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Ernst-Ulrich Gekeler

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**Ernst-Ulrich Gekeler**

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
gekeler@math.uni-sb.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

# FROBENIUS ACTIONS ON THE DE RHAM COHOMOLOGY OF DRINFELD MODULES

ERNST-ULRICH GEKELER

Abstract. We study the action of endomorphisms of a Drinfeld $A$-module $\phi$ on its de Rham cohomology $H_{DR}(\phi, L)$ and related modules, in the case where $\phi$ is defined over a field $L$ of finite $A$-characteristic $\mathfrak{p}$. Among others, we find that the nilspace $H_0$ of the total Frobenius $Fr_{DR}$ on $H_{DR}(\phi, L)$ has dimension $h$ = height of $\phi$. We define and study a pairing between the $\mathfrak{p}$-torsion ${}_{\mathfrak{p}}\phi$ of $\phi$ and $H_{DR}(\phi, L)$, which becomes perfect after dividing out $H_0$.

## Introduction.

The theory of Drinfeld modules (introduced in 1974 by V. G. Drinfeld as "elliptic modules" [4]) forms the core of the modern arithmetic of function fields. Work of many researchers (see the bibliography in [11] for an early account) contributes to establish deep results about Drinfeld modules and their moduli theory and connections with e.g. automorphic forms, Galois representations, the Langlands program, arithmetic groups, abelian varieties, transcendence theory, as well as to various applications.

One feature is the existence of "cohomology theories", which associate to each Drinfeld $A$-module $\phi$ over a suitable $A$-field $L$ (see sect. 1 for definitions and requirements) vector spaces analogous with the Betti, the $\ell$-adic, and the de Rham (co-)homology of an abelian variety. Rightly speaking, these are only first (co-)homology modules in the Drinfeld module framework (so we don't dispose of "true" cohomology theories), but these are provided with all the structure (functoriality, comparison isomorphisms, GAGA-type theorems, formalism of vanishing cycles) expected from the analogy with the first cohomologies of abelian varieties [4, 3, 7, 8].

In the present paper, we continue the study of the de Rham module $H_{DR}(\phi, L)$ in the case where the field $L$ of definition of $\phi$ has finite $A$-characteristic. Here we dispose of different Frobenius actions (geometric, arithmetic, total Frobenius) on $H_{DR}(\phi, L)$ and related modules. We prove two basic results:

**Theorem A** (Proposition 2.7, Theorem 2.9). *Let $\phi$ be defined over the $A$-field $L$ of $A$-characteristic $\mathfrak{p}$, let $\mathrm{End}_L(\phi)$ be its endomorphism ring and $D(\phi, L)$ the $L \otimes A$-module of biderivations of $\phi$. Then the natural action of $L \otimes \mathrm{End}_L(\phi)$ on $D(\phi, L)$ is faithful, and for each $u \in \mathrm{End}_L(\phi)$ the characteristic polynomial $\chi_{1\otimes u, D(\phi,L)}(X)$ of $1 \otimes u$ on $D(\phi, L)$ agrees with the characteristic polynomial $\chi_u(X)$ of $u$ (and in particular has coefficients in $A$).*

An important consequence is Corollary 2.10, which states that the characteristic polynomial $\chi_{u,H_{DR}(\phi,L)}(X)$ of $u$ on $H_{DR}(\phi, L)$ is the reduction of $\chi_u(X)$ modulo $\mathfrak{p}$.

**Theorem B** (Theorem 3.6). *Suppose that the definition field $L$ of $\phi$ is perfect, and let $Fr_{DR}$ be the semi-linear total Frobenius endomorphism of $H_{DR}(\phi, L)$. Then the nilkernel $H_{DR}(\phi, L)_0$ of $Fr_{DR}$ has dimension $h$ = heigth of $\phi$, and for each $k$ $(0 \leq k \leq h)$, the kernel of $Fr_{DR}^k$ has dimension $k$.*

These results englobe and sharpen those of Anglès [1], who gives similar statements in the case of finite fields $L$. Anglès' proofs make essential use of the known structure of $\mathrm{End}_L(\phi)$ over finite fields, and are limited to that case.

We further define a natural pairing between the $\mathfrak{p}$-torsion ${}_{\mathfrak{p}}\phi$ of $\phi$ and $H_{DR}(\phi, L)$, which is trivial on $H_{DR}(\phi, L)_0$ and becomes perfect as a pairing between ${}_{\mathfrak{p}}\phi$ and $H_{DR}(\phi, L)/H_{DR}(\phi, L)_0$ (Theorem 4.7). Up to dualizing, which is hidden in the self-duality of Jacobians of curves, this pairing is reminiscent of the familiar map, first defined in [15] and [16], from the $p$-torsion of the Jacobian $J(X)$ of an algebraic curve $X/L$ in characteristic $p$ to its de Rham cohomology$H^1_{DR}(X, L)$.

Finally, we work out the action of $Fr_{DR}$ on $H_{DR}(\phi, \mathbb{F}_{\mathfrak{p}})$ in the case where the Drinfeld ring $A$ is a polynomial ring $\mathbb{F}_q[T]$, $\phi$ has rank two, and the definition field $L$ is $\mathbb{F}_{\mathfrak{p}} := A/\mathfrak{p}$ with a prime $\mathfrak{p}$ of $A$. In that case, $H_{DR}(\phi, \mathbb{F}_{\mathfrak{p}})$ is endowed with a canonical basis, and so $Fr_{DR}$ determines a $2 \times 2$-matrix $M_\phi \in Mat(2, \mathbb{F}_{\mathfrak{p}})$ (and not merely a conjugacy class), whose coefficients are related to modular forms. We find a description of $M_\phi$ through a simple recursion formula (Proposition 5.8, Corollary 5.10), which also yields the eigenvectors of $Fr_{DR}$ and in particular allows to decide whether $\phi$ ist ordinary or supersingular.

## 1. Notations and background

(1.1) We consider a finite field $\mathbb{F} = \mathbb{F}_q$ with $q$ elements and a function field $K$ in one variable over $K$. That is, $K$ is finitely generated of transcendence degree one over $\mathbb{F}$, and $\mathbb{F}$ is algebraically closed in $K$. We fix a place "$\infty''$ of $K$ and let $A$ be the Dedekind ring of elements of $K$ with no poles off $\infty$. The degree function

$$\deg: \ A \longrightarrow \mathbb{N}_0 \cup \{-\infty\}$$

is defined by $\deg n = \log_q \ \#(A/n)$ for $0 \neq n \in A$ and $\deg 0 = -\infty$. Typical examples of such "Drinfeld rings" $A$ are given by

(1.2) $K = \mathbb{F}(T)$, the field of rational functions in an indeterminate $T$, "$\infty$" = the usual place at infinity, thus $A = \mathbb{F}_q[T]$; and

(1.3) $K = \mathbb{F}(X, Y)$, where $\mathrm{char}(\mathbb{F}) \neq 2$ and the indeterminates $X, Y$ are subject to $Y^2 = f(X)$ with a squarefree polynomial $f(X)$ of odd degree $d$. If again "$\infty$" is the place at infinity of the associated projective hyperelliptic curve, then $A = \mathbb{F}[X, Y]/(Y^2 - f(X))$.

In both cases "deg" is the natural degree function, where $\deg \ X = 2$, $\deg \ Y = d$ in (1.3). An *$A$-field* $L$ is some field $L$ provided with an $\mathbb{F}$-algebra homomorphism $\gamma : \ A \longrightarrow L$. Thus either $\gamma$ is injective and $L$ an extension of $K$, or $\ker(\gamma)$ is a maximal ideal $\mathfrak{p}$. We write $\mathrm{char}_A(L) = \mathfrak{p}$ in the latter and $\mathrm{char}_A(L) = \infty$ in the former case. For an $A$-field $L$, we denote the operator $x \longmapsto x^q$ on commutative $L$-algebas $B$ by $\tau$, i.e.,

$$\tau x = x^q \tau \quad \text{for } x \in B, \tag{1.4}$$

and write $B\{\tau\}$ for the non-commutative polynomial ring in $\tau$ over $B$ subject to the commutation rule (1.4).

