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Toeplitz Operators on Hardy Spaces

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Abstract

In the present thesis we establish Banach space counterparts for several results known for Toeplitz operators on $\mathrm{H}^2(\partial D)$ for suitable domains $D \subset \mathbb{C}^d$.

A classical result of Brown and Halmos, stating that the Toeplitz operators on the Hardy space $\mathrm{H}^2(\mathbb{T})$ with bounded measurable symbol are precisely the solutions X of the operator equation $T_z^* X T_z = X$, inspired Didas, Eschmeier and Everard to construct Toeplitz projections for Toeplitz operators associated with regular A-isometries. We use their methods in a suitable Banach space setting to construct Toeplitz projections for Toeplitz operators acting on a general class of Hardy-type spaces $\mathrm{H}^p(\partial D)$ $(1 < p < \infty)$ over suitable domains $D \subset \mathbb{C}^d$. These Toeplitz projections provide a general framework for Brown-Halmos type characterizations of Toeplitz operators and allow us to prove a Banach space version of a classical spectral inclusion theorem of Hartman and Wintner.

In the final chapter we show that a multivariable spectral mapping theorem of Eschmeier remains true for Toeplitz tuples on Hardy spaces $\mathrm{H}^p(\partial D)$ over strictly pseudoconvex domains $D \subset \mathbb{C}^d$ with smooth boundary. As an application we derive a spectral mapping theorem for truncated Toeplitz systems which generalizes a one dimensional spectral mapping theorem proved by Bessonov for the Hardy space $\mathrm{H}^2(\mathbb{T})$ on the unit disc.

Zusammenfassung

In der vorliegenden Arbeit zeigen wir, dass einige ausgewählte Sätze über Toeplitzoperatoren auf Hardy-Hilberträumen auch in einer entsprechenden Banachraumsituation richtig bleiben.
Inspiriert durch ein klassisches Resultat von Brown und Halmos, welches Toeplitzoperatoren auf dem Hardyraum $\mathrm{H}^2(\mathbb{T})$ mit beschränktem messbaren Symbol genau als die Lösungen X der Operatorgleichung $T_z^* X T_z = X$ identifiziert, konstruierten Didas, Eschmeier und Everard Toeplitzprojektionen für eine allgemeine Klasse mehrdimensionaler Toeplitzoperatoren. Wir benutzen die gleichen Methoden in einer geeigneten Banachraumsituation, um Toeplitzprojektionen auf Hardyräumen $\mathrm{H}^p(\partial D)$ $(1 < p < \infty)$ über geeigneten Gebieten $D \subset \mathbb{C}^d$ zu konstruieren. Diese Projektionen liefern einen allgemeinen Rahmen für Sätze vom Brown-Halmos Typ und ermöglichen es uns, einen klassischen spektralen Inklusionssatz von Hartman und Wintner in unserer Situation zu formulieren und zu beweisen.
Im zweiten Teil der Arbeit zeigen wir, dass ein mehrdimensionaler spektraler Abbildungssatz von Eschmeier auch für Tupel von Toeplitzoperatoren auf Hardyräumen $\mathrm{H}^p(\partial D)$ über streng pseudokonvexen Gebieten $D \subset \mathbb{C}^d$ mit glattem Rand richtig bleibt. Als Anwendung leiten wir einen spektralen Abbildungssatz für Tupel von trunkierten Toeplitzoperatoren her. Letzterer verallgemeinert ein eindimensionales Resultat von Bessonov.

CONTENTS

Introduction

During the last century Toeplitz operators on Hardy spaces have evolved into one of the most prominent example for the fruitful interplay between complex analysis, operator theory and the theory of Banach algebras. A Toeplitz operator on the Hardy space $\mathrm{H}^2(\mathbb{T})$ on the unit circle $\mathbb{T} \subset \mathbb{C}$ with bounded measurable symbol $f \in \mathrm{L}^\infty(\mathbb{T})$ is defined as the compression

$$X = P_{\mathrm{H}^2(\mathbb{T})} M_f|_{\mathrm{H}^2(\mathbb{T})} \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}))$$

of the multiplication operator $M_f \colon \mathrm{L}^2(\mathbb{T}) \to \mathrm{L}^2(\mathbb{T})$, $M_f g = fg$. A classical result of M. Riesz [16, Theorem 3.3.5] shows that, for $1 < p < \infty$, the restriction of the orthogonal projection $P_{\mathrm{H}^2(\mathbb{T})} \colon \mathrm{L}^2(\mathbb{T}) \to \mathrm{H}^2(\mathbb{T})$ to the subspace $\mathbb{C}[z, \overline{z}]\,|_{\mathbb{T}} \subset \mathrm{L}^2(\mathbb{T})$ of all trigonometric polynomials is L^p-continuous and extends to a continuous projection $\mathrm{L}^p(\mathbb{T}) \to \mathrm{H}^p(\mathbb{T})$ onto the Hardy space $\mathrm{H}^p(\mathbb{T}) \subset \mathrm{L}^p(\mathbb{T})$. Hence the definition of a Toeplitz operator for $\mathrm{H}^2(\mathbb{T})$ given above can naturally be extended to define Toeplitz operators $T_f^p \in \mathrm{L}(\mathrm{H}^p(\mathbb{T}))$ on the Hardy spaces $\mathrm{H}^p(\mathbb{T}) \subset \mathrm{L}^p(\mathbb{T})$ $(1 < p < \infty)$. This class of operators has been studied extensively not only on the unit disc, but also on more general single and multidimensional domains. A detailed exposition of the analysis of Toeplitz operators on the Hardy spaces $\mathrm{H}^p(\mathbb{T})$ with many historical remarks can be found in the monograph [13] of A. Böttcher and B. Silbermann.

Compared with the extremely rich theory of Toeplitz operators on Hardy-Hilbert spaces much less is known in the Banach space case. Therefore, a natural question is whether results known in the Hilbert space setting remain true in the more general context of Banach spaces.

We shall address some of these questions using as basic tools the existence of Toeplitz projections on Hardy spaces $\mathrm{H}^p(\partial D)$ for suitable multivariable domains $D \subset \mathbb{C}^d$ and results from the interpolation theory for compact operators. Our approach is mainly based on recent articles of J. Eschmeier and K. Everard [34] and J. Eschmeier [31] in which the Hilbert space case is treated.

A classical result of A. Brown and P. R. Halmos [14] states that the Toeplitz operators on $\mathrm{H}^2(\mathbb{T})$ are precisely the solutions $X \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}))$ of the algebraic operator equation

$$T_z^* X T_z = X,$$

where $T_z \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}))$ denotes the unilateral shift on $\mathrm{H}^2(\mathbb{T})$. If one replaces T_z and T_z^* on $\mathrm{H}^2(\mathbb{T})$ by T_z^p and $T_{\overline{z}}^p$ on $\mathrm{H}^p(\mathbb{T})$, one obtains a well known characterization of Toeplitz operators on $\mathrm{H}^p(\mathbb{T})$. Thus the classical Brown-Halmos characterization has a counterpart for $\mathrm{H}^p(\mathbb{T})$. On the other hand the Brown-Halmos condition described above inspired B. Prunaru [61] and M. Didas, J. Eschmeier, K. Everard [26], [34] to study more general classes of Toeplitz operators that are defined as fixed points of operator equations

$$T_{\overline{\theta}} X T_\theta = X,$$

where the symbols θ run through a sufficiently large class of inner functions. The general framework described in the latter articles contains a general notion of Hardy-Hilbert spaces which can be similarly defined for arbitrary values of $1 < p < \infty$.

Fix a positive integer $d \geq 1$, let $K \subset \mathbb{C}^d$ be a compact set and let $A \subset \mathrm{C}(K)$ be a closed subalgebra of the space $\mathrm{C}(K)$ of all complex-valued, continuous functions on K that contains the restrictions $\mathbb{C}[z]\,|_K$ of all holomorphic polynomials in d complex variables $z = (z_1, \dots, z_d)$. Furthermore, fix a positive regular Borel measure $\mu \in \mathrm{M}^+(\partial_A)$ on the Shilov boundary $\partial_A \subset K$ of A. For $1 < p < \infty$, we denote by $(\mathrm{L}^p(\partial_A), \|\cdot\|_p)$ the Banach space of (equivalence classes) of all p-integrable functions with respect to μ. We call the closure

$$\mathrm{H}^p(\partial_A) = \overline{A}^{\|\cdot\|_p} \subset \mathrm{L}^p(\partial_A)$$

in $\mathrm{L}^p(\partial_A)$ of A a Hardy-type space induced by A, provided that (A, ∂_A, μ) is regular in the sense of A. B. Aleksandrov [3] and that there exists a family $(P_p)_{1<p<\infty}$ of bounded linear projections $P_p \colon \mathrm{L}^p(\partial_A) \to \mathrm{L}^p(\partial_A)$ onto $\mathrm{H}^p(\partial_A)$ $(1 < p < \infty)$ that is compatible with the inclusion mappings $i_{pq} \colon \mathrm{L}^p(\partial_A) \to \mathrm{L}^q(\partial_A)$ $(1 < q < p < \infty)$ in the sense that $i_{pq} P_p = P_q i_{pq}$ for all $1 < q < p < \infty$. Natural examples of such Hardy-type spaces are the Hardy spaces $\mathrm{H}^p(\partial\mathbb{B})$, $\mathrm{H}^p(\mathbb{T}^d)$, $\mathrm{H}^p(\partial D)$ over the unit ball $\mathbb{B} \subset \mathbb{C}^d$, the unit polydisc $\mathbb{D}^d \subset \mathbb{C}^d$ or a strictly pseudoconvex domain $D \subset \mathbb{C}^d$ with C^2-boundary (see Section 1.3).

For $f \in \mathrm{L}^\infty(\partial_A)$, the Toeplitz operator $T_f^p \in \mathrm{L}(\mathrm{H}^p(\partial D))$ with symbol f is defined as the compression

$$T_f^p = P_p M_f^p|_{\mathrm{H}^p(\partial_A)}$$

of the multiplication operator $M_f^p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$. In the case $p = 2$ we use the shorter notation $T_f = T_f^2$. We write $\mathrm{H}^\infty(\partial_A) = \overline{A}^{\tau_{w^*}} \subset \mathrm{L}^\infty(\partial_A)$ for the weak* closure of A in $\mathrm{L}^\infty(\partial_A) = \mathrm{L}^1(\partial_A)'$ and denote by

$$I_\mu = \{\theta \in \mathrm{H}^\infty(\partial_A) \mid |\theta| = 1 \quad \mu\text{-almost everywhere on } \partial_A\}$$

the family of all μ-inner functions on ∂_A. Under a suitable regularity assumption on the triple (A, ∂_A, μ) due to A. B. Aleksandrov [3] which ensures the existence of sufficiently many μ-inner functions, it is shown in [26] that the Brown-Halmos condition

$$T_\theta^* X T_\theta = X \qquad \text{for every } \theta \in I_\mu$$

characterizes the family $\mathcal{T} = \left\{T_f \in \mathrm{L}(\mathrm{H}^2(\partial_A)) \;\middle|\; f \in \mathrm{L}^\infty(\partial_A)\right\}$ among all operators $X \in \mathrm{L}(\mathrm{H}^2(\partial_A))$ and that, under the same condition, there exists a unital continuous projection $\Phi\colon\ \mathrm{L}(\mathrm{H}^2(\partial_A)) \to \mathrm{L}(\mathrm{H}^2(\partial_A))$ onto the subspace $\mathcal{T}$. In [34] the projection Φ is constructed as the pointwise ultraweak limit of operators of the form

$$\Phi_k(X) = \frac{1}{k^k} \sum_{1 \le i_1, \ldots, i_k \le k} T^*_{\theta_k^{i_k} \cdots \theta_1^{i_1}} X T_{\theta_1^{i_1} \cdots \theta_k^{i_k}}$$

in $\mathrm{L}(\mathrm{H}^2(\partial_A))$, where $I = (\theta_k)_{k \ge 1}$ is a sequence in I_μ with the property that

$$\mathrm{L}^\infty(\partial_A) = \overline{\mathrm{alg}(I \cup I^*)}^{\tau_{w^*}}$$

is the weak* closure of the subalgebra $\mathrm{alg}(I \cup I^*) \subset \mathrm{L}^\infty(\partial_A)$ generated by the functions in I and their complex conjugates. If one replaces the Hilbert space $\mathrm{H}^2(\partial_A)$ by the Banach space $\mathrm{H}^p(\partial_A) \subset \mathrm{L}^p(\partial_A)$, a suitable modification of the above construction can be used to prove the existence of a Toeplitz projection on $\mathrm{H}^p(\partial_A)$ (see Chapter 2):

Theorem 2.9.
For every $1 < p < \infty$ and every family $I = (\theta_k)_{k\geq 1}$ of μ-inner functions on ∂_A with the property that $\mathrm{L}^\infty(\partial_A) = \overline{\mathrm{alg}(I \cup I^*)}^{\tau_{w^*}}$, there exists an associated unital continuous projection $\Phi\colon \mathrm{L}(\mathrm{H}^p(\partial_A)) \to \mathrm{L}(\mathrm{H}^p(\partial_A))$ onto the subspace

$$\mathcal{T}^p = \left\{T_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A)) \;\middle|\; f \in \mathrm{L}^\infty(\partial_A)\right\} \subset \mathrm{L}(\mathrm{H}^p(\partial_A)).$$

The projection in the last theorem will be called the Toeplitz projection for $\mathrm{H}^p(\partial_A)$. This result has a number of consequences which we will summarize now. Our proof of Theorem 2.9 shows at the same time that

$$\mathcal{T}^p = \left\{X \in \mathrm{L}(\mathrm{H}^p(\partial_A)) \;\middle|\; T_{\overline{\theta}}^p X T_\theta^p = X \text{ for all } \theta \in I\right\}.$$

Thus we obtain H^p-versions of the Brown-Halmos characterization of Toeplitz operators on Hardy-type spaces. Since the Hardy spaces $\mathrm{H}^p(\partial\mathbb{B})$, $\mathrm{H}^p(\mathbb{T}^d)$, $\mathrm{H}^p(\partial D)$ are Hardy-type spaces in the sense described above, one regains results proved for $p = 2$ on these spaces (see, e.g., [77] for the case of Toeplitz operators on $\mathrm{H}^2(\partial\mathbb{B})$). In particular, if one applies Theorem 2.9 to the family

$$I = \{z_1, \ldots, z_d, z_1, \ldots, z_d, \ldots\}$$

of coordinate functions $z_i\colon \mathbb{C}^d \to \mathbb{C}$, one obtains the following Brown-Halmos theorem for the Hardy space $\mathrm{H}^p(\mathbb{T}^d)$ on the unit polydisc in $\mathbb{C}^d$. For the Hilbert space case $p = 2$, this result was proved by A. Maji, J. Sarkar and S. Sarkar in [51, Theorem 3.1].

Corollary 2.11.
For $1 < p < \infty$, an operator $X \in \mathrm{L}(\mathrm{H}^p(\mathbb{T}^d))$ is a Toeplitz operator if and only if $T_{\overline{z}_i}^p X T_{z_i}^p = X$ for $i = 1, \ldots, d$.

For a unital closed subalgebra $\mathcal{B} \subset \mathrm{L}^\infty(\partial_A)$, denote by

$$\mathcal{T}^p(\mathcal{B}) = \overline{\mathrm{alg}}\left\{T_f^p \;\middle|\; f \in \mathcal{B}\right\} \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

the smallest norm-closed subalgebra containing all Toeplitz operators $T_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ with symbol $f \in \mathcal{B}$. Using the Toeplitz projection we show that the algebra $\mathcal{T}^p(\mathcal{B})$ has the direct sum decomposition

$$\mathcal{T}^p(\mathcal{B}) = \left\{T_f^p \mid f \in \mathcal{B}\right\} \oplus \mathcal{SC}^p(\mathcal{B}),$$

where $\mathcal{SC}^p(\mathcal{B})$ is the norm-closed ideal in $\mathcal{T}^p(\mathcal{B})$ generated by all semi-commutators $T_f^p T_g^p - T_{fg}^p$ $(f, g \in \mathcal{B})$.

As a last application we prove a version of the spectral inclusion formula due to P. Hartman and A. Wintner [41] and I. B. Simonenko [69] in our setting. Our theorem reads as follows:

Theorem 2.21.
Assume that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms. Then, for every $1 < p < \infty$ and every $f \in \mathrm{L}^\infty(\partial_A)$, the spectral inclusion formula

$$R(f) \subset \sigma_{\mathrm{e}}(T_f^p)$$

holds.

Here, for $f \in \mathrm{L}^\infty(\partial_A)$,

$$R(f) = \{z \in \mathbb{C} \mid \mu\left(\{x \in X \mid |f(x) - z| < \varepsilon\}\right) > 0 \text{ for all } \varepsilon > 0\}$$

denotes the essential range of the symbol f and $\sigma_{\mathrm{e}}(T_f^p)$ is the essential spectrum of the Toeplitz operator $T_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A))$.

In Chapter 3 we use results from complex interpolation theory for compact operators on Banach spaces to prove compactness results for commutators of Toeplitz operators on Hardy-type spaces $\mathrm{H}^p(\partial_A)$. For $1 < p < \infty$, let us denote by $C_f^p = [M_f^p, P_p] = M_f^p P_p - P_p M_f^p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ the commutator of the multiplication operator $M_f^p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ and the projection $P_p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ onto $\mathrm{H}^p(\partial_A)$. Furthermore, we write $H_f^p = (1 - P_p)M_f^p|_{\mathrm{H}^p(\partial_A)} \in \mathrm{L}(\mathrm{H}^p(\partial_A), \mathrm{L}^p(\partial_A))$ for the Hankel operator with symbol $f \in \mathrm{L}^\infty(\partial_A)$. Using a classical interpolation theorem of M. A. Krasnoselski (see Theorem 3.1) we prove that the set

$$\begin{aligned} QC &= \{f \in \mathrm{L}^\infty(\partial_A) \mid C_f^p \in \mathrm{L}(\mathrm{L}^p(\partial_A)) \text{ is compact}\} \\ &= \{f \in \mathrm{L}^\infty(\partial_A) \mid H_f^p \text{ and } H_{\bar{f}}^p \text{ are compact}\} \end{aligned}$$

is a unital C*-subalgebra of $\mathrm{L}^\infty(\partial_A)$ that does not depend on the choice of the exponent $p \in (1, \infty)$. The observation that, for $f \in QC$ and $g \in \mathrm{L}^\infty(\partial_A)$, the semi-commutator

$$T_g^p T_f^p - T_{gf}^p = -P_p M_g^p H_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A))$$

is compact, leads us to the following corollary.

Corollary 3.4.
For $1 < p < \infty$ and $f \in QC$, the commutators $[T_f^p, T_g^p] \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ are compact for every function $g \in \mathrm{L}^\infty(\partial_A)$.

It is an obvious question whether conversely, for $f \in \mathrm{L}^\infty(\partial_A)$, the condition that $[T_f^p, T_g^p] \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ is compact for all $g \in \mathrm{L}^\infty(\partial_A)$ is also sufficient for f to belong to QC. Under certain additional assumptions (see Theorem 3.5 and Corollary 3.7) on the Hardy-type spaces $\mathrm{H}^p(\partial_A)$ ($1 < p < \infty$), which are needed in order to apply an extrapolation result of M. Cwikel (see Theorem A.3), we give an affirmative answer to this question in Corollary 3.7.

In Chapter 4 we use Corollary 3.4 to extend a multidimensional spectral mapping theorem due to J. Eschmeier [31] to the case of Toeplitz tuples $T_f^p = (T_{f_1}^p, \ldots, T_{f_m}^p) \in \mathrm{L}(\mathrm{H}^p(\partial D))^m$ with symbol f in the closed subalgebra

$$\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D) = \{f + g \mid f \in \mathrm{H}^\infty(\partial D), g \in \mathrm{C}(\partial D)\} \subset \mathrm{L}^\infty(\partial D)$$

on strictly pseudoconvex domains $D \subset \mathbb{C}^d$ with C^∞-boundary. Since, for such domains $D \subset \mathbb{C}^d$, the continuous functions $\mathrm{C}(\partial D)$ on ∂D are contained in the C*-algebra QC, Corollary 3.4 shows that the Toeplitz tuple T_f^p essentially commutes. By a standard construction in multivariable operator theory (see Section 1.2), one can associate with the tuple T_f^p a commuting tuple $(T_f^p)^e \in \mathrm{L}(\mathrm{H}^p(\partial D)^e)^m$ in a canonical way. The essential

Taylor spectrum $\sigma_{\mathrm{e}}(T_f^p)$ of T_f^p is defined as the ordinary Taylor spectrum $\sigma((T_f^p)^e)$ of the tuple $(T_f^p)^e$. If $F = (\mathcal{P}[f_1], \dots, \mathcal{P}[f_m])\colon D \to \mathbb{C}^m$ denotes the tuple of real Poisson transforms of the components f_i of a tuple $f = (f_1, \dots, f_m) \in \mathrm{L}^\infty(\partial D)^m$, then we obtain the following H^p-version of Theorem 4 in [31].

Theorem 4.6.
For $f \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$, the formula

$$\sigma_{\mathrm{e}}(T_f^p) = \bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \textit{ open} \right)$$

holds.

The proof of the inclusion

$$\sigma_{\mathrm{e}}(T_f^p) \subset \bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \text{ open } \right)$$

follows the same lines as the one given in [31]. We use general Gelfand theory together with Corollary 3.4 and a spectral mapping theorem due to M. Andersson and S. Sandberg [6, Theorem 1.2]. The verification of the reverse inclusion turns out to be more involved and requires a different approach, since the arguments of J. Eschmeier (see Lemma 2 in [31]) rely heavily on Hilbert space techniques. In spite of this difficulty, we obtain the desired inclusion by a duality argument.

Assume that $0 \in \mathbb{C}^m$ is contained in the intersection on the right-hand side of the formula claimed in Theorem 4.6. By the definition of the essential Taylor spectrum of T_f^p, it suffices to show that the row operator $(\mathrm{H}^p(\partial D)^e)^m \to \mathrm{H}^p(\partial D)^e$ induced by the multioperator $(T_f^p)^e$ is not onto. Let $q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. By using general duality theory for Banach spaces one sees that the above assertion follows if one can show that the column operator $\mathrm{H}^q(\partial D) \to \mathrm{H}^q(\partial D)^m$ induced by the operator tuple $T_{\overline{f}}^q \in \mathrm{L}(\mathrm{H}^q(\partial D))^m$ has infinite dimensional kernel or non-closed range. The latter is verified exhibiting a joint approximate eigensequence for the tuple $T_{\overline{f}}^q$ converging weakly to zero.

In the last section of Chapter 4 we apply Theorem 4.6 to calculate essential Taylor spectra for a general class of truncated Toeplitz tuples. Truncated Toeplitz operators on the Hardy space $\mathrm{H}^2(\mathbb{T})$ were first systematically studied by D. Sarason [68] in 2007. These operators arise as compressions of ordinary Toeplitz operators $T_f \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}))$ to the orthogonal complement $K_\theta = \mathrm{H}^2(\mathbb{T}) \ominus \theta\, \mathrm{H}^2(\mathbb{T})$ of an invariant subspace $\theta\, \mathrm{H}^2(\mathbb{T}) \subset \mathrm{H}^2(\mathbb{T})$ of the unilateral shift $T_z \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}))$, given by an inner function $\theta \in \mathrm{H}^\infty(\mathbb{T})$. In 2015 R. V. Bessonov [10] proved the following spectral mapping theorem for truncated Toeplitz operators.

Theorem (Bessonov, 2015).
Let $\theta \in \mathrm{H}^\infty(\mathbb{T})$ be an inner function. For $f \in \mathrm{H}^\infty(\mathbb{T}) + \mathrm{C}(\mathbb{T})$, the formula

$$\sigma_{\mathrm{e}}(T_f^\theta) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \textit{there exists } \mathbb{D} \ni z_k \xrightarrow{k} z \in \mathbb{T} \textit{ such that} \\ \lim_{k\to\infty} F(z_k) = \lambda \textit{ and } \lim_{k\to\infty} \Theta(z_k) = 0 \end{array} \right\}$$

holds.

Here F and Θ are the Poisson transforms of the symbol $f \in \mathrm{L}^\infty(\mathbb{T})$ and the inner function $\theta \in \mathrm{H}^\infty(\mathbb{T})$, respectively. An application of Theorem 4.6 leads to a multivariable version of Bessonov's result for truncated Toeplitz systems with symbols in $\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D)$.

Let $D \subset \mathbb{C}^d$ be a strictly pseudoconvex domain with C^∞-boundary and let $\theta \in \mathrm{H}^\infty(\partial D)$ be an inner function. The Toeplitz operator $S = T_\theta^p \in \mathrm{L}(\mathrm{H}^p(\partial D))$ is an isometry with left inverse $R = T_{\overline{\theta}}^p$. Then $P_\theta = SR$ is a projection onto the closed subspace $\operatorname{Im} S \subset \mathrm{H}^p(\partial D)$. We identify the topological direct complement $\operatorname{Im}(1 - P_\theta) \subset \mathrm{H}^p(\partial D)$ of $\operatorname{Im} S$ with the quotient space $\mathrm{H}^p_\theta(\partial D) = \mathrm{H}^p(\partial D)/\theta\, \mathrm{H}^p(\partial D)$ via the topological isomorphism $\rho\colon \operatorname{Im} S \to \mathrm{H}^p_\theta(\partial D), x \longmapsto [x]$. For $f \in \mathrm{L}^\infty(\partial D)$, we call the bounded linear operator

$$T_f^{p,\theta} = \rho(1 - P_\theta) T_f^p \rho^{-1} \in \mathrm{L}(\mathrm{H}^p_\theta(\partial D))$$

the truncated Toeplitz operator with symbol f. According to Corollary 3.4, for $f = (f_1, \ldots, f_m) \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$ the tuples $T_f^p = (T_{f_1}^p, \ldots, T_{f_m}^p) \in \mathrm{L}(\mathrm{H}^p(\partial D))^m$ and $(T_f^p, T_\theta^p) = (T_{f_1}^p, \ldots, T_{f_m}^p, T_\theta^p) \in \mathrm{L}(\mathrm{H}^p(\partial D))^{m+1}$ essentially commute. It follows that also the truncated Toeplitz tuple $T_f^{p,\theta} = (T_{f_1}^{p,\theta}, \ldots, T_{f_m}^{p,\theta}) \in \mathrm{L}(\mathrm{H}^p_\theta(\partial D))^m$ essentially commutes. As an application of Theorem 4.6 and standard methods from homological algebra (see Lemma 4.10) we calculate the essential spectrum of the essentially commuting tuple $T_f^{p,\theta}$.

Theorem 4.13.
Let $\theta \in \mathrm{H}^\infty(\partial D)$ be an inner function and $f \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$. Then the formula

$$\sigma_{\mathrm{e}}(T_f^{p,\theta}) = \left\{ \lambda \in \mathbb{C}^m \;\middle|\; \begin{array}{c} \textit{there exists } D \ni z_k \xrightarrow{k} z \in \partial D \textit{ such that} \\ \lim_{k\to\infty} F(z_k) = \lambda \textit{ and } \lim_{k\to\infty} \Theta(z_k) = 0 \end{array} \right\}$$

holds.

The latter theorem can be slightly improved if one observes that the arguments used to prove Theorem 4.13 only require $\theta \in \mathrm{H}^\infty(\partial D)$ to be invertible in $\mathrm{L}^\infty(\partial D)$, since then T_θ^p is left invertible. This even slightly improves the one dimensional result of Bessonov stated above.

1 Preliminaries

The aim of the present chapter is to introduce the notation and to provide preliminary results and basic constructions that are used throughout this thesis. First of all, for normed vector spaces $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$, we denote by $\mathrm{L}(X,Y)$ the normed vector space of all bounded, linear operators $T\colon X \to Y$, where the norm on $\mathrm{L}(X,Y)$ is given by the usual operator norm that is induced by $\|\cdot\|_1$ and $\|\cdot\|_2$. In the case $(X, \|\cdot\|_1) = (Y, \|\cdot\|_2)$ we will write $\mathrm{L}(X)$ instead of $\mathrm{L}(X,X)$. In addition we use the notation X' for the topological dual space of X.

1.1 Multiplication Operators on L^p-Spaces

For an arbitrary set M and a bounded function $f\colon M \to \mathbb{C}$, we denote by $\|f\|_M = \sup_{z\in M} |f(z)|$ the supremum norm of f. Let $(X, \mathcal{A}, \mu)$ be a measure space. For $1 \le p \le \infty$ we write $\mathrm{L}^p(X) = \mathrm{L}^p(X, \mathcal{A}, \mu)$ for the Banach space of (equivalence classes[1]) of all p-integrable functions if $p < \infty$, and of all essentially bounded measurable functions if $p = \infty$, equipped with the norms

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \quad (p < \infty), \qquad \|f\|_\infty = \inf_{\mu(N)=0} \|f\|_{X\setminus N} \,.$$

For $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we use the sesquilinear form

$$\langle \cdot, \cdot \rangle : \; \mathrm{L}^p(X) \times \mathrm{L}^q(X) \longrightarrow \mathbb{C}, \qquad (f,g) \longmapsto \int_X f\overline{g} \, d\mu$$

to identify $\mathrm{L}^q(X)$ with the topological dual space of $\mathrm{L}^p(X)$.

[1]Two measurable functions $f\colon X \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$ are called equivalent if $f = g$ μ-a.e. on X

Proposition 1.1.
For $1 < p < \infty$, the mapping

$$\rho\colon \mathrm{L}^q(X) \longrightarrow \mathrm{L}^p(X)', \qquad h \longmapsto \langle \cdot, h\rangle$$

defines an antilinear isometric isomorphism.

Proof. This follows directly from the definition of the sesquilinear form $\langle \cdot, \cdot \rangle$ and standard $L^p - L^q$ duality (Theorem 4.5.1 in [17]). □

Using the above sesquilinear form we define the adjoint $T^* \in \mathrm{L}(\mathrm{L}^q(X))$ of an operator $T \in \mathrm{L}(\mathrm{L}^p(X))$. For $T \in \mathrm{L}(\mathrm{L}^p(X))$, we denote by $T' \in \mathrm{L}(\mathrm{L}^p(X)')$ the Banach space adjoint of T.

Lemma 1.2.
For $T \in \mathrm{L}(\mathrm{L}^p(X))$, there exists a unique continuous linear operator $T^* \in \mathrm{L}(\mathrm{L}^q(X))$ which makes the diagram

$$\begin{array}{ccc} \mathrm{L}^q(X) & \xrightarrow{\rho} & \mathrm{L}^p(X)' \\ \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T'} \\ \mathrm{L}^q(X) & \xrightarrow{\rho} & \mathrm{L}^p(X)' \end{array}$$

commutative, that is, such that

$$\langle f, T^* g\rangle = \langle Tf, g\rangle \tag{1.1}$$

for all $f \in \mathrm{L}^p(X)$, $g \in \mathrm{L}^q(X)$. The map

$$\mathrm{L}(\mathrm{L}^p(X)) \longrightarrow \mathrm{L}(\mathrm{L}^q(X)), \qquad T \longmapsto T^*$$

is isometric, antilinear and satisfies

$$(TS)^* = S^*T^* \qquad \text{and} \qquad (T^*)^* = T \qquad \text{for } T, S \in \mathrm{L}(\mathrm{L}^p(X)).$$

Proof. The map defined by

$$T^* = \rho^{-1} T' \rho \colon \mathrm{L}^q(X) \longrightarrow \mathrm{L}^q(X)$$

is continuous linear with $\|T^*\| = \|T\|$ and

$$\langle f, T^* g\rangle = \rho(T^* g)(f) = T'(\rho(g))(f) = \rho(g)(Tf) = \langle Tf, g\rangle$$

for all $f \in \mathrm{L}^p(X)$, $g \in \mathrm{L}^q(X)$. Obviously T^* is uniquely determined by (1.1) and the map $T \mapsto T^*$ has the stated properties. □

With respect to the sesquilinear form $\langle \cdot, \cdot \rangle : \mathrm{L}^p(X) \times \mathrm{L}^q(X) \to \mathbb{C}$ we define orthogonal complements of sets $M \subset \mathrm{L}^p(X)$, $N \subset \mathrm{L}^q(X)$ in the usual way

$$M^\perp = \{h \in \mathrm{L}^q(X) \mid \langle g, h\rangle = 0 \text{ for all } g \in M\},$$
$${}^\perp N = \{g \in \mathrm{L}^p(X) \mid \langle g, h\rangle = 0 \text{ for all } h \in N\}.$$

Proposition 1.3.
For a closed subspace $M \subset \mathrm{L}^p(X)$, the mapping

$$\mathrm{L}^q(X)/M^\perp \longrightarrow M', \qquad [h] \longmapsto \rho(h)|_M$$

defines an antilinear isometric isomorphism of Banach spaces and for $T \in \mathrm{L}(\mathrm{L}^p(X))$, the identity

$$(\operatorname{Im} T)^\perp = \ker T^*$$

holds.

Proof. Obviously $\rho(M^\perp) = \{u \in \mathrm{L}^p(X)' \mid u|_M = 0\}$. By basic Banach space duality the composition

$$\mathrm{L}^q(X)/M^\perp \xrightarrow{\hat{\rho}} \mathrm{L}^p(X)'/\{u \in \mathrm{L}^p(X)' \mid u|_M = 0\} \xrightarrow{u \mapsto u|_M} M'$$

is an antilinear isometric isomorphism. Since $\langle Tg, f\rangle = \langle g, T^* f\rangle$ for all $g \in \mathrm{L}^p(X)$, $f \in \mathrm{L}^q(X)$, we have $f \in \ker T^*$ if and only if $f \in (\operatorname{Im} T)^\perp$. □

For $f \in \mathrm{L}^\infty(X)$, the multiplication operator with symbol f

$$M_f^p\colon\ \mathrm{L}^p(X) \longrightarrow \mathrm{L}^p(X), \qquad g \longmapsto fg$$

defines a bounded linear operator $M_f^p \in \mathrm{L}(\mathrm{L}^p(X))$ with $\left\|M_f^p\right\| = \|f\|_\infty$. In the Hilbert space case $p = 2$ we will always suppress the exponent and write $M_f = M_f^2$ for $f \in \mathrm{L}^\infty(X)$.

