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Cheater Detection for Randomized Response-Techniques

Derivation, Analyses and Application

Sascha Feth
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Introduction

Quantitative methods for analysing response behaviour in the event of sensitive questions in surveys are extremely important for social science research. They provide insight into research fields that may simply be difficult to access by other means. The Randomized Response-Technique (RRT) provides a tried and tested instrument in various variants for such analyses. The instrument is used in survey studies, e.g. on addictive behaviours, attempts to cheat at universities, the extent of the black economy, the use of doping in sports or the voting behaviour by the general population. All of these research fields have in common that a true answer can be embarrassing, unpleasant or even burdened by criminal consequences for the respondents. The answers must be graded as sensitive and therefore the data that can be assigned to an individual must either be robustly encrypted.

From a social science perspective, we are not interested in tracking the actions of an individual but rather quantifying the proportion of addicts, cheats and fraudsters, illegal workers and dopers in a population. This is precisely what RRT provides. This volume demonstrates the development of various RRT methods, provides an overview of their statistical characteristics, is dedicated in particular to variants of so-called Cheater Detection (i.e. the group of respondents who in spite of the assured encryption do not comply with the RRT instructions) and illustrates practical applications of RRT using numerical simulations. To the interested reader, this offers a more concise and yet in-depth overview of the aspects and problems to be taken into account when applying RRT. This should result in better design for survey studies and therefore to the derivation of more reliable empirical results. RRT has been used repeatedly in the past as part of the research projects by the European Institute for Socioeconomics (EIS). This volume summarises the experience of EIS researchers when dealing with RRT and makes it accessible to a wide range of readers.

The EIS Series is available as an Open Access Publication so that interested readers are not tempted to create a pirated copy or to obtain a copy of the volume by other illegal channels. Therefore, readers of our series of publications may face future RRT-surveys on the frequency of copying books illegally in a relaxed way.

Christian Pierdzioch
Eike Emrich

Authors' foreword

This book moves along the border between social science methodology and mathematical-statistical method development. Studying behaviour in sport that varies from the norm created a need to expand the available statistical model for analysing delinquent behaviour. This double characteristic can also be seen clearly in the book. Starting with the practical questions on the sociology of deviance, it applies mathematical principles of newly developed methods to then return to more practical questions by analysing and numerically simulating the limits of using the method and their practical application. We hope to therefore both satisfy the interested reader from the individual discipline and to also present an example of successful cooperation across the limits of the disciplines.

Sascha Feth
Monika Frenger
Werner Pitsch
Patrick Schmelzeisen

1 Introduction

Finding out how often an embarrassing characteristic exists in a population is accompanied by numerous problems that can be counteracted with particular survey models or compensated for in the calculations. The idea on which the development of indirect survey methods is based is the well-known fact in research that a significant distortion exists for socially undesirable answers if questions are asked about embarrassing issues (cf. Lee, 1993; Holbrook & Krosnik, 2010, 2012). In order to counteract this distortion resulting from a high number of deniers (Rasinski, Willis, Baldwin, Yeh & Lee, 1999) and to get closer to the true score, various techniques are applied, such as unobtrusive research that is free of consequences for those being surveyed (cf. Lee, 2000), social desirability scales (e.g. Stoeber, 2001; Thompson & Phua, 2005) or techniques designed to increase the trust of those answering such that they themselves admit to the socially undesirable behaviour (e.g. the unmatched count or item count technique, refer to Ahart & Sackett, 2004; Coutts & Jann, 2011; Tourangeau & Yan, 2007; randomized count technique, cf. Frenger, Pitsch & Emrich, 2013 or single sample count technique, cf. James, Nepusz, Naughton & Petróczi, 2013). The Randomized Response-Technique (RRT) was the first of these techniques stated above. It currently remains the most widely used, evaluated and researched technique when undertaking research on sensitive issues.

It has been demonstrated repeatedly that these methods significantly increase the willingness to admit to an embarrassing characteristic when compared with direct surveys. To prove this, the results of direct questions are compared with the responses to randomized response questions under the assumption that the embarrassing characteristic is frequently denied in direct questions due to a social desirability bias. Thus a higher estimated prevalence of an embarrassing characteristic in randomized response questions compared to direct questions indicates that it has been possible to at least partially compensate for this distortion. Such studies were conducted for example

- on illegal abortions (Abernathy, Greenberg & Horvitz, 1970; Greenberg, Kuebler, Abernathy & Horvitz, 1971; Krotki & Fox, 1974; Rider, Harper, Chow & I-Cheng, 1976; I-Cheng, Chow & Rider, 1972),

- on the consumption of legal and illegal drugs (Barth & Sandler, 1976; Brewer, 1981; Akers, Massey, Clarke & Lauer, 1983; Goodstadt, Cook & Gruson, 1987; Danermark & Swensson, 1987),

- on fraud relating to claiming social security benefits (Böckenholt & van der Heijden, 2007),

- on complying with the rules on environmental and species protection (Chaloupka, 1985),

- on plagiarism (Coutts, Jann, Krumpal & Näher, 2011; Jerke & Krumpal, 2013; Krumpal, Jerke & Voss, 2016) or other forms of student cheating (Kerkvliet, 1994; Shotland & Yankowski, 1982; Scheers & Dayton, 1987),

- on improper behaviour at work (Reckers, Wheeler & Wong-On-Wing, 1997; Soeken & Macready, 1986) and

- on sexual behaviour (Fidler & Kleinknecht, 1977; Williams & Suen Hoi, 1994).

On various occasions several embarrassing characteristics were also surveyed within one instrument (Coutts & Jann, 2011; Edgell, Himmelfarb & Duchan, 1982; Fisher, Kupferman & Lesser, 1992). With a few exceptions (Duffy & Waterton, 1988; Krotki & Fox, 1974) the RRT resulted in higher estimates of the prevalence of the embarrassing characteristic(s) than the use of direct questions. This demonstrated for a variety of sensitive questions that the RRT can reduce the response bias for socially undesirable behaviours. As Wolter (2012, 96) found, however, there is significant incongruity between a number of methodological projects undertaken with RRT (see above) and the very few applications dealing with academic issues (on doping in sports cf. Pitsch & Emrich, 2012; Simon, Striegel, Aust, Dietz & Ulrich, 2006, on match fixing in football cf. Pitsch, Emrich & Pierdzioch, 2013, on competition fraud in elite sports cf. Pitsch, Frenger, Emrich & Pierdzioch, 2015).

The concept and development of RRT will be described below in order to clarify how this method can generate reliable answers yet maintain discretion. We will only provide a summary on the development of the method without detailing each variation in the mathematical background. We recommend that the reader refers here to the relevant developments and the associated mathematical basis in the original literature (e.g. Lensvelt-Mulders, Hox, van der Heijden & Maas, 2005; Lee, 1993; Coutts & Yann, 2011; Antonak & Livneh, 1995; Wolter, 2012).

1.1 Theoretical Background and Development of the Randomized Response-Technique

In the first RRT model by Warner (1965) two different questions are asked (regarding agreement with the statement) in order to find out whether the person being questioned is part of the group X or not (cf. Figure 1.1 using doping as an example).

Based on the results of a random process the respondents were asked with a known probability to answer question 1 or with the counter probability to answer the inverse question 2. Since the result of the random process is unknown to the researcher, answering yes cannot be equivalent to the person questioned actually having the embarrassing characteristic X. Only the difference in the rate of yes responses can be used to estimate the prevalence π_X of the characteristic X in the studied population. Warner (1965) was able to demonstrate that the calculated estimate $\hat{\pi}_X$ is an unbiased and normally distributed estimate for a true proportion of X in the population.

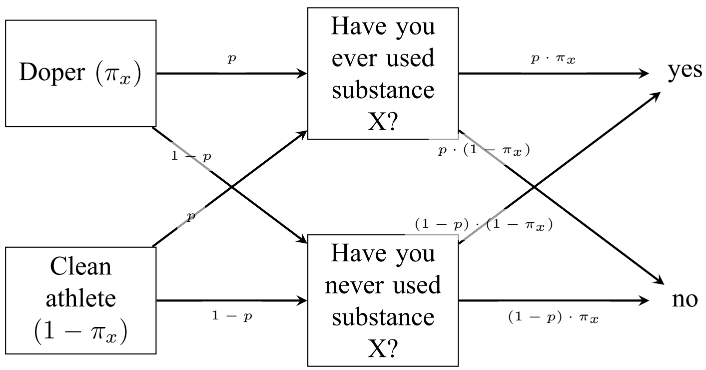

Figure 1.1: Warner's model using illicit doping substances as an example (cf. Warner, 1965).

Since Warner (1965) introduced the method there have been numerous improvements and new developments. Three methods that later proved to be effective in many areas of socially deviant behaviour were

- the Unrelated Question Model,
- the Forced Answer Model and
- the Cheater Detection Model.

These three RRT models will be briefly described below.

1.2 The Unrelated Question Model

The fact that in Warner's model all participants answer the embarrassing question (worded positively or negatively) leads to the well-founded assumption that respondents will answer more honestly if there are two different questions. One question should refer to the embarrassing characteristic and the second to a completely independent characteristic that is not embarrassing (Horvitz, Shah & Simmons, 1967). Due to the randomisation effect neither a yes nor a no answer can clearly indicate that the person providing the answer has or does not have the embarrassing characteristic. The theoretical and mathematical background for this model goes back to Greenberg, Abul-Ela, Simmons and Horvitz (1969). If this model is used the participants are asked two different questions (cf. Figure 1.2). They are requested to use a random generator (e.g. coin or dice) without informing the study leader of the result of the random process. Depending on the result, the participant is requested with a probability of p to answer the first question or with a probability of $1 - p$ to answer the second question.

Here X represents the embarrassing characteristic and Y the unrelated characteristic that is not embarrassing (this question could alternatively also be: "Have you visited New York?" or "Were you born in North Carolina?"). Greenberg et al. (1969) have shown that the Unrelated Question Model leads to better results than Warner's original model. This applies if the (unknown) prevalence π_X of X is significantly different from 0.5, some restrictions are observed for p and the unrelated characteristic Y is selected such that π_X and π_Y are both either higher or lower than 0.5. The researchers could simulate the relative efficiency of the Unrelated Question Model even if the occurrence of the embarrassing characteristic was not fully reported truthfully. In later applications, Unrelated Question Models were also used for frequent occurrences of the characteristic that was not embarrassing Y (e.g. "Were you born in the month of April?" Scheers, 1992) as well as for characteristics with an unknown prevalence ("Have you ever visited New York?"; Tracy & Fox, 1981). Figure 1.3 shows a schematic depiction of this model.

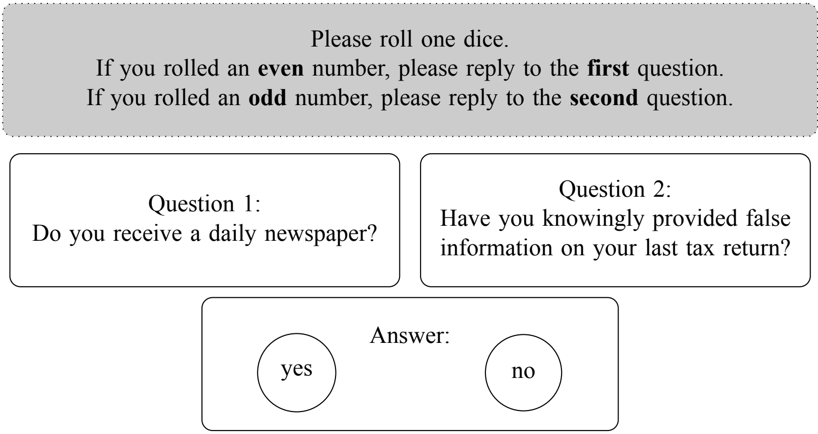

Figure 1.2: Sample case for the Unrelated Question Model.

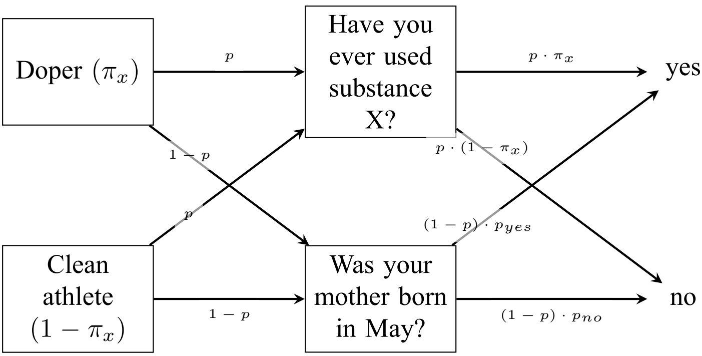

Figure 1.3: Unrelated Question Model (without Cheater Detection) according to Greenberg et al. (1969).

1.3 The Forced Answer Model

The Forced Answer Model (also called the Forced Response Model) was explicitly stated for the first time by Greenberg et al. in 1969 but was not formally developed. Fidler and Kleinknecht (1977) reported on its first use whereby both the yes and the no responses were set as forced answers. In contrast to this and without referring to these sources, Dawes and Moore (1980) later proposed an asymmetric variant in which only the embarrassing answer, so usually the yes, was set as a forced answer with the aim of reducing the variance of the estimator π_X. A sample for how this model could be applied is shown in Figure 1.4.

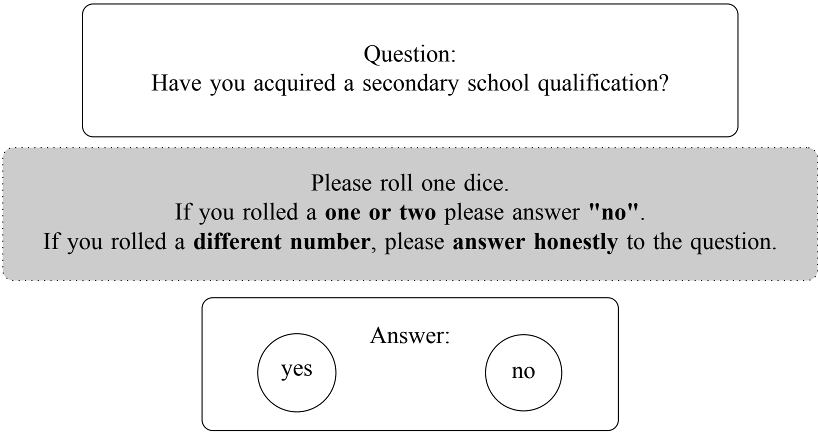

Figure 1.4: Forced Answer Model without Cheater Detection for an embarrassing no answer.

With this model a yes response cannot be clearly assigned to the existence of the embarrassing characteristic X and the anonymity of the participant is completely assured. Alternatively using the instruction "..., please say no", the no answer can be protected if it was the embarrassing question that was answered (e.g. for the question "Do you have qualifications?" or for surveys in delinquent sub-cultures according to Cohen & Short, 1958). Although the embarrassing characteristic X is perfectly protected by this method, the perceived security is however not as high as that of the symmetrical Forced Answer Models (cf. Bourke, 1984). In similar depiction forms to the Warner's model (Figure 1.1) and the Unrelated Question Model (Figure 1.3), Figure 1.5 shows the schematic depiction of the Dawes and More (1980) Forced Answer Model.

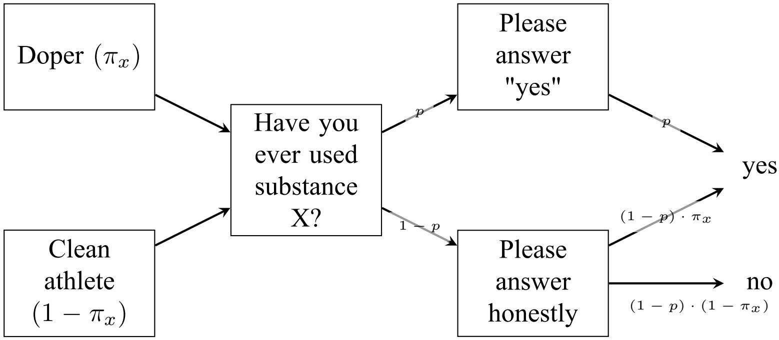

Figure 1.5: Forced Answer Model without Cheater Detection

1.4 The Cheater Detection Model

The fact that for RRT surveys there are still a certain number of participants who feel threatened was already discussed by Greenberg, Kuebler, Abernathy and Horvitz (1977) and demonstrated empirically by Tracy and Fox (1981), by Edgell, Himmelfarb and Duchan (1982) and by van der Heijden, van Gils, Bouts and Hox (2000). It may occur that individual people do not trust the anonymity of the method and therefore do not answer truthfully in spite of the special technique used. In addition, participants may consciously attempt to manipulate the outcome of the survey (for the effects of this behaviour cf. Becker, 2010 and Böckenholt, Barlas & van der Heijden, 2009; Edgell, Himmelfarb & Duchan, 1982 as well as the summary in Wolter, 2012). In addition to many other ideas on developing RRT that should result in higher trust by participants, Clark and Desharnais (1998) developed the Cheater Detection Model (cf. schematic depiction in Figure 1.6). Here cheating means that the RRT instruction is not applied correctly no matter whether the embarrassing characteristic exists or not. It is important to note that by addressing the problem of cheating in RRT, the logic of the method shifts from estimating the prevalence of an embarrassing property π_X to estimating proportions of honest yes and honest no answers.

To estimate the proportion of cheaters, the sample is randomly divided into two equally sized subsamples. Each subsample receives a Forced Answer question with different probabilities for this question. The increased degrees of freedom not only enables the researcher to estimate the proportions of two groups (π_X and $1-\pi_X$) in the population but also to estimate the proportions of the following three groups:

- π_X – the proportion of honest yes answers, which states the lowest possible prevalence of the embarrassing characteristic X,

- β – the proportion of no answers which shows the lowest possible rate of the population members who do not have the characteristic X and

- γ – the proportion of cheaters, i.e. those who do not answer as per the instructions in the RRT question.

