let k be any field. Let $R = \operatorname{Sym}(V)$ be the weighted polynomial ring as above in 1.1.6 with $V = (x_1, \dots, x_n)$ and $\deg x_l = d_l > 0$. Let W be another k-vectorspace such that $V = \operatorname{Hom}_k(W,k)$. Choose a basis $(\chi_1, \dots, \chi_n)$ of W such that the (χ_l) are a dual to (x_l) with $\deg \chi_l = d_l$.

We obtain the following direct Corollary of Proposition 1.2.1 as R is Noetherian.

Corollary 1.2.2. *Let M be any finitely generated R-module. Then M is of finite length if and only if M is Artinian.*

Remark 1.2.3 (Top degree)**.** If M is any graded R-module of finite length we understand by *top degree* the maximal j such that $M_j \neq 0$.

We state two free resolutions of a graded finite length module which are very strongly related. Note that they are non minimal in general. We need the constructions as a basic tool for our main theorem. They are both based on the construction of Nielsen ([Nie81]).

Theorem and Construction 1.2.4 (Nielsen I)**.** *Let M be a graded R-module of finite length. Let $A_i(M) = R \otimes_k \bigwedge^i W \otimes_k M$ be viewed as a graded R-module by left multiplication (i.e. $r' \cdot r \otimes w \otimes m = (r'r) \otimes w \otimes m$). Define the j-th graded part $(A_i(M))_j := \bigoplus_{j_1+j_2+j_3=j} R_{j_1} \otimes_k (\bigwedge^i W)_{j_2} \otimes_k M_{j_3}$.*
We define $\phi_i : A_i(M) \to A_{i-1}(M)$, $i = 1, \dots, n$, on generators by

$$r \otimes w \otimes \mathsf{m} \mapsto \sum_{l=1}^{n} x_l r \otimes (x_l \neg w) \otimes \mathsf{m} - \sum_{l=1}^{n} r \otimes (x_l \neg w) \otimes (x_l \cdot \mathsf{m}),$$

for all $r \in R, w \in \bigwedge^i W$ and $\mathsf{m} \in M$. Note that we skip any reference to M in the notation ϕ_i. Moreover it is clear from the context anyways.
$x_l \neg w$ denotes the usual contraction defined in 1.1.8. By $x_l \cdot \mathsf{m}$ is meant the multiplication in M.
Then

$$A_0(M) \xleftarrow{\phi_1} A_1(M) \leftarrow \dots \leftarrow A_{n-1}(M) \xleftarrow{\phi_n} A_n(M) \leftarrow 0$$

is a graded complex of free R-modules.

Proof. We split the complex from an schematic point of view into parts (note that $\phi_{i,0}(A_i(M)) \cap \phi_{i,1}(A_i(M)) = \emptyset$):

$$\begin{array}{ccccc} & & A_{i-1}(M) & \xleftarrow{\phi_{i,0}} & A_i(M) \\ & \swarrow^{\phi_{i,1}} & & \swarrow^{\phi_{i,1}} & \\ A_{i-2}(M) & \xleftarrow{\phi_{i-1,0}} & A_{i-1}(M) & & \end{array}$$

$\phi_{i,0}$ is defined by the special part of ϕ_i from above given by

$$r \otimes w \otimes \mathsf{m} \mapsto \sum_{l=1}^{n} (x_l r) \otimes (x_l \neg w) \otimes \mathsf{m},$$

and $\phi_{i,1}$ by

$$r \otimes w \otimes \mathsf{m} \mapsto -\sum_{l=1}^{n} r \otimes (x_l \neg w) \otimes (x_l \cdot \mathsf{m}).$$

Let us show that the above construction really gives a complex. At first we have to see that $\phi_{i,0} \circ \phi_{i+1,0} = 0$, this is because of the corresponding property of the Koszul complex. The same holds for $\phi_{i,1} \circ \phi_{i+1,1} = 0$:

$$\phi_{i,1} \circ \phi_{i+1,1}(r \otimes w_1 \wedge \ldots \wedge w_{i+1} \otimes \mathsf{m}) = \\ \sum_{l_2=1}^{n} \sum_{l_1=1}^{n} r \otimes (x_{l_2} \neg (x_{l_1} \neg w)) \otimes (x_{l_2} x_{l_1} \cdot \mathsf{m}) = 0.$$

This is true $(x_{l_2} \neg (x_{l_1} \neg w)) = -(x_{l_1} \neg (x_{l_2} \neg w))$ because of

$$\sum_{k_1=1, k_1 \neq k_2}^{i+1} \sum_{k_2=1}^{i+1} \delta_{(k_1,k_2)} x_{l_2}(w_{k_1}) x_{l_1}(w_{k_2}) (-1)^{k_1-1} (-1)^{k_2-1} w_1 \wedge \ldots \wedge \widehat{w_{k_1}} \wedge \ldots \wedge \widehat{w_{k_2}} \wedge \ldots \wedge w_{i+1} = \\ - \sum_{k_2=1, k_2 \neq k_1}^{i+1} \sum_{k_1=1}^{i+1} \delta_{(k_2,k_1)} x_{l_2}(w_{k_1}) x_{l_1}(w_{k_2}) (-1)^{k_1-1} (-1)^{k_2-1} w_1 \wedge \ldots \wedge \widehat{w_{k_1}} \wedge \ldots \wedge \widehat{w_{k_2}} \wedge \ldots \wedge w_{i+1},$$

with $\delta_{(k_1,k_2)} := \begin{cases} 1 & \text{if } k_1 < k_2 \\ -1 & \text{if } k_1 > k_2. \end{cases}$

Moreover we should see that $\phi_{i,0} \circ \phi_{i+1,1} = -\phi_{i,1} \circ \phi_{i+1,0}$.

$$\phi_{i,0} \circ \phi_{i+1,1}(r \otimes w \otimes \mathsf{m}) = - \sum_{l_2=1}^{n} \sum_{l_1=1}^{n} (x_{l_2} r) \otimes (x_{l_2} \neg (x_{l_1} \neg w)) \otimes (x_{l_1} \cdot \mathsf{m}) = \\ \sum_{l_2=1}^{n} \sum_{l_1=1}^{n} (x_{l_1} r) \otimes (x_{l_2} \neg (x_{l_1} \neg w)) \otimes (x_{l_1} \mathsf{m}) = -\phi_{i,1} \circ \phi_{i+1,0}(r \otimes w \otimes \mathsf{m}),$$

as $(x_{l_2} \neg (x_{l_1} \neg w)) = -(x_{l_1} \neg (x_{l_2} \neg w))$. □

Remark 1.2.5. If R is the trivially weighted polynomial ring, one can define $A_{(-j-i,j)}(M) := R \otimes \bigwedge^i W \otimes M_j$. Then the built complex is the total complex of the double complex obtained from

$$\ldots \leftarrow A_{(-j-i+1,j)}(M) \overset{\phi_{i,0}}{\leftarrow} A_{(-j-i,j)}(M) \overset{\phi_{i+1,0}}{\leftarrow} A_{(-j-i-1,j)}(M) \leftarrow \ldots$$

and

$$\ldots \leftarrow A_{(-j-i,j+1)}(M) \overset{\phi_{i,1}}{\leftarrow} A_{(-j-i,j)}(M) \overset{\phi_{i+1,1}}{\leftarrow} A_{(-j-i,j-1)}(M) \leftarrow \ldots.$$

(Compare [Nie81, 1.5])

The second complex construction respects the R-module structure of M in a different way. Before stating it we need to show a lemma at first. The setup is the following:

Notation 1.2.6 (Diagonal Action)**.** Let M be a graded module of finite length over R. Let F be a graded R-module. Let $A = F \otimes_k M$ be the k-vectorspace product.
Then we consider the R-module ${}_{\Delta}A = {}_{\Delta}(F \otimes_k M)$ with the *diagonal action*

$$\Delta(x): \quad x \cdot e \otimes \mathsf{m} := (1 \otimes x) + (x \otimes 1)\, e \otimes \mathsf{m} := e \otimes x\mathsf{m} + xe \otimes \mathsf{m}$$

for all $x \in V \setminus 0$, $e \in F$, $\mathsf{m} \in M$, and its linear extension. If the context is clear then the delta ${}_{\Delta}$ should indicate that we mean a module built with respect to this diagonal action. ${}_{\Delta}A$ is graded with respect to $({}_{\Delta}A)_j = \bigoplus_{j_1+j_2=j} F_{j_1} \otimes M_{j_2}$.

If F is free ${}_{\Delta}A$ is again a free module:

Lemma 1.2.7. *Let F be a graded free R-module, and let $(f_1, \ldots, f_\mu)$ be a homogeneous basis of F. Chosen a homogeneous k-vectorspace basis $(\mathsf{m}_1, \ldots, \mathsf{m}_\nu)$ of M we get that*

$$B = (f_i \otimes \mathsf{m}_j)_{(i,j)}$$

is a homogeneous basis of ${}_{\Delta}A$,i.e. ${}_{\Delta}A$ is free R-module of rank ${}_{\Delta}A = \mu \cdot \nu$.

Proof. As the tensor product commutes with direct sums it is enough to show that the module ${}_{\Delta}A = {}_{\Delta}(R \otimes_k M)$ is free.

Let $r \otimes \mathsf{m} \in {}_{\Delta}A$. Let $r = x^\alpha$ and $|\alpha| = \sum_{l=1}^n \alpha_l$. We show: There is an expression $r \otimes \mathsf{m} = \sum_j r_j(1 \otimes \mathsf{m}_j)$ with $r_j \in R$. The proof is an induction on $|\alpha|$. If $|\alpha| = 0$ we are done. If $|\alpha| > 0$ let $\alpha_l > 0$. Then $r \otimes \mathsf{m} = x_l \cdot (x^{(\alpha_1,\ldots,\alpha_l-1,\ldots,\alpha_n)} \otimes \mathsf{m}) - x^{(\alpha_1,\ldots,\alpha_l-1,\ldots,\alpha_n)} \otimes (x_l\mathsf{m})$. Use induction on $x^{(\alpha_1,\ldots,\alpha_{l-1}-1,\ldots,\alpha_n)} \otimes \mathsf{m}$ and on $x^{(\alpha_1,\ldots,\alpha_l-1,\ldots,\alpha_n)} \otimes (x_l\mathsf{m})$. If r is arbitrary use linearity. Hence the $(1 \otimes \mathsf{m}_i)$ generate ${}_{\Delta}A$.
Now assume $\sum_i r_i \cdot (1 \otimes \mathsf{m}_i) = 0$ with $r_i \in R$. And let $I \subset \{1, \ldots, \nu\}$ be the subset such that the total degree $\deg r_i \geq \deg r_j$ for all $i \in I$ and all $j \in \{1, \ldots, \nu\}$.
Moreover let $r_i = \tilde{r}_i + \tilde{\tilde{r}}_i$, such that $\tilde{r}_i$ is the part of the highest degree terms.
Now we have

$$0 = \sum_i r_i(1 \otimes \mathsf{m}_i) = \sum_{i \in I} \tilde{r}_i \otimes \mathsf{m}_i + \ldots + 1 \otimes (\tilde{r}_i\mathsf{m}_i) + \sum_{i \in I} \tilde{\tilde{r}}_i \otimes \mathsf{m}_i + \ldots + 1 \otimes (\tilde{\tilde{r}}_i\mathsf{m}_i) + \\ + \sum_{i \notin I} r_i \otimes \mathsf{m}_i + \ldots + 1 \otimes (r_i\mathsf{m}_i).$$

The $\{\tilde{r}_i \otimes \mathsf{m}_i\}_{i \in I}$ are of the highest total degree in the first factor of the tensor product. Hence the equation gives

$$\sum_{i \in I} \tilde{r}_i \otimes \mathsf{m}_i = 0.$$

As the m_i are linearly independent we have $\tilde{r}_i = 0$ for all $i \in I$. By the choice of I it follows that $r_j = 0$ for all j. Hence the claim follows. □

We present another complex construction. Its differentials are totally based on the Koszul complex. It is in the spirit of the Nielsen I construction but with respect to the diagonal action.
We have a free resolution of M as graded R-module:

Theorem and Construction 1.2.8 (Nielsen II)**.** *Let M be a graded R-module of finite length. Let ${}_\Delta A_i(M) = {}_\Delta(R \otimes \bigwedge^i W \otimes M)$ be the graded R-module with the diagonal action $x \cdot (r \otimes w \otimes \mathsf{m}) := (xr) \otimes w \otimes \mathsf{m} + r \otimes w \otimes (x\mathsf{m})$ for all $x \in V \setminus \{0\}$ and its linear extension. Then*

$$0 \leftarrow M = {}_\Delta(k \otimes M) \leftarrow^{\mathrm{pr}} {}_\Delta(A_0(M)) \leftarrow^{{}_\Delta(\phi_1)} {}_\Delta(A_1(M)) \leftarrow \ldots \leftarrow^{{}_\Delta(\phi_n)} {}_\Delta(A_n(M)) \leftarrow 0$$

is a graded free resolution of M where ${}_\Delta(\phi_i)$ is defined as in the Koszul complex by

$$r \otimes w \otimes \mathsf{m} \mapsto \sum_{l=1}^{n} (x_l r) \otimes (x_l \neg w) \otimes \mathsf{m}.$$

Note that we skip any reference to M in the ${}_\Delta(\phi_i)$, otherwise we would overstress the notation.

Proof. First of all the differentials are linear: ${}_\Delta(\phi_i)(x \cdot (r \otimes w \otimes \mathsf{m})) = {}_\Delta(\phi_i)(xr \otimes w \otimes \mathsf{m} + r \otimes w \otimes (x\mathsf{m})) = \sum_{l=1}^n x_l xr \otimes (x_l \neg w) \otimes \mathsf{m} + \sum_{l=1}^n x_l r \otimes (x_l \neg w) \otimes (x\mathsf{m}) = x \cdot {}_\Delta(\phi_i)(r \otimes w \otimes \mathsf{m})$ for all $x \in V \setminus \{0\}$.
The exactness follows by the exactness of the Koszul complex $K(x)$, as the differential behaves as $\dim_k M$ copies of the Koszul differentials.
Moreover we have on M the usual action as $x \cdot \mathsf{m} = (1 \otimes x) + (x \otimes 1)(1 \otimes \mathsf{m}) = 1 \otimes x\mathsf{m} + x \otimes \mathsf{m} = 1 \otimes x\mathsf{m}$ in ${}_\Delta(k \otimes_k M) \cong {}_\Delta A_0 / \operatorname{Im} {}_\Delta(\phi_1)$ for all $x \in V$. □

Let us separately consider the construction using the dual Koszul complex.

Corollary 1.2.9 (Nielsen IIa)**.** *As in the Construction 1.2.8 (Nielsen II) we consider free modules via the diagonal action. Let all assumptions be as in 1.2.8, and let $(\quad)^\vee = \operatorname{Hom}_R(\quad, R \otimes \bigwedge^n W)$. By $B_i(M) = (R \otimes \bigwedge^i W)^\vee \otimes M$, with $\bigoplus_{j_1+j_2=j} (R \otimes \bigwedge^i W)^\vee_{j_1} \otimes M_{j_2}$, we denote the graded free module with the left multiplication, i.e. $r' \cdot (\pi \otimes \mathsf{m}) = (r'\pi) \otimes \mathsf{m}$.*

$${}_\Delta(B_i(M)) = {}_\Delta((R \otimes_k \bigwedge^i W)^\vee \otimes_k M),$$

is the module together with the diagonal action. Let $\delta_i^\vee : (R \otimes_k \bigwedge^{i-1} W)^\vee \to (R \otimes_k \bigwedge^i W)^\vee$ be the differentials from the dual Koszul complex (see 2.3.3). We denote by ${}_\Delta(\varphi_i)$ the homomorphisms ${}_\Delta(B_{i-1}(M)) \to {}_\Delta(B_i(M))$ defined by

$$\pi \otimes \mathsf{m} \mapsto \delta_i^\vee(\pi) \otimes \mathsf{m},$$

for all $\pi \in (R \otimes \bigwedge^{i-1} W)^\vee$ *and* $\mathsf{m} \in M$. *We do not refer to* M *in the notation of* ${}_{\Delta}(\varphi_i)$*: The dependence is clear from the context. Moreover fix the canonical* k*-vectorspace isomorphism*

$$f : \operatorname{Hom}_k(\bigwedge^n W, \bigwedge^n W) \xrightarrow{\cong} k, \mathrm{id} \mapsto 1.$$

Then

$$0 \leftarrow M = {}_{\Delta}(k \otimes M) \overset{f \otimes \mathrm{id}}{\longleftarrow} {}_{\Delta}(B_n(M)) \overset{{}_{\Delta}(\varphi_n)}{\longleftarrow} \ldots \leftarrow {}_{\Delta}(B_1(M)) \overset{{}_{\Delta}(\varphi_1)}{\longleftarrow} {}_{\Delta}(B_0(M)) \leftarrow 0$$

is a graded free resolution of M.

Proof. The freeness of the ${}_{\Delta}(B_i(M))$ is guaranteed by Lemma 1.2.7. Moreover it is clear that the only non vanishing homology of the complex is H_0.
$f \otimes \mathrm{id}$ is meant as follows: We know that $R \otimes_k \operatorname{Hom}_k(\bigwedge^n W, \bigwedge^n W) \cong \operatorname{Hom}_R(R \otimes \bigwedge^n W, R \otimes \bigwedge^n W)$ given by $r \otimes \pi \mapsto (r' \otimes w \mapsto rr' \otimes \pi(w))$. Call this map γ. Then consider

$${}_{\Delta}(B_n(M)) \overset{\gamma^{-1} \otimes \mathrm{id}}{\rightarrow} {}_{\Delta}(R \otimes_k \operatorname{Hom}_k(\bigwedge^n W, \bigwedge^n W) \otimes_k M) \overset{\mathrm{id} \otimes f \otimes \mathrm{id}}{\rightarrow} {}_{\Delta}(R \otimes_k k \otimes_k M).$$

In this sense the complex is a graded free resolution of M. □

Now we need to understand the connection between both complex constructions.

Remark 1.2.10. Both complex contructions (Nielsen I and Nielsen II) are isomorphic in a canonical way. Let $i \geq 1$, M a graded R-module of finite length, and let the notation be as in the above constructions. Then the following diagrams commute for all i:

$$\begin{array}{ccc} A_i(M) & \xrightarrow{\epsilon_i} & {}_{\Delta}(A_i(M)) \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle {}_{\Delta}(\phi_i)} \\ A_{i-1}(M) & \xrightarrow{\epsilon_{i-1}} & {}_{\Delta}(A_{i-1}(M)), \end{array}$$

where $\epsilon_i : A_i(M) \rightarrow {}_{\Delta}(A_i(M))$ is defined by $r \otimes w \otimes \mathsf{m} \mapsto r(1 \otimes w \otimes \mathsf{m})$ for all $\mathsf{m} \in M, w \in \bigwedge^i W$ and $r \in R$. The diagram is commutative as on the one hand

$$r \otimes w \otimes \mathsf{m} \overset{\epsilon_i}{\mapsto} r(1 \otimes w \otimes \mathsf{m}) \overset{{}_{\Delta}(\phi_i)}{\mapsto} r \sum_{l=1}^{n} x_l \otimes (x_l \neg w) \otimes \mathsf{m},$$

and on the other hand

$$r \otimes w \otimes \mathsf{m} \overset{\phi_i}{\mapsto} \sum_{l=1}^{n} (r x_l) \otimes (x_l \neg w) \otimes \mathsf{m} - \sum_{l=1}^{n} r \otimes (x_l \neg w) \otimes (x_l \mathsf{m}) \overset{\epsilon_{i-1}}{\mapsto}$$

$$\sum_{l=1}^{n} (r x_l)(1 \otimes (x_l \neg w) \otimes \mathsf{m}) - \sum_{l=1}^{n} r(1 \otimes (x_l \neg w) \otimes (x_l \mathsf{m})) = r \sum_{l=1}^{n} x_l \otimes (x_l \neg w) \otimes \mathsf{m}.$$

Moreover the ϵ_i are isomorphisms as they map bases to bases by Lemma 1.2.7.

Corollary 1.2.11 (Nielsen I)**.** *The first complex construction (Nielsen I),*

$$0 \leftarrow M \leftarrow A_0(M) \stackrel{\phi_1}{\leftarrow} A_1(M) \leftarrow \ldots \leftarrow A_{n-1}(M) \stackrel{\phi_n}{\leftarrow} A_n(M) \leftarrow 0,$$

is a graded free resolution of the given graded finite length module M.

Proof. By the remark the construction Nielsen I 1.2.4 is isomorphic to 1.2.8 which resolves M. □

Chapter 2

Zero Dimensional Modules over the Polynomial Ring and Gorenstein Modules

Macaulay [Mac16] laid the foundations of the bijection between the set of graded Artinian Gorenstein quotients of the polynomial ring and the set of homogeneous divided power polynomials modulo the action of the base field (see A.27).
In this Chapter we use matrices with entries in divided powers to construct Gorenstein modules of finite length. The first two sections are devoted to general finite length modules and to Gorenstein modules, which will be defined in Section 2.2. Sections 2.3 and 2.5 provide the statement and proof of our main theorem: A finite length module over the polynomial ring is Gorenstein if and only if its resolution is selfdual in a strong sense.
As the construction in 2.3 has a bunch of applications these are described in 2.4. For example the proof of the dependence of the monoid of Betti tables of graded Cohen-Macaulay modules over the polynomial ring on the characteristic of the base field can be found there.

2.1 Graded Modules of Finite Length via Divided Powers

The aim of this section is to introduce and develop the theory of modules defined by matrices in divided powers. Troughout this section, let k be a field of arbitrary characteristic and $V = \langle x_1, \ldots, x_n \rangle$ be a k-vectorspace such that $\deg x_l = d_l \geq 1$. Moreover, let $R = \mathrm{Sym}(V)$ and let $\mathcal{D} = \mathrm{grHom}_k(R, k)$. As k-vectorspace $\mathcal{D}$ can be generated by divided powers $(X^{(u)})$ which form a dual basis to $(x^u = x_1^{u_1} \cdots x_n^{u_n})$. The theory of divided powers is recalled in Appendix A.2. We use the notations fixed there. Moreover, we know from Appendix A.2 that $\mathcal{D}$ carries via contraction the structure of a graded R-module.

Obviously $\mathcal{D}$ is not finitely generated as R-module. Nevertheless we consider direct sums of $\mathcal{D}$s and then concentrate on finitely generated submodules of depth $= 0$.

Note that in the category of graded rings and modules we always mean by a homomorphism a homogeneous homomorphism.

Definition 2.1.1. Let b_j, $j = 1, \ldots, p$, be integers and let $\mathcal{F} = \bigoplus_{j=1}^{p} \mathcal{D}(b_j)$ be the direct sum of p shifted R-modules $\mathcal{D}$. Let $G = \bigoplus_{j=1}^{p} R(-b_j)$. Let $\phi \in G$ and $f \in \mathcal{F}$ then

$$\langle f, \phi \rangle := \sum_{j=1}^{p} \phi_j \cdot f_j.$$

Recall that $\phi_j \cdot f_j$ means the contraction action defined in A.24.

Our first aim is to define quotients of annihilators of homogeneous matrices over $\mathcal{D}$. Now let the a_i and b_j be non-negative integers and let

$$P \in \mathrm{Hom}_R \left(\bigoplus_{j=1}^{p} R(b_j), \bigoplus_{i=1}^{q} \mathcal{D}(a_i) \right)$$

a homogeneous homomorphism, represented by a homogeneous matrix with entries such that $\deg P_{i,j} = -b_j + a_i$. That means the a column P^j represents $P(e_j) \in \bigoplus_{i=1}^q \mathcal{D}(a_i)$, where e_j denotes the j-th coordinate vector of $\bigoplus_{j=1}^p R(b_j)$. Moreover if e^i denotes the i-th coordinate vector of $\bigoplus_{i=1}^q \mathcal{D}(a_i)$, then $P(e_j) = \sum_{i=1}^q P_{i,j} e^i$, where $P_{i,j} \in \mathcal{D}_{-b_j+a_i}$.

The big advantage of the *P-notation* is that we are able to describe (naturally finitely generated) finite length submodules of the infinitely generated R-module $\bigoplus_{i=1}^q \mathcal{D}(a_i)$.

Definition 2.1.2. We define the *annihilator of P in R* to be

$$\operatorname{Ann}_R(P) := \left\{ \phi \in \bigoplus_{j=1}^p R(b_j) \;\middle|\; \langle P_i, \phi \rangle = 0 \text{ for all } i \right\}$$

where the P_i denote the rows of P considered as $P_i \in \bigoplus_{j=1}^p \mathcal{D}(-b_j)$.

Remark 2.1.3. Clearly the annihilator is an R-module as $\langle P_i, r\phi \rangle = r \langle P_i, \phi \rangle$.
Moreover $\operatorname{Ann}_R(P)$ is up to isomorphism independent from the matrix representation of P: Let $A \in \operatorname{Aut}_R(\langle e^1, \ldots, e^q \rangle \subset \bigoplus_{i=1}^q \mathcal{D}(a_i))$ and $B \in \operatorname{Aut}_R(\bigoplus_{j=1}^p R(b_j))$ then if $\langle P_i, \phi \rangle = 0$ for all i, then $\langle (AP)_i, \phi \rangle = 0$ for all i.
Moreover $\langle (PB)_i, B^{-1}(\phi) \rangle = 0$, hence $\operatorname{Ann}_R(PB) = B^{-1}(\operatorname{Ann}_R(P)) \cong \operatorname{Ann}_R(P)$.

We denote by $\langle f, \phi \rangle(0) = 0$ that via the projection $\mathcal{D} \xrightarrow{pr} \mathcal{D}_0 \cong k$ we have $pr(\langle f, \phi \rangle) = 0$.

Definition 2.1.4. Let $N \subset \bigoplus_{i=1}^q R(-a_i)$ be an R-submodule. Then let

$$N^\perp := \left\{ f \in \bigoplus_{i=1}^q \mathcal{D}(a_i) \;\middle|\; \langle f, \phi \rangle(0) = 0 \text{ for all } \phi \in N \right\}.$$

In fact we can express any graded R-module M of finite length by a matrix in divided powers. Consider the following situation: Let $d = \sum_{l=1}^n d_l$, and let p be minimal such that there are integers $(b_1, \ldots, b_p)$ with

$$\bigoplus_{j=1}^p R(b_j) \xrightarrow{\alpha} M \to 0.$$

As M if of finite length $\operatorname{Hom}_k(M, k) \cong \operatorname{Ext}_R^n(M, R(-d))$ (Lemma A.6). Hence especially $\operatorname{Hom}_k(M, k)$ is finitely generated as R-module. Again choose q minimal such that for some integers $(a_1, \ldots, a_q)$

$$\bigoplus_{i=1}^q R(-a_i) \xrightarrow{\beta} \operatorname{Hom}_k(M, k) \to 0.$$

We apply $\operatorname{grHom}_k(\quad, k)$ to the presentation and obtain as above a diagram

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \downarrow \\ M & \xrightarrow{\text{id}} & M \\ \alpha \uparrow & & \downarrow \beta^* \\ \oplus_{j=1}^{p} R(b_j) & \xrightarrow{P} & \oplus_{i=1}^{q} \mathcal{D}(a_i) \end{array} \qquad (1)$$

The matrix P here is defined to be $\beta^* \circ \alpha$. Hence the kernel of the matrix P considered as a map of graded R-module is $\mathrm{Ann}_R(P)$.

That motivates a new definition for the construction of modules of finite length. It is in the spirit of the Theorem of Macaulay, more exactly of the modern version (see A.27). Our central objects are quotients of annihilators $\mathrm{Ann}_R(P)$:

Definition 2.1.5. Let the quotient of P in R be

$$M(P) := \bigoplus_{j=1}^{p} R(b_j) \Big/ \mathrm{Ann}_R(P).$$

From the above diagram (1) one derives immediately the following theorem.