It is well known (see e.g. [44, Example 5.1.6] or [40, Problem 65]) that the von Neumann algebra

$$\left\{M_f\colon\ \mathrm{L}^2(X) \to \mathrm{L}^2(X) \mid f \in \mathrm{L}^\infty(X)\right\} \subseteq \mathrm{L}(\mathrm{L}^2(X))$$

consisting of all multiplication operators on $\mathrm{L}^2(X)$ with essentially bounded symbol is maximal abelian. The same arguments show that also in the case $p \neq 2$ the subalgebra

$$\mathcal{M}(\mathrm{L}^p(X)) = \left\{M_f^p\colon\ \mathrm{L}^p(X) \to \mathrm{L}^p(X) \mid f \in \mathrm{L}^\infty(X)\right\} \subset \mathrm{L}(\mathrm{L}^p(X)).$$

coincides with its commutant in $\mathrm{L}(\mathrm{L}^p(X))$.

Proposition 1.4.
Let $1 \leq p < \infty$. If $T \in \mathrm{L}(\mathrm{L}^p(X))$ commutes with all multiplication operators $M \in \mathcal{M}(\mathrm{L}^p(X))$, then $T \in \mathcal{M}(\mathrm{L}^p(X))$.

Proof. From the equation

$$M_f^p T = T M_f^p \qquad (f \in \mathrm{L}^\infty(X))$$

we get

$$fT(1) = M_f^p T(1) = T M_f^p(1) = Tf \tag{1.2}$$

for all $f \in \mathrm{L}^\infty(X)$. We show that the function $g = T(1) \in \mathrm{L}^p(X)$ is essentially bounded with $|g| \leq \|T\|$ μ-almost everywhere. Let $\varepsilon > 0$ and assume that there exists a set $M \in \mathcal{B}(X)$ with $\mu(M) > 0$ and $|g(x)| > \|T\| + \varepsilon$ for all $x \in M$. Then $\chi_M \in \mathrm{L}^\infty(X) \subset \mathrm{L}^p(X)$

and

$$\|T\chi_M\|_p^p = \int_X |\chi_M g|^p \, d\mu = \int_M |g|^p \, d\mu \geq (\|T\| + \varepsilon)^p \, \mu(M) > \|T\|^p \, \|\chi_M\|_p^p .$$

This contradiction implies that $|g| \leq \|T\| + \varepsilon$ μ-almost everywhere for every $\varepsilon > 0$. Hence $|g| \leq \|T\|$ μ-almost everywhere. By (1.2) we know that $M_g^p = T$ on the dense subspace $\mathrm{L}^\infty(X) \subset \mathrm{L}^p(X)$. As both M_g^p and T are bounded, the assertion follows. □

We finish this section with the observation that, under the additional assumption that the measure μ possesses no atoms[2], there exist no non-trivial compact multiplication operators on $\mathrm{L}^p(X)$.

Proposition 1.5.
Assume that μ *has no atoms and let* $f \in \mathrm{L}^\infty(X)$. *Then the multiplication operator* $M_f^p \in \mathcal{M}(\mathrm{L}^p(X))$ *is compact if and only if* $f = 0$.

Proof. Assume that $f\colon X \to \mathbb{C}$ is a bounded measurable function such that $M_f \in \mathrm{L}(\mathrm{L}^p(X)) \setminus \{0\}$ is compact. Choose $\varepsilon > 0$ such that the set

$$S = \{x \in X \mid |f(x)| > \varepsilon\} \in \mathcal{B}(X)$$

has positive μ-measure. Then

$$U = \{g \in \mathrm{L}^p(X) \mid g = 0 \ \mu - \text{a.e. on } X \setminus S\} \subset \mathrm{L}^p(X)$$

is an infinite-dimensional, closed subspace of $\mathrm{L}^p(X)$ that is invariant under M_f^p. Indeed, since μ has no atoms, there is a decreasing sequence $(S_i)_{i\geq 0}$ of sets in $\mathcal{A}$ such that $S_0 = S$ and $0 < \mu(S_{i+1}) < \mu(S_i)$ for all $i \geq 0$. Then an elementary argument shows that the family $(\chi_{S_i})_{i\geq 0}$ is linearly independent in $U \subset \mathrm{L}^p(X)$. With U defined as above the restriction $M_f^p|_U \in \mathrm{L}(U)$ is a compact operator. Furthermore it has the bounded left

[2] A set $A \in \mathcal{B}(X)$ is called an atom if $\mu(A) > 0$ and for every subset $B \in \mathcal{B}(X)$ of A we have $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

inverse $M_h^p|_U \in \mathrm{L}(U)$, where $h \in \mathrm{L}^\infty(X)$ is given by

$$h(x) = \begin{cases} \frac{1}{f(x)}, & x \in S, \\ 0, & x \in X \setminus S, \end{cases}$$

which is a contradiction. □

We finish this section with the remark that, in the case that the space $\mathrm{L}^p(X)$ is separable, it possesses a Schauder basis[3] (see [12, p. 296]). The last fact will be crucial in the proof of Theorem 2.18.

[3] see Appendix A of this thesis.

1.2 Taylor's joint Spectrum

In this section we recall the notion of the joint spectrum for a commuting tuple $T = (T_1, \ldots, T_n) \in \mathrm{L}(X)^n$ of bounded linear operators $T_i \colon X \to X$ $(i = 1, \ldots, n)$ on a complex Banach space X due to J. L. Taylor [72].

For each $k = 0, \ldots, n$, we set $\Lambda_n^k X = X \otimes \Lambda^k \mathbb{C}^n$, where $\Lambda^k \mathbb{C}^n$ denotes the k-th exterior power with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. Since the vector space $\Lambda_n^k X$ is isomorphic to the direct sum $X^{\binom{n}{k}}$, it becomes a Banach space in a canonical way.

Since T is commuting, the sequence

$$0 \longrightarrow \Lambda_n^0 X \xrightarrow{\delta_T^0} \Lambda_n^1 X \xrightarrow{\delta_T^1} \ldots \xrightarrow{\delta_T^{n-2}} \Lambda_n^{n-1} X \xrightarrow{\delta_T^{n-1}} \Lambda_n^n X \longrightarrow 0$$

with differentials $\delta_T^p \colon \Lambda_n^p X \to \Lambda_n^{p+1} X$ acting as

$$\sum_{1 \le i_1 < \ldots < i_p \le n} x_{i_1 \ldots i_p} e_{i_1} \wedge \ldots \wedge e_{i_p} \longmapsto \sum_{1 \le i_1 < \ldots < i_{p+1} \le n} \sum_{\nu=1}^{p+1} (-1)^{\nu-1} T_{i_\nu} x_{i_1 \ldots \hat{i}_\nu \ldots i_{p+1}} e_{i_1} \wedge \ldots \wedge e_{i_{p+1}}$$

is a well-defined complex of Banach spaces called the Koszul complex $\mathrm{K}^\bullet(T, X)$ of T. We shall write the elements in the spaces $\Lambda_n^k X$ as $(x_I)_{|I|=k}$ where $x_I \in X$ and $I = (i_1, \ldots, i_k) \in \mathbb{N}^k$ ranges over all tuples of non-negative integers with $1 \le i_1 < i_2 < \ldots < i_k \le n$. The quotient vector spaces

$$\mathrm{H}^p(T, X) = \ker \delta_T^p \,/\, \mathrm{Im}\, \delta_T^{p-1} \qquad (p = 0, \ldots, n)$$

are called the cohomology groups of the Koszul complex of T. Here by definition $\delta_T^{-1} = 0 = \delta_T^n$.

We write $z - T = (z_1 - T_1, \ldots, z_n - T_n) \in \mathrm{L}(X)^n$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $T = (T_1, \ldots, T_n) \in \mathrm{L}(X)^n$.

Definition 1.6.
Let $T \in \mathrm{L}(X)^n$ be a commuting tuple of bounded linear operators on a Banach space X.

(a) The set

$$\sigma(T) = \{z \in \mathbb{C}^n \mid \mathrm{K}^\bullet(z - T, X) \text{ is not exact}\}$$

is called the ***(Taylor) spectrum*** *of T.*

(b) The set

$$\sigma_{\mathrm{e}}(T) = \{z \in \mathbb{C}^n \mid \dim \mathrm{H}^p(z - T, X) = \infty \textit{ for at least one } p = 0, \dots, n\}$$

is called the ***essential (Taylor) spectrum*** *of T.*

The study of essential Taylor spectra $\sigma_{\mathrm{e}}(T)$ can be reduced to the study of the Taylor spectra $\sigma(\tilde{T})$ of suitably defined commuting tuples $\tilde{T} \in \mathrm{L}(\tilde{X})^n$. We recall one such construction. For a Banach space X, denote by X^∞ the Banach space of all bounded sequences in X equipped with the supremum norm and write X^{pc} for the closed subspace of X^∞ consisting of all precompact sequences in X, i.e., all sequences $(x_k)_{k\in\mathbb{N}}$ in X with the property that each subsequence of $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence. Then the quotient $X^e = X^\infty / X^{pc}$ equipped with the quotient norm is a Banach space. Let Y be another Banach space. For every $T \in \mathrm{L}(X, Y)$ the mapping

$$T^e \colon X^e \longrightarrow Y^e, \qquad [(x_k)_{k\in\mathbb{N}}] \longmapsto [(Tx_k)_{k\in\mathbb{N}}]$$

defines an operator $T^e \in \mathrm{L}(X^e, Y^e)$. For $T = (T_1, \dots, T_n) \in \mathrm{L}(X)^n$ write $T^e = ({T_1}^e, \dots, {T_n}^e)$ and observe that $T^e \in \mathrm{L}(X^e)^n$ is a commuting tuple whenever T is. An iterated application of [36, Lemma 2.6.5] yields the following characterization of the essential Taylor spectrum.

Lemma 1.7.
Let $T \in \mathrm{L}(X)^n$ be a commuting tuple on the Banach space X. Then $\sigma_{\mathrm{e}}(T) = \sigma(T^e)$.

For an arbitrary tuple $T = (T_1, \dots, T_n) \in \mathrm{L}(X)^n$, the induced tuple $T^e \in \mathrm{L}(X^e)^n$ is commuting if and only if all commutators $[T_i, T_j] = T_iT_j - T_jT_i$ $(i, j = 1, \dots n)$ are compact. In this case we define the essential Taylor spectrum of T by $\sigma_{\mathrm{e}}(T) = \sigma(T^e)$.

1.3 Hardy-type Spaces

Fix a positive integer $d \geq 1$. We write

$$\langle z, w\rangle = \sum_{k=1}^{d} z_k \overline{w}_k, \qquad (z = (z_k)_{k=1}^d, w = (w_k)_{k=1}^d \in \mathbb{C}^d)$$

for the scalar product on $\mathbb{C}^d$ and denote by

$$|\cdot| : \mathbb{C}^d \longrightarrow [0, \infty), \qquad z \longmapsto \sqrt{\langle z, z\rangle},$$

the Euclidean norm on $\mathbb{C}^d$. We write

$$\mathbb{B} = \mathbb{B}_d = \left\{ z \in \mathbb{C}^d \;\middle|\; |z| < 1 \right\} \qquad \text{and} \qquad \mathbb{D}^d = \left\{ z \in \mathbb{C}^d \;\middle|\; \max_{i=1,\ldots,d} |z_i| < 1 \right\}$$

for the unit ball and unit polydisc in $\mathbb{C}^d$.

Let $K \subset \mathbb{C}^d$ be a compact set and $A \subset \mathrm{C}(K)$ a closed subalgebra that contains the restrictions $\mathbb{C}[z]\,|_K$ of all holomorphic polynomials in d complex variables $z = (z_1, \ldots, z_d)$. We denote by $\partial_A \subset K$ the Shilov boundary of A, that is, the smallest closed subset of K such that $\|f\|_K = \|f\|_{\partial_A}$ for all $f \in A$. Now let $\mu \in \mathrm{M}^+(\partial_A)$ be a positive, regular Borel measure[4] such that (A, ∂_A, μ) is a regular triple in the sense of A. B. Aleksandrov [3]. Let $\mathrm{L}^p(\partial_A)$ be the L^p-space formed with respect to the measure μ. We define $\mathrm{H}^p(\partial_A) = \overline{A}^{L^p} \subset \mathrm{L}^p(\partial_A)$ as the L^p- closure of the algebra $A \subset \mathrm{L}^p(\partial_A)$ and make the additional assumption that there is a family $(P_p)_{1<p<\infty}$ of projections $P_p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ with $\operatorname{Im} P_p = \mathrm{H}^p(\partial_A)$ and

$$i_{sr} P_s = P_r i_{sr}$$

for all $1 < r < s < \infty$, where the mappings $i_{sr} \colon \mathrm{L}^s(\partial_A) \to \mathrm{L}^r(\partial_A)$ are the canonical inclusion mappings. Furthermore, we suppose that $P = P_2 \in \mathrm{L}(\mathrm{L}^2(\partial_A))$ is the orthogonal projection onto $\mathrm{H}^2(\partial_A)$. We define $\mathrm{H}^\infty(\partial_A) = \overline{A}^{\tau_{w^*}} \subset \mathrm{L}^\infty(\partial A)$ as the weak* closure of A in $\mathrm{L}^\infty(\partial A) = \mathrm{L}^1(\partial_A)'$. We call $\mathrm{H}^p(\partial_A)$ a **Hardy-type space induced by** A. The regularity assumption of the triple (A, ∂_A, μ) is a sufficient condition ensuring $\mathrm{H}^\infty(\partial_A)$ to have a rich supply of inner functions. We shall use this fact in later sections.

[4] Observe that, for measures of this kind, the corresponding L^p-spaces are separable (use, for example, Proposition 3.4.5 in [17]) and hence, as indicated at the end of Section 1.1, possess Schauder bases.

Lemma 1.8.
For $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$P_p^* = P_q.$$

Proof. By Stone-Weierstraß and standard measure theory (see Proposition 7.4.3 in [17]), the subspace $\mathbb{C}[z, \overline{z}]\,|_{\partial_A} \subset \mathrm{L}^r(\partial_A)$ is dense for all $1 \leq r < \infty$. Let us denote by $\gamma_r \colon \mathbb{C}[z, \overline{z}] \to \mathrm{L}^r(\partial_A)$ the inclusion mappings. Note that, for $p > 2$ and $f \in \mathrm{L}^p(\partial_A)$, $g \in \mathrm{L}^2(\partial_A)$,

$$\langle i_{p2} f, g \rangle_{\mathrm{L}^2(\partial_A)} = \int_{\partial_A} f \overline{g} \, d\mu = \langle f, i_{2q} g \rangle .$$

Hence, for $f, g \in \mathbb{C}[z, \overline{z}]\,|_{\partial_A}$, we obtain

$$\begin{aligned}
\langle P_p \gamma_p(f), \gamma_q(g) \rangle &= \langle P_p \gamma_p(f), i_{2q} \gamma_2(g) \rangle = \langle i_{p2} P_p \gamma_p(f), \gamma_2(g) \rangle_{\mathrm{L}^2(\partial_A)} \\
&= \langle P i_{p2} \gamma_p(f), \gamma_2(g) \rangle_{\mathrm{L}^2(\partial_A)} = \langle i_{p2} \gamma_p(f), P \gamma_2(g) \rangle_{\mathrm{L}^2(\partial_A)} \\
&= \langle \gamma_p(f), i_{2q} P \gamma_2(g) \rangle = \langle \gamma_p(f), P_q i_{2q} \gamma_2(g) \rangle \\
&= \langle \gamma_p(f), P_q \gamma_q(g) \rangle .
\end{aligned}$$

Since the sesquilinear form $\langle \cdot, \cdot \rangle : \mathrm{L}^p(\partial_A) \times \mathrm{L}^q(\partial_A) \to \mathbb{C}$ is continuous, it follows that $P_p^* = P_q$ in this case. To prove the assertion in the case $p < 2$, note that by arguing as above with the roles of p and q exchanged, one obtains

$$\overline{\langle \gamma_p(f), P_q \gamma_q(g) \rangle} = \langle P_q \gamma_q(g), \gamma_p(f) \rangle = \langle \gamma_q(g), P_p \gamma_p(f) \rangle = \overline{\langle P_p \gamma_p(f), \gamma_q(g) \rangle}$$

for all $f, g \in \mathbb{C}[z, \overline{z}]\,|_{\partial_A}$. □

The compatibility condition

$$i_{sr} P_s = P_r i_{sr} \qquad (1 < r < s < \infty)$$

is only needed in Section 2.3. Up to Section 2.2 it suffices to know that there is a family of projections $P_r \in \mathrm{L}(\mathrm{L}^r(\partial_A))$ $(1 < r < \infty)$ onto the subspaces $\mathrm{H}^r(\partial_A) \subset \mathrm{L}^r(\partial_A)$ satisfying the duality relation

$$P_p^* = P_q \qquad \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$$

from Lemma 1.8.

Proposition 1.9.
Let $1 < p < q < \infty$. Then the inclusions $\mathrm{H}^\infty(\partial_A) \subset \mathrm{H}^q(\partial_A) \subset \mathrm{H}^p(\partial_A)$ hold. Furthermore $\mathrm{H}^\infty(\partial_A) \subset \mathrm{L}^\infty(\partial_A)$ is a subalgebra with the property that, for every $\theta \in \mathrm{H}^\infty(\partial_A)$, the inclusion

$$\theta\, \mathrm{H}^p(\partial_A) = \{\theta \cdot f \mid f \in \mathrm{H}^p(\partial_A)\} \subset \mathrm{H}^p(\partial_A)$$

holds.

Proof. The inclusion $\mathrm{H}^q(\partial_A) \subset \mathrm{H}^p(\partial_A)$ directly follows from the definition of $\mathrm{H}^q(\partial_A)$ and $\mathrm{H}^p(\partial_A)$, respectively. Obviously $\mathrm{H}^\infty(\partial_A) \subset \mathrm{L}^\infty(\partial_A)$ is a subalgebra. Let $\theta \in \mathrm{H}^\infty(\partial_A)$ and $(\theta_\alpha)_\alpha$ be a net in A that converges to θ in the weak* topology of $\mathrm{L}^\infty(\partial_A) = \mathrm{L}^1(\partial_A)'$. Since $\mathrm{L}^q(\partial_A) \subset \mathrm{L}^1(\partial_A)$ the net $(\theta_\alpha)_\alpha$ converges to θ in the weak* topology of $\mathrm{L}^p(\partial_A) = \mathrm{L}^q(\partial_A)'$. As a norm closed convex subset of the reflexive Banach space $\mathrm{L}^p(\partial_A)$ the subspace $\mathrm{H}^p(\partial_A)$ is also closed with respect to the weak* topology. Hence $\theta \in \mathrm{H}^p(\partial_A)$. Since $\theta f \in \mathrm{H}^\infty(\partial_A) \subset \mathrm{H}^p(\partial_A)$ for every $f \in A$ another straight forward approximation argument shows that $\theta\, \mathrm{H}^p(\partial_A) \subset \mathrm{H}^p(\partial_A)$. □

The $L^p - L^q$-duality can be used to calculate the dual spaces of the Hardy spaces $\mathrm{H}^p(\partial_A)$ for $1 < p < \infty$.

Proposition 1.10.
Let $p, q \in (1, \infty)$ be real numbers. The mapping

$$\rho_H\colon\ \mathrm{H}^q(\partial_A) \longrightarrow \mathrm{H}^p(\partial_A)', \qquad h \longrightarrow \langle \cdot, h\rangle_{\mathrm{H}^p,\mathrm{H}^q}\,,$$

where $\langle \cdot, \cdot\rangle_{\mathrm{H}^p,\mathrm{H}^q} :\ \mathrm{H}^p(\partial_A) \times \mathrm{H}^q(\partial_A) \to \mathbb{C}$ is the sesquilinear form defined by

$$\langle g, h\rangle_{\mathrm{H}^p,\mathrm{H}^q} = \int_{\partial_A} g\overline{h}\, d\mu,$$

is an antilinear topological isomorphism.

Proof. By the bounded inverse theorem, we know that

$$\Phi\colon\ \mathrm{L}^q(\partial_A)/\ker P_q \longrightarrow \operatorname{Im} P_q = \mathrm{H}^q(\partial_A), \qquad [h] \longmapsto P_q h$$

is a topological isomorphism. By Proposition 1.3 we have

$$\ker P_q = (\operatorname{Im} P_p)^{\perp} = \mathrm{H}^p(\partial_A)^{\perp}.$$

Using the antilinear isometric isomorphism from Proposition 1.3 with $M = \mathrm{H}^p(\partial_A)$, we see that the composition

$$\sigma\colon\ \mathrm{H}^q(\partial_A) \xrightarrow{\Phi^{-1}} \mathrm{L}^q(\partial_A)/\,\mathrm{H}^p(\partial_A)^{\perp} \longrightarrow \mathrm{H}^p(\partial_A)'$$

is an antilinear topological isomorphism. Checking the definitions we find that

$$\sigma(P_q h) = \rho(h)|_{\mathrm{H}^p(\partial_A)} = \langle\,\cdot\,, h\rangle\,|_{\mathrm{H}^p(\partial_A)}$$

for $h \in \mathrm{L}^q(\partial_A)$. Since $h - P_q h \in \ker P_q = \mathrm{H}^p(\partial_A)^{\perp}$, we have that $\sigma(P_q h) = \langle\,\cdot\,, P_q h\rangle\,|_{\mathrm{H}^p(\partial_A)}$ for every $h \in \mathrm{L}^q(\partial_A)$. Thus $\rho_H = \sigma$ is an antilinear topological isomorphism. □

As in Lemma 1.2 we can use the identification $\rho_H\colon \mathrm{H}^q(\partial_A) \to \mathrm{H}^p(\partial_A)'$ to interprete the Banach space adjoint of an operator $T \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ as an operator $T^* \in \mathrm{L}(\mathrm{H}^q(\partial_A))$.

Lemma 1.11.
For $T \in \mathrm{L}(\mathrm{H}^p(\partial_A))$, there is a unique continuous linear operator $T^* \in \mathrm{L}(\mathrm{H}^q(\partial_A))$ for which the diagram

$$\begin{array}{ccc} \mathrm{H}^q(\partial_A) & \xrightarrow{\rho_H} & \mathrm{H}^p(\partial_A)' \\ \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T'} \\ \mathrm{H}^q(\partial_A) & \xrightarrow{\rho_H} & \mathrm{H}^p(\partial_A)' \end{array}$$

is commutative, or equivalently, which satisfies the identity

$$\langle f, T^* g\rangle_{\mathrm{H}^p,\mathrm{H}^q} = \langle Tf, g\rangle_{\mathrm{H}^p,\mathrm{H}^q}$$

for all $f \in \mathrm{H}^p(\partial_A)$, $g \in \mathrm{H}^q(\partial_A)$. The mapping $T \mapsto T^*$ is antilinear and satisfies

(i) $(TS)^* = S^*T^*$, $(T^*)^* = T$

(ii) $\frac{1}{\|P_p\|}\|T\| \leq \|T^*\| \leq \|P_p\|\,\|T\|$

for all $S, T \in \mathrm{L}(\mathrm{H}^p(\partial_A))$.

Proof. For $T \in \mathrm{L}(\mathrm{H}^p(\partial_A))$, define $T^* = \rho_H^{-1} T' \rho_H$. We only indicate why the estimates claimed in (ii) hold for T^*. The remaining properties are easily checked. For $T \in \mathrm{L}(\mathrm{H}^p(\partial_A))$, the adjoint of the operator $TP_p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$, formed in the sense of Lemma 1.2, satisfies

$$(1 - P_q)(TP_p)^* = (1 - P_q)((TP_p)P_p)^* = (1 - P_q)P_q(TP_p)^* = 0.$$

Thus $(TP_p)^*\,\mathrm{H}^q(\partial_A) \subset \mathrm{H}^q(\partial_A)$. Since

$$\langle f, (TP_p)^* g\rangle_{\mathrm{H}^p,\mathrm{H}^q} = \langle f, (TP_p)^* g\rangle = \langle (TP_p)f, g\rangle = \langle (TP_p)f, g\rangle_{\mathrm{H}^p,\mathrm{H}^q}$$

for all $f \in \mathrm{H}^p(\partial_A), g \in \mathrm{H}^q(\partial_A)$, it follows that $T^* = (TP_p)^*|_{\mathrm{H}^q(\partial_A)}$. But then

$$\|T^*\| \leq \|(TP_p)^*\| = \|TP_p\| \leq \|P_p\|\,\|T\|$$

and the remaining estimate follows by reversing the roles of T and T^*.

□

We briefly discuss two particular examples. Let $K = \overline{\mathbb{B}}$ be the closed unit ball in $\mathbb{C}^d$ and $A = A(\mathbb{B}) \subset \mathrm{C}(K)$ the ball algebra, i.e., the closed subalgebra consisting of all functions $f \in \mathrm{C}(K)$ that are holomorphic on $\mathbb{B}$. An elementary exercise using the maximum principle shows that the Shilov boundary of $A(K)$ is the topological boundary $S = \partial\mathbb{B}$ of $\mathbb{B}$. Let σ be the surface measure on S. By results of A. B. Aleksandrov [3] the triple $(A(K), S, \sigma)$ is regular.

The Hardy space $\mathrm{H}^p(\mathbb{B})$ $(1 \leq p < \infty)$ on the unit ball of $\mathbb{C}^d$ consists of all holomorphic functions $f\colon \mathbb{B} \to \mathbb{C}$ such that

$$\sup_{0<r<1} \int_S |f_r|^p \, d\sigma < \infty,$$

where $f_r\colon S \to \mathbb{C}$ is given by $f_r(\zeta) = f(r\zeta)$. The norm

$$\|\cdot\|_{\mathrm{H}^p} :\ \mathrm{H}^p(\mathbb{B}) \longrightarrow [0,\infty), \qquad f \longmapsto \sup_{0<r<1} \|f_r\|_p$$

turns $\mathrm{H}^p(\mathbb{B})$ into a Banach space.

Let $\mathrm{H}^p(S)$ be the closure of $\mathrm{A}(\mathbb{B})|_S \subset \mathrm{L}^p(S)$ in the p-norm. A classical result from the theory of Hardy spaces (see Theorem 5.6.8 in [64]) states that the map

$$(\mathrm{H}^p(\mathbb{B}), \|\cdot\|_{\mathrm{H}^p}) \longrightarrow (\mathrm{H}^p(S), \|\cdot\|_p), \qquad f \longmapsto f^*$$

associating with each function $f \in \mathrm{H}^p(\mathbb{B})$ its Koranyi limit f^* (see Section 5.4 in [64]), is an isometric isomorphism of Banach spaces. Let

$$C\colon \mathbb{B} \times S \longrightarrow \mathbb{C}, \qquad (z,\zeta) \longmapsto \frac{1}{(1-\langle z,\zeta\rangle)^d}$$

be the Cauchy-Szegő kernel of the unit ball, and for $f \in \mathrm{L}^p(S)$, denote by $C[f]$ the Cauchy transform of f, that is, the analytic function on $\mathbb{B}$ defined by

$$C[f](z) = \int_S C(z,\xi) f(\xi)\, d\sigma(\xi) \qquad (z \in \mathbb{B}).$$

For $1 < p < \infty$, Corollary 6.3.1 in [64] implies that $C[f] \in \mathrm{H}^p(\mathbb{B})$ and that the map

$$P_p\colon\ \mathrm{L}^p(S) \longrightarrow \mathrm{L}^p(S), \qquad f \longmapsto C[f]^*,$$

defines a bounded linear projection from $\mathrm{L}^p(S)$ onto the closed subspace $\mathrm{H}^p(S) \subset \mathrm{L}^p(S)$. The map $P = P_2\colon\ \mathrm{L}^2(S) \to \mathrm{L}^2(S)$ is the orthogonal projection of $\mathrm{L}^2(S)$ onto $\mathrm{H}^2(S)$ (Theorem 5.6.9 in [64]).

Obviously, the family $(P_p)_{1<p<\infty}$ satisfies the compatibility condition

$$i_{sr} P_s = P_r i_{sr} \qquad (1 < r < s < \infty)$$

demanded in the section leading to Lemma 1.8. In particular, we have

$$P_p^* = P_q$$

for $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The unit ball $\mathbb{B}$ serves as a prototype for the class of strictly pseudoconvex domains of $\mathbb{C}^d$. We make a few remarks concerning those domains. Let $D \subset \mathbb{C}^d$ be a strictly pseudoconvex domain with C^2-boundary and let σ be the normalized surface measure on ∂D. A well known result on the existence of peaking functions ([33, Korollar 11.24]) shows that $\partial D = \partial_A$ is the Shilov boundary of the domain algebra

$$A = \mathrm{A}\,(D) = \left\{ f \in \mathrm{C}(\overline{D}) \;\middle|\; f|_D \in \mathcal{O}(D) \right\}$$

of D. By results of A. B. Aleksandrov [4, Theorem 3] and E. Løw [50, Theorem 3] the triple $(A, \partial D, \mu)$ is regular for each positive regular Borel measure μ on ∂D (for details, see the proof of Corollary 2.1.3 in [23]). Let $\omega\colon \partial D \to (0, \infty)$ be a positive continuous function and let $\omega d\sigma$ be the regular Borel measure with Radon-Nikodym density ω on ∂D. For $1 < p < \infty$, define the associated Hardy space by

$$\mathrm{H}^p(\partial D, \omega d\sigma) = \overline{A}^{\mathrm{L}^p(\partial D, \omega d\sigma)}.$$

The Cauchy-Szegő projection $P\colon \mathrm{L}^2(\partial D, \omega d\sigma) \to \mathrm{L}^2(\partial D, \omega d\sigma)$ is defined as the orthogonal projection of the Hilbert space $\mathrm{L}^2(\partial D, \omega d\sigma)$ onto the closed subspace $\mathrm{H}^2(\partial D, \omega d\sigma)$. A recent result of L. Lanzani and E. M. Stein ([47], [48]) shows that, for $1 < p < \infty$, there is a (unique) bounded linear operator $P_p\colon \mathrm{L}^p(\partial D, \omega d\sigma) \to \mathrm{L}^p(\partial D, \omega d\sigma)$ that extends the Cauchy-Szegő projection in the sense that

$$i_{p2} P_p = P i_{p2} \text{ for } p > 2 \quad \text{and} \quad P_p i_{2p} = i_{2p} P \text{ for } p < 2$$

(Theorem 16 in [48]). Furthermore, the operators $P_p \in \mathrm{L}(\mathrm{L}^p(\partial D, \omega d\sigma))$ $(1 < p < \infty)$, define continuous linear projections onto the closed subspaces $\mathrm{H}^p(\partial D, \omega d\sigma) \subset \mathrm{L}^p(\partial D, \omega d\sigma)$ (Proposition 13 in [47]). From these two intertwining relations one can easily deduce the compatibility condition

$$i_{sr} P_s = P_r i_{sr} \qquad \text{for } 1 < r < s < \infty.$$

In particular, we obtain again the duality relation $P_p^* = P_q$ for $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\sigma_d = \bigotimes_{i=1}^{d} \sigma$ be the d-fold product measure of the normalized Lebesgue measure σ on the unit circle $\mathbb{T} \subset \mathbb{C}$. The Hardy spaces $\mathrm{H}^p(\mathbb{D}^d)$ and $\mathrm{H}^p(\mathbb{T}^d)$ $(1 < p < \infty)$ are defined in an analogous way as the spaces $\mathrm{H}^p(\mathbb{B})$ and $\mathrm{H}^p(S)$, respectively. It is also known (see Chapters 2 and 3 in [63]) that $\mathrm{H}^p(\mathbb{T}^d)$ consists precisely of the non-tangential boundary values $f^* \in \mathrm{L}^p(\mathbb{T}^d)$ of the functions $f \in \mathrm{H}^p(\mathbb{D}^d)$ and that the corresponding boundary mapping $*\colon\ \mathrm{H}^p(\mathbb{D}^d) \to \mathrm{H}^p(\mathbb{T}^d)$ is an isometric isomorphism of Banach spaces. We now briefly outline how the family $(P_p)_{1<p<\infty}$ of projections is constructed.

For given functions $f_1, \ldots, f_d \colon \mathbb{T} \to \mathbb{C}$, we denote by $f_1 \otimes \ldots \otimes f_d \colon \mathbb{T}^d \to \mathbb{C}$ the function defined by

$$(f_1 \otimes \ldots \otimes f_d)(z_1, \ldots, z_d) = f_1(z_1) \cdot \ldots \cdot f_d(z_d).$$

Clearly, if $[f_1], \ldots, [f_d] \in \mathrm{L}^p(\mathbb{T}^d)$, then the equivalence class of $f_1 \otimes \ldots \otimes f_d$ in $\mathrm{L}^p(\mathbb{T}^d)$ only depends on $[f_1], \ldots, [f_d]$. Hence we can define $[f_1] \otimes \ldots \otimes [f_d] = [f_1 \otimes \ldots \otimes f_d]$.