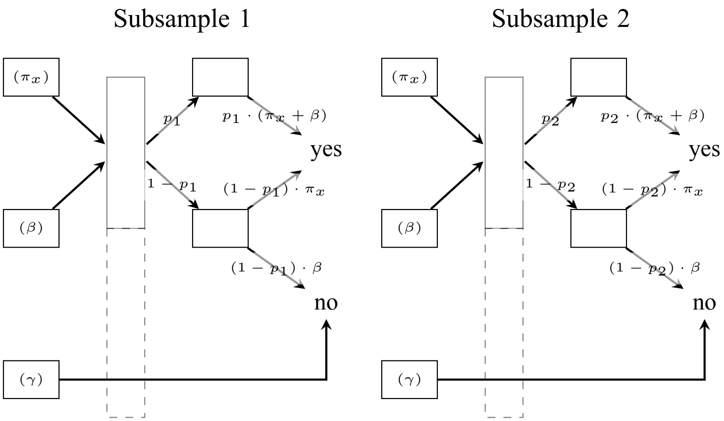

Figure 1.6: Schematic depiction of the Forced Answer Model with Cheater Detection: a doubled Forced Answer Model with $p_1 \neq p_2$ *and* $\pi_X + \beta + \gamma = 1$.

In this regard, it is important that cheating occurs when those who have the characteristic X answer no in cases where they should honestly answer yes or should do so on the basis of a forced answer. But cheating can also occur if persons without the characteristic X answer no although the randomisation led to the result that they should answer yes. Not following the RRT instructions may also be due to misunderstanding the instructions, the randomisation result or distrusting the anonymity of the response. The method therefore provides no information on the cheaters no matter the reason for cheating or the presence or absence of the embarrassing characteristic.

In the first publication of the method, Clark and Desharnais (1998) offer a χ^2-test of the working hypothesis $\gamma > 0$ and propose excluding the results if the test reveals a significant proportion of cheaters. In later applications, the estimated proportion of honest yes responses $\hat{\pi_X}$ was reported as the lower limit of a prevalence interval. For this the cheater proportion was added to obtain the upper limit $\hat{\pi}_X + \hat{\gamma}$ of the interval in which the true score for the prevalence π_X in the population is assumed (e.g. Musch, Bröder & Klauer, 2001; Pitsch & Emrich, 2012). In contrast to other models to detect cheating in RRT surveys, this technique works with only one RRT question while the other models published so far detect cheating based on an impossible pattern of honest answers to (at least) two different RRT questions (e.g. Cruyff, van den Hout, van der Heijden, & Böckenholt, 2007) or to the same RRT question if it was asked (at least) twice (Krishnamoorthy & Raghavarao, 1993).

In the logic of classical RRT, it is always clear which characteristic is embarrassing and will result in distortions as a result of social undesirability. In certain sub-cultures the non-existence of the characteristic X may however be embarrassing. One example of this would be the consumption of illegal drugs by young people. In general when studying deviant sub-cultures there is always the possibility of over-estimating the prevalence of the characteristic which is embarrassing in the main culture due to sub-cultural norms that deviate from the main culture (Cohen & Short, 1958; Cohen, 1957; Mays, 1957). In this case, respondents incorrectly ascribe the characteristic X to themselves and contrary to the instructions always answer with yes. In addition, these distortions can also occur for strategic response behaviour if respondents sabotage a survey hoping for advantages arising from biased results or for any other reason. These considerations resulted in the generalisation of the idea of Clark and Desharnais (1998) to estimate the proportion of "yes" cheaters, i.e. those who falsely assigned the embarrassing characteristic to themselves. As a result of these considerations, this problem is currently being handled mathematically. The aim is to develop and validate a Total Cheater Detection so that both forms of cheating can be depicted.

The notation for the coming chapters and basic modelling will now be explained in order to derive the new, extended method building on this. The estimation is then analysed, a comparison is made of the different methods and finally explanations are provided for practical applications. In the final outlook section, there is a brief discussion on potential developments of the procedure beyond the status described here.

2 Modeling and Notation

2.1 General Notation

For $v \in \mathbb{R}^n$ let v_i denote the ith component of v.

For $n \in \mathbb{N}$ we define $0_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ and $1_n := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$.

Let $M \in \mathbb{R}^{n \times m}$ be a matrix with full column rank. Then the columns of M are linearly independent and the pseudo inverse $M^+ = \left(M^T M\right)^{-1} M^T$ does exist. In the case of square M the inverse M^{-1} exists and due to

$$\left(M^T M\right)^{-1} M^T = M^{-1} \left(M^T\right)^{-1} M^T = M^{-1},$$

the inverse matrix M^{-1} is identical to the pseudo inverse M^+. Pseudo inverse matrices are used for solving overdetermined systems of linear equations. For a given coefficient matrix $M \in \mathbb{R}^{n \times m}$ with $n \geq m$ and the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, the solution of $Mx = y$ is approximated by $\hat{x} = M^+ y$. The approximation minimizes the sum of squared errors $\sum_{i=1}^{n} \left(M\hat{x} - y_i\right)^2$.

2.2 Population

We divide the population into four groups, according to their answer to the embarrassing question:

- α - the proportion of persons with property X who will honestly answer yes (honest yes).
- β - the proportion of persons who will answer no, no matter if they have property X or if they do not ("no" cheaters).
- γ - the proportion of persons without property X who will honestly answer no (honest no).

- δ - the proportion of persons who will always answer yes, no matter if they have property X or if they do not ("yes" cheaters).

Let the group name also denote the proportions within the population. This yields the canonical conditions:

$$\alpha + \beta + \gamma + \delta = 1 \qquad \text{and} \qquad 0 \leq \alpha, \beta, \gamma, \delta \leq 1.$$

2.3 Randomized Response-Technique

RRT with Forced Answer (see Chapter 1.3) will serve as our basic questioning technique. It is now desirable to protect the yes answers as well as the no answers. Similar to the technique of Clark and Desharnais (1998), all respondents are separated into several subsamples, differing in their Forced Answer probabilities. In general, to obtain s population parameters, at least $s - 1$ subsamples are needed. Each RRT subsample yields one claim for the population parameters. Together with $\alpha + \beta + \gamma + \delta = 1$ we have $s = 4$ equations, to be solved simultaneously.
For an RRT with R subsamples we denote

- $P^{\mathrm{Y}} \in \mathbb{R}^R$ the forced yes-probabilities,
- $P^{\mathrm{N}} \in \mathbb{R}^R$ the forced no-probabilities and
- $P^{\mathrm{E}} \in \mathbb{R}^R$ the probability of the embarrassing question.

The canonical boundary conditions are

$$0 \leq P_i^{\mathrm{Y}}, P_i^{\mathrm{N}}, P_i^{\mathrm{E}} \leq 1, \quad 1 \leq i \leq R \qquad \text{and} \qquad P^{\mathrm{Y}} + P^{\mathrm{N}} + P^{\mathrm{E}} = 1_R.$$

Another RRT model is based on the Unrelated Question method (see Chapter 1.2). A random number generator takes over the selection of either the embarrassing or an innocuous question like "Were you born in a month with an odd number?". The answering probabilities of the innocuous question are known. The mathematical transformation of Unrelated Question to Forced Answer is very easy. Let P denote the probability of posing the innocuous question and ϕ the probability of answering it with yes, then: $P^{\mathrm{Y}} = P\phi$, $P^{\mathrm{N}} = P(1 - \phi)$ and $P^{\mathrm{E}} = 1 - P$.

2.4 Overview of Methods

In the following, three methods will be derived:

1. Total Cheater Detection (TCD), allowing the estimation of all population parameters above.
2. "No" Cheater Detection (NCD), assuming $\delta = 0$ and estimating the remaining parameters.
3. "Yes" Cheater Detection (YCD), assuming $\beta = 0$ and estimating the remaining parameters.

The latter two items will be summarized as Directed Cheater Detection (DCD). Both methods do not estimate the individual parameter assumed to be zero and are inaccurate if this assumption is violated in practice. However, they can deliver good results compared to TCD for smaller samples if this assumption holds.

3 Derivation of the Method

3.1 Derivation of the Estimators

The estimators are derived for all methods using the same scheme. We firstly assume in very general terms an RRT survey with R different subsamples and a method that estimates the s population parameters and then use the general solution to derive each method. As already noted in Section 2.3, R must be greater or equal to $s-1$ in order to carry out the estimation. Typically, one will chose $R = s-1$ for reasons of economy. When deriving the method to estimate four proportions we must also consider various sub-estimators.

Each method has precisely one estimator that calculates $s-1$ population parameters. The remaining parameter is determined via $\alpha+\beta+\gamma\ldots = 1$. We call this estimation the **native estimation** or the **native solution** due to the parameters estimated by it. We summarise these parameters as $\Theta = (\alpha, \beta, \gamma, \ldots)^T$.

In addition, we have to consider marginal cases where a sub-quantity of the population parameters takes on the value zero. The number of parameters assumed to be zero is denoted by t.

Stated differently, for each t-element sub-set of the population parameter with $0 \leq t < s$ we must consider a special estimator. The required number of special estimators is

$$\sum_{t=0}^{s-1} \binom{s}{t} = \sum_{t=0}^{s} \binom{s}{t} - 1 = 2^s - 1$$

for each method.

For each estimator, the probability of a yes answer in each RRT subsample can be calculated from the population parameters and the Forced Answer probabilities. This creates an equation system in the form of $\lambda = M\Theta$, whereby a $M \in \mathbb{R}^{R\times(s-t-1)}$ (typically: $M \in \mathbb{R}^{R\times(R-1)}$) coefficient matrix is dependent on the Forced Answer probability, $\Theta \in \mathbb{R}^{(s-t-1)}$ is the vector of the population parameter to be estimated and $\lambda \in \mathbb{R}^R$ is the vector of calculated yes probabilities.

Example

We consider the marginal case $\alpha = 0$ of an RRT with TCD with $R = 3$ RRT subsamples. The TCD estimates $s = 4$ parameters $(\alpha, \beta, \gamma, \delta)$. Accordingly for this marginal case $t = 1$. As there are no persons with property X in the population ($\alpha = 0$), the contributions to the yes answers are provided by honest persons without property X who follow the instruction to answer yes and by "yes" cheaters who always answer yes. Therefore we obtain $\lambda = \gamma P^Y + \delta \in \mathbb{R}^R = \mathbb{R}^3$. The parameters to be estimated are $\Theta = \gamma, \delta \in \mathbb{R}^2$, since $\beta = 1 - \alpha - \gamma - \delta$ and $\alpha = 0$ as per the marginal case. From $\lambda = M\Theta$ we can now derive

$$M = \begin{pmatrix} P_1^Y \\ P_2^Y \\ P_3^Y \end{pmatrix} \in \mathbb{R}^{R \times (s-t-1)} = \mathbb{R}^{3 \times 2}.$$

3.1.1 ML Equation for the Probabilities of a Yes Answer

If the number of yes answers in the subsamples $k \in \mathbb{N}^R$ and the subsample sizes $N \in \mathbb{N}^R$ are given, we can calculate the probability of a yes answer in each RRT subsample using the maximum likelihood method. Within each subsample there is a binomial distribution for the number of yes answers. The subsamples are also surveyed independently of each other. If X_i denotes the number of yes answers in the subsample i the likelihood can be shown to be

$$\begin{aligned} L_\lambda(\lambda) &= \mathbb{P}_\lambda(X_1 = k_1, \ldots, X_R = k_R) \\ &= \prod_{i=1}^{R} \mathrm{P}_\lambda(X_i = k_i) \text{ (independence)} \\ &= \prod_{i=1}^{R} \begin{pmatrix} N_i \\ k_i \end{pmatrix} \lambda_i^{k_i} (1 - \lambda_i)^{N_i - k_i} \text{ (binomial distribution)}. \end{aligned}$$

The use of the logarithmic likelihood

$$l_\lambda(\lambda) := \ln(L_\lambda(\lambda)) = \sum_{i=1}^{R} \ln \begin{pmatrix} N_i \\ k_i \end{pmatrix} + k_i \ln(\lambda_i) + (N_i - k_i) \ln(1 - \lambda_i)$$

is appropriate for further calculation.

3.1.2 Solution of the ML Equation

***Lemma* 1.** *The logarithmic likelihood l_λ has its global maximum at $\hat{\lambda}_i = \frac{k_i}{N_i}$.*

Proof. We firstly determine the partial derivatives

$$\frac{\partial}{\partial \lambda_i} l_\lambda(\lambda) = \frac{k_i}{\lambda_i} - \frac{N_i - k_i}{1 - \lambda_i}.$$

$\lambda_i = \frac{k_i}{N_i}$ arises as the zeros for the partial derivatives. In order to demonstrate that this really is the maximum, it is enough to show that the Hesse matrix $\mathrm{D}^2 l_\lambda$ is negative definite at this point. The second partial derivatives are

$$\frac{\partial^2}{\partial \lambda_i \partial \lambda_j} l_\lambda(\lambda) = \begin{cases} -\frac{k_i}{\lambda_i^2} - \frac{N_i - k_i}{(1-\lambda_i)^2} & i = j \\ 0 & i \neq j \end{cases}.$$

The following also applies

$$\operatorname{sgn}\left(-\frac{k_i}{\lambda_i^2} - \frac{N_i - k_i}{(1-\lambda_i)^2}\right) = -1$$

because $k_i, N_i - k_i \geq 0$ and $N_i > 0$. So it follows that $k_i \neq 0 \vee N_i - k_i \neq 0$. The Hesse matrix therefore has the form

$$\mathrm{D}^2 l_\lambda(\lambda) = \begin{pmatrix} -\frac{k_1}{\lambda_1^2} - \frac{N_1 - k_1}{(1-\lambda_1)^2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{k_R}{\lambda_R^2} - \frac{N_R - k_R}{(1-\lambda_R)^2} \end{pmatrix}$$

with exclusively negative elements on the main diagonals and otherwise zeros everywhere. Therefore the leading main minors have alternating preliminary signs and the Hesse matrix is negative definite. The logarithmic likelihood l_λ therefore takes on its maximum at $\hat{\lambda}_i = \frac{k_i}{N_i}$.

In order to now determine an ML estimator for Θ, we also have to demand that the matrix M has the full column rank $s - t - 1$. For further derivation we assume that this claim has been met. In Chapter 4.3 we will investigate how the Forced Answer likelihoods are to be selected so that this assumption in fact holds. We now have to distinguish two cases:

Case 1: Square M

For square $M \neq 0$ the mapping $\Theta \mapsto M\Theta$ is bijective (because M has full column rank) and we obtain $l(\Theta) := l_\lambda(M\Theta)$ as the likelihood function. Since the mapping is bijective, we then obtain the ML estimator for the population parameter from $\hat{\Theta} = M^{-1}\hat{\lambda}$.

Case 2: Rectangular M

If M is not square, $\hat{\Theta} = M\hat{\lambda}$ does not create a pure ML estimator. Although $\hat{\lambda}$ was calculated using the ML method, the use of the pseudo-inverses matches here the least squares solution of the equation system $M\hat{\Theta} = \hat{\lambda}$. The least squares method only provides an ML estimation here if the right side of the equation system is normally distributed. In our case however the $\hat{\lambda}_i$ has a binomial distribution.

So we only get a real ML estimator for a regular coefficient matrix. This is only possible for the $R = s - t - 1$ case. Since $R \geq s - 1$ it then follows that this is only possible for native solutions. So that at least the native solution is a real ML estimator, the method must be selected according to the number of RRT subsamples $R = s - 1$.

3.1.3 Calculation of the Population Parameters

We will define the coefficient matrix of the estimator as the **process matrix**. The approach to calculating the population parameters depends on whether the process matrix M is square (native solution) or not (marginal solution).

Case 1: Square M

Although the solution stated above $\hat{\Theta} = M^{-1}\hat{\lambda}$ maximises the likelihood function it only considers the condition $\alpha + \beta + \gamma + \delta = 1$. As these parameters only deal with parts of the population, only $\alpha, \beta, \gamma, \delta \in]0, 1[$ are permissible. If this requirement is not met, all marginal solutions and their likelihoods must be calculated. The best estimation for the population parameters is then provided by the marginal solution with the highest likelihood that meets the requirements for α, β, γ and δ.

Case 2: Rectangular M

In this case the likelihood of the native solution $\hat{\Theta} = M\hat{\lambda}$ must also be compared with all of the marginal solutions no matter whether $\hat{\Theta}$ meets the marginal conditions as the native solution in this case is not based on an ML estimator.

3.1.4 ML Equation for the Probabilities of a No Answer

When deriving the method, we will occasionally in place of the equation system $\lambda = M\Theta$ for the likelihoods of a yes response refer to an equation system $1_R - \lambda = M^{'}\Theta$ depicting the likelihoods of a no response.

Lemma 2. *By* $\hat{\mu} := 1_R - \hat{\lambda} = 1_R - \frac{k_i}{N_i}$ *an ML estimator is given for the probability of a no response.*

Proof. The likelihood is calculated with k and N as above

$$\begin{aligned}
L_\mu(\mu) &= \mathbb{P}_\lambda (X_1 = N_1 - k_1, \ldots, X_R = N_R - k_R) \\
&= \prod_{i=1}^{R} \mathbb{P}_\lambda (X_i = N_i - k_i) \\
&= \prod_{i=1}^{R} \binom{N_i}{N_i - k_i} \mu_i^{N_i - k_i} (1 - \mu_i)^{k_i} \\
&= \prod_{i=1}^{R} \binom{N_i}{k_i} (1 - \lambda_i)^{N_i - k_i} \lambda_i^{k_i} \\
&= L_\lambda(\lambda)
\end{aligned}$$

and we find the global maximum as per Lemma 1 for $\lambda_i = \frac{k_i}{N_i}$.

Below we will write $1 - \hat{\lambda}$ instead of $\hat{\mu}$ because the results of the calculations suggest this syntax and in order to keep the notation as short as possible.

3.2 Derivation of the TCD

To derive the TCD method, we will first have to calculate the likelihood of a yes answer in order to find estimators for the population parameters. For TCD, there will be one native solution as well as marginal solutions in three different orders.

3.2.1 Calculation of the Yes Likelihoods

We use three RRT subsamples for this method. More subsamples are also possible but as we have seen above, we can at least ensure that for three RRT subsamples the native solution is based on an ML estimator.

We have the following proportions for the possibility of a yes answer:

- A person from group α always answers the sensitive question with yes.
- A person from group α follows the Forced Answer instruction (see Chapter 1.3) and answers yes. This answer yes is less embarrassing than answering yes to the incriminating question. Therefore, this assumption is highly plausible.