A *Drinfeld $A$-module* $\phi$ over the $A$-field $L$ is a morphism of $\mathbb{F}$-algebras

$$\begin{array}{rccl} \phi: & A & \longrightarrow & L\{\tau\} \\ & a & \longmapsto & \phi_a = \sum \ell_i(a)\tau^i \end{array} \tag{1.5}$$

such that for each $a \in A$, $\ell_0(a) = \gamma(a)$, but $\phi \neq \gamma$. It is known (see [3], [11] or [14] for the next statements, and for more about the elementary theory of Drinfeld modules) that there exists a natural number $r$, the *rank* $rk(\phi)$ of $\phi$, such that for each $a \in A$ the rule $\deg_\tau(\phi_a) = r \cdot \deg(a)$ holds, where "$\deg_\tau$" is the natural degree function on $L\{\tau\}$. Since the homomorphism $\phi$ is necessarily injective, we often identify $A$ with its image $\phi(A)$ in $L\{\tau\}$. Via $\phi$, $L$ and the algebraic (resp. separable) closure $\overline{L}$ (resp. $L^{sep}$) of $L$ are provided with structures as $A$-modules different from the tautological structures defined by $\gamma$. If $r = rk(\phi)$ and $a \in A$ is non-constant and coprime with $\mathrm{char}_A(L)$, then

$$_a\phi := \{x \in \overline{L} \mid \phi_a(x) = 0\} \tag{1.6}$$

is a free module of rank $r$ over the finite ring $A/a$. A *homomorphism* of Drinfeld modules $u : \phi \longrightarrow \psi$ over $L$ is some $u \in L\{\tau\}$ that satisfies $u \circ \phi_a = \psi_a \circ u$ for $a \in A$. In particular, the *endomorphism ring* $\mathrm{End}(\phi) = \mathrm{End}_L(\phi)$ of $\phi$ is the centralizer of $A \xrightarrow{\cong} \phi(A)$ in $L\{\tau\}$. It is a projective module over $A$ of rank a divisor of $r^2$, where $r = rk(\phi)$.

Let $\mathfrak{l}$ be a prime ideal of $A$ different from $\mathrm{char}_A(L)$, and $\mathfrak{l}^n = (\ell)$ for some $n \in \mathbb{N}$, $\ell \in A$. From the torsion modules ${}_{\mathfrak{l}^k}\phi$ one constructs the $\mathfrak{l}$-adic Tate module $T_\mathfrak{l}(\phi)$, which is a free module of rank $r$ over the $\mathfrak{l}$-adic completion $A_\mathfrak{l}$ of $A$ (see [4], [3]). It is provided with actions of

(a) the absolute Galois group $\mathrm{Gal}(L^{\mathrm{sep}}|L)$ of $L$;
(b) the endomorphism ring $\mathrm{End}_L(\phi)$,

which mutually commute. (See [17], [5], [13] for recent deep results.) We restrict to point out: the attached representation

$$i_\mathfrak{l} : \mathrm{End}_L(\phi) \otimes_A A_\mathfrak{l} \longrightarrow \mathrm{End}_{A_\mathfrak{l}}(T_\mathfrak{l}(\phi))$$

is faithful (i.e., injective) and to some extent independent of $\mathfrak{l}$:

(1.7) Let $u \in \mathrm{End}_L(\phi)$ be given. The characteristic polynomial $\chi_u(X)$ of $i_\mathfrak{l}(u)$ has coefficients in $A \subset A_\mathfrak{l}$ and is independent of $\mathfrak{l}$. We briefly call it the *characteristic polynomial of* $u$.
(This is implicit in [4, 3] and explicit in [9] sect. 3 in the crucial case of a finite $L$.) The above allows to define a norm map

$$\begin{array}{rcl} N : \mathrm{End}_L(\phi) & \longrightarrow & A \\ u & \longmapsto & \det(i_\mathfrak{l}(u)) \end{array}$$

that satisfies $N(uv) = N(u)N(v)$; $N(u) = 0 \Leftrightarrow u = 0$; $N(a) = a^r$ for $u, v \in \mathrm{End}_L(\phi)$ and $a \in A$. An alternative way of defining $N$ was via reduced algebra norms of $\mathrm{End}_L(\phi) \otimes_A K$ or $\mathrm{End}_L(\phi) \otimes_A K_\infty$, both of which are known to be division algebras over $K$ or $K_\infty$, respectively (*loc. cit.*).

We next recall the definition of the de Rham cohomology $H_{DR}(\phi, L)$ of $\phi$ (see [7, 8]). First, let $N(\phi, L)$ be the ideal $L\{\tau\}\tau$ of $L\{\tau\}$, regarded as a left $L$-module and a right $A$-module (under $(n, a) \longmapsto n\phi_a$ for $n \in N(\phi, L)$, $a \in A$). Since the multiplications with elements of the common subfield $\mathbb{F}$ of $L$ and $A$ agree, $N(\phi, L)$ is naturally a module under

$$A_L := L \otimes A, \tag{1.8}$$

where "$\otimes$" always denotes the tensor product "$\otimes_{\mathbb{F}}$" over $\mathbb{F}$. The ring $A_L$ is Dedekind with a distinguished maximal ideal

$$I_L := ker(\begin{array}{ccc} A_L & \longrightarrow & L \\ \ell \otimes a & \longmapsto & \ell \cdot \gamma(a) \end{array}) .$$

An $\mathbb{F}$-linear biderivation (*derivation* for short) from $A$ to $N(\phi, L)$ is an $\mathbb{F}$-linear map

$$\begin{array}{cccc} \eta : & A & \longrightarrow & N(\phi, L) \\ & a & \longmapsto & \eta_a \end{array}$$

subject to the rule $\eta_{ab} = \gamma(a)\eta_b + \eta_a\phi_b$ $(a, b \in A)$. It is inner (strictly inner) if $\eta$ is of shape $\eta^{(n)}$ with $n \in L\{\tau\}$ (resp. $n \in N(\phi, L)$), where

$$\eta_a^{(n)} := \gamma(a)n - n\phi_a. \tag{1.9}$$

We let $D_{si}(\phi, L) \subset D_i(\phi, L) \subset D(\phi, L)$ be the $A_L$-modules of strictly inner, inner, all derivations from $A$ to $N(\phi, L)$, respectively. As is shown in [7], sect. 4, $N(\phi, L)$ and $D(\phi, L)$ are projective $A_L$-modules of rank $r = rk(\phi)$. Furthermore,

$$I_L \cdot D(\phi, L) = D_{si}(\phi, L), \tag{1.10}$$

and thus the *de Rham cohomology module*

$$H_{DR}(\phi, L) := D(\phi, L)/D_{si}(\phi, L) = D(\phi, L) \otimes_{A_L} L \tag{1.11}$$

is an $L$-vector space of dimension $r$. W write $[\eta]$ for the class of $\eta \in D(\phi, L)$. Then $H_{DR}(\phi, L)$ contains the distinguished one-dimensional subspace $D_i(\phi, L)/D_{si}(\phi, L)$ with basis vector $[\eta^{(1)}]$, where $\eta_a^{(1)} = \gamma(a) - \phi_a$.

**1.12 Remarks.** In our definition of Drinfeld module, we have implicitly chosen a coordinate on the additive group $\mathbb{G}_a/L$. In the well-known analogy between Drinfeld modules and elliptic curves, this corresponds to choosing a Weierstraß equation for an elliptic curve. The definitions both of Drinfeld modules themselves and of the quantities $N(\phi, L), \ldots, H_{DR}(\phi, L)$ can be given in a coordinate-free way, and may be generalized to arbitrary commutative algebras $B$ over $A$ (or even to non-affine $A$-schemes $X$) as domains of $\phi$, instead of $A$-fields $L$ only. Then $N(\phi, B), \ldots, H_{DR}(\phi, B)$ become functors, covariant in $B$ and contravariant in $\phi$, see [7], sect. 3/4.

## 2. The action of endomorphisms.

Let the Drinfeld $A$-module $\phi$ be defined over the $A$-field $L$ as before. The endomorphism ring $\mathrm{End}_L(\phi)$ acts from the right on $N(\phi, L)$

(through multiplication) and on $D(\phi, L)$ through

$$\eta \longmapsto \eta \circ u, \quad (\eta \circ u)_a := \eta_a \circ u$$

($\eta \in D(\phi, L)$, $u \in \mathrm{End}_L(\phi)$, $a \in A$), an action which commutes with left multiplication by elements of $L$. There result structures of (right) $L \otimes \mathrm{End}_L(\phi)$-modules on $N(\phi, L)$ and $D(\phi, L)$ compatible with their $A_L$-structures ($A_L = L \otimes A$; recall that "$\otimes$"= "$\otimes_{\mathbb{F}}$"). In view of the formula

$$\eta^{(m)} \circ u = \eta^{(m \circ u)} \tag{2.1}$$

for $m \in L\{\tau\}$, $D_{si}(\phi, L)$, $D_i(\phi, L)$ resp. $H_{DR}(\phi, L)$ are $L \otimes \mathrm{End}_L(\phi)$ submodules resp. an $L \otimes \mathrm{End}_L(\phi)$-quotient module.