For $d = 1$ and $1 < p < \infty$, the Cauchy-Szegő projection

$$P\colon\ \mathrm{L}^p(\mathbb{T}) \longrightarrow \mathrm{L}^p(\mathbb{T}), \qquad f \longmapsto C[f]^*$$

is a well defined continuous linear projection from $\mathrm{L}^p(\mathbb{T})$ onto $\mathrm{H}^p(\mathbb{T})$. An inductive application of Theorem 7.9 in [22] shows that there is a unique continuous linear map $P_p\colon\ \mathrm{L}^p(\mathbb{T}^d) \to \mathrm{L}^p(\mathbb{T}^d)$ with

$$P_p(f_1 \otimes \ldots \otimes f_d) = P(f_1) \otimes \ldots \otimes P(f_d) \qquad (f_1, \ldots, f_d \in \mathrm{L}^p(\mathbb{T})).$$

Let

$$C\colon \mathbb{D}^d \times \overline{\mathbb{D}}^d \longrightarrow \mathbb{C}, \qquad (z, \xi) \longmapsto \prod_{\nu=1}^{d} (1 - z_\nu \overline{\xi}_\nu)^{-1}$$

be the Cauchy kernel on the unit polydisc. For $f \in \mathrm{L}^1(\mathbb{T}^d)$, we write

$$C[f]\colon \mathbb{D}^d \longrightarrow \mathbb{C}, \qquad z \longmapsto \int_{\mathbb{T}^d} f(\xi) C(z, \xi)\, d\sigma_d(\xi)$$

for the Cauchy transform of f. Since, for $z \in \mathbb{D}^d$, the series expansion

$$C(z, \xi) = \sum_{\alpha \in \mathbb{N}^d} z^\alpha \overline{\xi}^\alpha$$

converges uniformly for $\xi \in \mathbb{T}^d$, we obtain for $j, k \in \mathbb{N}^d$ and $z \in \mathbb{D}^d$,

$$C[\xi^j \overline{\xi}^k](z) = \sum_{\alpha \in \mathbb{N}^d} \left(\int_{\mathbb{T}^d} \xi^j \overline{\xi}^{k+\alpha} \, d\sigma_d(\xi) \right) z^\alpha = z^{j-k}.$$

Here by definition $z^{j-k} = 0$ if $j-k \notin \mathbb{N}^d$. Hence the Cauchy transform of each polynomial $f \in \mathbb{C}\big[\xi, \overline{\xi}\big] \,|_{\mathbb{T}^d}$ in ξ and $\overline{\xi}$ extends to an analytic polynomial on $\mathbb{C}^d$ which we again denote by $C[f]$. In particular,

$$P_p(\xi^j \overline{\xi}^k) = P(\xi_1^{j_1} \overline{\xi}_1^{k_1}) \otimes \ldots \otimes P(\xi_d^{j_d} \overline{\xi}_d^{k_d}) = C[\xi^j \overline{\xi}^k]|_{\mathbb{T}^d}$$

for all $j, k \in \mathbb{N}^d$. Since $\mathbb{C}\big[\xi, \overline{\xi}\big] \,|_{\mathbb{T}^d} \subset \mathrm{L}^p(\mathbb{T}^d)$ is dense, we have

$$P_p(\mathrm{L}^p(\mathbb{T}^d)) \subset \overline{\mathbb{C}[z] \,|_{\mathbb{T}^d}}^{\mathrm{L}^p(\mathbb{T}^d)} = \mathrm{H}^p(\mathbb{T}^d).$$

Since $P_p(f) = C[f]|_{\mathbb{T}^d} = f$ for each polynomial $f \in \mathbb{C}[z] \,|_{\mathbb{T}^d}$, it follows that P_p is a continuous linear projection from $\mathrm{L}^p(\mathbb{T}^d)$ onto $\mathrm{H}^p(\mathbb{T}^d)$.

Let $f \in \mathrm{L}^p(\mathbb{T}^d)$ and define $F = P_p(f) \in \mathrm{H}^p(\mathbb{T}^d)$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{C}\big[\xi, \overline{\xi}\big] \,|_{\mathbb{T}^d}$ with $f = \lim_{k \to \infty} f_k$ in $\mathrm{L}^p(\mathbb{T}^d)$. Then $F = \lim_{k \to \infty}(C[f_k]|_{\mathbb{T}^d})$ in $\mathrm{L}^p(\mathbb{T}^d)$ and

$$C[f](z) = \lim_{k \to \infty} C[f_k](z) = \lim_{k \to \infty} C[C[f_k]|_{\mathbb{T}^d}](z) = C[F](z)$$

for any $z \in \mathbb{D}^d$. But since each function in $\mathrm{H}^p(\mathbb{T}^d)$ is the radial limit almost everywhere of its Cauchy integral (see [63]), it follows that $C[f]^* = C[F]^* = F = P_p(f)$. Thus exactly as in the case of the unit ball we have constructed a family $(P_p)_{1<p<\infty}$ of projections $P_p \in \mathrm{L}(\mathrm{L}^p(\mathbb{T}^d))$ from $\mathrm{L}^p(\mathbb{T}^d)$ onto $\mathrm{H}^p(\mathbb{T}^d)$ which act as

$$P_p(f) = C[f]^* \qquad (p \in (1, \infty), f \in \mathrm{L}^p(\mathbb{T}^d))$$

and are compatible with the inclusion mappings $i_{sr} \colon \mathrm{L}^s(\mathbb{T}^d) \to \mathrm{L}^r(\mathbb{T}^d)$ in the sense that $i_{sr} P_s = P_r i_{sr}$ for $1 < r < s < \infty$. As in the case of the unit ball (Theorem 5.6.9 in [64]), it follows that $P_2 \in \mathrm{L}(\mathrm{L}^2(\mathbb{T}^d))$ is the orthogonal projection onto $\mathrm{H}^2(\mathbb{T}^d)$. We repeat the elementary argument. For $z \in \mathbb{D}^d$, the function

$$u_z \colon \overline{\mathbb{D}}^d \longrightarrow \mathbb{C}, \qquad \xi \longmapsto \prod_{\nu=1}^{d} (1 - \xi_\nu \overline{z}_\nu)^{-1}$$

belongs to the polydisc algebra $\mathrm{A}\left(\mathbb{D}^d\right)$. Let $f \in \mathrm{L}^2(\mathbb{T}^d)$ be given. Then $f = g + h$ with $g \in \mathrm{H}^2(\mathbb{T}^d)$, $h \in \mathrm{H}^2(\mathbb{T}^d)^{\perp}$. Since

$$C[h](z) = \int_{\mathbb{T}^d} h\overline{u}_z \, d\sigma_d = 0 \quad (z \in \mathbb{D}^d),$$

we find that

$$P_2(f) = C[f]^* = C[g]^* = g.$$

In particular, the projections P_p satisfy the duality relations $P_p^* = P_q$ for $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We want to point out that for general bounded symmetric domains $D \subset \mathbb{C}^d$, the Szegő projection need not be bounded on $\mathrm{L}^p(\partial D)$(see, e.g., [7]). These domains will not be taken into account in this thesis.

2 Toeplitz Projections on Hardy-type Spaces

2.1 Toeplitz Operators on Hardy-type Spaces

In this section we give an overview over some classical results concerning Toeplitz operators on Hardy spaces. First we give the definition of a Toeplitz operator in the general setting that was expounded in Section 1.3.

Definition 2.1.
Fix $1 < p < \infty$ and a compact set $K \subset \mathbb{C}^d$ and let $\mathrm{H}^p(\partial_A)$ be a Hardy-type space induced by the closed algebra $A \subset \mathrm{C}(K)$, which contains the holomorphic polynomials. For $f \in \mathrm{L}^\infty(\partial_A)$ the compression

$$T_f^p \colon \mathrm{H}^p(\partial_A) \longrightarrow \mathrm{H}^p(\partial_A), \qquad g \longmapsto P_p M_f^p g$$

*of the multiplication operator $M_f^p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ is called the **Toeplitz operator with symbol** f.*

In the case $p = 2$, for $f \in \mathrm{L}^\infty(\partial_A)$, we will always write $T_f = T_f^2$. The following basic algebraic relations between Toeplitz operators hold.

Lemma 2.2.
For $f \in \mathrm{L}^\infty(\partial_A)$ and $\theta \in \mathrm{H}^\infty(\partial_A)$, we have

$$(T_f^p)^* = T_{\overline{f}}^q, \quad T_f^p T_\theta^p = T_{f\theta}^p \quad \text{and} \quad T_{\overline{\theta}}^p T_f^p = T_{\overline{\theta} f}^p.$$

In the year 1963 A. Brown and P. R. Halmos gave in [14] an algebraic characterization of Toeplitz operators on the Hardy-Hilbert space $\mathrm{H}^2(\mathbb{T})$ over the unit circle $\mathbb{T} \subset \mathbb{C}$. They showed that an operator $T \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}))$ is Toeplitz if and only if $T_z^* X T_z = X$. An analogue of this result is true for Toeplitz operators acting on $\mathrm{H}^p(\mathbb{T})$ for $p \in (1, \infty)$ (see, e.g. Theorem 2.7 in [13]).

Theorem 2.3 (Brown, Halmos).
For $p \in (1, \infty)$, an operator $X \in \mathrm{L}(\mathrm{H}^p(\mathbb{T}))$ is a Toeplitz operator if and only if $T^p_{\bar{z}} X T^p_z = X$.

As a corollary we get

Corollary 2.4.
For $p \in (1, \infty)$, an operator $X \in \mathrm{L}(\mathrm{H}^p(\mathbb{T}))$ is a Toeplitz operator if and only if $T^p_{\bar{\theta}} X T^p_\theta = X$ for every inner function $\theta \in \mathrm{H}^\infty(\mathbb{T})$.

The Brown-Halmos condition has undergone extensive research and proved to be the starting point for further applications. We want to mention here the perturbation theory of Toeplitz operators that was developed in [76] and [27]. Theorem 2.3 also prompted the attempt to leave the setting of classical Hardy spaces towards more abstract notions of Toeplitz operators (see, for example Murphy [57], Sz.-Nagy and Foias [71], Tolokonnikov [73], Karlovich and Shargorodsky [45], Eschmeier and Everard [34] or Eschmeier and Langendörfer [35]). The purpose of the next section is to show that one can adapt the methods used in [34] to get a version of the Brown-Halmos Theorem for Toeplitz operators acting on Hardy-type spaces $\mathrm{H}^p(\partial_A)$.

The methods presented in [34] also faciliate the formulation of a spectral inclusion formula for Toeplitz operators acting on the Hardy-type spaces $\mathrm{H}^p(\partial_A)$. In the case of the Hardy space $\mathrm{H}^2(\mathbb{T})$ over the unit circle $\mathbb{T} \subset \mathbb{C}$ this goes back to P. Hartman and A. Wintner [41] and I. B. Simonenko [69] (see also Theorem 2.30 in [13]).

Let $(X, \mathcal{A}, \mu)$ be a measure space. For $f \in \mathrm{L}^\infty(X)$, we denote by

$$R(f) = \{z \in \mathbb{C} \mid \mu\left(\{x \in X \mid |f(x) - z| < \varepsilon\}\right) > 0 \text{ for all } \varepsilon > 0\}$$

the essential range of f. This set coincides with the spectrum of the function f in the C*-algebra $\mathrm{L}^\infty(X)$.

Theorem 2.5 (Hartman/Wintner, 1954).
Let $f \in \mathrm{L}^\infty(\mathbb{T})$ *and* $p \in (1, \infty)$. *Then the spectral inclusion*

$$R(f) \subset \sigma_e(T_f^p)$$

holds.

We show in Theorem 2.21 that this result is true for arbitrary Hardy-type spaces $\mathrm{H}^p(\partial_A)$ $(1 < p < \infty)$ where the underlying measure is non-atomic.

2.2 The Construction of a Toeplitz Projection on Hardy-type Spaces

If not stated otherwise, we make the following assumptions. Let $K \subset \mathbb{C}^d$ be a compact set and let $A \subset \mathrm{C}(K)$ be a closed subalgebra containing the restrictions $p|_K$ of all holomorphic polynomials $p \in \mathbb{C}[z]$ in d complex variables. Recall that the Shilov boundary of A is denoted by $\partial_A \subset K$. Let $\mu \in \mathrm{M}^+(\partial_A)$ be a positive regular Borel measure such that (A, ∂_A, μ) is a regular triple in the sense of Aleksandrov. Fix $1 < p < \infty$ and let $\mathrm{H}^p(\partial_A)$ be a Hardy-type space as explained at the beginning of Section 1.3. Building on the work of Prunaru [61], Eschmeier and Everard constructed in [34] Toeplitz projections for regular A-isometries $T \in \mathrm{L}(\mathcal{H})^n$ on a Hilbert space $\mathcal{H}$ and used these projections to extend various classical results on Toeplitz operators to the setting of abstract $T-$Toeplitz operators (see Definition 3.1 in [26] for the notion of abstract $T-$Toeplitz operators). We adapt the methods developed in [34] to show that in an analogous way one can associate a Toeplitz projection $\Phi_{T_z} \colon \mathrm{L}(\mathrm{H}^p(\partial_A)) \to \mathrm{L}(\mathrm{H}^p(\partial_A))$ with the tuple $T_z^p = (T_{z_1}^p, \dots, T_{z_d}^p)$ of multiplication operators

$$T_{z_i}^p \colon \mathrm{H}^p(\partial_A) \longrightarrow \mathrm{H}^p(\partial_A), \qquad f \longmapsto z_i f.$$

We apply this result to prove H^p-versions of the Brown-Halmos characterization of Toeplitz operators and to derive extensions of the spectral inclusion formulas of Hartman-Wintner and Simonenko (see, e.g. Theorem 2.30 in [13]). First we recall some results from Banach space theory concerning the spaces $\mathrm{L}^p(\partial_A)$ $(1 \leq p < \infty)$.

The space $\mathrm{L}(\mathrm{L}^p(\partial_A))$ can be isometrically identified with the topological dual of a canonically formed Banach space Z. We briefly recall this construction in a more general setting (see Chapter 7 in [60]). Let X, Y be Banach spaces. For $x \in X, y \in Y$, define

$$x \otimes y \colon \mathrm{L}(X, Y') \longrightarrow \mathbb{C}, \qquad T \longmapsto \langle y, Tx \rangle$$

and denote by

$$Z = \overline{\mathrm{LH}} \{ x \otimes y \mid x \in X \text{ and } y \in Y \} \subseteq \mathrm{L}(X, Y')'$$

the smallest closed subspace containing the functionals $x \otimes y \in \mathrm{L}(X, Y')'$. One can show that the bilinear mapping

$$\langle\cdot,\cdot\rangle : Z \times \mathrm{L}(X, Y') \longrightarrow \mathbb{C}, \qquad (u, T) \longmapsto \langle u, T\rangle = u(T)$$

induces an isometric isomorphism

$$\Phi\colon\ \mathrm{L}(X, Y') \longrightarrow Z', \qquad T \longmapsto \langle\,\cdot\,, T\rangle\,.$$

By definition the BW-topology τ_{BW} of $\mathrm{L}(X, Y')$ is the topology τ on $\mathrm{L}(X, Y')$ for which the mapping

$$\Phi\colon\ (\mathrm{L}(X, Y'), \tau) \longrightarrow (Z', \tau_{w^*})$$

is a homeomorphism. It is well known that a norm-bounded net $(T_\alpha)_\alpha$ in $\mathrm{L}(X, Y')$ is τ_{BW}-convergent to an operator $T \in \mathrm{L}(X, Y')$ if and only if $T_\alpha(x) \overset{\alpha}{\to} Tx$ in (Y', τ_{w^*}) for every $x \in X$. By the theorem of Alaoglu-Bourbaki each norm-closed ball

$$B_r = \{T \in \mathrm{L}(X, Y') \mid \|T\| \leq r\} \subset \mathrm{L}(X, Y')$$

is τ_{BW}-compact. Since Φ is norm-continuous, the norm topology on $\mathrm{L}(X, Y')$ is finer than the BW-topology.

Let X be a reflexive Banach space. Then the canonical embedding $j\colon X \to X''$ is an isometric isomorphism. Hence also the induced map

$$J\colon\ \mathrm{L}(X) \longrightarrow \mathrm{L}(X, (X')'), \qquad T \longmapsto j \circ T$$

is an isometric isomorphism. We equip the space on the right-hand side with the BW-topology explained above and define the BW-topology τ_{BW} on $\mathrm{L}(X)$ as the unique topology for which the map

$$J\colon (\mathrm{L}(X), \tau_{\mathrm{BW}}) \longrightarrow (\mathrm{L}(X, (X')'), \tau_{\mathrm{BW}})$$

is a homeomorphism. Then a norm-bounded net $(T_\alpha)_\alpha$ in $\mathrm{L}(X)$ is τ_{BW}-convergent to an operator $T \in \mathrm{L}(X)$ if and only if

$$\langle u, jT_\alpha(x)\rangle \overset{\alpha}{\longrightarrow} \langle u, jT(x)\rangle$$

for all $x \in X$ and $u \in X'$ or, equivalently, if

$$u(T_\alpha x) \xrightarrow{\alpha} u(Tx)$$

for all $x \in X$ and $u \in X'$. The Alaoglu-Bourbaki theorem implies that each norm-closed ball

$$B_r = \{T \in \mathrm{L}(X) \mid \|T\| \leq r\}$$

is τ_{BW}-compact.

Remark 2.6.
By construction there is an isometric isomorphism $\tau\colon \mathrm{L}(X) \to Z'$ *onto the dual of a Banach space* Z *such that* $\tau\colon (\mathrm{L}(X), \tau_{BW}) \to (Z', \tau_{w^*})$ *is a homeomorphism. If* X *and* Y *are reflexive Banach spaces, then a linear map* $A\colon (\mathrm{L}(X), \tau_{BW}) \to (\mathrm{L}(Y), \tau_{BW})$ *is continuous if and only if* $\tau_{BW} - \lim_\alpha A(T_\alpha) = A(T)$ *for each norm-bounded net* $(T_\alpha)_\alpha$ *in* $\mathrm{L}(X)$ *with* $\tau_{BW} - \lim_\alpha T_\alpha = T$. *This follows from a well-known characterization of weak* continuous linear maps between Banach spaces (see e.g. Lemma 1.16 in [37]).*

Lemma 2.7.
Let X, Y *be reflexive Banach spaces and let* $A \in \mathrm{L}(X)$, $B \in \mathrm{L}(Y, X), C \in \mathrm{L}(X, Y)$. *Then the mappings*

$$L_A\colon (\mathrm{L}(X), \tau_{BW}) \longrightarrow (\mathrm{L}(X), \tau_{BW}), \qquad T \longmapsto AT,$$

$$R_A\colon (\mathrm{L}(X), \tau_{BW}) \longrightarrow (\mathrm{L}(X), \tau_{BW}), \qquad T \longmapsto TA$$

and

$$M_{C,B}\colon (\mathrm{L}(X), \tau_{BW}) \longrightarrow (\mathrm{L}(Y), \tau_{BW}), \qquad T \longmapsto CTB$$

are continuous.

Proof. Let $(T_\alpha)_\alpha$ be a norm-bounded net in $\mathrm{L}(X)$ with $\tau_{\mathrm{BW}}\text{-}\lim_\alpha T_\alpha = T$. Then $(AT_\alpha)_\alpha$ and $(T_\alpha A)_\alpha$ are norm-bounded nets in $\mathrm{L}(X)$ with

$$u(AT_\alpha x) = (u \circ A)(T_\alpha x) \xrightarrow{\alpha} (u \circ A)(Tx) = u(ATx)$$

and

$$u(T_\alpha Ax) \xrightarrow{\alpha} u(TAx)$$

for all $x \in X, u \in X'$. Hence $\tau_{\mathrm{BW}}\text{-}\lim_\alpha AT_\alpha = AT$ and $\tau_{\mathrm{BW}}\text{-}\lim_\alpha T_\alpha A = TA$. Similarly, we obtain

$$u(CT_\alpha By) = (u \circ C)(T_\alpha(By)) \xrightarrow{\alpha} (u \circ C)(T(By)) = u(CTBy)$$

for all $y \in Y$ and $u \in Y'$. Hence $\tau_{\mathrm{BW}}\text{-}\lim_\alpha CT_\alpha B = CTB$. □

We will now define a projection $\Phi_{M_z}\colon \mathrm{L}(\mathrm{L}^p(\partial_A)) \to \mathrm{L}(\mathrm{L}^p(\partial_A))$ onto the subspace $\mathcal{M}(\mathrm{L}^p(\partial_A))$ consisting of all multiplication operators $M_f^p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ with bounded measurable symbols. We write

$$I_\mu = \{\theta \in \mathrm{H}^\infty(\partial_A) \mid |\theta| = 1 \ \mu\text{-almost everywhere}\}$$

for the set of μ-inner functions. By results of A. B. Aleksandrov [3] (see also Corollary 2.5 in [24]) the set I_μ generates $\mathrm{L}^\infty(\partial_A)$ as a von Neumann algebra. More precisely, we have

$$\mathrm{L}^\infty(\partial_A) = W^*(I_\mu) = \overline{\mathrm{LH}}^{w*}\left(\{\overline{\eta} \cdot \theta \mid \eta, \theta \in I_\mu\}\right).$$

By the Alaoglu-Bourbaki theorem and since $\mathrm{L}^1(\mu)$ is separable, the closed unit ball

$$B_{\mathrm{L}^\infty(\partial_A)} = \left\{f \in \mathrm{L}^\infty(\partial_A) \;\middle|\; \|f\|_{\mathrm{L}^\infty(\partial_A)} \leq 1\right\}$$

equipped with the relative weak* topology of $\mathrm{L}^\infty(\partial_A) = \mathrm{L}^1(\mu)'$ is a compact metrizable space. Hence by elementary topology $B_{\mathrm{L}^\infty(\partial_A)}$ and its subset I_μ are separable metrizable spaces in the relative weak* topology. For a given subset $I \subset \mathrm{L}^\infty(\partial_A)$, we define $I^* = \left\{\overline{f} \mid f \in I\right\}$ and we denote by $\mathrm{alg}(I) \subset \mathrm{L}^\infty(\partial_A)$ the unital subalgebra generated by I. In the following let $I = \{\theta_k \mid k \in \mathbb{N}^*\} \subset I_\mu$ be a fixed countable subset such that

$$\mathrm{L}^\infty(\partial_A) = \overline{\mathrm{alg}(I \cup I^*)}^{\tau_{w*}}.$$

Note that each countable weak* dense subset of I_μ satisfies this condition. For $X \in$

$\mathrm{L}(\mathrm{L}^p(\partial_A))$, the averages

$$\Phi_{M_z,k}(X) = \frac{1}{k^k} \sum_{1 \le i_1,\ldots,i_k \le k} M^p_{\overline{\theta}_k^{i_k} \cdot \ldots \cdot \overline{\theta}_1^{i_1}} X M^p_{\theta_k^{i_k} \cdot \ldots \cdot \theta_1^{i_1}} \in \mathrm{L}(\mathrm{L}^p(\partial_A))$$

form a sequence $(\Phi_{M_z,k}(X))_{k\ge1}$ in $B_{\|X\|} = \{T \in \mathrm{L}(\mathrm{L}^p(\partial_A)) \mid \|T\| \le \|X\|\}$. Since by the Alaoglu-Bourbaki theorem the ball $B_{\|X\|}$ equipped with the relative topology of the τ_{BW} topology of $\mathrm{L}(\mathrm{L}^p(\partial_A))$ is a compact topological space, Tychonoff's theorem yields that the topological product $\prod_{X\in\mathrm{L}(\mathrm{L}^p(\partial_A))}(B_{\|X\|}, \tau_{\mathrm{BW}})$ is compact. Convergence in the product topology is equivalent to componentwise convergence. Hence there is a subnet $(\Phi_{M_z,k_\alpha})_\alpha$ of the sequence $(\Phi_{M_z,k})_{k\ge1}$ such that the τ_{BW} limits

$$\Phi_{M_z}(X) = \tau_{\mathrm{BW}}\text{-}\lim_\alpha \Phi_{M_z,k_\alpha}(X) \in \mathrm{L}(\mathrm{L}^p(\partial_A))$$

exist simultaneously for every $X \in \mathrm{L}(\mathrm{L}^p(\partial_A))$. Let us fix such a subnet $(\Phi_{M_z,k_\alpha})_\alpha$ and let us consider the induced map

$$\Phi_{M_z}\colon\ \mathrm{L}(\mathrm{L}^p(\partial_A)) \longrightarrow \mathrm{L}(\mathrm{L}^p(\partial_A)), \qquad X \longmapsto \tau_{\mathrm{BW}}\text{-}\lim_\alpha \Phi_{M_z,k_\alpha}(X).$$

By construction Φ_{M_z} is a linear contraction.

Theorem 2.8.
The mapping

$$\Phi_{M_z}\colon\ \mathrm{L}(\mathrm{L}^p(\partial_A)) \longrightarrow \mathrm{L}(\mathrm{L}^p(\partial_A)), \qquad X \longmapsto \Phi_{M_z}(X)$$

is a unital projection with

$$\operatorname{Im}\Phi_{M_z} = \mathcal{M}(\mathrm{L}^p(\partial_A)). \tag{2.1}$$

Proof. Obviously the mappings $\Phi_{M_z,k}$ and hence also Φ_{M_z} act as the identity operator on $\mathcal{M}(\mathrm{L}^p(\partial_A)) \subset \mathrm{L}(\mathrm{L}^p(\partial_A))$. Thus to show that Φ_{M_z} defines a unital projection onto this closed subalgebra, it suffices to check that $\operatorname{Im}\Phi_{M_z} \subset \mathcal{M}(\mathrm{L}^p(\partial_A))$. Now fix $X \in \mathrm{L}(\mathrm{L}^p(\partial_A))$. We have to show that $\Phi_{M_z}(X)$ is a multiplication operator. According to Proposition 1.4 it suffices to show that $\Phi_{M_z}(X)$ commutes with all multiplication operators M_f^p $(f \in \mathrm{L}^\infty(\partial_A))$. For $1 \le j \le k$ and $i = (i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k) \in$

$\{1,\ldots,k\}^{k-1}$, we write

$$R_{ij} = M^p_{\theta_1^{i_1}\ldots\theta_{j-1}^{i_{j-1}}\theta_{j+1}^{i_{j+1}}\ldots\theta_k^{i_k}}$$

and

$$\overline{R}_{ij} = M^p_{\overline{\theta}_1^{i_1}\ldots\overline{\theta}_{j-1}^{i_{j-1}}\overline{\theta}_{j+1}^{i_{j+1}}\ldots\overline{\theta}_k^{i_k}}.$$

Then for $X \in \mathrm{L}(\mathrm{L}^p(\partial_A))$, the estimate

$$\begin{aligned}\left\| M^p_{\overline{\theta}_j}\Phi_{M_z,k}(X)M^p_{\theta_j} - \Phi_{M_z,k}(X)\right\| &= \frac{1}{k^k}\left\|\sum_i \overline{R}_{ij}\left(\sum_{\mu=1}^k M^p_{\overline{\theta}_j^{\mu+1}}XM^p_{\theta_j^{\mu+1}} - M_{\overline{\theta}_j^{\mu}}XM_{\theta_j^{\mu}}\right)R_{ij}\right\| \\ &\leq \frac{k^{k-1}}{k^k}2\left\|X\right\| = \frac{2\left\|X\right\|}{k}\end{aligned}$$

holds for every $k \geq 1$ and $1 \leq j \leq k$. Using Lemma 2.7 we obtain

$$M^p_{\overline{\theta}_j}\Phi_{M_z}(X)M^p_{\theta_j} = \tau_{\mathrm{BW}}\text{-}\lim_\alpha M^p_{\overline{\theta}_j}\Phi_{M_z,k_\alpha}(X)M^p_{\theta_j} = \Phi_{M_z}(X).$$

Thus $\Phi_{M_z}(X)$ commutes with all multiplication operators of the form $M^p_{\overline{\theta}_j}$ and $M^p_{\theta_j}$ for $j \geq 1$. Since the commutant of $\Phi_{M_z}(X)$ in $\mathrm{L}(\mathrm{L}^p(\partial_A))$ is a unital τ_{WOT}-closed subalgebra containing the operators $M^p_{\overline{\theta}_j}$ and $M^p_{\theta_j}$ for all $j \geq 1$ and since the mapping

$$(\mathrm{L}^\infty(\partial_A), \tau_{w^*}) \longrightarrow (\mathrm{L}(\mathrm{L}^p(\partial_A)), \tau_{\mathrm{WOT}}), \qquad f \longmapsto M^p_f$$

is a unital continuous algebra homomorphism, our hypothesis that

$$\mathrm{L}^\infty(\partial_A) = \overline{\mathrm{alg}(I \cup I^*)}^{\tau_{w^*}}$$

implies that $\Phi_{M_z}(X)$ commutes with all multiplication operators M^p_f with $f \in \mathrm{L}^\infty(\partial_A)$. This observation completes the proof. □

For $k \in \mathbb{N}$, define $N_k = \{1,\ldots,k\}^k$. To simplify the notation, we use the abbreviations

$$\theta(i) = \theta_1^{i_1}\cdot\ldots\cdot\theta_k^{i_k} \qquad \text{and} \qquad \overline{\theta}(i) = \overline{\theta}_1^{i_1}\cdot\ldots\cdot\overline{\theta}_k^{i_k}$$

for $i = (i_1,\ldots,i_k) \in N_k$. For $k \geq 1$, let $\Phi_{T_z,k}\colon \mathrm{L}(\mathrm{H}^p(\partial_A)) \to \mathrm{L}(\mathrm{H}^p(\partial_A))$ be the bounded linear operator defined by

$$\Phi_{T_z,k}(X) = \frac{1}{k^k} \sum_{i \in N_k} T^p_{\overline{\theta}(i)} X T^p_{\theta(i)}.$$

For $X \in \mathrm{L}(\mathrm{H}^p(\partial_A))$, we define $\tilde{X} = XP_p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$. Then

$$\begin{aligned}
\Phi_{T_z,k}(X) &= \frac{1}{k^k} \sum_{i \in N_k} T^p_{\overline{\theta}(i)} X T^p_{\theta(i)} \\
&= \frac{1}{k^k} \sum_{i \in N_k} P_p M^p_{\overline{\theta}(i)} X P_p M^p_{\theta(i)}|_{\mathrm{H}^p(\partial_A)} \\
&= P_p \left(\frac{1}{k^k} \sum_{i \in N_k} M^p_{\overline{\theta}(i)} \tilde{X} M^p_{\theta(i)} \right)\Bigg|_{\mathrm{H}^p(\partial_A)} \\
&= P_p \Phi_{M_z,k}(\tilde{X})|_{\mathrm{H}^p(\partial_A)}
\end{aligned}$$

for $X \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ and $k \geq 1$. Using Lemma 2.7 we find that the τ_{BW}-limit

$$\begin{aligned}
\Phi_{T_z}(X) &= \tau_{\mathrm{BW}}\text{-}\lim\nolimits_\alpha \Phi_{T_z,k_\alpha}(X) \\
&= \tau_{\mathrm{BW}}\text{-}\lim\nolimits_\alpha P_p \Phi_{M_z,k_\alpha}(\tilde{X})|_{\mathrm{H}^p(\partial_A)} \\
&= P_p \Phi_{M_z}(\tilde{X})|_{\mathrm{H}^p(\partial_A)}
\end{aligned} \tag{2.2}$$

exists in $\mathrm{L}(\mathrm{H}^p(\partial_A))$. Our next aim is to show that the set

$$\mathcal{T} = \left\{ X \in \mathrm{L}(\mathrm{H}^p(\partial_A)) \;\middle|\; T^p_{\overline{\theta}} X T^p_\theta = X \text{ for all } \theta \in I_\mu \right\}$$

consists precisely of all Toeplitz operators on $\mathrm{H}^p(\partial_A)$. It follows from Lemma 2.2 that $\left\{ T^p_f \mid f \in \mathrm{L}^\infty(\partial_A) \right\} \subset \mathcal{T}$. We use Theorem 2.8 to prove the reverse inclusion.

Theorem 2.9.
The mapping

$$\Phi_{T_z} \colon \mathrm{L}(\mathrm{H}^p(\partial_A)) \longrightarrow \mathrm{L}(\mathrm{H}^p(\partial_A)), \qquad X \longmapsto \Phi_{T_z}(X)$$

is a unital continuous projection with

$$\mathrm{Im}(\Phi_{T_z}) = \mathcal{T} = \left\{ T^p_f \mid f \in \mathrm{L}^\infty(\partial_A) \right\}. \tag{2.3}$$

Proof. By Theorem 2.8 and (2.2) we obtain that $\operatorname{Im}\Phi_{T_z} \subset \left\{T_f^p \mid f \in \mathrm{L}^\infty(\partial_A)\right\} \subset \mathcal{T}$. From the definition of $\mathcal{T}$ it follows that $\Phi_{T_z}(X) = X$ for all $X \in \mathcal{T}$. Combining these two remarks we find that (2.3) holds and that Φ_{T_z} is a unital projection. Since

$$\|\Phi_{T_z}(X)\| \leq \|P_p\| \, \|\tilde{X}\| \leq \|P_p\|^2 \, \|X\| \qquad (X \in \mathrm{L}(\mathrm{H}^p(\partial_A))),$$

the projection Φ_{T_z} is continuous with $\|\Phi_{T_z}\| \leq \|P_p\|^2$. □

Remark 2.10.
An analysis of the proof of Theorem 2.9 shows that

$$\left\{T_f^p \mid f \in \mathrm{L}^\infty(\partial_A)\right\} = \left\{X \in \mathrm{L}(\mathrm{H}^p(\partial_A)) \;\middle|\; T_{\overline{\theta}}^p X T_\theta^p = X \text{ for all } \theta \in I\right\}.$$

Indeed by the construction of the projection Φ_{T_z} it easily follows that every operator X contained in the set on the right-hand side, satisfies the fixed point equation $\Phi_{T_z}(X) = X$.

In the case of the regular triple $(A(S), S, \sigma)$ (see Section 1.3), Theorem 2.9 yields the multivariable version of Corollary 2.4. Since for a long time even the existence of non-trivial inner functions on the unit ball has been an open problem[1], this generalization of Corollary 2.4 is of course of a more theoretical nature. In contrast, in the case of the unit polydisc in $\mathbb{C}^d$ it is possible to derive a concrete characterization of Toeplitz operators from Theorem 2.9 that is closer in spirit to the original result of Brown and Halmos. In the case $p = 2$ the following Corollary 2.11 was already observed by A. Maji, J. Sarkar and S. Sarkar in [51, Theorem 3.1]. Alternatively one can derive it from Theorem 3.1 of [34] with arguments similar to those used here.

Corollary 2.11.
For $1 < p < \infty$, an operator $X \in \mathrm{L}(\mathrm{H}^p(\mathbb{T}^d))$ is a Toeplitz operator if and only if $T_{\overline{z}_i}^p X T_{z_i}^p = X$ for $i = 1, \ldots, d$.

Proof. Since the polynomials $\mathbb{C}[z, \overline{z}]$ in $z = (z_1, \ldots, z_d)$ and $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_d)$ form a weak*

[1]This problem is called the inner function problem and it was solved simultaneously by A. B. Aleksandrov [3] and E. Løw [49] in 1982.

dense subset of $\mathrm{L}^\infty(\mathbb{T}^d)$, the set $I = \{\theta_k \mid k \in \mathbb{N}^*\}$ given by the sequence

$$(\theta_k)_{k\in\mathbb{N}^*} = (z_1, \dots, z_d, z_1, \dots, z_d, \dots)$$

satisfies our hypothesis that

$$\mathrm{L}^\infty(\mathbb{T}^d) = \overline{\mathrm{alg}(I \cup I^*)}^{\tau_{w^*}}.$$

Thus the assertion follows as an application of Remark 2.10. □

We want to point out that it also would have been possible to prove the last corollary by following the lines in the proof of [51, Theorem 3.1]. One simply has to replace the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{L}^2(\mathbb{T}^d)}$ by the corresponding sesquilinear form $\langle\cdot,\cdot\rangle$ on $\mathrm{L}^p(\mathbb{T}^d)\times\mathrm{L}^q(\mathbb{T}^d)$. There exists yet another algebraic characterization of Toeplitz operators on the Hardy space $\mathrm{H}^2(S)$ on the unit sphere $S \subset \mathbb{C}^d$ that was proven by A. M. Davie and N. P. Jewell in [21, Theorem 2.6] which states that an operator $T \in \mathrm{L}(\mathrm{H}^2(S))$ is a Toeplitz operator on $\mathrm{H}^2(S)$ if and only if the identity $\sum_{k=1}^d T_{\overline{z}_k} T T_{z_k} = T$ holds. Since, for $d > 1$, the coordinate functions $z_1, \dots, z_d$ are no inner functions for the sphere $S \subset \mathbb{C}^d$, the theory of Toeplitz projections on Hardy-type spaces that we presented in this section seems not to be applicable to derive an H^p version of this theorem. We do not know if such a generalization exists.