Theorem 2.1.6. *Let M be a graded R-module of finite length. Then there are integers $(a_1, \ldots, a_q)$, $(b_1, \ldots, b_p)$ and a $P \in \mathrm{Hom}_R(\bigoplus_{j=1}^{p} R(b_j), \bigoplus_{i=1}^{q} \mathcal{D}(a_i))$ such that*

$$M \cong M(P)$$

as graded R-modules.

Definition 2.1.7 (Associated matrix in divided powers)**.** Let M be a graded R-module of finite length. Let p and q be minimal integers such that there are integers $(a_1, \ldots, a_q)$ and $(b_1, \ldots, b_p)$ with $\oplus_{j=1}^{p} R(b_j) \to M \to 0$, and with $\oplus_{i=1}^{q} R(-a_i) \to \mathrm{Hom}_k(M, k) \to 0$.

Then we call any homogeneous matrix in divided powers

$$P \in \mathrm{Hom}_R(\bigoplus_{j=1}^{p} R(b_j), \bigoplus_{i=1}^{q} \mathcal{D}(a_i))$$

an *(to M) associated matrix in divided powers*, if $M(P) \cong M$ as homogeneous R-homomorphism. By Thereom 2.1.6 such a matrix always exists. Its uniqueness is examined later on.

Again let $P \in \mathrm{Hom}_R(\bigoplus_{j=1}^{p} R(b_j), \bigoplus_{i=1}^{q} \mathcal{D}(a_i))$ be any homogeneous matrix in divided powers.

Lemma 2.1.8. *The quotient of P in R, the above defined R-module $M(P)$, is of finite length.*

Proof. Over a Noetherian ring it is equivalent to show that $R/\operatorname{ann}(M(P))$ is Artinian ([Eis94, Corollary 2.17]). Every descending chain in $R/\operatorname{ann}(M(P))$ can be seen as chain of descending k-vectorspaces. Hence it is enough to see that $R/\operatorname{ann}(M(P))$ is finite dimensional as k-vectorspace. Let $N = |\min_{(i,j)} \deg P_{i,j}|$. Then for all $r \in R_l$ with $l > N$ we have that $re_j \in \operatorname{Ann}_R(P)$ for all $j = 1, \ldots, p$. Hence $r \in \operatorname{ann}(M(P))$ and therefore $\overline{r} = 0$ in $R/\operatorname{ann}(M(P))$. □

Proposition 2.1.9. *The following isomorphism of graded R-modules is canonical:*

$$M(P) \cong \operatorname{Im}_R(P).$$

Proof. Consider the following exact sequence:

$$\bigoplus_{i=1}^{q} \mathcal{D}(a_i) \xleftarrow{P} \bigoplus_{j=1}^{p} R(b_j) \longleftarrow \operatorname{Ann}_R(P) \longleftarrow 0 \tag{2.1}$$

Hence we obtain for the image $\operatorname{Im}_R(P) \subset \bigoplus_{i=1}^{q} \mathcal{D}(a_i)$ that

$$\operatorname{Im}_R(P) \cong \bigoplus_{j=1}^{p} R(b_j) \Big/ \operatorname{Ann}_R(P) = M(P).$$

□

Remark 2.1.10. On can say P is the *multiplication matrix* of $M(P)$. Because multiplying the columns of P is the same as multiplying the generators of $M(P)$. The result is given in terms of divided powers and translated back by the isomorphism 2.1.9.

Theorem 2.1.11. *There is an isomorphism*

$$M(P^t) \cong \operatorname{Hom}_k(M(P), k)$$

as graded R-modules. Here P^t denotes the transposed matrix of P, defining an element in $\operatorname{Hom}_R(\bigoplus_{i=1}^{q} R(-a_i), \bigoplus_{j=1}^{p} \mathcal{D}(-b_j))$.

Proof. Now we consider the first part of the sequence 2.1 :

$$0 \longleftarrow \operatorname{Coker} P \xleftarrow{pr} \bigoplus_{i=1}^{q} \mathcal{D}(a_i) \xleftarrow{P} \bigoplus_{j=1}^{p} R(b_j) \longleftarrow \operatorname{Ann}_R(P) \longleftarrow 0 \tag{2.2}$$

Using $\operatorname{grHom}_k(\quad, k)$, which is left exact ([BH93]), we dualize the sequence to

$$0 \longrightarrow \operatorname{grHom}_k(\operatorname{Coker} P, k) \longrightarrow \bigoplus_{i=1}^{q} R(-a_i) \xrightarrow{P^* = \operatorname{Hom}_k(P,k)} \bigoplus_{j=1}^{p} \mathcal{D}(-b_j).$$

Note that we have used the canonical isomorphism implied by the definition of $\operatorname{grHom}_k(\quad, k)$ $\alpha : \oplus_{i=1}^q R(-a_i) \xrightarrow{\cong} \operatorname{grHom}_k(\oplus_{i=1}^q \mathcal{D}(a_i), k), r \mapsto (f \mapsto f(r))$.
Hence we have $(P^*)_{\mu,\nu} = \langle P^*(e_\nu), e_\mu\rangle = \langle \alpha(e_\nu) \circ P, e_\mu\rangle = \langle \sum_{j=1}^p \langle P^j, e_\nu\rangle e_j^*, e_\mu\rangle = \langle P^\mu, e_\nu\rangle = (P)_{\nu,\mu} = (P^t)_{\mu,\nu}$.
So $\operatorname{grHom}_k(\operatorname{Coker} P, k) \cong \ker P^* = \ker P^t = \operatorname{Ann}_R P^t$. That is why dualizing sequence 2.2 we obtain (note that $M(P)$ is of finite length and hence finite dimensional as k-vector space)

$$0 \longrightarrow \operatorname{Ann}_R(P^t) \longrightarrow \bigoplus_{i=1}^q R(-a_i) \longrightarrow \operatorname{grHom}_k(M(P), k) \longrightarrow 0.$$

Note that $\operatorname{Ext}_k^1((\operatorname{Coker} P)_\nu, k) = 0$ for all ν, hence the sequence is exact.
As $\operatorname{grHom}_k(N, k) = \operatorname{Hom}_k(N, k)$ for N finite dimensional as k-vector space we have

$$\operatorname{Hom}_k(M(P), k) \cong M(P^t).$$

□

Example 2.1.12. Let $R = k[x_1, x_2]$ with all weights 1.

Let
$$P = \begin{pmatrix} X_1^{(3)} & X_1^{(1)}X_2^{(1)} + X_2^{(2)} \end{pmatrix} \in \operatorname{Hom}_R(R(3) \oplus R(2), \mathcal{D}).$$

Then the R-module $R(3) \oplus R(2) \supset \operatorname{Ann}_R(P) =$

$$\left\langle \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ x_1 - x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix} \right\rangle.$$

As a k-vector space $M(P)$ can be represented by the following basis:

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \right).$$

Considering $\operatorname{Im}(P) \subset \mathcal{D}$, we give also a basis as k-vector space:

$$\left(\begin{pmatrix} X_1^{(3)} \end{pmatrix}, \begin{pmatrix} X_1^{(2)} \end{pmatrix}, \begin{pmatrix} X_1^{(1)} \end{pmatrix}, \begin{pmatrix} X_1^{(1)}X_2^{(1)} + X_2^{(2)} \end{pmatrix}, \begin{pmatrix} X_1^{(1)} + X_2^{(1)} \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix} \right)$$

We have the graded R-isomorphism $\phi : \operatorname{Im}(P) \to M(P)$ given by

$$\begin{pmatrix} X_1^{(3)} \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} X_1^{(1)}X_2^{(1)} + X_2^{(2)} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that for example

$$\phi(1) = \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix} = \begin{pmatrix} x_1^3 \\ 0 \end{pmatrix}$$

in $M(P)$ and that in $\mathrm{Im}(P)$

$$x_1^3 \cdot \left(X_1^{(3)} \right) = x_1 x_2 \cdot \left(X_1^{(1)} X_2^{(1)} + X_2^{(2)} \right).$$

For $\mathrm{Ann}_R(P^t) \subset R$ we obtain as generators:

$$\left\langle \left(x_1^4 \right), \left(x_1 x_2 - x_2^2 \right), \left(x_1^2 x_2 \right), \left(x_2^3 \right) \right\rangle.$$

One sees two things: $\mathrm{Ann}_R(P^t)^\perp = \mathrm{Im}(P)$ and $M(P^t)$ can be represented as k-vector space by the following basis:

$$\left(\left(1 \right), \left(x_1 \right), \left(x_1^2 \right), \left(x_1^3 \right), \left(x_1 x_2 \right), \left(x_2 \right) \right)$$

We can define a graded R-module isomorphism $\mathrm{Im}(P) \to \mathrm{Hom}_k(M(P^t), k)$ given by

$$\left(X_1^{(3)} \right) \longmapsto \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \left(X_1^{(3)} \right) \right\rangle (0) \right), \text{ and}$$

$$\left(X_1^{(1)} X_2^{(1)} + X_2^{(2)} \right) \longmapsto \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \left(X_1^{(1)} X_2^{(1)} + X_2^{(2)} \right) \right\rangle (0) \right)$$

for all $\mathsf{m} \in M(P^t)$.

We have also implemented an algorithm computing $M(P)$ for a given matrix P in [Macaulay2]. One finds it in B.1. Here we use this implementation to compute some more examples:

Examples 2.1.13. We compute four examples and check whether the Betti tables verify our theorem.

```
load "dualModule.m2"
kk=QQ;
R1=kk[x_1,x_2];
P1=map(R1^1,R1^{-3,-2},matrix {{x_1^3, x_1*x_2+x_2^2}});
P1t= (transpose P1)**R1^{-3};
MP1=dualModule(P1);
MP1t=dualModule(P1t);
betti res MP1
```

```
o1 = total: 1 3 2
         0: 1 . .
         1: . 1 .
         2: . 1 1
         3: . 1 1
```

```
betti res MP1t
```

```
o3 = total: 2 3 1
         0: 1 1 .
         1: 1 1 .
         2: . 1 .
         3: . . 1
```

Now we check a random matrix.

```
P2=random(R1^{0,-2,-3},R1^{-5,-8});
P2t= (transpose P2)**R1^{-8};
MP2=dualModule(P2);
MP2t=dualModule(P2t);
betti res MP2
```

```
o4 = total: 3 5 2
         0: 1 . .
         1: . . .
         2: 1 . .
         3: 1 3 .
         4: . 2 .
         5: . . 1
         6: . . .
         7: . . .
         8: . . 1
```

```
betti res MP2t
```

```
o6 = total: 2 5 3
         0: 1 . .
         1: . . .
         2: . . .
         3: 1 . .
         4: . 2 .
         5: . 3 1
         6: . . 1
         7: . . .
         8: . . 1
```

This time we consider more variables.

```
R2=kk[x_1,x_2,x_3];
P3=random(R2^{0},R2^{-3,-4});
P3t= (transpose P3)**R2^{-4};
MP3=dualModule(P3);
MP3t=dualModule(P3t);
betti res MP3
betti res MP3t
```

```
o6 = total: 1 6 7 2
         0: 1 . . .
         1: . . . .
         2: . 6 4 .
         3: . . 3 1
         4: . . . 1

o7 = total: 2 7 6 1
         0: 1 . . .
```

```
          1: 1 3 . .
          2: . 4 6 .
          3: . . . .
          4: . . . 1
```

Here is an example over a not trivially weighted polynomial ring.

```
R3=kk[x_1,x_2,x_3, Degrees=>{1,2,3}];
P4=random(R3^{0},R3^{-3,-4});
P4t= (transpose P4)**R3^{-4}
MP4=dualModule(P4);
MP4t=dualModule(P4t);
betti res MP4
betti res MP4t
```

```
o9 = total: 1 4 5 2
          0: 1 . . .
          1: . . . .
          2: . 1 . .
          3: . 2 . .
          4: . 1 1 .
          5: . . 3 .
          6: . . 1 1
          7: . . . 1

o10 = total: 2 5 4 1
          0: 1 . . .
          1: 1 1 . .
          2: . 3 . .
          3: . 1 1 .
          4: . . 2 .
          5: . . 1 .
          6: . . . .
          7: . . . 1
```

Let us describe now R-module homomorphisms from M to $\mathrm{Hom}(M,k)$. They can also be identified with matrices in divided powers.

Lemma and Definition 2.1.14. *Let $\beta : M \to \mathrm{Hom}(M,k)(-s)$ be a homogeneous R-module homomorphism for some integer s. Then we define an to β associated matrix in divided powers as follows:*
Let p be minimal such that $\alpha : \oplus_{j=1}^{p} R(b_j) \to M \to 0$ projects on the generators of M, then also $0 \to \mathrm{Hom}_k(M,k) \overset{\alpha^*=\mathrm{Hom}_k(\alpha,k)}{\longrightarrow} \oplus_{j=1}^{p}\mathcal{D}(-b_j)$ is exact because the functor $\mathrm{grHom}_k(\quad,k)$ is left exact and $\mathrm{grHom}_k(N,k) = \mathrm{Hom}_k(N,k)$ for N finite dimensional k-vectorspace ([BH93]).

Now let us define $P \in \mathrm{Hom}_R\left(\bigoplus_{j=1}^{p} R(b_j), \bigoplus_{j=1}^{p}\mathcal{D}(-b_j-s)\right)$ as follows:

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \downarrow \\ M & \xrightarrow{\beta} & \mathrm{Hom}_k(M,k)(-s) \\ \alpha \uparrow & & \downarrow \alpha^* \\ \oplus_{j=1}^{p} R(b_j) & \xrightarrow{P} & \oplus_{j=1}^{p} \mathcal{D}(-b_j - s) \end{array}$$

A matrix associated to P represents β with respect to a chosen basis of M and its dual in $\mathrm{Hom}(M,k)$*.*

If β is injective, hence an isomorphism, then moreover $M \cong M(P)$ as $\ker P = \ker \alpha^* \circ \beta \circ \alpha = \ker \alpha$*.*

About the uniqueness of an associated P to a module of finite length one can say the following:

Theorem 2.1.15. *Let P and $\tilde{P} \in \mathrm{Hom}_R(\bigoplus_{i=1}^{q} R(-a_i), \bigoplus_{j=1}^{p} \mathcal{D}(-b_j))$, such that $M(P) \cong M(\tilde{P})$. Let $(\quad)^* = \mathrm{grHom}_k(\quad, k)$.*
Then there are $\delta_1 \in \mathrm{Aut}_R(\bigoplus_{i=1}^{q} R(-a_i))$ and $\delta_2 \in \mathrm{Aut}_R(\bigoplus_{j=1}^{p} R(b_j))$, such that

$$P = \delta_2^* \circ \tilde{P} \circ \delta_1.$$

If the numbers of generators of $M(P)$ and $M(\tilde{P})$ are chosen minimally then δ_1 and δ_2 are isomorphisms.

Proof. Let $\gamma : M(\tilde{P}) \overset{\cong}{\to} M(P)$, hence $\gamma^* : \mathrm{Hom}_k(M(P), k) \overset{\cong}{\to} \mathrm{Hom}_k(M(\tilde{P}), k)$. By definition we have two diagrams

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \downarrow \\ M(P) & \xrightarrow{\mathrm{id}} & M(P) \\ \alpha_1 \uparrow & & \downarrow \beta_1^* \\ \oplus_{i=1}^{q} R(-a_i) & \xrightarrow{P} & \oplus_{j=1}^{p} \mathcal{D}(-b_j), \end{array} \qquad \begin{array}{ccc} 0 & & 0 \\ \uparrow & & \downarrow \\ M(\tilde{P}) & \xrightarrow{\mathrm{id}} & M(\tilde{P}) \\ \alpha_2 \uparrow & & \downarrow \beta_2^* \\ \oplus_{i=1}^{q} R(-a_i) & \xrightarrow{\tilde{P}} & \oplus_{j=1}^{p} \mathcal{D}(-b_j). \end{array}$$

As $\bigoplus_{i=1}^{q} R(-a_i)$ is projective we gain the following lifting δ_1:

$$\begin{array}{ccc} \oplus_{i=1}^{q} R(-a_i) & \xrightarrow{\gamma \circ \alpha_2} & M(P) \\ \delta_1 \uparrow & \nearrow_{\alpha_1} & \\ \oplus_{i=1}^{q} R(-a_i) & & \end{array} .$$

As γ is an isomorphism it follows that δ_1 can be chosen as an isomorphism.

If we consider $\mathrm{grHom}_k(\quad, k)$ we can also lift and define δ_2 as follows:

$$\begin{array}{ccc} \oplus_{j=1}^{p} R(b_j) & \xrightarrow{\gamma^{*-1} \circ \beta_2} & \mathrm{Hom}_k(M(P), k) \\ \delta_2 \uparrow & \nearrow_{\beta_1} & \\ \oplus_{j=1}^{p} R(b_j) & & \end{array} \text{, hence}$$

$$\begin{array}{ccc} \oplus_{j=1}^{p}\mathcal{D}(-b_j) & \xleftarrow{\beta_2^*\circ\gamma^{-1}} & M(P) \\ \downarrow{\scriptstyle \delta_2^*} & \swarrow{\scriptstyle \beta_1^*} & \\ \oplus_{j=1}^{p}\mathcal{D}(-b_j) & & \end{array} .$$

That means now $\delta_2^* \circ \tilde{P} \circ \delta_1 = \delta_2^* \circ \beta_2^* \circ \alpha_2 \circ \delta_1 = \beta_1^* \circ \gamma \circ \gamma^{-1} \circ \alpha_1 = P$. By definition the δ_i fulfill the mentioned property as isomorphism, because modulo the annihilators the α_i respectively the β_i are isomorphisms, too. □

We have implemented in [Macaulay2] an algorithm for computing P for a given module of finite length. On finds it in Appendix B.2. We use it for computing some examples.

Examples 2.1.16. The procedure `computeP` computes one P defining a given module of finite length. It uses the idea of the proof of 2.1.6.

```
load "dualModule.m2";
load "computeP.m2";
R1=QQ[x_1,x_2];
P=map(R1^{1:0,1:-1},R1^{-3,-2},matrix{{x_1^3,x_1*x_2+x_2^2},
{x_2^2,x_1}});
MP=dualModule(P);
Pn=computeP(MP);
MPn=dualModule(Pn);
HM=Hom(MP,MPn);
homomorphism(HM_{0})

o1 = {0} | 1 0 |
     {1} | 0 1 |
```

The isomorphism between `MP` and `MPn` is induced by the identity map. That is because `MP` comes with a system of generators, fixed by `P`. `dualModule` uses the same generators.

In the next example we compute via `computePsymm` a symmetric P for a module given by a symmetric P. It is the same P up to a scalar multiplication.

```
R=ZZ/101[x_0..x_2];
P=random(R^{2:0,2:-2},R^{2:-5,2:-3});
P2=P+map(R^{2:0,2:-2},,transpose P);
P3=map(R^{2:0,2:-2},R^{2:-5,2:-3},P2);
MP=dualModule(P3);
P4=computePsymm(MP);
MP2=dualModule(P4);
HM=Hom(MP,MP2);
homomorphism HM_{0}
```

```
o2 = {0} | 1 0 0 0 |
     {0} | 0 1 0 0 |
     {2} | 0 0 1 0 |
     {2} | 0 0 0 1 |

P3-(sub(leadCoefficient(P3_(0,0))/leadCoefficient(P4_(0,0)),R))*P4

o3 = 0
```

Here P3 and P4 are the same up to a scalar multiple. Using 2.1.15 we know that there is an isomorphism δ_1 such that $\texttt{P3} = \delta_1^* \circ \texttt{P4} \circ \delta_1$ as both P_s are symmetric. As again all generators are fixed δ_1 is multiple of the identity map.

We compute again a symmetric P, this time in four variables.

```
R=ZZ/101[x_0..x_3];
P=random(R^{2:0,4:-2},R^{2:-5,4:-3});
P2=P+map(R^{2:0,4:-2},,transpose P);
P3=map(R^{2:0,4:-2},R^{2:-5,4:-3},P2);
MP=dualModule(P3);
P4=computePsymm(MP);
P3-(sub(leadCoefficient(P3_(0,0))/leadCoefficient(P4_(0,0)),R))*P4

o4 = 0
```

2.2 The Gorenstein Case

We focus now on the Gorenstein case: We define within this section what we mean by calling a module Gorenstein. The main theorem of this section says basically that $M(P)$ is Gorenstein if and only if P is symmetric. Within this section R denotes again $\mathrm{Sym}(V)$ with $V = \langle x_1, \ldots, x_n \rangle_k$, $\deg x_l = d_l > 0$, k an arbitrary field, if not stated differently. Also, let $\mathcal{D} = \mathrm{grHom}_k(R, k)$.

Theorem 2.2.1. *Let M be a graded R-module of finite length. Moreover let $\tau : M \longrightarrow \mathrm{Hom}_k(M, k)(-s)$ be an isomorphism of graded R-modules for some integer s with the following property:*

$$\tau^* := \mathrm{Hom}_k(\tau, k) : M(s) \longrightarrow \mathrm{Hom}_k(M, k)$$

is such that

$$\tau^*(-s) = \tau.$$

Note that in the definition of τ^* we use the canonical isomorphism $\mathrm{Hom}_k(\mathrm{Hom}_k(M, k)(-s), k) \cong M(s)$.
Then there exists an

$$P \in \mathrm{Hom}_R\left(\bigoplus_{i=1}^{p} R(-a_i), \bigoplus_{i=1}^{p} \mathcal{D}(a_i - s)\right)$$

for some integers $p, a_1, \dots, a_p$ such that the transposed matrix P^t has the property

$$P^t(-s) = P$$

and $M \cong M(P)$.

Proof. We define P as in 2.1.14. Choose p minimal such that $\alpha : \oplus_{i=1}^p R(-a_i) \to M \to 0$ projects on the generators of M, then also $0 \to \mathrm{Hom}_k(M,k)(-s) \overset{\alpha^* = \mathrm{Hom}_k(\alpha,k)}{\longrightarrow} \oplus_{i=1}^p \mathcal{D}(a_i - s)$. P is defined by the diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \downarrow \\
M & \xrightarrow{\tau} & \mathrm{Hom}_k(M,k)(-s) \\
{\scriptstyle \alpha}\uparrow & & \downarrow{\scriptstyle \alpha^*} \\
\oplus_{i=1}^p R(-a_i) & \xrightarrow{P} & \oplus_{i=1}^p \mathcal{D}(a_i - s)
\end{array}$$

We consider the dual of the diagram (note: M is of finite length and by definition $\mathrm{grHom}_k(\mathcal{D}, k) \cong R$):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \uparrow \\
\mathrm{Hom}_k(M,k) & \xleftarrow{\tau^*} & M(s) \\
{\scriptstyle \alpha^*}\downarrow & & \uparrow{\scriptstyle \alpha} \\
\oplus_{i=1}^p \mathcal{D}(a_i) & \xleftarrow{P^*} & \oplus_{i=1}^p R(s - a_i).
\end{array}$$

Now we have

$$< P^*(e_j), e_i >=< \alpha^* \tau^* \alpha(e_j), e_i >=< \alpha^* \tau \alpha(e_j), e_i >= P_{i,j}$$

by assumption. Here e_i denotes the i-th unit vector of $\bigoplus_{i=1}^p R(-a_i)$, and e_l^* the l-th of $\bigoplus_{i=1}^p \mathcal{D}(a_i)$. Moreover

$$< P^*(e_j), e_i >=< e_j^{**} \circ P, e_i >=< \sum_{l=1}^p < P(e_l), e_j > e_l^*, e_i >=< P(e_i), e_j >= P_{j,i},$$

hence $P = P^t(-s)$. And as in Lemma 2.1.14 $M = M(P)$. □

Example 2.2.2. We begin with an example, coming already from a symmetric matrix, just to follow the steps in the proof.

Let $R = k[x_1, x_2]$, weights $\deg x_1 = \deg x_2 = 1$ and $\mathrm{char}\, k \neq 2$.

Let

$$P = \begin{pmatrix} X_1^{(2)} X_2^{(1)} & X_2^{(2)} \\ X_2^{(2)} & X_1^{(1)} \end{pmatrix} \in \mathrm{Hom}_R(R(3) \oplus R(2), \mathcal{D} \oplus \mathcal{D}(1)).$$

Then the R-module $R(3) \oplus R(2) \supset \mathrm{Ann}_R(P) =$

$$\left\langle \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ -x_1 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ -x_2 \end{pmatrix} \right\rangle.$$

As a k-vector space $M(P)$ can be represented by the following basis:

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}x_2\\0\end{pmatrix},\begin{pmatrix}x_1x_2\\0\end{pmatrix},\begin{pmatrix}x_1^2x_2\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\x_1\end{pmatrix},\begin{pmatrix}0\\x_2\end{pmatrix}\right)$$

We fix the isomorphism $\tau : M(P) \to M(P)^*(3) := \operatorname{Hom}_k(M(P), k)(3)$ given by

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix} \mapsto \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_1^{(2)}X_2^{(1)} \\ X_2^{(2)} \end{pmatrix} \right\rangle (0)\right) \text{ and } e_2 = \begin{pmatrix}0\\1\end{pmatrix} \mapsto \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_2^{(2)} \\ X_1^{(1)} \end{pmatrix} \right\rangle (0)\right).$$

Here m denotes an arbitrary element of $M(P)$.

Now let us check, if $\tau^*(3) = \tau$: First of all we are using the canonical isomorphism

$$\begin{array}{ccc} M(P)^{**}(-3) & \xrightarrow{\tau^*} & M(P)^* \\ \cong \uparrow & \nearrow & \\ M(P)(-3) & & \end{array}$$

and consider τ^* as starting from $M(P)(3)$.

Then

$$\tau^*(e_1) = \begin{cases} M \to k \\ \mathsf{m} \mapsto \tau(\mathsf{m})(e_1) \end{cases}.$$

So we have to check whether $\tau(\mathsf{m})(e_1) = \tau(e_1)(\mathsf{m})$ for all $\mathsf{m} \in M(P)$.

This is true, for example

$$\tau\begin{pmatrix}0\\x_1\end{pmatrix}(e_1) = \left\langle e_1, \begin{pmatrix}0\\1\end{pmatrix}\right\rangle(0) = 0 = \left\langle \begin{pmatrix}0\\x_1\end{pmatrix}, \begin{pmatrix} X_1^{(2)}X_2^{(1)} \\ X_2^{(2)} \end{pmatrix}\right\rangle(0) = \tau(e_1)\begin{pmatrix}0\\x_1\end{pmatrix},$$

and for $\tau^*(e_2)$ we obtain here

$$\tau\begin{pmatrix}0\\x_1\end{pmatrix}(e_2) = \left\langle e_2, \begin{pmatrix}0\\1\end{pmatrix}\right\rangle(0) = 1 = \left\langle \begin{pmatrix}0\\x_1\end{pmatrix}, \begin{pmatrix} X_2^{(2)} \\ X_1^{(1)} \end{pmatrix}\right\rangle(0) = \tau(e_2)\begin{pmatrix}0\\x_1\end{pmatrix}.$$

By the definition of the associated P we obtain the above P back, as

$$\alpha^*\left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_1^{(2)}X_2^{(1)} \\ X_2^{(2)} \end{pmatrix}\right\rangle(0)\right) = \begin{pmatrix} X_1^{(2)}X_2^{(1)} \\ X_2^{(2)} \end{pmatrix}$$

and

$$\alpha^*\left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_2^{(2)} \\ X_1^{(1)} \end{pmatrix}\right\rangle(0)\right) = \begin{pmatrix} X_2^{(2)} \\ X_1^{(1)} \end{pmatrix}.$$

Remark 2.2.3. Let again M be a graded R-module of finite length. Then we have already used the fact that $\operatorname{Hom}_k(M, k) \cong \operatorname{Ext}_R^n(M, R(-\sum_{l=1}^n d_l))$ canonically as graded R-modules.