- A person from group γ follows the instruction and answers yes. If persons from group γ were to answer no although instructed to answer yes, they were "no" cheaters (β) as they would always answer no.

- A person from group δ always answers yes.

***Corollary* 1.** *For TCD the probabilities of the yes answers are given by*

$$\begin{aligned}\lambda &= \alpha\left(P^{E}+P^{Y}\right)+\beta\cdot 0_3+\gamma P^{Y}+\delta\cdot 1_3\\&=\alpha\left(1_3-P^{N}\right)+\gamma P^{Y}+\delta\cdot 1_3.\end{aligned}$$

3.2.2 Calculation of the Population Parameter Estimators

We now want to interpret the permissible value range of the estimators. We see that β can only be estimated implicitly via $\beta = 1-\alpha-\gamma-\delta$ as a result of the yes probabilities. The permissible value range of the estimators can therefore be shown via the remaining three parameters in the $\mathbb{R}^3$. For α, γ and δ the conditions $\alpha, \gamma, \delta \geq 0$ and $\alpha+\gamma+\delta \leq 1$ always apply. Each of these four terms defines a closed half-space in the $\mathbb{R}^3$. The valid range for α, γ, δ arises where these half-spaces intersect. This creates an asymmetric tetrahedron (cf. Figure 3.1).

Native Solution

Permissible values of the native solution are precisely inside the tetrahedron. With $M = \left(1_3 - P^{\mathrm{N}}, P^{\mathrm{Y}}, 1_3\right)$ we end up with the parameters

$$\hat{\Theta} = \begin{pmatrix}\hat{\alpha}\\ \hat{\gamma}\\ \hat{\delta}\end{pmatrix} = M^{-1}\hat{\lambda} \quad \text{and} \quad \hat{\beta} = 1-\hat{\alpha}-\hat{\gamma}-\hat{\delta}.$$

First Order Marginal Solutions

For these marginal solutions exactly one of the population parameters is equal to zero. The other three add up to one and this is why the permissible solutions are located precisely on the faces of the tetrahedron.

1st Marginal Case $\alpha = 0$

The approach simplifies to

$$\lambda = \gamma P^{\mathrm{Y}} + \delta \cdot 1_3.$$

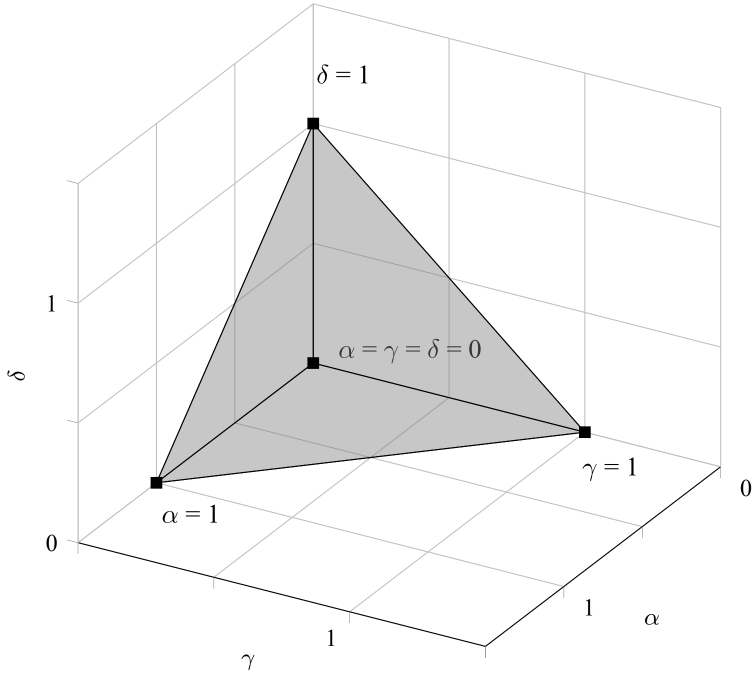

Figure 3.1: Depiction of the definition range from α, γ and δ. All permissible points $(\alpha, \gamma$ and $\delta)$ are inside and on the facets of the tetrahedron.

With $M = \left(P^{\mathrm{Y}}, 1_3\right)$, we obtain the parameters

$$\hat{\Theta} = \begin{pmatrix} \hat{\gamma} \\ \hat{\delta} \end{pmatrix} = M^{+}\hat{\lambda}, \quad \hat{\alpha} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\gamma} - \hat{\delta}.$$

2nd Marginal Case $\beta = 0$

Due to $\delta = 1 - \alpha - \gamma$ the approach may be rewritten as

$$\lambda = \alpha(1_3 - P^{\mathrm{N}}) + \gamma P^{\mathrm{Y}} + 1_3 - \alpha \cdot 1_3 - \gamma \cdot 1_3 \Leftrightarrow 1_3 - \lambda = \alpha P^{\mathrm{N}} + \gamma \left(1_3 - P^{\mathrm{Y}}\right).$$

With $M = \left(P^{\mathrm{N}}, 1_3 - P^{\mathrm{Y}}\right)$, we obtain the parameters

$$\hat{\Theta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = M^{+}\left(1 - \hat{\lambda}\right), \quad \hat{\beta} = 0 \quad \text{and} \quad \hat{\delta} = 1 - \hat{\alpha} - \hat{\gamma}.$$

3rd Marginal Case $\gamma = 0$

Due to $\delta = 1 - \alpha - \beta$ the approach may be rewritten as

$$\lambda = \alpha(1_3 - P^{\mathrm{N}}) + 1_3 - \alpha \cdot 1_3 - \beta \cdot 1_3 \Leftrightarrow 1_3 - \lambda = \alpha P^{\mathrm{N}} + \beta \cdot 1_3.$$

With $M = \left(P^{\mathrm{N}}, 1_3\right)$, we obtain the parameters

$$\hat{\Theta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = M^+ \left(1 - \hat{\lambda}\right), \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 1 - \hat{\alpha} - \hat{\beta}.$$

4th Marginal Case $\delta = 0$

The approach simplifies to

$$\lambda = \alpha(1_3 - P^{\mathrm{N}}) + \gamma P^{\mathrm{Y}}.$$

With $M = \left(1_3 - P^{\mathrm{N}}, P^{\mathrm{Y}}\right)$, we obtain the parameters

$$\hat{\Theta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = M^+ \hat{\lambda}, \quad \hat{\delta} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\alpha} - \hat{\gamma}.$$

Second Order Marginal Solutions

For these solutions two population parameters equal zero. This leads to a point on the intersection of two facets of the tetrahedron, respectively on one of its edges.

5th Marginal Case $\alpha = \beta = 0$

Due to $\delta = 1 - \gamma$ the approach may be rewritten as

$$\lambda = \gamma P^{\mathrm{Y}} + 1_3 - \gamma \cdot 1_3 \Leftrightarrow 1_3 - \lambda = \gamma \left(1_3 - P^{\mathrm{Y}}\right).$$

With $M = \left(1_3 - P^{\mathrm{Y}}\right)$, we obtain the parameters

$$\hat{\gamma} = M^+ \left(1 - \hat{\lambda}\right), \quad \hat{\alpha} = \hat{\beta} = 0 \quad \text{and} \quad \hat{\delta} = 1 - \hat{\gamma}.$$

6th Marginal Case $\alpha = \gamma = 0$

The approach simplifies to

$$\lambda = \delta.$$

With $M = 1_3$, we obtain the parameters

$$\hat{\delta} = M^+ \hat{\lambda}, \quad \hat{\alpha} = \hat{\gamma} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\delta}.$$

7th Marginal Case $\alpha = \delta = 0$

The approach simplifies to

$$\lambda = \gamma P^{\mathrm{Y}}.$$

With $M = P^{\mathrm{Y}}$, we obtain the parameters

$$\hat{\gamma} = M^{+}\hat{\lambda}, \quad \hat{\alpha} = \hat{\delta} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\gamma}.$$

8th Marginal Case $\beta = \gamma = 0$

Due to $\delta = 1 - \alpha$ the approach may be rewritten as

$$\lambda = \alpha(1_3 - P^{\mathrm{N}}) + 1_3 - \alpha \cdot 1_3 \Leftrightarrow 1_3 - \lambda = \alpha P^{\mathrm{N}}.$$

With $M = P^{\mathrm{N}}$, we obtain the parameters

$$\hat{\alpha} = M^{+}\left(1 - \hat{\lambda}\right), \quad \hat{\beta} = \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 1 - \hat{\alpha}.$$

9th Marginal Case $\beta = \delta = 0$

Due to $\gamma = 1 - \alpha$ the approach may be rewritten as

$$\lambda = \alpha(1_3 - P^{\mathrm{N}}) + (1 - \alpha)\, P^{\mathrm{Y}} \Leftrightarrow \lambda = \alpha\left(1_3 - P^{\mathrm{Y}} - P^{\mathrm{N}}\right) + P^{\mathrm{Y}}.$$

With $M = 1_3 - P^{\mathrm{Y}} - P^{\mathrm{N}}$, we obtain as an intermediate step

$$\hat{\lambda} = M\alpha + P^{\mathrm{Y}} \Leftrightarrow \hat{\lambda} - P^{\mathrm{Y}} = M\alpha.$$

Thus, we obtain the parameters

$$\hat{\alpha} = M^{+}\left(\hat{\lambda} - P^{\mathrm{Y}}\right), \quad \hat{\beta} = \hat{\delta} = 0 \quad \text{and} \quad \hat{\gamma} = 1 - \hat{\alpha}.$$

10th Marginal Case $\gamma = \delta = 0$

The approach simplifies to

$$\lambda = \alpha(1_3 - P^{\mathrm{N}}).$$

With $M = 1_3 - P^{\mathrm{N}}$, we obtain the parameters

$$\hat{\alpha} = M^{+}\hat{\lambda}, \quad \hat{\gamma} = \hat{\delta} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\alpha}.$$

Third Order Marginal Solutions

These solutions are located in the corners of the tetrahedron.

11th Marginal Case $\alpha = \beta = \gamma = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\beta} = 0, \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 1.$$

12th Marginal Case $\alpha = \beta = \delta = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\beta} = 0, \quad \hat{\gamma} = 1 \quad \text{and} \quad \hat{\delta} = 0.$$

13th Marginal Case $\alpha = \gamma = \delta = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\beta} = 1, \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 0.$$

14th Marginal Case $\beta = \gamma = \delta = 0$

The parameters are

$$\hat{\alpha} = 1, \quad \hat{\beta} = 0, \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 0.$$

3.3 Derivation of the NCD

As for TCD, the derivation of the NCD method builds on the calculation of the yes likelihoods to derive the estimators for the population parameters. Due to the lower number of parameters, there are only first and second order marginal solutions to calculate.

3.3.1 Calculation of the Yes Likelihoods

With this method only the "no" cheaters are detected. We assume that $\delta = 0$ in the population parameters. If this assumption is not met, we will obtain a biased estimator (see Chapter 4.5.3). For this method we require two RRT subsamples. There are the following options for a yes answer:

- A person from group α always answers the sensitive question with yes.
- A person from group α follows the Forced Answer instruction (see Chapter 1.3) and answers yes. This yes answer is less embarrassing than a yes answer to the incriminating question. Therefore, this assumption is highly plausible.
- A person from group γ follows the Forced Answer instruction and answers yes. If persons from group γ were to answer no although instructed to answer yes, they were "no" cheaters (β) as they would always answer no.

***Corollary* 2.** *For NCD the probabilities of the yes answers are given by*

$$\begin{aligned}\lambda &= \alpha\left(P^E + P^Y\right) + \beta \cdot 0_2 + \gamma P^Y \\ &= \alpha\left(1_2 - P^N\right) + \gamma P^Y.\end{aligned}$$

3.3.2 Calculation of the Population Parameter Estimators

In the same way as for TCD, the area of the permissible solutions can be interpreted geometrically. It is shown here as a triangle in $\mathbb{R}^3$. The native solution is inside the triangle and the permissible marginal solutions are on the sides or vertices.

Native Solution

With $M = \left(1_2 - P^{\mathrm{N}}, P^{\mathrm{Y}}\right)$ this yields the parameters

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\gamma}\end{pmatrix} = M^{-1}\hat{\lambda} \quad \text{and} \quad \hat{\beta} = 1 - \hat{\alpha} - \hat{\gamma}.$$

First Order Marginal Solutions

1st Marginal Case $\alpha = 0$

The approach simplifies to

$$\lambda = \gamma P^{\mathrm{Y}}.$$

With $M = P^{\mathrm{Y}}$, we obtain the parameters

$$\hat{\gamma} = M^{+}\hat{\lambda}, \quad \hat{\alpha} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\gamma}.$$

2nd Marginal Case $\beta = 0$

Due to $\gamma = 1 - \alpha$ the approach may be rewritten as

$$\lambda = \alpha(1_2 - P^{\mathrm{N}}) + P^{\mathrm{Y}} - \alpha P^{\mathrm{Y}} \Leftrightarrow \lambda - P^{\mathrm{Y}} = \alpha(1_2 - P^{\mathrm{N}} - P^{\mathrm{Y}}).$$

With $M = 1_2 - P^{\mathrm{Y}} - P^{\mathrm{N}}$, we obtain the parameters

$$\hat{\alpha} = M^{+}\left(\hat{\lambda} - P^{\mathrm{Y}}\right), \quad \hat{\beta} = 0 \quad \text{and} \quad \hat{\gamma} = 1 - \hat{\alpha}.$$

3rd Marginal Case $\gamma = 0$

The approach simplifies to

$$\lambda = \alpha\left(1_2 - P^{\mathrm{N}}\right).$$

With $M = 1_2 - P^{\mathrm{N}}$, we obtain the parameters

$$\hat{\alpha} = M^{+}\hat{\lambda}, \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\beta} = 1 - \hat{\alpha}.$$

Second Order Marginal Solutions

4th Marginal Case $\alpha = \beta = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\beta} = 0 \quad \text{and} \quad \hat{\gamma} = 1.$$

5th Marginal Case $\alpha = \gamma = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\beta} = 1 \quad \text{and} \quad \hat{\gamma} = 0.$$

6th Marginal Case $\beta = \gamma = 0$

The parameters are

$$\hat{\alpha} = 1, \quad \hat{\beta} = 0 \quad \text{and} \quad \hat{\gamma} = 0.$$

3.4 Derivation of the YCD

In contrast to TCD and NCD, we will use the no likelihoods to derive the estimators for the YCD. As the number of parameters equals the NCD estimators, there will also be one native solution as well as six marginal cases.

3.4.1 Calculation of the No Likelihoods

Now the "yes" cheaters will be detected. As for the NCD, we assume that $\beta = 0$ in the population parameters. We also will obtain a biased estimator if this assumption is not met (see Chapter 4.6.3). As for the NCD, for the YCD we require two RRT subsamples. There are the following options for a yes answer:

- A person from group α always answers the sensitive question with yes.
- A person from group α follows the Forced Answer instruction (see Chapter 1.3) and answers yes. This yes answer is less embarrassing than a yes answer to the incriminating question. Therefore, this assumption is highly plausible.
- A person from group γ follows the Forced Answer instruction and answers yes. If persons from group γ were to answer no although instructed to answer yes, they were "no" cheaters (β) as they would always answer no.
- A person from group δ always answers yes.

***Corollary* 3.** *For YCD the probabilities of the yes answers are given by*

$$\lambda = \alpha \left(P^E + P^Y\right) + \gamma P^Y + \delta \cdot 1_2.$$

With $\delta = 1 - \alpha - \gamma$, we obtain

$$1_2 - \lambda = \alpha P^N + \gamma \left(1_2 - P^Y\right)$$

as the probabilities for no answers.

3.4.2 Calculation of the Population Parameter Estimators

As the YCD and NCD estimate the same number of parameters, we obtain here precisely the same geometric interpretation as for NCD.

Native Solution

With $M = \left(P^{\mathrm{N}}, 1_2 - P^{\mathrm{Y}}\right)$ this yields the parameters

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = M^{-1} \left(1_2 - \hat{\lambda}\right) \quad \text{and} \quad \hat{\delta} = 1 - \hat{\alpha} - \hat{\gamma}.$$

First Order Marginal Solutions

1st Marginal Case $\alpha = 0$

The approach simplifies to

$$1_2 - \lambda = \gamma \left(1_2 - P^{\mathrm{Y}}\right).$$

With $M = 1_2 - P^{\mathrm{Y}}$, we obtain the parameters

$$\hat{\gamma} = M^{+} \left(1_2 - \hat{\lambda}\right), \quad \hat{\alpha} = 0 \quad \text{and} \quad \hat{\delta} = 1 - \hat{\gamma}.$$

2nd Marginal Case $\gamma = 0$

The approach simplifies to

$$1_2 - \lambda = \alpha P^{\mathrm{N}}.$$

With $M = P^{\mathrm{N}}$, we obtain the parameters

$$\hat{\alpha} = M^{+} \left(1_2 - \hat{\lambda}\right), \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 1 - \hat{\alpha}.$$

3rd Marginal Case $\delta = 0$

Due to $\gamma = 1 - \alpha$ the approach may be rewritten as

$$\lambda = \alpha(1_2 - P^{\mathrm{N}}) + P^{\mathrm{Y}} - \alpha P^{\mathrm{Y}} \Leftrightarrow \lambda - P^{\mathrm{Y}} = \alpha(1_2 - P^{\mathrm{N}} - P^{\mathrm{Y}}).$$

With $M = 1_2 - P^{\mathrm{Y}} - P^{\mathrm{N}}$, we obtain the parameters

$$\hat{\alpha} = M^{+} \left(\hat{\lambda} - P^{\mathrm{Y}}\right), \quad \hat{\delta} = 0 \quad \text{and} \quad \hat{\gamma} = 1 - \hat{\alpha}.$$

Second Order Marginal Solutions

4th Marginal Case $\alpha = \gamma = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 1.$$

5th Marginal Case $\alpha = \delta = 0$

The parameters are

$$\hat{\alpha} = 0, \quad \hat{\gamma} = 1 \quad \text{and} \quad \hat{\delta} = 0.$$

6th Marginal Case $\gamma = \delta = 0$

The parameters are

$$\hat{\alpha} = 1, \quad \hat{\gamma} = 0 \quad \text{and} \quad \hat{\delta} = 0.$$

3.5 Application

The use of this method is now described using TCD as an example (refer to Chapter 3.1.3). This approach applies accordingly for the NCD and YCD.