Proposition 2.12.
For $1 < p < \infty$, the identity

$$\mathrm{L}^p(\partial_A) = \overline{\mathrm{LH}}^{\mathrm{L}^p(\partial_A)}\left(\left\{\overline{\theta}\cdot\eta \;\middle|\; \theta,\eta \in I_\mu\right\}\right)$$

holds.

Proof. Define $V = \overline{\mathrm{LH}}^{\mathrm{L}^p(\partial_A)}\left(\left\{\overline{\theta}\cdot\eta \;\middle|\; \theta,\eta \in I_\mu\right\}\right)$. Let $f \in \mathrm{L}^\infty(\partial_A)$ and $(f_\alpha)_\alpha$ be a net in $\mathrm{LH}\left(\left\{\overline{\theta}\cdot\eta \;\middle|\; \theta,\eta \in I_\mu\right\}\right) \subseteq \mathrm{L}^\infty(\partial_A) \subseteq \mathrm{L}^p(\partial_A)$ that converges to f in the weak* topology of $\mathrm{L}^\infty(\partial_A) = \mathrm{L}^1(\partial_A)'$. Since $\mathrm{L}^q(\partial_A) \subseteq \mathrm{L}^1(\partial_A)$ the net $(f_\alpha)_\alpha$ converges to f in the weak* topology of $\mathrm{L}^p(\partial_A) = \mathrm{L}^q(\partial_A)'$. As a norm closed convex subset of the reflexive Banach space $\mathrm{L}^p(\partial_A)$ the subspace V is also closed with respect to the weak* topology. Hence $\mathrm{L}^\infty(\partial_A) \subseteq V$. The density of $\mathrm{L}^\infty(\partial_A)$ in $\mathrm{L}^p(\partial_A)$ now yields the assertion. □

Define as before $\tilde{X} = XP_p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ for $X \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ and

$$\hat{\pi}\colon\ \mathrm{L}(\mathrm{H}^p(\partial_A)) \longrightarrow \mathrm{L}(\mathrm{L}^p(\partial_A)), \qquad X \longmapsto \Phi_{M_z}(\tilde{X}),$$

$$C\colon\ \mathrm{L}(\mathrm{L}^p(\partial_A)) \longrightarrow \mathrm{L}(\mathrm{H}^p(\partial_A)), \qquad X \longmapsto P_p X|_{\mathrm{H}^p(\partial_A)}.$$

Lemma 2.13.
The compression mapping C induces a topological isomorphism

$$\varphi\colon\ \mathcal{M}(\mathrm{L}^p(\partial_A)) \longrightarrow \mathcal{T}, \qquad X \longmapsto P_p X|_{\mathrm{H}^p(\partial_A)}$$

with inverse given by

$$\mathcal{T} \longrightarrow \mathcal{M}(\mathrm{L}^p(\partial_A)), \qquad X \longmapsto \hat{\pi}(X).$$

Proof. Obviously the mapping φ is continuous linear. Let $X \in \ker\varphi$. Then, for $\theta_1, \theta_2, \eta_1, \eta_2 \in I_\mu$, we have

$$\left\langle X(\overline{\theta}_1\eta_1), \overline{\theta_2}\eta_2\right\rangle = \langle X(\theta_2\eta_1), \theta_1\eta_2\rangle = \langle P_p X(\theta_2\eta_1), \theta_1\eta_2\rangle = 0.$$

Hence Proposition 2.12 shows that $X = 0$, thus φ is injective. Using Theorem 2.9 and the definition of Φ_{T_z} we obtain

$$\varphi(\hat{\pi}(T_f^p)) = P_p \Phi_{M_z}(\tilde{T}_f^p)|_{\mathrm{H}^p(\partial_A)} = \Phi_{T_z}(T_f^p) = T_f^p$$

for all $f \in \mathrm{L}^\infty(\partial_A)$. Thus the injective map φ is also surjective and $\hat{\pi}\colon \mathcal{T} \to \mathcal{M}(\mathrm{L}^p(\partial_A))$ defines the inverse of $\varphi\colon\ \mathcal{M}(\mathrm{L}^p(\partial_A)) \to \mathcal{T}$. □

The preceding lemma implies in particular that $\hat{\pi}(T_f^p) = M_f^p$ for all $f \in \mathrm{L}^\infty(\partial_A)$.

Theorem 2.14.
The continuous linear mapping

$$\hat{\pi}\colon\ \mathrm{L}(\mathrm{H}^p(\partial_A)) \longrightarrow \mathrm{L}(\mathrm{L}^p(\partial_A)), \qquad X \longmapsto \Phi_{M_z}(\tilde{X})$$

has image $\operatorname{Im}\hat{\pi} = \mathcal{M}(\mathrm{L}^p(\partial_A))$*. It is multiplicative in the sense that the identity*

$$\hat{\pi}\left(XT_f^p\right) = \hat{\pi}(X)\hat{\pi}\left(T_f^p\right)$$

holds for $X \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ *and* $f \in \mathrm{L}^\infty(\partial_A)$*. Furthermore we have:*

(i) $P_p\left(\hat{\pi}(X)\right)|_{\mathrm{H}^p(\partial_A)} = X$ *for every* $X \in \mathcal{T}$*,*

(ii) $\hat{\pi}\left(T_{f_1}^p \dots T_{f_r}^p\right) = M_{f_1\dots f_r}^p$ *for all* $f_1, \dots, f_r \in \mathrm{L}^\infty(\partial_A)$*.*

In particular, for $f_1, \dots, f_r \in \mathrm{L}^\infty(\partial_A)$*, we have the identity*

$$\Phi_{T_z}(T_{f_1}^p \dots T_{f_r}^p) = T_{f_1\dots f_r}^p.$$

Proof. By Theorem 2.8 and the remark preceding Theorem 2.14 we have

$$\operatorname{Im}\hat{\pi} = \mathcal{M}(\mathrm{L}^p(\partial_A))$$

and $\hat{\pi}(T_f^p) = M_f^p$ for all $f \in \mathrm{L}^\infty(\partial_A)$. Lemma 2.13 implies that

$$P_p\hat{\pi}(X)|_{\mathrm{H}^p(\partial_A)} = X$$

for $X \in \mathcal{T}$. Let $X \in \mathrm{L}(\mathrm{H}^p(\partial_A))$, $f \in \mathrm{L}^\infty(\partial_A)$, $h \in \mathrm{H}^p(\partial_A)$, $g \in \mathrm{L}^q(\partial_A)$ and $\eta \in I_\mu$ be given. Using the characterization of norm-bounded τ_{BW}-convergent nets given at the beginning of Section 2.2 and the fact that $\hat{\pi}(XT_f^p) \in \mathcal{M}(\mathrm{L}^p(\partial_A))$, we obtain

$$\left\langle \hat{\pi}(XT_f^p)M_{\overline{\eta}}^p h, g\right\rangle = \lim_\alpha \left\langle M_{\overline{\eta}}^p \Phi_{M_z,k_\alpha}((XT_f^p)^\sim)h, g\right\rangle.$$

For $\theta \in I_\mu$, we have

$$M_{\overline{\theta}}^p(XT_f^p)^\sim M_\theta^p h = M_{\overline{\theta}}^p X P_p M_f^p M_\theta^p h = M_{\overline{\theta}}^p X P_p M_\theta^p \hat{\pi}(T_f^p)h.$$

Using the definition of $\hat{\pi}(X) = \Phi_{M_z}(\tilde{X})$ and using the same arguments as above, we find that

$$\left\langle \hat{\pi}(XT_f^p)M_{\overline{\eta}}^p h, g\right\rangle = \left\langle M_{\overline{\eta}}^p \hat{\pi}(X)\hat{\pi}(T_f^p)h, g\right\rangle = \left\langle \hat{\pi}(X)\hat{\pi}(T_f^p)M_{\overline{\eta}}^p h, g\right\rangle.$$

Using Proposition 2.12 we obtain that $\hat{\pi}(XT_f^p) = \hat{\pi}(X)\hat{\pi}(T_f^p)$ holds. For finitely many

$f_1, \ldots, f_r \in \mathrm{L}^\infty(\partial_A)$, it follows that

$$\hat{\pi}(T^p_{f_1} \ldots T^p_{f_r}) = \hat{\pi}(T^p_{f_1} \ldots T^p_{f_{r-1}})\hat{\pi}(T^p_{f_r}) = \ldots = \hat{\pi}(T^p_{f_1}) \ldots \hat{\pi}(T^p_{f_r}) = M^p_{f_1 \cdots f_r}$$

and hence that $\Phi_{T_z}(T^p_{f_1} \ldots T^p_{f_r}) = P_p\hat{\pi}(T^p_{f_1} \ldots T^p_{f_r})|_{\mathrm{H}^p(\partial_A)} = T^p_{f_1 \cdots f_r}$. □

For a unital norm-closed subalgebra $\mathcal{B} \subset \mathrm{L}^\infty(\partial_A)$, we denote by

$$\mathcal{T}^p(\mathcal{B}) = \overline{\mathrm{alg}}\left\{T^p_f \;\middle|\; f \in \mathcal{B}\right\} \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

the smallest norm-closed subalgebra containing all Toeplitz operators T^p_f with symbol $f \in \mathcal{B}$. We define the semi-commutator ideal $\mathcal{SC}^p(\mathcal{B})$ of $\mathcal{T}^p(\mathcal{B})$ as the norm-closed ideal in $\mathcal{T}^p(\mathcal{B})$ generated by the operators of the form

$$T^p_f T^p_g - T^p_{fg} \qquad (f, g \in \mathcal{B}).$$

Let $\rho\colon \mathcal{M}(\mathrm{L}^p(\partial_A)) \to \mathrm{L}^\infty(\partial_A)$ be the inverse of the isometric algebra isomorphism $\mathrm{L}^\infty(\partial_A) \to \mathcal{M}(\mathrm{L}^p(\partial_A)), f \mapsto M^p_f$.

Corollary 2.15.
Let $\mathcal{B} \subset \mathrm{L}^\infty(\partial_A)$ be a unital closed subalgebra. Then

$$\Phi_{T_z}(\mathcal{T}^p(\mathcal{B})) = \left\{T^p_f \;\middle|\; f \in \mathcal{B}\right\} \subset \mathcal{T}^p(\mathcal{B}).$$

The map

$$\sigma\colon\ \mathcal{T}^p(\mathcal{B}) \longrightarrow \mathcal{B}, \qquad X \longmapsto \rho \circ \hat{\pi}(X)$$

is a surjective continuous morphism of unital Banach algebras with $\sigma(T^p_f) = f$ for $f \in \mathcal{B}$ and

$$\ker(\sigma) = (1 - \Phi_{T_z})(\mathcal{T}^p(\mathcal{B})) = \ker \Phi_{T_z}|_{\mathcal{T}^p(\mathcal{B})} = \mathcal{SC}^p(\mathcal{B}).$$

In particular, we obtain the direct sum decomposition

$$\mathcal{T}^p(\mathcal{B}) = \left\{T^p_f \;\middle|\; f \in \mathcal{B}\right\} \oplus \mathcal{SC}^p(\mathcal{B}).$$

Proof. Note that $\mathcal{T}^p(\mathcal{B})$ is the closed linear span of operators of the form $T^p_{f_1} \ldots T^p_{f_r}$

$(r \in \mathbb{N}, f_1, \ldots, f_r \in \mathcal{B})$. The multiplicativity of $\hat{\pi}$ proven in Theorem 2.14 shows that σ is a surjective continuous morphism of unital Banach algebras. By Lemma 2.13 the Toeplitz operators with symbol in $\mathcal{B}$ form a norm-closed subalgebra $\{T_f^p \mid f \in \mathcal{B}\} \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$. By the last part of Theorem 2.14 we know that

$$\Phi_{T_z}(\mathcal{T}^p(\mathcal{B})) = \{T_f^p \mid f \in \mathcal{B}\} \subset \mathcal{T}^p(\mathcal{B}).$$

It follows from Theorem 2.9 that the restriction $\Phi_0\colon \mathcal{T}^p(\mathcal{B}) \to \mathcal{T}^p(\mathcal{B})$ of Φ_{T_z} to $\mathcal{T}^p(\mathcal{B})$ is a unital projection. Next, from the injectivity of the compression mapping

$$\{M_f^p \mid f \in \mathcal{B}\} \longrightarrow \{T_f^p \mid f \in \mathcal{B}\}, \qquad X \longmapsto P_p X|_{\mathrm{H}^p(\partial_A)}$$

and the mapping ρ, we conclude

$$\ker \sigma = \ker \hat{\pi}|_{\mathcal{T}^p(\mathcal{B})} = \ker \Phi_0 = (1 - \Phi_{T_z})(\mathcal{T}^p(\mathcal{B})).$$

Since $\ker \sigma$ is a norm-closed ideal in $\mathcal{T}^p(\mathcal{B})$ containing all semi-commutators $T_f^p T_g^p - T_{fg}^p$ $(f, g \in \mathcal{B})$ we have $\mathcal{SC}^p(\mathcal{B}) \subset \ker \sigma$. To prove the reverse inclusion we check that $X - \Phi_0(X) \in \mathcal{SC}^p(\mathcal{B})$ for every $X \in \mathcal{T}^p(\mathcal{B})$. By definition of the algebra $\mathcal{T}^p(\mathcal{B})$ and the continuity of Φ_0 we only have to verify the above claim for finite products $X = T_{f_1}^p \ldots T_{f_r}^p$ with $r \in \mathbb{N}^*$ and $f_1, \ldots, f_r \in \mathcal{B}$. We assume $r \geq 3$ since the cases $r = 1$ and $r = 2$ are obvious. By inserting suitable summands and by using Theorem 2.14, we expand

$$\begin{aligned}
X - \Phi_0(X) &= T_{f_1}^p \ldots T_{f_r}^p - T_{f_1 \cdots f_r}^p \\
&= T_{f_1}^p \ldots T_{f_{r-2}}^p (T_{f_{r-1}}^p T_{f_r}^p - T_{f_{r-1} f_r}^p) \\
&\quad + T_{f_1}^p \ldots T_{f_{r-3}}^p \left(T_{f_{r-2}}^p T_{f_{r-1} f_r}^p - T_{f_{r-2} f_{r-1} f_r}^p \right) \\
&\quad + \ldots + T_{f_1}^p \ldots T_{f_{r-l}}^p \left(T_{f_{r-l+1}}^p T_{f_{r-l+2} \cdots f_r}^p - T_{f_{r-l+1} f_{r-l+2} \cdots f_r}^p \right) \\
&\quad + \ldots + T_{f_1}^p T_{f_2 \cdots f_r}^p - T_{f_1 \cdots f_r}^p .
\end{aligned}$$

Since all summands of the latter expression are clearly included in $\mathcal{SC}^p(\mathcal{B})$ we have $X - \Phi_0(X) \in \mathcal{SC}^p(\mathcal{B})$. From what we have shown so far we get the direct sum decomposition

$$\mathcal{T}^p(\mathcal{B}) = \{T_f^p \mid f \in \mathcal{B}\} \oplus \mathcal{SC}^p(\mathcal{B}).$$

□

2.3 The Spectral Inclusion Theorem of Hartman-Wintner

For an arbitrary Banach space X, we denote by $\mathcal{K}(X)$ the closed two-sided ideal of the compact operators on X. We write $[T]$ for the image of an operator $T \in \mathrm{L}(X)$ in the Calkin algebra

$$\mathcal{C}(X) = \mathrm{L}(X)/\mathcal{K}(X)$$

on X under the canonical quotient mapping $\mathrm{L}(X) \to \mathcal{C}(X)$. Denote by $r([T])$ the spectral radius with respect to the algebra $\mathcal{C}(X)$ and write $\|T\|_e = \|[T]\|$ for the essential operator norm of T, that is, the quotient norm of $[T]$ in $\mathcal{C}(X)$.

Proposition 2.16.
For $f \in \mathrm{L}^\infty(\partial_A)$ it holds

$$r\left([M_f^p]\right) = \left\|M_f^p\right\|_e = \left\|M_f^p\right\|.$$

Proof. Fix $f \in \mathrm{L}^\infty(\partial_A)$. The inequality $\left\|M_f^p\right\|_e \leq \left\|M_f^p\right\|$ follows immediately from the definition of the norm of $\mathcal{C}(\mathrm{L}^p(\partial_A))$. To show the reverse inequality let $u \in \mathrm{L}^p(\partial_A)$ be an arbitrary unit vector and $K \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ be a compact operator. By a well known approximation theorem of Aleksandrov (see [3, Corollary 2.9] or [24, Proposition 2.1]) there exists a weak* zero sequence $(\theta_k)_{k\in\mathbb{N}}$ of μ-inner functions. Then $(M_{\theta_k}^p u)_{k\in\mathbb{N}}$ is a weak zero sequence and therefore the compactness of K implies $\lim_{k\to\infty} \left\|KM_{\theta_k}^p u\right\|_p = 0$. The estimate

$$\left\|M_f^p - K\right\| \geq \left\|M_{\overline{\theta}_k}^p (M_f^p - K) M_{\theta_k}^p u\right\|_p = \left\|M_f^p u - M_{\overline{\theta}_k}^p K M_{\theta_k}^p u\right\|_p \qquad (k \in \mathbb{N})$$

yields

$$\left\|M_f^p - K\right\| \geq \lim_{k\to\infty} \left\|M_f^p u - M_{\overline{\theta}_k}^p K M_{\theta_k}^p u\right\|_p = \left\|M_f^p u\right\|_p.$$

Since this holds true for every unit vector $u \in \mathrm{L}^p(\partial_A)$ and every compact operator $K \in \mathrm{L}(\mathrm{L}^p(\partial_A))$, we conclude that $\left\|M_f^p\right\| = \left\|M_f^p\right\|_e$. To verify the spectral radius equation $\left\|M_f^p\right\|_e = r\left([M_f^p]\right)$, observe that

$$\left\|\left(M_f^p\right)^k\right\|_e = \left\|M_{f^k}^p\right\|_e = \left\|M_{f^k}^p\right\| = \left\|f^k\right\|_\infty = \|f\|_\infty^k = \left\|M_f^p\right\|^k = \left\|M_f^p\right\|_e^k$$

and use the spectral radius formular to compute

$$r\left([M_f^p]\right) = \lim_{k\to\infty} \sqrt[k]{\left\| \left(M_f^p\right)^k \right\|_e} = \left\| M_f^p \right\|_e .$$

□

Our proof of the essential spectral inclusion formula will be based on the observation that the Toeplitz projection Φ_{T_z} annihilates the compact operators. The arguments that we use to prove $\Phi_{T_z}|_{\mathcal{K}(\mathrm{H}^p(\partial_A))} \equiv 0$ allow us to calculate at the same time the essential commutant of the algebra $\mathcal{M}(\mathrm{L}^p(\partial_A)) \subset \mathrm{L}(\mathrm{L}^p(\partial_A))$.

For an arbitrary subset $M \subset \mathrm{L}(X)$ of the algebra of all bounded linear operators on a Banach space X, we define the essential commutant of M by

$$M^{\mathrm{ec}} = \{A \in \mathrm{L}(X) \mid AT - TA \in \mathcal{K}(X) \text{ for all } T \in M\} .$$

Proposition 2.17.
Let $T \in \mathcal{M}(\mathrm{L}^p(\partial_A))^{ec}$ be given. Define

$$F\colon\ \mathrm{L}^\infty(\partial_A) \longrightarrow \mathrm{L}(\mathrm{L}^p(\partial_A)), \qquad f \longmapsto M_f^p T - \Phi_{M_z}(T) M_f^p .$$

Then the following holds:

(P1) F is pointwise boundedly SOT*-continuous, that is, for every bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathrm{L}^\infty(\partial_A)$ converging pointwise μ-almost everywhere to some function $f \in \mathrm{L}^\infty(\partial_A)$, we have $F(f) = \tau_{\mathrm{SOT}}\text{-}\lim_{k\to\infty} F(f_k)$,*

(P2) If $F(\mathrm{L}^\infty(\partial_A)) \not\subset \mathcal{K}(\mathrm{L}^p(\partial_A))$, then there exist a positive real number $\rho > 0$ and a sequence $(f_k)_{k\geq 1}$ in $\mathrm{C}(\partial_A, [0,1])$ of functions with pairwise disjoint supports such that

$$\|F(f_k)\| > \rho$$

for all $k \geq 1$.

Proof. To prove (P1) fix a bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathrm{L}^\infty(\partial_A)$ that converges pointwise μ-almost everywhere to some function $f \in \mathrm{L}^\infty(\partial_A)$. By Theorem 2.8 we can choose a function $g \in \mathrm{L}^\infty(\partial_A)$ with $\Phi_{M_z}(T) = M_g^p$. Then the dominated convergence theorem implies that

$$\begin{aligned}\lim_{k\to\infty} \|(F(f_k)-F(f))h\|_p^p &= \lim_{k\to\infty} \|(f_k-f)(Th-gh)\|_p^p \\ &= \lim_{k\to\infty} \int_{\partial_A} |(f_k-f)(Th-gh)|^p \, d\mu = 0\end{aligned}$$

for every function $h \in \mathrm{L}^p(\partial_A)$. To prove (P2), suppose that there exists a function $f \in \mathrm{L}^\infty(\partial_A)$ such that $F(f)$ is not a compact operator. Since f can be uniformly approximated by step functions, there is a characteristic function $\chi \in \mathrm{L}^\infty(\partial_A)$ of some Borel set in ∂_A such that $F(\chi)$ is not compact. Hence the number $\rho = \|F(\chi)\|_e/2$ is strictly positive. To prove condition (P2) we shall use a version of the Allan-Douglas localization principle due to Simonenko. We use [62] as a reference.

First let us observe that $\mathrm{L}^p(\partial_A)$ is a Banach space of local type in the sense of Simonenko (see Definition 2.5.4 together with Example 2.5.5 in [62]). Since $T \in \mathcal{M}(\mathrm{L}^p(\partial_A))^{\mathrm{ec}}$, it follows that

$$\begin{aligned}F(\chi)M_f^p - M_f^p F(\chi) &= M_\chi^p T M_f^p - M_g^p M_\chi^p M_f^p - M_f^p M_\chi^p T + M_f^p M_g^p M_\chi^p \\ &= M_\chi^p [T, M_f^p] \in \mathcal{K}(\mathrm{L}^p(\partial_A)).\end{aligned}$$

for all $f \in \mathrm{L}^\infty(\partial_A)$. Hence by Theorem 2.5.6 in [62] the operator $F(\chi)$ is of local type (see Definition 2.5.3 in [62]). Thus we can apply Theorem 2.5.12 together with Lemma 2.5.9 of [62] to $F(\chi)$ to obtain the identity

$$\|F(\chi)\|_e = \max_{\zeta\in\partial_A} \inf \left\{ \left\|M_f^p F(\chi)\right\|_e \;\middle|\; f \in \mathrm{C}(\partial_A), f(\zeta) = 1 \right\}.$$

In the next step we prove the existence of a sequence $(g_k)_{k\geq 1}$ of continuous functions $g_k \in \mathrm{C}(\partial_A, [0,1])$ with pairwise disjoint supports such that $\|F(g_k\chi)\| > \rho$ for all $k \geq 1$.

The above observations allow us to choose a point $z_0 \in \partial_A$ such that

$$\left\|M_f^p F(\chi)\right\|_e > \rho \tag{2.4}$$

for all $f \in \mathrm{C}(\partial_A)$ with $f(z_0) = 1$. We construct the sequence $(g_k)_{k\geq 1}$ inductively and only focus on the induction step. Suppose that $g_1, \ldots, g_k \in \mathrm{C}(\partial_A, [0,1])$ are functions with pairwise disjoint supports such that $\|F(g_j\chi)\| > \rho$ and $z_0 \notin \operatorname{supp} g_j$ for $j = 1, \ldots, k$.

Using Urysohn's lemma we can choose a function $f \in \mathrm{C}(\partial_A, [0,1])$ with $f(z_0) = 1$ and $\operatorname{supp} f \cap \operatorname{supp} g_j = \emptyset$ for all $j = 1, \ldots, k$ as well as a sequence of functions $(\kappa_j)_{j\geq 1}$ in $\mathrm{C}(\partial_A, [0,1])$ with $z_0 \notin \operatorname{supp}(\kappa_j)$ for all $j \geq 1$ and such that $\lim_{j\to\infty} \kappa_j(z) = 1$ for all $z \in \partial_A \setminus \{z_0\}$. By construction $(\kappa_j f\chi)_{j\geq 1}$ is a bounded sequence in $\mathrm{L}^\infty(\partial_A)$ which converges pointwise μ-almost everywhere to the function $f\chi$. Using (P1) we see that

$$M_f^p F(\chi) = F(f\chi) = \tau_{\mathrm{SOT}}\text{-}\lim_{j\to\infty} F(\kappa_j f\chi).$$

Hence there is an integer $j \geq 1$ with $\|F(\kappa_j f\chi)\| > \rho$. Define $g_{k+1} = \kappa_j f$. Inductively we obtain a sequence $(g_k)_{k\geq 1}$ in $\mathrm{C}(\partial_A)$ with the desired properties. Next we apply Lusin's theorem (cf. Theorem 7.4.4 and Proposition 3.1.3 in [17]) to get a sequence $(h_j)_{j\geq 1}$ of continuous functions $h_j \in \mathrm{C}(\partial_A, [0,1])$ converging to χ μ-almost everywhere. Again using (P1) we deduce

$$F(g_k\chi) = \tau_{\mathrm{SOT}}\text{-}\lim_{j\to\infty} F(g_k h_j)$$

for every $k \geq 1$. Hence for every $k \geq 1$, there is an integer $j_k \in \mathbb{N}$ such that $\|F(g_k h_{j_k})\| > \rho$. The functions $f_k = g_k h_{j_k}$ $(k \in \mathbb{N}^*)$ are as desired. □

In the next theorem we will make use of the fact that the spaces $\mathrm{L}^p(\partial_A)$ $(1 < p < \infty)$ possess Schauder bases (see [12, p. 296]). This fact will allow us to apply Lemma A.2 with $X = \mathrm{L}^p(\partial_A)$. In the proof we use fixed bounded measurable representatives of the μ-inner functions $\theta_k \in \mathrm{H}^\infty(\partial_A)$ such that $\|\theta_k\|_{\partial_A} \leq 1$ and define the products $\theta(i) = \theta_1^{i_1} \cdot \ldots \cdot \theta_k^{i_k}$ for $i = (i_1, \ldots, i_k) \in N_k$.

Theorem 2.18.
Let $T \in \mathcal{M}(\mathrm{L}^p(\partial_A))^{ec}$. Then $T - \Phi_{M_z}(T) \in \mathcal{K}(\mathrm{L}^p(\partial_A))$.

Proof. Let $F\colon \mathrm{L}^\infty(\partial_A) \to \mathrm{L}(\mathrm{L}^p(\partial_A))$ be the mapping defined in Proposition 2.17. Since $T - \Phi_{M_z}(T) = F(1)$, it suffices to show that $F(\mathrm{L}^\infty(\partial_A)) \subset \mathcal{K}(\mathrm{L}^p(\partial_A))$. Assume that this inclusion does not hold and choose a sequence $(f_k)_{k\geq 1}$ as in Proposition 2.17. By construction the operator $F(f_j)$ is the τ_{BW}-limit of a net of operators of the form

$$M_{f_j}^p T - \frac{1}{k^k} \sum_{i\in N_k} M_{\overline{\theta(i)}}^p T M_{\theta(i)}^p M_{f_j}^p = \frac{1}{k^k} \sum_{i\in N_k} M_{\overline{\theta(i)}}^p \left(M_{f_j\theta(i)}^p T - T M_{f_j\theta(i)}^p \right)$$

with suitable inner functions $\theta(i) \in \mathrm{H}^\infty(\partial_A)$. Thus there exist $u \in \mathrm{L}^p(\partial_A)$ and $v \in \mathrm{L}^q(\partial_A)$ with $\|u\|_p \leq 1$, $\|v\|_q \leq 1$ and an integer $k \geq 1$ such that

$$\begin{aligned}\rho &< \left| \left\langle \frac{1}{k^k} \sum_{i \in N_k} M^p_{\overline{\theta(i)}} \left(M^p_{f_j\theta(i)} T - T M^p_{f_j\theta(i)} \right) u, v \right\rangle_{L^p, L^q} \right| \\ &\leq \frac{1}{k^k} \sum_{i \in N_k} \left\| M^p_{f_j\theta(i)} T - T M^p_{f_j\theta(i)} \right\| .\end{aligned}$$

Hence we find a multiindex $i_j \in N_k$ such that $\theta(i_j) \in I_\mu$ fulfils

$$\left\| M^p_{f_j\theta(i_j)} T - T M^p_{f_j\theta(i_j)} \right\| > \rho.$$

In other words, for each $j \geq 1$ the function $h_j = f_j\theta(i_j) \in \mathcal{L}^\infty(\partial_A)$ satisfies

$$\|h_j\|_{\partial_A} \leq 1, \quad h_j \equiv 0 \text{ on } \partial_A \setminus \operatorname{supp}(f_j), \quad \left\| \left[M^p_{h_j}, T \right] \right\| > \rho$$

and the commutators $K_j = \left[M^p_{h_j}, T \right] \in \mathrm{L}(\mathrm{L}^p(\partial_A))$ are compact. Since $\rho < \|K_j\| \leq 2\,\|T\|$, by passing to a subsequence, we can achieve that the limit

$$c = \lim_{j \to \infty} \|K_j\| \in [\rho, 2\,\|T\|]$$

exists. Fix a subsequence $\left(K_{j(k)}\right)_{k \in \mathbb{N}}$ of $(K_j)_{j \geq 1}$ and set $\varphi_N = \sum_{k=0}^N h_{j(k)} \in \mathcal{L}^\infty(\partial_A)$ for $N \in \mathbb{N}$. We already know that the functions $h_{j(k)} \in \mathcal{L}^\infty(\partial_A)$ ($k \in \mathbb{N}$) have disjoint supports and fulfil $\left\|h_{j(k)}\right\|_{\partial_A} \leq 1$. Therefore $\|\varphi_N\|_{\partial_A} \leq 1$ for all $N \in \mathbb{N}$ and the sequence $(\varphi_N)_{N \in \mathbb{N}}$ converges pointwise on ∂_A to the function $\varphi = \sum_{k=0}^\infty h_{j(k)} \in \mathcal{L}^\infty(\partial_A)$. The dominated convergence theorem implies that

$$\mathrm{SOT} - \sum_{k=0}^\infty K_{j(k)} = \tau_{\mathrm{SOT}}\text{-}\lim_{N \to \infty} \left[M^p_{\varphi_N}, T \right] = \left[M^p_\varphi, T \right] .$$

Since $K_j^* = \left[T^*, M^q_{\overline{h_j}} \right]$ for all $j \geq 1$, it follows in exactly the same way that

$$\mathrm{SOT} - \sum_{k=0}^\infty K^*_{j(k)} = \left[T^*, M^q_{\overline{\varphi}} \right].$$

In particular,

$$\tau_{\mathrm{SOT}}\text{-}\lim_{j\to\infty} K_j^* = 0.$$

We apply Lemma A.2 to the sequence $(K_j)_{j\geq 1}$ to see that, by passing to a subsequence, one can achieve that $K = \mathrm{SOT} - \sum_{j=0}^{\infty} K_j$ is not compact. On the other hand, we already know, as shown above, that $K = \left[M_h^p, T\right] \in \mathcal{K}(\mathrm{L}^p(\partial_A))$ with a suitable function $h \in \mathrm{L}^\infty(\partial_A)$. This contradiction finishes the proof. □

As an obvious consequence of the preceding theorem we obtain the announced description of the essential commutant of the subalgebra $\mathcal{M}(\mathrm{L}^p(\partial_A)) \subset \mathrm{L}(\mathrm{L}^p(\partial_A))$.

Corollary 2.19.
For $1 < p < \infty$, the identity

$$\mathcal{M}(\mathrm{L}^p(\partial_A))^{ec} = \mathcal{M}(\mathrm{L}^p(\partial_A)) + \mathcal{K}(\mathrm{L}^p(\partial_A))$$

holds.

As a second application we show that Φ_{M_z} and Φ_{T_z} annihilate the compact operators.

Corollary 2.20.
Suppose that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms. Then we have

$$\Phi_{M_z}(\mathcal{K}(\mathrm{L}^p(\partial_A))) = \{0\} \qquad \text{and} \qquad \Phi_{T_z}(\mathcal{K}(\mathrm{H}^p(\partial_A))) = \{0\}.$$

Proof. Let $T \in \mathcal{K}(\mathrm{L}^p(\partial_A))$. Then Theorem 2.18 yields $\Phi_{M_z}(T) = T - F(1) \in \mathcal{K}(\mathrm{L}^p(\partial_A))$. Together with Theorem 2.8 we get that $\Phi_{M_z}(T)$ is a compact multiplication operator on $\mathrm{L}^p(\partial_A)$. As an application of Proposition 1.5 we find that $\Phi_{M_z}(T) = 0$. If $T \in \mathcal{K}(\mathrm{H}^p(\partial_A))$, then $\tilde{X} = XP_p \in \mathcal{K}(\mathrm{L}^p(\partial_A))$ and

$$\Phi_{T_z}(X) = P_p \Phi_{M_z}(\tilde{X})|_{\mathrm{H}^p(\partial_A)} = 0.$$

□

We are now ready to prove a multivariable version of the spectral inclusion formula of Hartman-Wintner and Simonenko.

Theorem 2.21.
Assume that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms. Then, for $f \in \mathrm{L}^\infty(\partial_A)$, the spectral inclusion formula

$$R(f) \subset \sigma_{\mathrm{e}}(T_f^p)$$

holds.