This can be seen by using local cohomology at the maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. As M is of finite length $H_{\mathfrak{m}}^0(M) = M$. On the other hand using theorem A.6 we have in general $\operatorname{Hom}_k(H_{\mathfrak{m}}^i(M), k) = \operatorname{Ext}_R^{n-i}(M, R(-\sum_{l=1}^n d_l))$.

Now let us come up with a definition motivated by Theorem 2.2.1.

Definition 2.2.4 (Gorenstein)**.** Let M be a graded R-module of finite length as above. Then we call M *weakly Gorenstein* if there exists a graded isomorphism of R-modules

$$\tau : M \xrightarrow{\cong} \mathrm{Hom}_k(M, k)(-s)$$

for some integer s.
Let M be weakly Gorenstein, and let τ have the property that

$$\tau^* = \mathrm{Hom}_k(\tau, k) : M(s) \to \mathrm{Hom}_k(M, k)$$

is such that

$$\tau^*(-s) = \pm\tau.$$

Then we call M *strongly Gorenstein* or simply *Gorenstein*.

Theorem 2.2.5. *Let $P \in \mathrm{Hom}_R\left(\bigoplus_{i=1}^p R(-a_i), \bigoplus_{i=1}^p \mathcal{D}(a_i - s)\right)$ be symmetric, i.e. $P^t(-s) = P$, then there exists $\tau : M(P) \xrightarrow{\cong} \mathrm{Hom}_k(M(P), k)(-s)$ such that $\tau^*(-s) = \tau$.*

Proof. Let $\alpha : \bigoplus_{i=1}^p R(-a_i) \to M(P) \to 0$ be the projection and let $\alpha^* : 0 \to \mathrm{Hom}_k(M(P), k)(-s) \to \bigoplus_{i=1}^p \mathcal{D}(a_i - s)$ be the dual map.
Then define τ as follows:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \downarrow \\
M(P) & \xrightarrow{\tau} & \mathrm{Hom}_k(M(P), k)(-s) \\
\alpha \uparrow & & \downarrow \alpha^* \\
\oplus_{i=1}^p R(-a_i) & \xrightarrow{P} & \oplus_{i=1}^p \mathcal{D}(a_i - s).
\end{array}$$

Let $\mathsf{m} \in M(P)$ then choose an $\tilde{\mathsf{m}} \in \bigoplus_{i=1}^p R(-a_i)$ such that $\alpha(\tilde{\mathsf{m}}) = \mathsf{m}$. $P(\tilde{\mathsf{m}})$ is well-defined, as $\alpha(\tilde{\mathsf{m}} - \bar{\mathsf{m}}) = 0$ if and only if $\tilde{\mathsf{m}} - \bar{\mathsf{m}} \in \mathrm{Ann}_R(P)$, hence we have $P(\tilde{\mathsf{m}} - \bar{\mathsf{m}}) = 0$. Moreover we have $\langle P(\tilde{\mathsf{m}}), b\rangle = 0$ for all $b \in \mathrm{Ann}_R(P)$ as P is symmetric. Hence $P(\tilde{\mathsf{m}}) \in \mathrm{Im}\,\alpha^*$. Therefore we can define $\tau(\mathsf{m}) := (\alpha^*)^{-1}(P(\tilde{\mathsf{m}}))$.
τ is injective, because $P(\tilde{\mathsf{m}}) = 0$ if and only if $\tilde{\mathsf{m}} \in \mathrm{Ann}_R(P)$, but this is only the case if $\alpha(\tilde{\mathsf{m}}) = 0$ in M(P).
τ is surjective as it is as map of finite dimensional vectorspaces of the same dimension.
As τ^* is defined by the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \uparrow \\
\mathrm{Hom}_k(M(P), k) & \xleftarrow{\tau^*} & M(P)(s) \\
\alpha^* \downarrow & & \uparrow \alpha \\
\oplus_{i=1}^p \mathcal{D}(a_i) & \xleftarrow{P^* = P} & \oplus_{i=1}^p R(s - a_i)
\end{array}$$

we obtain $\tau^*(-s) = \tau$. □

Example 2.2.6. In this example we want to see, what happens in the case, in which P is non reduced. This means that a column of P is a combination of derivatives of the other columns.

Let $R = k[x_1, x_2]$ with weights $\deg x_1 = \deg x_2 = 1$, $\operatorname{char} k \neq 2$ and

$$P = \begin{pmatrix} X_1^{(3)} & X_1^{(2)} & X_2^{(1)} \\ X_1^{(2)} & X_1^{(1)} & 2 \\ X_2^{(1)} & 2 & 0 \end{pmatrix} \in \operatorname{Hom}_R(R(3) \oplus R(2) \oplus R(1), \mathcal{D} \oplus \mathcal{D}(1) \oplus \mathcal{D}(2)).$$

Here we have $(x_1 + 2x_2) \cdot P_1 = P_2$. The R-module $R(3) \oplus R(2) \oplus R(1) \supset \operatorname{Ann}_R(P) =$

$$\left\langle \begin{pmatrix} x_1 + 2x_2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \\ -x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix} \right\rangle.$$

We represent $M(P)$ by the following basis:

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

Let $\mathsf{m} \in M(P)$ be arbitrary. Hence $\tau : M(P) \to M(P)^*(3)$ can be described by

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_1^{(3)} \\ X_1^{(2)} \\ X_2^{(1)} \end{pmatrix} \right\rangle (0) \right) \text{ and } e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_2^{(1)} \\ 2 \\ 0 \end{pmatrix} \right\rangle (0) \right).$$

We consider τ^* and want to see $\tau^* = \tau$, for example we have to check $\tau(\mathsf{m})(x_2 \cdot e_1) = \tau(x_2 \cdot e_1)(\mathsf{m})$ for all $\mathsf{m} \in M(P)$ as in Example 2.2.2.

For e_3 we have

$$\tau(e_3)(x_2 \cdot e_1) = \left\langle \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} X_2^{(1)} \\ 2 \\ 0 \end{pmatrix} \right\rangle (0) = 1 = \left\langle e_3, x_2 \cdot \begin{pmatrix} X_1^{(3)} \\ X_1^{(2)} \\ X_2^{(1)} \end{pmatrix} \right\rangle (0) = \tau(x_2 \cdot e_1)(e_3).$$

Using τ let us compute a symmetric $\tilde{P}$ defining τ as in 2.1.14. Note that we have a minimal projection $\alpha : R(3) \oplus R(1) \to M(P) \to 0$ defined by $e_1 \mapsto \bar{e}_1 \in M(P)$ and $e_2 \mapsto \bar{e}_3 \in M(P)$. As in example 2.2.2 we have to compute

$$\alpha^* \left(\mathsf{m} \mapsto \left\langle \mathsf{m}, \begin{pmatrix} X_1^{(3)} \\ X_1^{(2)} \\ X_2^{(1)} \end{pmatrix} \right\rangle (0) \right) = \begin{pmatrix} X_1^{(3)} \\ X_2^{(1)} \end{pmatrix}$$

and

$$\alpha^*\left(\mathsf{m}\mapsto\left\langle \mathsf{m},\begin{pmatrix} X_2^{(1)}\\ 2\\ 0\end{pmatrix}\right\rangle(0)\right)=\begin{pmatrix} X_2^{(1)}\\ 0\end{pmatrix}.$$

Hence we obtain $\tilde{P}=\begin{pmatrix} X_1^{(3)} & X_2^{(1)}\\ X_2^{(1)} & 0\end{pmatrix}\in \mathrm{Hom}(R(3)\oplus R(1),\mathcal{D}\oplus\mathcal{D}(1))$.

Corollary 2.2.7. *Let M be a graded R-module of finite length. Assume there is a homogeneous isomorphism $\tau : M\to \mathrm{Hom}_k(M,k)(-s)$ for some integer s such that $\tau^*(-s)=-\tau$. Then there exists a $P\in\mathrm{Hom}_R(\bigoplus_{i=1}^p R(s-a_i),\bigoplus_{i=1}^p\mathcal{D}(a_i))$ for some integers $p,a_1,\dots,a_p$ with the properties $P^t(-s)=-P$ and $M\cong M(P)$.*

On the other hand if there exists a $P\in\mathrm{Hom}_R(\bigoplus_{i=1}^p R(s-a_i),\bigoplus_{i=1}^p\mathcal{D}(a_i))$ for some integers $p,a_1,\dots,a_p,s$ such that $P^t(-s)=-P$ and $M\cong M(P)$, then there exists a homogeneous R-isomorphism $\tau : M\to\mathrm{Hom}_k(M,k)(-s)$ such that $\tau^*(-s)=-\tau$.

Proof. The proof is analogues to the proofs of the Theorems 2.2.1 and 2.2.5. □

Remark 2.2.8. From Theorem 2.1.11 we know that any symmetric or skew symmetric matrix P provides an isomorphism $M(P)\cong\mathrm{Hom}(M(P),k)(s)$ for some s given by P.

Moreover in a second we see that weakly Gorensteiness does not imply Gorensteiness. More explicitly we see that Gorensteiness implies a kind of a symmetry for all multiplication forms at once. An example shows that this is not the case for a general weakly Gorenstein module of finite length.

The next lemma gives us a strong tool in the case that M is Gorenstein. Let $\tau : M\to \mathrm{Hom}_k(M,k)(-s)$ with $\tau^*=\mathrm{Hom}_k(\tau,k) : M(s)\to\mathrm{Hom}_k(M,k)$ such that $\tau^*(-s)=\pm\tau$. Let $M^*=\mathrm{Hom}_k(M,k)$. Assume that M is based in degree 0, i.e. $M_0\neq 0$ and $M_\iota=0$ for all $\iota<0$. Note that s is the top degree of M in the sense that the Hilbert function $\mathrm{HF_M}$ has the property $\mathrm{HF_M}(\iota)=0$ for all $\iota\geq s+1$ and $\mathrm{HF_M}(\iota)=\mathrm{HF_M}(s-\iota)$ for all $0\leq\iota\leq\lfloor\frac{s}{2}\rfloor$. The last property is true as $\dim_k M_\iota=\dim_k M^*(-s)_\iota=\dim_k M^*_{\iota-s}=\dim_k M_{s-\iota}$. It is naturally already true in the weakly Gorenstein case.

Lemma 2.2.9. *Let M be a graded R-module, of finite length, which is Gorenstein with two cases $\tau^*(-s)=\pm\tau$. Let $B=(b_i)$ be any homogeneous k-vector space basis of M. There is a homogeneous basis $\widetilde{B}=(\widetilde{b_i})$ of M such that for all integers $0\leq h\leq s$ and all $\iota\leq s-h$ the following property is satisfied: For all $a\in R_\iota$ the multiplication maps $M_h\xrightarrow{M^B_{\widetilde{B}}(\cdot a)_h}M_{h+\iota}$ and $M_{s-h-\iota}\xrightarrow{M^B_{\widetilde{B}}(\cdot a)_{s-h}}M_{s-h}$ have the property $M^B_{\widetilde{B}}(\cdot a)_h=\pm M^B_{\widetilde{B}}(\cdot a)^t_{s-h}$.*

Here B and $\widetilde{B}$ are restricted to the subspaces M_h and $M_{h+\iota}$, respectively $M_{s-h-\iota}$ and M_{s-h}.

Proof. Let τ^{-1} be the given R-module isomorphism $M^*(-s)\to M$. Choose $\widetilde{B}$ to be $(\widetilde{b_i}:=\tau^{-1}(b_i^*))$, where (b_i^*) denotes the dual k-vectorspace basis to B. Restricting τ^{-1}

we obtain a vectorspace isomorphism $\tau_h^{-1} : M_{-h}^* \oplus M_{-s+h+\iota}^* \xrightarrow{\cong} M_{s-h} \oplus M_{h+\iota}$. Hence $(\widetilde{b_i})_{b_i \in M_h \oplus M_{s-h-\iota}}$ is a vectorspace basis for $M_{s-h} \oplus M_{h+\iota}$. Restrict B and $\widetilde{B}$ in this way.

Define $M_{\widetilde{B}}^B(\cdot a)$ to be the vector space morphism $M_h \oplus M_{s-h-\iota} \xrightarrow{\cong} M_{s-h} \oplus M_{h+\iota}$ given by

$$\begin{pmatrix} 0 & M_{\widetilde{B}}^B(\cdot a)_{s-h} \\ M_{\widetilde{B}}^B(\cdot a)_h & 0 \end{pmatrix}.$$

Then $M_{\widetilde{B}}^B(\cdot a)_{ij} = \widetilde{b_i}^*(a \cdot b_j) = b_i^{**} \circ \tau(a \cdot b_j) = b_i^{**} \circ (a \cdot \tau)(b_j) = \pm b_i^{**} \circ (a \cdot \tau^*)(b_j) = \pm b_j^{**} \circ (a \cdot \tau)(b_i) = \pm b_j^{**} \circ \tau(a \cdot b_i) = \pm \widetilde{b_j}^*(a \cdot b_i) = M_{\widetilde{B}}^B(\cdot a)_{ji}$. □

Remark 2.2.10. In fact in the Gorenstein case the module M has a symmetric respectively skew symmetric dual form, defined by $M \times M \to k$ with $m \times m' \mapsto \tau(m)(m')$. So $\widetilde{B}$ is the dual base of B with respect to this form.

For a chosen vector space basis $B = (b_i)$ of M in the lemma we have defined $\widetilde{B} = (\tau^{-1}(b_i^*))$. For exactly the same result we can also define $\widetilde{B} = (\tau(b_i)^*)$. This means if we have chosen already B define $\widetilde{B}$ in the first way, call it B'. Fixing B' and applying the lemma again with the second definition we obtain $\widetilde{B'} = B$.

Let us give an example for a weakly Gorenstein, but not Gorenstein module.

Example 2.2.11. Let $R = k[x_0, x_1, x_2]$ with all weights 1 with $\operatorname{char} k \neq 2$, and

$$P = \begin{pmatrix} X_0^{(1)} & X_1^{(1)} & 0 & 0 & 0 \\ X_1^{(1)} & X_2^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1^{(1)} & X_2^{(1)} \\ 0 & 0 & -X_1^{(1)} & 0 & X_0^{(1)} \\ 0 & 0 & -X_2^{(1)} & -X_0^{(1)} & 0 \end{pmatrix} \in \operatorname{Hom}_R(R(1)^5, \mathcal{D}^5).$$

We have

$$\operatorname{Ann}_R(P) = \left\langle \begin{pmatrix} x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ -x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_0 \\ -x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -x_2 \\ x_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1 \\ 0 \\ x_0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 \\ -x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2 \\ 0 \end{pmatrix} \right\rangle,$$

and $\operatorname{Ann}_R(P^t) = \operatorname{Ann}_R(P)$.

Using $\operatorname{Hom}_k(M(P), k) \cong M(P^t)$ we can give an isomorphism $\tau : M \xrightarrow{\tau_1} \operatorname{Hom}_k(M(P), k)(-1)$ by $\tau_1 : M(P) \to M(P^t)(-1), \quad e_i \mapsto e_i$. Hence $M(P)$ is weakly Gorenstein.

Consider the basis B_0 of $M(P)_0$ given by the representatives of the standard vectors $B_0 = (e_1, e_2, e_3, e_4, e_5)$ and $\tilde{B}_1$ of $M(P)_1$ given by $(x_1e_2, x_2e_2, x_1e_3, x_2e_3, x_2e_5)$. We compute $M_{\tilde{B}_1}^{B_0}(x_i)$ for all i. Then by the above lemma and remark it is enough to show that there is no $G_5 \in Gl_5(k)$ such that $G_5 \cdot M_{\tilde{B}_1}^{B_0}(x_i)$ is symmetric respectively skew symmetric for all i.

$$M_{\tilde{B}_1}^{B_0}(x_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, M_{\tilde{B}_1}^{B_0}(x_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } M_{\tilde{B}_1}^{B_0}(x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If there were such a G_5 it would have to be of the form

$$\begin{pmatrix} * & 0 & * & 0 & 0 \\ * & g_{2,2} & * & g_{2,4} & g_{2,5} \\ * & \pm g_{2,4} & * & g_{3,4} & g_{3,5} \\ * & 0 & * & 0 & 0 \\ * & \pm g_{2,5} & * & \pm g_{3,5} & g_{5,5} \end{pmatrix},$$

because of $M_{\tilde{B}_1}^{B_0}(x_2)$. Considering for example $M_{\tilde{B}_1}^{B_0}(x_1)$ one sees that already $g_{4,1} = g_{2,5} = g_{5,1} = g_{5,3} = 0$. From $M_{\tilde{B}_1}^{B_0}(x_2)$ finally we get $g_{2,4} = g_{3,1} = g_{3,3} = g_{1,3} = g_{3,4} = g_{2,1} = g_{3,5} = 0$. That gives a zero column in G_5. That means it is impossible to choose G_5 from $Gl_5(k)$. Hence $M(P)$ is not Gorenstein.

2.3 A Structure Theorem

At the beginning of this section we recall and restate some symmetry properties of the Koszul complex. In this sense we continue our discussion concerning it from Chapter one. We use these properties in our symmetric resolution construction for Gorenstein modules of finite length over the weighted polynomial ring.
More explicitly: We prove our main Theorem 0.2 and one direction of Theorem 0.3 within this section, namely that any Gorenstein module has a selfdual minimal resolution in a strong sense.

Notation 2.3.1. In the following — if not stated differently — let k be any field. Let $R = k[x_1, \dots, x_n] = \mathrm{Sym}(V)$, $V = \mathrm{Hom}_k(W, k)$, where $W = (\chi_1, \dots, \chi_n)$, and let (x_l) be the dual vectorspace basis of (χ_l) with $\deg \chi_l = \deg x_l = d_l > 0$. Let $d = \sum_{l=1}^n d_l$.

Lemma and Definition 2.3.2 (Selfduality of the Koszul Complex I)**.** *Let $1 \le i \le n$.*
Consider $\alpha_i : R \otimes \bigwedge^i W \to (R \otimes \bigwedge^{n-i} W)^\vee$:

$$r \otimes w \mapsto \begin{cases} R \otimes \bigwedge^{n-i} W \to R \otimes \bigwedge^n W \\ (r' \otimes w') \mapsto (r' \cdot r) \otimes (w' \wedge w), \end{cases}$$

where $(\quad)^\vee = \mathrm{Hom}_R(\quad, R \otimes \bigwedge^n W)$. We denote by $(K(x))^\vee$ the dual Koszul complex, i.e. the complex resulting from applying $(\quad)^\vee$ to $K(x)$. Let $\ell(i) = \lfloor \frac{i-1}{2} \rfloor$. Then the following diagram is graded commutative (as the coherent diagram for $i = 1, \dots, n$):

$$\begin{array}{llllll} K(x): & \cdots \longrightarrow R\otimes \bigwedge^i W & \xrightarrow{\delta_i} & R\otimes \bigwedge^{i-1} W \longrightarrow \cdots \\ & \downarrow (-1)^{n-i}(-1)^{\ell(i)}\alpha_i & & \downarrow (-1)^{\ell(i)}\alpha_{i-1} \\ K(x)^\vee: & \cdots \longrightarrow (R\otimes \bigwedge^{n-i} W)^\vee & \xrightarrow{\delta^\vee_{n-i+1}} & (R\otimes \bigwedge^{n-i+1} W)^\vee \longrightarrow \cdots. \end{array}$$

Proof. We prove the claim on generators $R\otimes \bigwedge^i W \ni r\otimes w$:

$$r\otimes w \overset{\alpha_i}{\mapsto} \left\{ \begin{array}{c} R\otimes \bigwedge^{n-i} W \to R\otimes \bigwedge^n W \\ r'\otimes w' \mapsto r'r\otimes w'\wedge w \end{array} \right\} \overset{\delta^\vee_{n-i+1}}{\longmapsto} \left\{ \begin{array}{c} R\otimes \bigwedge^{n-i+1} W \to R\otimes \bigwedge^n W \\ r''\otimes w'' \mapsto \sum_{l=1}^n r''rx_l\otimes (x_l\neg w'')\wedge w \end{array} \right\}.$$

And the other way around:

$$(r\otimes w) \overset{\delta_i}{\mapsto} \sum_{l=1}^n rx_l\otimes(x_l\neg w) \overset{\alpha_{i-1}}{\mapsto} \left\{ \begin{array}{c} R\otimes \bigwedge^{n-i+1} W \to R\otimes \bigwedge^n W \\ (r''\otimes w'') \mapsto \sum_{l=1}^n r''rx_l\otimes w''\wedge (x_l\neg w). \end{array} \right.$$

Now the claim follows: Because of the linearity we may assume $w = \chi_1\wedge\ldots\wedge\chi_i$ and $w'' = \chi_{i+1}\wedge\ldots\wedge\chi_{\nu-1}\wedge\chi_l\wedge\chi_\nu\wedge\ldots\wedge\chi_n$ with $i+2\le\nu\le n$, $1\le l\le i$.

We show that $w''\wedge(x_l\neg w) = (-1)^{n-i}(x_l\neg w'')\wedge w$:

$$\begin{aligned} w''\wedge(x_l\neg w) = (-1)^{l-1}\chi_{i+1}\wedge\ldots\wedge\chi_l\wedge\ldots\wedge\chi_n\wedge\chi_1\wedge\ldots\wedge\widehat{\chi_l}\wedge\ldots\wedge\chi_i = \\ (-1)^{l-1}(-1)^{l-1}(-1)^{n-\nu+1}\chi_{i+1}\wedge\ldots\wedge\widehat{\chi_l}\wedge\ldots\wedge\chi_n\wedge\chi_1\wedge\ldots\wedge\chi_l\wedge\ldots\wedge\chi_i = \\ (-1)^{n-\nu+1}(-1)^{(\nu-1)-i}(x_l\neg w'')\wedge w = \\ (-1)^{n-i}(x_l\neg w'')\wedge w. \end{aligned}$$

This explains the $(-1)^{n-i}$ in the complex map. That means in the complex map we have a sign change at every second down arrow. The $(-1)^{\ell(i)}$ give the actual sign at the first down arrow of a square. □

This leads directly to some symmetry result of the Koszul complex:

Lemma 2.3.3 (Selfduality of the Koszul Complex II)**.** *Let n be odd and $m = \frac{n-1}{2}$. Consider the complex*

$$K:\ 0\to (R\otimes \bigwedge^0 W)^\vee \overset{\delta_1^\vee}{\to} \ldots (R\otimes \bigwedge^m W)^\vee \overset{\delta_{m+1}\circ\alpha^{-1}_{m+1}}{\to} R\otimes \bigwedge^m W \to R\to \ldots \overset{\delta_1}{\to} R\otimes \bigwedge^0 W\to 0$$

with $(\ \ )^\vee = \mathrm{Hom}_R(\ \ , R\otimes \bigwedge^n W)$.
If $n\equiv 3 \mod 4$ then the complex K is skew symmetric. That is if one chooses a homogeneous basis B for $R\otimes\bigwedge^m W$ and a dual basis $B^\vee$ for $(R\otimes \bigwedge^m W)^\vee$ then $\delta_{m+1}\circ\alpha^{-1}_{m+1}$ is skew with respect to these bases.
In the same manner K is symmetric if $n\equiv 1 \mod 4$.

Proof. At first we have to show that $\alpha_{m+1}: R\otimes \bigwedge^{m+1} W \overset{\alpha_{m+1}}{\to} (R\otimes \bigwedge^{n-m-1} W)^\vee$ and $\alpha_m: R\otimes \bigwedge^m W \overset{\alpha_m}{\to} (R\otimes\bigwedge^{n-m} W)^\vee$ have the property that $\alpha^\vee_{m+1} = \alpha_m$.

We have $n-m-1=m$, hence we have for

$$\alpha_{m+1}^{\vee}: R\otimes\bigwedge^{m} W \overset{\cong}{\to} (R\otimes\bigwedge^{m} W)^{\vee\vee} \overset{\alpha_{m+1}^{\vee}}{\to} (R\otimes\bigwedge^{m+1} W)^{\vee},$$

$$r'\otimes w' \mapsto (\gamma\mapsto\gamma(r'\otimes w')) \mapsto (r\otimes w\mapsto (r'\cdot r)\otimes(w'\wedge w)).$$

As $w'\wedge w=(-1)^{m(n-m)}w\wedge w'=w\wedge w'$ the first claim follows.

Now we use Lemma 2.3.2 for $i=m+1$. We have the commutative diagram:

$$\begin{array}{ccc} R\otimes\bigwedge^{m+1}W & \xrightarrow{\delta_{m+1}} & R\otimes\bigwedge^{m}W \\ \downarrow{\scriptstyle\alpha_{m+1}} & & \downarrow{\scriptstyle\alpha_m} \\ (R\otimes\bigwedge^{m}W)^{\vee} & \xrightarrow{(-1)^m\delta_{m+1}^{\vee}} & (R\otimes\bigwedge^{m+1}W)^{\vee}. \end{array}$$

Choose a basis B in $R\otimes\bigwedge^{m}W$ and a dual basis $B^{\vee}$ in $(R\otimes\bigwedge^{m}W)^{\vee}$. Then as $\alpha_{m+1}^{\vee}=\alpha_m$ we have

$$(M_B^{B^\vee}(\delta_{m+1}\circ\alpha_{m+1}^{-1}))^t = M_B^{B^\vee}((\delta_{m+1}\circ\alpha_{m+1}^{-1})^{\vee}) = M_B^{B^\vee}((\alpha_{m+1}^{-1})^{\vee}\circ\delta_{m+1}^{\vee}) = M_B^{B^\vee}(\alpha_m^{-1}\circ\delta_{m+1}^{\vee}) = (-1)^m M_B^{B^\vee}(\delta_{m+1}\circ\alpha_{m+1}^{-1}).$$

□

The above result is the basement for our later construction. But now let us recall some well known fact in our case: Even in the weakly Gorenstein case we have a symmetry of the Betti table.

Lemma 2.3.4. *Let $d=\sum_{l=1}^{n}d_l$. Let M be a graded weakly Gorenstein R-module of finite length such that $M\cong \operatorname{Ext}_R^n(M,R(-d))(-s)$ for some integer s. If F_i is an R-module, we write for the moment $F_i^{\vee}=\operatorname{Hom}_R(F_i,R)$. Let*

$$0\leftarrow M\leftarrow F_0=\bigoplus_{j\geq 0}R(-j)^{\beta_{0,j}}\leftarrow\ldots\leftarrow F_n=\bigoplus_{j\geq 0}R(-j)^{\beta_{n,j}}\leftarrow 0 \quad (*)$$

be its graded free resolution. Let $i\leq n$. Then $F_i\cong F_{n-i}^{\vee}(-s-d)$, i.e. $\beta_{i,j}=\beta_{n-i,d+s-j}$.

Proof. We apply the functor $\operatorname{Hom}_R(\quad,R(-d))$ to $(*)$ and obtain

$$0\leftarrow \operatorname{Ext}_R^n(M,R(-d))\leftarrow \operatorname{Hom}_R(F_n,R(-d))\leftarrow\ldots\leftarrow \operatorname{Hom}_R(F_0,R(-d))\leftarrow 0$$

which is a free resolution of $\operatorname{Ext}_R^n(M,R(-d))$ as all other Ext-groups vanish (seen by A.4 and A.6 as M is of finite length). By assumption $M\cong\operatorname{Ext}_R^n(M,R(-d))(-s)$. This extends to an isomorphism of complexes. Hence $F_i\cong\operatorname{Hom}_R(F_{n-i},R(-d))(-s)$. But

$$\operatorname{Hom}_R(F_{n-i},R(-d))(-s)=\operatorname{Hom}_R(F_{n-i},R)(-d-s)=F_{n-i}^{\vee}(-d-s).$$

For the equality $\beta_{i,j}=\beta_{n-i,d+s-j}$ note that

$$\operatorname{Hom}_R(F_{n-i},R)(-d-s)=\operatorname{Hom}_R(\bigoplus_{j\geq 0}R(-j)^{\beta_{n-i,j}},R)(-d-s)\cong\bigoplus_{j\geq 0}R(j-d-s)^{\beta_{n-i,j}}.$$

□

We are able to formulate and restate the one direction of our main Theorem 0.3 within this section now. It makes use of the first complex Construction 1.2.4 (Nielsen I) and gives a selfduality result for the resolution of a graded Gorenstein module M in a strong sense. That is for any free resolution of M the middle matrix with respect to dual bases is symmetric respectively skew symmetric.