Suppose that $\mathrm{char}_A(L) = \mathfrak{p} \neq \{0\}$, and let $a, p, \eta$ be elements of $A, \mathfrak{p}, D(\phi, L)$, respectively. We have

$$\gamma(a)\eta_p + \eta_a\phi_p = \eta_{ap} = \eta_{pa} = \eta_p\phi_a$$

(since $\gamma(p) = 0$), thus

$$(\eta \circ \phi_p)_a = \eta_a \circ \phi_p = \eta_p \circ \phi_a - \gamma(a)\eta_p = -\eta_a^{(\eta_p)}. \tag{2.2}$$

This means that $\mathfrak{p} \hookrightarrow A \hookrightarrow \mathrm{End}_L(\phi)$ acts trivially on $H_{DR}(\phi, L)$, which therefore is an $L \otimes (\mathrm{End}_L(\phi) \otimes_A \mathbb{F}_{\mathfrak{p}})$-module. It is obvious that for $u \in \mathrm{End}_L(\phi)$ the coefficientwise congruence of characteristic polynomials

$$\chi_{u,D(\phi,L)}(X) \equiv \chi_{\overline{u},H_{DR}(\phi,L)}(X)(\mathrm{mod} I_L) \tag{2.3}$$

holds, where $u = 1 \otimes u$ on the left hand side is regarded as an endomorphism of the $A_L$-module $D(\phi, L)$, with reduction $\overline{u} \in \mathrm{End}_L(H_{DR}(\phi, L))$. Our objective is to show (Theorem 2.9 below) that the left hand side of (2.3) agrees with the characteristic polynomial $\chi_u(X)$ of $u$ in the sense of (1.7). We need some preparations.

**2.4 Proposition.** *Let $L$ be an algebraically closed field that contains $\mathbb{F} = \mathbb{F}_q$, $V$ an $n$-dimensional $L$-vector space, and $F : V \longrightarrow V$ a $\tau$-linear map, i.e., $f$ is additive and satisfies*

$$f(\ell x) = \ell^q f(x) \quad (\ell \in L,\ x \in V). \tag{$*$}$$

*Put*

$$\begin{aligned} V_0 &:= \{x \in V \mid f^n(x) = 0\} \\ V_1' &:= \{x \in V \mid f(x) = x\}. \end{aligned}$$

*Then*
*(i) $V_0$ is an $L$-subspace, (ii) $V_1'$ an $\mathbb{F}$-subspace of $V$, and (iii) $V = V_0 \oplus V_1$, where $V_1 = L \otimes V_1'$.*
*The same statement holds if $f$ is $\tau^{-1}$-linear ($f(\ell x) = \ell^{q^{-1}} f(x)$).*

*Proof* (see [12]). (i) and (ii) are obvious. To prove (iii), we make $V$

an $L\{\tau\}$-module by decreeing $\tau x = f(x)$ for $x \in V$, which is possible in view of $(*)$. Then $V_0$ and $LV_1'$ are sub-$L\{\tau\}$-modules. Let $\{x_1, \ldots, x_{n_1}\}$ be an $L$-basis of $LV_1'$. For $x = \sum \ell_i x_i \in LV_1'$ ($\ell_i \in L$), we have $f(x) = x \Leftrightarrow \ell_i \in \mathbb{F}$ for $i = 1, 2, \ldots, n_1$, which shows that $LV_1' = L \otimes V_1' = V_1$. Since obviously $V_0 \cap V_1 = \{0\}$, we have $n \geq n_1 + n_0$ ($n_0 := \dim V_0$), and it remains to show equality.

For some $x \in V$, let $\overline{V} := L\{\tau\}x$ be the $L\{\tau\}$-submodule generated by $x$, and consider the similarly defined submodules $\overline{V}_0$, $\overline{V}_1$ of $\overline{V}$. Their dimensions satisfy (with obvious terminology) the equality $\overline{n} \geq \overline{n}_1 + \overline{n}_0$. Here $\overline{n} = \deg_\tau g$, where $g \in L\{\tau\}$ generates the annihilator of $x$. Write $g = \tau^h \cdot g_0$, where $g_0$ has non-zero constant term. Then the equation $f(y) = y$ has at least $q^{\deg_\tau g_0} = q^{\overline{n}-h}$ solutions in $\overline{V}$, namely $y = kx$, where $k \in L\{\tau\}$ is specified in Lemma 2.5 below. This yields $\overline{n}_1 \geq \overline{n} - h$.

Moreover, the elements $g_0 x, \tau g_0 x, \ldots, \tau^{h-1} g_0 x$ of $\overline{V}_0$ are $L$-linearly independent, thus $\overline{n}_0 \geq h$. Together, we find $\overline{n} = \overline{n}_1 + \overline{n}_0$, which implies $\overline{V} = \overline{V}_0 \oplus \overline{V}_1$, and thus finally $V = V_0 \oplus V_1$.

The proof for a $\tau^{-1}$-linear map $f : V \longrightarrow V$ is identical, except that we have to replace $\tau$ everywhere (including Lemma 2.5) by $\tau^{-1}$. $\square$

**2.5 Lemma.** *Assume $L$ algebraically closed, and let $g \in L\{\tau\}$ of degree $n > 0$ be such that $g = \tau^h g_0$, where $g_0$ has non-vanishing constant term. Then there are precisely $q^{n-h}$ elements $k$ of $L\{\tau\}$ that satisfy*

$$(1-\tau)k = c \cdot g$$

*with some $c \in L$.*

*Proof.* This is an exercise in calculating in $L\{\tau\}$, and will be omitted (see [12] Satz 12). $\square$

We recall the following fact from [7] (4.2):

(2.6) There is a canonical and functorial isomorphism of $A_L$-modules

$$\mathrm{Hom}_{A_L}(I_L, N(\phi, L)) \xrightarrow{\cong} D(\phi, L).$$

Hence the relevant properties of the $A_L$-module $D(\phi, L)$ follow from those of $N(\phi, L)$.

**2.7 Proposition.** *The actions of $L \otimes \mathrm{End}_L(\phi)$ on $N(\phi, L)$ and on $D(\phi, L)$ are faithful.*

*Proof.* It suffices to prove the statement for $N(\phi, L)$. $\mathrm{End}_L(\phi)$ acts through right multiplication on $N(\phi, L) = L\{\tau\}\tau$, which yields a faithful representation of $\mathrm{End}_L(\phi)$. We are therefore reduced to show:

Let $\{u_i \mid 1 \leq i \leq k\}$ be an $\mathbb{F}$-linearly independent set in $\mathrm{End}_L(\phi)$. Then the set $\{1 \otimes u_i\}$ of operators on $N(\phi, L)$ remains $L$-linearly independent.

Without restriction we may assume that $L$ is algebraically closed. Thus, suppose

$$(*) \qquad 0 = n(\textstyle\sum_i \ell_i \otimes u_i) = \sum_i \ell_i n u_i$$

for some $\ell_1, \ldots, \ell_k \in L$ and all elements $n \in N(\phi, L)$.
Inserting $n = \tau^j$ and multiplying by $\tau^{-j}$ from the left yields

$$\textstyle\sum_i \ell_i^{q^{-j}} u_i = 0, \quad j = 1, 2, 3, \ldots$$

That is, the relation space

$$V = \{(\ell_1, \ldots, \ell_k) \in L^k \mid \textstyle\sum_i \ell_i n u_i = 0 \text{ for all } n \in N(\phi, L)\}$$

is stable under the $\tau^{-1}$-linear map $(\ell_1, \ldots, \ell_k) \longmapsto (\ell_1^{q^{-1}}, \ldots, \ell_k^{q^{-1}})$. From (2.4) we get that $V$ has an $\mathbb{F}$-structure ($\mathbb{F} = \mathbb{F}_q$), which in view of the $\mathbb{F}$-linear independence of $\{u_i\}$ implies that $V = \{0\}$. $\square$

Before we come to the main result of this section, we make the following observation:

(2.8) Let $F$ be any field and $G$ a commutative subalgebra of dimension $s$ of the endomorphism ring $\mathrm{End}_F(V)$ of an $r$-dimensional $F$-vector space such that $V$ is a free $G$-module (necessarily of dimension $t := r/s$). For $u \in G$ we have the formula

$$\det(u) = N_F^G(u)^t,$$

where $N_F^G(u)$ is the algebra norm from $G$ to $F$, i.e., the determinant of the $F$-linear map $x \longmapsto ux$ on $G$.

**2.9 Theorem.** *Let $u \in \mathrm{End}_L(\phi)$ be given, where the Drinfeld $A$-module $\phi$ over $L$ has rank $r$ and $\mathrm{char}_A(L) = \mathfrak{p} \neq \{0\}$. The characteristic polynomial $\chi_{u,D(\phi,L)}(X)$ of the $A_L$-endomorphism $u = 1 \otimes u$ of $D(\phi, L)$ equals $\chi_u(X)$ (cf.* 1.7*). In particular, it has coefficients in $A$.*

*Proof.* (1) Again, it suffices in view of (2.6) to show the corresponding statement for the $A_L$-module $N(\phi, L)$.

(2) It even suffices to show that for each $u \in \mathrm{End}_L(\phi)$, the determinant $\det_{N(\phi,L),A_L}(1 \otimes u)$ of the $A_L$-endomorphism $1 \otimes u$ of $N(\phi, L)$ equals $1 \otimes N(u)$, where $N : \mathrm{End}_L(\phi) \longrightarrow A$ is the norm map of (1.7).

(3) Let $u \in \mathrm{End}_L(\phi)$ be contained in a maximal commutative $K$-subalgebra (i.e., subfield) $G$ of the division algebra $\mathrm{End}_L(\phi) \otimes_A K$. Then $s := [G : K]$ is a divisor of $r$, the $A$-order $B := G \cap \mathrm{End}_L(\phi)$ in

$G$ is projective of rank $s$ over $A$, and

$$N(u) = N_A^B(u)^t = N_K^G(u)^t$$

with $t := r/s$, as follows from considering the faithful representations $i_{\mathfrak{l}}$ of (1.7) and (2.8).