Proof. It suffices to prove that a function $f \in \mathrm{L}^\infty(\partial_A)$ is invertible as an element in the C*-algebra $\mathrm{L}^\infty(\partial_A)$ whenever T_f^p is invertible in the Calkin algebra $\mathcal{C}(\mathrm{H}^p(\partial_A))$. Suppose that $Y \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ is a linear operator such that $YT_f^p - 1 \in \mathcal{K}(\mathrm{H}^p(\partial_A))$. Using Theorem 2.14 we get

$$\hat{\pi}(Y)M_f^p = \hat{\pi}(Y)\hat{\pi}(T_f^p) = \hat{\pi}(YT_f^p) = 1 + \hat{\pi}(YT_f^p - 1).$$

By Corollary 2.20 we know that

$$\hat{\pi}(YT_f^p - 1) = \Phi_{M_z}\left((YT_f^p - 1)P_p\right) = 0.$$

By Theorem 2.8 there is a function $g \in \mathrm{L}^\infty(\partial_A)$ with

$$\hat{\pi}(Y) = \phi_{M_z}(\tilde{Y}) = M_g^p.$$

Thus we find that $M_g^p M_f^p = \hat{\pi}(Y)M_f^p = 1$, or equivalently, that $gf = 1$ μ-almost everywhere on ∂_A. □

The proof of Theorem 2.21 shows that the function $f \in \mathrm{L}^\infty(\partial_A)$ is invertible in $\mathrm{L}^\infty(\partial_A)$ whenever $T_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ is left invertible in the Calkin algebra $\mathcal{C}(\mathrm{H}^p(\partial_A))$.

Corollary 2.22.
Suppose that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms. Let $f \in \mathrm{L}^\infty(\partial_A)$ be given. Then

$$\left\|T_f^p\right\|_e \geq \|f\|_\infty .$$

For $f \in \mathrm{H}^\infty(\partial_A)$,

$$\left\|T_f^p\right\| = \left\|T_f^p\right\|_e = \|f\|_\infty .$$

Proof. By Theorem 2.21 we have

$$\left\|T_f^p\right\|_e \geq \left\{|w| \;\middle|\; w \in \sigma_{\mathrm{e}}(T_f^p)\right\} \geq \{|w| \mid w \in R(f)\} = \|f\|_\infty .$$

For $f \in \mathrm{H}^\infty(\partial_A)$, it follows that

$$\|f\|_\infty = \left\|M_f^p\right\| \geq \left\|T_f^p\right\| \geq \left\|T_f^p\right\|_e \geq \|f\|_\infty .$$

□

3 Compactness of Commutators of Toeplitz Operators

This section is devoted to the study of commutators of Toeplitz operators. The results stated here will be used in the subsequent section which deals with multidimensional spectral mapping theorems for Toeplitz tuples with symbol in suitable subalgebras of $\mathrm{L}^\infty(\partial_A)$. Let $\mathrm{H}^p(\partial_A)$ be a Hardy-type space as defined at the beginning of Section 1.3. We start with an application of an interpolation theorem for compact operators going back to M. A. Krasnoselski (see Theorem 4.2.9 in [8]). We restate it in a version that is suitable for our purpose.

Let $i\colon Y \to X$ be an injective continuous linear operator between Banach spaces. Let $T \in \mathrm{L}(X)$ be a bounded operator with $T(\operatorname{Im} i) \subset \operatorname{Im} i$. Then the closed graph theorem shows that the induced operator $S\colon Y \to Y$, that is, the unique map with $iS = Ti$, is continuous again.

Theorem 3.1 (M. A. Krasnoselski).
Let $1 < p_0 < p < p_1$ be real numbers and let

$$i_1\colon\ \mathrm{L}^{p_1}(\partial_A) \longrightarrow \mathrm{L}^{p_0}(\partial_A) \qquad \text{and} \qquad i\colon\ \mathrm{L}^{p}(\partial_A) \longrightarrow \mathrm{L}^{p_0}(\partial_A)$$

be the inclusion mappings. Suppose that $T_0\colon\ \mathrm{L}^{p_0}(\partial_A) \to \mathrm{L}^{p_0}(\partial_A)$ is a bounded linear operator such that $T_0(\operatorname{Im} i_1) \subset \operatorname{Im} i_1$ and such that T_0 or the induced operator $T_1\colon\ \mathrm{L}^{p_1}(\partial_A) \to \mathrm{L}^{p_1}(\partial_A)$ is compact. Then $T_0(\operatorname{Im} i) \subset \operatorname{Im} i$ and the induced operator $T\colon\ \mathrm{L}^{p}(\partial_A) \to \mathrm{L}^{p}(\partial_A)$ is compact again.

Let $1 < p_0 < p < p_1$ be real numbers and let $f \in \mathrm{L}^\infty(\partial_A)$ be given. Since the commutator $C_f^{p_0} = [M_f^{p_0}, P_{p_0}]\colon\ \mathrm{L}^{p_0}(\partial_A) \to \mathrm{L}^{p_0}(\partial_A)$ maps the linear subspace $\mathrm{L}^{p_1}(\partial_A) \subset$

$\mathrm{L}^{p_0}(\partial_A)$ into itself, it follows from Theorem 3.1 that the compactness of $C_f^{p_0}$ or $C_f^{p_1}$ implies the compactness of C_f^p for every real number $p \in (p_0, p_1)$. For simplicity, we write C_f, P, M_f, T_f instead of C_f^2, P_2, M_f^2, T_f^2. As a straightforward application one obtains the following consequence.

Theorem 3.2.
The set

$$QC = \left\{ f \in \mathrm{L}^\infty(\partial_A) \;\middle|\; C_f^p \in \mathcal{K}(\mathrm{L}^p(\partial_A)) \right\} \subset \mathrm{L}^\infty(\partial_A)$$

is a unital C-subalgebra that does not depend on the choice of* $p \in (1, \infty)$.

Proof. The remarks preceding the theorem show that the definition of QC does not depend on the choice of p. Since the map

$$\mathrm{L}^\infty(\partial_A) \longrightarrow \mathrm{L}(\mathrm{L}^2(\partial_A)), \qquad g \longmapsto C_g$$

is continuous linear, the set $QC \subset \mathrm{L}^\infty(\partial_A)$ is a closed linear subspace. For $\phi, \psi \in QC$, the identity

$$[M_{\phi\psi}, P] = M_\phi(M_\psi P - PM_\psi) + (M_\phi P - PM_\phi)M_\psi$$

shows that $\phi\psi \in QC$. Finally the identity

$$C_f^* = (M_f P - PM_f)^* = -(M_{\overline{f}} P - PM_{\overline{f}}) = -C_{\overline{f}}$$

shows that $QC \subset \mathrm{L}^\infty(\partial_A)$ is a C*-subalgebra. □

For $f \in \mathrm{L}^\infty(\partial_A)$, we define the Hankel operator with symbol f on $\mathrm{H}^p(\partial_A)$ as the operator

$$H_f^p = (1 - P_p)M_f^p \colon\ \mathrm{H}^p(\partial_A) \longrightarrow \mathrm{L}^p(\partial_A).$$

We simply write $H_f = H_f^2$ for the Hankel operator with symbol f on $\mathrm{H}^2(\partial_A)$.

Corollary 3.3.
For $1 < p < \infty$, *we have*

$$QC = \left\{ f \in \mathrm{L}^\infty(\partial_A) \;\middle|\; H_f^p \text{ and } H_{\overline{f}}^p \text{ are compact.} \right\}.$$

Proof. Fix a function $f \in QC$. Then the operator

$$(1 - P_p)M_f^p|_{\mathrm{H}^p(\partial_A)} = M_f^p|_{\mathrm{H}^p(\partial_A)} - (M_f^p P_p)|_{\mathrm{H}^p(\partial_A)} + C_f^p|_{\mathrm{H}^p(\partial_A)} = C_f^p|_{\mathrm{H}^p(\partial_A)}$$

is compact. Since also $\overline{f} \in QC$, we have shown that H_f^p and $H_{\overline{f}}^p$ are compact. To prove the converse, note first that, for a fixed function $f \in \mathrm{L}^\infty(\partial_A)$ and for $1 < p_0 < p_1$, the bounded operator

$$\mathrm{L}^{p_0}(\partial_A) \longrightarrow \mathrm{L}^{p_0}(\partial_A), \qquad g \longmapsto f P_{p_0}(g) - P_{p_0}(f P_{p_0}(g))$$

maps $\mathrm{L}^{p_1}(\partial_A)$ into itself. Thus exactly as before, Krasnoselski's interpolation theorem implies that the compactness of the operator $H_f^p P_p = (1 - P_p)M_f^p P_p \colon \mathrm{L}^p(\partial_A) \to \mathrm{L}^p(\partial_A)$ does not depend on the choice of $p \in (1, \infty)$. Suppose that $f \in \mathrm{L}^\infty(\partial_A)$ is a function such that H_f^p and $H_{\overline{f}}^p$ are both compact. Let $q \in (1, \infty)$ be the conjugate exponent of p. Then the operator

$$(P_p M_f^p (1 - P_p))^* = (1 - P_q)M_{\overline{f}}^q P_q = H_{\overline{f}}^q P_q \in \mathrm{L}(\mathrm{L}^q(\partial_A))$$

and hence also the operator

$$C_f^p = (1 - P_p)M_f^p P_p - P_p M_f^p (1 - P_p)$$

is compact. □

Corollary 3.4.
For $1 < p < \infty$ and $f \in QC$, the commutators $[T_f^p, T_g^p] \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ are compact for every function $g \in \mathrm{L}^\infty(\partial_A)$.

Proof. By the preceding corollary the Hankel operator $H_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A), \mathrm{L}^p(\partial_A))$ is compact. Hence for $g \in \mathrm{L}^\infty(\partial_A)$, the semi-commutator

$$\begin{aligned} T_g^p T_f^p - T_{gf}^p &= P_p M_g^p T_f^p - P_p M_{gf}^p|_{\mathrm{H}^p(\partial_A)} \\ &= P_p M_g^p (P_p M_f^p - M_f^p)|_{\mathrm{H}^p(\partial_A)} \\ &= -P_p M_g^p H_f^p \in \mathrm{L}(\mathrm{H}^p(\partial_A)) \end{aligned}$$

is compact. Since also $\overline{f} \in QC$, we obtain the compactness of

$$T_f^p T_g^p - T_{fg}^p = (T_{\overline{g}}^q T_{\overline{f}}^q - T_{\overline{g}\overline{f}}^q)^* \in \mathrm{L}(\mathrm{H}^p(\partial_A)),$$

where as usual $q \in (1, \infty)$ denotes the conjugate exponent of p. Hence $[T_f^p, T_g^p]$ is compact for every $g \in \mathrm{L}^\infty(\partial_A)$. □

If $K = \overline{D} \subset \mathbb{C}^d$ is the closure of a bounded strictly pseudoconvex domain with smooth boundary, $A = \{f \in \mathrm{C}(K) \mid f|_D \text{ is holomorphic}\}$ is the domain algebra of D and μ is the normalized surface measure on $\partial D = \partial_A$, then $\mathrm{H}^2(\partial_A) = \mathrm{H}^2(\partial D)$ is the usual Hardy space on D. In this case $T_z = (T_{z_1}, \dots, T_{z_d}) \in \mathrm{L}(\mathrm{H}^2(\partial_A))^d$ is essentially normal[1] (Theorem 4.2.24 in [74]), all Hankel operators $H_f = (1 - P)M_f|_{\mathrm{H}^2(\partial_A)}$ with symbol $f \in \mathrm{C}(\partial_A)$ are compact (Lemma 3.9 in [26]) and hence $\mathrm{C}(\partial_A) \subset QC$. For $D = \mathbb{B}_d \subset \mathbb{C}^d$, it is well known (Section 6 in [75]) that QC consists precisely of all bounded measurable functions $f\colon \partial\mathbb{B}_d \to \mathbb{C}$ with vanishing mean oscillation[2]. For $d = 1$, the set QC is the largest C*-algebra contained in $\mathrm{H}^\infty(\mathbb{T}) + \mathrm{C}(\mathbb{T})$ (see [66]). A result of A. M. Davie and N. P. Jewell [21, Proposition 4.1] shows that QC is not contained in $\mathrm{H}^\infty(\partial\mathbb{B}_d) + \mathrm{C}(\partial\mathbb{B}_d)$ for $d > 1$.

For $A = \mathrm{A}\left(\mathbb{D}^d\right)$, the Shilov boundary is the distinguished boundary $\partial_A = \mathbb{T}^d$. If $\mu = \sigma_d$ is the d-fold product measure of the normalized Lebesgue measure σ on $\mathbb{T}$, then $\mathrm{H}^2(\partial_A) = \mathrm{H}^2(\mathbb{T}^d)$, the tuple $T_z \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}^d))^d$ is essentially normal if and only if $d = 1$. For $d > 1$, the C*-algebra QC consists just of the constant functions, that is, $QC = \mathbb{C}$.[3]

Using compactness results of commutators of singular integral operators on spaces of homogeneous type, S. G. Krantz and S-Y. Li [46] proved that also on bounded strictly pseudoconvex domains $D \subset \mathbb{C}^d$ with smooth boundary, the commutators $[M_f^p, P_p]$ are compact for all functions $f \in \mathrm{L}^\infty(\partial D) \cap \mathrm{VMO}(\partial D)$.[4]

[1] A tuple $T = (T_1, \dots, T_m) \in \mathrm{L}(H)^m$ of operators $T_i \in \mathrm{L}(H)$ on a Hilbert space H is called essentially normal if $T_i T_i^* - T_i^* T_i$ is compact for $i = 1, \dots, m$.

[2] The notion of a function with vanishing mean oscillation was first introduced by D. Sarason [67] in 1975. See [75] for a definition that matches our setting.

[3] This result is well known. It follows, for instance, from a result of M. Cotlar and C. Sadosky (see [18, Corollary 5] or [1, Theorem 2.5]), stating that the symbol $f \in \mathrm{L}^\infty(\mathbb{T}^d)$ of a compact Hankel operator $H_f \in \mathrm{L}(\mathrm{H}^2(\mathbb{T}^d))$ must lie in $\mathrm{H}^\infty(\mathbb{T}^d)$.

[4] Here $\mathrm{VMO}(\partial D)$ denotes the space of all functions on ∂D with vanishing mean oscillation.

Using the identity

$$H^p_{fg} = H^p_f T^p_g + (1 - P_p) M^p_f H^p_g \qquad (f, g \in \mathrm{L}^\infty(\partial_A)),$$

one sees that the symbol class

$$\mathcal{A} = \left\{ f \in \mathrm{L}^\infty(\partial_A) \mid H^p_f \text{ is compact} \right\} \subset \mathrm{L}^\infty(\partial_A)$$

forms a norm-closed subalgebra of $\mathrm{L}^\infty(\partial_A)$. Since $H^p_f = (1 - P_p) M^p_f|_{\mathrm{H}^p(\partial_A)}$ is compact if and only if its trivial extension

$$H^p_f P_p = (1 - P_p) M^p_f P_p \in \mathrm{L}(\mathrm{L}^p(\partial_A))$$

is compact, an application of Krasnoselski's interpolation theorem shows exactly as before that also the definition of the symbol class $\mathcal{A}$ does not depend on the choice of the exponent $p \in (1, \infty)$. For the Hardy spaces on the unit circle $\mathbb{T}$, a theorem of Hartman (see Theorem 2.2.5 in [58]) shows that $\mathcal{A} = \mathrm{H}^\infty(\mathbb{T}) + \mathrm{C}(\mathbb{T})$. By the result of A. M. Davie and N. P. Jewell cited above, the inclusion $\mathrm{H}^\infty(\partial\mathbb{B}_d) + \mathrm{C}(\partial\mathbb{B}_d) \subset \mathcal{A}$ is strict in the ball case for $d > 1$.

It is an obvious question whether, for $f \in \mathrm{L}^\infty(\partial_A)$, the condition that $[T^p_f, T^p_g] \in \mathcal{K}(\mathrm{H}^p(\partial_A))$ for all $g \in \mathrm{L}^\infty(\partial_A)$ is also sufficient for f to belong to QC. Under certain additional conditions on the Hardy-type spaces $\mathrm{H}^p(\partial_A)$ we can give an affirmative answer to this question. In the sequel we use standard notation from interpolation theory as explained in the appendix. For a more elaborate exposition of interpolation theory on Banach spaces we refer the reader to [9].

The space $\mathrm{L}^0(\partial_A) = \{f \mid f \colon \partial_A \to \mathbb{C} \text{ is measurable}\} / N$, where measurable means Borel measurable and N consists of all measurable functions vanishing μ-almost everywhere, equipped with the topology induced by the metric

$$d([f], [g]) = \inf \{r > 0 \mid \mu(\{|f - g| \geq r\}) \leq r\}$$

is a Hausdorff topological vector space. A sequence $([f_n])_{n \in \mathbb{N}}$ in $\mathrm{L}^0(\partial_A)$ converges to $[f] \in \mathrm{L}^0(\partial_A)$ if and only if $(f_n)_{n \in \mathbb{N}} \xrightarrow{n} f$ in measure (cf. Proposition A.2.4 in [43]). For $1 < p_0 < p_1 < \infty$, the pair $(\mathrm{L}^{p_0}(\partial_A), \mathrm{L}^{p_1}(\partial_A))$ together with the inclusion mappings

$i_j \colon \mathrm{L}^{p_j}(\partial_A) \to \mathrm{L}^0(\partial_A)$ $(j = 0,1)$ is a compatible couple of Banach spaces. It is well known that their complex interpolation space with parameter $\theta \in (0,1)$ is given by

$$L_\theta = [\mathrm{L}^{p_0}(\partial_A), \mathrm{L}^{p_1}(\partial_A)]_\theta = \mathrm{L}^p(\partial A) \qquad \left(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right) \tag{3.1}$$

with equality of norms (see Theorem 5.1.1 in [9]). Since the measure μ is finite, we have that $\mathrm{L}^{p_0}(\partial_A) + \mathrm{L}^{p_1}(\partial_A) = \mathrm{L}^{p_0}(\partial_A)$ and $\mathrm{L}^{p_0}(\partial_A) \cap \mathrm{L}^{p_1}(\partial_A) = \mathrm{L}^{p_1}(\partial_A)$ as Banach spaces with equivalent norms.

Let $1 < p_0 < p_1 < \infty$. Then the pair $(\mathrm{H}^{p_0}(\partial_A), \mathrm{H}^{p_1}(\partial_A))$ together with the restrictions $\iota_j = i_j|_{\mathrm{H}^{p_j}(\partial_A)}$ of the inclusion mappings i_j from above is a compatible couple of Banach spaces. From Proposition 1.9 we see $\mathrm{H}^{p_0}(\partial_A) + \mathrm{H}^{p_1}(\partial_A) = \mathrm{H}^{p_0}(\partial_A)$ and $\mathrm{H}^{p_0}(\partial_A) \cap \mathrm{H}^{p_1}(\partial_A) = \mathrm{H}^{p_1}(\partial_A)$ as Banach spaces with equivalent norms. Now we know (see the Appendix) that the interpolation spaces $H_\theta = [\mathrm{H}^{p_0}(\partial_A), \mathrm{H}^{p_1}(\partial_A)]_\theta$ and L_θ of the compatible couples $(\mathrm{H}^{p_0}(\partial_A), \mathrm{H}^{p_1}(\partial_A))$ and $(\mathrm{L}^{p_0}(\partial_A), \mathrm{L}^{p_1}(\partial_A))$, respectively, form an interpolation pair of exponent θ for every real number $\theta \in (0,1)$. Since the inclusion operator

$$j \colon (\mathrm{H}^{p_0}(\partial_A), \|\cdot\|_{p_0}) \longrightarrow (\mathrm{L}^{p_0}(\partial_A), \|\cdot\|_{p_0})$$

is an admissible operator for the couples $(\mathrm{H}^{p_0}(\partial_A), \mathrm{H}^{p_1}(\partial_A)), (\mathrm{L}^{p_0}(\partial_A), \mathrm{L}^{p_1}(\partial_A))$, it follows that $j(H_\theta) \subset L_\theta$ and $\|j\|_{\mathrm{L}(H_\theta, L_\theta)} \le 1$.

The following theorem is a slight generalization of Theorem 4.38 in [78].

Theorem 3.5.
Suppose that

$$\mathrm{H}^r(\partial_A) \cap \mathrm{L}^s(\partial_A) = \mathrm{H}^s(\partial_A) \tag{3.2}$$

for $1 < r < s < \infty$. *Then, for* $1 < p_0 < p < p_1 < \infty$ *with* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *for some* $\theta \in (0,1)$,

$$H_\theta = \mathrm{H}^p(\partial_A)$$

with equivalent norms.

Proof. In view of (3.1), for $f \in H_\theta$, we have $f = j(f) \in L_\theta = \mathrm{L}^p(\partial_A)$. Thus $f \in \mathrm{H}^{p_0}(\partial_A) \cap \mathrm{L}^p(\partial_A) = \mathrm{H}^p(\partial_A)$ and $\|f\|_{\mathrm{H}^p} = \|f\|_p \le \|f\|_{H_\theta}$. To prove the reverse inclusion, fix $f \in \mathrm{H}^p(\partial_A) \subset \mathrm{L}^p(\partial_A)$. Then by (3.1) there exists a function $g \in A(\mathrm{L}^{p_0}(\partial_A), \mathrm{L}^{p_1}(\partial_A))$

such that $g(\theta) = f$. Since $g\colon \overline{S} \to \mathrm{L}^{p_0}(\partial_A) + \mathrm{L}^{p_1}(\partial_A) = \mathrm{L}^{p_0}(\partial_A)$ is continuous, also the map

$$h\colon \overline{S} \longrightarrow \mathrm{H}^{p_0}(\partial_A) + \mathrm{H}^{p_1}(\partial_A) = \mathrm{H}^{p_0}(\partial_A), \qquad \xi \longmapsto P_{p_0}(g(\xi))$$

is well-defined and continuous. From the properties of the map g and the identities

$$\begin{aligned} h(it) &= P_{p_0}(g(it)) \in \mathrm{H}^{p_0}(\partial_A), \\ h(1+it) &= P_{p_0}(g(1+it)) = P_{p_0}(i_{p_1 p_0}(g(1+it))) = i_{p_1 p_0} P_{p_1}(g(1+it)) \in \mathrm{H}^{p_1}(\partial_A), \end{aligned}$$

where $t \in \mathbb{R}$, we conclude that $h \in A(\mathrm{H}^{p_0}(\partial_A), \mathrm{H}^{p_1}(\partial_A))$. Moreover, we have

$$h(\theta) = P_{p_0}(g(\theta)) = P_{p_0}(f) = P_{p_0}(i_{pp_0}(f)) = i_{pp_0} P_p(f) = f.$$

Hence $f \in H_\theta$. That the norms $\|\cdot\|_{H_\theta}$ and $\|\cdot\|_p$ on $H_\theta = \mathrm{H}^p(\partial_A)$ are equivalent, follows from the bounded inverse theorem. □

Condition (3.2) of Theorem 3.5 is fulfilled for Hardy spaces $\mathrm{H}^p(\partial D)$ over strictly pseudoconvex domains $D \subset \mathbb{C}^d$ with C^2-boundary (see Corollary 2 in [47]) and for the Hardy spaces $\mathrm{H}^p(\mathbb{T}^d)$ over the unit polydisc $\mathbb{D}^d \subset \mathbb{C}^d$ (see Theorem 2.1.3 (c) and Exercise 3.4.4 (c) in [63]). Cwikels extrapolation result (see Theorem A.3) has the following consequences.

Corollary 3.6.
Suppose that condition (3.2) *from Theorem 3.5 holds. Then, for any symbols* $f_{ji} \in \mathrm{L}^\infty(\partial_A)$ *(*$j = 1, \ldots, s, i = 1, \ldots, p_j$*), the compactness of the operator*

$$T^p = \sum_{j=1}^{s} \prod_{i=1}^{p_j} T^p_{f_{ji}} \in \mathrm{L}(\mathrm{H}^p(\partial_A))$$

does not depend on the choice of the exponent $p \in (1, \infty)$*. If* $\mu \in \mathrm{M}^+(\partial_A)$ *has no atoms and* $T^p \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ *is compact, then* $\sum_{j=1}^{s} \prod_{i=1}^{p_j} f_{ji} = 0$*.*

Proof. Fix $1 < p_0 < p_1 < \infty$ and symbols $f_{ji} \in \mathrm{L}^\infty(\partial_A)$ $(j = 1, \ldots, s, i = 1, \ldots, p_j)$. It suffices to show that the compactness of the operator

$$T^p = \sum_{j=1}^{s} \prod_{i=1}^{p_j} T^p_{f_{ji}} \in \mathrm{L}(\mathrm{H}^p(\partial_A))$$

does not depend on $p \in (p_0, p_1)$. Since the compactness of linear operators is preserved when passing to equivalent norms, the assertion follows from Theorem 3.5 and Theorem A.3. If $T^p \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ is compact, then by Corollary 2.15 and Corollary 2.20 it follows that

$$\sum_{j=1}^{s} \prod_{i=1}^{p_j} f_{ji} = \sigma(T^p) = 0.$$

□

Corollary 3.7.
Suppose that condition (3.2) *from Theorem 3.5 holds,* $\mu \in \mathrm{M}^+(\partial_A)$ *has no atoms and that* $T_z = (T_{z_1}, \ldots, T_{z_d}) \in \mathrm{L}(\mathrm{H}^2(\partial_A))^d$ *is essentially normal. Then, for* $1 < p < \infty$,

$$QC = \left\{ f \in \mathrm{L}^\infty(\partial_A) \;\middle|\; [T^p_f, T^p_g] \in \mathcal{K}(\mathrm{H}^p(\partial_A)) \text{ for all } g \in \mathrm{L}^\infty(\partial_A) \right\}$$

and

$$\mathcal{A} = \left\{ f \in \mathrm{L}^\infty(\partial_A) \;\middle|\; [T^p_f, T^p_g] \in \mathcal{K}(\mathrm{H}^p(\partial_A)) \text{ for all } g \in \mathrm{H}^\infty(\partial_A) \right\}.$$

Proof. Suppose that $f \in \mathrm{L}^\infty(\partial_A)$ satisfies $[T^p_f, T^p_g] \in \mathcal{K}(\mathrm{H}^p(\partial_A))$ for all $g \in \mathrm{L}^\infty(\partial_A)$. By Corollary 3.6 the same holds for $p = 2$. Since $T_z \in \mathrm{L}(\mathrm{H}^2(\partial_A))^d$ is an essentially normal regular A-isometry in the sense of [26] and since in particular $[T_f, T_g] \in \mathcal{K}(\mathrm{H}^2(\partial_A))$ for all $g \in \mathrm{H}^\infty(\partial_A)$, it follows from Corollary 5.1 in [34] that there exist a function $g \in \mathcal{A}$ and an operator $K \in \mathcal{K}(\mathrm{H}^2(\partial_A))$ such that $T_f = T_g + K$. Using Theorem 2.9 and Corollary 2.20 we find that

$$T_f = \Phi_{T_z}(T_g + K) = T_g.$$

But then Theorem 2.21 shows that $f = g \in \mathcal{A}$. Choose $q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. since

$$[T^q_{\overline{f}}, T^q_{\overline{g}}] = -[T^p_f, T^p_g]^* \in \mathcal{K}(\mathrm{H}^q(\partial_A))$$

for all $g \in \mathrm{L}^\infty(\partial_A)$, also $\overline{f} \in \mathcal{A}$. It follows from Corollary 3.3 that $f \in QC$. The second assertion follows from the first part of the proof. □

For a strictly pseudoconvex domain $D \subset \mathbb{C}^d$ with smooth boundary and $\mu \in \mathrm{M}^+(\partial D)$ equal to the surface measure of ∂D the tuple $T_z = (T_{z_1}, \ldots, T_{z_d}) \in \mathrm{L}(\mathrm{H}^2(\partial D))^d$ is essentially normal and therefore all assumptions of Corollary 3.7 are fulfilled for these types of Hardy spaces.

As in Section 2.2, for a unital closed subalgebra $\mathcal{B} \subset \mathrm{L}^\infty(\partial_A)$, let us denote by

$$\mathcal{T}^p(\mathcal{B}) = \overline{\mathrm{alg}}\left\{T^p_f \;\middle|\; f \in \mathcal{B}\right\} \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

the smallest norm-closed subalgebra containing $\left\{T^p_f \;\middle|\; f \in \mathcal{B}\right\}$.

Theorem 3.8.
Suppose that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms.

(a) The subset

$$\mathcal{T}^p(\mathcal{A}) + \mathcal{K}(\mathrm{H}^p(\partial_A)) = \left\{T^p_f \;\middle|\; f \in \mathcal{A}\right\} \oplus \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

is the smallest norm-closed subalgebra containing $\left\{T^p_f \;\middle|\; f \in \mathcal{A}\right\} \cup \mathcal{K}(\mathrm{H}^p(\partial_A))$.

(b) The mapping

$$\rho_p\colon \mathcal{A} \longrightarrow \left(\mathcal{T}^p(\mathcal{A}) + \mathcal{K}(\mathrm{H}^p(\partial_A))\right)/\mathcal{K}(\mathrm{H}^p(\partial_A)), \qquad f \longmapsto [T^p_f]$$

is a topological algebra isomorphism onto an inverse closed[5] Banach subalgebra of the Calkin algebra $\mathcal{C}(\mathrm{H}^p(\partial_A))$.

Proof. Fix a real number $p \in (1, \infty)$. Obviously the map

$$\rho\colon\ \mathrm{L}^\infty(\partial_A) \longrightarrow \mathcal{C}(\mathrm{H}^p(\partial_A)), \qquad f \longmapsto [T^p_f]$$

is well-defined and continuous linear. It follows from Theorem 2.21 that ρ is bounded below

$$\left\|[T^p_f]\right\| \geq r([T^p_f]) \geq \|f\|_\infty \qquad (f \in \mathrm{L}^\infty(\partial_A)).$$

Using the identity (see the proof of Corollary 3.4)

$$T^p_g T^p_f - T^p_{gf} = -P_p M^p_g H^p_f \qquad (f, g \in \mathrm{L}^\infty(\partial_A)),$$

[5]A subalgebra $B \subset A$ of an unital algebra A is called inverse closed if $f^{-1} \in B$ holds whenever $f \in B$ has an inverse $f^{-1} \in A$

we find that ρ induces a topological isomorphism of Banach algebras

$$\mathcal{A} \longrightarrow \rho(\mathcal{A}), \qquad f \longmapsto \rho(f) = [T_f^p].$$

Let $\pi\colon \mathrm{L}(\mathrm{H}^p(\partial_A)) \to \mathcal{C}(\mathrm{H}^p(\partial_A))$ be the quotient map. Then

$$B = \pi^{-1}(\rho(\mathcal{A})) = \big\{T_f^p \;\big|\; f \in \mathcal{A}\big\} + \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

is a norm-closed unital subalgebra. Since by Corollary 2.22 there exist no compact Toeplitz operators T_f^p with $f \in \mathrm{L}^\infty(\partial_A) \setminus \{0\}$, the latter sum is direct. It follows that

$$B = \mathcal{T}^p(\mathcal{A}) + \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

is the smallest norm-closed subalgebra containing $\big\{T_f^p \;\big|\; f \in \mathcal{A}\big\} \cup \mathcal{K}(\mathrm{H}^p(\partial_A))$ and that

$$\rho_p\colon \mathcal{A} \longrightarrow B/\mathcal{K}(\mathrm{H}^p(\partial_A)), \qquad f \longmapsto \rho(f) = [T_f^p]$$

is a topological isomorphism of Banach algebras. It remains to be shown that

$$B/\mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathcal{C}(\mathrm{H}^p(\partial_A))$$

is inverse closed. Let $[T] \in B/\mathcal{K}(\mathrm{H}^p(\partial_A))$. Then there is a function $f \in \mathcal{A}$ with $[T] = [T_f^p]$. Suppose that there is an operator $X \in \mathrm{L}(\mathrm{H}^p(\partial_A))$ such that $XT_f^p - 1 \in \mathcal{K}(\mathrm{H}^p(\partial_A))$. Using the definitions of the mappings Φ_{T_z} and $\hat{\pi}$ given in Section 2.2 we find that

$$\Phi_{T_z}(X) = P_p\hat{\pi}(X)|_{\mathrm{H}^p(\partial_A)}.$$

Hence an application of Theorem 2.14 yields that

$$\begin{aligned}
\Phi_{T_z}(XT_f^p) &= P_p\hat{\pi}(X)\hat{\pi}(T_f^p)|_{\mathrm{H}^p(\partial_A)} \\
&= P_p\hat{\pi}(X)P_p\hat{\pi}(T_f^p)|_{\mathrm{H}^p(\partial_A)} + P_p\hat{\pi}(X)(1-P_p)\hat{\pi}(T_f^p)|_{\mathrm{H}^p(\partial_A)} \\
&= \Phi_{T_z}(X)T_f^p + P_p\hat{\pi}(X)H_f^p.
\end{aligned}$$

Since Φ_{T_z} annihilates the compact operators and since H_f^p is compact, we obtain

$$1 - \Phi_{T_z}(X)T_f^p = \Phi_{T_z}(XT_f^p) - \Phi_{T_z}(X)T_f^p \in \mathcal{K}(\mathrm{H}^p(\partial_A)).$$

Choose a function $g \in \mathrm{L}^\infty(\partial_A)$ with $\Phi_{T_z}(X) = T_g^p$. Then by the last part of Theorem 2.14

$$T_{gf}^p = \Phi_{T_z}(T_{gf}^p + (T_g^p T_f^p - T_{gf}^p)) = \Phi_{T_z}(T_g^p T_f^p + (1 - \Phi_{T_z}(X) T_f^p)) = 1$$

and hence $gf = 1$. The identity

$$0 = H_{gf}^p = H_g^p T_f^p + (1 - P_p) M_g^p H_f^p$$

shows that $H_g^p T_f^p$ is compact. Now let us suppose in addition that $[T] \in B/\mathcal{K}(\mathrm{H}^p(\partial_A))$ is invertible in $\mathcal{C}(\mathrm{H}^p(\partial_A))$. Then also $K = T_f^p X - 1 \in \mathcal{K}(\mathrm{H}^p(\partial_A))$ and therefore

$$H_g^p = H_g^p T_f^p X - H_g^p K$$

is compact. It follows that $[T_g^p] \in B/\mathcal{K}(\mathrm{H}^p(\partial_A))$ and

$$[T_g^p][T] = [\Phi_{T_z}(X)][T_f^p] = [1].$$

Hence $[T]$ is also invertible in $B/\mathcal{K}(\mathrm{H}^p(\partial_A))$. □

Combining Corollary 2.15 and Theorem 3.8 we get the following corollary.