Theorem 2.3.5 (Selfduality of the Resolution)**.** *Let n be odd, $m = \frac{n-1}{2}$.
Let M be a graded Gorenstein R-module with $\tau = \pm\tau^*(-s)$. Recall that $(\quad)^* = \mathrm{Hom}_k(\quad, k)$, and that $A_i(M) = R \otimes_k \bigwedge^i W \otimes_k M$. Define ϕ_i as in Theorem 1.2.4 (Nielsen I). Let $\beta_i : A_i(M) \to (A_{n-i}(M))^\vee$, with $(\quad)^\vee = \mathrm{Hom}_R(\quad, R \otimes \bigwedge^n W)(-s)$, be defined by*

$$r \otimes w \otimes \mathsf{m} \mapsto \begin{cases} A_{n-i}(M) \to R \otimes \bigwedge^n W \\ (r' \otimes w' \otimes \mathsf{m}') \mapsto (r'r) \otimes (w' \wedge w) \otimes \tau(\mathsf{m})(\mathsf{m}') \end{cases},$$

*for all $r, r' \in R, w \in \bigwedge^i W, w' \in \bigwedge^{n-i} W$ and $\mathsf{m}, \mathsf{m}' \in M$.
Then the graded free resolution of M*

$$K(M): \; 0 \to (A_0(M))^\vee \overset{\phi_1^\vee}{\to} \dots (A_m(M))^\vee \overset{\phi_{m+1}\circ\beta_{m+1}^{-1}}{\to} A_m(M) \to \dots \overset{\phi_1}{\to} A_0(M) \to M \to 0$$

is symmetric in the following sense: Let B be any homogeneous basis of $A_m(M)$ and $B^\vee$ its dual basis of $(A_m(M))^\vee$. Then the matrix representation of $\phi_{m+1} \circ \beta_{m+1}^{-1}$ has the property:

$$M_B^{B^\vee}(\phi_{m+1} \circ \beta_{m+1}^{-1}) = (-1)^m \pm (M_B^{B^\vee}(\phi_{m+1} \circ \beta_{m+1}^{-1}))^t.$$

(Here the $\pm$ denotes the sign of $\tau = \pm\tau^*(-s)$.)

Proof. As in the proof of Lemma 2.3.3 we have to see at first that $\beta_{m+1} : A_{m+1}(M) \overset{\beta_{m+1}}{\to} (A_m(M))^\vee$ and $\beta_m : A_m(M) \overset{\beta_m}{\to} (A_{m+1}(M))^\vee$ have the property $\beta_{m+1}^\vee = \pm\beta_m$.

We have again

$$\beta_{m+1}^\vee : A_m(M) \overset{\cong}{\to} (A_m(M))^{\vee\vee} \overset{\beta_{m+1}^\vee}{\to} (A_{m+1}(M))^\vee,$$

$$r' \otimes w' \otimes \mathsf{m}' \mapsto (\gamma \mapsto \gamma(r' \otimes w' \otimes \mathsf{m}')) \mapsto \begin{cases} A_{m+1}(M) \to R \otimes \bigwedge^n W, \\ r \otimes w \otimes \mathsf{m} \mapsto (r' \cdot r) \otimes (w' \wedge w) \otimes \tau(\mathsf{m})(\mathsf{m}'). \end{cases}$$

On the other hand β_m maps as follows:

$$r' \otimes w' \otimes \mathsf{m}' \mapsto \begin{cases} A_{m+1}(M) \to R \otimes \bigwedge^n W, \\ (r \otimes w \otimes \mathsf{m}) \mapsto (r' \cdot r) \otimes (w \wedge w') \otimes \tau(\mathsf{m}')(\mathsf{m}). \end{cases}$$

As $w \wedge w' = (-1)^{m(m+1)} w' \wedge w = w' \wedge w$ and $\tau(\mathsf{m}')(\mathsf{m}) = \pm\tau(\mathsf{m})(\mathsf{m}')$ by the Gorenstein property the claim follows. Note that $\tau(\mathsf{m}')(\mathsf{m}) = \pm\tau^*(\mathsf{m}'^{**})(\mathsf{m}) = \pm\mathsf{m}'^{**} \circ \tau(\mathsf{m}) = \pm\tau(\mathsf{m})(\mathsf{m}')$.
Here m'^{**} denotes the image of m' under the canonical isomorphism $M \to M^{**}$.

Let us show that the following diagram is commutative:

$$\begin{array}{ccc} R\otimes\bigwedge^{m+1}W\otimes M & \xrightarrow{\beta_{m+1}} & (R\otimes\bigwedge^{m}W\otimes M)^\vee \\ \downarrow{\scriptstyle\phi_{m+1}} & & \downarrow{\scriptstyle(-1)^m\phi_{m+1}^\vee} \\ R\otimes\bigwedge^{m}W\otimes M & \xrightarrow{\beta_m} & (R\otimes\bigwedge^{m+1}W\otimes M)^\vee. \end{array} \qquad (*)$$

On the one hand we map as follows:

$$r\otimes w\otimes \mathsf{m} \overset{\beta_{m+1}}{\mapsto} \begin{cases} A_{m+1}(M)\to R\otimes\bigwedge^n W, \\ (r'\otimes w'\otimes\mathsf{m}')\mapsto (r'r)\otimes(w'\wedge w)\otimes\tau(\mathsf{m})(\mathsf{m}'), \end{cases}$$

which is mapped further by $\phi_{m+1}^\vee$ to an element of $(A_{m+1}(M))^\vee$. This is the map $A_{m+1}(M)\to R\otimes\bigwedge^n W$:

$$(r''\otimes w''\otimes\mathsf{m}'')\mapsto\sum_{l=1}^n(x_lr''r)\otimes((x_l\neg w'')\wedge w)\otimes\tau(\mathsf{m})(\mathsf{m}'')-\sum_{l=1}^n(r''r)\otimes((x_l\neg w'')\wedge w)\otimes\tau(\mathsf{m})(x_l\mathsf{m}'').$$

On the other way we get

$$r\otimes w\otimes\mathsf{m}\overset{\phi_{m+1}}{\mapsto}\sum_{l=1}^n(x_lr)\otimes(x_l\neg w)\otimes\mathsf{m}-\sum_{l=1}^n r\otimes(x_l\neg w)\otimes(x_l\mathsf{m})\overset{\beta_m}{\mapsto}\begin{cases} A_{m+1}(M)\to R\otimes\bigwedge^n W, \\ (r''\otimes w''\otimes\mathsf{m}'') \end{cases}$$

$$\begin{cases} \\ \mapsto\sum_{l=1}^n(x_lrr'')\otimes(w''\wedge(x_l\neg w))\otimes\tau(\mathsf{m})(\mathsf{m}'')-\sum_{l=1}^n rr''\otimes(w''\wedge(x_l\neg w))\otimes\tau(x_l\mathsf{m})(\mathsf{m}''). \end{cases}$$

Hence because of $\tau(\mathsf{m})(x_l\mathsf{m}'')=\tau(x_l\mathsf{m})(\mathsf{m}'')$ the commutativity comes down to see that

$$(w''\wedge(x_l\neg w))=(-1)^m((x_l\neg w'')\wedge w).$$

But that is exactly the same computation as in the proof of Theorem 2.3.2 (the selfduality of the Koszul complex).

Now it follows with the first claim and the commutativity of (*) that

$$\begin{aligned} M_B^{B^\vee}(\phi_{m+1}\circ\beta_{m+1}^{-1})^t &= M_B^{B^\vee}((\phi_{m+1}\circ\beta_{m+1}^{-1})^\vee)=M_B^{B^\vee}((\beta_{m+1}^{-1})^\vee\circ\phi_{m+1}^\vee)= \\ &\pm M_B^{B^\vee}((\beta_m^{-1})\circ\phi_{m+1}^\vee))=\pm(-1)^m M_B^{B^\vee}(\phi_{m+1}\circ\beta_{m+1}^{-1}). \end{aligned}$$

□

In the following we compute an example. Our motivation to do so is to show that the proof is constructive. That means it leads to an algorithm for the computation of a selfdual resolution.

Example 2.3.6. We compute the middle skew symmetric map $\phi_2\circ\beta_2^{-1}$ in a concrete example:

Let $R=k[x_1,x_2,x_3]$ with $\deg x_i=1$ and $P=\begin{pmatrix} X_1^{(2)} & X_2^{(1)} \\ X_2^{(1)} & 1\end{pmatrix}\in\mathrm{Hom}_R(R(2)\oplus R(1),\mathcal{D}\oplus\mathcal{D}(1))$. Let $M=M(P)$.

As a first basis B of M we fix representatives:

$$B=(\mathsf{m}_1,\dots,\mathsf{m}_5)=\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}x_2\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}x_1^2\\0\end{pmatrix}\right).$$

Hence a dual basis $B^* = (\mu_1, \ldots, \mu_5)$ in $M^* = \operatorname{Im} P$ is given by

$$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}X_1^{(1)}\\0\end{pmatrix}, \begin{pmatrix}X_2^{(1)}\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}X_1^{(2)}\\X_2^{(1)}\end{pmatrix}\right).$$

Note that within this notation an element $\mu \in \mathcal{D} \oplus \mathcal{D}(1)$ stays for the functional

$$\begin{cases} M \to k \\ R(2) \oplus R(1) \ni \mathsf{m} \mapsto \langle \mu, \mathsf{m} \rangle(0). \end{cases}$$

Here $R(2) \oplus R(1) \ni \mathsf{m}$ denotes a representative of an element $\mathsf{m} \in M$.

Now we have to compute

$$\beta_2 : A_2(M) = R \otimes \bigwedge^2 W \otimes M \to (A_1(M))^\vee = (R \otimes \bigwedge^1 W \otimes M)^\vee :$$

$$1 \otimes \chi_i \wedge \chi_j \otimes \mathsf{m}_j \mapsto 1 \otimes (\_ \wedge \chi_i \wedge \chi_j) \otimes \tau(\mathsf{m}_j).$$

Note that the last expression stands for the functional

$$\begin{cases} R \otimes \bigwedge^1 W \otimes M \to R \otimes \bigwedge^3 W \\ r \otimes w \otimes \mathsf{m} \mapsto r \otimes (w \wedge \chi_i \wedge \chi_j) \otimes \tau(\mathsf{m}_j)(\mathsf{m}). \end{cases}$$

The map $\tau : M \to M^*(-2)$ can be computed by $\mathsf{m} \mapsto P(\mathsf{m})$.
We represent β_2 with respect to the bases

$$\widetilde{B} = (1 \otimes \chi_1 \wedge \chi_2 \otimes \mathsf{m}_1, 1 \otimes \chi_1 \wedge \chi_3 \otimes \mathsf{m}_1, 1 \otimes \chi_2 \wedge \chi_3 \otimes \mathsf{m}_1, \ldots, 1 \otimes \chi_2 \wedge \chi_3 \otimes \mathsf{m}_5)$$

and

$$A_1^\vee \supset \widehat{B} = (1 \otimes (\_ \wedge \chi_1 \wedge \chi_2) \otimes \mu_1, 1 \otimes (\_ \wedge \chi_1 \wedge \chi_3) \otimes \mu_1, 1 \otimes (\_ \wedge \chi_2 \wedge \chi_3) \otimes \mu_1, \ldots, 1 \otimes (\_ \wedge \chi_2 \wedge \chi_3) \otimes \mu_5).$$

We use the same abbreviatory notation as above. Hence we have the following matrix representation

$$M_{\widehat{B}}^{\widetilde{B}}(\beta_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1_3 \\ 0 & 1_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_3 & 0 \\ 0 & 0 & 1_3 & 1_3 & 0 \\ 1_3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where 0 is 3×3 zero matrix and 1_3 stands for the 3×3 unit matrix.

Now let us represent $\phi_2 : A_2(M(P)) \to A_1(M(P))$ again with respect to $\widetilde{B}$ and the dual basis to $\widehat{B}$,

$$\widehat{B}^* = (1 \otimes \chi_3 \otimes \mathsf{m}_1, 1 \otimes (-\chi_2) \otimes \mathsf{m}_1, 1 \otimes \chi_1 \otimes \mathsf{m}_1, \ldots, 1 \otimes \chi_1 \otimes \mathsf{m}_5).$$

We use the following abbreviations

$$K_3 = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}, C_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

In this notation we obtain

$$M^{\widetilde{B}}_{\widehat{B}^*}(\phi_2) = \begin{pmatrix} K_3 & 0 & 0 & 0 & 0 \\ C_1 & K_3 & 0 & 0 & 0 \\ C_2 & 0 & K_3 & 0 & 0 \\ 0 & 0 & 0 & K_3 & 0 \\ 0 & C_1 & 0 & C_2 & K_3 \end{pmatrix},$$

where 0 is again a 3×3 zero matrix.

Finally we obtain the skew symmetric matrix

$$M^{\widehat{B}}_{\widehat{B}^*}(\phi_2 \circ \beta_2^{-1}) = M^{\widetilde{B}}_{\widehat{B}^*}(\phi_2)(M^{\widetilde{B}}_{\widehat{B}}(\beta_2))^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & K_3 \\ 0 & K_3 & 0 & 0 & C_1 \\ 0 & 0 & -K_3 & K_3 & C_2 \\ 0 & 0 & K_3 & 0 & 0 \\ K_3 & C_1 & C_2 & 0 & 0 \end{pmatrix}.$$

The blocks can be seen as follows (for example the K_3 in the first column represents $\phi_{2,0,2} \circ \beta_2^{-1}$):

$$\begin{array}{ccc}
\longleftarrow A_{(-1,0)}(M) & \xleftarrow{\phi_{2,0}\circ\beta_2^{-1}} & (A_{(-3,2)}(M))^\vee \longleftarrow \\
& \swarrow_{\phi_{2,1}\circ\beta_2^{-1}} & \\
\longleftarrow A_{(-2,1)}(M) & \xleftarrow{\phi_{2,0}\circ\beta_2^{-1}} & (A_{(-2,1)}(M))^\vee \longleftarrow \\
& \swarrow_{\phi_{2,1}\circ\beta^{-1}} & \\
\longleftarrow A_{(-3,2)}(M) & \xleftarrow{\phi_{2,0}\circ\beta_2^{-1}} & (A_{(-1,0)}(M))^\vee \longleftarrow
\end{array}.$$

We can easily obtain a minimal free resolution with the same symmetry properties as in Theorem 2.3.5:

Corollary 2.3.7. *Let $n \geq 3$ be an odd integer, and let $m = \frac{n-1}{2}$. Let the characteristic of the base field k be* $\operatorname{char} k \neq 2$. *Let $d = \sum_{l=1}^{n} d_l$ the sum of the variable degrees, and let $f : \bigwedge^n W \to k(-d)$ be a fixed isomorphism. Let M be a graded Gorenstein module of finite length over R, the weighted polynomial ring. Let s be the top degree of M and*

$(\quad)^\vee = \mathrm{Hom}_R(\quad, R(-d-s))$.
Then there is a minimal graded free resolution of M,

$$0 \leftarrow M \leftarrow F_0 \stackrel{\psi_1}{\leftarrow} F_1 \leftarrow \ldots \leftarrow F_m \stackrel{\psi_{m+1}}{\leftarrow} (F_m)^\vee \leftarrow \ldots \leftarrow \stackrel{\psi_1^\vee}{\leftarrow} (F_0)^\vee \leftarrow 0,$$

with the same symmetry properties as in Theorem 2.3.5 under the same assumptions.

Proof. The proof is constructive: We generate a graded free resolution with the symmetry properties as in 2.3.5.

$$K(M) : 0 \leftarrow M \leftarrow A_0(M) \stackrel{\phi_1}{\leftarrow} \ldots A_m(M) \stackrel{\phi_{m+1}\circ\beta_{m+1}^{-1}}{\longleftarrow} (A_m(M))^\vee \ldots \stackrel{\phi_1^\vee}{\leftarrow} (A_0(M))^\vee \leftarrow 0$$

Consider now all maps as matrices with respect to arbitrary bases in $A_0(M)$ up to $A_m(M)$ and the corresponding dual bases in $(A_0(M))^\vee$ up to $(A_m(M))^\vee$. That means $\phi_i^\vee$ is the transposed matrix of ϕ_i for all $1 \leq i \leq m$.
ϕ_1 is not minimal if and only if at least one entry $a_{ij} := (\phi_1)_{ij}$ is a unit. We perform row and column operations to produce a matrix, which has only zeros in the ith row and the jth column except for position (i, j). That means we multiply ϕ_1 with an invertible matrices G_1 from the left hand side and G_1' from the right. In the same way using the transposed matrices $G_1'^t\phi_1^\vee G_1^t$ has zeros in the jth row and ith column except for an unit in (j, i).
Now $(G_1')^{-1}\phi_2$ must have a zero jth row, and $\phi_2^\vee G_1'^{t^{-1}}$ a zero jth column. Cancel the ith row and jth column from $G_1\phi_1G_1'$ and call the new map $\widetilde{\phi}_1$. Do the same with the jth row from ${G_1'}^{-1}\phi_2$, call the new map $\widetilde{\phi}_2$, this gives again an exact complex at this point. In the same manner we cancel the corresponding summand of the free modules of the resolution. We perform the transposed operations to obtain $\widetilde{\phi_2^\vee}$ and $\widetilde{\phi_1^\vee}$. Continue this process with the new complex defined by $\phi_1 := \widetilde{\phi}_1, \phi_2 := \widetilde{\phi}_2$, $\phi_1^\vee := \widetilde{\phi_1^\vee}$ and $\phi_2^\vee := \widetilde{\phi_2^\vee}$. Once ϕ_1 is minimal continue with ϕ_2.
Finally one has to consider the middle matrix $D := \phi_{m+1} \circ \beta_{m+1}^{-1}$. By construction D is homogeneous with respect to dual bases. Assume the bases to be ordered by the (total) degree (that can always be achieved easily). That means there are no other elements besides zeros and units at the right hand side of a unit, and there are no other elements in the same column under a unit. We start with the lowest row containing a unit and take its most right one, in position (i, j) say. Assume $i \neq j$. Clear the whole ith row with column operations. By symmetry we can perform the same operations on rows to clear the ith column over the symmetric unit $B_{j,i}$. Now clear the jth column over position (i, j). This does not affect the ith column as the ith row has zeros there. Do the same with column operations on the jth row. The only point where one has to pay attention to is the position (j, j). If it is not zero then add both, the ith row and the ith column, $\frac{-1}{2}B_{j,j}$-times within the last operations.
Finally continue with the row and column deleting process as before. □

Remark 2.3.8. Using a computer algebra system we can compute the above selfdual symmetric resolution of a given Gorenstein Module of finite length. For this purpose we use implemented algorithms in order to obtain the first part and make a symmetric ansatz for the middle matrix. A syzygy computation gives the desired matrix. See B.3 for a the code written in [MACAULAY2].

Some examples using [MACAULAY2] follow here.

Examples 2.3.9. The procedures `resolutionsymm` und `resolutionskew` try to find a resolution with a skew respectively symmetric middle matrix. The symmetric respectively skew symmetric ansatz mentioned in the remark is made with respect to the lowest possible degree of the middle matrix. It leads to equations which we try to solve using the syzygy command. If there is no sufficient solution the matrix given back contains zero blocks. They can be recognized in the Betti tables as the unexpected numbers in upper rows, which are not in a symmetric position.
`randomskewP` and `randomsymmP` chose random skew and symmetric matrices with respect to a list and a positive integer. In the case of $M(P)$ the list can be thought as the generator degrees of $M(P)$ and s as the top degree.
The first example is in three variables with a skew τ.

```
load "resolutionskew.m2"
R = ZZ/101[x_0..x_2];
gen={2:0,2:-1};
s=3;
P=randomskewP(R,s,gen);
MP=dualModule(P);
betti (C = resolutionsymm(MP))
```

From our theorem we know, a symmetric resolution must exist:

```
total: 4 10 10 4
    0: 2  .  . .
    1: 2 10  . .
    2: .  . 10 2
    3: .  .  . 2
```

In general there is no skew resolution.

```
betti (C = resolutionskew(MP))
```

```
o1  = total: 4 10 10 10
          0: 2  .  .  .
          1: 2 10  . 10
          2: .  . 10  .
```

In the same manner a symmetric random matrix in divided powers leads to a skew symmetric resolution in three variables, but not to a symmetric resolution:

```
gen={0,-1,-2};
s=4;
P=randomsymmP(R,s,gen);
MP=dualModule(P);
betti (C = resolutionsymm(MP))

o2 = total: 3 15 15 15
         0: 1  .  .  .
         1: 1  .  . 15
         2: 1 15 15  .

betti (C = resolutionskew(MP))

o3 = total: 3 15 15 3
         0: 1  .  . .
         1: 1  .  . .
         2: 1 15 15 1
         3: .  .  . 1
         4: .  .  . 1
```

In five variables the opposite behavior is true: A general symmetric matrix in divided powers leads to a symmetric resolution, but in general no skew symmetric.

```
R = ZZ/101[x_0..x_4];
s=1;
gen={2:0};
P=randomsymmP(R,s,gen);
MP=dualModule(P);
betti (C = resolutionsymm(MP))

o4 = total: 2 8 14 14 8 2
         0: 2 8 10  4 . .
         1: . .  4 10 8 2

betti (C = resolutionskew(MP))

o5 = total: 2 8 14 14 12 2
        -1: . .  .  .  4 .
         0: 2 8 10  4  . .
         1: . .  4 10  8 2
```

The whole approach is independent from the weights of the variables. Again in the case of three weighted variables and a general matrix in divided powers we gain a symmetric but not a skew symmetric resolution.

```
R = ZZ/101[x_1,x_2,x_3,Degrees=>{1,2,3}];
s=4;
gen={0,-1,-1,-1};
P=randomskewP(R,s,gen);
MP=dualModule(P);
betti (C = resolutionsymm(MP))
```

```
o6 = total: 4 11 11 4
         0: 1  .  . .
         1: 3  1  . .
         2: .  4  . .
         3: .  6  . .
         4: .  .  6 .
         5: .  .  4 .
         6: .  .  1 3
         7: .  .  . 1
```

```
  betti (C = resolutionskew(MP))
```

```
o7 = total: 4 11 11 11
         0: 1  .  .  .
         1: 3  1  .  .
         2: .  4  .  .
         3: .  6  .  6
         4: .  .  6  4
         5: .  .  4  1
         6: .  .  1  .
```

Corollary 2.3.10 (Zero Dimensional Gorenstein Ideals)**.** *Let n be odd, $m = \frac{n-1}{2}$, and* char $k \neq 2$*. Let I be a homogeneous zero dimensional Gorenstein ideal in $R = k[x_1, \dots, x_n]$ with* $\deg x_l = d_l > 0$*, $d = \sum_{l=1}^{n} d_l$ and top degree s. Then there is a minimal graded free resolution of R/I,*

$$0 \leftarrow R/I \leftarrow R \overset{\psi_1}{\leftarrow} F_1 \leftarrow \dots F_m \overset{\psi_{m+1}}{\leftarrow} F_m^\vee \leftarrow \dots F_1^\vee \overset{\psi_1^\vee}{\leftarrow} R(-d-s)(= R^\vee) \leftarrow 0,$$

such that ψ_m is skew symmetric if m is odd and symmetric if m is even with respect to dual bases. Here $(\quad)^\vee = \mathrm{Hom}_R(\quad, R(-d-s))$.

Proof. We want to apply our main Theorems 2.3.5 respectively 2.3.7 that means we have to show that a Gorenstein ideal with top degree s induces an isomorphism $\tau : R/I \to \mathrm{Hom}_k(R/I, k)(-s)$ such that $\tau^*(-s) = \tau$. By the Theorem of Macaulay A.27 we have that $I^\perp$ is a simple submodule of $\mathcal{D}$, generated by a homogeneous element f of degree $-s$ (This is also clear from the definition of the associated P 2.1.7). Moreover $I = \mathrm{Ann}(f)$. Hence $P := (f) \in \mathrm{Hom}_R(R(-s), \mathcal{D})$ is the symmetric matrix of Theorem 2.2.1, which gives the existence of τ. □

Remark 2.3.11. There are intersectional cases with the famous Theorem of Buchsbaum and Eisenbud (A.23 or [BE77, Theorem 2.1]). We do not use the structure as a differential graded algebra on the resolution of R/I, but only apply the above theorem. On the other hand our approach is restricted to the very special case of zero dimensional ideals I of R. Nevertheless we are not restricted on the codimension.

2.4 Some Applications

The structure Theorem 2.3.5 has a bunch of natural applications. In this section let k be any field if not stated differently. Moreover let $R = k[x_1, \dots, x_n]$ be the usual (trivially weighted) polynomial ring. Recall that in the context of a graded Gorenstein R-module of finite length $(\ \ )^*$ stands for the functor $\mathrm{Hom}_k(\ \ , k)$. Moreover we use the notation from the previous section fixed in 2.3.1 — in the case $d_1 = \dots = d_n = 1$.
We start with a proposition.

Proposition 2.4.1. *Let n be odd, and let $m = \frac{n-1}{2}$. Let M be a graded Gorenstein R-module of finite length with $\tau : M \to (M)^*(-s)$ such that $\tau = \pm(\tau)^*(-s)$ for some $s \in \mathbb{Z}$. Let the Hilbert function of M be $(a_0, \dots, a_p, a_p, \dots, a_0) := (\dim_k M_0, \dots, \dim_k M_{2p+1})$. Let $(\ \ )^\vee = \mathrm{Hom}_R(\ \ , R \otimes \bigwedge^n W)(-s)$. Applying the Nielsen I construction, respectively Theorem 2.3.5, we can choose a homogeneous basis B of $A_m(M)$ and its dual $B^\vee$ of $(A_m(M))^\vee$ such that we gain a resolution of type*

$$\begin{array}{llllllll}
a_0 & \dots & a_0\binom{n}{m-1} & a_0\binom{n}{m} & a_0\binom{n}{m+1} & a_0\binom{n}{m+2} & \dots & a_0 \\
\vdots & & & & & & & \vdots \\
a_{p-1} & \dots & a_{p-1}\binom{n}{m-1} & a_{p-1}\binom{n}{m} & a_{p-1}\binom{n}{m+1} & a_{p-1}\binom{n}{m+2} & \dots & a_{p-1} \\
a_p & \dots & a_p\binom{n}{m-1} & a_p\binom{n}{m} & a_p\binom{n}{m+1} & a_p\binom{n}{m+2} & \dots & a_p \\
a_p & \dots & a_p\binom{n}{m-1} & a_p\binom{n}{m} & a_p\binom{n}{m+1} & a_p\binom{n}{m+2} & \dots & a_p \\
a_{p-1} & \dots & a_{p-1}\binom{n}{m-1} & a_{p-1}\binom{n}{m} & a_{p-1}\binom{n}{m+1} & a_{p-1}\binom{n}{m+2} & \dots & a_{p-1} \\
\vdots & & & & & & & \vdots \\
a_0 & \dots & a_0\binom{n}{m-1} & a_0\binom{n}{m} & a_0\binom{n}{m+1} & a_0\binom{n}{m+2} & \dots & a_0,
\end{array}$$

with the following property: The restriction B_{p+1} of B to $R \otimes \bigwedge^m W \otimes M_{p+1}$ and $B^\vee_{p+1}$ of $B^\vee$ to $(R \otimes \bigwedge^m W \otimes M_{p+1})^\vee$ guarantees for the constant matrix $f := M^{B^\vee_{p+1}}_{B_{p+1}}(\phi_{m+1} \circ \beta^{-1}_{m+1})$ of size $a_p\binom{n}{m}$ that $f = \pm(-1)^m f^t$ and $f_{(i,i)} = 0$ for all i.