(4) Let $Q$ be the quotient field of the Dedekind ring $A_L = L \otimes A$. For any $A_L$-module or -algebra $M$, we write $M_Q$ for $M \otimes_{A_L} Q$.

Suppose for the moment that the algebraic closure $\mathbb{F}'$ of $\mathbb{F}$ in $B$ (or in $G$, which is the same) equals $\mathbb{F}$. Then $L \otimes B$ is an integral domain that operates faithfully on the $A_L$-module $N(\phi, L)$. Upon applying $\otimes_{A_L} Q$, the field $(L \otimes B)_Q = \mathrm{Quot}(L \otimes B)$, a $Q$-algebra of dimension $s$, is contained in $\mathrm{End}_Q(N(\phi, L)_Q)$. Hence $N(\phi, L)_Q$ has dimension $t$ over $Q$, and from (2.8) we get

$$\begin{aligned}\det{}_{N(\phi,L),A_L}(1 \otimes u) = \det{}_{N(\phi,L)_Q,Q}(1 \otimes u) = \\ N_Q^{(L\otimes B)_Q}(1 \otimes u)^t = N_A^B(u)^t,\end{aligned}$$

which equals $N(u)$ by (3). Hence we are done in this case.

(5) We now deal with the general case, where the constant field $\mathbb{F}'$ of $B$ has degree $f \geq 1$, say, over $\mathbb{F}$. Let $A' = \mathbb{F}'A \xrightarrow{\cong} \mathbb{F}' \otimes A$ be the subring of $B$ generated by $\mathbb{F}'$ and $A$. Under the embedding of $\mathrm{End}_L(\phi)$ into $L\{\tau\}$, $\mathbb{F}'$ maps to $L$. Moreover, since the elements of $\mathbb{F}'^*$ provide automorphisms, $\phi$ takes its values in the subring $L\{\tau^f\}$ of $L\{\tau\}$. We may regard $\phi$ as a Drinfeld $A'$-module $\phi'$ over the $A'$-field $L$, of rank $r' = r/f$. Furthermore, $B \subset \mathrm{End}_L(\phi') \subset L\{\tau^f\}$. As an $A'$-module, $B$ is projective of rank $s' = s/f$; in particular, $f$ divides $s$.

(6) There are canonical isomorphisms

$$A_L = L \otimes A \xrightarrow{\cong} L \otimes_{\mathbb{F}'} A'$$

and

$$\begin{array}{rcl} L \otimes B & \xrightarrow{\cong} & \prod_{0 \leq i < f} B_i, \\ B_i & := & L \otimes_{\mathbb{F}', \tau^i} B, \end{array}$$

where $\mathbb{F}'$ is embedded into $L$ via $\tau^i : \mathbb{F}' \longrightarrow L$, $x \longmapsto x^{q^i}$. The rings $B_i$ are integral domains of projective rank $s' = s/f$ over $A_L$.

(7) For $0 \leq i < f$, let $N_i := \{\sum \ell_k \tau^k \mid k \equiv i \bmod f,\ k > 0\} \subset N(\phi, L) = L\{\tau\}\tau$. Then

$$N(\phi, L) = \bigoplus_{0 \leq i < f} N_i$$

as $A_L$-modules and even as $L \otimes B$-modules, and $N_0 = N(\phi', L)$ as $L \otimes_{\mathbb{F}'} A'$-module.

(8) For $n\tau^i \in N_i$ and $c \in \mathbb{F}'$, we have $n\tau^i c = c^{q^i} n\tau^i$, that is, $c^{q^i} \otimes 1 = 1 \otimes c$ as operators on $N_i$. Therefore, the action of $L \otimes B$ on $N_i$ is via its quotient $B_i$, and $N_i$ is a faithful $B_i$-module.

(9) Each of the $A_L$-modules $N_i$ is projective of rank $r' = r/f$. For $N_0$ this follows from (7); for general $i$, this may be proved as in [7], Proposition 4.6 (reducing to the case where $A = \mathbb{F}_q[T]$ by comparing projective ranks; then $\{\tau^k \mid 1 \leq k \leq r,\ k \equiv i \bmod f\}$ is an $A_L$-basis for $N_i$).

(10) As follows from (8) and (9), $N_{i,Q}$ is a $Q$-vector space of dimension $r'$, provided with an action of the field $B_{i,Q}$, where $[B_{i,Q} : Q] = s'$ by (6). Hence $\dim_{B_{i,Q}}(N_{i,Q}) = r'/s' = r/s = t$ independently of $i$. Therefore $N(\phi, L)_Q$ is free of rank $t$ over $(L \otimes B)_Q = \prod B_{i,Q}$, which is a $Q$-algebra of dimension $s$ in $\mathrm{End}_Q(N(\phi, L)_Q)$. Again we may apply (2.8), and the sequence of equalities in (4) yields the wanted result $\det_{N(\phi,L),A_L}(1 \otimes u) = N(u)$. □

**2.10 Corollary.** Let the Drinfeld $A$-module $\phi$ be defined over the $A$-field $L$ of $A$-characteristic $\mathfrak{p}$, and let $u$ be an endomorphism of $\phi$. Its characteristic polynomial on $H_{DR}(\phi, L)$ equals $\chi_u(X) \bmod \mathfrak{p}$.

*Proof.* Combine (2.9) with (2.3) and note that $I_L \cap A = \mathfrak{p}$ holds in $A_L$. □

## 3. Frobenius actions.

As with schemes over finite fields (see the exposition in [2], pp. 77–81), there are different Frobenius auto- and endomorphisms in the context of Drinfeld modules, which must be carefully distinguished.

Let $\phi$ be a Drinfeld $A$-module defined over an $A$-field $L$ of $A$-characteristic $\mathfrak{p}$, where $\deg \mathfrak{p} = d$. We always denote the effect of applying the $q^d$-th power map

$$\varphi = \varphi_{\mathfrak{p}} = \tau^d$$

to the coefficients of a polynomial $f$ by $f^{(\varphi)}$. There results a Drinfeld module $\phi^{(\varphi)}$, where $\phi_a^{(\varphi)} = (\phi_a)^{(\varphi)}$, endowed with an isogeny, labelled $F = F_{\mathfrak{p}}$:

$$F : \phi \longrightarrow \phi^{(\varphi)}. \tag{3.1}$$

As an element of $L\{\tau\}$, $F$ is nothing else but $\varphi = \tau^d$ (see (1.6)); we call it the *geometric* Frobenius map. By functoriality, we get maps $F_D : D(\phi^{(\varphi)}, L) \longrightarrow D(\varphi, L)$ and $F_{DR} : H_{DR}(\phi^{(\varphi)}, L) \longrightarrow H_{DR}(\phi, L)$. If $L$ happens to be finite, of degree $m$ over $\mathbb{F}_{\mathfrak{p}}$, $\varphi_L = \varphi_{\mathfrak{p}}^m = \tau^{dm}$ yields an endomorphism $F_L$ of $\phi$ and endomorphisms $F_{L,D}$ of $D(\phi, L)$, $F_{L,DR}$

of $H_{DR}(\phi, L)$, respectively. According to the general device, $F_{DR}$ is obtained from

$$\begin{array}{rrcl} F_D: & D(\phi^{(\varphi)}, L) & \longrightarrow & D(\phi, L) \\ & \eta & \longmapsto & \eta \circ \varphi \end{array} \tag{3.2}$$

(where $(\eta \circ \varphi)_a = \eta_a \circ \varphi$ for $a \in A$) by passing to the quotient. On the other hand, there is the map

$$\begin{array}{rrcl} \varphi_D: & D(\phi, L) & \longrightarrow & D(\phi^{(\varphi)}, L), \\ & \eta & \longmapsto & \eta^{(\varphi)} \end{array} \tag{3.3}$$

which is a homomorphism of $\mathbb{F}_\mathfrak{p} \otimes A$-modules, and is $\varphi$-linear, i.e., $\varphi_D(\ell x) = \varphi(\ell)\varphi_D(x)$ for $\ell \in L$. It maps $D_{si}(\phi, L)$ to $D_{si}(\phi^{(\varphi)}, L)$, and thus induces a $\varphi$-linear map $\varphi_{DR}: H_{DR}(\phi, L) \longrightarrow H_{DR}(\phi^{(\varphi)}, L)$. We refer to $\varphi_D$, $\varphi_{DR}$ as the *arithmetic* Frobenius maps. Finally, we consider the composition (the *total* Frobenius map)

$$\begin{array}{rrcl} Fr_D: & D(\phi, L) & \xrightarrow{\varphi_D} & D(\phi^{(\varphi)}, L) \xrightarrow{F_D} D(\phi, L), \\ & \eta & \longmapsto & \eta^{(\varphi)} \circ \varphi = \varphi \circ \eta \end{array} \tag{3.4}$$

which on the $H_{DR}$-level induces

$$\begin{array}{rrcl} Fr_{DR}: & H_{DR}(\phi, L) & \longrightarrow & H_{DR}(\phi, L). \\ & [\eta] & \longmapsto & [\varphi \circ \eta] \end{array}$$

Since $Fr_{DR}$ is $\varphi$-linear, Proposition 2.4 applies.