Corollary 3.9.
Suppose that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms and let $\mathcal{B} \subset \mathcal{A}$ be a unital closed subalgebra. Then the subset

$$\mathcal{T}^p(\mathcal{B}) + \mathcal{K}(\mathrm{H}^p(\partial_A)) = \left\{T_f^p \;\middle|\; f \in \mathcal{B}\right\} \oplus \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$$

is the smallest norm-closed subalgebra containing $\left\{T_f^p \;\middle|\; f \in \mathcal{B}\right\} \cup \mathcal{K}(\mathrm{H}^p(\partial_A))$ and the equality

$$\mathcal{SC}^p(\mathcal{B}) = \mathcal{T}^p(\mathcal{B}) \cap \mathcal{K}(\mathrm{H}^p(\partial_A))$$

holds.

Proof. If one replaces $\mathcal{A}$ by $\mathcal{B}$ in the proof of part (a) of Theorem 3.8 one gets the first assertion. Then by Corollary 2.15 we have the direct sum inclusion

$$\mathcal{T}^p(\mathcal{B}) = \left\{T_f^p \;\middle|\; f \in \mathcal{B}\right\} \oplus \mathcal{SC}^p(\mathcal{B}) \subset \left\{T_f^p \;\middle|\; f \in \mathcal{B}\right\} \oplus \mathcal{K}(\mathrm{H}^p(\partial_A)).$$

Thus, for fixed $T \in \mathcal{SC}^p(\mathcal{B})$, there exist $f \in \mathcal{B}$, $K \in \mathcal{K}(\mathrm{H}^p(\partial_A))$ such that $T = T_f^p + K$. Then $K = T - T_f^p \in \mathcal{T}^p(\mathcal{B})$ and since $\ker \Phi_{T_z}|_{\mathcal{T}^p(\mathcal{B})} = \mathcal{SC}^p(\mathcal{B})$ (see Corollary 2.15) Corollary 2.20 yields $K \in \mathcal{SC}^p(\mathcal{B})$. This means $T_f^p = 0$, that is $T \in \mathcal{T}^p(\mathcal{B}) \cap \mathcal{K}(\mathrm{H}^p(\partial_A))$. By Corollary 2.20 we have

$$\mathcal{T}^p(\mathcal{B}) \cap \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \ker \Phi_{T_z}|_{\mathcal{T}^p(\mathcal{B})} = \mathcal{SC}^p(\mathcal{B}),$$

which proves the reverse inclusion. □

On a strictly pseudoconvex domain $D \subset \mathbb{C}^d$ with smooth boundary ∂D (equipped with its surface measure) the algebra $\mathcal{T}(\mathcal{A}) = \overline{\mathrm{alg}}\,\{T_f \mid f \in \mathcal{A}\} \subset \mathrm{L}(\mathrm{H}^2(\partial D))$ contains all compact operators and $\mathcal{T}(\mathcal{A})/\mathcal{K}(\mathrm{H}^2(\partial D)) \subset \mathcal{C}(\mathrm{H}^2(\partial D))$ is a maximal abelian subalgebra (see [25]). Although we have not been able to decide wether the subalgebra $\mathcal{T}^p(\mathcal{A}) + \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathcal{C}(\mathrm{H}^p(\partial_A))$ is maximal abelian, the weaker result proved in Theorem 3.8 still yields the following spectral mapping results.

Corollary 3.10.
Assume that $\mu \in \mathrm{M}^+(\partial_A)$ has no atoms and let $f \in \mathcal{A}$ and $g \in QC$ be arbitrary functions. Then we have:

(a) $\sigma_{\mathrm{e}}(T_f^p) = \sigma_{\mathcal{A}}(f)$,

(b) $\sigma_{\mathrm{e}}(T_g^p) = \sigma_{\mathcal{A}}(g) = \sigma_{QC}(g) = R(g)$.

Proof. By Theorem 3.8 $B = \mathcal{T}^p(\mathcal{A}) + \mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathrm{L}(\mathrm{H}^p(\partial_A))$ is a closed subalgebra such that the quotient $B/\mathcal{K}(\mathrm{H}^p(\partial_A)) \subset \mathcal{C}(\mathrm{H}^p(\partial_A))$ is inverse closed and topologically isomorphic to the Banach algebra $\mathcal{A}$ via the map

$$\rho_p \colon \mathcal{A} \longrightarrow B/\mathcal{K}(\mathrm{H}^p(\partial_A)), \qquad h \longmapsto [T_h^p].$$

Thus, for $f \in \mathcal{A}$, $\sigma_{\mathrm{e}}(T_f^p) = \sigma([T_f^p]) = \sigma_{\mathcal{A}}(f)$. Using Theorem 2.21 and the fact that $QC \subset \mathcal{A}$ is a C*-subalgebra of $\mathrm{L}^\infty(\partial_A)$, we find that

$$R(g) \subset \sigma_{\mathrm{e}}(T_g^p) = \sigma_{\mathcal{A}}(g) \subset \sigma_{QC}(g) = \sigma_{\mathrm{L}^\infty(\partial_A)}(g) = R(g).$$

Thus also the second assertion has been proved. □

In the particular case of the unit ball, part (a) of Corollary 3.10 can be used to improve the essential spectral inclusion theorem (Theorem 2.21) for Toeplitz operators T_f^p with symbol $f \in \mathcal{A}$. For $f \in \mathrm{L}^\infty(S)$, let $F = P[f]\colon \mathbb{B} \to \mathbb{C}$,

$$P[f](z) = \int_S f(\xi) \frac{(1-|z|^2)^d}{|1-\langle z, \xi \rangle|^{2d}} \, d\sigma(\xi)$$

be its Poisson-Szegő transform. The set

$$\mathrm{Cl}(F) = \bigcap \left(\overline{F(U \cap \mathbb{B})}; U \supset S \text{ open} \right),$$

known as the cluster set of F, is easily seen to contain the essential range of f. Making explicit use of the rich conformal group of the unit ball it was shown in [32] that $\mathrm{Cl}(F) \subset \sigma_{\mathcal{A}}(f)$ for all $f \in \mathcal{A}$. Thus Corollary 3.10 shows that

$$\mathrm{Cl}(F) \subset \sigma_{\mathrm{e}}(T_f^p) \qquad (f \in \mathcal{A}).$$

Let $f \in \mathrm{H}^\infty(S)$ be a non-constant inner function and $F = P[f] \in \mathrm{H}^\infty(\mathbb{B})$. For $d > 1$, it is well known (see Theorem 1.2 in [65]) that $\mathrm{Cl}(F) = \overline{\mathbb{D}}$ while $R(f) \subset \mathbb{T}$. In this case, the Toeplitz operator T_f^p is an isometry and $R(f) \subsetneq \sigma_{\mathrm{e}}(T_f^p) = \overline{\mathbb{D}}$. This example also shows that the algebra $\mathcal{A} \subset \mathrm{L}^\infty(S)$ is not inverse closed.

One can use Corollary 3.10 and arguments from [32] to calculate the maximal ideal space of the commutative C*-algebra $QC \subset \mathrm{L}^\infty(S)$. Let $\beta(\mathbb{B})$ be the Stone-Čech compactification[6] of the unit ball $\mathbb{B} \subset \mathbb{C}^d$. For $f \in \mathrm{L}^\infty(S)$, the Poisson-Szegő transform $F = P[f]\colon \mathbb{B} \to \mathbb{C}$ has a unique extension to a continuous function $F^\beta\colon \beta(\mathbb{B}) \to \mathbb{C}$. In [32] it is shown that every point $\Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}$ gives rise to a multiplicative linear functional

$$\delta_\Lambda\colon \mathcal{A} \longrightarrow \mathbb{C}, \qquad f \longmapsto F^\beta(\Lambda)$$

and that $\{\delta_\Lambda \mid \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\} \subset \Delta_{\mathcal{A}}$ is a closed subset of the maximal ideal space $\Delta_{\mathcal{A}}$ of the commutative Banach algebra $\mathcal{A}$ such that

$$\mathrm{Cl}(F) = \{\delta_\Lambda(f) \mid \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\}$$

[6] see §38 in [56]

for all $f \in \mathcal{A}$ (Theorem 6 in [32]). For $d = 1$, the equality

$$\Delta_{\mathcal{A}} = \{\delta_\Lambda \mid \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\}$$

holds (Theorem 7 in [32]). It is an open question whether this equality holds for $d > 1$. We conclude this section by showing that the above multiplicative linear functionals at least determine the maximal ideal space of the commutative C*-algebra QC.

Corollary 3.11.
The maximal ideal space of the commutative C*-*algebra QC is given by*

$$\Delta_{QC} = \{\delta_\Lambda|_{QC} \mid \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\}\,.$$

Proof. We endow the maximal ideal spaces $\Delta_{\mathcal{A}}$ and Δ_{QC} with their corresponding Gelfand topologies. In the proof of Theorem 6 from [32] it is shown that the set $\beta(\mathbb{B}) \setminus \mathbb{B} \subset \beta(\mathbb{B})$ is compact and that the mapping

$$j\colon\ \beta(\mathbb{B}) \setminus \mathbb{B} \longrightarrow \Delta_{\mathcal{A}}, \qquad \Lambda \longmapsto \delta_\Lambda$$

is continuous. Since the restriction mapping

$$r\colon \Delta_{\mathcal{A}} \longrightarrow \Delta_{QC}, \qquad \delta \longmapsto \delta|_{QC}$$

is also continuous, the set

$$\Delta = \{\delta_\Lambda|_{QC} \mid \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\} = (r \circ j)(\beta(\mathbb{B}) \setminus \mathbb{B})$$

is a compact subset of the maximal ideal space Δ_{QC}. Using Theorem 6 from [32] and Corollary 3.10 we find that

$$\hat{f}(\Delta_{QC}) = \sigma_{QC}(f) = R(f) \subset \mathrm{Cl}(F) = \hat{f}(\Delta) \subset \hat{f}(\Delta_{QC})$$

for all $f \in QC$, where $\hat{f} \in \mathrm{C}(\Delta_{QC})$ denotes the Gelfand transform of f with respect to the C*-algebra QC. Since $\{\hat{f} \mid f \in QC\} = \mathrm{C}(\Delta_{QC})$, an application of Urysohn's Lemma now yields $\Delta_{QC} = \Delta$. □

4 Spectral Mapping Theorems for Toeplitz Tuples

4.1 Essential Spectra of Toeplitz Tuples with Symbol in $H^\infty + C$

In [31] Eschmeier proved the following spectral mapping formula for essentially commuting tuples of Toeplitz operators on the Hardy space $\mathrm{H}^2(\partial D)$ over a strictly pseudoconvex domain $D \subseteq \mathbb{C}^d$ with C^∞-boundary.

Theorem (Eschmeier, 2012).
For $f \in (H^\infty + C)^m$, the formula

$$\sigma_{\mathrm{e}}(T_f) = \bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \text{ open} \right)$$

holds.

Here $F = (F_1, \ldots, F_m) = (P[f_1], \ldots, P[f_m])$ denotes the Poisson-Szegő transform of a tuple $f = (f_1, \ldots, f_m) \in \mathrm{L}^\infty(\partial D)^m$. By using Proposition 3.4 and by imitating the proofs of [31] and [52], respectively, we will show that the above spectral formula still holds true for Toeplitz tuples with symbols in $\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D)$ on the Banach space $\mathrm{H}^p(\partial D)$ $(1 < p < \infty)$.

Let $D \subset \mathbb{C}^d$ be a strictly pseudoconvex domain with C^∞-boundary. Then there is a strictly plurisubharmonic defining function for D, that is, a strictly plurisubharmonic function $r \in \mathrm{C}^\infty(\mathbb{C}^d, \mathbb{R})$ with $\operatorname{grad} r(z) \neq 0$ for all $z \in \partial D$ and

$$D = \left\{ z \in \mathbb{C}^d \;\middle|\; r(z) < 0 \right\}.$$

Let $G_z : \overline{D} \backslash \{z\} \to \mathbb{R}$ be the Green's function[1] for D with pole $z \in D$. For $y \in \partial D$, denote by n_y the outward unit normal to ∂D at y. Let $U \supset \partial D$ be an open neighbourhood such that $\operatorname{grad} r(z) \neq 0$ for all $z \in U$. Then $D \cap U^c = \overline{D} \cap U^c$ is compact and $t_0 = -\sup\{r(z) \mid z \in D \cap U^c\} > 0$. For $0 < t < t_0$, the set

$$D_t = \left\{z \in \mathbb{C}^d \;\middle|\; r(z) < -t\right\} \subset \mathbb{C}^d$$

is open with $\overline{D}_t \subset D$ and $\partial D_t \subset U$. Thus $D_t \subset \mathbb{C}^d$ is a strictly pseudoconvex domain with C^∞-boundary and defining function $r_t = r + t$. In particular, for $0 < t < t_0$, the boundary $\partial D_t \subset \mathbb{C}^d$ is a compact real hypersurface of class C^∞. Let us denote by $\sigma_t \in \mathrm{M}^+(\partial D_t)$ and $\sigma \in \mathrm{M}^+(\partial D)$ the normalized surface measures.

For $1 \leq p < \infty$, we define the Hardy space $\mathrm{H}^p(D)$ as the set of all functions $f \in \mathcal{O}(D)$ for which

$$\sup_{0<t<t_0} \int_{\partial D_t} |f(\xi)|^p \, d\sigma_t(\xi) < \infty.$$

Then $\mathrm{H}^p(D) \subset \mathcal{O}(D)$ is a linear subspace independent of the choices of r and t_0. Via a suitable boundary map the vector space $\mathrm{H}^p(D)$ is isomorphic to a norm-closed subspace $\mathrm{H}^p(\partial D) \subset \mathrm{L}^p(\partial D, \sigma)$. More precisely, for $\alpha > 1$ and $w \in \partial D$, define the approach regions

$$\Gamma_\alpha(w) = \{z \in D \mid |z - w| < \alpha \operatorname{dist}(z, \partial D)\}.$$

For each $f \in \mathrm{H}^p(D)$, there is a function $f^* \in \mathcal{L}^p(\partial D, \sigma)$ such that

$$\lim_{\substack{z \to w \\ z \in \Gamma_\alpha(w)}} f(z) = f^*(w)$$

for σ-almost every $w \in \partial D$. The subset $\mathrm{H}^p(\partial D) \subset \mathrm{L}^p(\partial D, \sigma)$ consisting of all equivalence classes of functions f^* arising in this way is a norm-closed subspace and the map

$$r^p : \ \mathrm{H}^p(D) \longrightarrow \mathrm{H}^p(\partial D), \qquad f \longmapsto [f^*]$$

is a vector-space isomorphism. Equipped with the norm $\|f\|_{\mathrm{H}^p(D)} = \|f^*\|_{\mathrm{L}^p(\partial D, \sigma)}$ the spaces $\mathrm{H}^p(D)$ become Banach spaces such that convergence in $\mathrm{H}^p(D)$ implies uniform convergence on all compact subsets. The inverse of $r^p : \mathrm{H}^p(D) \to \mathrm{H}^p(\partial D)$ is given by

[1] see [42] for a description of potential theory in $\mathbb{R}^n$.

the Poisson integral from real potential theory

$$(r^p)^{-1}(f)(z) = \mathcal{P}[f](z) = \int_{\partial D} f(w)\mathcal{P}(z,w)\, d\sigma(w) \qquad (f \in \mathrm{H}^p(\partial D), z \in D),$$

where

$$\mathcal{P}\colon D \times \partial D \longrightarrow \mathbb{R}, \qquad (z,w) \longmapsto \frac{\partial G_z}{\partial n_w}(w)$$

denotes the Poisson kernel of D. If $f \in \mathrm{C}(\partial D)$ and $u(z) = \mathcal{P}[f](z)$ $(z \in D)$, then u is harmonic on D and extends to a continuous function $u\colon \overline{D} \to \mathbb{C}$ with $u|_{\partial D} = f$. By Theorem 10 in [47] the subspace $\mathrm{A}\,(D)\,|_{\partial D} \subset \mathrm{H}^p(\partial D)$ is dense. As explained in Section 1.3 the spaces $\mathrm{H}^p(\partial D)$ are Hardy-type spaces satisfying all conditions listed in the section leading to Proposition 1.9.

The space $\mathrm{H}^2(D)$ equipped with the norm $\|f\|_2 = \|f^*\|_{\mathrm{L}^2(\partial D,\sigma)}$ is a functional Hilbert space. Its reproducing kernel is given by the Szegő kernel $S \in \mathrm{C}(D \times D)$ of D. Since every boundary point $p \in \partial D$ of D is of finite type in the sense of J. P. D'Angelo (see Definition 2.18 and Corollary 5.8 in [20]) the following proposition is an immediate consequence of a result of H. P. Boas [11, Corollary 5.2].

Proposition 4.1.
For every $p \in \partial D$, there exists an open neighbourhood $U \subset \mathbb{C}^d$ of p such that $S|_{(U\cap D)\times D}$ extends to a continuous function

$$S \in \mathrm{C}((U \cap \overline{D}) \times \overline{D} \setminus \{(w,z) \in \partial D \times \partial D \mid w = z\}).$$

From this we see that the Szegő kernel $S \in \mathrm{C}(D \times D)$ extends to a continuous function $S\colon \overline{D} \times D \to \mathbb{C}$. It follows that, for every $z \in D$, $S_z = S(\,\cdot\,,z)|_{\partial D} \in \mathrm{A}\,(D)\,|_{\partial D}$. We need the following additional properties of the Szegő kernel.

Lemma 4.2.

(i) For each $\lambda \in \partial D$, there is a real number $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$,

$$\sup\left\{|S(w,z)| \;\middle|\; (z,w) \in (\mathrm{B}_{\frac{\delta}{2}}(\lambda) \cap D) \times (\mathrm{B}_\delta(\lambda)^c \cap \partial D)\right\} < \infty.$$

(ii) For every $1 < q < \infty$, we have $\lim_{z\to\partial D} \|S_z\|_q = \infty$.

Proof. For $\lambda \in \partial D$, use Proposition 4.1 to choose an open neighbourhood $U \subset \mathbb{C}^d$ of λ such that $S \in \mathrm{C}((U \cap \overline{D}) \times \overline{D} \setminus \{(w,z) \in \partial D \times \partial D \mid w = z\})$. The observation that the compact set $\overline{\mathrm{B}_{\frac{\delta}{2}}(\lambda) \cap D} \times (\mathrm{B}_\delta(\lambda)^c \cap \partial D)$ is contained in the domain of the continuous function S for all sufficiently small numbers $\delta > 0$ finishes the proof of part (i).

Fix $1 < q < \infty$ and assume that there exists a sequence $(z_k)_{k\in\mathbb{N}}$ in D converging to some boundary point $z \in \partial D$ such that $\sup_{k\in\mathbb{N}} \|S_{z_k}\|_q < \infty$. Then by the theorem of Alaoglu-Bourbaki and the separability of $\mathrm{L}^p(\partial D)$, where $p \in (1,\infty)$ is chosen such that $\frac{1}{p} + \frac{1}{q} = 1$, we may suppose that the sequence $(S_{z_k})_{k\in\mathbb{N}}$ converges to some function $u \in \mathrm{L}^q(\partial D)$ in the weak topology of $\mathrm{L}^q(\partial D)$. Then, for $g \in \mathrm{H}^\infty(D)$, we have

$$\begin{aligned}\langle r^p(g), u\rangle &= \lim_{k\to\infty} \langle r^p(g), S_{z_k}\rangle = \lim_{k\to\infty} \left\langle r^2(g), r^2(S(\,\cdot\,, z_k))\right\rangle_{L^2,L^2} \\ &= \lim_{k\to\infty} \langle g, S(\,\cdot\,, z_k)\rangle = \lim_{k\to\infty} g(z_k).\end{aligned}$$

Since every boundary point of the smoothly bounded strictly pseudoconvex domain D is a peak point[2], there is a function $f \in \mathrm{H}^\infty(D)$ with $f(D) \subset \mathbb{D}$ and $\lim_{w\to z} f(w) = 1$. By passing to a subsequence we can achieve that $(f(z_k))_{k\in\mathbb{N}}$ is an interpolating sequence for $\mathrm{H}^\infty(\mathbb{D})$[3], that is,

$$\{((g \circ f)(z_k))_{k\in\mathbb{N}} \mid g \in \mathrm{H}^\infty(\mathbb{D})\} = l^\infty(\mathbb{N}).$$

But then

$$\{(g(z_k))_{k\in\mathbb{N}} \mid g \in \mathrm{H}^\infty(D)\} = l^\infty(\mathbb{N})$$

which is impossible, since as shown above $\lim_{k\to\infty} g(z_k) = \langle r^p(g), u\rangle$ for all $g \in \mathrm{H}^\infty(D)$. □

The Poisson-Szegő transform $F = P[f] \colon D \to \mathbb{C}$ of a function $f \in \mathrm{L}^2(\partial D)$ is defined by

$$P[f](z) = \int_{\partial D} P(z,\xi) f(\xi)\, d\sigma(\xi) \qquad (z \in D),$$

where

$$P \colon D \times \partial D \to \mathbb{R}, \qquad (z,\xi) \longmapsto \frac{|S_z(\xi)|^2}{S(z,z)}$$

[2] see Theorem 2.3 in [59]
[3] see Theorem 9.2 in [30]

denotes the Poisson-Szegő kernel of D. The Poisson-Szegő transform reproduces functions in $\mathrm{H}^2(D)$, that is $P[r^2(f)] = f$ for all $f \in \mathrm{H}^2(D)$ (see [70, p. 19]). Therefore $P[r^2(f)] = \mathcal{P}[r^2(f)]$ for every $f \in \mathrm{H}^2(D)$. We want to point out that, for arbitrary dimension $d > 1$, the kernels P and $\mathcal{P}$ are different from each other even in the case of the unit ball $\mathbb{B} \subset \mathbb{C}^d$ (see [70] for more details on this topic). We use the notation $F = (F_1, \dots, F_m) = (\mathcal{P}[f_1], \dots, \mathcal{P}[f_m])$ to denote the Poisson transform of a tuple $f = (f_1 \dots, f_m) \in \mathrm{L}^\infty(\partial D)^m$ $(m \geq 1)$. Let $1 < p < \infty$ be given. For $z \in D$, we define continuous linear functionals

$$\varepsilon_z^{(p)} \colon \mathrm{H}^p(\partial D) \to \mathbb{C}, \qquad f \longmapsto F(z).$$

Theorem 4.3.
Let $p, q \in (1, \infty)$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. For $z \in D$, the functions $k_z^{(q)} = \left\|\varepsilon_z^{(p)}\right\|^{-1} S_z \in \mathrm{H}^q(\partial D)$ satisfy:

(a) There is a constant $c_q > 0$ such that

$$c_q^{-1} \leq \left\|k_z^{(q)}\right\|_q \leq c_q$$

for all $z \in D$.

(b) For $f \in \mathrm{H}^\infty(\partial D)$ and $F = \mathcal{P}[f] \colon D \to \mathbb{C}$, we have

$$(T_f^p)^* k_z^{(q)} = \overline{F(z)} k_z^{(q)} \qquad (z \in D).$$

(c) The functions $k_z^{(q)}$ converge to 0 weakly in $\mathrm{H}^q(\partial D)$ as $z \to \partial D$.

Proof. Since, for $f \in \mathrm{A}(D)$,

$$\begin{aligned} \rho_H(S_z)(f|_{\partial D}) &= \int_{\partial D} f(\xi)\overline{S_z(\xi)}\, d\sigma(\xi) = \langle f|_{\partial D}, S_z\rangle_{\mathrm{H}^2(\partial D)} \\ &= \left\langle r^2(f|_D), r^2(S(\,\cdot\,, z))\right\rangle_{\mathrm{H}^2(\partial D)} = \langle f|_D, S(\,\cdot\,, z)\rangle_{\mathrm{H}^2(D)} \\ &= f(z) = \varepsilon_z^{(p)}(f|_{\partial D}), \end{aligned}$$

it follows that $\rho_H(S_z) = \varepsilon_z^{(p)}$. Since $\rho_H \colon \mathrm{H}^q(\partial D) \to \mathrm{H}^p(\partial D)'$ is a topological isomorphism, there is a constant $c_q > 0$ such that

$$\frac{1}{c_q}\left\|\varepsilon_z^{(p)}\right\| \leq \|S_z\|_q \leq c_q\left\|\varepsilon_z^{(p)}\right\|.$$

This shows part (a). For $f \in \mathrm{H}^\infty(\partial D)$ and $g \in \mathrm{A}\,(D)\,|_{\partial D}$, the computation

$$\begin{aligned}\left\langle g, (T_f^p)^* k_z^{(q)}\right\rangle_{\mathrm{H}^p,\mathrm{H}^q} &= \left\langle fg, k_z^{(q)}\right\rangle_{\mathrm{H}^p,\mathrm{H}^q} = \frac{1}{\left\|\varepsilon_z^{(p)}\right\|}\left\langle fg, S_z\right\rangle_{\mathrm{H}^p,\mathrm{H}^q} \\ &= \frac{1}{\left\|\varepsilon_z^{(p)}\right\|}\varepsilon_z^{(p)}(fg) = \frac{1}{\left\|\varepsilon_z^{(p)}\right\|}\mathcal{P}[fg](z) = \frac{1}{\left\|\varepsilon_z^{(p)}\right\|}\mathcal{P}[f](z)\mathcal{P}[g](z) \\ &= \frac{1}{\left\|\varepsilon_z^{(p)}\right\|}F(z)G(z) = \left\langle g, \overline{F(z)}k_z^{(q)}\right\rangle\end{aligned}$$

shows that

$$(T_f^p)^* k_z^{(q)} = \overline{F(z)}k_z^{(q)}.$$

Furthermore, for $f \in \mathrm{A}\,(D)$ and any sequence $(z_k)_{k\in\mathbb{N}}$ in D converging to some boundary point $z \in \partial D$, part (ii) of Lemma 4.2 yields

$$\left\langle f|_{\partial D}, k_{z_k}^{(q)}\right\rangle = \frac{1}{\left\|\varepsilon_{z_k}^{(p)}\right\|}\left\langle f|_{\partial D}, S_{z_k}\right\rangle = \frac{f(z_k)}{\left\|\varepsilon_{z_k}^{(p)}\right\|} \xrightarrow{k\to\infty} 0.$$

Since $\mathrm{A}\,(D)\,|_{\partial D} \subset \mathrm{H}^p(\partial D)$ is dense, it follows that $\left(k_{z_k}^{(q)}\right)_{k\in\mathbb{N}}$ is a weak zero sequence in $\mathrm{H}^q(\partial D)$. □

Since

$$\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D) \subset \mathcal{A} = \{f \in \mathrm{L}^\infty(\partial D) \mid H_f \text{ is compact}\},$$

Theorem 3.8 implies that the tuple $T_f^p = (T_{f_1}^p, \ldots, T_{f_m}^p) \in \mathrm{L}(\mathrm{H}^p(\partial D))^m$ is essentially commuting for every symbol tuple $f = (f_1, \ldots, f_m) \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$. Hence its essential Taylor spectrum is defined as

$$\sigma_{\mathrm{e}}(T_f^p) = \sigma((T_f^p)^e),$$

where $(T_f^p)^e \in \mathrm{L}(\mathrm{H}^p(\partial D)^e)^m$ is a commuting tuple on the Banach space $\mathrm{H}^p(\partial D)^e = \mathrm{H}^p(\partial D)^\infty/\mathrm{H}^p(\partial D)^{pc}$ (cf. Section 1.2). In a first step, we show the following inclusion formula.

Lemma 4.4.
For $f \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$, the inclusion formula

$$\bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \text{ open} \right) \subseteq \sigma_{\mathrm{e}}(T_f^p) \tag{4.1}$$

holds.

Proof. Suppose that 0 is contained in the intersection on the left-hand side. We show that $0 \in \sigma_{\mathrm{e}}(T_f^p) = \sigma((T_f^p)^e)$. Since the last map in the Koszul complex of the operator tuple $(T_f^p)^e$ is given by the row operator

$$(\mathrm{H}^p(\partial D)^e)^m \longrightarrow \mathrm{H}^p(\partial D)^e,$$

it suffices to show that this map is not onto. To see this, let $q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and fix a sequence $(\lambda_j)_{j \in \mathbb{N}}$ in D such that $\lim_{j \to \infty} \lambda_j = \lambda \in \partial D$ and $\lim_{j \to \infty} F(\lambda_j) = 0$ and assume for a moment that

$$\lim_{j \to \infty} T_{\overline{f}}^{(q)} k_{\lambda_j}^{(q)} = 0, \tag{4.2}$$

where $T_{\overline{f}}^{(q)}$ denotes the column operator

$$\mathrm{H}^q(\partial D) \longrightarrow \mathrm{H}^q(\partial D)^m, \qquad h \longmapsto \left(T_{\overline{f}_i}^q h \right)_{i=1}^m.$$

In part (a) and (c) of Theorem 4.3 we have shown that $\inf_{j \in \mathbb{N}} \|k_{\lambda_j}^{(q)}\|_q > 0$ and that $(k_{\lambda_j}^{(q)})_{j \in \mathbb{N}}$ converges to 0 weakly in $\mathrm{H}^q(\partial D)$. Then by Lemma A.1 the operator $T_{\overline{f}}^{(q)}$ has infinite dimensional kernel or non-closed range. Since $\rho_H \colon \mathrm{H}^q(\partial D) \to (\mathrm{H}^p(\partial D))'$ is a topological isomorphism, the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^q(\partial D) & \xrightarrow{T_{\overline{f}}^{(q)}} & \mathrm{H}^q(\partial D)^m \\ \downarrow \rho_H & & \downarrow \bigoplus \rho_H \\ \mathrm{H}^p(\partial D)' & \longrightarrow & \left(\mathrm{H}^p(\partial D)'\right)^m \end{array}$$

yields that the lower horizontal map

$$\mathrm{H}^p(\partial D)' \longrightarrow (\mathrm{H}^p(\partial D)')^m, \qquad \varphi \longmapsto \left(\left(T^p_{f_i}\right)' (\varphi)\right)_{i=1}^m$$

has infinite dimensional kernel or non-closed range. Since $(\mathrm{H}^p(\partial D)^m)' \cong \left(\mathrm{H}^p(\partial D)'\right)^m$ the same holds for the adjoint of the row operator

$$\mathrm{H}^p(\partial D)^m \xrightarrow{T^p_f} \mathrm{H}^p(\partial D).$$

By general duality theory for Banach spaces it follows that

$$\dim \mathrm{H}^p(\partial D)/\operatorname{Im} T^p_f = \infty.$$

But then Proposition 2.6.4 in [36] shows that

$$(\mathrm{H}^p(\partial D)^e)^m \cong (\mathrm{H}^p(\partial D)^m)^e \xrightarrow{\left(T^p_f\right)^e} \mathrm{H}^p(\partial D)^e$$

is not onto. It remains to show that (4.2) holds. Write $f = g + h$ with $g \in \mathrm{H}^\infty(\partial D)^m$ and $h = (h_1, \dots, h_m) \in \mathrm{C}(\partial D)^m$. Then $\beta = \lim_{j\to\infty} H(\lambda_j) = H(\lambda)$ and $-\beta = \lim_{j\to\infty} G(\lambda_j)$ exist and

$$\begin{aligned} T^q_{\overline{f}} k^{(q)}_{\lambda_j} &= T^q_{\overline{h}} k^{(q)}_{\lambda_j} + T^q_{\overline{g}} k^{(q)}_{\lambda_j} \\ &= T^q_{\overline{h}} k^{(q)}_{\lambda_j} + \overline{H(\lambda_j)} k^{(q)}_{\lambda_j} - \overline{H(\lambda_j)} k^{(q)}_{\lambda_j} + T^q_{\overline{g}} k^{(q)}_{\lambda_j} \\ &= T^q_{\overline{h - H(\lambda_j)}} k^{(q)}_{\lambda_j} + (\overline{H(\lambda_j)} + T^q_{\overline{g}}) k^{(q)}_{\lambda_j}. \end{aligned}$$

By Lemma 2.2 and Theorem 4.3 (a) + (b) we have

$$(\overline{H(\lambda_j)} + T^q_{\overline{g}}) k^{(q)}_{\lambda_j} = \left(\overline{H(\lambda_j)} + \left(T^p_g\right)^*\right) k^{(q)}_{\lambda_j} = \left(\overline{H(\lambda_j)} + \overline{G(\lambda_j)}\right) k^{(q)}_{\lambda_j} \xrightarrow{j\to\infty} 0. \tag{4.3}$$

Now fix an index $i \in \{1, \dots, m\}$ and let $\varepsilon > 0$. Use part (i) of Lemma 4.2 and the continuity of H_i on $\overline{D}$ to choose $\delta > 0$ such that at the same time

$$K = \sup \left\{ |S(w,z)| \;\middle|\; (z,w) \in (\mathrm{B}_{\frac{\delta}{2}}(\lambda) \cap D) \times (\mathrm{B}_\delta(\lambda)^c \cap \partial D) \right\} < \infty$$

and

$$|H_i(\lambda) - H_i(\eta)| < \frac{\varepsilon}{2}$$

for all $\eta \in \overline{D} \cap \mathrm{B}_\delta(\lambda)$. Let $N \in \mathbb{N}$ be large enough such that $\lambda_j \in \mathrm{B}_{\frac{\delta}{2}}(\lambda)$ for all $j \geq N$. Then we get

$$|h_i(\xi) - H_i(\lambda_j)| \leq |H_i(\xi) - H_i(\lambda)| + |H_i(\lambda) - H_i(\lambda_j)| < \varepsilon$$

for all $\xi \in \partial D \cap \mathrm{B}_\delta(\lambda)$ and all $j \geq N$ and the estimate

$$\left|k_{\lambda_j}^{(q)}(\xi)\right| = \frac{1}{\left\|\varepsilon_{\lambda_j}^{(p)}\right\|} |S(\xi, \lambda_j)| \leq \frac{c_q K}{\left\|S_{\lambda_j}\right\|_q} \quad (j \geq N, \xi \in \partial D \cap \mathrm{B}_\delta(\lambda)^c)$$

shows that $\left(k_{\lambda_j}^{(q)}\right)_{j \geq N}$ converges to zero uniformly on $\partial D \cap \mathrm{B}_\delta(\lambda)^c$ as $j \to \infty$. Hence by another application of Theorem 4.3 (a) we get

$$\begin{aligned}
\left\|T^q_{\overline{h_i - H_i(\lambda_j)}} k_{\lambda_j}^{(q)}\right\|_q^q &\leq \|P_q\|^q \left\|\overline{h_i - H_i(\lambda_j)} k_{\lambda_j}^{(q)}\right\|_q^q \\
&= \|P_q\|^q \int_{\partial D} |h_i - H_i(\lambda_j)|^q \left|k_{\lambda_j}^{(q)}\right|^q d\sigma \\
&= \|P_q\|^q \left(\int_{\partial D \cap \mathrm{B}_\delta(\lambda)} |h_i - H_i(\lambda_j)|^q \left|k_{\lambda_j}^{(q)}\right|^q d\sigma \right. \\
&\qquad \left. + \int_{\partial D \cap \mathrm{B}_\delta(\lambda)^c} |h_i - H_i(\lambda_j)|^q \left|k_{\lambda_j}^{(q)}\right|^q d\sigma \right) \\
&\leq \|P_q\|^q \left(\varepsilon^q \left\|k_{\lambda_j}^{(q)}\right\|_q^q + 2^q \|H_i\|_{\partial D}^q \varepsilon^q\right) \\
&\leq \|P_q\|^q \left(c_q^q + 2^q \|H_i\|_{\partial D}^q\right) \varepsilon^q
\end{aligned}$$

for all sufficiently large $j \geq N$. This together with (4.3) shows that

$$\lim_{j \to \infty} T_{\overline{f}}^{(q)} k_{\lambda_j}^{(q)} = 0.$$

Thus the proof is complete. □

We recall some results from Gelfand theory. Consider a unital algebra homomorphism $\Phi \colon \mathcal{M} \to \mathrm{L}(X)$ from a unital commutative Banach algebra $\mathcal{M}$ into the algebra of all bounded operators on a Banach space X. A spectral system on $B = \overline{\Phi(\mathcal{M})}$ is a rule σ

that assigns to each finite tuple $a \in B^r$ a compact subset $\sigma(a) \subset \mathbb{C}^r$ which is contained in the joint spectrum $\sigma_B(a) = \{z \in \mathbb{C}^r \mid 1_X \notin \sum_{i=1}^r (z_i - a_i)B\}$ of a in B and which is compatible with projections in the sense that

$$p(\sigma(a,b)) = \sigma(a) \quad \text{and} \quad q(\sigma(a,b)) = \sigma(b),$$

where p and q are the projections of $\mathbb{C}^{r+s}$ onto its first r and last s coordinates.