Remark 2.4.2. In the notation of Remark 1.2.5 f represents the map $A_{(-p-1-m,p+1)}(M) \overset{\phi_{m+1,1} \circ \beta^{-1}_{m+1}|}{\longleftarrow} (A_{(-p-1-m,p+1)}(M))^\vee$.

Proof of the proposition. Let $W_m = (1 \otimes \chi_{i_1} \wedge \dots \wedge \chi_{i_m})_{(i_1,\dots,i_m)}$ be the canonical basis of $R \otimes \bigwedge^m W$, and let $\widetilde{B}$ be any homogeneous basis of M. Let $B = W_m \otimes \widetilde{B}$. By Theorem 2.3.5 we know that f has the property $f = \pm(-1)^m f^t$, because f is the restriction of the representation matrix of the middle map $\phi_{m+1} \circ \beta^{-1}_{m+1}$ to the elements B_{p+1} of B in degree $p + 1 + m$ and their duals $B^\vee_{p+1}$ in $B^\vee$.
Let $\mathsf{b} = 1 \otimes \chi_{i_1} \wedge \dots \wedge \chi_{i_m} \otimes \mathsf{m}$ be any element of B_{p+1}, and let β be its dual element in $B^\vee_{p+1}$. Then β is up to sign of the form $\beta = (\_ \wedge \chi_{i_{m+1}} \wedge \dots \wedge \chi_{i_n}) \otimes \mu :=$

$$\begin{cases} R \otimes \bigwedge^m W \otimes M_{p+1} \to R \otimes \bigwedge^n W \\ r \otimes w \otimes \mathsf{m} \mapsto r \otimes (w \wedge \chi_{i_{m+1}} \wedge \dots \wedge \chi_{i_n}) \otimes \mu(\mathsf{m}), \end{cases}$$

such that $i_1, \ldots, i_n \in \{1, \ldots, n\}$ are pairwise different.
Then we obtain $\phi_{m+1} \circ \beta_{m+1}^{-1}(\beta) = \phi_{m+1}(\chi_{i_{m+1}} \wedge \ldots \wedge \chi_{i_n} \otimes \tau^{-1}(\mu)) = x_{i_{m+1}} \otimes \chi_{i_{m+2}} \wedge \ldots \wedge \chi_{i_n} \otimes \tau^{-1}(\mu) + \ldots + (-1)^{n-m-1} x_{i_n} \otimes \chi_{i_{m+1}} \wedge \ldots \wedge \chi_{i_{n-1}} \otimes \tau^{-1}(\mu) - 1 \otimes \chi_{i_{m+2}} \wedge \ldots \wedge \chi_{i_n} \otimes (x_{i_{m+1}} \tau^{-1}(\mu)) + (-1)^{n-m-1} \otimes \chi_{i_{m+1}} \wedge \ldots \wedge \chi_{i_{n-1}} \otimes (x_{i_n} \tau^{-1}(\mu))$. Hence the linear representation of $\phi_{m+1} \circ \beta_{m+1}^{-1}(\beta)$ does not involve $\chi_{i_1} \wedge \ldots \wedge \chi_{i_m}$, i.e. not b. Therefore the diagonal elements $f_{(j,j)} = 0$ for all j. □

Lemma 2.4.3. *Let c be a positive integer with $c \geq 2$. Let $A \in k^{c \times c}$ be a skew symmetric matrix with constant entries, i.e. $A^t = -A$. If $\operatorname{char} k = 2$ we require moreover that $A_{(j,j)} = 0$ for all $j \in \{1, \ldots, c\}$.*
Then A is of even rank.

Proof. The proof is immediate taking into account that A is equivalent to a matrix with zeros everywhere and blocks of form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the diagonal. □

Corollary 2.4.4. *Let $n \equiv 3 \bmod 4$, and let $m = \frac{n-1}{2}$. Let $I \subset R$ be an Artinian Gorenstein ideal with Hilbert function $(1, n, n, 1)$.*
Then the graded minimal free resolution of R/I has a Betti table of type

	0	1		m	$m+1$		n
	1	$-$					
	$-$	$\beta_{1,2}$	$\cdots$	$\beta_{m,m+1}$	$2a$	$\cdots$ $\beta_{1,3}$	$-$
	$-$	$\beta_{1,3}$	$\cdots$	$2a$	$\beta_{m,m+1}$	$\cdots$ $\beta_{1,2}$	$-$
	$-$					$-$	1

for some $a \in \{0, \ldots, \lfloor \frac{1}{2} n \binom{n}{m} \rfloor - \binom{n}{m-1}\}$.

Proof. First of all note that $\binom{n}{m-1}$ and $\binom{n}{m}$ are of equal parity. This is the case as $\binom{n}{m-1} \cdot \frac{m+2}{m} = \binom{n}{m}$, and m is odd.
By the Theorem of Macaulay A.27 I is given by a homogeneous element f in divided powers, such that $I = \operatorname{Ann}_R(f)$. Let $P := (f)$. It defines a symmetric matrix, which gives a $\tau : R/I \to \operatorname{Hom}_k(R/I, k)$, such that τ^* is τ up to twist. Hence R/I is a Gorenstein module of finite length in the sense of our definition. We apply the Nielsen Construction of Theorem 2.3.5. By Proposition 2.4.1 we obtain a resolution of R/I with constant middle

maps of type

$$\begin{array}{cccccccc}
1 & \cdots & \binom{n}{m-1} & \binom{n}{m} & \binom{n}{m} & \binom{n}{m-1} & \cdots & 1 \\
 & & & & \swarrow^{g^t} & & & \\
n & \cdots & n\binom{n}{m-1} & n\binom{n}{m} & n\binom{n}{m} & n\binom{n}{m-1} & \cdots & n \\
 & & & \swarrow^{f} & & & & \\
n & \cdots & n\binom{n}{m-1} & n\binom{n}{m} & n\binom{n}{m} & n\binom{n}{m-1} & \cdots & n \\
 & & \swarrow^{g} & & & & & \\
1 & \cdots & \binom{n}{m-1} & \binom{n}{m} & \binom{n}{m} & \binom{n}{m-1} & \cdots & n,
\end{array}$$

such that f is skew symmetric, hence by the lemma of even rank. By the assumption on the Hilbert function g and g^t are of full rank. Hence the size $n\binom{n}{m}$ of the matrix f and the rank $\binom{n}{m-1}$ of g are of the same parity. Therefore we can reduce f by symmetric operations from the left and right to $\tilde{f}$, a skew symmetric matrix of even size. $\tilde{f}$ is of even rank. Hence a minimization gives the free submodules of even rank $2a$. □

We obtain the following special case in characteristic 2:

Corollary 2.4.5. *Let $n \equiv 1 \bmod 4$ with $n \geq 5$, and let $m = \frac{n-1}{2}$. Let k be a field of characteristic 2. Let $I \subset R$ be an Artinian Gorenstein ideal with Hilbert function $(1, n, n, 1)$. Assume that $\binom{n}{m-1}$ is of opposite parity as $\binom{n}{m}$. Then the graded minimal free resolution of R/I has a Betti table of type*

	0	1		m	$m+1$			n
	1	–						
	–	$\beta_{1,2}$	$\cdots$	$\beta_{m,m+1}$	$2a+1$	$\cdots$	$\beta_{1,3}$	–
	–	$\beta_{1,3}$	$\cdots$	$2a+1$	$\beta_{m,m+1}$	$\cdots$	$\beta_{1,2}$	–
	–						–	1

for some $a \in \{0, \ldots, \lfloor \frac{1}{2} n\binom{n}{m} \rfloor - \binom{n}{m-1} - 1\}$.

Proof. As in Corollary 2.4.4 we gain by Proposition 2.4.1 a resolution of R/I with constant middle maps of type

$$\begin{array}{cccccccc}
1 & \cdots & \binom{n}{m-1} & \binom{n}{m} & \binom{n}{m} & \binom{n}{m-1} & \cdots & 1 \\
 & & & & \swarrow^{g^t} & & & \\
n & \cdots & n\binom{n}{m-1} & n\binom{n}{m} & n\binom{n}{m} & n\binom{n}{m-1} & \cdots & n \\
 & & & \swarrow^{f} & & & & \\
n & \cdots & n\binom{n}{m-1} & n\binom{n}{m} & n\binom{n}{m} & n\binom{n}{m-1} & \cdots & n \\
 & & \swarrow^{g} & & & & & \\
1 & \cdots & \binom{n}{m-1} & \binom{n}{m} & \binom{n}{m} & \binom{n}{m-1} & \cdots & n,
\end{array}$$

such that f is symmetric with $f_{(j,j)} = 0$ for all j. That means in characteristic 2 that f is also skew symmetric, hence of even rank. By the assumption on the Hilbert function g and g^t are of full rank. Therefore the size $n\binom{n}{m}$ of the matrix f and the rank $\binom{n}{m-1}$ of g are of opposite parity. We reduce f by symmetric operations from the left and right to $\tilde{f}$, a skew symmetric matrix of odd size. $\tilde{f}$ is still of even rank. That means a minimization leads to the free submodules of even rank $2a+1$. □

Corollary 2.4.6. *Let k be a field of characteristic 2. Let $n = 2^\ell - 3$ for some $\ell \geq 3$. Let $I \subset R$ be an Artinian Gorenstein ideal with Hilbert function $(1, n, n, 1)$. Then the minimal free resolution of R/I is of type as in Corollary 2.4.5.*

Proof. It is well known that $\binom{2^\ell-1}{i}$ is odd for all $0 \leq i \leq 2^\ell - 1$. These are the horizontal lines of Sierpinski's gasket one gets from Pascal's triangle. We know $\binom{2^\ell-1}{i} = \binom{2^\ell-2}{i-1} + \binom{2^\ell-2}{i}$ for all $i \geq 1$, and $\binom{2^\ell-1}{0} = \binom{2^\ell-2}{0}$. Hence the $\binom{2^\ell-2}{i}$, $0 \leq i \leq 2^\ell - 2$, alternate with respect to their parity, starting with odd.
Applying the formula from above again leads to the fact that the $\binom{2^\ell-3}{i}$ change parity in every second step. That means $\binom{2^\ell-3}{2^{\ell-1}-3}$ is odd and $\binom{2^\ell-3}{2^{\ell-1}-2}$ is even as $2^{\ell-1} - 3 \equiv 1 \mod 4$. Hence we can apply Corollary 2.4.5. □

Remark 2.4.7. Sierpinski's gasket also shows that all binomial coefficients in the interior of the "triangle" $\binom{2^{\ell-1}-1}{0}$, $\binom{2^{\ell-1}}{0}$, $\binom{2^\ell-1}{2^{\ell-1}-1}$, $\binom{2^\ell-1}{2^{\ell-1}}$ and $\binom{2^{\ell-1}}{2^{\ell-1}}$, $\binom{2^{\ell-1}-1}{2^{\ell-1}-1}$ are even. Hence the only case of Corollary 2.4.5 is the case $n = 2^\ell - 3$ from 2.4.6.

There are two corollaries of special interest. The first one concerns Green's Conjecture in characteristic 2 for curves of genus $g = 2^\ell - 1$. This case was already determined for smooth curves by Schreyer in [Sch86] and [Sch91].

Corollary 2.4.8 (Green's Conjecture in Characteristic 2)**.** *The obvious extension of the Green's Conjecture to positive characteristic fails for general curves of genus $g = 2^\ell - 1$ for all $\ell \geq 3$ in characteristic 2.*

Proof. In this case Green's conjecture would mean that minimal free resolution of canonical model $X \subset \mathbb{P}^{g-1}$ of the curve has a selfdual pure Betti table of type:

0	1	...	$(g-3)/2$	$(g-3)/2+1$	...	$g-2$
1	–					
–	$\beta_{1,2}$	$\cdots$	$\beta_{(g-3)/2,(g-3)/2+1}$	–		
–			–	$\beta_{(g-3)/2,(g-3)/2+1}$	$\cdots$	$\beta_{1,2}$
–					–	1.

Modulo two regular elements this would lead to an Artinian Gorenstein factor ring with Hilbert function $(1, g-2, g-2, 1)$ over the polynomial ring $k[x_1, \ldots, x_{g-2}]$ with a pure resolution. That would be a contradiction to Corollary 2.4.6. □

Remark 2.4.9. In the following by a *degree sequence* is meant a sequence of integers $d = (d_0 < d_1 < \ldots < d_c)$. A graded minimal free resolution of an R-module M is called *pure with degree sequence* d if $\beta_{i,j}(M) = 0$ except when $j = d_i$.
Note that for a given degree sequence d the Betti table of a pure minimal graded free resolution with d is uniquely determined by the Herzog-Kühl equations up to a rational multiple ([HK84, Theorem 1]).
It is clear that the Betti tables of graded minimal free resolutions of R-modules form a monoid with respect to addition (take the direct sum of the corresponding modules).

The second Corollary concerns the Boji-Söderberg Conjectures ([BS06]) on the existence of Cohen-Macaulay modules over $R = k[x_1, \ldots, x_n]$ with pure resolutions having any given degree sequence. The recent paper of Eisenbud and Schreyer [ES08] gives an introduction and a proof of a strengthened form of the Boji-Söderberg Conjectures. We can prove an experimentally verified conjecture of Eisenbud and Schreyer from page 7 of [ES08] : They give an algorithm which expresses every Betti table of a finitely generated graded Cohen-Macaulay module as a positive rational linear combination of the Betti tables of Cohen-Macaulay modules with pure resolutions. That means the Betti tables of Cohen-Macaulay modules over R lie inside a rational cone with Betti tables of pure resolutions as extremal rays. By the way this result is generalized to the non-Cohen-Macaulay case by Boji and Söderberg in their recent paper [BS08].
However it is not clear which Betti tables really are in the monoid of actual resolutions. Eisenbud and Schreyer conjecture that the monoid of resolutions depends on the characteristic of the base field k.

Corollary 2.4.10 (Monoid of Resolutions of Cohen-Macaulay Modules)**.** *The monoid of resolutions of Cohen-Macaulay graded R-modules depends on the characteristic of k.*

Proof. Let $R = k[x_1, \ldots, x_5]$. Consider again the case of Artinian Gorenstein factor rings with Hilbert function $(1, 5, 5, 1)$. If $\mathrm{char}(k) = 0$ it is easy to construct such a module with betti table

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. \end{matrix}$$

By Corollary 2.4.5 we know there is no Artinian Gorenstein ideal $I \subset R$ over $\mathrm{char}(k) = 2$ with such a resolution. Moreover any Cohen-Macaulay module over a polynomial ring in more variables with this Betti table comes modulo a regular sequence down to this situation: Especially the Gorenstein property follows as the associated matrix in divided powers is symmetric (it is a 1×1-matrix). □

As mentioned above Eisenbud and Schreyer also proved that there exists a Cohen-Macaulay module with a pure resolution for any given degree sequence ([ES08, Theorem 0.1]). However it is not clear if the lowest possible multiple of the by the Herzog-Kühl equations determined Betti table occurs as a resolution of a module. In the above corollary we have seen examples for characteristic 2. Here are some characteristic independent examples:

Corollary 2.4.11. *Let $\ell \geq 2$. Consider the degree sequence*

$$(0, 2^{\ell-1}+1, 2^{\ell-1}+2, \ldots, 2^{\ell}-1, 2^{\ell}+1, \ldots, 2^{\ell}+2^{\ell-1}-2, 2^{\ell}+2^{\ell-1}-1, 2^{\ell+1})$$

of length $2^{\ell}-1$.
Then there is no graded Cohen-Macaulay factor ring over the polynomial ring R of codimension $2^{\ell}-1$ with a pure minimal free resolution having this degree sequence.

Proof. Assume there is such a factor ring $M = R/I$. Modulo a regular sequence we obtain an Artinian Gorenstein factor ring of a polynomial ring in $n = 2^{\ell}-1$ variables (the associated matrix in divided powers is symmetric). By abuse of notation we call the polynomial ring again R and the factor ring $M = R/I$.
The Hilbert function is $\mathrm{HF_M} = (a_0, a_1, \ldots, a_p, a_p, \ldots, a_1, a_0) =$

$$\left(1, \binom{2^{\ell}-1}{1}, \binom{2^{\ell}}{2}, \ldots, \binom{2^{\ell}+2^{\ell-1}-2}{2^{\ell-1}}, \binom{2^{\ell}+2^{\ell-1}-2}{2^{\ell-1}}, \ldots, \binom{2^{\ell}}{2}, \binom{2^{\ell}-1}{1}, 1\right)$$

with $p = 2^{\ell-1}$. By Sierpinski's gasket it follows that $a_2, \ldots, a_p$ are even: As in Remark 2.4.7 we see that these binomial coefficients are in the interior of the triangle with odd sides.
We apply Proposition 2.4.1 and obtain that the constant "middle" matrix f is skew symmetric, as $n = 2^{\ell}-1 \equiv 3 \bmod 4$. Hence f is of even rank. We know that the minimization of the resolution is set up by the minimization of the constant complexes as C_{2p}:

$$0 \leftarrow A_{(-2p,2p)}(M) \overset{\phi_{(1,1)}}{\longleftarrow} A_{(-2p,2p-1)}(M) \overset{\phi_{(2,1)}}{\longleftarrow} \ldots \overset{\phi_{(p-1,1)}}{\longleftarrow} A_{(-2p,p+1)}(M) \overset{f}{\longleftarrow} (A_{(-2p,p+1)}(M))^{\vee} \ldots$$

(Recall by Remark 1.2.5 that the resolution is the total complex of the double complex made by these constant complexes and the linear part.). We know that $\operatorname{rank} A_{(-2p,2p-i)}(M) = a_{i+1}\binom{2^{\ell}-1}{i}$ for $0 \leq i \leq 2^{\ell-1}-1$. Hence $\operatorname{rank} A_{(-2p,2p-i)}(M)$ is odd for $i = 0$ and even else.
Moreover by our assumption on the degree sequence the complex C_{2p} is exact at all $A_{(-2p,2p-i)}(M)$ for $0 \leq i \leq p-2$. Therefore $\phi_{(1,1)}$ is of odd rank. By the rank behavior of the $A_{(-2p,2p-i)}(M)$ we finally gain that $\phi_{(p-1,1)}$ is of odd rank. With symmetric operations we cancel within the minimization process a summand of odd rank from $A_{(-2p,p+1)}(M)$ and $(A_{(-2p,p+1)}(M))^{\vee}$. As f is of even rank in the last minimization step a summand of odd rank remains. □

Remark 2.4.12. The first case of Corollary 2.4.11 is $(0, 3, 5, 8)$. Here the statement follows also from the Theorem of Buchsbaum and Eisenbud. The next case is $(0, 5, 6, 7, 9, 10, 11, 16)$. Computer experiments seem to show that there are Cohen-Macaulay factor rings in any characteristic with this degree sequence and nearly pure resolutions such that $\beta_{i,j} = 0$ except when $j = d_j$ and $\beta_{3,8} = \beta_{4,8} = 1$.

2.5 Selfdual Resolution implies the Gorenstein Property

In this section we want to verify that whenever a graded module M of finite length over the polynomial ring R has a selfdual resolution, there is an R-module isomorphism $\tau : M \to \mathrm{Hom}_k(M,k)(-s)$ such that $\tau^* := \mathrm{Hom}_k(\tau,k) = \pm\tau(s)$ for some $s \in \mathbb{Z}$. We state the main theorem first and give the proof at the end of the section. The theorem implies the second direction of 0.3.
For the proof the language of category theory comes more intensively into play. Especially we work out some natural equivalences between the functors $\mathrm{Ext}^n_R(\quad, R \otimes \bigwedge^n W)$ and $\mathrm{Hom}_k(\quad, k)$.
Again throughout this section let n be a positive integer, $W = \langle \chi_1, \dots, \chi_n \rangle_k$, $V = \mathrm{Hom}_k(W,k)$, with (x_l) a dual basis to (χ_l). Let $\deg \chi_l = \deg x_l = d_l > 0$, and $d = \sum_{l=1}^n d_l$. Moreover let $R = \mathrm{Sym}(V)$, the weighted polynomial ring.
Frequently in this section we need notations for arbitrary elements in graded R-modules of finite length M, N and their vectorspace duals $M^* = \mathrm{Hom}_k(M,k)$, $N^* = \mathrm{Hom}_k(N,k)$. We use $\mathsf{m} \in M$, $\mathsf{n} \in N$, $\mu \in M^*$ and $\nu \in N^*$ to denote these elements if not stated differently.

Theorem 2.5.1. *Let n be odd and let $m = \frac{n-1}{2}$. Let M be a module of finite length over R. Let $(\quad)^\vee = \mathrm{Hom}_R(\quad, R(-d))$. We assume that M has a symmetric minimal resolution of the form*

$$(**)\ \ 0 \leftarrow M \leftarrow F_0 \stackrel{\psi_1}{\leftarrow} F_1 \leftarrow \dots \leftarrow F_m \stackrel{\psi_{m+1}}{\leftarrow} F_m^\vee(-s) \leftarrow \dots \leftarrow F_1^\vee(-s) \stackrel{\psi_1^\vee(-s)}{\leftarrow} F_0^\vee(-s) \leftarrow 0$$

such that $\psi_{m+1}^\vee = \pm\psi_{m+1}$ up to twist. Then there exists a graded R-module isomorphism

$$\tau : M \to M^*(-s) := \mathrm{Hom}_k(M,k)(-s)$$

with $\tau^*(-s) = \pm(-1)^m\tau$ (if we identify $M = M^{**}$).

The proof of the theorem follows at the end of this section.
The following lemma is an essential tool for our machinery within this section.

Lemma 2.5.2. *Let F_1 and F_2 be graded free R-modules, and let M be any graded R-module of finite length. Let $M^* = \mathrm{Hom}_k(M,k)$. Let $A = F_1 \otimes_k M$ and $B = \mathrm{Hom}_R(F_1,F_2) \otimes_k M^*$. Recall that ${}_\Delta A = {}_\Delta(F_1 \otimes_k M)$ and ${}_\Delta B = {}_\Delta(\mathrm{Hom}_R(F_1,F_2) \otimes_k M^*)$ are modules with respect to the diagonal action as defined in 1.2.6. Then there is a canonical R-module isomophism*

$$\alpha_B : {}_\Delta B \cong \mathrm{Hom}_R({}_\Delta A, F_2).$$

Proof. As the tensor product, $\operatorname{Hom}_R(\ \ , F_2)$ and $\operatorname{Hom}_R(F_1, \ \ )$ commute with direct sums, it is enough to see the claim for $F_1 = F_2 = R$. Moreover let $A = R \otimes_k M$ and let $B = \operatorname{Hom}_R(R, R) \otimes_k M^*$ be equipped with the usual (left-)module structure, i.e. for all $r \otimes m \in A$ and $r' \in R$: $r' \cdot r \otimes m := (r'r) \otimes m$ (analogously for B).
We define $g_1 : B \to {}_\Delta B$ such that

$$(r \operatorname{id}_R) \otimes \mu \mapsto r(\operatorname{id}_R \otimes \mu),$$

for all $r \in R, \mu \in M^*$. And define $g_2 : A \to {}_\Delta A$ by

$$r \otimes \mathsf{m} \mapsto r(1 \otimes \mathsf{m}),$$

for all $\mathsf{m} \in M$.
By arguments on bases both maps are obviously isomorphisms. Moreover consider the linear map $\gamma : B \to \operatorname{Hom}_R(A, R)$ defined by

$$(r \operatorname{id}_R) \otimes \mu \mapsto \begin{cases} {}_\Delta(R \otimes M) \to R \\ (r' \otimes \mathsf{m}) \mapsto r \operatorname{id}_R(r') \cdot \mu(\mathsf{m}) = rr' \cdot \mu(\mathsf{m}). \end{cases}$$

Note that the last element is a functional $F_1 \otimes M \to F_2$. γ is again an isomorphism.
Hence we can define $\alpha_B : {}_\Delta B \to \operatorname{Hom}({}_\Delta A, R)$ via

$$\begin{array}{ccc} {}_\Delta B & \xrightarrow{\alpha_B} & \operatorname{Hom}_R({}_\Delta A, R) \\ {\scriptstyle g_1}\uparrow & & \downarrow{\scriptstyle \operatorname{Hom}_R(g_2, R)} \\ B & \xrightarrow{\gamma} & \operatorname{Hom}_R(A, R) \end{array}$$

by

$$\mathsf{b} \mapsto \operatorname{Hom}_R(g_2^{-1}, R) \circ \gamma \circ g_1^{-1}(\mathsf{b}).$$

$\square$

Example 2.5.3. Let the M be defined as in Example 2.3.6. Let $A = A_1(M) = R \otimes \bigwedge^1 W \otimes M$, i.e. $F_1 = R \otimes \bigwedge^1 W$, and let $F_2 = R \otimes \bigwedge^3 W$. Let $(\ \ )^\vee = \operatorname{Hom}_R(\ \ , R \otimes \bigwedge^3 W)$.
We use the following abbreviation: $x_1 \otimes (\_ \wedge \chi_2 \wedge \chi_3)$ stands for the functional

$$\begin{cases} R \otimes \bigwedge^1 W \to R \otimes \bigwedge^3 W \\ r \otimes w \mapsto r x_1 \otimes w \wedge \chi_2 \wedge \chi_3. \end{cases}$$

Moreover in this example a vector $v \in \bigoplus_i \mathcal{D}(a_i)$ in divided powers denotes the k-functional

$$\begin{cases} M \to k \\ \mathsf{m} \mapsto \langle v, \mathsf{m} \rangle (0). \end{cases}$$

Then let ${}_\Delta B \ni x_1 \otimes (\_ \wedge \chi_2 \wedge \chi_3) \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}$. Via g_1^{-1} we map it to

$$x_1 \otimes (\_ \wedge \chi_2 \wedge \chi_3) \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} - 1 \otimes (\_ \wedge \chi_2 \wedge \chi_3) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{\gamma}{\mapsto}$$

$$\omega = \begin{cases} A \to R \otimes \bigwedge^3 W \\ (r \otimes \chi_j \otimes \mathsf{m}) \mapsto x_1 r \cdot (\chi_j \wedge \chi_2 \wedge \chi_3) \cdot \left\langle \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}, \mathsf{m} \right\rangle (0) - r \cdot (\chi_j \wedge \chi_2 \wedge \chi_3) \cdot \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathsf{m} \right\rangle (0). \end{cases}$$

The last functional is an element of $\operatorname{Hom}_R(A, F_2)$, let $r \in R$, $\chi_j \in \bigwedge^1 W$ and $\mathsf{m} \in M$ be arbitrary.
Then $\operatorname{Hom}_R(g_2^{-1}, F_2)$ maps it to an element in $\operatorname{Hom}_R({}_{\Delta}A, F_2)$, which behaves for example as follows

$$x_1 \otimes \chi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{g_2^{-1}}{\mapsto} x_1 \otimes \chi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \otimes \chi_1 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \mapsto$$
$$-x_1 \chi_1 \wedge \chi_2 \wedge \chi_3 - x_1 \chi_1 \wedge \chi_2 \wedge \chi_3 = -2x_1 \chi_1 \wedge \chi_2 \wedge \chi_3.$$

In the following we use the language of categories. It is necessary to state later on our main theorem correctly. We define three isomorphisms of functors involving the functors $(\quad)^*$ and $\operatorname{Ext}^n_R(\quad, R \otimes \bigwedge^n W)$. We have to fix some abbreviatory notations. Let us give exactly our definitions:

Remark and Definition 2.5.4. We denote by $\underline{\text{grMFL}}$ the category of graded R-modules of finite length. Denote by $R \otimes \bigwedge^n W$ the free R-module sitting in degree d.
We consider the functors

$$(\quad)^* := \operatorname{Hom}_k(\quad, k), \quad (\quad)^\vee := \operatorname{Hom}_R(\quad, R \otimes \bigwedge^n W), \text{ and } \operatorname{Ext}^n(\quad) := \operatorname{Ext}^n_R(\quad, R \otimes \bigwedge^n W):$$
$$\underline{\text{grMFL}} \to \underline{\text{grMFL}}.$$

The last one is the n-th right derived functor of the contravariant left exact functor $(\quad)^\vee$.
Let $M \in \operatorname{Obj}(\underline{\text{grMFL}})$. $\operatorname{Ext}^n(M)$ can be computed by taking any graded free resolution

$$F : \ldots \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$$

of M as $\operatorname{Ext}^n(M) = H_n(\operatorname{Hom}_R(F, R \otimes \bigwedge^n W))$, the n-th homology group. Especially it is independent from the choice of the resolution.