- First, one calculates the native solution. If all of the parameters are in the permissible range this solution provides the estimation of the population parameters.
- If the native solution is not permissible, one has to calculate all of the marginal solutions and their likelihoods (or log likelihoods). From all of the permissible marginal solutions, the one with the highest likelihood provides the estimation of the population parameter. Such a solution exists as the 3rd level marginal solutions are always permissible.

To determine the likelihood of a solution, one first calculates $\lambda = M\Theta$, whereby M is the process matrix (coefficient matrix of the native estimator) and $\Theta = (\alpha, \gamma, \delta)^T$ are the population parameters estimated by the solution. This λ is then used with N and k in the formula derived above for the likelihood or log likelihood. The calculation of the binomial coefficient is not required because N and k are the same for all special estimators and therefore stretch the likelihoods by a fixed factor or add a fixed summand to the log likelihood.

The next chapter, in particular Chapter 4.3, investigates how to select the Forced Answer probabilities and thus the process matrix.

4 Analysis of the Estimators

In this chapter we investigate the estimators first mathematically by deriving the variances and expected values of the solutions analytically. We restrict the calculation of the variances to the native solutions. For each method it is shown that the individual sub-estimators of the native and marginal solutions are unbiased. The assembled estimators, i.e. the methods themselves, are however biased close to the marginal cases. Close to the marginal case, $\alpha = 0$, the native estimator $\hat{\alpha}$ can over-estimate the parameter, but for under-estimation with $\hat{\alpha} < 0$ the calculated estimate applies to the marginal case which delivers $\hat{\alpha} = 0$. On average this results in an over-estimation. Due to its complexity we will investigate this bias close to the marginal cases using Monte Carlo simulation.

4.1 Calculating the Variances of the Estimators

In order to show the variance of the estimators we use the relationship

$$M^{-1} = \frac{1}{\det M} \operatorname{adj}(M)$$

as this clearly shows the interaction with the determinant of the process matrix. The general form of the adjugates for (3×3) matrices is shown by

$$\operatorname{adj} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}.$$

We also need the variances $\operatorname{Var}(\hat{\lambda}_i)$ and covariances $\operatorname{Cov}(\hat{\lambda}_i, \hat{\lambda}_j)$ of the estimators. For $i \neq j$ is $\operatorname{Cov}(\hat{\lambda}_i, \hat{\lambda}_j) = 0$, because the samplings in the RRT subsamples are independent of each other. When deriving the estimators, we received $\hat{\lambda}_i = \frac{k_i}{N_i}$, whereby k_i are the absolute frequencies of the yes responses in the RRT subsamples and N_i are the subsample sizes. Here k_i are binomially distributed random variables with N_i repetitions and λ_i probabilities of success.

***Lemma* 3.** *We therefore receive*

$$\mathrm{Var}(\hat{\lambda_i}) = \mathrm{Var}\left(\frac{k_i}{N_i}\right) = \frac{\mathrm{Var}\,(k_i)}{N_i^2} = \frac{N_i \lambda_i \,(1-\lambda_i)}{N_i^2} = \frac{\lambda_i \,(1-\lambda_i)}{N_i}$$

whereby the probabilities of success λ_i depend on the selected method, the forced answer probabilities and the population parameters (refer to Corollary 1, 2 and 3).

4.1.1 Variances of the TCD

The process matrix of the TCD is

$$M = \begin{pmatrix} 1-P_1^{\mathrm{N}} & P_1^{\mathrm{Y}} & 1 \\ 1-P_2^{\mathrm{N}} & P_2^{\mathrm{Y}} & 1 \\ 1-P_3^{\mathrm{N}} & P_3^{\mathrm{Y}} & 1 \end{pmatrix}.$$

Therefore we obtain

$$\begin{aligned} M^{-1} &= \frac{1}{\det M}\mathrm{adj}\,(M) = \frac{1}{\det M}\,(m_{ij})_{1\le i,j\le 3} \\ &= \frac{1}{\det M}\begin{pmatrix} P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}} & P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}} & P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}} \\ P_2^{\mathrm{N}}-P_3^{\mathrm{N}} & P_3^{\mathrm{N}}-P_1^{\mathrm{N}} & P_1^{\mathrm{N}}-P_2^{\mathrm{N}} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} m_{31} &= \left(1-P_2^{\mathrm{N}}\right)P_3^{\mathrm{Y}} - P_2^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right) \\ m_{32} &= P_1^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right) - \left(1-P_1^{\mathrm{N}}\right)P_3^{\mathrm{Y}} \\ m_{33} &= \left(1-P_1^{\mathrm{N}}\right)P_2^{\mathrm{Y}} - P_1^{\mathrm{Y}}\left(1-P_2^{\mathrm{N}}\right). \end{aligned}$$

The probabilities for yes answers are calculated by

$$\lambda_i = \alpha\left(1-P_i^{\mathrm{N}}\right) + \gamma P_i^{\mathrm{Y}} + \delta.$$

Variance of $\hat{\alpha}$

Using the representation of the inverse process matrix from above, the estimator $\hat{\alpha}$ may be written as

$$\hat{\alpha} = \frac{1}{\det M}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)\hat{\lambda_1} + \left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)\hat{\lambda_2} + \left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)\hat{\lambda_3}\right).$$

For the variance of the estimator we obtain

$$\begin{aligned}\operatorname{Var}(\hat{\alpha}) &= \operatorname{Var}\left(\frac{1}{\det M}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)\hat{\lambda_1}+\left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)\hat{\lambda_2}\right.\right.\\ &\qquad\left.\left.+\left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)\hat{\lambda_3}\right)\right)\\ &= \frac{1}{(\det M)^2}\operatorname{Var}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)\hat{\lambda_1}+\left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)\hat{\lambda_2}+\left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)\hat{\lambda_3}\right).\end{aligned}$$

As $\operatorname{Cov}(\hat{\lambda_i},\hat{\lambda_j})=0$ for $i\neq j$

$$\begin{aligned}\operatorname{Var}(\hat{\alpha}) &= \frac{1}{(\det M)^2}\left(\operatorname{Var}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)\hat{\lambda_1}\right)+\operatorname{Var}\left(\left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)\hat{\lambda_2}\right)\right.\\ &\qquad\left.+\operatorname{Var}\left(\left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)\hat{\lambda_3}\right)\right)\\ &= \frac{1}{(\det M)^2}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)^2\operatorname{Var}\left(\hat{\lambda_1}\right)+\left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)^2\operatorname{Var}\left(\hat{\lambda_2}\right)\right.\\ &\qquad\left.+\left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)^2\operatorname{Var}\left(\hat{\lambda_3}\right)\right)\\ &= \frac{1}{(\det M)^2}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)^2\frac{\lambda_1\left(1-\lambda_1\right)}{N_1}+\left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)^2\frac{\lambda_2\left(1-\lambda_2\right)}{N_2}\right.\\ &\qquad\left.+\left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)^2\frac{\lambda_3\left(1-\lambda_3\right)}{N_3}\right)\\ &\overset{(*)}{\approx} \frac{3}{N\cdot(\det M)^2}\left(\left(P_2^{\mathrm{Y}}-P_3^{\mathrm{Y}}\right)^2\lambda_1\left(1-\lambda_1\right)+\left(P_3^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)^2\lambda_2\left(1-\lambda_2\right)\right.\\ &\qquad\left.+\left(P_1^{\mathrm{Y}}-P_2^{\mathrm{Y}}\right)^2\lambda_3\left(1-\lambda_3\right)\right),\end{aligned}$$

whereby the last approximation $(*)$ is only valid if all three RRT subsamples are nearly equally sized, i.e. $N_1, N_2, N_3 \approx \frac{1}{3}N$. This allows us to set up a rule of thumb to develop the variance relating to sample size in the following chapters.

Variance of $\hat{\gamma}$

We get

$$\hat{\gamma}=\frac{1}{\det M}\left(\left(P_2^{\mathrm{N}}-P_3^{\mathrm{N}}\right)\hat{\lambda_1}+\left(P_3^{\mathrm{N}}-P_1^{\mathrm{N}}\right)\hat{\lambda_2}+\left(P_1^{\mathrm{N}}-P_2^{\mathrm{N}}\right)\hat{\lambda_3}\right)$$

as a new depiction of the estimator. Analogously to $\mathrm{Var}\left(\hat{\alpha}\right)$, we can calculate

$$\begin{aligned}\mathrm{Var}(\hat{\gamma}) &= \frac{1}{\left(\det M\right)^2}\left(\left(P_2^{\mathrm{N}} - P_3^{\mathrm{N}}\right)^2 \frac{\lambda_1\left(1-\lambda_1\right)}{N_1}\right.\\ &\qquad + \left(P_3^{\mathrm{N}} - P_1^{\mathrm{N}}\right)^2 \frac{\lambda_2\left(1-\lambda_2\right)}{N_2}\\ &\qquad \left.+ \left(P_1^{\mathrm{N}} - P_2^{\mathrm{N}}\right)^2 \frac{\lambda_3\left(1-\lambda_3\right)}{N_3}\right)\\ &\approx \frac{3}{N\cdot\left(\det M\right)^2}\left(\left(P_2^{\mathrm{N}} - P_3^{\mathrm{N}}\right)^2 \lambda_1\left(1-\lambda_1\right)\right.\\ &\qquad + \left(P_3^{\mathrm{N}} - P_1^{\mathrm{N}}\right)^2 \lambda_2\left(1-\lambda_2\right)\\ &\qquad \left.+ \left(P_1^{\mathrm{N}} - P_2^{\mathrm{N}}\right)^2 \lambda_3\left(1-\lambda_3\right)\right).\end{aligned}$$

Variance of $\hat{\delta}$

Analogously to $\hat{\alpha}$ and $\hat{\gamma}$, we get

$$\begin{aligned}\hat{\delta} = \frac{1}{\det M}&\left(\left(\left(1-P_2^{\mathrm{N}}\right)P_3^{\mathrm{Y}} - P_2^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right)\right)\hat{\lambda_1}\right.\\ &+ \left(P_1^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right) - \left(1-P_1^{\mathrm{N}}\right)P_3^{\mathrm{Y}}\right)\hat{\lambda_2}\\ &\left.+ \left(\left(1-P_1^{\mathrm{N}}\right)P_2^{\mathrm{Y}} - P_1^{\mathrm{Y}}\left(1-P_2^{\mathrm{N}}\right)\right)\hat{\lambda_3}\right).\end{aligned}$$

Here, we can calculate as above.

$$\begin{aligned}\mathrm{Var}(\hat{\delta}) &= \frac{1}{\left(\det M\right)^2}\left(\left(\left(1-P_2^{\mathrm{N}}\right)P_3^{\mathrm{Y}} - P_2^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right)\right)^2 \frac{\lambda_1\left(1-\lambda_1\right)}{N_1}\right.\\ &\qquad + \left(P_1^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right) - \left(1-P_1^{\mathrm{N}}\right)P_3^{\mathrm{Y}}\right)^2 \frac{\lambda_2\left(1-\lambda_2\right)}{N_2}\\ &\qquad \left.+ \left(\left(1-P_1^{\mathrm{N}}\right)P_2^{\mathrm{Y}} - P_1^{\mathrm{Y}}\left(1-P_2^{\mathrm{N}}\right)\right)^2 \frac{\lambda_3\left(1-\lambda_3\right)}{N_3}\right)\\ &\approx \frac{3}{N\cdot\left(\det M\right)^2}\left(\left(\left(1-P_2^{\mathrm{N}}\right)P_3^{\mathrm{Y}} - P_2^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right)\right)^2 \lambda_1\left(1-\lambda_1\right)\right.\\ &\qquad + \left(P_1^{\mathrm{Y}}\left(1-P_3^{\mathrm{N}}\right) - \left(1-P_1^{\mathrm{N}}\right)P_3^{\mathrm{Y}}\right)^2 \lambda_2\left(1-\lambda_2\right)\\ &\qquad \left.+ \left(\left(1-P_1^{\mathrm{N}}\right)P_2^{\mathrm{Y}} - P_1^{\mathrm{Y}}\left(1-P_2^{\mathrm{N}}\right)\right)^2 \lambda_3\left(1-\lambda_3\right)\right).\end{aligned}$$

Variance of $\hat{\beta}$

The variance is calculated by

$$\begin{aligned}\mathrm{Var}(\hat{\beta}) &= \mathrm{Var}(1-\hat{\alpha}-\hat{\gamma}-\hat{\delta}) \\ &= \mathrm{Var}(\hat{\alpha})+\mathrm{Var}(\hat{\gamma})+\mathrm{Var}(\hat{\delta}) \\ &\quad + 2\left(\mathrm{Cov}(\hat{\alpha},\hat{\gamma})+\mathrm{Cov}(\hat{\alpha},\hat{\delta})+\mathrm{Cov}(\hat{\gamma},\hat{\delta})\right).\end{aligned}$$

But calculating the covariances is however complicated and the resulting expression cannot be summarised in a compact manner. We therefore want to describe only the rough method here. In Chapter 4.4 we will introduce an empirical aid that we can use to abstract from the complicated form of the variances for TCD.

Let $\chi, \psi \in \{\alpha, \gamma, \delta\}$. $\mathrm{E}(\hat{\chi}) = \chi$ applies (refer to Chapter 4.2). We define

$$i(\chi) := \begin{cases} 1 & \text{for } \chi = \alpha \\ 2 & \text{for } \chi = \gamma \\ 3 & \text{for } \chi = \delta. \end{cases}$$

The covariances can be calculated using

$$\mathrm{Cov}(\hat{\chi},\hat{\psi}) = \mathrm{E}(\hat{\chi}\cdot\hat{\psi}) - \mathrm{E}(\hat{\chi})\mathrm{E}(\hat{\psi}) = \mathrm{E}(\hat{\chi}\cdot\hat{\psi}) - \chi\psi.$$

$$\begin{aligned}\mathrm{E}(\hat{\chi}\cdot\hat{\psi}) &= \frac{1}{(\det M)^2}\mathrm{E}\left(\left(m_{i(\chi),1}\hat{\lambda_1}+m_{i(\chi),2}\hat{\lambda_2}+m_{i(\chi),3}\hat{\lambda_3}\right)\right. \\ &\qquad \left.\cdot\left(m_{i(\psi),1}\hat{\lambda_1}+m_{i(\psi),2}\hat{\lambda_2}+m_{i(\psi),3}\hat{\lambda_3}\right)\right) \\ &= \frac{1}{(\det M)^2}\sum_{k=1}^{3}\sum_{l=1}^{3} m_{i(\chi),k}m_{i(\psi),l}\mathrm{E}(\hat{\lambda_k}\cdot\hat{\lambda_l})\end{aligned}$$

with $m_{i,j}$ being the elements of $\mathrm{adj}(M)$. Here

$$\mathrm{E}(\hat{\lambda_k}\hat{\lambda_l}) = \begin{cases} \mathrm{E}(\hat{\lambda_k})\cdot\mathrm{E}(\hat{\lambda_l}) = \lambda_k\lambda_l & \text{for } k \neq l \text{ (see Chapter 4.2)} \\ \mathrm{Var}(\hat{\lambda_k}) + \left(\mathrm{E}(\hat{\lambda_k})\right)^2 & \text{for } k = l\text{, because} \\ & \mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2. \end{cases}$$

Under the assumption $N_1, N_2, N_3 \approx \frac{1}{3}N$ arises the expression of the form

$$\mathrm{Var}(\hat{\beta}) = \frac{3}{N\cdot(\det M)^2}A\left(\alpha,\gamma,\delta,P^{\mathrm{N}},P^{\mathrm{Y}},N\right),$$

for the variance of $\hat{\beta}$ whereby A only depends on the population parameters, Forced Answer probabilities and sample size.

4.1.2 Variances of the NCD

The process matrix is

$$M = \begin{pmatrix} 1 - P_1^{\mathrm{N}} & P_1^{\mathrm{Y}} \\ 1 - P_2^{\mathrm{N}} & P_2^{\mathrm{Y}} \end{pmatrix}.$$

The general form of adjugates for (2×2) matrices is given by

$$\text{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and we obtain

$$M^{-1} = \frac{1}{\det M} \text{adj}\,(M) = \frac{1}{\det M} \begin{pmatrix} P_2^{\mathrm{Y}} & -P_1^{\mathrm{Y}} \\ -\left(1 - P_2^{\mathrm{N}}\right) & 1 - P_1^{\mathrm{N}} \end{pmatrix}$$

as the inverse of the process matrix. The probabilities for yes answers result from

$$\lambda_i = \alpha \left(1 - P_i^{\mathrm{N}}\right) + \gamma P_i^{\mathrm{Y}}.$$

Variance of $\hat{\alpha}$

Now we can rewrite $\hat{\alpha}$ as:

$$\hat{\alpha} = \frac{1}{\det M} \left(P_2^{\mathrm{Y}} \hat{\lambda_1} - P_1^{\mathrm{Y}} \hat{\lambda_2}\right).$$

Similar computations as in the TCD case are performed as

$$\begin{aligned} \text{Var}(\hat{\alpha}) &= \frac{1}{(\det M)^2} \left(\left(P_2^{\mathrm{Y}}\right)^2 \frac{\lambda_1 (1 - \lambda_1)}{N_1} + \left(P_1^{\mathrm{Y}}\right)^2 \frac{\lambda_2 (1 - \lambda_2)}{N_2} \right) \\ &\approx \frac{2}{N \cdot (\det M)^2} \left(\left(P_2^{\mathrm{Y}}\right)^2 \lambda_1 (1 - \lambda_1) + \left(P_1^{\mathrm{Y}}\right)^2 \lambda_2 (1 - \lambda_2) \right). \end{aligned}$$

Here as for the following calculation of $\text{Var}(\hat{\gamma})$, we also assume for the last approximation that the RRT subsamples are nearly equally sized.