For a given set M, let us denote by $c(M)$ the set of all finite tuples of elements in M. Standard results going back to J. L. Taylor (see, e.g., Proposition 2.6.1 in [36]) show that, for a spectral system σ as above, the set

$$\Delta_{\Phi,\sigma} = \left\{ \lambda \in \Delta_{\mathcal{M}} \;\middle|\; \hat{f}(\lambda) \in \sigma(\Phi(f)) \text{ for all } f \in c(\mathcal{M}) \right\}$$

is the unique closed subset of the maximal ideal space $\Delta_{\mathcal{M}}$ of $\mathcal{M}$ with $\hat{f}(\Delta_{\Phi,\sigma}) = \sigma(\Phi(f))$ for all $f \in c(\mathcal{M})$. Here $\Phi(f) = (\Phi(f_1), \ldots, \Phi(f_r))$ and the Gelfand transform $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_r)$ are formed componentwise for $f \in \mathcal{M}^r$.

Let $\Phi_0 \colon \mathcal{M}_0 \to \mathrm{L}(X)$ be the restriction of Φ to a unital closed subalgebra $\mathcal{M}_0 \subset \mathcal{M}$ and let σ_0 denote the spectral system on $B_0 = \overline{\Phi(\mathcal{M}_0)}$ obtained by restricting σ. An elementary exercise, using the uniqueness property of Δ_{Φ_0,σ_0}, shows that the restriction map

$$r \colon \Delta_{\Phi,\sigma} \longrightarrow \Delta_{\Phi_0,\sigma_0}, \qquad \lambda \longmapsto \lambda|_{\mathcal{M}_0}$$

is well-defined, surjective and continuous (relative to the Gelfand topologies). We apply the above remarks to the Banach algebras $\mathcal{M}_0 = \mathrm{H}^\infty(\partial D)$, $\mathcal{M} = \mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D)$ and the algebra homomorphism

$$\Phi \colon \mathcal{M} \longrightarrow \mathrm{L}(\mathrm{H}^p(\partial D)^e), \qquad f \longmapsto (T_f^p)^e$$

(see Theorem 3.8 and the remarks at the end of Section 1.2). Let σ be the spectral system on $B = \overline{\Phi(\mathcal{M})}$ associating with each tuple $a \in B^r$ its Taylor spectrum as a commuting tuple of bounded operators on $\mathrm{H}^p(\partial D)^e$. Write σ_0 for the restriction of σ to $B_0 = \overline{\Phi(\mathcal{M}_0)}$.

It was shown by M. Andersson and S. Sandberg [6, Theorem 1.2] that the spectral mapping formula

$$\sigma(\Phi(f)) = \sigma_e(T_f^p) = \bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \text{ open}\right)$$

holds for every tuple $f \in c(\mathrm{H}^\infty(\partial D))$. Let $\pi = (\pi_1, \ldots, \pi_d)$ be the tuple of coordinate functions. Using Theorem 1 in [28] we obtain that

$$\hat{f}(\lambda) \in \bigcap \left(\overline{F(U \cap D)}; U \text{ open neighbourhood of } \hat{\pi}(\lambda)\right)$$

for $f \in c(\mathrm{H}^\infty(\partial D))$ and every functional $\lambda \in \Delta_{\Phi_0,\sigma_0}$.

Proposition 4.5.
For $g \in \mathrm{H}^\infty(\partial D)^r$, $h \in \mathrm{C}(\partial D)^s$ and $f = (g, h)$, the spectral inclusion formula

$$\sigma_e(T_f^p) \subset \bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \textit{ open}\right)$$

holds.

Proof. Suppose that $0 \in \sigma_e(T_f^p)$. It suffices to show that 0 is contained in the intersection on the right-hand side. By the remarks preceding the proposition there is a functional $\lambda \in \Delta_{\phi,\sigma}$ with $0 = \hat{f}(\lambda) = (\hat{g}(\lambda), \hat{h}(\lambda))$. Since $\lambda|_C \in \Delta_C$, there is a point $z_0 \in \partial D$ with

$$\lambda(\varphi) = \varphi(z_0) \quad (\varphi \in \mathrm{C}(\partial D))$$

(see, e.g., Corollary 3.4.2 in [44]). In particular it follows that $\lim_{z \to z_0} H(z) = h(z_0) = 0$. The above cited results from [6] and [28] imply that

$$0 = \hat{g}(\lambda) \in \bigcap \left(\overline{G(U \cap D)}; U \text{ open neighbourhood of } z_0 = \hat{\pi}(\lambda)\right).$$

Hence there is a sequence $(z_k)_{k \geq 1}$ in D with $\lim_{k \to \infty} z_k = z_0$ and

$$\lim_{k \to \infty} (G(z_k), H(z_k)) = 0.$$

This observation completes the proof. □

Now we show that the last proposition remains true for arbitrary symbols $f \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$.

Theorem 4.6.
For $f \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$, the formula

$$\sigma_{\mathrm{e}}(T_f^p) = \bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \text{ open} \right) \tag{4.4}$$

holds.

Proof. Let $f = g + h \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$ be given with $g \in \mathrm{H}^\infty(\partial D)^m$ and $h \in \mathrm{C}(\partial D)^m$. Using a particular case of the analytic spectral mapping theorem for the Taylor spectrum (see, e.g., Theorem 2.5.10 in [36]), we obtain that

$$\begin{aligned} \sigma_e(T_f^p) &= \sigma_e(T_g^p + T_h^p) = \sigma((T_g^p)^e + (T_h^p)^e) \\ &= \left\{ z + w \mid (z, w) \in \sigma((T_g^p)^e, (T_h^p)^e) \right\} \\ &= \left\{ z + w \mid (z, w) \in \sigma_e(T_g^p, T_h^p) \right\}. \end{aligned}$$

If $(z, w) \in \sigma_e(T_g^p, T_h^p)$, then by Proposition 4.5 there is a sequence $(u_k)_{k \in \mathbb{N}}$ in D converging to some point $u \in \partial D$ such that

$$(z, w) = \lim_{k \to \infty} (G, H)(u_k).$$

But then

$$z + w = \lim_{k \to \infty} (G + H)(u_k) = \lim_{k \to \infty} F(u_k).$$

Hence $\sigma_e(T_f^p)$ is contained in the intersection on the right-hand side of (4.4). The reverse inclusion has been shown in Lemma 4.4. □

The following corollary is contained in the last theorem as a special case.

Corollary 4.7.
For $f \in \mathrm{C}(\partial D)^m$ the spectral formula $\sigma_{\mathrm{e}}(T_f^p) = f(\partial D)$ holds.

Proof. Using the continuity of $F \colon \overline{D} \longrightarrow \mathbb{C}^m$ one obtains that

$$\bigcap \left(\overline{F(U \cap D)}; U \supset \partial D \text{ open} \right) = F(\partial D) = f(\partial D).$$

Thus the assertion follows from Theorem 4.6. □

General interpolation results for the Taylor spectrum of commuting m-tuples of Banach space operators have been proved by E. Albrecht in [2]. In particular it is shown that the Taylor spectrum and essential Taylor spectrum of an admissible commuting tuple is upper semicontinuous with respect to the parameter $\theta \in (0,1)$ for the complex interpolation method.

4.2 A Remark on the Maximal Ideal Space of $\mathrm{H}^\infty(\partial\mathbb{B}) + \mathrm{C}(\partial\mathbb{B})$

In Chapter 3 (Corollary 3.11) we proved that on the unit ball $\mathbb{B} \subset \mathbb{C}^d$ the maximal ideal space of the C*-algebra

$$QC = \left\{ f \in \mathrm{L}^\infty(S) \;\middle|\; H_f^p \text{ and } H_{\bar{f}}^p \text{ are compact} \right\}$$

is given by the point evaluations of the Poisson-Szegő transform of functions $f \in QC$ at points of the Stone-Čech corona $\beta(\mathbb{B}) \setminus \mathbb{B}$ of $\mathbb{B}$:

$$\Delta_{QC} = \{\delta_\Lambda|_{QC} \mid \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\}\,.$$

We show that the question whether the corresponding result holds for the Banach algebra $\mathrm{H}^\infty(S) + \mathrm{C}(S) \subset \mathrm{L}^\infty(S)$ is equivalent to the corona problem[4] on the unit ball $\mathbb{B} \subset \mathbb{C}^d$. For simplicity, we define $\mathrm{H}^\infty = \mathrm{H}^\infty(S)$, $\mathrm{C} = \mathrm{C}(S)$ and write $\mathrm{Co}(\mathbb{B}) = \beta(\mathbb{B}) \setminus \mathbb{B}$ for the Stone-Čech corona of $\mathbb{B}$. For $f \in \mathrm{L}^\infty(S)$, we denote by $F = P[f]\colon \mathbb{B} \to \mathbb{C}$ its Poisson-Szegő transform. The map

$$\mathbb{B} \longrightarrow \Delta_{\mathrm{H}^\infty}, \qquad \lambda \longmapsto \varepsilon_\lambda,$$

where $\varepsilon_\lambda\colon \mathrm{H}^\infty \to \mathbb{C}$ is defined by $\varepsilon_\lambda(f) = F(\lambda)$, is easily seen to be a homeomorphism onto an open subset of $\Delta_{\mathrm{H}^\infty}$. By general properties of the Stone-Čech compactification[5] this map has a unique continuous extension $q\colon \beta(\mathbb{B}) \to \Delta_{\mathrm{H}^\infty}$. The image $\operatorname{Im} q \subset \Delta_{\mathrm{H}^\infty}$ is given by

$$K = \overline{\{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}}^{\tau_{w^*}} \subset \Delta_{\mathrm{H}^\infty}$$

and the corona problem on the unit ball is the question whether $K = \Delta_{\mathrm{H}^\infty}$.

Let $\Lambda \in \mathrm{Co}(\mathbb{B})$. Choose a net $(\lambda_\alpha)_\alpha$ in $\mathbb{B}$ with $\Lambda = \lim_\alpha \lambda_\alpha$. Then $q(\Lambda) = \lim_\alpha q(\lambda_\alpha) = \tau_{w^*}\text{-}\lim_\alpha \varepsilon_{\lambda_\alpha}$. Since $q\colon \mathbb{B} \to q(\mathbb{B})$ is a homeomorphism, we have $q(\Lambda) \notin \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}$. By definition

$$\delta_\Lambda(f) = F^\beta(\Lambda) = \lim_\alpha F(\lambda_\alpha) = \lim_\alpha \varepsilon_{\lambda_\alpha}(f)$$

[4]see [29] for an exposition of the corona problem
[5]see Theorem 38.4 in [56]

for every $f \in \mathrm{H}^\infty$. Thus it follows that

$$K \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\} = q(\mathrm{Co}(\mathbb{B})) = \{\delta_\Lambda|_{\mathrm{H}^\infty} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\}\,.$$

A result of G. McDonald (Theorem 4 in [53]) shows that the restriction map

$$\Delta_{\mathrm{H}^\infty+\mathrm{C}} \longrightarrow \Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}\,, \qquad \delta \longmapsto \delta|_{\mathrm{H}^\infty}$$

is a well defined homeomorphism.

For $z \in \mathbb{B}$, denote by $\varphi_z \colon \mathbb{B} \to \mathbb{B}$ the usual involutional, conformal mapping of the unit ball with $\varphi_z(0) = z$[6]. As in Chapter 3 let $F^\beta \in \mathrm{C}(\beta(\mathbb{B}))$ be the unique continuous extension of the Poisson-Szegő transform $F \colon \mathbb{B} \to \mathbb{C}$ of a function $f \in \mathrm{L}^\infty(S)$.

Suppose that $(\lambda_\alpha)_\alpha$ is a net in $\mathbb{B}$ such that the limit $\lambda = \lim_\alpha \lambda_\alpha \in S$ exists. By Tychonoff's theorem the space

$$\beta(\mathbb{B})^{\mathbb{B}} = \prod_{\mathbb{B}} \beta(\mathbb{B})$$

equipped with the product topology is compact. By passing to a subnet, we can achieve that the net $(\varphi_{\lambda_\alpha})_\alpha$ converges pointwise to some $\varphi \in \beta(\mathbb{B})^{\mathbb{B}}$. Since $\overline{\mathbb{B}} \subset \mathbb{C}^d$ is a compactification of $\mathbb{B}$, there exists a surjective continuous map $h\colon\ \beta(\mathbb{B}) \to \overline{\mathbb{B}}$ with $h(z) = z$ for all $z \in \mathbb{B}$. It follows that

$$\lambda = \lim_\alpha \varphi_{\lambda_\alpha}(z) = \lim_\alpha h(\varphi_{\lambda_\alpha}(z)) = h(\varphi(z))$$

for every $z \in \mathbb{B}$. In particular, we find that $\varphi(\mathbb{B}) \subset \mathrm{Co}(\mathbb{B})$.

The following proposition is a multivariable version of Theorem 6 in [32] which can be proven exactly in the same way.

Proposition 4.8.
For $m \in \mathbb{N}^*$ *and* $f = (f_1, \ldots, f_m) \in \mathcal{A}^m$ *let* $F = (F_1, \ldots, F_m)$ *and* $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_m) \in \mathrm{C}(\Delta_{\mathcal{A}})^m$*. Then the identity*

$$\mathrm{Cl}(F) = \hat{f}(\{\delta_\Lambda \mid \Lambda \in \mathrm{Co}(\mathbb{B})\})$$

[6]see Chapter 2.2 in [64]

holds.

Proof. Let $\Lambda \in \mathrm{Co}(\mathbb{B})$ and choose a net $(\lambda_\alpha)_\alpha$ in $\mathbb{B}$ with $\lim_\alpha \lambda_\alpha = \Lambda$ in $\beta(\mathbb{B})$. Then the limit $\lambda = \lim_\alpha \lambda_\alpha = \lim_\alpha h(\lambda_\alpha) = h(\Lambda)$ belongs to S. For every open set $U \subset \mathbb{C}^d$ containing S there exists $\alpha_0 \in A$ such that $\lambda_\alpha \in U \cap \mathbb{B}$ for all $\alpha \geq \alpha_0$. It follows that

$$\hat{f}(\delta_\Lambda) = (\delta_\Lambda(f_1), \ldots, \delta_\Lambda(f_m)) = (F_1^\beta(\Lambda), \ldots, F_m^\beta(\Lambda)) = \lim_\alpha F(\lambda_\alpha) \in \overline{F(U \cap \mathbb{B})},$$

thus $\hat{f}(\delta_\Lambda) \in \mathrm{Cl}(F)$.

For $z \in \mathrm{Cl}(F)$ use the remarks preceding Proposition 4.8 to choose a net $(\lambda_\alpha)_\alpha$ in $\mathbb{B}$ with $z = \lim_\alpha F(\lambda_\alpha)$ such that $\lambda = \lim_\alpha \lambda_\alpha \in S$ exists and such that $(\varphi_{\lambda_\alpha})_\alpha$ converges pointwise on $\mathbb{B}$ to some function $\varphi \in \beta(\mathbb{B})^{\mathbb{B}}$. Bearing in mind that $\varphi(0) \in \mathrm{Co}(\mathbb{B})$, we get

$$z = \lim_\alpha F(\lambda_\alpha) = \lim_\alpha F(\varphi_{\lambda_\alpha}(0)) = \lim_\alpha (F_1^\beta(\varphi_{\lambda_\alpha}(0)), \ldots, F_m^\beta(\varphi_{\lambda_\alpha}(0))) = \hat{f}(\varphi(0)).$$

□

Theorem 4.9.
Let $1 < p < \infty$ be arbitrary. The following are equivalent:

(i) $K = \Delta_{\mathrm{H}^\infty}$,

(ii) $\Delta_{\mathrm{H}^\infty + \mathrm{C}} = \{\delta_\Lambda|_{\mathrm{H}^\infty + \mathrm{C}} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\}$,

(iii) $\Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\} = \{\delta_\Lambda|_{\mathrm{H}^\infty} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\}$,

(iv) $\sigma_{\mathrm{e}}(T_f^p) = \sigma_{\mathrm{H}^\infty + \mathrm{C}}(f)$ *for all* $f \in c(\mathrm{H}^\infty + \mathrm{C})$,

(v) $\sigma_{\mathrm{e}}(T_f^p) = \hat{f}(\Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\})$ *for all* $f \in c(\mathrm{H}^\infty)$.

Proof. Suppose that (i) is true and let $\delta \in \Delta_{\mathrm{H}^\infty + \mathrm{C}}$. From the remarks preceding Proposition 4.8 and the assumption we get $\delta|_{\mathrm{H}^\infty} = \delta_\Lambda|_{\mathrm{H}^\infty}$ for some $\Lambda \in \mathrm{Co}(\mathbb{B})$. Since $\mathrm{H}^\infty + \mathrm{C} \subset \mathcal{A}$, we have $\delta = \delta_\Lambda|_{\mathrm{H}^\infty + \mathrm{C}}$. The reverse inclusion in (ii) is obvious.

Now assume that (ii) holds. Together with the fact that the restriction map $\Delta_{\mathrm{H}^\infty + \mathrm{C}} \longrightarrow \Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}$ is a homeomorphism (see Theorem 4 in [53]), we see that every $\delta \in \Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}$ is of the form $\delta = \delta_\Lambda|_{\mathrm{H}^\infty}$ for some $\Lambda \in \mathrm{Co}(\mathbb{B})$. Conversely, we have already established that

$$\{\delta_\Lambda|_{\mathrm{H}^\infty} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\} = K \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\} \subset \Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}\,.$$

Hence (ii) implies (iii). If we suppose (iii) to be true, then according to the remarks preceding Proposition 4.8, we know that $\Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\} = K \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}$ and thus $K = \Delta_{\mathrm{H}^\infty}$. Therefore the first three statements are equivalent.

Next observe that by Proposition 4.8 we have

$$\mathrm{Cl}(F) = \hat{f}(\{\delta_\Lambda \mid \Lambda \in \mathrm{Co}(\mathbb{B})\})$$

for all $f \in c(\mathcal{A})$, where $F = (F_1, \ldots, F_m)$ and $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_m) \in \mathrm{C}(\Delta_{\mathcal{A}})^m$ for $f = (f_1, \ldots, f_m) \in \mathcal{A}^m$. Theorem 4.6 yields that statement (iv) is equivalent to

$$\begin{aligned} \hat{f}(\Delta_{\mathrm{H}^\infty + \mathrm{C}}) &= \sigma_{\mathrm{H}^\infty + \mathrm{C}}(f) = \sigma_{\mathrm{e}}(T_f^p) \\ &= \mathrm{Cl}(F) = \hat{f}(\{\delta_\Lambda \mid \Lambda \in \mathrm{Co}(\mathbb{B})\}) \\ &= \hat{f}(\{\delta_\Lambda|_{\mathrm{H}^\infty + \mathrm{C}} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\}) \end{aligned} \tag{4.5}$$

for all $f \in c(\mathrm{H}^\infty + \mathrm{C})$ and that (v) is equivalent to

$$\hat{f}(\Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\}) = \sigma_{\mathrm{e}}(T_f^p) = \hat{f}(\{\delta_\Lambda|_{\mathrm{H}^\infty} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\}) \tag{4.6}$$

for all $f \in c(\mathrm{H}^\infty)$. Observe that $\Delta_{\mathrm{H}^\infty} \setminus \{\varepsilon_\lambda \mid \lambda \in \mathbb{B}\} \subset \Delta_{\mathrm{H}^\infty}$ is closed. In Theorem 6 in [32] it is shown that $\{\delta_\Lambda \mid \Lambda \in \mathrm{Co}(\mathbb{B})\} \subset \Delta_{\mathcal{A}}$ is closed, hence compact. Since the restriction mappings

$$\Delta_{\mathcal{A}} \longrightarrow \Delta_{\mathrm{H}^\infty}, \quad \delta \longmapsto \delta|_{\mathrm{H}^\infty} \qquad \text{and} \qquad \Delta_{\mathcal{A}} \longrightarrow \Delta_{\mathrm{H}^\infty + \mathrm{C}}, \quad \delta \longmapsto \delta|_{\mathrm{H}^\infty + \mathrm{C}}$$

are continuous, the sets $\{\delta_\Lambda|_{\mathrm{H}^\infty} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\} \subset \Delta_{\mathrm{H}^\infty}$ and $\{\delta_\Lambda|_{\mathrm{H}^\infty + \mathrm{C}} \mid \Lambda \in \mathrm{Co}(\mathbb{B})\} \subset \Delta_{\mathrm{H}^\infty + \mathrm{C}}$ are closed. By general Gelfand theory (see the remarks preceding Proposition 4.5)

$$\Delta = \left\{\lambda \in \Delta_{\mathrm{H}^\infty + \mathrm{C}} \;\middle|\; \hat{f}(\lambda) \in \sigma_{\mathrm{e}}(T_f^p) \text{ for all } f \in c(\mathrm{H}^\infty + \mathrm{C})\right\}$$

is the unique closed subset of $\Delta_{\mathrm{H}^\infty + \mathrm{C}}$ with $\hat{f}(\Delta) = \sigma_{\mathrm{e}}(T_f^p)$ for all $f \in c(\mathrm{H}^\infty + \mathrm{C})$. This together with (4.5) shows the equivalence of the statements (ii) and (iv). Using the uniqueness property of the set

$$\left\{\lambda \in \Delta_{\mathrm{H}^\infty} \;\middle|\; \hat{f}(\lambda) \in \sigma_{\mathrm{e}}(T_f^p) \text{ for all } f \in c(\mathrm{H}^\infty)\right\}$$

together with (4.6) one obtains that (iii) and (v) are equivalent. □

4.3 Essential Spectra of Truncated Toeplitz Systems

In this chapter we will illustrate how Theorem 4.6 can be used to obtain a spectral mapping theorem for certain quotient Toeplitz tuples $T_f^{p,\theta} \in \mathrm{L}(\mathrm{H}^p(\partial D)/\theta\,\mathrm{H}^p(\partial D))^m$ induced by an inner function $\theta \in \mathrm{H}^\infty(\partial D)$ over a strictly pseudoconvex domain $D \subset \mathbb{C}^d$ with C^∞-boundary. This will lead to a generalization of a spectral mapping theorem of R. V. Bessonov [10, Theorem 1] for truncated Toeplitz operators on the Hardy space $\mathrm{H}^2(\mathbb{T})$ over the unit circle $\mathbb{T} \subset \mathbb{C}$.

In the following we will use some methods from homological algebra. We refer the reader to [36, Appendix 2] for the relevant definitions and results. Let $T = (T_1, \dots, T_m) \in \mathrm{L}(X)^m$ be a commuting tuple on a Banach space X. Recall that we denote by $\mathrm{H}^i(T, X)$ $(i = 0, \dots, m)$ the cohomology groups of the Koszul complex $\mathrm{K}^\bullet(T, X)$ of T and as in the preliminaries we define $T^e = ({T_1}^e, \dots, {T_m}^e) \in \mathrm{L}(X^e)^m$. Let $S \in \{T_1, \dots, T_m\}'$ be an operator in the commutant of T. Then T induces a commuting tuple $T_S = (T_{S,1}, \dots, T_{S,m})$ on the quotient vector space $X_S = X/SX$ with components

$$T_{S,j} \colon X_S \longrightarrow X_S, \qquad [x] \longmapsto [T_j x].$$

Lemma 4.10.
Let $T = (T_1, \dots, T_m) \in \mathrm{L}(X)^m$ be a commuting tuple on a Banach space X and $S \in \{T_1, \dots, T_m\}'$ be an injective operator. Then there are vector space isomorphisms

$$\mathrm{H}^{i-1}(T_S, X) \longrightarrow \mathrm{H}^i((T, S), X) \qquad (i = 0, \dots, m+1).$$

Proof. Denote by $K = (K_{l,k})_{l,k\in\mathbb{Z}}$ the double complex

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Lambda_m^0 X & \xrightarrow{\delta_T^0} & \Lambda_m^1 X & \xrightarrow{\delta_T^1} & \dots & \xrightarrow{\delta_T^{m-1}} & \Lambda_m^m X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^0} & & \downarrow{\scriptstyle \partial^1} & & & & \downarrow{\scriptstyle \partial^m} & & \\
0 & \longrightarrow & \Lambda_m^0 X & \xrightarrow{\delta_T^0} & \Lambda_m^1 X & \xrightarrow{\delta_T^1} & \dots & \xrightarrow{\delta_T^{m-1}} & \Lambda_m^m X & \longrightarrow & 0,
\end{array}$$

i.e., $K_{l,k} = \Lambda_m^l X$ for $l = 0, \dots, m$ and $k = 0, 1$ and $K_{l,k} = 0$ otherwise, where the rows $(k = 0, 1)$ are given by the Koszul complex of the operator tuple $T \in \mathrm{L}(X)^m$ and the vertical maps ∂^l $(l = 0, \ldots, m)$ are the injections

$$\partial^l \colon \Lambda^l_m X \longrightarrow \Lambda^l_m X, \qquad (x_I)_{|I|=l} \longmapsto (-1)^l (Sx_I)_{|I|=l}$$

induced by $S \in \mathrm{L}(X)$. Let us denote by $K^\bullet = \mathrm{Tot}(K)$ the total complex of the double complex K. By definition the spaces in the complex $K^\bullet$ are

$$K^l = \Lambda^{l-1}_m X \oplus \Lambda^l_m X \qquad (l = 0, \ldots, m+1)$$

and the differentials act as

$$K^l \longrightarrow K^{l+1}, \qquad (y_{l-1}, x_l) \longmapsto (\delta^{l-1}_T(y_{l-1}) + \partial^l(x_l), \delta^l_T(x_l)).$$

Modulo the isomorphisms

$$\Lambda^l_{m+1} X \longrightarrow K^l, \qquad (x_I)_{|I|=l} \longmapsto ((x_{i_1 \ldots i_{l-1}, m+1})_{1 \leq i_1 < \ldots < i_{l-1} \leq m}, (x_{i_1 \ldots i_l})_{1 \leq i_1 < \ldots < i_l \leq m})$$

the complex $K^\bullet$ coincides with the Koszul complex of the $(m+1)$-tuple $(T, S) \in \mathrm{L}(X)^{m+1}$ and therefore we have

$$\mathrm{H}^l((T, S), X) \cong \mathrm{H}^l(K^\bullet) \tag{4.7}$$

for $l = 0, \ldots, m+1$. Standard double complex methods (see [36, Lemma A2.6]) or elementary diagram chasing can be used to show that the mappings

$$\mathrm{H}^l(K^\bullet) \longrightarrow \mathrm{H}^{l-1}(T_S, X_S), \qquad [(y_{l-1}, x_l)] \longmapsto [\tilde{y}_{l-1}],$$

where $\tilde{y}_{l-1} \in \Lambda^{l-1}_m X_S$ is the $(l-1)$-form obtained from $y_{l-1} \in \Lambda^{l-1}_m X$ by replacing its coefficients by their equivalence classes in $X_S = X/SX$, are vector space isomorphisms. Thus we obtain vector space isomorphisms $\mathrm{H}^l((T, S), X) \cong \mathrm{H}^{l-1}(T_S, X_S)$ for $l = 0, \ldots, m+1$. □

Let $S \in \mathrm{L}(X)$ be a left invertible operator. In the following let $R \in \mathrm{L}(X)$ be a fixed left inverse of S. Then $P = SR$ is a projection onto the closed subspace $\mathrm{Im}\, S \subset X$ and $Q = 1 - P$ defines a projection onto a topological direct complement of $\mathrm{Im}\, S$ in X. For $T \in \mathrm{L}(X)$, we form the compression

$$T^Q = QT|_{\mathrm{Im}\, Q} \in \mathrm{L}(\mathrm{Im}\, Q).$$

Lemma 4.11.
If $T_1, T_2 \in \mathrm{L}(X)$ are bounded operators such that T_1, T_2, S essentially commute, then also the operators T_1^Q and T_2^Q essentially commute.

Proof. The assertion follows from the identities

$$\begin{aligned}[T_1^Q, T_2^Q] &= QT_1QT_2|_{\operatorname{Im} Q} - QT_2QT_1|_{\operatorname{Im} Q} \\ &= Q[T_1, T_2]|_{\operatorname{Im} Q} - QT_1PT_2|_{\operatorname{Im} Q} + QT_2PT_1|_{\operatorname{Im} Q} \\ &= Q[T_1, T_2]|_{\operatorname{Im} Q} - Q[T_1, S]RT_2|_{\operatorname{Im} Q} + Q[T_2, S]RT_1|_{\operatorname{Im} Q}.\end{aligned}$$

□

Modulo the topological isomorphism

$$\rho\colon \operatorname{Im} Q \longrightarrow X/\operatorname{Im} S, \qquad x \longmapsto [x]$$

the operator T^Q is similar[7] to the operator $T_S = \rho T^Q \rho^{-1} \in \mathrm{L}(X/\operatorname{Im} S)$. The reader should be aware that T_S depends on the choice of the fixed left inverse R of S.

Let $T = (T_1, \ldots, T_m) \in \mathrm{L}(X)^m$ be a tuple of bounded operators on X such that the $(m+1)$-tuple $(T, S) = (T_1, \ldots, T_m, S) \in \mathrm{L}(X)^{m+1}$ is essentially commuting. Let us consider the short exact sequence

$$0 \longrightarrow \operatorname{Im} S \xrightarrow{\;i\;} X \xrightarrow{\;q\;} X/\operatorname{Im} S \longrightarrow 0,$$

where i is the inclusion and q is the quotient map. Since the e-functor preserves exactness (see the remarks preceding Corollary 2.6.9 in [36]), the induced sequence

$$0 \longrightarrow (\operatorname{Im} S)^e \xrightarrow{\;i^e\;} X^e \xrightarrow{\;q^e\;} (X/\operatorname{Im} S)^e \longrightarrow 0,$$

remains exact. Writing S as the composition

$$X \xrightarrow{\;S\;} \operatorname{Im} S \xrightarrow{\;i\;} X,$$

[7] Two bounded linear operators $T_1 \in \mathrm{L}(X)$ and $T_2 \in \mathrm{L}(Y)$ on Banach spaces X and Y, respectively, are called similar if there exists a topological isomorphism $\rho\colon X \to Y$ such that $\rho T_1 = T_2 \rho$.

and using the fact that the e-functor preserves topological isomorphisms, we find that $i^e(\operatorname{Im} S)^e = \operatorname{Im}(S^e)$. Hence, by the bounded inverse theorem, the topological epimorphism $q^e \colon X^e \to (X/\operatorname{Im} S)^e$ induces a topological isomorphism

$$X^e/\operatorname{Im}(S^e) = X^e/i^e(\operatorname{Im} S)^e = X^e/\ker(q^e) \cong (X/\operatorname{Im} S)^e$$

that is given by

$$\Phi \colon X^e/\operatorname{Im}(S^e) \longrightarrow (X/\operatorname{Im} S)^e, \qquad [x] \longmapsto q^e(x).$$

Modulo this topological isomorphism the quotient tuple $T^e/\operatorname{Im} S^e \in \mathrm{L}(X^e/\operatorname{Im}(S^e))^m$ with components

$$T_i{}^e/\operatorname{Im} S^e \colon X^e/\operatorname{Im}(S^e) \longrightarrow X^e/\operatorname{Im}(S^e), \qquad [x] \longmapsto [T_i{}^e(x)] \qquad (i = 1, \ldots, m)$$

and the tuple $(T_S)^e = ((T_{S,1})^e, \ldots, (T_{S,m})^e) \in \mathrm{L}((X/\operatorname{Im} S)^e)^m$ are similar. To verify this assertion, note that the operators

$$\begin{aligned} qT_i - qQT_iQ &= qT_i - q(1 - SR)T_i(1 - SR) \\ &= qT_i - qT_i(1 - SR) \\ &= qT_iSR = q[T_i, S]R \end{aligned}$$

are compact for $i = 1, \ldots, m$. Hence the intertwining relations

$$(T_{S,i})^e q^e = (T_{S,i} \circ q)^e = \left(\rho QT_i\rho^{-1}q\right)^e = (qQT_iQ)^e = (qT_i)^e = q^e T_i{}^e$$

hold for $i = 1, \ldots, m$. Thus Φ defines a similarity between $T^e/\operatorname{Im} S^e \in \mathrm{L}(X^e/\operatorname{Im}(S^e))^m$ and $(T_S)^e \in \mathrm{L}((X/\operatorname{Im} S)^e)^m$.