Let $P_i = R \otimes_k \bigwedge^i W$ be the free R-module by left multiplication (i.e. $r' \cdot r \otimes w = (r'r) \otimes w$). Recall ${}_{\Delta}(P_i \otimes M)$ means the free R-module with the diagonal action as in Notation 1.2.6. We want to use the complex construction Nielsen IIa from 1.2.9. Fix the canonical isomorphism $f : \operatorname{Hom}_k(\bigwedge^n W, \bigwedge^n W) \cong k, \mathrm{id} \mapsto 1$.
We define a map

$$r_M : M^* \to \operatorname{Ext}^n(M)$$

as follows:
We apply the complex construction Nielsen II to M (using the Koszul complex (1.2.8)) and Nielsen IIa to M^* (using the dual of the Koszul complex (1.2.9)):

$$0 \leftarrow M^* \leftarrow {}_{\Delta}B_n(M^*) \overset{\Delta(\varphi_n)}{\longleftarrow} {}_{\Delta}B_{n-1}(M^*) \leftarrow \ldots \leftarrow {}_{\Delta}B_0(M^*) \leftarrow 0.$$

We define r_M via the following diagram using the canonical isomorphisms from Lemma 2.5.2. We set $\alpha_{M,i} := \alpha_{P_i^\vee \otimes M^*}$. Recall that $B_i(M^*) = P_i^\vee \otimes M^*$ and $A_i(M) = P_i \otimes M$:

$$\begin{array}{ccccccccc} 0 \leftarrow M^* & \leftarrow & {}_\Delta B_n(M^*) & \overset{\Delta(\varphi_n)}{\longleftarrow} & {}_\Delta B_{n-1}(M^*) & \dots & \overset{\Delta(\varphi_1)}{\longleftarrow} & {}_\Delta B_0(M^*) & \leftarrow 0 \\ \downarrow r_M & & \downarrow \alpha_{M,n} & & \downarrow \alpha_{M,n-1} & & & \downarrow \alpha_{M,0} & \\ 0 \leftarrow \mathrm{Ext}^n(M) & \leftarrow & ({}_\Delta(A_n(M))^\vee & \overset{({}_\Delta\phi_n)^\vee}{\longleftarrow} & ({}_\Delta(A_{n-1}(M))^\vee & \leftarrow \dots & \overset{({}_\Delta\phi_1)^\vee}{\longleftarrow} & ({}_\Delta(A_0(M))^\vee & \leftarrow 0. \end{array}$$

Note that there is no index shift by our definition of the differentials here. It makes sense to use the Koszul complex and its dual in the definition: Only in this way we are able to apply the canonical isomorphisms from Lemma 2.5.2.

Lemma 2.5.5. *Then r_M is well defined as the diagram is commutative.*

Proof. Let $w \in \bigwedge^i W$ and $w' \in \bigwedge^{i-1} W$ be arbitrary elements. Let $\mu \in M^*$, $\mathsf{m} \in M$ be arbitrary, and let for simplicity $\pi \in (P_{i-1}^\vee)$ be homogeneous such that $1 \otimes w \mapsto 1 \otimes w \wedge \tilde{w}$ for some $\tilde{w} \in \bigwedge^{n-i+1} W$.
The diagram commutes because if ${}_\Delta B_{i-1}(M^*) \ni \pi \otimes \mu$, then

$$c\pi \otimes \mu \overset{\Delta(\varphi_i)}{\mapsto} \left\{ \begin{array}{c} P_i \to R \otimes \bigwedge^n W \\ 1 \otimes w \mapsto \pi(\sum_{l=1}^n x_l \otimes (x_l \neg w)) = \sum_{l=1}^n x_l \pi(1 \otimes (x_l \neg w)) \end{array} \right\} \otimes \mu \overset{\alpha_{M,i}}{\mapsto}$$
$$\left\{ \begin{array}{c} {}_\Delta A_{i-1}(M) \to R \otimes \bigwedge^n W \\ 1 \otimes w \otimes \mathsf{m} \mapsto \sum_{l=1}^n x_l \pi(x_l \neg w) \cdot \mu(\mathsf{m}) - \sum_{l=1}^n \pi(x_l \neg w) \cdot (x_l \mu)(\mathsf{m}) \end{array} \right\},$$

and on the other hand

$$\pi \otimes \mu \overset{\alpha_{M,i-1}}{\mapsto} \left\{ \begin{array}{c} {}_\Delta A_{i-1}(M) \to R \otimes \bigwedge^n W \\ (1 \otimes w' \otimes \mathsf{m}) \mapsto \pi(1 \otimes w') \cdot \mu(\mathsf{m}) \end{array} \right\} \overset{({}_\Delta\phi_i)^\vee}{\mapsto}$$
$$\left\{ \begin{array}{c} {}_\Delta A_i(M) \to R \otimes \bigwedge^n W \\ (1 \otimes w \otimes \mathsf{m}) \mapsto \sum_{l=1}^n x_l \pi(x_l \neg w) \cdot \mu(\mathsf{m}) - \sum_{l=1}^n \pi(x_l \neg w) \cdot \mu(x_l \mathsf{m}). \end{array} \right\}$$

□

Now we can define the following natural equivalence of the functors $(\quad)^*$ and $\mathrm{Ext}^n(\quad)$. Let in the following - if not stated differently - $(\quad)^\vee = \mathrm{Hom}_R(\quad, R \otimes \bigwedge^n W)$. It is central in order to understand the relationship between $(\quad)^*$ and $\mathrm{Ext}^n(\quad)$.

Theorem and Definition 2.5.6 (Equivalences of Functors I)**.** *Consider the category $\underline{\mathrm{grMFL}}$. The collection of isomorphism $\{M \mapsto r_M \,|\, M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})\}$ as defined in Lemma 2.5.4 gives an isomorphism of the functors $(\quad)^*$ and $\mathrm{Ext}^n(\quad)$, i.e. for all $M, N \in \mathrm{Obj}(\underline{\mathrm{grMFL}})$ and all $\tau \in \mathrm{Mor}(\underline{\mathrm{grMFL}})$, $\tau : M \to N$, the diagram*

$$\begin{array}{ccc} N^* & \xrightarrow{\tau^*} & M^* \\ \downarrow r_N & & \downarrow r_M \\ \mathrm{Ext}^n(N) & \xrightarrow{\mathrm{Ext}^n(\tau)} & \mathrm{Ext}^n(M) \end{array}$$

commutes.

Proof. We resolve M and N using the Construction Nielsen II (1.2.8), as r_N and r_M are defined via these resolutions. Let $\mathsf{n} \in N$, $\nu \in N^*$, $\mathsf{m} \in M$, $\mu \in M^*$, $\mathsf{p} \in P_i$ and $\pi \in P_i^\vee$ be arbitrary. Then we resolve τ by

$$\begin{array}{ccccccccc}
0 \leftarrow M & \leftarrow & {}_\triangle A_0(M) & \overset{\triangle(\phi_1)}{\longleftarrow} & {}_\triangle A_1(M) & \dots & \overset{\triangle(\phi_n)}{\longleftarrow} & {}_\triangle A_n(M) & \leftarrow 0 \\
\downarrow^{\tau} & & \downarrow^{\tau_0} & & \downarrow^{\tau_1} & & & \downarrow^{\tau_n} & \\
0 \leftarrow N & \leftarrow & {}_\triangle A_0(N) & \overset{\triangle(\phi_1)}{\longleftarrow} & {}_\triangle A_1(N) & \leftarrow \dots & \overset{\triangle(\phi_n)}{\longleftarrow} & {}_\triangle A_n(N) & \leftarrow 0,
\end{array}$$

where $\tau_i : {}_\triangle(P_i \otimes M) \to {}_\triangle(P_i \otimes N)$, $\mathsf{p} \otimes \mathsf{m} \mapsto \mathsf{p} \otimes \tau(\mathsf{m})$. Moreover we resolve M^* and N^* via Nielsen IIa (1.2.9) and τ^* by

$$\begin{array}{ccccccccc}
0 \leftarrow M^* & \leftarrow & {}_\triangle B_n(M^*) & \overset{\triangle(\varphi_n)}{\longleftarrow} & {}_\triangle B_{n-1}(M^*) & \dots & \overset{\triangle(\varphi_1)}{\longleftarrow} & {}_\triangle B_0(M^*) & \leftarrow 0 \\
\uparrow^{\tau^*} & & \uparrow^{\tau_n^*} & & \uparrow^{\tau_{n-1}^*} & & & \uparrow^{\tau_0^*} & \\
0 \leftarrow N^* & \leftarrow & {}_\triangle B_n(N^*) & \overset{\triangle(\varphi_n)}{\longleftarrow} & {}_\triangle B_{n-1}(N^*) & \leftarrow \dots & \overset{\triangle(\varphi_1)}{\longleftarrow} & {}_\triangle B_0(N^*) & \leftarrow 0.
\end{array}$$

Here $\tau_i^* : {}_\triangle B_i(N^*) \to {}_\triangle B_i(M^*), \pi \otimes \nu \mapsto \pi \otimes \tau^*(\nu)$. We dualize the above diagram with $(\quad)^\vee = \mathrm{Hom}_R(\quad, R \otimes \bigwedge^n W)$. Then we connect the dual of the upper diagram with the lower one via the isomorphisms $\alpha_{M,i} = \alpha_{B_i(M^*)}$ and $\alpha_{N,i} = \alpha_{B_i(N^*)}$ from 2.5.2. In total we have nearly gained the commutativity of

$$\begin{array}{ccccccc}
 & (A_n(M))^\vee & \leftarrow \dots & & \leftarrow (A_0(M))^\vee \leftarrow 0 & & \\
 & \nearrow_{\alpha_{M,n}} & & \nearrow_{\alpha_{M,i}} & & & \\
B_n(M^*) & \longleftarrow & \dots & \leftarrow B_0(M^*) & \leftarrow 0 & & \\
\tau_n^* \uparrow & \tau_n^\vee \uparrow & & \tau_i^* \uparrow & \tau_i^\vee \uparrow & & \\
 & (A_n(N))^\vee & & \dots \leftarrow (A_0(N))^\vee & \leftarrow 0 & & \\
 & \nearrow_{\alpha_{N,n}} & & \nearrow_{\alpha_{N,i}} & & & \\
B_n(N^*) & \longleftarrow & \dots & \leftarrow B_0(N^*) & \leftarrow 0. & &
\end{array}$$

Note that we skip the ${}_\triangle(\quad)$ in order not to overstress the diagram. The only leftover is to see the commutativity of the diagrams

$$\begin{array}{ccc}
{}_\triangle B_i(M^*) & \xrightarrow{\alpha_{M,i}} & ({}_\triangle A_i(M))^\vee \\
\tau_i^* \uparrow & & \uparrow \tau_i^\vee \\
{}_\triangle B_i(N^*) & \xrightarrow{\alpha_{N,i}} & ({}_\triangle A_i(N))^\vee.
\end{array}$$

But that is obviously true. Finally we derive the commutativity of

$$\begin{array}{ccc}
M^* & \xrightarrow{r_M} & \mathrm{Ext}^n(M) \\
\tau^* \uparrow & & \uparrow \mathrm{Ext}^n(\tau) \\
N^* & \xrightarrow{r_N} & \mathrm{Ext}^n(N).
\end{array}$$

□

We need to define two other equivalences of functors. They are both used in the main tool of this section, Theorem 2.5.9.

Lemma and Definition 2.5.7 (Equivalences of Functors II)**.** *We consider on* $\underline{\mathrm{grMFL}}$ *two collections of maps. Let* $M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})$*, then we define*

$$s_M : M \to \mathrm{Ext}^n(\mathrm{Ext}^n(M))$$

to be the canonical isomorphism. We can compute it as follows: Let

$$0 \leftarrow M \leftarrow F_0 \leftarrow \ldots \leftarrow F_n \leftarrow 0$$

be any graded free resolution of M*. Then* s_M *is resolved by:*

$$\begin{array}{ccccccccc} 0 \leftarrow \mathrm{Ext}^n(\mathrm{Ext}^n(M)) & \leftarrow ((F_0)^\vee)^\vee & \leftarrow ((F_1)^\vee)^\vee & \ldots & \leftarrow ((F_n)^\vee)^\vee & \leftarrow 0 \\ \uparrow^{s_M} & \uparrow^{s_{M0}} & \uparrow^{s_{M1}} & & \uparrow^{s_{Mn}} & \\ 0 \leftarrow M & \leftarrow F_0 & \leftarrow F_1 & \leftarrow \ldots & \leftarrow F_n & \leftarrow 0, \end{array}$$

where

$$s_{Mi} : \begin{cases} F_i \to ((F_i)^\vee)^\vee \\ \mathsf{a} \mapsto (\alpha \mapsto \alpha(\mathsf{a})) \end{cases}.$$

s_M *is independent from the resolution.*

Moreover we define $u_M : M \to M^{**}$ *by* $\mu \mapsto (\phi \mapsto \phi(\mu))$*.*

Both collections $\{M \mapsto s_M \mid M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})\}$ *and* $\{M \mapsto u_M \mid M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})\}$ *give obviously isomorphisms of the functors* id *and* $\mathrm{Ext}^n(\mathrm{Ext}^n(\quad))$*, respectively* id *and* $((\quad)^*)^*$*.*

Corollary 2.5.8 (Equivalences of Functors III)**.** *In* $\underline{\mathrm{grMFL}}$ *the collection* $\{M \mapsto t_M \mid M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})\}$*, with*

$$t_M := \mathrm{Ext}^n(r_M) \circ s_M : M \to \mathrm{Ext}^n(M^*),$$

is an isomorphism of functors: $\mathrm{id} \to \mathrm{Ext}^n((\quad)^*)$*.*

Proof. Let $M, N \in \mathrm{Obj}(\underline{\mathrm{grMFL}})$, and let $\tau \in \mathrm{Mor}(\underline{\mathrm{grMFL}})$, $\tau : M \to N$, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ext}^n(N^*) & \xleftarrow{\mathrm{Ext}^n(\tau^*)} & \mathrm{Ext}^n(M^*) \\ {\scriptstyle \mathrm{Ext}^n(r_N)}\uparrow & & \uparrow{\scriptstyle \mathrm{Ext}^n(r_M)} \\ \mathrm{Ext}^n(\mathrm{Ext}^n(N)) & \xleftarrow{\mathrm{Ext}^n(\mathrm{Ext}^n(\tau))} & \mathrm{Ext}^n(\mathrm{Ext}^n(M)) \\ {\scriptstyle s_N}\uparrow & & \uparrow{\scriptstyle s_M} \\ N & \xleftarrow{\tau} & M. \end{array}$$

This is true as the upper part is just $\mathrm{Ext}^n(\quad)$ of the diagram from 2.5.6. □

The following theorem describes the connection between the two functors $(\quad)^* = \mathrm{Hom}_k(\quad, k)$ and $\mathrm{Ext}^n(\quad) = \mathrm{Ext}^n_R(\quad, R \otimes_k \bigwedge^n W)$ in $\underline{\text{grMFL}}$. It is central for our main theorem. That is for the part of Theorem 0.3 we prove within this section. We state it in the language of the categories from the above isomorphisms of functors (2.5.6, 2.5.7, 2.5.8).

Theorem 2.5.9. *Let n be odd, and let $m = \frac{n-1}{2}$, and let $M, N \in \mathrm{Obj}(\underline{\text{grMFL}})$ and $\tau \in \mathrm{Mor}(\underline{\text{grMFL}})$, $\tau : M \to N$. Using the isomorphisms of functors from above the following diagram commutes:*

$$\begin{array}{ccc} M^* & \xrightarrow{r_M} & \mathrm{Ext}^n(M) \\ {\scriptstyle (-1)^m \tau^*}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Ext}^n(\tau)\circ \mathrm{Ext}^n(u_N)} \\ N^* & \xrightarrow{t_{N^*}} & \mathrm{Ext}^n(N^{**}). \end{array}$$

Remark 2.5.10. Besides the technical details the central point of this theorem is the following: For the definition of r_M we need to resolve M via the Koszul complex, and M^* via the dual Koszul complex. Moreover t_{N^*} — at least in the case $N = M^*$ — is roughly $\mathrm{Ext}^n(\quad)$ of r_M. That is why we have to resolve this time N^* via the Koszul complex and N^{**} via its dual. The nature of the Koszul complex finally gives the sign.

Let us continue with the detailed proof.

Proof of Theorem 2.5.9. Let again $P_i = R \otimes_k \bigwedge^i W$. Recall that ${}_{\Delta}A_i(M) = {}_{\Delta}(R \otimes \bigwedge^i W \otimes M)$ and ${}_{\Delta}B_i(M) = {}_{\Delta}((R \otimes \bigwedge^i W)^\vee \otimes M)$. r_M and t_{N^*} are defined in 2.5.6 and 2.5.8 using certain resolutions. We resolve now τ^* and $\mathrm{Ext}^n(\tau)$ via these resolutions. The resolutions use the complex constructions Nielsen II (1.2.8) and Nielsen IIa (1.2.9).

At first we resolve τ. Let $\ell(i) = \lfloor \frac{i-1}{2} \rfloor$, then we obtain the following commutative diagram:

$$\begin{array}{cccccc}
0 \leftarrow M \leftarrow & {}_{\Delta}A_0(M) & \ldots \leftarrow {}_{\Delta}A_m(M) & \xleftarrow{\Delta(\phi_{m+1})} & {}_{\Delta}A_{m+1}(M)\ldots & \leftarrow {}_{\Delta}A_n(M) \leftarrow 0 \\
\downarrow{\scriptstyle \tau} & \downarrow{\scriptstyle \tilde{\tau}_0} & \downarrow{\scriptstyle (-1)^{\ell(m)}\tilde{\tau}_m} & & \downarrow{\scriptstyle (-1)^{(m+\ell(m))}\tilde{\tau}_{m+1}} & \downarrow{\scriptstyle (-1)^m\tilde{\tau}_n} \\
0 \leftarrow N \leftarrow & {}_{\Delta}B_n(N) & \ldots \leftarrow {}_{\Delta}B_{m+1}(N) & \leftarrow & {}_{\Delta}B_m(N)\ldots & \leftarrow {}_{\Delta}B_0(N) \leftarrow 0 \\
\downarrow{\scriptstyle u_N} & \downarrow{\scriptstyle u_{Nn}} & \downarrow{\scriptstyle u_{N\,m+1}} & & \downarrow{\scriptstyle u_{Nm}} & \downarrow{\scriptstyle u_{N0}} \\
0 \leftarrow N^{**} \leftarrow & {}_{\Delta}B_n(N^{**}) & \ldots \leftarrow {}_{\Delta}B_{m+1}(N^{**}) & \xleftarrow{\Delta(\varphi_{m+1})} & {}_{\Delta}B_m(N^{**})\ldots & \leftarrow {}_{\Delta}B_0(N^{**}) \leftarrow 0,
\end{array}$$

where $\tilde{\tau}_i : {}_{\Delta}(R \otimes \bigwedge^i W \otimes M) \to {}_{\Delta}((R \otimes \bigwedge^{n-i} W)^\vee \otimes N)$ is given by

$$r \otimes w \otimes \mathsf{m} \mapsto \left\{ \begin{array}{c} R \otimes \bigwedge^{n-i} W \to R \otimes \bigwedge^n W \\ (r' \otimes w') \mapsto (rr') \otimes w' \wedge w \end{array} \right\} \otimes \tau(\mathsf{m}),$$

for all $\mathsf{m} \in M$, $w \in \bigwedge^i W$, $w' \in \bigwedge^{n-i} W$ and $r, r' \in R$. Moreover $u_{Ni} : P_i^\vee \otimes N \to P_i^\vee \otimes N^{**}$ is defined by $\mathrm{id} \otimes u_N$.

The diagram commutes by Lemma 2.3.2, because the $\tilde{\tau}_i$ behave exactely as the map between the Koszul complex and its dual.
Applying $(\quad)^\vee = \mathrm{Hom}_R(\quad, R \otimes \bigwedge^n W)$ to the diagram we obtain

$$\begin{array}{ccccccc}
0 \leftarrow \mathrm{Ext}^n(M) \leftarrow & ({}_\Delta A_n(M))^\vee & \ldots & ({}_\Delta A_{m+1}(M))^\vee & \overset{({}_\Delta\phi_{m+1})^\vee}{\longleftarrow} & ({}_\Delta A_m(M))^\vee & \ldots ({}_\Delta A_0(M))^\vee \leftarrow 0 \\
\uparrow {\scriptstyle \mathrm{Ext}^n(\tau)\circ \mathrm{Ext}^n(u_N)} & \uparrow {\scriptstyle (-1)^m(\tilde{\tau}_n)^\vee \circ u_{N0}^\vee} & & \uparrow {\scriptstyle (-1)^{(m+\ell(m))}(\tilde{\tau}_{m+1})^\vee \circ u_{Nm}^\vee} & & \uparrow {\scriptstyle (-1)^{\ell(m)}(\tilde{\tau}_m)^\vee \circ u_{Nm+1}^\vee} & \uparrow {\scriptstyle (\tilde{\tau}_0)^\vee \circ u_{Nn}^\vee} \\
0 \leftarrow \mathrm{Ext}^n(N^{**}) \leftarrow & ({}_\Delta B_0(N^{**}))^\vee & \ldots & ({}_\Delta B_m(N^{**}))^\vee & \overset{({}_\Delta\varphi_{m+1})^\vee}{\longleftarrow} & ({}_\Delta(B_{m+1}(N^{**}))^\vee & \ldots ({}_\Delta B_n(N^{**}))^\vee \leftarrow 0
\end{array}$$

In the same manner as above we resolve $(-1)^m\tau^*$ (also using Nielsen II and Nielsen IIa):

$$\begin{array}{cccccc}
0 \leftarrow M^* & \leftarrow {}_\Delta B_n(M^*) & \ldots \leftarrow {}_\Delta B_{m+1}(M^*) & \overset{{}_\Delta(\varphi_{m+1})}{\longleftarrow} & {}_\Delta B_m(M^*) \ldots & \leftarrow {}_\Delta B_0(M^*) \leftarrow 0 \\
\uparrow {\scriptstyle (-1)^m\tau^*} & \uparrow {\scriptstyle (-1)^m\widetilde{\tau_0^*}} & \uparrow {\scriptstyle (-1)^{m+\ell(m)}\widetilde{\tau_m^*}} & & \uparrow {\scriptstyle (-1)^{\ell(m)}\widetilde{\tau_{m+1}^*}} & \uparrow {\scriptstyle \widetilde{\tau_n^*}} \\
0 \leftarrow N^* & \leftarrow {}_\Delta A_0(N^*) & \ldots \leftarrow {}_\Delta A_m(N^*) & \overset{{}_\Delta(\phi_{m+1})}{\longleftarrow} & {}_\Delta A_{m+1}(N^*) \ldots & \leftarrow {}_\Delta A_n(N^*) \leftarrow 0,
\end{array}$$

where $\widetilde{\tau_i^*} : {}_\Delta A_i(N^*) \to {}_\Delta B_{n-i}(M^*)$ is defined by

$$(r \otimes w \otimes \nu) \mapsto \left\{ \begin{array}{c} R \otimes \bigwedge^{n-i} W \to R \otimes \bigwedge^n W \\ r' \otimes w' \mapsto (rr') \otimes w' \wedge w \end{array} \right\} \otimes \tau^*(\nu)$$

for all $\nu \in N^*$, $w \in \bigwedge^i W$, $w' \in \bigwedge^{n-i} W$ and $r, r' \in R$.

Consider the canonical isomorphism from 2.5.2, respectively 2.5.4: $\alpha_{M,i} : {}_\Delta B_i(M^*) \overset{\cong}{\to} ({}_\Delta A_i(M))^\vee$. Hence $(\alpha_{N^*,n-i})^\vee : {}_\Delta A_{n-i}(N^*) \overset{\cong}{\to} (({}_\Delta A_{n-i}(N^*))^\vee)^\vee \overset{\cong}{\to} ({}_\Delta B_{n-i}(N^{**}))^\vee$. We connect the two diagrams using them.
Let us show the commutativity of the following diagram for all $n \geq i \geq 0$:

$$\begin{array}{ccc}
{}_\Delta B_i(M^*) & \xrightarrow{\alpha_{M,i}} & ({}_\Delta A_i(M))^\vee \\
{\scriptstyle \widetilde{\tau_{n-i}^*}} \uparrow & & \uparrow {\scriptstyle (\tilde{\tau}_i)^\vee \circ (u_{N,n-i})^\vee} \\
{}_\Delta A_{n-i}(N^*) & \xrightarrow{(\alpha_{N^*,n-i})^\vee} & ({}_\Delta B_{n-i}(N^{**}))^\vee.
\end{array}$$

Let $r \in R, \nu \in N^*, \mathsf{n}^{**} \in N^{**}, w \in \bigwedge^{n-i} W, \tilde{w} \in \bigwedge^i W$ and $\mathsf{m} \in M$ be arbitrary. Moreover let $\pi \in P_{n-i}^\vee$, such that $1 \otimes w \mapsto 1 \otimes w \wedge w'$ for some fixed $w' \in \bigwedge^i W$.

$${}_\Delta A_{n-i}(N^*) \ni 1 \otimes w \otimes \nu \mapsto \left\{ \begin{array}{c} B_{n-i}(N^{**}) \to R \otimes \bigwedge^n W \\ \pi \otimes \mathsf{n}^{**} \mapsto \pi(w) \cdot \nu(\mathsf{n}) \end{array} \right\} \overset{\tilde{\tau}_i^\vee \circ u_{N,n-i}^\vee}{\longmapsto}$$
$$\left\{ \begin{array}{c} A_i(M) \to R \otimes \bigwedge^n W \\ (1 \otimes \tilde{w} \otimes \mathsf{m}) \mapsto (w \wedge \tilde{w}) \cdot \nu(\tau(\mathsf{m})) = (-1)^{i(n-i)}(\tilde{w} \wedge w) \cdot \nu(\tau(\mathsf{m})) \end{array} \right\},$$

and first applying $\widetilde{\tau_{n-i}^*}$ we have

$$1 \otimes w \otimes \nu \mapsto (1 \otimes ({}_{-} \wedge w) \otimes \nu \circ \tau) \mapsto \left\{ \begin{array}{c} A_i(M) \to R \otimes \bigwedge^n W \\ (1 \otimes \tilde{w} \otimes \mathsf{m}) \mapsto (\tilde{w} \wedge w) \cdot \nu(\tau(\mathsf{m})) \end{array} \right\},$$

which equals to the above expression as n is odd. Note again that in the last row $1\otimes(\_\wedge w)$ stands for the functional

$$R\otimes\bigwedge^{i} W \to R\otimes\bigwedge^{n} W,\quad r\otimes\tilde{w}\mapsto r\otimes(\tilde{w}\wedge w).$$

Taking into account the commutativity of the diagrams from Lemma 2.5.4 we have that the right part of the next diagram commutes (note that for the commutativity we need n to be odd). On the other hand we derive by the definition of the considered equivalences of functors on the left the diagram of the theorem. For clarity we use a reduced notation.

$$\begin{array}{ccccccccc}
0\leftarrow & \operatorname{Ext}^n(M) & \leftarrow & (A_n(M))^\vee & \leftarrow\dots & \leftarrow & (A_0(M))^\vee & \leftarrow 0 \\
 & \nearrow r_M \uparrow & & \nearrow \alpha_{M,n}\uparrow & & & \nearrow \alpha_{M,0}\uparrow & \\
0\leftarrow & M^* & \longleftarrow & B_n(M^*) & \leftarrow\dots & \longleftarrow & B_0(M^*) & \leftarrow 0 \\
 & \operatorname{Ext}^n(\tau)\uparrow\Big| & & \uparrow\quad\Big| & & & \uparrow\quad\Big| & \\
(-1)^m\tau^* & \operatorname{Ext}^n(N^{**}) & \leftarrow & (B_0(N^{**}))^\vee & \leftarrow\dots & \leftarrow & (B_n(N^{**}))^\vee & \leftarrow 0 \\
 & \Big|\nearrow t_{N^*} & & \Big|\nearrow (\alpha_{N^*,0})^\vee & & & \Big|\nearrow (\alpha_{N^*,n})^\vee & \\
0\leftarrow & N^* & \longleftarrow & A_0(N^*) & \leftarrow\dots & \leftarrow & A_n(N^*) & \leftarrow 0
\end{array}$$

□

The following example shows a little more concretely the appearance of the sign.