Variance of $\hat{\gamma}$

With

$$\hat{\gamma} = \frac{1}{\det M} \left(- \left(1 - P_2^{\mathrm{N}}\right) \hat{\lambda_1} + \left(1 - P_1^{\mathrm{N}}\right) \hat{\lambda_2} \right),$$

it holds that

$$\begin{aligned}
\mathrm{Var}(\hat{\gamma}) &= \frac{1}{(\det M)^2} \left(\left(1 - P_2^{\mathrm{N}}\right)^2 \frac{\lambda_1 \left(1 - \lambda_1\right)}{N_1} + \left(1 - P_1^{\mathrm{N}}\right)^2 \frac{\lambda_2 \left(1 - \lambda_2\right)}{N_2} \right) \\
&\approx \frac{2}{N \cdot (\det M)^2} \left(\left(1 - P_2^{\mathrm{N}}\right)^2 \lambda_1 \left(1 - \lambda_1\right) + \left(1 - P_1^{\mathrm{N}}\right)^2 \lambda_2 \left(1 - \lambda_2\right) \right).
\end{aligned}$$

Variance of $\hat{\beta}$

Similar to the TCD, we obtain an expression of the form

$$\mathrm{Var}(\hat{\beta}) = \frac{1}{N \cdot (\det M)^2} A \left(\alpha, \gamma, P^{\mathrm{N}}, P^{\mathrm{Y}}, N\right).$$

4.1.3 Variances of the YCD

The process matrix is given by

$$M = \begin{pmatrix} P_1^{\mathrm{N}} & 1 - P_1^{\mathrm{Y}} \\ P_2^{\mathrm{N}} & 1 - P_2^{\mathrm{Y}} \end{pmatrix}.$$

Using the above representation of (2×2)-adjugate matrices, we get

$$M^{-1} = \frac{1}{\det M} \mathrm{adj}\left(M\right) = \frac{1}{\det M} \begin{pmatrix} 1 - P_2^{\mathrm{Y}} & -\left(1 - P_1^{\mathrm{Y}}\right) \\ -P_2^{\mathrm{N}} & P_1^{\mathrm{N}} \end{pmatrix}$$

as the inverse of the process matrix. Further

$$1 - \lambda_i = \alpha P_i^{\mathrm{N}} + \gamma \left(1 - P_i^{\mathrm{Y}}\right)$$

as the probabilities of a no answer. The variance is computed analogously to the NCD.

Variance of $\hat{\alpha}$

Similarly to the NCD we get

$$\hat{\alpha} = \frac{1}{\det M}\left(\left(1 - P_2^{\mathrm{Y}}\right)\left(1 - \hat{\lambda_1}\right) - \left(1 - P_1^{\mathrm{Y}}\right)\left(1 - \hat{\lambda_2}\right)\right)$$

and we compute

$$\begin{aligned}
\mathrm{Var}(\hat{\alpha}) &= \frac{1}{(\det M)^2}\left(\left(1 - P_2^{\mathrm{Y}}\right)^2 \frac{\lambda_1\left(1-\lambda_1\right)}{N_1} + \left(1 - P_1^{\mathrm{Y}}\right)^2 \frac{\lambda_2\left(1-\lambda_2\right)}{N_2}\right) \\
&\approx \frac{2}{N \cdot (\det M)^2}\left(\left(1 - P_2^{\mathrm{Y}}\right)^2 \lambda_1\left(1-\lambda_1\right)\right. \\
&\qquad\qquad \left. + \left(1 - P_1^{\mathrm{Y}}\right)^2 \lambda_2\left(1-\lambda_2\right)\right),
\end{aligned}$$

assuming that $N_1, N_2 \approx \frac{1}{2}N$.

Variance of $\hat{\gamma}$

Analogous computations yield

$$\hat{\gamma} = \frac{1}{\det M}\left(-P_2^{\mathrm{N}}\left(1 - \hat{\lambda_1}\right) + P_1^{\mathrm{N}}\left(1 - \hat{\lambda_2}\right)\right)$$

and

$$\begin{aligned}
\mathrm{Var}(\hat{\gamma}) &= \frac{1}{(\det M)^2}\left(\left(P_2^{\mathrm{N}}\right)^2 \frac{\lambda_1\left(1-\lambda_1\right)}{N_1} + \left(P_1^{\mathrm{N}}\right)^2 \frac{\lambda_2\left(1-\lambda_2\right)}{N_2}\right) \\
&\approx \frac{2}{N \cdot (\det M)^2}\left(\left(P_2^{\mathrm{N}}\right)^2 \lambda_1\left(1-\lambda_1\right) + \left(P_1^{\mathrm{N}}\right)^2 \lambda_2\left(1-\lambda_2\right)\right),
\end{aligned}$$

also under the assumption that $N_1, N_2 \approx \frac{1}{2}N$.

Variance of $\hat{\delta}$

Here, we also get the form

$$\mathrm{Var}(\hat{\delta}) = \frac{1}{N \cdot (\det M)^2} A\left(\alpha, \gamma, P^{\mathrm{N}}, P^{\mathrm{Y}}, N\right).$$

4.1.4 Consideration to Optimise the Method

For a population parameter $\chi \in \{\alpha, \beta, \gamma, \delta\}$, we set the mean squared error of $\hat{\chi}$ as $\mathrm{MSE}(\hat{\chi}) := \mathrm{Var}(\hat{\chi}) + (\mathrm{Bias}(\hat{\chi}))^2$. In order to minimise this, it is sufficient to minimise $\mathrm{Var}(\hat{\chi})$ because the estimators are unbiased (as we shall see in the next chapter). When using the formulae derived above for this, two problems arise. On the one hand, the expression in the numerator is very complex. On the other hand, unknown population parameters on which the experimenter has no influence are used. The rules of thumb

$$\mathrm{Std}\left(\mathrm{Bias}\left(\hat{\chi}\right)_{\mathrm{TCD}}\right) \approx \frac{1}{\sqrt{|\det M|\, N}}$$

and

$$\mathrm{Std}\left(\mathrm{Bias}\left(\hat{\chi}\right)_{\mathrm{DCD}}\right) \approx \frac{1}{|\det M|\, \sqrt{N}}$$

for the standard deviations of the estimation error $\mathrm{Bias}\left(\hat{\chi}\right), \chi \in \{\alpha, \beta, \gamma, \delta\}$ provide assistance here. These are very rough estimates of the proportionality which may have variations of ±30% in the proportionality constants, depending on the forced answer probabilities and population parameters. Nevertheless, for practical use they provide a detachment of the population parameters from the forced answer probabilities that is appropriate to optimise the estimators.

Tables 4.1, 4.2 and B.2 show the connection between the standard deviation of the estimation errors and the determinant of the process matrix. For this 1,000 random populations were generated and 1,000 Monte Carlo simulations were conducted with sample size $N = 1{,}500$. Then the standard deviations of the estimation errors and the 10%, 50% and 90% quantiles were calculated for each of these populations. The interaction is shown with $\frac{1}{\sqrt{N}}$ in Figures 4.4, 4.7 and 4.12.

4.2 Expected Values of the Estimators

With the considerations stated above on Lemma 3, we can calculate $\mathrm{E}(\hat{\lambda}_i)$.

***Corollary* 4.**

$$\mathrm{E}(\hat{\lambda}_i) = \mathrm{E}\left(\frac{k_i}{N_i}\right) = \frac{\mathrm{E}\left(k_i\right)}{N_i} = \frac{N_i \lambda_i}{N_i} = \lambda_i.$$

In order to show that all estimators are unbiased, we can use the method as for calculating the variances. Instead, we will prove the first three special cases ($1 \leq n_0 \leq 3$) of the following assumption 1 because this will also provide the

Table 4.1: Interaction of the standard deviations of the estimation errors for TCD with the determinant of the process matrix. The determinant is exact. The quantiles are scaled by a factor of 1,000 and rounded.

χ	$\lvert\det M\rvert$	Std (Bias ($\hat{\chi}$))			Std (Bias ($\hat{\chi}$)) $\cdot \sqrt{\lvert\det M\rvert}$		
		10%	50%	90%	10%	50%	90%
α	0.36	38.97	49.91	62.22	23.38	29.95	37.33
	0.20	51.53	69.37	88.71	23.04	31.02	39.67
	0.11	65.94	92.67	120.70	21.87	30.74	40.03
	0.06	87.39	123.48	164.03	21.41	30.25	40.18
β	0.36	25.86	32.19	34.94	15.52	19.31	20.96
	0.20	35.48	42.86	46.52	15.87	19.17	20.81
	0.11	45.88	64.10	71.97	15.22	21.26	23.87
	0.06	49.27	66.91	75.77	12.07	16.39	18.56
γ	0.36	39.79	49.13	52.69	23.88	29.48	31.61
	0.20	46.68	59.30	63.47	20.88	26.52	28.38
	0.11	78.97	112.99	127.25	26.19	37.47	42.20
	0.06	89.25	125.85	147.26	21.86	30.83	36.07
δ	0.36	25.20	32.53	42.69	15.12	19.52	25.62
	0.20	34.67	45.14	60.29	15.51	20.19	26.96
	0.11	34.15	47.08	65.02	11.33	15.62	21.56
	0.06	47.72	66.76	94.58	11.69	16.35	23.17

proof that all estimators (including the estimators from the marginal solutions) are unbiased for all three methods.

***Assumption* 1.** *Let $n, n_0 \in \mathbb{N}$ with $n \geq n_0 > 0$ and let $\chi_1, \ldots, \chi_{n_0+1} \in \mathbb{R}$. Let $v_1, \ldots, v_{n_0} \in \mathbb{R}^n$ so that $M := (v_1, \ldots, v_{n_0})$ has full column rank n_0 and let $\begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_{n_0} \end{pmatrix} := \hat{\lambda}$ be estimators with $\mathrm{E}(\hat{\lambda}) = \sum_{i=1}^{n_0} v_i \chi_i$.*
The estimators

$$\begin{pmatrix} \hat{\chi}_1 \\ \vdots \\ \hat{\chi}_{n_0} \end{pmatrix} := M^+ \hat{\lambda}$$

are then unbiased, i.e. $\mathrm{E}\left(\hat{\chi}_i\right) = \chi_i$ applies to $1 \leq i \leq n_0$.
If $\sum_{i=1}^{n_0+1} \chi_i = 1$ also applies, it is clear that $\hat{\chi}_{n_0+1} := 1 - \sum_{i=1}^{n_0} \hat{\chi}_i$ is also unbiased.

Table 4.2: Interaction of the standard deviations of the estimation errors for NCD with the determinant of the process matrix. The determinant is exact. The quantiles are scaled by a factor of 1,000 and rounded.

χ	$\lvert\det M\rvert$	Std (Bias ($\hat{\chi}$))			Std (Bias ($\hat{\chi}$)) $\cdot$ $\lvert\det M\rvert$		
		10%	50%	90%	10%	50%	90%
α	0.54	16.33	21.96	23.97	8.82	11.86	12.94
	0.40	18.35	24.90	27.56	7.34	9.96	11.02
	0.22	30.87	40.29	43.77	6.79	8.86	9.63
	0.16	43.89	60.35	65.78	7.02	9.66	10.52
β	0.54	22.80	26.70	28.22	12.31	14.42	15.24
	0.40	27.00	31.92	33.75	10.80	12.77	13.50
	0.22	41.68	55.61	59.65	9.17	12.23	13.12
	0.16	44.67	61.29	66.25	7.15	9.81	10.60
γ	0.54	32.78	40.04	42.74	17.70	21.62	23.08
	0.40	38.23	47.88	51.44	15.29	19.15	20.58
	0.22	67.05	91.42	98.52	14.75	20.11	21.67
	0.16	84.01	117.47	128.14	13.44	18.79	20.50

We will now demonstrate the assumption for the cases $n_0 = 1$, $n_0 = 2$ and $n_0 = 3$. As usual let the real standard scalar product be referred to as $\langle\cdot,\cdot\rangle$.

***Lemma* 4.** *The assumption applies for $n_0 = 1$.*

Proof. Let $M^+ = \left(M^T M\right)^{-1} M^T = \left(v_1^T v_1\right)^{-1} v_1^T = \frac{1}{\langle v_1, v_1\rangle} v_1^T$. We get $\mathrm{E}(\hat{\chi}_1) = M^+ \mathrm{E}(\hat{\lambda}) = \frac{1}{\langle v_1, v_1\rangle} \cdot \chi_1 \cdot v_1^T v_1 = \chi_1 \frac{\langle v_1, v_1\rangle}{\langle v_1, v_1\rangle} = \chi_1$.

***Lemma* 5.** *The assumption applies for $n_0 = 2$.*

Proof. Let

$$\begin{aligned}
M^+ = \left(M^T M\right)^{-1} M^T &= \left(\begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}\right)^{-1} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle \\ \langle v_1, v_2\rangle & \langle v_2, v_2\rangle \end{pmatrix}^{-1} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} \langle v_2, v_2\rangle & -\langle v_1, v_2\rangle \\ -\langle v_1, v_2\rangle & \langle v_1, v_1\rangle \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{D} \begin{pmatrix} \langle v_2, v_2 \rangle v_1^T - \langle v_1, v_2 \rangle v_2^T \\ -\langle v_1, v_2 \rangle v_1^T + \langle v_1, v_1 \rangle v_2^T \end{pmatrix}$$

with

$$D = \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle - \langle v_1, v_2 \rangle^2.$$

We get

$$\begin{aligned}
\mathrm{E}\begin{pmatrix} \hat{\chi}_1 \\ \hat{\chi}_2 \end{pmatrix} &= M^+ \mathrm{E}(\hat{\lambda}) = M^+ \left(\chi_1 v_1 + \chi_2 v_2 \right) \\
&= \frac{1}{D} \begin{pmatrix} \chi_1 \left(\langle v_2, v_2 \rangle \langle v_1, v_1 \rangle - \langle v_1, v_2 \rangle \langle v_2, v_1 \rangle \right) \\ \quad + \chi_2 \left(\langle v_2, v_2 \rangle \langle v_1, v_2 \rangle - \langle v_1, v_2 \rangle \langle v_2, v_2 \rangle \right) \\ \\ \chi_1 \left(-\langle v_1, v_2 \rangle \langle v_1, v_1 \rangle + \langle v_1, v_1 \rangle \langle v_2, v_1 \rangle \right) \\ \quad + \chi_2 \left(-\langle v_1, v_2 \rangle \langle v_1, v_2 \rangle + \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle \right) \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} \chi_1 \cdot D + \chi_2 \cdot 0 \\ \chi_1 \cdot 0 + \chi_2 \cdot D \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.
\end{aligned}$$

***Lemma* 6.** *The assumption applies for $n_0 = 3$.*

Proof. Let

$$\begin{aligned}
\left(M^T M \right)^{-1} &= \left(\begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle \\ \langle v_1, v_3 \rangle & \langle v_2, v_3 \rangle & \langle v_3, v_3 \rangle \end{pmatrix}^{-1}
\end{aligned}$$

$$= \frac{1}{D} \begin{pmatrix} \langle v_2, v_2 \rangle \langle v_3, v_3 \rangle & \langle v_1, v_3 \rangle \langle v_2, v_3 \rangle & \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \\ \quad -\langle v_2, v_3 \rangle^2 & \quad -\langle v_1, v_2 \rangle \langle v_3, v_3 \rangle & \quad -\langle v_1, v_3 \rangle \langle v_2, v_2 \rangle \\ \\ \langle v_2, v_3 \rangle \langle v_1, v_3 \rangle & \langle v_1, v_1 \rangle \langle v_3, v_3 \rangle & \langle v_1, v_2 \rangle \langle v_1, v_3 \rangle \\ \quad -\langle v_1, v_2 \rangle \langle v_3, v_3 \rangle & \quad -\langle v_1, v_3 \rangle^2 & \quad -\langle v_1, v_1 \rangle \langle v_2, v_3 \rangle \\ \\ \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle & \langle v_1, v_2 \rangle \langle v_1, v_3 \rangle & \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle \\ \quad -\langle v_1, v_3 \rangle \langle v_2, v_2 \rangle & \quad -\langle v_1, v_1 \rangle \langle v_2, v_3 \rangle & \quad -\langle v_1, v_2 \rangle^2 \end{pmatrix}$$

with

$$\begin{aligned}D &= \langle v_1, v_1\rangle\langle v_2, v_2\rangle\langle v_3, v_3\rangle + 2\langle v_1, v_2\rangle\langle v_1, v_3\rangle\langle v_2, v_3\rangle \\ &\quad - \langle v_1, v_3\rangle^2\langle v_2, v_2\rangle - \langle v_2, v_3\rangle^2\langle v_1, v_1\rangle - \langle v_1, v_2\rangle^2\langle v_3, v_3\rangle.\end{aligned}$$

Therefore we get

$$\begin{aligned}M^+ &= \left(M^T M\right)^{-1}\begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} \\ &= \frac{1}{D}\begin{pmatrix} \left(\langle v_2, v_2\rangle\langle v_3, v_3\rangle - \langle v_2, v_3\rangle^2\right) v_1^T \\ \quad + \left(\langle v_1, v_3\rangle\langle v_2, v_3\rangle - \langle v_1, v_2\rangle\langle v_3, v_3\rangle\right) v_2^T \\ \quad + \left(\langle v_1, v_2\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_2\rangle\right) v_3^T \\ \left(\langle v_2, v_3\rangle\langle v_1, v_3\rangle - \langle v_1, v_2\rangle\langle v_3, v_3\rangle\right) v_1^T \\ \quad + \left(\langle v_1, v_1\rangle\langle v_3, v_3\rangle - \langle v_1, v_3\rangle^2\right) v_2^T \\ \quad + \left(\langle v_1, v_2\rangle\langle v_1, v_3\rangle - \langle v_1, v_1\rangle\langle v_2, v_3\rangle\right) v_3^T \\ \left(\langle v_1, v_2\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_2\rangle\right) v_1^T \\ \quad + \left(\langle v_1, v_2\rangle\langle v_1, v_3\rangle - \langle v_1, v_1\rangle\langle v_2, v_3\rangle\right) v_2^T \\ \quad + \left(\langle v_1, v_1\rangle\langle v_2, v_2\rangle - \langle v_1, v_2\rangle^2\right) v_3^T \end{pmatrix}\end{aligned}$$

and because

$$\mathrm{E}\begin{pmatrix} \hat{\chi}_1 \\ \hat{\chi}_2 \\ \hat{\chi}_3 \end{pmatrix} = M^+\mathrm{E}(\hat{\lambda}) = M^+\left(\chi_1 v_1 + \chi_2 v_2 + \chi_3 v_3\right)$$

this results in

$$\begin{aligned}\mathrm{E}\left(\hat{\chi}_1\right) = \frac{1}{D}\Big(&\chi_1\big(\langle v_2, v_2\rangle\langle v_3, v_3\rangle\langle v_1, v_1\rangle - \langle v_2, v_3\rangle^2\langle v_1, v_1\rangle \\ &\quad + \langle v_1, v_3\rangle\langle v_2, v_3\rangle\langle v_2, v_1\rangle - \langle v_1, v_2\rangle\langle v_3, v_3\rangle\langle v_2, v_1\rangle \\ &\quad + \langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle - \langle v_1, v_3\rangle\langle v_2, v_2\rangle\langle v_3, v_1\rangle\big) \\ &+ \chi_2\big(\langle v_2, v_2\rangle\langle v_3, v_3\rangle\langle v_1, v_2\rangle - \langle v_2, v_3\rangle^2\langle v_1, v_2\rangle \\ &\quad + \langle v_1, v_3\rangle\langle v_2, v_3\rangle\langle v_2, v_2\rangle - \langle v_1, v_2\rangle\langle v_3, v_3\rangle\langle v_2, v_2\rangle \\ &\quad + \langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_2\rangle - \langle v_1, v_3\rangle\langle v_2, v_2\rangle\langle v_3, v_2\rangle\big) \\ &+ \chi_3\big(\langle v_2, v_2\rangle\langle v_3, v_3\rangle\langle v_1, v_3\rangle - \langle v_2, v_3\rangle^2\langle v_1, v_3\rangle \\ &\quad + \langle v_1, v_3\rangle\langle v_2, v_3\rangle\langle v_2, v_3\rangle - \langle v_1, v_2\rangle\langle v_3, v_3\rangle\langle v_2, v_3\rangle \\ &\quad + \langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_2\rangle\langle v_3, v_3\rangle\big)\Big)\end{aligned}$$

$$= \frac{1}{D}\left(\chi_1 \cdot D + \chi_2 \cdot 0 + \chi_3 \cdot 0\right) = \chi_1.$$

One can calculate $\mathrm{E}\left(\hat{\chi}_2\right) = \chi_2$ and $\mathrm{E}\left(\hat{\chi}_3\right) = \chi_3$ analogously.