By applying Lemma 4.10 to the commuting tuple $(T^e, S^e) \in \mathrm{L}(X^e)^{m+1}$ we obtain that

$$\mathrm{H}^i((T,S)^e, X^e) \cong \mathrm{H}^{i-1}(T^e/\operatorname{Im}(S^e), X^e/\operatorname{Im}(S^e)) \cong \mathrm{H}^{i-1}(T_S{}^e, (X/\operatorname{Im} S)^e)$$

for $i = 0, \ldots, m+1$. Thus $0 \in \sigma_{\mathrm{e}}(T_S)$ if and only if $0 \in \sigma_{\mathrm{e}}(T, S)$. Since, for $\lambda \in \mathbb{C}^m$, everything remains true with T replaced by $\lambda - T$, one can calculate the essential Taylor spectrum of the truncated tuple $T_S \in \mathrm{L}(X/\operatorname{Im} S)^m$ using the essential spectrum of $(T,S) \in \mathrm{L}(X)^{m+1}$.

Corollary 4.12.
With the notations from above, the essential Taylor spectrum of $T_S \in \mathrm{L}(X/\operatorname{Im} S)^m$ is given by

$$\sigma_{\mathrm{e}}(T_S) = \{\lambda \in \mathbb{C}^m \mid (\lambda,0) \in \sigma_{\mathrm{e}}(T,S)\}\,.$$

Let $D \subset \mathbb{C}^d$ be a strictly pseudoconvex domain with smooth boundary. We apply the above results to calculate the essential spectra of truncated Toeplitz tuples with symbols in $\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D)$. Let $\theta \in \mathrm{H}^\infty(\partial D)$ be a fixed inner function. Then $S = T^p_\theta \in \mathrm{L}(\mathrm{H}^p(\partial D))$ is an isometry with left inverse $R = T^p_{\bar\theta}$. According to Corollary 3.4, for $f = (f_1,\ldots,f_m) \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$ the Toeplitz tuples $T^p_f = (T^p_{f_1},\ldots,T^p_{f_m}) \in \mathrm{L}(\mathrm{H}^p(\partial D))^m$ and $(T^p_f, T^p_\theta) = (T^p_{f_1},\ldots,T^p_{f_m}, T^p_\theta) \in \mathrm{L}(\mathrm{H}^p(\partial D))^{m+1}$ essentially commute. By applying the above constructions to the Toeplitz tuple T^p_f (relative to $S = T^p_\theta$ and its left inverse $R = T^p_{\bar\theta}$) one obtains the "truncated" Toeplitz tuples

$$T^{p,\theta}_f = (T^p_f)_S \in \mathrm{L}(\mathrm{H}^p_\theta(\partial D))^m,$$

where $\mathrm{H}^p_\theta(\partial D) = \mathrm{H}^p(\partial D)/\theta\,\mathrm{H}^p(\partial D)$. As an application of Theorem 4.6 and Corollary 4.12 we can calculate the essential spectrum of the essentially commuting tuple $T^{p,\theta}_f$.

Theorem 4.13.
Let $\theta \in \mathrm{H}^\infty(\partial D)$ be an inner function and $f \in (\mathrm{H}^\infty(\partial D) + \mathrm{C}(\partial D))^m$. Then the formula

$$\sigma_{\mathrm{e}}(T^{p,\theta}_f) = \left\{ \lambda \in \mathbb{C}^m \;\middle|\; \begin{array}{l} \textit{there exists } D \ni z_k \xrightarrow{k} z \in \partial D \textit{ such that} \\ \lim_{k\to\infty} F(z_k) = \lambda \textit{ and } \lim_{k\to\infty} \Theta(z_k) = 0 \end{array} \right\}$$

holds.

Proof. By Corollary 4.12 we have

$$\sigma_{\mathrm{e}}(T^{p,\theta}_f) = \left\{\lambda \in \mathbb{C}^m \;\middle|\; (\lambda,0) \in \sigma_{\mathrm{e}}(T^p_f, T^p_\theta)\right\}$$

and Theorem 4.6 yields

$$\sigma_{\mathrm{e}}(T_f^p, T_\theta^p) = \bigcap \left(\overline{(F,\Theta)(U \cap D)}; U \supset \partial D \text{ open} \right).$$

Suppose now that $(\lambda, 0) \in \mathbb{C}^m \times \mathbb{C}$ is an element in the right-hand side of the last equation. Since

$$U_k = \left\{ z \in \mathbb{C}^d \;\middle|\; \operatorname{dist}(z, \partial D) < 1/k \right\} \qquad (k \in \mathbb{N}^*)$$

are open supersets of ∂D whose intersections with D are not empty, there exist points $z_k \in U_k \cap D$ such that $|(F,\theta)(z_k) - (\lambda, 0)| < 1/k$. Then the sequence $(z_k)_{k\geq 1}$ fulfils $\lim_{k\to\infty} F(z_k) = \lambda$ and $\lim_{k\to\infty} \Theta(z_k) = 0$. Since $(z_k)_{k\geq 1}$ is contained in the compact set $\overline{D}$, it has a convergent subsequence, say with limit $z \in \overline{D}$, that will again be denoted by $(z_k)_{k\geq 1}$. It follows that

$$\operatorname{dist}(z, \partial D) = \lim_{k\to\infty} \operatorname{dist}(z_k, \partial D) = 0,$$

hence $z \in \partial D$. Thus the inclusion

$$\sigma_{\mathrm{e}}(T_f^{p,\theta}) \subset \left\{ \lambda \in \mathbb{C}^m \;\middle|\; \begin{array}{l} \text{there exists } D \ni z_k \xrightarrow{k} z \in \partial D \text{ such that} \\ \lim_{k\to\infty} F(z_k) = \lambda \text{ and } \lim_{k\to\infty} \Theta(z_k) = 0 \end{array} \right\}$$

holds. Since the reverse inclusion is obvious, the proof is finished. □

Remark 4.14.

Let $\theta \in \mathrm{H}^\infty(\partial D)$ be an inner function and let $X = \mathrm{H}^p(\partial D)/\theta\, \mathrm{H}^p(\partial D)$. Using the definitions one sees that, for $f = (f_1, \ldots, f_m) \in \mathrm{H}^\infty(\partial D)^m$, the truncated Toeplitz tuple $T_f^{p,\theta} \in \mathrm{L}(X)^m$ coincides with the quotient tuple $T_f^p/\theta\, \mathrm{H}^p(\partial D) \in \mathrm{L}(X)^m$ with components

$$T_{f_i}^p/\theta\, \mathrm{H}^p(\partial D) \colon X \longrightarrow X, \qquad [g] \longmapsto [T_{f_i}^p g] \qquad (i = 1, \ldots, m).$$

A result of M. Andersson and H. Carlsson [5, Corollary 2.3] shows that

$$\sigma(T_f^p, T_\theta^p) = \overline{(F,\Theta)(D)}.$$

Thus an application of Lemma 4.10 yields the equality

$$\begin{aligned}\sigma(T_f^p/\theta\,\mathrm{H}^p(\partial D)) &= \big\{\lambda\in\mathbb{C}^m \;\big|\; (\lambda,0)\in\sigma(T_f^p,T_\theta^p)\big\}\\ &= \left\{\lambda\in\mathbb{C}^m \;\middle|\; \begin{array}{l}\textit{there exists a sequence } (z_k)_{k\in\mathbb{N}} \textit{ in } D \textit{ with}\\ \lim_{k\to\infty}F(z_k)=\lambda \textit{ and } \lim_{k\to\infty}\Theta(z_k)=0\end{array}\right\}.\end{aligned}$$

Note that Theorem 4.13 and the result stated in Remark 4.14 remain valid if the inner function $\theta\in\mathrm{H}^\infty(\partial D)$ is replaced by a function $\theta\in\mathrm{H}^\infty(\partial D)$ that is invertible in $\mathrm{L}^\infty(\partial D)$. Indeed, if $g\in\mathrm{L}^\infty(\partial D)$ is the inverse of θ in $\mathrm{L}^\infty(\partial D)$, then $T_g^pT_\theta^p=T_{g\theta}^p=1$ and hence T_θ^p is left invertible. This is the only condition that is needed to apply Corollary 4.12. By the remark following Theorem 2.21, for $\theta\in\mathrm{H}^\infty(\partial_A)$, the left invertibility of T_θ^p is equivalent to the invertibility of the symbol θ in $\mathrm{L}^\infty(\partial_A)$.

For an inner function $\theta\in\mathrm{H}^\infty(\partial D)$, the operator $P=T_\theta T_{\overline{\theta}}\in\mathrm{L}(\mathrm{H}^2(\partial D))$ is the orthogonal projection onto the closed subspace $\theta\,\mathrm{H}^2(\partial D)$ of $\mathrm{H}^2(\partial D)$. Let $P_\theta\in\mathrm{L}(\mathrm{H}^2(\partial D))$ be the projection onto the orthogonal complement $K_\theta=\mathrm{H}^2(\partial D)\ominus\theta\,\mathrm{H}^2(\theta)$ of $\theta\,\mathrm{H}^2(\partial D)$ in $\mathrm{H}^2(\partial D)$. Then the truncated Toeplitz operator[8] $T_f^\theta\in\mathrm{L}(K_\theta)$ is defined as the compression $T_f^\theta=P_\theta T_f|_{K_\theta}$ of the Toeplitz operator $T_f\in\mathrm{L}(\mathrm{H}^2(\partial D))$. By the construction preceding Corollary 4.12 the operator $T_f^{2,\theta}\in\mathrm{L}(\mathrm{H}^2(\partial D)/\theta\,\mathrm{H}^2(\partial D))$ is unitarily equivalent to T_f^θ via the unitary operator

$$\rho\colon K_\theta\longrightarrow\mathrm{H}^2(\partial D)/\theta\,\mathrm{H}^2(\partial D),\qquad f\longmapsto[f].$$

Therefore, Theorem 4.13 has the following consequence.

Corollary 4.15.
Let $\theta\in\mathrm{H}^\infty(\partial D)$ be an inner function. For $f\in(\mathrm{H}^\infty(\partial D)+\mathrm{C}(\partial D))^m$, the formula

$$\sigma_{\mathrm{e}}(T_f^\theta)=\left\{\lambda\in\mathbb{C}^m \;\middle|\; \begin{array}{l}\textit{there exists } D\ni z_k\xrightarrow{k} z\in\partial D \textit{ such that}\\ \lim_{k\to\infty}F(z_k)=\lambda \textit{ and } \lim_{k\to\infty}\Theta(z_k)=0\end{array}\right\}$$

holds.

[8] Truncated Toeplitz operators on the Hardy space $\mathrm{H}^2(\mathbb{T})$ were first introduced by D. Sarason in [68]. See also the survey article [38] of S. R. Garcia and W. T. Ross.

The last corollary contains results of R. V. Bessonov [10, Theorem 1] and K. Guo and Y. Duan [39, Theorem 3.4] as special cases.

Let $\theta \in \mathrm{H}^\infty(\partial D)$ be an inner function. We finish this section with the observation that a truncated Toeplitz operator $T_f^{p,\theta} \in \mathrm{L}(\mathrm{H}^p_\theta(\partial D))$ with symbol $f \in \mathcal{A}$ is compact if and only if $f \in \theta\mathcal{A}$. For $p = 2$ and $D = \mathbb{D}$, this was proven by R. V. Bessonov in [10, Proposition 2.1]. Let $P_p^\theta = T_\theta^p T_{\overline{\theta}}^p \in \mathrm{L}(\mathrm{H}^p(\partial D))$.

Proposition 4.16.
Let $\theta \in \mathrm{H}^\infty(\partial D)$ be an inner function and $1 < p < \infty$. Then the mapping

$$\rho_{p,\theta}\colon \mathcal{A} \longrightarrow \mathcal{C}(\mathrm{H}^p_\theta(\partial D)), \qquad f \longmapsto [T_f^{p,\theta}]$$

is a continuous unital algebra homomorphism with

$$\ker \rho_{p,\theta} = \theta\mathcal{A}.$$

Proof. It follows from Theorem 3.8 and Lemma 4.11 that $\rho_{p,\theta}$ is a unital continuous algebra homomorphism. Obviously $\theta \in \ker \rho_{p,\theta}$. Hence the inclusion $\theta\mathcal{A} \subset \ker \rho_{p,\theta}$ follows. Let $f \in \mathcal{A}$ be a function such that the truncated Toeplitz operator $T_f^{p,\theta}$ is compact. Since also the Hankel operator H_f^p is compact and

$$P_p M_{\overline{\theta}}^p P_p = (P_q M_\theta^q P_q)^* = (M_\theta^q P_q)^* = P_p M_{\overline{\theta}}^p,$$

the computation

$$\begin{aligned} M_\theta^p H_{\overline{\theta}f}^p - H_f^p &= M_\theta^p (1 - P_p) M_{\overline{\theta}f}^p|_{\mathrm{H}^p(\partial D)} - (1 - P_p) M_f^p|_{\mathrm{H}^p(\partial D)} \\ &= P_p M_f^p|_{\mathrm{H}^p(\partial D)} - M_\theta^p P_p M_{\overline{\theta}f}^p|_{\mathrm{H}^p(\partial D)} \\ &= P_p M_f^p|_{\mathrm{H}^p(\partial D)} - P_p M_\theta^p P_p M_{\overline{\theta}}^p P_p M_f^p|_{\mathrm{H}^p(\partial D)} \\ &= P_p M_f^p|_{\mathrm{H}^p(\partial D)} - T_\theta^p T_{\overline{\theta}}^p P_p M_f^p|_{\mathrm{H}^p(\partial D)} \\ &= (1 - P_p^\theta) T_f^p \end{aligned}$$

shows that the operator $M_\theta^p H_{\overline{\theta}f}|_{(1-P_p^\theta)\,\mathrm{H}^p(\partial D)}$ is compact. But then also the operator

$$H_{\overline{\theta}f}^p (1 - P_p^\theta) = H_{\overline{\theta}f}^p - (1 - P_p) M_{\overline{\theta}f}^p T_\theta^p T_{\overline{\theta}}^p = H_{\overline{\theta}f}^p - (1 - P_p) M_f^p T_{\overline{\theta}}^p = H_{\overline{\theta}f}^p - H_f^p T_{\overline{\theta}}^p$$

and hence $H^p_{\bar{\theta} f}$ is compact. It follows that $f = \theta\overline{\theta} f \in \theta\mathcal{A}$. $\square$

Appendix A

Tools from Banach space theory

Lemma A.1.
Let $T \in \mathrm{L}(X,Y)$ be a bounded linear operator between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ such that there exists a weak zero sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\inf_{n\in\mathbb{N}} \|x_n\|_X > 0$ and $\lim_{n\to\infty} \|Tx_n\|_Y = 0$. Then T has infinite dimensional kernel or non-closed range.

Proof. Let $T \in \mathrm{L}(X,Y)$ and $(x_n)_{n\in\mathbb{N}}$ be given as above and assume that T has finite dimensional kernel and closed range. By the last assumption an application of the inverse mapping theorem yields a constant $C > 0$ such that

$$\inf \{\|x - y\|_X \mid y \in \ker T\} \leq C \|Tx\|_Y$$

for all $x \in X$. Therefore, since $\lim_{n\to\infty} \|Tx_n\|_Y = 0$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\ker T$ with

$$\lim_{n\to\infty} \|x_n - y_n\|_X = 0. \tag{A.1}$$

As a weak zero sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in norm by the uniform boundedness principle. But then from (A.1) it follows that $(y_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence in the finite dimensional Banach space $\ker T$ and hence possesses a convergent subsequence $(y_{n_k})_{k\in\mathbb{N}}$, say with limit $y \in \ker T$. From (A.1) we see that $\lim_{k\to\infty} \|x_{n_k} - y\|_X = 0$. Bearing in mind that $(x_{n_k})_{k\in\mathbb{N}}$ is also a weak zero sequence, we have $y = 0$, which contradicts the hypothesis that $\inf_{k\in\mathbb{N}} \|x_{n_k}\|_X > 0$. □

We now formulate a Banach space version of Lemma 2.1 in [55]. Recall that a Schauder basis of a complex Banach space $(X, \|\cdot\|_X)$ is a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that, for every $x \in X$, there exists a unique sequence $(\alpha_k)_{k\in\mathbb{N}}$ of scalars such that the series

$\sum_{k=0}^{\infty} \alpha_k x_k$ converges in the norm $\|\cdot\|_X$ to x. If $(x_k)_{k\in\mathbb{N}}$ is a Schauder basis for X with coefficient functionals

$$x_k^* \colon X \longrightarrow \mathbb{C}, \qquad \sum_{i=0}^{\infty} \alpha_i x_i \longmapsto \alpha_k \qquad (k \in \mathbb{N}),$$

then the n^{th} natural projection $P_n \in \mathrm{L}(X)$ associated with the basis $(x_k)_{k\in\mathbb{N}}$ is defined by

$$P_n(x) = \sum_{k=0}^{n} \langle x, x_k^* \rangle \, x_k,$$

where $\langle \cdot, \cdot \rangle : X \times X' \to \mathbb{C}$ denotes the usual dual pairing between X and its topological dual X'. The number $M = \sup_{n\in\mathbb{N}} \|P_n\|$ is finite and is called the basis constant for the basis $(x_k)_{k\in\mathbb{N}}$ of X. If, in addition, X is reflexive, the sequence $(x_k^*)_{k\in\mathbb{N}}$ is a Schauder basis of X' with coefficient functionals

$$\langle x_k, \cdot \rangle : X' \longrightarrow \mathbb{C}, \qquad u \longmapsto \langle x_k, u \rangle \qquad (k \in \mathbb{N})$$

and associated natural projections $P_n^* = \sum_{k=0}^{n} \langle x_k, \cdot \rangle \, x_k^*$ (see, e.g., Corollary 4.4.16 in [54]). We refer the reader to [54] for a detailed exposition of the concept of Schauder bases.

Lemma A.2.
Let X be a reflexive Banach space such that X possesses a Schauder basis $(x_k)_{k\in\mathbb{N}}$ with basis constant $M > 0$. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(X)$ such that

(i) $\tau_{\mathrm{SOT}}\text{-}\lim_{n\to\infty} B_n^* = 0$ *in* $\mathrm{L}(X^*)$,

(ii) $c = \lim_{n\to\infty} \|B_n\|$ *exists,*

(iii) $\mathrm{SOT} - \sum_{k=0}^{\infty} B_{n(k)} \in \mathrm{L}(X)$ *exists for each subsequence* $\left(B_{n(k)}\right)_{k\in\mathbb{N}}$ *of* $(B_n)_{n\in\mathbb{N}}$.

Then there is a strictly increasing sequence $(\mu(k))_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for each subsequence $(n(k))_{k\in\mathbb{N}}$ of $(\mu(k))_{k\in\mathbb{N}}$, we have

$$\left\| \mathrm{SOT} - \sum_{k=0}^{\infty} B_{n(k)} \right\|_e \geq \frac{c}{2M}.$$

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a Schauder basis of X with basis constant $M > 0$ and coefficient functionals $(x_k^*)_{k\in\mathbb{N}}$. Let $P_n = \sum_{k=0}^n \langle \,\cdot\,, x_k^*\rangle\, x_k$ and $P_k^* = \sum_{k=0}^n \langle x_k, \cdot\rangle\, x_k^*$ $(n \in \mathbb{N})$ be the natural projections associated to the Schauder bases $(x_k)_{k\in\mathbb{N}}$ and $(x_k^*)_{k\in\mathbb{N}}$, respectively. In a first step we show that there are strictly increasing sequences $(\mu(k))_{k\in\mathbb{N}}$ and $(N(k))_{k\in\mathbb{N}}$ of natural numbers such that the projections

$$E_0 = P_{N(0)}, \quad E_k = P_{N(k)} - P_{N(k-1)} \quad (k \geq 1)$$

satisfy

$$\left\|(1-E_k)B_{\mu(k)}\right\| + \left\|B_{\mu(k)}(1-E_k)\right\| < \frac{2^{-k}}{2M} \tag{A.2}$$

for $k \in \mathbb{N}$. Define $\mu(0) = 0$. Since

$$\|(1-P_k)B_0\| + \|B_0(1-P_k)\| = \|(1-P_k)B_0\| + \|(1-P_k^*)B_0^*\| \xrightarrow{k} 0,$$

there is a natural number $N(0)$ such that

$$\left\|(1-P_{N(0)})B_0\right\| + \left\|B_0(1-P_{N(0)})\right\| < \frac{1}{2M}.$$

Thus condition (A.2) holds for $k = 0$. Suppose that natural numbers $\mu(0) < \mu(1) < \ldots < \mu(n)$ and $N(0) < N(1) < \ldots < N(n)$ have been chosen such that condition (A.2) holds for $k = 0, \ldots, n$. Since

$$\left\|B_k P_{N(n)}\right\| + \left\|P_{N(n)}B_k\right\| = \left\|B_k P_{N(n)}\right\| + \left\|B_k^* P_{N(n)}^*\right\| \xrightarrow{k} 0,$$

there is an integer $\mu(n+1) > \mu(n)$ with

$$\left\|B_{\mu(n+1)}P_{N(n)}\right\| + \left\|P_{N(n)}B_{\mu(n+1)}\right\| < \frac{2^{-k-2}}{2M}.$$

Similarly, one finds an integer $N(n+1) > N(n)$ with

$$\left\|(1-P_{N(n+1)})B_{\mu(n+1)}\right\| + \left\|B_{\mu(n+1)}(1-P_{N(n+1)})\right\| < \frac{2^{-k-2}}{2M}.$$

The estimates

$$\begin{aligned}\left\|(1-E_{n+1})B_{\mu(n+1)}\right\| + \left\|B_{\mu(n+1)}(1-E_{n+1})\right\| &\leq \left\|(1-P_{N(n+1)})B_{\mu(n+1)}\right\| + \left\|B_{\mu(n+1)}(1-P_{N(n+1)})\right\| \\ &\quad + \left\|P_{N(n)}B_{\mu(n+1)}\right\| + \left\|B_{\mu(n+1)}P_{N(n)}\right\| \\ &< \frac{2\cdot 2^{-k-2}}{2M} = \frac{2^{-(k+1)}}{2M}\end{aligned}$$

show that condition (A.2) holds for $k = n+1$. This completes the inductive definition of the sequences $(\mu(k))_{k\in\mathbb{N}}$ and $(N(k))_{k\in\mathbb{N}}$. Using the identities

$$E_k B_{\mu(k)} E_k = B_{\mu(k)} - B_{\mu(k)}(1-E_k) - (1-E_k)B_{\mu(k)}E_k$$

we find that the sequences $(E_k)_{k\in\mathbb{N}}$ and $(B_{\mu(k)})_{k\in\mathbb{N}}$ satisfy the estimates

$$\begin{aligned}\left\|E_k B_{\mu(k)} E_k\right\| &\geq \left\|B_{\mu(k)}\right\| - 2M\left(\left\|B_{\mu(k)}(1-E_k)\right\| + \left\|(1-E_k)B_{\mu(k)}\right\|\right) \\ &> \left\|B_{\mu(k)}\right\| - 2^{-k}\end{aligned} \tag{A.3}$$

for all $k \in \mathbb{N}$. Let $(k_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Then, for $N \in \mathbb{N}$,

$$\sum_{i=0}^{N} B_{\mu(k_i)} = \sum_{i=0}^{N} K_i + \sum_{i=0}^{N} E_{k_i} B_{\mu(k_i)} E_{k_i}$$

with compact operators

$$K_i = B_{\mu(k_i)}(1-E_{k_i}) + (1-E_{k_i})B_{\mu(k_i)}E_{k_i}.$$

Since

$$\|K_i\| \leq 2M\left(\left\|B_{\mu(k_i)}(1-E_{k_i})\right\| + \left\|(1-E_{k_i})B_{\mu(k_i)}\right\|\right) < 2^{-k_i}$$

for all $i \in \mathbb{N}$, it follows that $K = \|\cdot\| - \sum_{i=0}^{\infty} K_i \in \mathcal{K}(X)$ is a well-defined compact operator. Since by hypothesis the series $\mathrm{SOT} - \sum_{i=0}^{\infty} B_{\mu(k_i)}$ converges, also the series

$$A = \mathrm{SOT} - \sum_{i=0}^{\infty} E_{k_i} B_{\mu(k_i)} E_{k_i}$$

yields a well-defined operator $A \in \mathrm{L}(X)$. The estimates proved in (A.3) allow us to choose, for each $i \in \mathbb{N}$, a unit vector $z_i \in X$ with

$$\left\|E_{k_i} B_{\mu(k_i)} E_{k_i} z_i\right\| > \left\|B_{\mu(k_i)}\right\| - 2^{-k_i}.$$

The latter estimates remain true if the vectors z_i are replaced by the vectors $h_i = E_{k_i} z_i$. Since $\|h_i\| \le 2M$ and

$$|\langle h_i, u\rangle| = \left|\left\langle h_i, E_{k_i}^* u\right\rangle\right| \le \|h_i\| \left\|P_{N(k_i)}^* u - P_{N(k_i-1)}^* u\right\|$$

for all $u \in X'$ and $i \ge 1$, the sequence $(h_i)_{i\in\mathbb{N}}$ is a weak zero sequence in X. For $L \in \mathcal{K}(X)$ and $j \in \mathbb{N}$,

$$\begin{aligned}
\|L + A\| &\ge \left\|(L + A)\tfrac{h_j}{2M}\right\| \\
&\ge \left\|\left(L + E_{k_j} B_{\mu(k_j)} E_{k_j}\right) \tfrac{h_j}{2M}\right\| \\
&\ge \frac{\left\|B_{\mu(j)}\right\| - 2^{-k_j}}{2M} - \left\|L\tfrac{h_j}{2M}\right\| \xrightarrow{j} \frac{c}{2M}.
\end{aligned}$$

We conclude that

$$\left\|\mathrm{SOT} - \sum_{i=0}^{\infty} B_{\mu(k_i)}\right\|_e = \|A\|_e \ge \frac{c}{2M}.$$

Thus the proof is complete. □

In the remainder of this section we recall some tools from interpolation theory for Banach spaces. First we recap the complex interpolation method due to A.-P. Calderón [15] and then we will formulate an extrapolation result for compact operators on Banach spaces that was proved by M. Cwikel in [19].

Let E be a Hausdorff topological vector space. We call a linear subspace $X \subset E$ a **Banach subspace** of E if X is equipped with a complete norm topology such that the inclusion mapping $X \to E$ is continuous.

A **compatible couple** (X_0, X_1) of Banach spaces consists of Banach spaces $(X_0, \|\cdot\|_{X_0}), (X_1, \|\cdot\|_{X_1})$ together with injective continuous linear maps $i_j \colon X_j \to E$ $(j = 0, 1)$ into a fixed Hausdorff topological vector space E. If (X_0, X_1) is a compatible couple of Banach spaces, then

$$X_0 + X_1 = i_0(X_0) + i_1(X_1) \subset E,$$

$$X_0 \cap X_1 = i_0(X_0) \cap i_1(X_1) \subset E$$

equipped with the norms

$$\begin{aligned} \|x\|_{X_0+X_1} &= \inf\left\{ \|x_0\|_{X_0} + \|x_1\|_{X_1} \;\middle|\; x_j \in X_j \ (j=0,1) \text{ with } x = i_0(x_0) + i_1(x_1) \right\}, \\ \|x\|_{X_0\cap X_1} &= \max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\} \text{ if } x = i_0(x_0) = i_1(x_1) \end{aligned}$$

are Banach subspaces of E. An **intermediate space** for (X_0, X_1) is by definition a Banach subspace $X \subset E$ such that the inclusion mappings

$$(X_0 \cap X_1, \|\cdot\|_{X_0\cap X_1}) \to X \to (X_0 + X_1, \|\cdot\|_{X_0+X_1})$$

are continuous. Let (X_0, X_1), (Y_0, Y_1) be compatible couples of Banach spaces. A linear map $T\colon X_0 + X_1 \to Y_0 + Y_1$ is called **admissible** if $Ti_j(X_j) \subset i_j(Y_j)$ for $j = 0, 1$ and if the unique operators $T_j\colon X_j \to Y_j$ $(j = 0, 1)$ with $i_jT_j = Ti_j$ are continuous. A pair (X, Y) of intermediate spaces X for (X_0, X_1) and Y for (Y_0, Y_1) is said to be an **interpolation pair** if $TX \subset Y$ for each admissible operator T. In this case the operator $T\colon (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous by the closed graph theorem. We denote its norm by $\|T\|_{\mathrm{L}(X,Y)}$. An interpolation pair (X, Y) as above is said to be of **exponent** $\theta \in (0, 1)$ if

$$\|T\|_{\mathrm{L}(X,Y)} \leq \|T_0\|^{1-\theta} \|T_1\|^{\theta}.$$

for each admissible operator $T\colon X_0 + X_1 \to Y_0 + Y_1$. Let (X_0, X_1) be a compatible couple of Banach spaces. Let $X_0 \cap X_1$, $X_0 + X_1$ be equipped with the complete norms explained above. Define

$$S = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}.$$

The space $A(X_0, X_1)$ consisting of all continuous functions $f\colon \overline{S} \to X_0 + X_1$ such that $f|_S \in \mathcal{O}(S, X_0 + X_1)$ and such that

$$\mathbb{R} \to X_0, t \mapsto f(it) \quad \text{and} \quad \mathbb{R} \to X_1, t \mapsto f(1 + it)$$

are well-defined bounded continuous functions is a Banach space with respect to the norm

$$\|f\| = \max\left\{ \sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1} \right\}.$$

Here we have identified X_0 and X_1 with the Banach subspaces $i_0(X_0), i_1(X_1) \subset E$ with norms $\|i_j(x)\| = \|x\|_{X_j}$ $(j = 0, 1)$. For $\theta \in (0, 1)$ the complex **interpolation space**

$$X_\theta = [X_0, X_1]_\theta \subset X_0 + X_1$$

is defined as the linear subspace of $X_0 + X_1$ consisting of all vectors x such that $f(\theta) = x$ for some $f \in A(X_0, X_1)$. Equipped with the norm

$$\|x\|_{X_\theta} = \inf \{\|f\| \mid f \in A(X_0, X_1) \text{ with } f(\theta) = x\}$$

the space X_θ is a Banach space such that the inclusion mappings

$$X_0 \cap X_1 \longrightarrow X_\theta \longrightarrow X_0 + X_1$$

are well-defined contractions. If (X_0, X_1) and (Y_0, Y_1) are compatible couples of Banach spaces, then (X_θ, Y_θ) is an interpolation pair of exponent θ for every real number $\theta \in (0, 1)$.

Theorem A.3 (M. Cwikel)**.**
Let (X_0, X_1) and (Y_0, Y_1) be compatible couples of Banach spaces. Let $T\colon X_0 + X_1 \to Y_0 + Y_1$ be a linear operator such that T induces bounded operators $T_j\colon X_j \to Y_j$ $(j = 0, 1)$ and such that $T\colon [X_0, X_1]_{\theta^*} \to [Y_0, Y_1]_{\theta^*}$ is compact for one $\theta^* \in (0, 1)$. Then $T\colon [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for all $\theta \in (0, 1)$.

List of Symbols

Common Notation

$\mathrm{B}_\delta(z)$ .... open ball in $\mathbb{C}^d$ with center z and radius δ
$\mathrm{A}(\mathbb{B}), \mathrm{A}(\mathbb{D}^d), \mathrm{A}(D)$ .... ball-, polydisc- and domain algebra (see Section 1.3)
$\mathbb{B}, \mathbb{B}_d$ .... open unit ball in $\mathbb{C}^d$
$|\cdot|$ .... Euclidean norm on $\mathbb{C}^d$
$\mathbb{C}$ .... field of complex numbers
$\mathcal{C}(X)$ .... Calkin algebra on a Banach space X
$\mathrm{C}(X)$ .... complex-valued, continuous functions on a topological space X
χ_M .... characteristic function of a set M
$\overline{\mathrm{LH}}^{w^*}(M)$ .... weak*-closed linear span of M
$\mathcal{K}(X)$ .... compact operators on a Banach space X
$\mathbb{D}$ .... unit disc in $\mathbb{C}$
Δ_B .... maximal ideal space of an unital Banach algebra B
dim .... dimension of a vector space
$\mathrm{dist}(z, M)$ .... distance of a number $z \in \mathbb{C}^d$ to a subset $M \subset \mathbb{C}^d$
M^{ec} .... essential commutant of M
$\mathcal{O}(U)$ .... holomorphic functions on an open set $U \subset \mathbb{C}^d$
Im .... range of a linear mapping
ker .... kernel of a linear mapping
$\mathrm{L}^p(X)$ .... L^p-space with respect to the measure space $(X, \mathcal{A}, \mu)$
$\mathrm{L}(X)$ .... bounded linear operators on a normed space X
$\mathrm{L}(X, Y)$ .... bounded linear operators $X \to Y$ for normed spaces X and Y
$\mathbb{N}, \mathbb{N}^*$ .... non negative integers $\{0, 1, 2, \dots\}$, positive integers $\{1, 2, 3, \dots\}$
$\|\cdot\|_p, \|\cdot\|_\infty$ .... p-norm $(1 \le p < \infty)$, essential supremum norm

∂M .... boundary of a set M
$\mathbb{C}[z]$ .... polynomials over $\mathbb{C}$ in d variables $z = (z_1, \ldots, z_d)$
$\mathbb{R}$ .... field of real numbers
$\sigma_B(b)$ .... spectrum of an element $b \in B$ in an unital Banach algebra B
$\mathbb{T}$ .... unit circle in $\mathbb{C}$
$\sigma_{\mathrm{e}}(T)$ .... essential (Taylor) spectrum of an operator (tuple) T
$\sigma(T)$ .... (Taylor) spectrum of an operator (tuple) T
$\tau_{w^*}, \tau_{\mathrm{SOT}}, \tau_{\mathrm{WOT}}$ .... weak* topology, strong- and weak operator topology
$l^\infty(\mathbb{N})$ .... Banach space of bounded sequences in $\mathbb{C}$
M^c .... complement of a set M
T' .... Banach space adjoint of an operator $T \in \mathrm{L}(X, Y)$
$V \oplus W$ .... direct sum of vector spaces V and W
X' .... topological dual space of a Banach space X

Specific Symbols

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