**3.5 Corollary.** *Suppose that $L$ is algebraically closed. Then $H_{DR}(\phi, L) = H_0 \oplus H_1$ as an $L$-vector space, where*

$$\begin{array}{rcccl} H_0 & = & H_{DR}(\phi, L)_0 & = & \{x \in H_{DR}(\phi, L) \mid Fr_{DR}^r(x) = 0\} \\ H_1 & = & H_{DR}(\phi, L)_1 & = & \{x \in H_{DR}(\phi, L) \mid Fr_{DR}(x) = x\} \otimes_{\mathbb{F}_\mathfrak{p}} L. \end{array}$$

*For general $L$, the subspaces $H_{DR}(\phi, \overline{L})_i$ $(i = 0, 1)$ of $H_{DR}(\phi, \overline{L})$ are galois-stable and therefore already defined over the perfect hull $L'$ of $L$ in its algebraic closure $\overline{L}$.*

We will show:

**3.6 Theorem.** *Let the Drinfeld $A$-module $\phi$ be defined over the $A$-field $L$ with* $\mathrm{char}_A(L) = \mathfrak{p}$. *Suppose that $L$ is perfect, and let $h$ be the height $ht(\phi)$ of $\phi$ (that is, $1 \leq h \leq r = rk(\phi)$ and $r - h$ is the rank of the $\mathfrak{p}$-adic Tate module $T_\mathfrak{p}(\phi)$ of $\phi$). Then* $\dim\ H_{DR}(\phi, L)_0 = h$ *and moreover, for $0 \leq k \leq h$ we have* $\dim\ \ker(Fr_{DR}^k) = k$.

**3.7 Remark.** The theorem covers the most important cases where $L$ is finite (where a slightly weaker form has been given in [1], Theorem 4.4) or algebraically closed. The perfectness assumption is necessary for the assertion as given; the case of imperfect $L$ is more complicated and needs further investigation.

**3.8 Corollary** (cf. [1], Cor. 4.3). *Let $\phi$ be defined over the finite $A$-field $L$ of characteristic $\mathfrak{p}$, and let $\chi_{\phi,L}(X) \in A[X]$ be the characteristic polynomial $\chi_{F_L}(X)$ of its Frobenius endomorphism $F_L$. Write $\chi_{\phi,L}(X) = \sum a_i X^i$ ($0 \le i \le r$, $a_r = 1$). Then* $\min\{i \mid a_i \not\equiv 0 \bmod \mathfrak{p}\} = ht(\phi) = \dim\ H_{DR}(\phi, L)$.

*Proof.* This follows from combining (3.6) with (2.10), taking into account that $F_{L,DR}$ is a power of $Fr_{DR}$. □

As a first step towards the proof of (3.6), we consider a critical special case.

**3.9 Proposition.** *Theorem* 3.6 *is valid in the case where $A = \mathbb{F}_q[T]$ is a polynomial ring and* $\mathrm{char}_A(L) = (T)$.

*Proof.* Under our assumptions, the Drinfeld module $\phi$ is uniquely determined by

$$\phi_T = \sum_{h \le i \le r} \ell_i \tau^i = f \cdot \tau^h \in L\{\tau\},$$

where $\ell_h \neq 0 \neq \ell_r$, $r = rk(\phi)$, $h = ht(\phi)$. Furthermore, specifying $\eta \in D(\phi, L)$ is the same as specifying $\eta_T \in N(\phi, L) = L\{\tau\}\tau$. It is easy to see ([7], Lemma 5.1) that the composite map

$$V := \{\eta_T \in N(\phi, L) \mid \deg_\tau \eta_\tau \le r\} \hookrightarrow D(\phi, L) \longrightarrow H_{DR}(\phi, L)$$

is an isomorphism, i.e., $V$ is a set of representatives for $H_{DR}(\phi, L)$. Thus for each $\eta \in D(\pi, L)$, there exists a unique $n \in N(\phi, L)$ such that

$$\eta_T - \eta_T^{(n)} = \eta_T + n \cdot \phi_T$$

belongs to $V$. (Recall that $\eta_a^{(n)} = \gamma(a)n - n \cdot \phi_a$, which is $-n \cdot \phi_a$ for $a \in A$ divisible by $T$.) The map $Fr_D$ on $D(\phi, L)$ is simply left multiplication by $\tau$. Hence the kernel of $Fr_{DR}^k$ on $H_{DR}(\phi, L)$ corresponds to the subspace of those $\eta_T \in V$ that satisfy

$$\tau^k \eta_T = -n \cdot \phi_T = -n f \tau^h \tag{1}$$

for some $n \in N(\phi, L)$. Suppose that $k > h$. Then (1) is equivalent with

$$\tau^{k-h} \eta_T^{(\tau^h)} = -nf. \tag{2}$$

Since $f$ has non-zero constant term and $L$ is perfect, $n$ is left divisible by $\tau^{k-h}$, and already $\eta_T^{(\tau^h)} = -\tilde{n} f$ for some $\tilde{n}$. This shows that $\ker(Fr_{DR}^k) = \ker(Fr_{DR}^h)$.

Let $U \subset V$ be the $h$-dimensional $L$-subspace

$$U := \{g \in V \mid g = -nf,\ n \in N(\phi, L)\}.$$

Then, again by the perfectness of $L$, the map

$$\begin{array}{ccccc} U & \longrightarrow & U' := \{g' \in V \mid g'^{(\tau^h)} \in U\} & \longrightarrow & \ker(Fr^h_{Dr}) \\ g & \longmapsto & g^{(\tau^{-h})} \qquad g' & \longmapsto & [\eta] \text{ such that } \eta_T = g' \end{array}$$

is $\tau^{-h}$-linear and bijective, which in particular gives the equality of dimensions. Similarly, for $k < h$, (1) becomes

$$\eta_T^{(\tau^k)} = -nf\tau^{h-k}, \tag{3}$$

where $\deg_\tau n \leq k$, and the $\eta_T$ thus described run through a $k$-dimensional subspace of $V$. $\square$

We now reduce the

*Proof of Theorem* 3.6 to the special case just treated.
Let $\phi$ of rank $r$ and height $h$ be defined over the perfect $A$-field $L$ of $A$-characteristic $\mathfrak{p}$, with $d := \deg \mathfrak{p}$ and structure homomorphism $\gamma : A \longrightarrow \mathbb{F}_\mathfrak{p} \hookrightarrow L$. Let $e$ be the order of $\mathfrak{p}$ in the class group of $A$, so $\mathfrak{p}^e = (T)$ with some $T \in A$, and let $A^0$ be the subring $\mathbb{F}_q[T]$ of $A$. The prime $(T)$ of $A^0$ splits in $A$ as

$$(T) = \mathfrak{p}^e, \tag{1}$$

and thus $A$ is a free $A^0$-module of rank $de$. Accordingly, the Dedekind ring $A_L = L \otimes A$ is a free module of rank $de$ over $A^0_L = L \otimes A^0 = L[T]$. The distinguished ideal $I^0_L$ of $A^0$ (see (1.8)) is the ideal generated by $T$; it splits in $A_L$ as

$$A_L \cdot I^0_L = (\prod_{1 \leq i \leq d} I^{(i)}_L)^e =: J^e, \tag{2}$$

where $I^{(i)}_L$ is the kernel of the ring homomorphism

$$\begin{array}{ccc} A_L & \longrightarrow & L \\ \ell \otimes a & \longmapsto & \ell\gamma(a)^{q^i} \end{array} . \tag{3}$$

In particular, $I^{(d)}_L$ is the distinguished maximal ideal $I_L$ of $A$. Restricting the ring homomorphism $\phi$ to $A^0$ yields a Drinfeld $A^0$-module $\phi^0$ over $L$. Its invariants are

$$\begin{array}{ccccc} rk(\phi^0) & = & rk_{A^0}(A)rk(\phi) & = & rde \\ ht(\phi^0) & = & rk_{A^0}(A)ht(\phi) & = & hde, \end{array} \tag{4}$$

as is easily seen by comparing the respective torsion submodules of $\phi$ and $\phi^0$. We regard $A$ as a subring of $\mathrm{End}_L(\phi^0)$; thus $D(\phi^0, L)$ and

$H_{DR}(\phi^0, L)$ are naturally $A_L$-modules. ¿From (2.6) we get

$$
\begin{array}{lrcl}
(5) & D(\phi, L) & = & \mathrm{Hom}_{A_L}(I_L, N(\phi, L)) \\
& \downarrow & & \\
& D(\phi^0, L) & = & \mathrm{Hom}_{A_L^0}(I_L^0, N(\phi^0, L)) = \mathrm{Hom}_{A_L}(J^e, N(\phi, L)),
\end{array}
$$

where the last equality comes from $A_L \otimes_{A_L^0} I_L^0 = A_L I_L^0 = J^e$ and Frobenius reciprocity. Both $D(\phi, L)$ and $D(\phi^0, L)$ are projective $A_L$-modules of rank $r$.