The unbiased nature of all estimators for each of the methods follows from the corollary (or the lemmas):

- **TCD**
  Lemma 6 provides the unbiased nature of the native estimators, Lemma 5 the unbiased nature of the estimators in the first order marginal solutions and Lemma 4 the unbiased nature of the second order marginal solutions. In the third order trivial cases where a population parameter equals one the unbiased nature is obvious.

- **NCD/YCD**
  The unbiased nature of the native estimators follows from Lemma 5, that of the estimators for the first order marginal solutions from Lemma 4.

4.3 Optimal Forced Answer Probabilities

When solving the ML equation, it is required that the process matrix has full row rank or in the case of a square matrix can be inverted. If its row rank is not at the maximum, redundant information enters the estimation due to the choice of the Forced Answer probability which of course fails. In addition, as noted above, the determinant of the process matrix should have the highest possible absolute value in order to minimise the average estimation errors. This purely mathematical optimisation of the estimators is however in conflict with the fundamental concept of the RRT.

As this cannot be immediately detected on the process matrices of the methods presented, we want to explain this situation using classical RRT without any subsampling as an example.
Let $\beta = \delta = 0$. With $\gamma = 1 - \alpha$, the probability of a yes answer is given as

$$\lambda = \alpha\left(P^{\mathrm{E}} + P^{\mathrm{Y}}\right) + \gamma P^{\mathrm{Y}} \Leftrightarrow \lambda - P^{\mathrm{Y}} = \alpha\left(1 - P^{\mathrm{Y}} - P^{\mathrm{N}}\right).$$

The process matrix is therefore $M = \left(1 - P^{\mathrm{Y}} - P^{\mathrm{N}}\right)$. This result can be seen above in general form for the marginal cases $\beta = \delta = 0$ of the TCD, NCD and YCD. Maximising the determinant obviously would mean leaving out the Forced Answer instructions which precisely matches a direct question.

This principle can be generalised to the methods presented here: the mathematical optimisation generates a solution which is equivalent to leaving out the

Forced Answer instruction in one subsample and enforcing the forced answer in the remaining subsample(s). This merely mathematical optimum therefore matches an abnormal practical solution. Maximising the determinants in the process matrices is achieved by the following Forced Answer probabilities:

- For TCD
$$P^{\mathrm{Y}}_{\mathrm{TCD}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}}_{\mathrm{TCD}} = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$
- For NCD
$$P^{\mathrm{Y}}_{\mathrm{NCD}} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}}_{\mathrm{NCD}} = \begin{pmatrix}0\\0\end{pmatrix}.$$
- For YCD
$$P^{\mathrm{Y}}_{\mathrm{YCD}} = \begin{pmatrix}0\\0\end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}}_{\mathrm{YCD}} = \begin{pmatrix}1\\0\end{pmatrix}.$$

This mathematical optimum reduces the RRT to absurdity. Appropriate P^{Y} and P^{N} for applications should therefore approach this mathematical optimum as far as respondents still believe in being protected from disclosure and from social sanctioning.

4.4 Analysis of the TCD

In order to analyse the estimation errors across possible population parameters, a series of Monte Carlo simulations were conducted. As the methods estimate three or four parameters, we cannot directly show the estimation errors or their standard deviations in a three-dimensional coordinate system. Nevertheless, coordinate transformation allows us to place the complete domain of three population parameters on a triangle in $\mathbb{R}^2$ such that we can plot the estimation errors and their standard deviations in a diagram along the third axis. This triangle represents the universe of all populations. Each point uniquely represents one population. Three parameters are varied to analyse the DCD. The transformation of the domain stated above shows a factor to be studied (such as the estimation error $\mathrm{Bias}\,(\hat{\alpha})$) completely in a single diagram. A detailed derivation of this transformation is found in Appendix A.

The interpretation will be illustrated using the NCD as an example: the population parameters in which we are interested are α, β and γ. For representation we use an equilateral triangle with vertices A, B and C. Each point P in this triangle can be uniquely written as $P = \alpha A + \beta B + \gamma C$. This representation is only unique because $\alpha, \beta, \gamma \in [0, 1]$ applies. This is also referred to by a convex combination of A, B and C with coefficients α, β and γ. So the closer P is to a vertex, the higher the coefficient relating to the vertex. For example, $P = A$ means the marginal case that P represents the population with $\alpha = 1$. So if P is on the side $\overline{AB}$, P represents a population with $\alpha + \beta \equiv 1 \Leftrightarrow \gamma \equiv 0$. Points on a straight line parallel to $\overline{AB}$ stand for populations where $\alpha + \beta \equiv 1 - \gamma \equiv const \Leftrightarrow \gamma \equiv const$ applies. The population with $\alpha = \beta = \gamma = \frac{1}{3}$ can be found at the triangle's centroid.

The TCD cannot be shown fully in a diagram by these means as there are four parameters to be estimated. For illustration using the procedure stated above graphs for some selected values of the parameter δ are viewed, i.e. the population parameter δ is fixed. The interpretation described above on the position of a point in the triangle can be transferred to it with the additional restriction $\alpha + \beta + \gamma = 1 - \delta$. In such a triangle the vertex A precisely represents the population where $\alpha = 1 - \delta$ and $\beta = \gamma = 0$ applies, whereby δ is fixed.

As already mentioned at the start of this chapter, we now want to investigate the complete method, i.e. the composition of all estimators, with the aid of Monte Carlo simulations. For the TCD, the analysis is based on 1,000 Monte Carlo simulations for potential populations. We selected the following Forced Answer probabilities

$$P^{\mathrm{Y}} = \begin{pmatrix} 0.7 \\ 0.1 \\ 0.1 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}} = \begin{pmatrix} 0.1 \\ 0.7 \\ 0.1 \end{pmatrix}.$$

First an example is used to explain the basic characteristics of the TCD before we systematically vary the parameters.

Figure 4.1 shows the average estimation error Bias $(\hat{\gamma})$ for $\delta = 0.20$. It can clearly be seen that the TCD is biased close to the marginal cases. The reason for this is the positive constraints on the parameters: close to the marginal cases where there is a population with $\gamma = 0$, i.e. on the side $\overline{AB}$, the estimator $\hat{\gamma}$ has a high positive bias. The comparatively high value of the estimation error close to point A on the side $\overline{AB}$ can be explained as follows: a no answer can be assigned both to the population proportion γ (to which this graphic refers) and to the population proportion β. A point close to A represents a population with very small proportions β and γ, whereby the estimator $\hat{\gamma}$ frequently incorrectly delivers an estimation from the marginal cases $\beta = 0$ or $\beta = \gamma = 0$ for the relevant simulation series. The marginal case $\beta = 0$ then always means overestimating $\hat{\gamma}$. If the native ML solution delivers a value $\gamma < 0$, (in nearly all cases) the marginal case $\gamma = 0$ has the highest likelihood of all marginal cases. Therefore on average no compensation can be made for a native ML solution

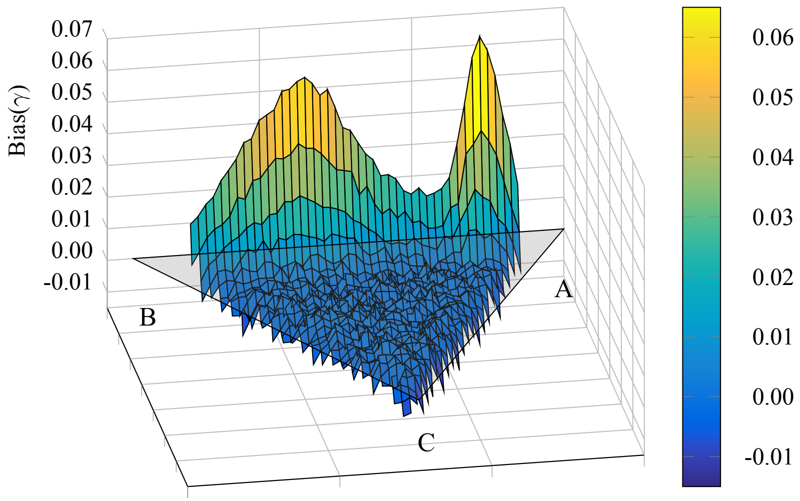

Figure 4.1: Estimation errors for γ in the TCD (N = 1,500, δ = 0.20, 1,000 repetitions).

with $\gamma > 0$ as opposed to, for example, in the middle of the triangle where all parameters are relatively far away from a marginal solution.

This is also reflected in the standard deviation of the estimation error, as can be clearly seen in Figure 4.2. Close to the marginal cases, the standard deviation generally decreases. Towards the centre of the triangle the standard deviation is higher because the estimation is not artificially biased by marginal cases and the TCD has space for large under- and over-estimations. In the same way as for the estimation error (Fig. 4.1) the standard deviation close to Point A is also very high. At this point the already discussed incorrect marginal solutions $\beta = 0$ and the correct marginal solution $\gamma = 0$ lead to this high standard deviation.

For the $\delta = 0$ case, one can easily detect the bias of the estimator close to the marginal cases in Figure B.1 and B.2. The apparent symmetry of the errors emphasises that yes answers can be false due both to the population proportions α and δ.

Outside the marginal cases the TCD can certainly be recognized as unbiased. This is particularly clear in Figure 4.3. It shows the same estimation errors as shown in Figure 4.1 but here only values for populations with $\alpha, \beta, \gamma > 0.05$ are shown. Here too, the effects of the marginal cases $\gamma = 0$ can still be seen clearly whereby the biggest error for $\gamma \approx 0.05$ with Bias $(\hat{\gamma}) \approx 0.01$ for the estimation of a population proportion is still at a very acceptable level.

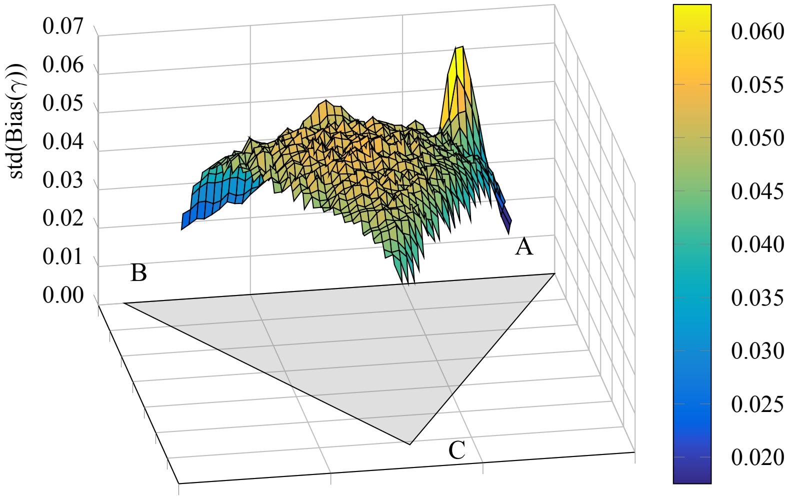

Figure 4.2: Standard deviation of the estimation error for γ in the TCD ($N = 1{,}500$, $\delta = 0.20$, $1{,}000$ repetitions)

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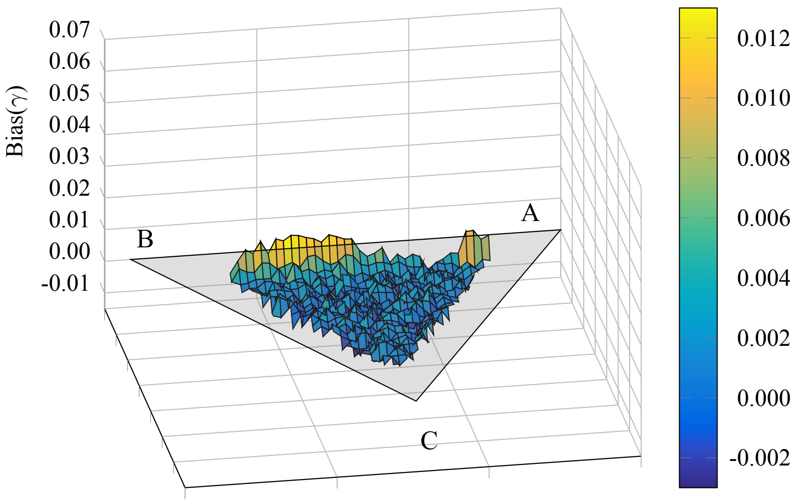

Figure 4.3: Estimation errors for γ in the TCD ($N = 1{,}500$, $\delta = 0.20$, $1{,}000$ repetitions) for populations with $\alpha, \beta, \gamma, \delta \leq 0.05$.

4.4.1 Mean Errors

The mean estimation errors and therefore the unbiased nature of the method will now be presented for the simulated data. We have already demonstrated analytically that the estimators are each unbiased but we have also seen that the composition of the TCD estimators close to the marginal cases is biased. For this the average value for all estimation errors in the simulation series was formed for each parameter, once including all populations and again omitting the populations close to the marginal cases ($\alpha, \beta, \gamma, \delta < 0.05$). For example, the total average values of the errors shown in Figure 4.1 and 4.3 are also found in Table 4.3, Column 3 or 4. The corresponding values for Bias($\hat{\beta}$) and Bias($\hat{\delta}$) are shown in Table B.1.

Table 4.3: Average TCD estimation errors for the simulated data scaled by a factor of 10^6.

δ	χ	Bias ($\hat{\chi}$) with marginal cases	Bias ($\hat{\chi}$) without marginal cases
0.00	α	-112,393	-
	γ	20,690	-
0.05	α	14,190	1,839
	γ	3,182	2,733
0.10	α	23,228	7,248
	γ	12,044	7,056
0.15	α	18,439	4,265
	γ	25,918	3,959
0.20	α	301	-770
	γ	42,919	10,060
0.35	α	-57,091	-14,791
	γ	61,993	14,216
0.50	α	-40,546	-33,231
	γ	41,423	24,200
0.65	α	-55,749	-19,506
	γ	69,690	15,936
0.80	α	-27,722	-16,765
	γ	35,223	8,020
0.95	α	14,861	-
	γ	-3,707	-

It can be seen that the estimation errors are significantly smaller if the majority of marginal cases are excluded. As demonstrated analytically above, we once again see that the native solution of the TCD is unbiased. The regularities in the signs of the estimation errors show that there is still some systematic error. This is because, in spite of the restriction $\alpha, \beta, \gamma, \delta > 0.05$, by far not all of the marginal cases were excluded. Figure 4.3 and Table 4.3 show however that the remaining errors are so small that the TCD can be considered unbiased for its purpose within these restrictions.

4.4.2 Standard Deviation of the Estimation Errors

To show how the standard deviation of the estimation errors develops as the sample size increases, we generated 1,000 populations. The standard deviation of the estimation error was calculated in 1,000 Monte Carlo simulations for each of the selected sample sizes and depicted as a box plot against the grid points of the sample size N. The medians produced a regression curve with the form $\sqrt{\frac{a}{N} + b}$ that illustrates the proportionality of the standard deviation for $\frac{1}{\sqrt{N}}$. The results for the population parameters α and γ are found in Figure 4.4 and 4.5 with regard to β and δ in Figure B.3 and B.4 in the appendix.

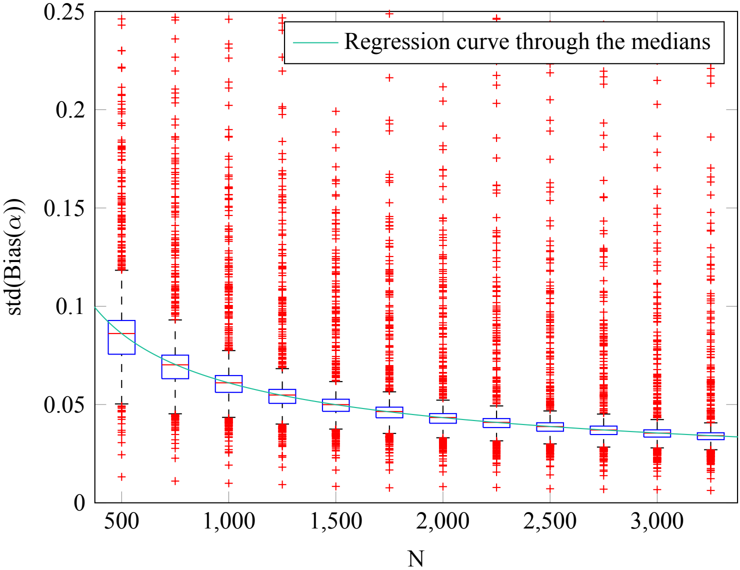

Figure 4.4: Development of the standard deviation of the estimation error of α in the TCD as per Chapter 4.4.2. The red line within the box marks the median of the data points. The box marks the limits for the upper and lower quartile. The end points of the whiskers set the location of the largest or smallest data point, which is located maximum 1.5 times of the inter-quartile distance above or below the limit for the upper or lower quartile.