Example 2.5.11. We give an example using the devided powers notation from the first section.
Let $R=k[x_1,x_2,x_3]$ with $\deg x_i=1$, we consider again the examples

$$P_M=\begin{pmatrix} X_1^{(2)}X_2^{(1)} & X_2^{(2)}\\ X_2^{(2)} & X_1^{(1)}\end{pmatrix},\qquad P_N=\begin{pmatrix} X_1^{(2)} & X_2^{(1)}\\ X_2^{(1)} & 1\end{pmatrix}.$$

As a k-vector space $M=M(P_M)$ can be represented by

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}x_2\\0\end{pmatrix},\begin{pmatrix}x_1x_2\\0\end{pmatrix},\begin{pmatrix}x_1^2\\0\end{pmatrix},\begin{pmatrix}x_1^2x_2\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\x_1\end{pmatrix}\right),$$

and $N=M(P_N)$ by

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}x_2\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}x_1^2\\0\end{pmatrix}\right).$$

Moreover there is a well-defined map $\tau: M\to N$, such that

$$\begin{pmatrix}1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\0\end{pmatrix},\ \text{and}\ \begin{pmatrix}0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\1\end{pmatrix}.$$

We consider the commutative diagram $((\quad)^\vee=\operatorname{Hom}_R(\quad,R\otimes\bigwedge^3 W))$:

$$\begin{array}{ccccccccc}
0\leftarrow M^*\leftarrow & {}_\Delta B_3(M^*) & \overset{\Delta(\varphi_3)}{\longleftarrow} & {}_\Delta B_2(M^*) & \overset{\Delta(\varphi_2)}{\longleftarrow} & {}_\Delta B_1(M^*) & \overset{\Delta(\varphi_1)}{\longleftarrow} & {}_\Delta B_0(M^*)\leftarrow 0 \\
\uparrow{-\tau^*}\qquad & \uparrow{-\widetilde{\tau_0^*}} & & \uparrow{-\widetilde{\tau_1^*}} & & \uparrow{\widetilde{\tau_2^*}} & & \uparrow{\widetilde{\tau_3^*}} \\
0\leftarrow N^*\leftarrow & {}_\Delta A_0(N^*) & \overset{\Delta(\phi_1)}{\longleftarrow} & {}_\Delta A_1(N^*) & \overset{\Delta(\phi_2)}{\longleftarrow} & {}_\Delta A_2(N^*) & \overset{\Delta(\phi_3)}{\longleftarrow} & {}_\Delta A_3(N^*)\leftarrow 0.
\end{array}$$

And connect it via the isomorphisms $\alpha_{M,i}$ and $(\alpha_{N^*,n-i})^\vee$ towards behind with

$$\begin{array}{ccccccccc}
0 \leftarrow \mathrm{Ext}^3(M) \leftarrow ({}_\Delta A_3(M))^\vee & \overset{({}_\Delta\phi_3)^\vee}{\longleftarrow} & ({}_\Delta A_2(M))^\vee & \overset{({}_\Delta\phi_2)^\vee}{\longleftarrow} & ({}_\Delta A_1(M))^\vee & \overset{({}_\Delta\phi_1)^\vee}{\longleftarrow} & ({}_\Delta A_0(M))^\vee \leftarrow 0 \\
\uparrow^{\mathrm{Ext}^3(\tau)} \quad \uparrow^{-(\tilde\tau_3)^\vee} & & \uparrow^{-(\tilde\tau_2)^\vee} & & \uparrow^{(\tilde\tau_1)^\vee} & & \uparrow^{(\tilde\tau_0)^\vee} \\
0 \leftarrow \mathrm{Ext}^3(N) \leftarrow ({}_\Delta B_0(N))^\vee & \overset{({}_\Delta\varphi_1)^\vee}{\longleftarrow} & ({}_\Delta B_1(N))^\vee & \overset{({}_\Delta\varphi_2)^\vee}{\longleftarrow} & ({}_\Delta B_2(N))^\vee & \overset{({}_\Delta\varphi_3)^\vee}{\longleftarrow} & ({}_\Delta B_3(N))^\vee \leftarrow 0.
\end{array}$$

As an abbreviation we denote by $\begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}$ the functional $\mathsf{m} \mapsto \left\langle \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}, \mathsf{m} \right\rangle (0)$. In the same manner $\begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} \circ \tau$ is meant.

Let $\mathsf{m} \in M$ and $\mathsf{n} \in N$ be arbitrary. For example if we map first via ${}_\Delta(\phi_2)$, then with $-\widetilde{\tau_1^*}$ and $\alpha_{M,2}$ we obtain

$$A_2(N^*) \ni 1 \otimes \chi_1 \wedge \chi_2 \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} \mapsto x_1 \otimes \chi_2 \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} - x_2 \otimes \chi_1 \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} \overset{-\widetilde{\tau_1^*}}{\mapsto}$$

$$\left\{ \begin{array}{c} R \otimes \bigwedge^2 W \to R \otimes \bigwedge^3 W \\ r' \otimes w' \mapsto -x_1 r' \otimes (w' \wedge \chi_2) \end{array} \right\} \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} \circ \tau + \left\{ \begin{array}{c} R \otimes \bigwedge^2 W \to R \otimes \bigwedge^3 W \\ r' \otimes w' \mapsto x_2 r' \otimes (w' \wedge \chi_1) \end{array} \right\} \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} \circ \tau$$

$$\overset{\alpha_{M,2}}{\mapsto} \left\{ \begin{array}{c} {}_\Delta(A_2(M)) \to R \otimes \bigwedge^3 W \\ 1 \otimes w' \otimes \mathsf{m} \mapsto -x_1 \otimes (w' \wedge \chi_2) \cdot \left\langle \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}, \tau(\mathsf{m}) \right\rangle (0) + \end{array} \right.$$

$$\left. (w' \wedge \chi_2) \cdot \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tau(\mathsf{m}) \right\rangle (0) + x_2 \otimes (w' \wedge \chi_1) \cdot \left\langle \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}, \tau(\mathsf{m}) \right\rangle (0) \right\}.$$

Let us see how this functional for example behaves on the element

$$1 \otimes \chi_1 \wedge \chi_3 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in {}_\Delta(P_2 \otimes M) = {}_\Delta A_2(M).$$

It is mapped to $-x_1 \cdot \chi_1 \wedge \chi_3 \wedge \chi_2 = x_1 \cdot \chi_1 \wedge \chi_2 \wedge \chi_3$.

If we first apply the isomorphism $(\alpha_{N^*,2})^\vee$ towards behind, map then via $(\tilde\tau_1)^\vee$ and finally $(\phi_2)^\vee$ we have

$$1 \otimes \chi_1 \wedge \chi_2 \otimes \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix} \mapsto \left\{ \begin{array}{c} {}_\Delta(B_2(N)) \to R \otimes \bigwedge^3 W \\ \pi \otimes \mathsf{n} \mapsto \pi(\chi_1 \wedge \chi_2) \left\langle \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}, \mathsf{n} \right\rangle (0) \end{array} \right\} \overset{(\tilde\tau_1)^\vee}{\mapsto}$$

$$\left\{ \begin{array}{c} A_1(N) \to R \otimes \bigwedge^3 W \\ 1 \otimes w'' \otimes \mathsf{m} \mapsto (\chi_1 \wedge \chi_2 \wedge w'') \left\langle \begin{pmatrix} X_1^{(1)} \\ 0 \end{pmatrix}, \tau(\mathsf{m}) \right\rangle (0) \end{array} \right\}.$$

To understand its image under $(\phi_2)^\vee$ it is the easiest to see how it behaves as a functional in $({}_{\Delta}A_2(M))^\vee$. We choose the same element of ${}_{\Delta}A_2(M)$ as above. Then we have:

$$1 \otimes \chi_1 \wedge \chi_3 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \overset{\Delta(\phi_2)}{\mapsto} x_1 \otimes \chi_3 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} - x_3 \otimes \chi_1 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} =$$

$$x_1 \cdot \left(1 \otimes \chi_3 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix}\right) - 1 \otimes \chi_3 \otimes \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix} - x_3 \cdot \left(1 \otimes \chi_1 \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix}\right) \mapsto x_1 \cdot \chi_1 \wedge \chi_2 \wedge \chi_3.$$

Hence we derive the commutativity of the diagram of Theorem 2.5.9:

$$\begin{array}{ccc} M^* & \xrightarrow{r_M} & \mathrm{Ext}^3_R(M, R \otimes \bigwedge^3 W) \\ \uparrow {\scriptstyle -\tau^*} & & \uparrow {\scriptstyle \mathrm{Ext}^3(\tau)} \\ N^* & \xrightarrow{t_{N^*}} & \mathrm{Ext}^3_R(N, R \otimes \bigwedge^3 W). \end{array}$$

Now we are able to prove the central theorem of the section, Theorem 2.5.1. Together with the main theorem of the last section, Theorem 2.3.5, we gain our main result 0.3. It basically says that our definition of Gorensteiness is actually equivalent to having a selfdual resolution.

Proof of Theorem 2.5.1. Let again $(\quad)^\vee = \mathrm{Hom}_R(\quad, R(-d))$.
Using the given selfdual resolution, we define an isomorphism $\tau' : M \to \mathrm{Ext}^n_R(M, R(-d))(-s)$ as follows: We consider the dual of the resolution and obtain an obvious map of complexes as seen in the diagram. By abuse of notation, we denote by $\mathrm{id} : F_i \overset{\cong}{\mapsto} F_i^{\vee\vee}$ the canonical isomorphism, too.

$$\begin{array}{ccccccccc} 0 \leftarrow M & & \leftarrow F_0 \overset{\psi_1}{\leftarrow} & \ldots \leftarrow F_m & \overset{\psi_{m+1}}{\leftarrow} & F_m^\vee(-s) & \leftarrow \ldots & \overset{\psi_1^\vee(-s)}{\leftarrow} & F_0^\vee(-s) \leftarrow 0 \\ \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle id} & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pm id} & & & \downarrow{\scriptstyle \pm id} \\ 0 \leftarrow \mathrm{Ext}^n_R(M, R(-d-s)) & & \leftarrow (F_0^\vee)^\vee \leftarrow & \ldots \leftarrow (F_m^\vee)^\vee & \overset{\pm\psi_{m+1}}{\leftarrow} & F_m^\vee(-s) & \leftarrow \ldots & \overset{\psi_1^\vee(-s)}{\leftarrow} & F_0^\vee(-s) \leftarrow 0. \end{array}$$

Applying $\mathrm{Hom}_R(\quad, R(-d))$ to the diagram, we obtain (here $\mathrm{Ext}^n_R(\quad)$ denotes $\mathrm{Ext}^n_R(\quad, R(-d))$:

$$\begin{array}{cccccc} 0 \leftarrow \mathrm{Ext}^n_R(M) & \leftarrow (F_0^\vee(-s))^\vee & \ldots (F_m^\vee(-s))^\vee & \overset{\pm\psi_{m+1}}{\leftarrow} F_m^\vee & \ldots F_0^\vee & \leftarrow 0 \\ \uparrow{\scriptstyle \mathrm{Ext}^n_R(\tau')} & \uparrow{\scriptstyle \pm id} & \uparrow{\scriptstyle \pm id} & \uparrow{\scriptstyle id} & \uparrow{\scriptstyle id} & \\ 0 \leftarrow \mathrm{Ext}^n_R(\mathrm{Ext}^n_R(M)(-s)) & \leftarrow (F_0^\vee(-s))^\vee & \ldots (F_m^\vee(-s))^\vee & \overset{\psi_{m+1}}{\leftarrow} ((F_m^\vee)^\vee)^\vee & \ldots ((F_0^\vee)^\vee)^\vee & \leftarrow 0. \end{array}$$

We want to make use of the machinery developed within this section. Especially our aim is to apply Theorem 2.5.9. Therefore in what follows we identify $R \otimes \bigwedge^n W \cong R(-d)$, via $r \otimes \chi_1 \wedge \ldots \wedge \chi_n \mapsto r$

Recall that $s_M : M \xrightarrow{\cong} \mathrm{Ext}^n_R(\mathrm{Ext}^n_R(M, R(-d)), R(-d))$ (see 2.5.7) is the derived isomorphism from the canonical isomorphism between the resolution of M and $\mathrm{Hom}_R(\mathrm{Hom}_R(\quad, R(-d)), R(-d))$. From the diagrams we obtain

$$\mathrm{Ext}^n_R(\tau')(-s) = \pm\tau' \circ s_M^{-1}. \ (*)$$

Finally we use the isomorphism $r_M : M^* \xrightarrow{\cong} \mathrm{Ext}^n_R(M, R(-d))$ from 2.5.4 to define $\tau : M \to M^*(-s)$ as

$$\tau := r_M^{-1}(-s) \circ \tau'. \ (**)$$

We apply Theorem 2.5.9 to the situation $N = M^*(-s)$. Hence we obtain (note that by definition $t_N = \mathrm{Ext}^n(r_N) \circ s_N$ (see 2.5.8)):

$$\begin{aligned}
\tau^* = (-1)^m r_M^{-1} \circ \mathrm{Ext}^n(\tau) \circ \mathrm{Ext}^n(u_{M^*})(s) \circ t_{M^{**}}(s) &\overset{(**)}{=} \\
(-1)^m r_M^{-1} \circ \mathrm{Ext}^n(\tau') \circ \mathrm{Ext}^n(r_M)^{-1}(s) \circ \mathrm{Ext}^n(u_{M^*})(s) \circ \mathrm{Ext}^n(r_{M^{**}})(s) \circ s_{M^{**}}(s) &\overset{(*)}{=} \\
\pm(-1)^m r_M^{-1} \circ \tau'(s) \circ s_M^{-1}(s) \circ \mathrm{Ext}^n(r_M)^{-1}(s) \circ \mathrm{Ext}^n(u_{M^*})(s) \circ \mathrm{Ext}^n(r_{M^{**}})(s) \circ s_{M^{**}}(s) &\overset{(1)}{=} \\
\pm(-1)^m r_M^{-1} \circ \tau'(s) \circ u_M^{-1}(s) &\overset{(**)}{=} \\
\pm(-1)^m \tau(s) \circ u_M^{-1}(s).&
\end{aligned}$$

For (1) it remains to show the commutativity of the following diagram:

$$\begin{array}{ccccc}
M^{**} & \xrightarrow{s_{M^{**}}} & \mathrm{Ext}^n\,\mathrm{Ext}^n(M^{**}) & \xrightarrow{\mathrm{Ext}^n(r_{M^{**}})} & \mathrm{Ext}^n(M^{***}) \\
\downarrow{\scriptstyle u_M^{-1}} & (2) & \downarrow{\scriptstyle \mathrm{Ext}^n\,\mathrm{Ext}^n(u_M)^{-1}} & (3) & \downarrow{\scriptstyle \mathrm{Ext}^n(u_{M^*})} \\
M & \xleftarrow{s_M^{-1}} & \mathrm{Ext}^n\,\mathrm{Ext}^n(M) & \xleftarrow{\mathrm{Ext}^n(r_M)^{-1}} & \mathrm{Ext}^n(M^*).
\end{array}$$

Diagram (2) commutes as $\{M \mapsto s_M \mid M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})\}$ is an isomorphism of functors (see 2.5.7): $\mathrm{id} \to \mathrm{Ext}^n(\mathrm{Ext}^n(\quad))$. Apply this property to the morphism $u_M^{-1} : M^{**} \to M$. Diagram(3) commutes as

$$\begin{array}{ccc}
M^{***} & \xrightarrow{u_{M^*}^{-1}} & M^* \\
\downarrow{\scriptstyle r_{M^{**}}} & & \downarrow{\scriptstyle r_M} \\
\mathrm{Ext}^n(M^{**}) & \xrightarrow{\mathrm{Ext}^n(u_M)} & \mathrm{Ext}^n(M)
\end{array} \qquad (3a)$$

commutes by using the isomorphism of functors property of $\{M \mapsto r_M \mid M \in \mathrm{Obj}(\underline{\mathrm{grMFL}})\}$ (see 2.5.6), $(\quad)^* \to \mathrm{Ext}^n(\quad)$, and applying it to the morphism: $u_M : M \to M^{**}$ (note that $u_M^* = u_{M^*}^{-1}$). Apply $\mathrm{Ext}^n(\quad)$ to $(3a)$ and gain

$$\begin{array}{ccc}
\mathrm{Ext}^n(M^{***}) & \xleftarrow{\mathrm{Ext}^n(u_{M^*})^{-1}} & \mathrm{Ext}^n(M^*) \\
\uparrow{\scriptstyle \mathrm{Ext}^n(r_{M^{**}})} & & \uparrow{\scriptstyle \mathrm{Ext}^n(r_M)} \\
\mathrm{Ext}^n\,\mathrm{Ext}^n(M^{**}) & \xleftarrow{\mathrm{Ext}^n\,\mathrm{Ext}^n(u_M)} & \mathrm{Ext}^n\,\mathrm{Ext}^n(M).
\end{array}$$

Inverting all maps on the right hand side gives (3). Hence we have finished the proof. □

Remark 2.5.12. In the proof — omitting for a second the technical details — we see the reason why we needed t_{M^*} to be mainly $\mathrm{Ext}^n(r_{M^*})$: It is because it vanishes together with $\mathrm{Ext}^n(r_M)^{-1}$ form the definition of $\mathrm{Ext}^n(\tau)$, and we can compute τ^* in terms of τ.

Appendix A

Some Algebraic Background

A.1 Local Cohomology and Homological Algebra

In this section we want to recall some well known notions and corollaries from the theory of local cohomology. Let k be any field. In the following, let — if not further specialized — R always be a Noetherian ring and N an R-module. We are interested in the case where N is a finitely generated graded module over $R = \mathrm{Sym}(V)$ with $V = \langle x_1, \dots, x_n \rangle_k$ (as k-vectorspace) with weights $\deg x_l = d_l \geq 1$. Let $\mathfrak{m} = (x_1, \dots, x_n)$, and let $X = \operatorname{Spec} R$. In this case we work out the connection between $H^j_{\mathfrak{m}}(\quad)$ and $\mathrm{Ext}^{n-j}_R(\quad, R(-\sum_{l=1}^n d_l))$. The second part excerpts some results around the famous Theorem of Buchsbaum and Eisenbud from "Algebra Structures for Finite Free Resolutions, and some Structure Theorems for Ideals of Codimension 3" ([BE77]).

Let us now recall the used definitions.

Definition A.1 (Local Cohomology)**.** Let $\mathfrak{m} \subset R$ be any ideal. We define the zeroth local cohomology module of N to be

$$H^0_{\mathfrak{m}}(N) = \{n \in N | \mathfrak{m}^d n = 0 \text{ for some } d\}.$$

This defines by restriction of any homomorphism $\phi : N \to M$ a left exact functor, its derived functors are called $H^j_{\mathfrak{m}}$ ([Eis05, A1A]).

Definition A.2 (Cohomology with Supports)**.** Let $\mathscr{F}$ be a sheaf of modules on a topological space X and let $Y \subset X$ be a closed subset. Then let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections with support on Y ([Har67]). This a left exact functor, its derived functors are called $H^j_Y(X, \quad)$. Sometimes we write $H^j_{\mathfrak{m}}(X, \quad)$, if $X = \operatorname{Spec} R$ and $\mathfrak{m} \subset R$ is an ideal such that $Y = V(\mathfrak{m})$.

Remark A.3. If $\mathfrak{m} \subset R$ and $X = \operatorname{Spec} R$, then local cohomology is compatible with sheafification:

$$H^j_{\mathfrak{m}}(X, \widetilde{N}) = H^j_{\mathfrak{m}}(N),$$

for all $j \geq 0$, see for example [Har77, Exercise III.3.3].

Lemma A.4. *Let $X = \mathrm{Spec}(R)$. Let $\mathfrak{m} \subset R$ be an ideal, and let $m \in \mathbb{N}_{>0}$. Moreover let N be a finitely generated R-module. Then are equivalent:*

$$(i) \quad H^j_{\mathfrak{m}}(X, \widetilde{N}) = 0 \quad \text{for all } j < m$$
$$(ii) \quad \mathrm{depth}(\mathfrak{m}, N) \geq m.$$

Proof. [Har67, Theorem 3.8]. □

Lemma A.5. *Let $\mathfrak{m} \subset R$ be any ideal and let $U = \mathrm{Spec}(R) - V(\mathfrak{m})$. Then there is an exact sequence:*

$$0 \to H^0_{\mathfrak{m}}(N) \to N \to H^0(U, \widetilde{N}|_U) \to H^1_{\mathfrak{m}}(N) \to 0$$

and there are isomorphisms

$$H^j(U, \widetilde{N}|_U) \cong H^{j+1}_{\mathfrak{m}}(N)$$

for all $j > 0$.
The same is true for the following setting on the weighted projective space: Let $R = \mathrm{Sym}(V)$ *and* $\mathfrak{m} = (x_1, \ldots, x_n)$. *Consider the projective space* $\mathbb{P}(d_1, \ldots, d_n) = \mathrm{Proj}(R)$. *Let* N *be a finitely generated graded* R*-module, and consider instead of* $H^j(U, \widetilde{N}|_U)$ *the direct sum* $\bigoplus_d H^j(\mathrm{Proj}(R), \widetilde{M(d)})$.

Proof. [Har67, Proposition 2.1] and [Eis94, Theorem A.4.1] □

Lemma A.6. *Let* $R = \mathrm{Sym}(V)$, *and let* $\mathfrak{m} = (x_1, \ldots, x_n)$. *Let* N *be a finitely generated graded* R*-module. Then for all* j *we have the following (homogeneous) isomorphism of* R*-modules*

$$(H^j_{\mathfrak{m}}(N))^* \cong \mathrm{Ext}^{n-j}_R(N, R(-\sum_{i=1}^{n} d_i)),$$

where $(\quad)^*$ *denotes* $\mathrm{Hom}_k(\quad, k)$.

Proof. [BH93, Theorem 3.6.29]. □

Lemma A.7. *Let* $R = \mathrm{Sym}(V)$, *and* $\mathfrak{m} = (x_1, \ldots, x_n)$. *Let* N *be a finitely generated graded* R*-module. Then if* $j < \mathrm{depth}(\mathfrak{m}, N)$ *or* $j > \dim N = \dim \mathrm{ann}\, N$, *we have*

$$H^j_{\mathfrak{m}}(N) = 0.$$

Proof. The assertion on depth is a special case of A.4 using A.3. For the second assertion we use A.6 and the Cohen-Macaulayness of R and can follow the proof of [Eis05, A 1.16] for this case. □

For completeness we recall the Auslander-Buchsbaum formula, which is valid in the graded and the local case.

Lemma A.8 (Auslander-Buchsbaum)**.** *Let* $(R, \mathfrak{m})$ *be any local (not necessarily Noetherian) ring (respectively a positively graded ring, such that* $\mathfrak{m} = \oplus_{i>0} R_i$*), and let* N *be a finitely generated (respectively graded)* R*-module. Let* N *be of finite projective dimension. Then we have the following connection between the projective dimension and the* depth *of* N

$$\mathrm{pd}(N) = \mathrm{depth}(\mathfrak{m}, R) - \mathrm{depth}(\mathfrak{m}, N).$$

Proof. [Eis94, Theorem 19.9 and Exercise 19.8]. □

Another very powerful lemma of Peskine and Szpiro is the following:

Lemma A.9 (Acyclicity Lemma)**.** *Let*

$$K : 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{\phi_1} F_0$$

be a complex of finitely generated modules F_j over a local ring R with maximal ideal $\mathfrak{m}$ such that $\operatorname{depth}(\mathfrak{m}, F_j) > j$*. If there is a homology module $H_jK \neq 0$ for some $j > 0$, then there is some j such that we have* $\operatorname{depth}(\mathfrak{m}, H_jK) \geq 1$*.*

Proof. See [PS72, Lemma 1.8] or [Eis94, Lemma 20.11]. □

Definition A.10. Let R be any ring, and let $l \geq 1$ be an integer. Let $\phi : F \to G$ be a morphism of free modules over R. Then denote by $I_l(\phi)$ the ideal of all $l \times l$-minors of ϕ. If $l = \operatorname{rank} \phi$, then define $I(\phi) := I_l(\phi)$.

Lemma A.11 (Acyclicity Lemma, Second Version)**.** *Let R be a Noetherian ring. Let*

$$K : 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{\phi_1} F_0$$

be a complex of finitely generated free R-modules F_j. Then K is exact if and only if for each j:

$$\operatorname{rank} \phi_j + \operatorname{rank} \phi_{j+1} = \operatorname{rank} F_j \text{ and}$$
$$\operatorname{depth} I(\phi_j) \geq j \text{ or } I(\phi_j) = R.$$

Proof. [BE77, Theorem 3.1]. □

Definition A.12 (alternating)**.** Let R be a any ring, and let F be a finitely generated free R-module. Let $(\quad)^\vee = \operatorname{Hom}_R(\quad, R)$. Then a morphism $\phi : F^\vee \to F$ is called *alternating* if ϕ is skew symmetric with respect to dual bases. I.e. for any basis B of F let $B^\vee$ be the dual basis of $F^\vee$. Then $M_B^{B^\vee}(\phi) = -(M_B^{B^\vee}(\phi))^t$ and $M_B^{B^\vee}(\phi)_{(i,i)}$ for all i.

Remark A.13 (Pfaffian)**.** Assume the situation of Definition A.12. If $\operatorname{rank} F$ is odd, then $\det(\phi) = 0$, if $\operatorname{rank} F$ is even, then there is some element $\operatorname{Pf}(\phi) \in R$, called the *Pfaffian* of ϕ (a polynomial in the entries of ϕ), such that

$$\det(\phi) = (\operatorname{Pf}(\phi))^2.$$

If $m = \operatorname{rank} F$ is odd one considers the ideal generated by the Pfaffians of the skew symmetric matrices ϕ_j resulting form deleting the j-th row and column (we identify ϕ and its matrix). We define $\operatorname{Pf}_{n-1}(\phi)$ to be the ideal generated by these submaximal Pfaffians of ϕ. If $l \geq 1$ is any integer, then call by $\operatorname{Pf}_l(\phi)$ the ideal of all Pfaffians of all skew symmetric $l \times l$-submatrices (obtained from ϕ by deleting rows and corresponding columns).

Remark A.14. Note the following for our usage of the term *depth*: Let I be an ideal in a ring R. Let N be any module over R. Then $\operatorname{depth}(I, N)$ denotes the length of a maximal N-sequence in I. Whereas by $\operatorname{depth} I$ we mean $\operatorname{depth}(I, R)$. Sometimes in the literature the last one is called $\operatorname{grade}(I)$.