Since a derivation $\eta \in D(\phi, L)$ is uniquely determined by $\eta_a$ for any $a \in A - \mathbb{F}_q$, we have $D(\phi, L) \cap D_{si}(\phi^0, L) = D_{si}(\phi, L)$, which implies that the canonical map

$$(6) \qquad H_{DR}(\phi, L) \longrightarrow H_{DR}(\phi^0, L)$$

induced from (5) is injective. Again from (5) we see that

$$(7) \qquad H_{DR}(\phi, L) = \{x \in H_{DR}(\phi^0, L) \mid I_L x = 0\}.$$

Let $B$ be the image of $A_L$ in $\mathrm{End}_L(H_{DR}(\phi^0, L))$, that is

$$(8) \qquad B = A_L \otimes_{A_L^0} A_L^0 / I_L^0 = A_L / J^e = \prod_{1 \le i \le d} A_L / I_L^{(i)e},$$

where we use the canonical isomorphisms as identifications. It is an Artin algebra of dimension $de$ over $L$. By (5) (since $D(\phi^0, L)$ is $A_L$-projective of rank $r$), $H_{DR}(\phi^0, L)$ is $B$-free of rank $r$. In view of $\dim_L\{x \in A_L/J^e \mid I_L x = 0\} = 1$, each $B$-free constituent of $H_{DR}(\phi^0, L)$ contributes by 1 to $\dim_L H_{DR}(\phi, L)$. We will see in a moment that a similar statement holds for the kernels of powers of $Fr_{DR}$.

According to (8), $H_{DR}(\phi^0, L)$ splits into $B$-isotypical components $H^{(i)}$ $(1 \le i \le d)$, each of $L$-dimension $re$ and free over $B^{(i)} = A_L / I_L^{(i)e}$ of dimension $r$. We have

$$(9) \qquad H^{(i)} = \{x \in H_{DR}(\phi^0, L) \mid I_L^{(i)e} x = 0\}$$

and

$$H_{DR}(\phi, L) = \text{socle of the } B^{(d)}\text{-module } H^{(d)}.$$

We write $Fr_{DR}$ for the map (3.4) on $H_{DR}(\phi, L)$, which is induced by left multiplication with $\varphi_{\mathfrak{p}} = \tau^d$ on $D(\phi, L)$, and $Fr_{DR}^{(0)}$ for the corresponding map on $H_{DR}(\phi^0, L)$, induced by left multiplication with $\tau$ on $D(\phi^0, L)$. Hence $Fr_{DR} = Fr_{DR}^{(0)d}$ restricted to $H_{DR}(\phi, L)$.

(10) We have

$$Fr_{DR}^{(0)}(H^{(i)}) \subset H^{(i-1)},$$

where the superscript $i$ should be read modulo $d$. In particular, $Fr_{DR}^{(0)d}$ respects the decomposition

$$H_{DR}(\phi^0, L) = \oplus H^{(i)}.$$

*Proof of* (10). Let $a = \sum \ell_k \otimes a_k$ be an element of $I_L^{(i)}$ ($\ell_k \in L,\ a_k \in A$), i.e., $\sum \ell_k \gamma(a_k)^{q^i} = 0$. Then $\sum \ell_k^{q^{-1}} \gamma(a_k)^{q^{i-1}} = 0$, so $a' := \sum \ell_k^{q^{-1}} \otimes a_k \in I_L^{(i-1)}$. Accordingly, $a \in I_L^{(i)e}$ implies $a' \in I_L^{(i-1)e}$.

Assume now that $x \in H^{(i)}$. For $a \in I_L^{(i)e}$ as above, $0 = xa = x(\sum \ell_k \otimes a_k) = \sum \ell_k x \phi_{a_k}$ and $(Fr_{DR}^{(0)}(x))a' = \sum \ell_k^{q^{-1}} \tau x \phi_{a_k} = \tau(\sum \ell_k x \phi_{a_k}) = Fr_{DR}^{(0)}(xa) = 0$.

That is, $Fr_{DR}^{(0)}(x)$ is annihilated by $I_L^{(i-1)e}$, and (10) is shown.

¿From (3.9) we read off that $\ker(Fr_{DR}^{(0)d})$ has dimension $d$; moreover,

$$\ker(Fr_{DR}^{(0)d}) = \bigoplus_{1 \leq i \leq d} (\ker(Fr_{DR}^{(0)d}) \cap H^{(i)}), \tag{11}$$

where each term has dimension one.

*Proof of* (11). The existence of the direct sum decomposition follows from (10). For the dimension statement, we distinguish the cases

(a) there exists $j$ such that the one-dimensional space $\ker(Fr_{DR}^{(0)})$ is contained in $H^{(j)}$;

(b) $\ker(Fr_{DR}^{(0)}) \cap H^{(i)} = \{0\}$ for all $i$.

In case (a), all the maps induced from $Fr_{DR}^{(0)}$:

$$H^{(i)} \longrightarrow H^{(i-1)} \longrightarrow \cdots \longrightarrow H^{(i-d)} = H^{(i)}$$

are bijective except for one, which has a kernel of dimension one, which gives the assertion. In case (b), it follows from the commutative diagram

$$\begin{array}{ccccc} & H^{(i)} & \xrightarrow{Fr_{DR}^{(0)}} & H^{(i-1)} & \\ Fr_{DR}^{(0)d} & \downarrow & & \downarrow & Fr_{DR}^{(0)d} \\ & H^{(i)} & \xrightarrow{Fr_{DR}^{(0)}} & H^{(i-1)}. & \end{array}$$

Viz, since the horizontal maps are bijective, the kernels of all vertical maps have the same dimensions, which sum up to $d$.

Similarly, we see that for exponents $k \in \mathbb{N}$,

$$\dim(\ker(Fr_{DR}^{(0)dk})) \cap H^{(i)} = \begin{array}{ll} k, & 1 \leq k \leq he \\ he, & k \geq he. \end{array} \tag{12}$$

We now restrict to considering the direct decomposition

$$H^{(d)} = H_0^{(d)} \oplus H_1^{(d)} \tag{13}$$

(see (2.4)) into the nilspace of $Fr_{DR}^{(0)d}$ on $H^{(d)}$ and its canonical complement. (As $L$ is perfect, the decomposition is defined over $L$ by Galois descent.) Both $H_0^{(d)}$ and $H_1^{(d)}$ are $B^{(d)}$-submodules and $B^{(d)}$-free since $B^{(d)} = A_L/I_L^e$ is local. From (8) and (9) we see that $H_0^{(d)} \cong (A_L I_L^e)^h$ with socle $S \cong (I_L^{e-1}/I_L^e)^h$ of dimension $h$. Since $S$ is the nilspace of $Fr_{DR}$ on $H_{DR}(\phi, L)$, we have established that $\dim H_{DR}(\phi, L)_0 = h$. Then $Fr_{DR}^h$ must vanish on $H_{DR}(\phi, L)_0$, which, together with (12) and $Fr_{DR}^{(0)d} = Fr_{DR}$ on $H_{DR}(\phi, L)$, implies $\dim\ker(Fr_{DR}^k) = k$ as long as $k \leq h$. Theorem 3.6 is proved. $\square$

## 4. A pairing between $\mathfrak{p}$-torsion and de Rham cohomology.

We let $A$ be any Drinfeld ring as described in (1.1), $L$ an $A$-field of $A$-characteristic $\mathfrak{p} \neq \{0\}$ with algebraic closure $\overline{L}$ and $\phi$ a Drinfeld $A$-module over $L$, of rank $r$ and height $h \leq r$. From (4.4) on we assume that $L$ is perfect.