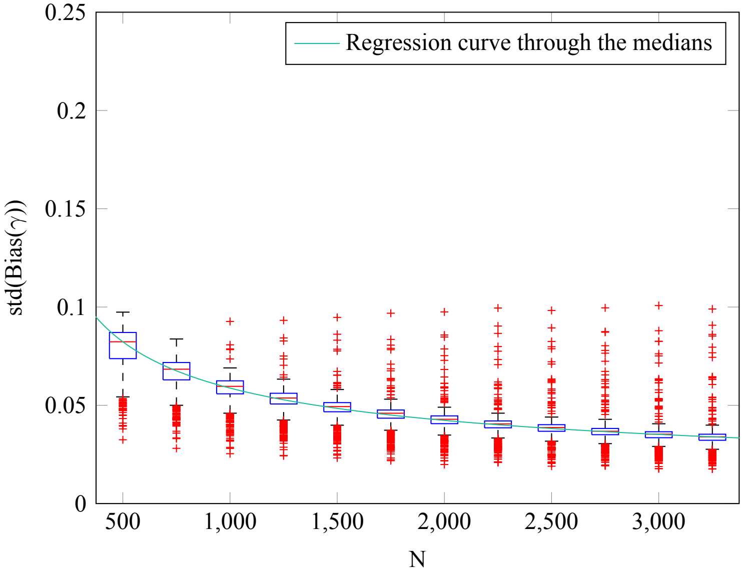

Figure 4.5: Development of the standard deviation of the estimation error of γ in the TCD as per Chapter 4.4.2. For notes on the depiction refer to Figure 4.4.

4.5 Analysis of the NCD

In principle, all of the results gained from the TCD can be transferred. The only exception here is the case that the parameter δ is incorrectly assumed to be zero (see Chapter 4.5.3). The following were used as Forced Answer probabilities

$$P^{\mathrm{Y}} = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}.$$

4.5.1 Mean Errors

It can be seen here that the NCD can be classified as unbiased except for the marginal cases. The estimation errors $\mathrm{Bias}(\hat{\gamma}), \mathrm{Bias}(\hat{\alpha})$ and $\mathrm{Bias}(\hat{\beta})$ are each shown in Figure 4.6, B.5 and B.6. The average estimation errors for the simulated data are shown in Table 4.4.

Table 4.4: Average NCD estimation errors for the simulated data scaled by a factor of 10^6*. As for the TCD the bias by the marginal cases can be clearly seen.*

	with marginal cases	without marginal cases
$\mathrm{Bias}(\hat{\alpha})$	-6,336	-392
$\mathrm{Bias}(\hat{\beta})$	-3,412	-307
$\mathrm{Bias}(\hat{\gamma})$	9,748	700

4.5.2 Standard Deviation of the Estimation Errors

The same approach as for TCD was selected. The results are found in Figure 4.7 for α, in Figure 4.8 for γ and for the cheater proportion in Figure B.8. The proportionality to $\frac{1}{\sqrt{N}}$ can also be seen clearly. Compared with the TCD, the NCD needs a smaller sample size to provide results of similar quality as the standard variations are lower throughout.

4.5.3 Effects of $\delta \neq 0$

If the population parameter $\delta \neq 0$, the NCD is biased. The calculation of the expected values is the same as was conducted above with the only difference that the probability of yes answers is calculated by

$$\lambda_i = \alpha \left(1 - P_i^{\mathrm{N}}\right) + \gamma P_i^{\mathrm{Y}} + \delta.$$

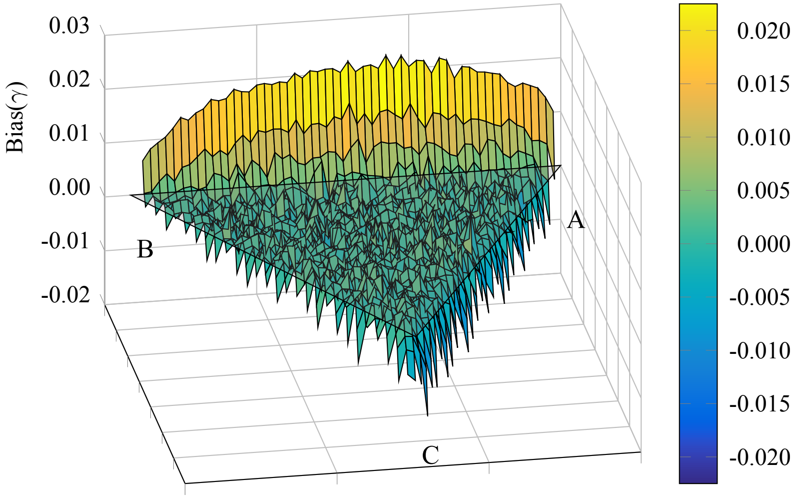

Figure 4.6: Estimation errors for γ in the NCD ($N = 1{,}000$, $\delta = 0.00$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

For the native solution we get the estimator for the population parameter α as

$$\hat{\alpha} = \frac{1}{\det M}\left(P_2^{\mathrm{Y}}\hat{\lambda_1} - P_1^{\mathrm{Y}}\hat{\lambda_2}\right).$$

The expected value of $\hat{\alpha}$ can be calculated as

$$\begin{aligned}
\mathrm{E}(\hat{\alpha}) &= \frac{1}{\det M}\left(P_2^{\mathrm{Y}}\left(\alpha\left(1-P_1^{\mathrm{N}}\right)+\gamma P_1^{\mathrm{Y}}+\delta\right)\right.\\
&\qquad\left.-P_1^{\mathrm{Y}}\left(\alpha\left(1-P_2^{\mathrm{N}}\right)+\gamma P_2^{\mathrm{Y}}+\delta\right)\right)\\
&= \frac{1}{\det M}\left(\alpha\det M+\gamma\cdot 0+\delta\left(P_2^{\mathrm{Y}}-P_1^{\mathrm{Y}}\right)\right)\\
&= \alpha+\delta\frac{P_2^{\mathrm{Y}}-P_1^{\mathrm{Y}}}{\det M}.
\end{aligned}$$

The population parameter γ is estimated by

$$\hat{\gamma} = \frac{1}{\det M}\left(-\left(1-P_2^{\mathrm{N}}\right)\hat{\lambda_1}+\left(1-P_1^{\mathrm{N}}\right)\hat{\lambda_2}\right)$$

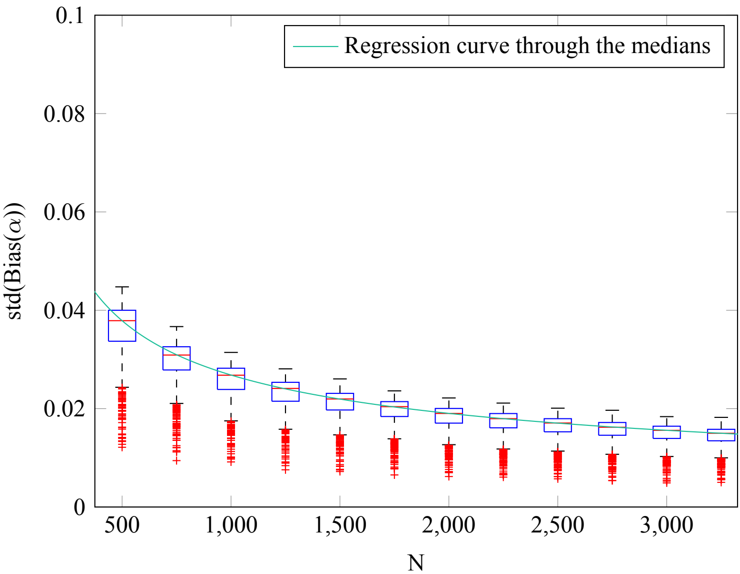

Figure 4.7: Development of the standard deviation of the estimation error of α in the NCD as per 4.5.2. For notes on the depiction refer to Figure 4.4. The scaling of the ordinate differs from Figures 4.4 and 4.5.

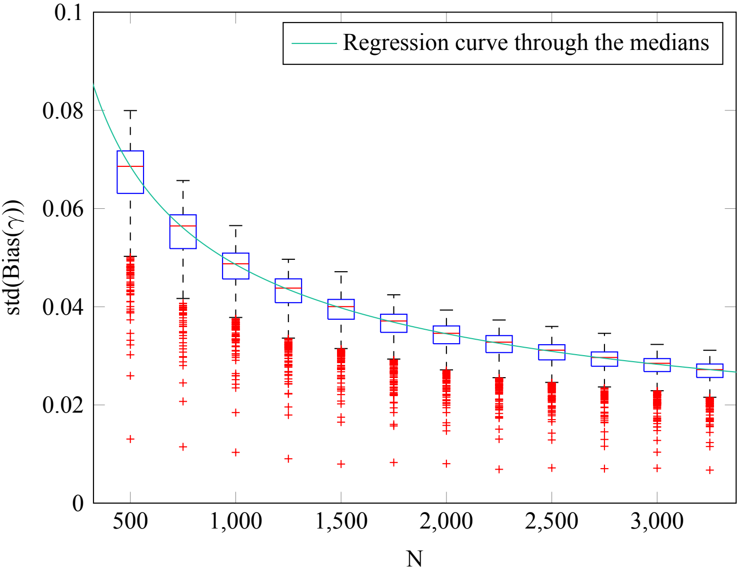

Figure 4.8: Development of the standard deviation of the estimation error of γ in the NCD as per 4.5.2. For notes on the depiction refer to Figure 4.4. The scaling of the ordinate differs from Figures 4.4 and 4.5.

and has the expected value

$$\begin{aligned}
\mathrm{E}(\hat{\gamma}) &= \frac{1}{\det M}\left(\alpha \cdot 0 + \gamma \det M + \delta\left(-1 + P_2^{\mathrm{N}} + 1 - P_1^{\mathrm{N}}\right)\right) \\
&= \gamma + \delta \frac{P_2^{\mathrm{N}} - P_1^{\mathrm{N}}}{\det M}.
\end{aligned}$$

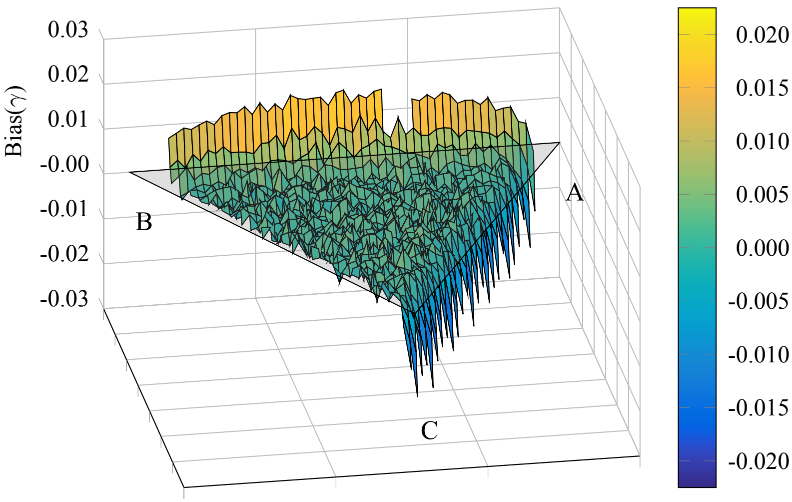

Figure 4.9: Estimation errors for γ in the NCD ($N = 1{,}000$, $\delta = 0.10$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4. The missing data points on the side $\overline{AB}$ are conditional on rounding errors in the coordinate transformation.

The value for $\hat{\beta}$ is calculated as

$$\hat{\beta} = 1 - \hat{\alpha} - \hat{\gamma}.$$

Thus it has the expected value

$$\begin{aligned}
\mathrm{E}(\hat{\beta}) &= \mathrm{E}(1 - \hat{\alpha} - \hat{\gamma}) \\
&= 1 - \alpha - \delta \frac{P_2^{\mathrm{Y}} - P_1^{\mathrm{Y}}}{\det M} - \gamma - \delta \frac{P_2^{\mathrm{N}} - P_1^{\mathrm{N}}}{\det M} \\
&= \beta + \delta\left(1 - \frac{P_2^{\mathrm{Y}} - P_1^{\mathrm{Y}} + P_2^{\mathrm{N}} - P_1^{\mathrm{N}}}{\det M}\right).
\end{aligned}$$

In the case of optimal forced answer probabilities P_1^{N} and P_2^{N} are both relatively small or approximately equal. If one chooses $P_1^{\mathrm{N}} = P_2^{\mathrm{N}}$ for a survey, this results in an unbiased estimator $\hat{\gamma}$, independently of δ. This can easily be seen when comparing the simulation results for $\delta \neq 0$ (Figures 4.9 and B.7) to the results for $\delta = 0$ (Figure 4.6). After removing the marginal cases (as for TCD) in Figure 4.9,the average estimation error $\mathrm{Bias}(\hat{\gamma}) = 0.000070$. As calculated, the native solution of $\hat{\gamma}$ is therefore unbiased.

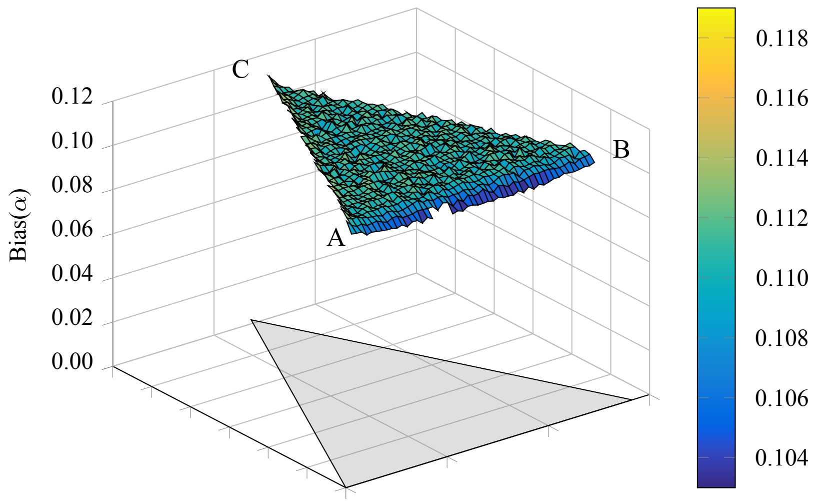

Figure 4.10: Estimation errors for α in the NCD ($N = 1{,}000$, $\delta = 0.10$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4. The missing data points on the side $\overline{AB}$ are conditional on rounding errors in the coordinate transformation.

For the population parameters α and β, the case $\delta \neq 0$ leads to biased estimators. With the forced answer probabilities used here, $\det M = -\frac{54}{100}$ and therefore $\mathrm{Bias}(\hat{\alpha}) = \frac{10}{9}\delta$ and $\mathrm{Bias}(\hat{\beta}) = -\frac{1}{9}\delta$. Simulations on this are shown in Figure 4.10 and B.7.

4.6 Analysis of the YCD

The results gained here correspond (within the limits of simulation precision) exactly to the NCD results if α is exchanged with γ and β with δ. The following were used

$$P^{\mathrm{Y}} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}} = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$$

as Forced Answer probabilities.

4.6.1 Average Errors

The estimation errors Bias $(\hat{\alpha})$, Bias $(\hat{\gamma})$ and Bias$(\hat{\delta})$ are shown in Figure 4.11 and Table 4.5, as well as in Figure B.9 and B.10. The TCD can also be viewed as unbiased with a sufficient distance to the marginal cases.

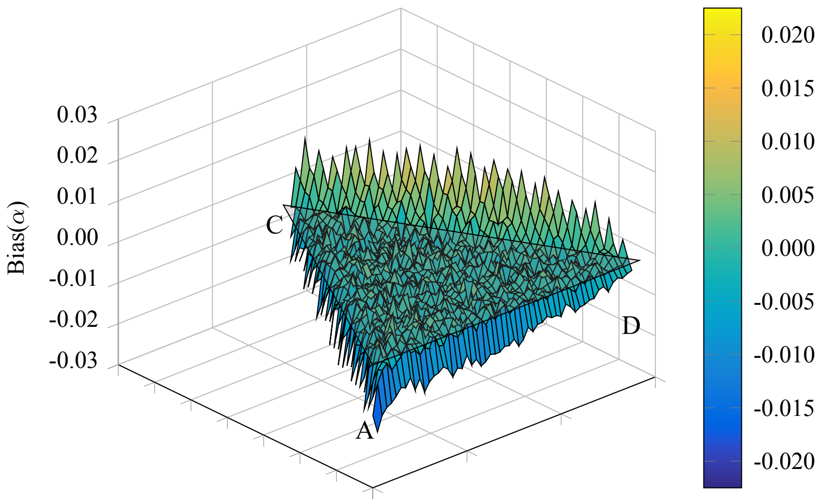

Figure 4.11: Estimation errors for α in the YCD ($N = 1{,}000$, $\beta = 0.00$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

4.6.2 Standard Deviation of the Estimation Errors

The development with an increasing sample size is shown in Figure 4.12, B.11 and B.12. The approach here is similar to the TCD and NCD. A comparison

Table 4.5: Average YCD estimation errors for the simulated data scaled by a factor of 10^6.

	with marginal cases	without marginal cases
Bias $(\hat{\alpha})$	-7,865	1,705
Bias $(\hat{\gamma})$	2,619	-708
Bias$(\hat{\delta})$	5,245	-997

with the figures relating to Chapter 4.5.2 also emphasises the symmetry of the two DCD methods and the results stated here for NCD also apply in this case.