Definition A.15 (Canonical Module — local case)**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. A finitely generated R-module ω_R is a *canonical module of R* if

$$\operatorname{Ext}^j_R(R/\mathfrak{m}, \omega_R) \cong \begin{cases} 0 & \text{for } j \neq d, \\ R/\mathfrak{m} & \text{for } j = d. \end{cases}$$

Definition A.16 (Canonical Module — graded case)**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay graded local ring of dimension d. A finitely generated graded R-module ω_R is a *canonical module of R* if there exist homogeneous isomorphisms

$$\operatorname{Ext}^j_R(R/\mathfrak{m}, \omega_R) \cong \begin{cases} 0 & \text{for } j \neq d, \\ R/\mathfrak{m} & \text{for } j = d. \end{cases}$$

Theorem A.17. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay graded local ring, and let ω_R be a graded canonical module of R. Then ω_R is uniquely determined up to homogeneous isomorphism.*

Proof. [BH93, Proposition 3.6.9] □

Theorem A.18 (Construction of Canonical Modules)**.** *Let $(S, \mathfrak{n})$ be a graded Cohen-Macaulay ring with canonical module ω_S. Let $(R, \mathfrak{m})$ be a graded Cohen-Macaulay S-algebra which is finitely generated as S-module. Then R has a canonical module. If $c = \dim S - \dim R$, then*

$$\omega_R \cong \operatorname{Ext}^c_S(R, \omega_S).$$

Proof. [BH93, Proposition 3.6.12] □

Example A.19. Let $R = k[x_1, \ldots, x_n]$ be the weighted polynomial ring with variables of degree $d_1, \ldots, d_n > 0$, and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal. Then the Koszul complex yields a graded free resolution of $R/\mathfrak{m}$ with last term $R(-\sum_{l=1}^n d_l)$. If we apply $\operatorname{Hom}(\quad, R)$ to the complex we obtain that $\operatorname{Ext}^n_R(R/\mathfrak{m}, R) = (R/\mathfrak{m})(\sum_{l=1}^n d_l)$. Hence $\omega_R = R(-\sum_{l=1}^n d_l)$.

Definition A.20 (Gorenstein Ring— local case)**.** A Cohen-Macaulay ring $(R, \mathfrak{m})$ is *Gorenstein* if ω_R exists and is isomorphic to R.

Definition A.21 (Gorenstein Ring— graded case)**.** A Cohen-Macaulay graded local ring $(R, \mathfrak{m})$ with canonical module ω_R is *Gorenstein* if $\omega_R \cong R(a)$ with a homogeneous isomorphism for some $a \in \mathbb{Z}$.

Example A.22. Let R be the weighted polynomial ring from A.19. Then R is Gorenstein. Moreover let $f_1, \ldots, f_c$ be weighted homogeneous polynomials of degrees $\delta_1, \ldots, \delta_c$ such that $f_1, \ldots, f_c$ form a regular sequence. Then $R' = R/(f_1, \ldots, f_c)$ is Gorenstein with canonical module $\omega_{R'} = R'(-\sum_{i=0}^n \epsilon_i + \sum_{j=1}^c \delta_j)$ (see [Eis94, Exercise 21.16]).

Let us continue with the famous Theorem of Buchsbaum and Eisenbud.

Theorem A.23 (Buchsbaum and Eisenbud)**.** *Let R be Noetherian and local with maximal ideal $\mathfrak{m}$.*

1. *Suppose $m \geq 3$ is an odd integer, and let F be a free R-module of rank m. Let $\phi : F^\vee \to F$ be an alternating map of rank $m-1$ whose image is contained in $\mathfrak{m}F^\vee$. Let $\mathrm{Pf}_{m-1}(\phi)$ be the ideal generated by the submaximal Pfaffians of ϕ. Then $\operatorname{depth}\mathrm{Pf}_{m-1}(\phi) \leq 3$, and if $\operatorname{depth}\mathrm{Pf}_{m-1}(\phi) = 3$, then $\mathrm{Pf}_{m-1}(\phi)$ is a Gorenstein ideal. Moreover in this case the complex*

$$R \overset{\bigwedge^{(m-1)/2}(\phi)}{\longleftarrow} F \overset{\phi}{\leftarrow} F^\vee \overset{(\bigwedge^{(m-1)/2}(\phi))^\vee}{\longleftarrow} R^\vee \leftarrow 0$$

   is exact. Here $\bigwedge^{(m-1)/2}(\phi)$ denotes the linear map defined by the submaximal Pfaffians of ϕ and an alternating sign (here ϕ is identified with a representing matrix).

2. *Conversely any Gorenstein ideal I of $\operatorname{depth} I = 3$ arises as in 1).*

Proof. [BE77, Theorem 2.1] □

Remark A.24 (Buchsbaum and Eisenbud)**.** Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k, and let F be a free R-module of odd rank m. Let $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, and let $(\quad)^\vee = \mathrm{Hom}_R(\quad, R(d))$ for some $d \in \mathbb{Z}$. Let $\phi : F^\vee \to F$ be a homogeneous alternating map of rank $m-1$. Under these assumptions the Theorem of Buchsbaum and Eisenbud A.23 is also valid. See for example [IK99, Theorem B.2.].

Lemma A.25. *Let R be any ring, and let $\phi : F^\vee \to F$ be an alternating morphism. Let $m \geq 1$ be an integer. Then*

1. *$I_{2m}(\phi) \subset \mathrm{Pf}_{2m}(\phi) \subset \mathrm{rad}(I_{2m}(\phi))$,*
2. *$I_{2m-1}(\phi) \subset \mathrm{Pf}_{2m}(\phi)$,*
3. *If $\operatorname{rank}\phi = 2m-1$ is odd, then $I_{2m-1}(\phi)$ is nilpotent. If $\operatorname{rank} F = 2m-1$, then $\det\phi = 0$.*

Proof. [BE77, Corollary 2.6] □

Lemma A.26. *Let $F_2 \overset{\phi_2}{\to} F_1 \overset{\phi_1}{\to} F_0$ be an exact sequence of free R-modules. Then*

$$\mathrm{rad}(I(\phi_2)) \supset \mathrm{rad}(I(\phi_1)).$$

Proof. [BE77, Proposition 3.2] □

A.2 Divided Powers

In this section let k be a field of arbitrary characteristic if not mentioned otherwise. We follow the notation of the books of Iarrobino and Kanev ([IK99]) and Eisenbud ([Eis94]). Let $R = k[x_1, \dots, x_n] = \bigoplus_{j\geq 0} R_j$ be the weighted polynomial ring, with weights $\deg x_i = d_i \geq 1$.

Let $\mathcal{D}$ be the graded dual of R, i.e.

$$\mathcal{D} = \mathrm{grHom}_k(R, k) = \bigoplus_{j\geq 0} \mathrm{Hom}_k(R_j, k) = \bigoplus_{j\geq 0} \mathcal{D}_{-j} \xrightarrow{pr} \mathcal{D}_0 = k.$$

We denote by $x^u = x_1^{u_1} \cdots x_n^{u_n}$, $u_i \in \mathbb{N}$ and $\|u\| = d_1 \cdot u_1 + \cdots + d_n \cdot u_n = j$, the standard monomial basis of R_j. By $X_1 = X_1^{(1)}, \dots, X_n = X_n^{(1)}$ we mean the basis dual to $x_1, \dots, x_n$ (here $V = \langle x_1, \dots, x_n\rangle$ can be viewed as sub vectorspace of R) and by $X^{(u)} = X_1^{(u_1)} \cdots X_n^{(u_n)}$ the basis elements of $\mathcal{D}_{-j}$ dual to the basis $\{x^u : \|u\| = j\}$.

There is a left action of $GL_n(k)$ on V given by $A \cdot x_i = \sum_{j=1}^n A_{ji} x_j$. It extends to an left action on $R = \mathrm{Sym}(V)$. By duality we have also a left action of $GL_n(k)$ on $\mathcal{D}$ defined by $GL_n(k) \times \mathcal{D} \to \mathcal{D} : (A, f) \mapsto f(A^{-1}\ )$.

Following [Eis94] one can define an algebra structure on $\mathcal{D}$. Consider the diagonal map $\tilde{\Delta} : V \to V \oplus V$, defined by $x \mapsto (x, x)$, and view V as k-vectorspace. Then we have the following map of symmetric algebras

$$R = \mathrm{Sym}(V) \to \mathrm{Sym}(V \oplus V) \xrightarrow{\cong} \mathrm{Sym}(V) \otimes_k \mathrm{Sym}(V) = R \otimes_k R,$$

defined for elements on V by

$$x \mapsto (x, x) \mapsto x \otimes 1 + 1 \otimes x$$

and by its linear extension. Call it Δ. The map $\tau : \mathcal{D} \otimes \mathcal{D} \to \mathrm{grHom}_k(R \otimes R, k)$ is defined by $\tau(d_1 \otimes d_2)(x \otimes y) = d_1(x) d_2(y)$. We obtain by dualizing Δ:

$$\psi = \Delta^* \circ \tau : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}.$$

With that multiplication $\mathcal{D}$ is called *divided power algebra.* It is infact an associative and commutative algebra ([Eis94]Proposition A.2.4).

One can treat the development of divided powers from a more axiomatic point of view which we do in the following.

Definition A.20 (System of Divided Powers)**.** Let A be graded commutative k-algebra with $A_0 = k$, then a *system of divided powers* in A consists of a collection of functions, one for each integer $d \geq 0 : X \mapsto X^{(d)}$, defined on $A_{>0} = \cup_{i>0} A_i$. These functions fullfill the following axioms for all $X, Y \in A_{>0}$:

$$\begin{aligned}
X^{(0)} = 1,\ X^{(1)} = X,\ \deg X^{(d)} &= d \cdot \deg X,\\
X^{(d)}X^{(e)} &= (d+e)!/(d!e!)X^{(d+e)},\\
(X^{(d)})^{(e)} &= ((de)!/e!(d!)^e)X^{(de)},\\
(XY)^{(d)} = d!X^{(d)}Y^{(d)} = X^dY^{(d)} &= X^{(d)}Y^d,\\
(aX)^{(d)} &= a^dX^{(d)} \quad \text{for} \quad a \in A_0, \quad \text{and}\\
(X+Y)^{(d)} &= \sum_{e=0}^{d} X^{(e)}Y^{(d-e)}.
\end{aligned}$$

Remark A.21. If k is of charateristic $\mathrm{char}(k) = 0$ then the existence is trivial for any algebra A as in Definition A.20 : Simply take $X^{(d)} = X^d/d!$. Moreover from the second relation of A.20 we get that this choice is unique.

More concrete in the case of $A = \mathcal{D}$ over k with $\mathrm{char}(k) = 0$ we can set

$$X^{(u)} = \frac{1}{u!}\frac{\partial}{\partial x^d} = \frac{1}{u_1! \cdot \ldots \cdot u_n!}\frac{\partial}{\partial x_1^{u_1}} \cdots \frac{\partial}{\partial x_n^{u_n}}.$$

The existence of a system of divided powers is true over any field:

Proposition A.22. *The algebra $\mathcal{D} = \mathrm{grHom}_k(\mathrm{Sym}(V), k)$ has a system of divided powers. Moreover $\mathcal{D}$ is freely generated as a vectorspace over k by the divided powers which form a dual basis to the basis (x^u) of $\mathrm{Sym}(V)$.*

Proof. [Eis94, Propositions A 2.6 and A 2.7] □

Example A.23. The dual basis to (x^u) at the beginning of this section, which we have also denoted by $(X^{(u)})$, consists of divided powers, which are generators of the divided power algebra $\mathcal{D}$. For example we show that $X_1^{(2)} \cdot X_1^{(1)} = \frac{(2+1)!}{2!1!}X_1^{(3)}$ with respect to the above defined multiplication ψ on $\mathcal{D}$.

$$X_1^{(2)}X_1^{(1)}(x_1^3) = \tau(X_1^{(2)} \otimes X_1^{(1)})(((x_1,x_1) \otimes_S (x_1,x_1)) \otimes_S (x_1,x_1)) =$$
$$\tau(X_1^{(2)} \otimes X_1^{(1)})(((x_1^2 \otimes 1) + 2(x_1 \otimes x_1) + (1 \otimes x_1^2)) \otimes_S (x_1 \otimes 1 + 1 \otimes x_1)) =$$
$$\tau(X_1^{(2)} \otimes X_1^{(1)})(((x_1^3 \otimes 1) + 2(x_1^2 \otimes x_1) + (x_1 \otimes x_1^2)) + ((x_1^2 \otimes x_1) + 2(x_1 \otimes x_1^2) + (1 \otimes x_1^3))) = 3.$$

Note that within this notation if we write for example $(x_1, x_1) \otimes_S (x_1, x_1)$, equivalently as $(x_1e_1 + x_1e_2) \otimes_S (x_1e_1 + x_1e_2)$, then $((x_1^2 \otimes 1) + 2(x_1 \otimes x_1) + (1 \otimes x_1^2))$ means $x_1^2 e_1^{\otimes_S 2} + 2x_1e_1 \otimes_S x_1e_2 + x_2^2e_2^{\otimes_S 2}$, where $\otimes_S$ denotes the symmetric algebra.

Definition A.24. For all $(i,j) \in \mathbb{N}_0^2$ we define a contraction map

$$R_i \times \mathcal{D}_{-j} \longrightarrow \mathcal{D}_{-j+i}$$

as follows: Let $\phi \in R_i$ and $f \in \mathcal{D}_{-j}$. If $j < i$ then let $\phi \cdot f$ be 0 and else the functional

$$\psi \longmapsto f(\phi\psi) \text{ for } \psi \in R_{j-i}.$$

Remark A.25. $\mathcal{D}$ has a graded R-module structure with the above contraction action. That is why we consider the negative grading of $\mathcal{D}$.
The induced map $R_i \times \mathcal{D}_{-i} \to \mathcal{D}_0 \cong k$ is a perfect pairing. All elements of $\mathcal{D}$ have negative degree, e.g. $X_1^{(2)} X_2^{(1)}$ has degree -3. In this context a graded homomorphism is defined as usually. Moreover in terms of basis one has

$$x_1^{u_1} \cdots x_n^{u_n} \cdot X_1^{(j_1)} \cdots X_n^{(j_n)} = X_1^{(j_1-u_1)} \cdots X_n^{(j_n-u_n)}.$$

Lemma A.26. *The contraction map is equivariant with respect to the left action of $A \in GL_n(k)$ mentioned above.*

Proof. We have that $A \cdot (\phi \cdot f) = (A \cdot \phi) \cdot (A \cdot f)$, because if $\phi \in R_i$, $f \in \mathcal{D}_{-j}$ and $\psi \in R_{j-i}$ then

$$A \cdot (\phi \cdot f)(\psi) = (\phi \cdot f)(A^{-1}\psi) = f(\phi(A^{-1}\psi)) = f(A^{-1}(A\phi)(A^{-1}\psi)) =$$
$$f(A^{-1}((A\phi)\psi)) = ((A \cdot \phi) \cdot (A \cdot f))(\psi)$$

□

Let us recall the famous Theorem of Macaulay ([Mac16, §60]). It states the correspondence between homogeneous forms in divided powers and graded Artinian Gorenstein quotients. Again we use the formulation of Iarrobino and Kanev.

Theorem A.27 (Macaulay)**.** *There is a bijective correspondence*

$$\{f \in \mathcal{D}_j \ mod\ k^*\} \xrightarrow{\sigma} \{A = R/I\}$$

with A a graded Artinian Gorenstein quotient of socle degree j. The correspondence is given by $\sigma(f) = R/\operatorname{Ann}_R(f)$, where $\operatorname{Ann}_R(\quad)$ is the annihilator of f under the contraction map.

Proof. [IK99, Lemma 2.12] □

Appendix B

Program Code for Gorenstein Modules

All the following code is written in [Macaulay2].
The next example is for computing the the module $M(P)$, see 2.1.5, where P is matrix given in divided powers. Moreover one might verify Theorem 2.1.11.

Example B.1. The procedure `dualModule(P)` computes the module $M(P)$ for any given matrix P in divided powers.

```
dualModule=(P)->(
     R:=ring P;
     v:= (max degrees R)#0;
     deg1:= degrees target P;
     deg2:= degrees source P;
     s := max flatten apply(#deg2,i-> max(apply(#deg1,j-> deg2#i-deg1#j)));
     t := min degrees target P;
     t1:= -(t#0);
     T:= target P;
     S:= source P;
     u:= -max degrees S;
     Sd:= degrees S;
     S1:= R^(apply(#Sd,i-> (Sd#i)#0));
     F1:= (T)**R^t;

--- possible basis of the annihilator
     As := apply((s+v+1),i->basis(i,F1));
     s1 := rank target P;
     s2 := rank source P;
     AAn:=map(F1,R^{},0);

--- compute the annihilator
     for j from 0 to (s+v) do (
     A:=As#j;
     while (rank source A == 0) and (j < (s+v)) do (j=j+1; A=As#j;);
     s3:= rank source A;

--- we make an ansatz A*P, simulating the contraction action of the
--- base ring on P
     A2a:= matrix(apply(s2,l->apply(s3,m -> sum(s1,k->
     contract((A_(k,m),P_(k,l)))))));
      A2:=map(S1**R^u,R^{s3:-s+j},A2a);

--- and compute the annihilator via constant syzygies
     syzA2:=syz(A2,DegreeLimit => (s-j));
     s4:= rank source syzA2;
     syzA22:=map(R^{s3:-j},R^{s4:-j},syzA2);
```

```
     AA := A*syzA22;

--- saving all annihilator vectors
     AAn=map(F1,directSum(source AAn, source AA),(AAn|AA)););
    coker AAn);
```

For a given module of finite length over the weighted polynomial ring we compute the associated matrix in divided powers.

Example B.2. `computeP(M)` computes for any module of finite length over the (weighted) polynomial ring an associated matrix in divided powers. That means $M(\mathtt{P}) \cong \mathtt{M}$. In general the computed matrix `P` is pretty redundant, in the sense that is much larger than it needs to be.

```
computeP=(M)->(
    R:=ring M;
    n:=dim R;
    s:=sum(n,i->(degrees vars R)#1#i#0);
    Rcan:=R^{-s};
    ResM:=res M;
    F0:=target(ResM.dd_1);
    Fn:=source(ResM.dd_(n));

--- in the free resolution F0 and Fn give information on the
--- socle degree of the module M, which restricts the
--- base vectors one has to consider for P
    sockeldegree:=(max degrees Fn)#0-s;
    Fnc:=Fn**(R^{s+sockeldegree-1});
    dFn:=unique degrees Fnc;
    Pa:={basis(sockeldegree-1+(dFn#0)#0,F0)};
    apply((#dFn-1),i->Pa=join(Pa,{basis(sockeldegree-1+(dFn#(i+1))#0,F0)}));

--- A is the presentation matrix of M, we make again
--- an ansatz and simulate the contraction action of
--- A on Pa
    A:= ResM.dd_1;
    cond:=apply(#Pa,l->(matrix apply((rank source A),i->
    apply((rank source Pa#l),j->sum((rank target Pa#l),k->
    contract(A_(k,i),(Pa#l)_(k,j))))))));
    cond2:=apply(#cond,l->(map(R^(-degrees source A),
    R^(-degrees source Pa#l),cond#l)));
    db:=apply(#cond2,l->max degrees source cond2#l);

--- the syzygy command gives a possible solution for P
    sol:=apply(#cond2,l->syz(cond2#l,DegreeLimit=>db#l));
```

```
    P1:=apply(#sol,l->Pa#l*sol#l);
    P11:=P1#0;
    apply((#P1-1),l->P11=(P11|P1#(l+1)));
    P:=map(F0,,P11);
    P);
```

It is more complicated to compute a associated symmetric matrix in divided powers for a given of finite length. Naturally this is only possible if the module is Gorenstein (see Chapter 3). `computePsymm(M)` gives an associated symmetric divided power matrix for `M` back, if possible. `M` should be defined over a (weighted) polynomial ring in an odd number of variables.

```
computePsymm=(M)->(
    R:=ring M;
    varsR:=apply(rank source vars R,i-> (vars R)_(0,i));
    dvarsR:= flatten (degrees vars R)#1;
    kk:=coefficientRing R;
    n:=dim R;
    s:=sum(n,i->(degrees vars R)#1#i#0);
    Rcan:=R^{-s};
    ResM:=res M;
    F0:=target(ResM.dd_1);
    Fn:=source(ResM.dd_(n));
    sockeldegree:=(max degrees Fn)#0-s;
    dF0:=flatten degrees F0;
    dF0u:=unique dF0;
    Pa:=apply(#dF0u,i->basis(sockeldegree-dF0u#i,F0));

--- as above Pa provides a basis of vectors which can
--- occur in P
    ddF0u:=apply(#dF0u,i->sum(#dF0,j-> if((dF0#j)==(dF0u#i)) then 1 else 0));
    ddF0u2:=apply(#ddF0u,i->sum(i,j->ddF0u#j));
    len:=#dF0;
    varnb:=sum(len,i->sum(len-i,j->(rank source
    basis((sockeldegree-dF0#(j+i)-dF0#i),R))));

--- S sets up a new polynomial ring in variables standing
--- for the coefficients of the entries of symmetric power matrix
--- in general forms
    varS:=join(varsR,apply(varnb,i->a_(i+1)));
    dvarS:=join(dvarsR,{varnb:1});
    S:=kk[varS,Degrees=>dvarS];
    dP:=matrix apply(len,i->apply(len,j->(rank source
    basis((sockeldegree-dF0#(j)-dF0#i),R))));
    dP2:=matrix apply(len,i->apply(len,j-> if (i<=j) then
```

```
    sum(i+1,k-> if (k<i) then sum(len-k,l->dP_(k,l+k)) else
    sum(j-i+1,l->dP_(l+i,i))) else 0));
    dP3:=matrix apply(len,i->apply(len,j-> if (i==j) then dP2_(i,j)
    else 0));
    dP4:= dP2+transpose dP2-dP3;
    dP5:=apply(#dF0u,i->dP4_{ddF0u2#i..(ddF0u2#i+ddF0u#i-1)});
    dP6:=apply(#dF0u,i->dP_{ddF0u2#i..(ddF0u2#i+ddF0u#i-1)});
    A:=apply(#dF0u,l-> matrix apply(ddF0u#l,i->flatten
    apply(len,j->apply((dP6#l)_(j,i),k->a_((dP5#l)_(j,i)-(dP6#l)_(j,i)+k+1)))));
    A2:=apply(#dF0u,l->map(S^(-degrees source Pa#l),,transpose A#l));
    A3:=apply(#dF0u,l->sub(Pa#l,S)*A2#l);
    A4:=A3#0;
    apply(#A3-1,l->A4=(A4|A3#(l+1)));

--- Ages is finally the symmetric ansatz in general forms in divided power
--- polynomials over S
    Ages:=map(S^(-degrees F0),,A4);
    M1:= sub(ResM.dd_1,S);
    len2:=rank source M1;

--- cond gives the conditions from the contraction action of the
--- presentation matrix M1 on Ages
    cond:= gens ideal matrix apply(len2,i->apply(len,j->sum(len,k->
    contract(M1_(k,i),(Ages)_(k,j)))));
    S2:=kk[apply(varnb,i->a_(i+1))];
    md:=max dF0u;
    mid:=min dF0u;
    xd:=md-mid;
    xs:=apply(xd,i->sub(basis(i+1,R),S));
    I:=ideal mingens ideal sub(sub(cond,S2),S);
    apply(xd,i->I=I+ideal mingens ideal sub(sub(contract(xs#i,cond),S2),S));

--- sol is the solution for the coefficients in this case
    sol:=vars S%I;
    I2:=sub(sol,S2);
    I22:=mingens ideal sub(I2,S);
    sol2:=apply(rank source I22,i-> I22_(0,i)=>random(0,S));
    Ages2:=sub(sub(Ages,sol),sol2);

--- Ages3 is finally the associated symmetric matrix in divided powers
    Ages3:=map(R^(-degrees F0),,sub(Ages2,R));
    Ages3);
```

The following code describes how to compute a symmetric respectively skew symmetric resolution for a Gorenstein module of finite length. We give in `resolutionsymm` the symmetric approach. The skew symmetric approach is analogous.

Example B.3.
```
resolutionsymm=(MP)->(
     R := ring MP;
     kk:= coefficientRing(R);
     n := rank source vars R;
     middlematrix:=floor(n/2)+1;
     resMPn:=res MP;
     deg1:=degrees target resMPn.dd_middlematrix;
     deg2:=degrees source resMPn.dd_middlematrix;
     sizemat:= #deg1;
     deg2i:=apply(sizemat,i->deg2#(sizemat-i-1));
     deg3:= flatten apply(sizemat,i->apply(sizemat,j-> if(deg2i#j-deg1#i > {0})
     then deg2i#j-deg1#i else {1}));
     S:=kk[d_(0,0)..d_((sizemat-1),(sizemat-1)),Degrees=>deg3];

--- degrees in d1, constant part to 0
     d1:=(matrix apply(sizemat,i->apply(sizemat,j-> if (deg2i#j-deg1#i > {0})
     then d_(i,j) else 0)));
     d2:=flatten matrix(d1-transpose d1);
     d3:=ideal d2;
     d4:=vars S%d3;

--- invert gradings
     deg1n:=apply(sizemat,i->-deg1#i);
     deg2in:=apply(sizemat,i->-deg2i#i);
     D:=map(S^deg1n,S^deg2in,substitute(d1,d4));

--- D is a homogeneous symmetric ansatz
     SR:=S**R;
     An:=resMPn.dd_(middlematrix-1);
     I:=gens ideal (sub(An,SR)*sub(D,SR));
     varsD:= mingens ideal sub(D,SR);
     Mdiff:=sub(transpose diff(transpose(varsD),I),R);

--- only degree 0 syzygies von are relevant
     DD:=syz(Mdiff,DegreeLimit => 0);

 --- chose a random solution
     r:= random(R^(rank source DD),R^1);
     DD2:=DD*r;
     ID:=ideal(varsD-(transpose(sub(DD2,SR))));
     SolD:=vars SR%ID;
```

```
     D1:=sub(sub(D,SR),SolD);
     Db:=sub(D1,R);

--- form new complex with respect to dual basis
     use R;
     resDb:=res image Db;
     C=new ChainComplex; C.ring=R;
     apply(middlematrix,i->C#i=resMPn_i);
     apply((middlematrix-1),i->C.dd#(i+1)=resMPn.dd_(i+1));
     apply(middlematrix,i->C#(i+middlematrix)=resDb_i);
     C.dd#(middlematrix)=Db;
     apply((middlematrix-1),i->C.dd#(i+middlematrix+1)=resDb.dd_(i+1));
     C)
```

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Glossary of Nonstandard Notations

α_B	canonical isomorphism	48
α_i	canonical isomorphism	31
$\alpha_{M,i}$	$= \alpha_{P_i^\vee \otimes M}$	51
$\mathrm{Ann}_R(P)$	annihilator of P	15
$A_i(M)$	$A_i(M) = R \otimes_k \bigwedge^i W \otimes_k M$	7
$A_{(-j-i,j)}(M)$	double complex of $A_i(M)$	8
β_i	Morphism $A_i(M) \to A_{n-i}^\vee(M)$	34
$\beta_{i,j}$	Betti number	3
$B_i(M)$	free module with dual Koszul complex	10
$\langle \phi, f \rangle$	contraction action	14
$\langle f, \phi \rangle(0)$	projection to $\mathcal{D}_0$	15
`computePsymm`	procedure in [MACAULAY2]	74
`computeP`	procedure in [MACAULAY2]	73
${}_\triangle(\quad)$	free module with diagonal action	9
δ_i	Koszul map	5
$\mathcal{D}$	divided powers	67
d	sum of the degrees	31
`dualModule`	procedure in [MACAULAY2]	72
$\mathrm{Ext}^n(\quad)$	$= \mathrm{Ext}_R^n(\quad, R \otimes \bigwedge^n W$	50
$\underline{\mathrm{grMFL}}$	category of graded R-modules of finite length	50

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