**4.1 Definition.** *We let*

$$_{\mathfrak{p}}\phi := \bigcap_{p \in \mathfrak{p}} {}_p\phi = \{x \in \overline{L} \mid \phi_p(x) = 0 \; \forall p \in \mathfrak{p}\}$$

*be the $\mathfrak{p}$-torison of $\phi$, an $\mathbb{F}_{\mathfrak{p}}$-module of dimension $r - h$.*
*For $0 \neq p \in \mathfrak{p}$ fixed, we define the pairing*

$$\begin{array}{llcl} \langle\ ,\ \rangle_p : & {}_{\mathfrak{p}}\phi \times D(\phi, L) & \longrightarrow & \overline{L} \\ & (x, \eta) & \longmapsto & \langle x, \eta\rangle_p := \eta_p(x) \end{array}.$$

**4.2 Proposition.** *The symbol $\langle\ ,\ \rangle$ has the following properties:*

(i) *$\langle\ ,\ \rangle_p$ is bi-additive;*
(ii) *$\langle x, \eta\rangle_p$ is $A$-linear in $x$;*
(iii) *$\langle x, \eta\rangle_p$ is $A_L$-linear in $\eta$;*
(iv) *$\langle x, \eta\rangle_p = 0$ if $\eta \in D_i(\phi, L)$; so $\langle\ ,\ \rangle_p$ defines a pairing*

$$_{\mathfrak{p}}\phi \times H_{DR}(\phi, L) \longrightarrow \overline{L}$$

*that vanishes on $D_i(\phi, L)/D_{si}(\phi, L) \hookrightarrow H_{DR}(\phi, L)$;*
(v) *for $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}|L)$ we have $\langle \sigma(x), \eta\rangle_p = \sigma(\langle x, \eta\rangle_p)$;*
(vi) *for $a \in A$, $\langle\ ,\ \rangle_{ap} = \gamma(a)\langle\ ,\ \rangle_p$;*
(vii) *let $u : \phi \longrightarrow \psi$ be an isogeny (non-trivial homomorphism) from $\phi$ to a Drinfeld $A$-module $\psi$ over $L$, $x \in {}_{\mathfrak{p}}\phi$, $\eta \in D(\psi, L)$. Then*

$$\langle u(x), \eta\rangle_p = \langle x, \eta \circ u\rangle_p,$$

*where the left hand side refers to the pairing for $\psi$.*

*Proof.* *(i) is obvious, as is $L$-linearity in the second variable $\eta$. For the other items, we first recall the identity for $a \in A$, $\eta \in D(\phi, L)$:*

$$\eta_p \circ \phi_a = \eta_{pa} = \eta_{ap} = \gamma(a)\eta_p + \eta_a \circ \phi_p. \tag{4.3}$$

*It immediately implies (ii), (iii) and (vi). Let now $n \in L\{\tau\}$ and $\eta^{(n)} \in D_i(\phi, L)$ be the associated derivation. For $x \in {}_{\mathfrak{p}}\phi$, $\langle x, \eta^{(n)}\rangle_p = \eta_p^{(n)}(x) = \gamma(p)n(x) - n(\phi_p(x)) = 0$ since $\gamma(p) = 0 = \phi_p(x)$, thus (iv). The remaining assertions (v) and (vii) are obivous, e.g. $\langle u(x), \eta\rangle_p = \eta_p(u(x)) = (\eta \circ u)_p(x) = \langle x, \eta \circ u\rangle_p$.* □

*¿From now on, we suppose that $L$ is perfect. Let*

$$H_{DR}(\phi, L) = H_{DR}(\phi, L)_0 \oplus H_{DR}(\phi, L)_1 = H_0 \oplus H_1 \tag{4.4}$$

*be the decomposition asserted by Corollary 3.5, with $\dim_L H_0 = h = ht(\phi)$.*

**4.5 Proposition.** *For any $p \in \mathfrak{p}$ we have $\langle {}_{\mathfrak{p}}\phi, H_0\rangle_p = 0$.*

*Proof.* *Let $x \in {}_{\mathfrak{p}}\phi$ and $\eta \in D(\phi, L)$ be such that its class $[\eta]$ belongs to $H_0$. Then $Fr_D^h(\eta) = \eta^{(n)}$ for some $n \in N(\phi, L)$. As $Fr_D$ is left multiplication with $\tau^d$ ($d := \deg \mathfrak{p}$), this means that $\tau^{hd}\eta_p = -n\phi_p$. Therefore, $\tau^{hd}\eta_p(x) = -n(\phi_p(x)) = 0$, which gives $0 = \eta_p(x) = \langle x, \eta\rangle_p$.* □

*In view of 4.2 (vi), the pairing $\langle\ ,\ \rangle_p$ depends on $p$ only via a factor in $\mathbb{F}_{\mathfrak{p}} \hookrightarrow L$. From now on, we choose $p \in \mathfrak{p}$ as a uniformizer, i.e., an element of $A$ with $\mathfrak{p}$-adic valuation one, and label the resulting pairing $\langle\ ,\ \rangle_p$ by $\langle\ ,\ \rangle$. (For the cost of some more notation, we could avoid that ambiguity and write a canonical pairing, using differentials.) By (4.5), we may restrict the second argument of $\langle\ ,\ \rangle$ to elements of $H_1 \xrightarrow{\cong} H_{DR}(\phi, L)/H_0$. Recall that ${}_{\mathfrak{p}}\phi$ is an $\mathbb{F}_{\mathfrak{p}}$-vector space of dimension $r - h$ ($r = rk(\phi)$, $h = ht(\phi)$), while $H_1$ has $L$-dimension $r - h$. The pairing $\langle\ ,\ \rangle$ has a unique $\overline{L}$-bilinear extension, labelled by the same symbol, to the corresponding $\overline{L}$-spaces:*

$$\langle\ ,\ \rangle : {}_{\mathfrak{p}}\phi \otimes_{\mathbb{F}_{\mathfrak{p}}} \overline{L} \times H_{DR}(\phi, \overline{L})_1 \longrightarrow \overline{L}. \tag{4.6}$$

*The result of this section is*

**4.7 Theorem.** *The pairing in (4.6) is perfect. That is, it identifies each of these vector spaces with the dual of the other one.*

*Proof.* *Since both spaces have the same dimension $r - h$, it suffices to show that the right kernel of $\langle\ ,\ \rangle$ is zero. Let $\eta \in D(\phi, L)$ be such that its class $[\eta]$ in $H_{DR}(\phi, L)$ pairs with ${}_{\mathfrak{p}}\phi$ to zero. We are going to show that $Fr_D^h(\eta) = \eta^{(n)}$ for some $n \in N(\phi, L)$, which will give the* *conclusion.*

*Suppose first that* $\mathfrak{p} = (p)$ *is principal, and that* $\langle\ ,\ \rangle = \langle\ ,\ \rangle_p$. *The polynomial* $\phi_p$ *is right (and left, since* $L$ *is perfect) divisible by* $\tau^{hd} = \varphi_{\mathfrak{p}}^h$. *Write*

$$\phi_p = u \circ \varphi^h = \varphi^h \circ v$$

*with* $\varphi = \varphi_{\mathfrak{p}}$*; both* $u$ *and* $v$ *have non-vanishing constant terms, and* $v = u^{(\varphi^{-h})}$ *has* ${}_{\mathfrak{p}}\phi$ *as its kernel. The fact that* $\langle {}_{\mathfrak{p}}\phi, \eta\rangle = 0$ *means* $\eta_p = n \circ v$ *for some* $n \in L\{\tau\}\tau = N(\phi, L)$*, i.e.,*

$$\varphi^h \circ \eta_p = \varphi^h \circ n \circ v = n^{(\varphi^h)}(\varphi^h \circ v) = n^{(\varphi^h)} \circ \phi_p.$$

*That is,* $\varphi^h \circ \eta_p = \eta_p^{(n')}$ *for* $n' = -n^{(\varphi^h)}$*, which in fact implies* $Fr_D^h(\eta) = \varphi^h \circ \eta = \eta^{(n')}$ *is strictly inner as wanted.*

*Essentially the same argument works in the general case, where* $\mathfrak{p}$ *fails to be principal. Let* $p \in \mathfrak{p} - \mathfrak{p}^2$*, and choose* $a \in A - \mathfrak{p}$ *congruent to zero modulo the ideal* $(p)\mathfrak{p}^{-1}$. *Then* $\phi_a$ *maps* ${}_p\phi = \{x \in \overline{L} \mid \phi_p(x) = 0\}$ *onto* ${}_{\mathfrak{p}}\phi$. *For* $x \in {}_p\phi$ *we have from (4.3) the relation*

$$\eta_p(\phi_a(x)) = \eta_{ap}(x) = \gamma(a)\eta_p(x),$$

*where the left hand side vanishes due to the choice of* $a$ *and the assumption* $\langle {}_{\mathfrak{p}}\phi, \eta\rangle = 0$. *Hence that assumption implies that* $\eta_p(x) = 0$ *even holds for* $x \in {}_p\phi \supset {}_{\mathfrak{p}}\phi$. *Repeating the argument from before, we find*

$$Fr_D^h(\eta) = \eta^{(n')},$$

*where* $n' = -n^{(\varphi^h)}$, $\eta_p = n \circ v$, $\phi_p = \varphi^h \circ v$. □

## 5. An example.

*In this section, we work out the case of rank-2 Drinfeld A-modules* $\phi$*, where* $A = \mathbb{F}_q[T]$ *is a polynomial ring and* $L$ *is the A-prime field* $\mathbb{F}_{\mathfrak{p}} = A/\mathfrak{p}$. *Here* $\mathfrak{p}$ *is a prime of degree* $d$, $\gamma : A \longrightarrow L = \mathbb{F}_{\mathfrak{p}}$ *is the reduction, and* $\phi$ *is given by*

$$\phi_T = \gamma(T) + g\tau + \Delta\tau^2, \tag{5.1}$$

*where* $g \in L$, $\Delta \in L^*$ *may be chosen without restrictions. Similarly, specifying* $\eta \in D(\phi, L)$ *is the same as specifying* $\eta_T \in N(\phi, L)$. *In that situation, a basis of* $H_{DR}(\phi, L)$ *may be described independently of* $\mathfrak{p}$ *and* $\phi$*, viz:*

*(5.2) Each* $\eta \in D(\phi, L)$ *is congruent modulo* $D_{si}(\phi, L)$ *to a unique* reduced *derivation, i.e., one that satisfies* $\deg_\tau \eta_T \leq 2$.

*Hence the classes* $[{}^i\eta]$ *of* ${}^i\eta$ *with* ${}^i\eta = \tau^i$ $(i = 1, 2)$ *form an* $L$*-basis of* $H_{DR}(\phi, L)$*. The Frobenius* $\varphi = \tau^d$ *acts on* $D(\phi, L)$ *by multiplication*

$$\eta \longmapsto \eta \circ \varphi,\ (\eta \circ \varphi)_a = \eta_a \circ \varphi = \varphi \circ \eta_a.$$

*Note that in this special situation we need not distinguish between left and right multiplication with* $\varphi$*, so* $Fr_D : D(\phi, L) \longrightarrow D(\phi, L)$, $\eta \longmapsto \eta \circ \varphi$ *is* $A_L$*-linear and* $Fr_{DR} : H_{DR}(\phi, L) \longrightarrow H_{DR}(\phi, L)$, $[\eta] \longmapsto [\eta \circ \varphi]$ *is* $L$*-linear.*

*(5.3) For* $k \geq 0$ *write*

$$\tau^{k+2} = a_k\tau + b_k\tau^2 + \gamma(T)n_k - n_k \circ \phi_T$$

*with uniquely determined* $n_k \in N(\phi, L)$, $a_k, b_k \in L$*. Then*

$$\begin{array}{rcl} Fr_{DR}([{}^1\eta]) & = & a_{d-1}[{}^1\eta] + b_{d-1}[{}^2\eta] \\ Fr_{DR}([{}^2\eta]) & = & a_d[{}^1\eta] + b_d[{}^2\eta], \end{array} \tag{5.4}$$

*that is, with respect to the basis* $\{[{}^1\eta], [{}^2\eta]\}$, $Fr_{DR}$ *is represented by the matrix*

$$M_\phi := \begin{pmatrix} a_{d-1} & a_d \\ b_{d-1} & b_d \end{pmatrix} \in \mathrm{Mat}(2, L).$$

*To get our hands on* $(a_k, b_k)$ *we perform calculations in the ring* $R = A[g, \Delta, \Delta^{-1}]$*, where the quantities* $g$ *and* $\Delta$ *are considered as formal indeterminates over* $A$*. Similar to (5.3), there exist unique elements* $\nu_k$ *of* $R\{\tau\}\tau$ *and* $\alpha_k, \beta_k \in R$ *such that*

$$\tau^{k+2} = \alpha_k\tau + \beta_k\tau^2 + T\nu_k - \nu_k \cdot \phi_T \tag{5.5}$$

*hold with* $\phi_T = T + g\tau + \Delta\tau^2$*. The coefficients* $n_{k,i}$ *of* $\nu_k = \sum_{1 \leq i \leq k} n_{k,i}\tau^i$ *are determined by comparing coefficients in (5.5). Putting* $[i]$ *for the element* $T^{q^i} - T$ *of* $A$*, we find the following equations, which allow to recursively solve for* $n_{k,i}$ *(in decreasing order in* $i$ *starting with* $n_{k,k}$*) and* $\alpha_k, \beta_k$*, where* $k \geq 2$*:*

$$\begin{array}{lcll} n_{k,k} & = & -\Delta^{-q^k} & (i = k+2) \\ g^{q^k} n_{k,k} + \Delta^{q^{k-1}} n_{k,k-1} & = & 0 & (i = k+1) \\ [i]n_{k,i} + g^{q^{i-1}} n_{k,i-1} + \Delta^{q^{i-2}} n_{k,i-2} & = & 0 & (3 \leq i \leq k) \\ [2]n_{k,2} + g^q n_{k,1} & = & \beta_k & (i = 2) \\ [1]n_{k,1} & = & \alpha_k & (i = 1) \end{array} \tag{5.6}$$

*For small* $k$*, we find directly:*

(5.7)

| $k$ | $\alpha_k$ | $\beta_k$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | $-\frac{[1]}{\Delta^q}$ | $-(\frac{g}{\Delta})^q$ |
| 2 | $[1]\frac{g^{q^2}}{\Delta^{q^2+q}}$ | $-\frac{[2]}{\Delta^{q^2}} + (\frac{g}{\Delta})^{q^2+q}$ |

*On a first view, (5.6) doesn't yield a direct relationship between the $\alpha_k, \beta_k$ for varying $k$. Such a relation is supplied by the next result.*

**5.8 Proposition.** *For $k \geq 2$ the $\alpha_k, \beta_k$ satisfy*

$$\begin{aligned}
\alpha_k &= -(\tfrac{g}{\Delta})^{q^k} \alpha_{k-1} - \tfrac{[k]}{\Delta^{q^k}} \alpha_{k-2} \\
\beta_k &= -(\tfrac{g}{\Delta})^{q^k} \beta_{k-1} - \tfrac{[k]}{\Delta^{q^k}} \beta_{k-2}.
\end{aligned}$$

*Proof (sketch). We may compare the different systems (5.6) for $k$ and $k+1$, which yields recursions for the $n_{k,i}$ with respect to $k$. From these and the last two equations of (5.6), we find the stated recursions for the $\alpha_k$ and $\beta_k$. We omit the complicated but elementary details.* $\square$

**5.9 Remark.** *The ring $R = A[g, \Delta, \Delta^{-1}]$ is a ring of "meromorphic modular forms for rank-two Drinfeld A-modules", see* [6] *sect. 5; "meromorphic" since denominators $\Delta$ are allowed. It is natural to assign $g$ and $\Delta$ the weights $q-1$ and $q^2-1$, respectively, which defines a grading on $R$. Then the above $\alpha_k$ and $\beta_k$ are modular forms of respective weights $w(\alpha_k) = -(q^{k+1}-1)q$, $w(\beta_k) = -(q^k-1)q^2$, as is immediate from (5.8) (or directly from (5.6)).*

*We collect the results.*

**5.10 Corollary.** *The Frobenius matrix $M_\phi$ of* (5.4) *is obtained from*

$$M_d = \begin{pmatrix} \alpha_{d-1} & \alpha_d \\ \beta_{d-1} & \beta_d \end{pmatrix} \in \mathrm{Mat}(2, R)$$

*by reduction modulo $\mathfrak{p}$ (and inserting values for $g, \Delta$). The modular forms $\alpha_k, \beta_k$ are homogeneous polynomials in $g$ and $\Delta$ divided by $\Delta^{n(k)}$, where $n(k) = q + q^2 + \cdots + q^k$, and may be determined by the recursion (5.8) along with the initial values given in* (5.7).

*Proof. Only the assertion about the denominator $\Delta^{n(k)}$ remains to be shown. It is straightforward from (5.8).* $\square$

**5.11 Corollary.**

(i) *The matrix $M_\phi$ is of shape*

$$\begin{pmatrix} a_{d-1}, & -\frac{g}{\Delta} a_{d-1} \\ b_{d-1}, & -\frac{g}{\Delta} b_{d-1} \end{pmatrix},$$

*of rank one.*

(ii) *It always has the eigenvector $\begin{pmatrix} g \\ \Delta \end{pmatrix}$ with eigenvalue* 0, *corresponding to the element $[g\ {}^1\eta + \Delta\ {}^2\eta]$ in the kernel of $Fr_{DR}$.*

(iii) *We have the equivalences*
$tr(M_\phi) = 0 \Leftrightarrow M_\phi$ *nilpotent* $\Leftrightarrow H_{DR}(\phi, L) = H_0$
$\Leftrightarrow \phi$ *supersingular (see* [9] *sect.* 4*).*

(iv) *Suppose* $\phi$ *is not supersingular. Then* $\begin{pmatrix} a_{d-1} \\ b_{d-1} \end{pmatrix}$ *is an eigenvector for the non-vanishing eigenvalue, corresponding to the element* $[a_{d-1}\ {}^1\eta + b_{d-1}\ {}^2\eta]$ *of* $H_{DR}(\phi, L)$*.*

*Proof. The shape of* $M_\phi$ *results from (5.10) and (5.8), since* $[d] \equiv 0(\mathrm{mod}\ \mathfrak{p})$*. On the other hand,* $M_\phi$ *cannot be the zero matrix by (3.6), thus (i). Items (ii) and (iv) are immediate from (i), and (iii) results from (i) together with the properties characterizing supersingularity given e.g. in* [9] *sect. 4.* □

**5.12 Remark.** *From (5.10) one can derive a recursive procedure for the determination of* $tr(M_\phi)$*, or, what amounts to the same, the* Hasse invariant $H(\phi)$ *of* $\phi$ *(see* [10] *sect. 3). However, that procedure is equivalent to the one described loc. cit. Prop. 3.7, for which the present considerations therefore provide a new proof.*

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