4.6.3 Effects of $\beta \neq 0$

If $\beta \neq 0$, we will also obtain a biased estimation here. To calculate the expected values we use the modified estimators of the no probability

$$1 - \hat{\lambda}_i = \alpha \left(1 - P_i^{\mathrm{N}}\right) - \beta + \gamma P_i^{\mathrm{Y}}.$$

For the native solution, we calculate the estimator for α as

$$\hat{\alpha} = \frac{1}{\det M} \left(\left(1 - P_2^{\mathrm{Y}}\right)\left(1 - \hat{\lambda}_1\right) - \left(1 - P_1^{\mathrm{Y}}\right)\left(1 - \hat{\lambda}_2\right)\right),$$

with the expected value

$$\begin{aligned} \mathrm{E}(\hat{\alpha}) &= \frac{1}{\det M} \left(\alpha \det M + \beta \left(1 - P_2^{\mathrm{Y}} - +P_1^{\mathrm{Y}}\right) + \gamma \cdot 0\right) \\ &= \alpha + \delta \frac{P_1^{\mathrm{Y}} - P_2^{\mathrm{Y}}}{\det M}. \end{aligned}$$

Then γ is estimated by calculating

$$\hat{\gamma} = \frac{1}{\det M} \left(-P_2^{\mathrm{N}}\left(1 - \hat{\lambda}_1\right) + P_1^{\mathrm{N}}\left(1 - \hat{\lambda}_2\right)\right).$$

The expected value for $\hat{\gamma}$ is

$$\begin{aligned} \mathrm{E}(\hat{\gamma}) &= \frac{1}{\det M} \left(\alpha \cdot 0 + \beta \left(P_1^{\mathrm{N}} - P_2^{\mathrm{N}}\right) + \gamma \det M\right) \\ &= \gamma + \frac{P_2^{\mathrm{N}} - P_1^{\mathrm{N}}}{\det M} \delta. \end{aligned}$$

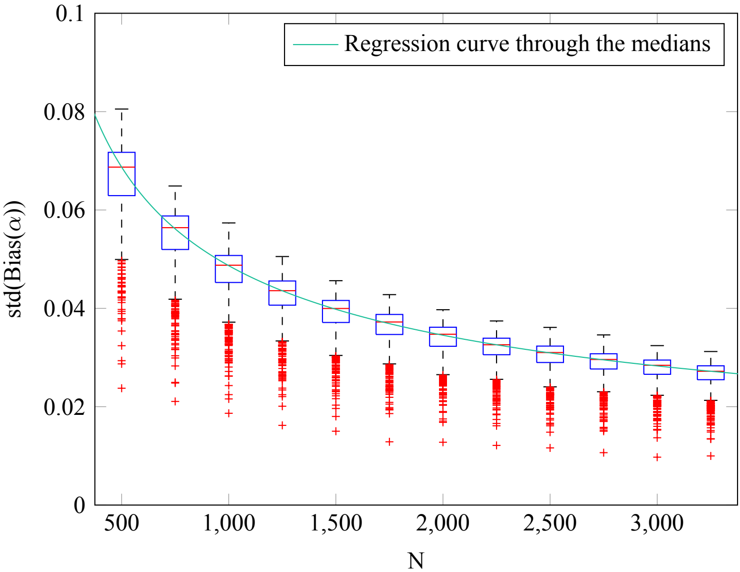

Figure 4.12: Development of the standard deviation of the estimation error of α in the YCD as per Chapter 4.6.2. For notes on the depiction refer to Figure 4.4. The scaling of the ordinate differs from Figures 4.4 and 4.5.

As the population parameter β is estimated via

$$\hat{\delta} = 1 - \hat{\alpha} - \hat{\gamma},$$

we get as its expected value

$$\begin{aligned}
\mathrm{E}(\hat{\delta}) &= \mathrm{E}(1 - \hat{\alpha} - \hat{\gamma}) \\
&= 1 - \alpha - \beta \frac{P_1^{\mathrm{Y}} - P_2^{\mathrm{Y}}}{\det M} - \gamma - \beta \frac{P_1^{\mathrm{N}} - P_2^{\mathrm{N}}}{\det M} \\
&= \delta + \beta \left(1 - \frac{P_1^{\mathrm{Y}} - P_2^{\mathrm{Y}} + P_1^{\mathrm{N}} - P_2^{\mathrm{N}}}{\det M} \right).
\end{aligned}$$

At first glance, the signs of the results look reversed to those of the NCD. Since for symmetrical Forced Answer probabilities the YCD process matrix provides the NCD process matrix by exchanging the columns, the sign of the determinant is reversed. The biases match those of the NCD. Here too, it is possible with $P_1^{\mathrm{Y}} = P_2^{\mathrm{Y}}$ to obtain an unbiased estimator $\hat{\alpha}$ even if $\beta \neq 0$.

5 Comparison of the Methods

5.1 Notation

When analysing the estimators, we used as Forced Answer probabilities

$$P^{\mathrm{Y}}_{\mathrm{TCD}} = \begin{pmatrix} 0.7 \\ 0.1 \\ 0.1 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}}_{\mathrm{TCD}} = \begin{pmatrix} 0.1 \\ 0.7 \\ 0.1 \end{pmatrix}$$

for TCD

$$P^{\mathrm{Y}}_{\mathrm{NCD}} = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}}_{\mathrm{NCD}} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$$

for NCD as well as

$$P^{\mathrm{Y}}_{\mathrm{YCD}} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}}_{\mathrm{YCD}} = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$$

for YCD. For the following analyses in this and the next chapter, we will retain these and give the associated RRT subsamples names:

- G^{T} refers to the RRT subsample with $P^{\mathrm{Y}} = 0.1$ and $P^{\mathrm{N}} = 0.1$. This subsample has the highest share of honest answers.
- G^{N} refers to the RRT subsample with $P^{\mathrm{Y}} = 0.7$ and $P^{\mathrm{N}} = 0.1$. This subsample with a high forced yes probability is useful for TCD and NCD.
- G^{Y} refers to the RRT subsample with $P^{\mathrm{Y}} = 0.1$ and $P^{\mathrm{N}} = 0.7$. This subsample with a high forced no probability is useful for TCD and YCD.

5.2 Connection between TCD and DCD

Under certain criteria, the estimations of TCD match those of DCD. A glance at the Forced Answer probabilities proposed above suggests an investigation of how e.g. the NCD behaves if it is applied to the two subsamples G^T and G^N of a TCD survey conducted. It became clear that the TCD estimation with the size of $1.5 \cdot N$ participants (whereby all three subsamples have the same size $0.5 \cdot N$) matches the DCD estimation with a sample size of N participants with reference to α and γ under particular criteria, even if the population parameter which is not estimated in the relevant DCD method is not equal to zero.

A necessary criterion for the matching seems to be that the TCD Forced Answer probabilities have the form

$$P^{\mathrm{Y}} = \begin{pmatrix} 1-\varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_2 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}} = \begin{pmatrix} \varepsilon_3 \\ 1-\varepsilon_4 \\ \varepsilon_3 \end{pmatrix},$$

but we could only observe this empirically. If this is met, one can observe the following regularities, except for the marginal cases $\beta = 1$ and $\delta = 1$:

- If the TCD results in the marginal case with $\beta = 0$ the NCD also results in this marginal case and both estimators differ only minimally in terms of γ. i.e. only absolute discrepancies $< 1\%$ were observed.
- For the marginal case $\delta = 0$ the same statement holds for the estimation of α in the YCD.
- If the TCD results in the marginal case $\alpha = 0$, the YCD also results in this and the estimations apparently match whereas the γ estimator of the NCD differs from that of the TCD.
- In the marginal case $\gamma = 0$ for the TCD, the statement applies accordingly for the estimation of γ in the NCD and of α in the YCD.

If the Forced Answer probabilities are not in the form stated above, there is a functional connection between the variances in the methods and (at least) the Forced Answer probabilities. Apparently this cannot be shown in a simple way but is not of key importance here.

5.3 Comparison with the Clark and Desharnais Method

The derivation gives rise to the expectation that the cheater detection method developed by Clark and Desharnais (1998) is a special case of the NCD presented here with $P^{\mathrm{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Simulations confirm this assumption. From the standpoint of trial planning the methods are however different as we assume that the non-existence of the sensitive characteristic could also be embarrassing. We will therefore not make any further comparisons with this method.

6 Applications of the Method

We now want to discuss a few possibilities and problems that arise when applying the method. We will restrict ourselves to the special characteristics of the method developed here and do not want to handle other general questions about applying RRT as these have already been investigated in detail (see especially the review papers by Lensvelt-Mulders et al., 2005 and Wolter, 2012).

6.1 Effects of Differently Sized RRT Subsamples

For trial planning one should consider how best to handle nonresponses to RRT questions, in particular if this occurs frequently in one of the RRT subsamples. Should one continue to assign the subsamples randomly or should one assign participants to the subsample with the fewest answers if this is possible like in online surveys which can be controlled dynamically?

From a mathematical perspective, the question of how different subsample shares affect the whole survey is appropriate as well. In an NCD, the mathematically optimal Forced Answer probabilities for $\mathrm{G^N}$ are $P^{\mathrm{Y}} = 1$ and $P^{\mathrm{N}} = 0$. With these probabilities the $\mathrm{G^N}$ subsample can be used without information from the subsample $\mathrm{G^T}$ to directly estimate the parameter β using $\lambda_{\mathrm{G^N}} = \alpha + \gamma = 1 - \beta$. In this case, which refers to a merely mathematical optimal while practically abnormal selection of P^{Y} and P^{N}, the estimation of β can be completely detached from the estimation of the other parameters. Nevertheless if the Forced Answer probabilities are chosen appropriately for practical conditions like, e.g. $P^{\mathrm{Y}} = 0.7$ and $P^{\mathrm{N}} = 0.1$, it is clear that the subsample $\mathrm{G^N}$ contributes more to the estimator of β than the subsample $\mathrm{G^T}$. Therefore a higher share of $\mathrm{G^N}$ should positively influence the quality of the estimator of β.

When presenting the results one is again faced with the problem of a clear depiction. As the development of the standard deviation of the estimation error with the survey size N is known, N was set for these simulations to $1{,}500$. We used the Monte Carlo method already described above, varying only the sizes of the RRT subsamples.

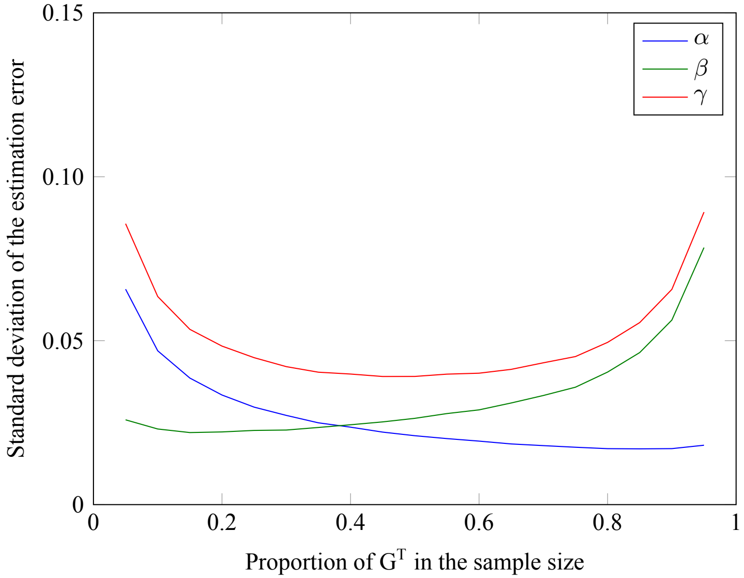

Figure 6.1: Effect of different subsample sizes for the NCD ($N = 1{,}500$).

For the NCD, it is $|G^N| + |G^T| = 1$ whereby $|\cdot|$ refers to the shares of the relevant subsamples in the whole survey sample. As expected, increasing $|G^N|$ (by lowering $|G^T|$) improves the estimator of β. The key parameter for the NCD γ is is best estimated when both RRT subsamples are approximately equally sized (see Figure 6.1 in this regard). It can also be seen that to estimate the occurrence of an extremely embarrassing characteristic (so embarrassing that there are no "yes" cheaters, i.e. $\delta = 0$, whereby the NCD is guaranteed to be unbiased), the NCD with a large $|G^T|$ is very well suited if one is primarily interested in the parameter α. Figure B.13 shows the simulation results for YCD which are again symmetrical to the NCD results.

For the TCD (Figure 6.2) $|G^N| = |G^Y|$ was assumed for further simplification. Therefore $|G^N| = |G^Y| = \frac{1-|G^T|}{2}$ applies. The ideal estimation of α and γ are around the range where all of the subsamples are approximately equally sized ($|G^T| \approx \frac{1}{3}$). Reducing $|G^T|$ results in a better estimation of β and δ, which in most cases are however the uninteresting parameters because they offer no information on the existence of the characteristic being studied.

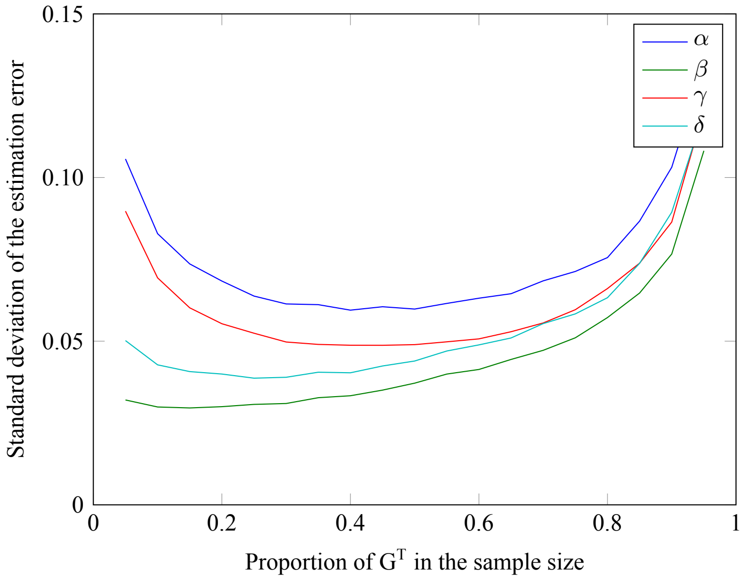

Figure 6.2: Effect of different subsample sizes for the TCD $(N = 1{,}500)$.

For the application of NCD and YCD these results show that the assignment of participants can be controlled in order to improve the quality of the estimator which is most interesting to the researcher. However for the TCD, in contrast to the two DCD methods it is not possible to selectively increase the quality of the estimators for α or γ by increasing $|\mathrm{G}^{\mathrm{T}}|$. But even in this case, the assignment of participants to the three subsamples should be controlled to ensure that $|\mathrm{G}^{\mathrm{T}}|$ lies between approximately 0.3 and 0.5 because this is the range where the average error for either α and γ has its least variability.

The knowledge about how the sizes of the subsamples affect the estimation can also be used when planning a study and trying to preserve the possibility to change from an ongoing NCD to a TCD survey (refer to Chapter 6.2).

6.2 Dynamic Survey Control

In online surveys, the option to analyse an ongoing survey before receiving the majority of the returns is convenient. This gives the opportunity to optimise the survey in terms of its objective or to use the remaining returns more efficiently.

If, for example, after starting an NCD survey it becomes clear that the population parameter $\delta = 0$ was assumed incorrectly, the survey should be converted to a TCD as the estimators are biased with regard to α and β. Therefore this chapter primarily deals with the question of how a DCD can be converted as efficiently as possible into a TCD.

In Chapter 6.1 we stayed with $|\mathrm{G}^{\mathrm{N}}| = |\mathrm{G}^{\mathrm{Y}}|$ when studying the TCD. We will do so here too, as the results available to date suggest that changing the RRT subsample sizes would apparently improve the estimation of one parameter but worsen the estimation of another parameter. Under this condition expanding a DCD into a TCD is possible with a few additional respondents if $|\mathrm{G}^{\mathrm{T}}|$ is relatively large. It was seen above that the TCD however more poorly estimates α and γ with a large $|\mathrm{G}^{\mathrm{T}}|$ and therefore with different $|\mathrm{G}^{\mathrm{N}}|$ and $|\mathrm{G}^{\mathrm{Y}}|$. Additionally, there is one more key issue when expanding the survey: the survey size N is larger when transferring to the TCD. So the efficiency relating to N must be included above all if the returns are lower than originally expected. We want to explain this using an example: answers have been received for an NCD with 1,500 participants. In the event of $|\mathrm{G}^{\mathrm{N}}| = |\mathrm{G}^{\mathrm{T}}| = \frac{1}{2}$ another 750 respondents are required so that the RRT subsamples are optimally sized for TCD. After arriving at optimal subsample shares for a TCD with 2,250 participants, one can distribute all other survey participants evenly over the three RRT subsamples. On the other hand, if there are 1,500 answers with $|\mathrm{G}^{\mathrm{T}}| = \frac{4}{5}$, only $\frac{1}{5} \cdot 1{,}500 = 300$ answers are still required before reaching a TCD with $|\mathrm{G}^{\mathrm{N}}| = |\mathrm{G}^{\mathrm{Y}}|$. After that all other survey participants can be assigned to G^{N} and G^{Y} until all three subsamples are sized equally.

Hence, with a restricted number of respondents there is the general problem of balancing the quality of the DCD and the costs of converting to a TCD. As the sample size of the TCD is affected by the size of $|\mathrm{G}^{\mathrm{T}}|$ in the DCD, the standard deviation of the estimation errors alone is not a suitable indicator for the conversion efficiency. Since the standard deviation is proportional to $\frac{1}{\sqrt{N}}$, we can estimate the quality of the resulting TCD in relation to the sample size via

$$\mathrm{Std}\left(\mathrm{Bias}\left(\chi_e\right), N_{\mathrm{TCD}}\right) \cdot \sqrt{N_{\mathrm{TCD}}}$$

with $N_{\mathrm{TCD}} = N_{\mathrm{DCD}}\left(1 + \left|\mathrm{G}^{\mathrm{N}}_{\mathrm{DCD}}\right|\right)$ and $\chi \in \{\alpha, \beta, \gamma, \delta\}$ whereby a lower value is better (see figure 6.3). The continuous lines show the standard deviations (scaled by a factor of 100) of the NCD as can also be seen in Figure 6.1. At the bottom along the horizontal axis the shares $|\mathrm{G}^{\mathrm{T}}_{\mathrm{NCD}}|$ are drawn, whereas the sample size of the resulting TCD is found at the top. The dotted lines are the calculated estimates for the efficiency as defined above.

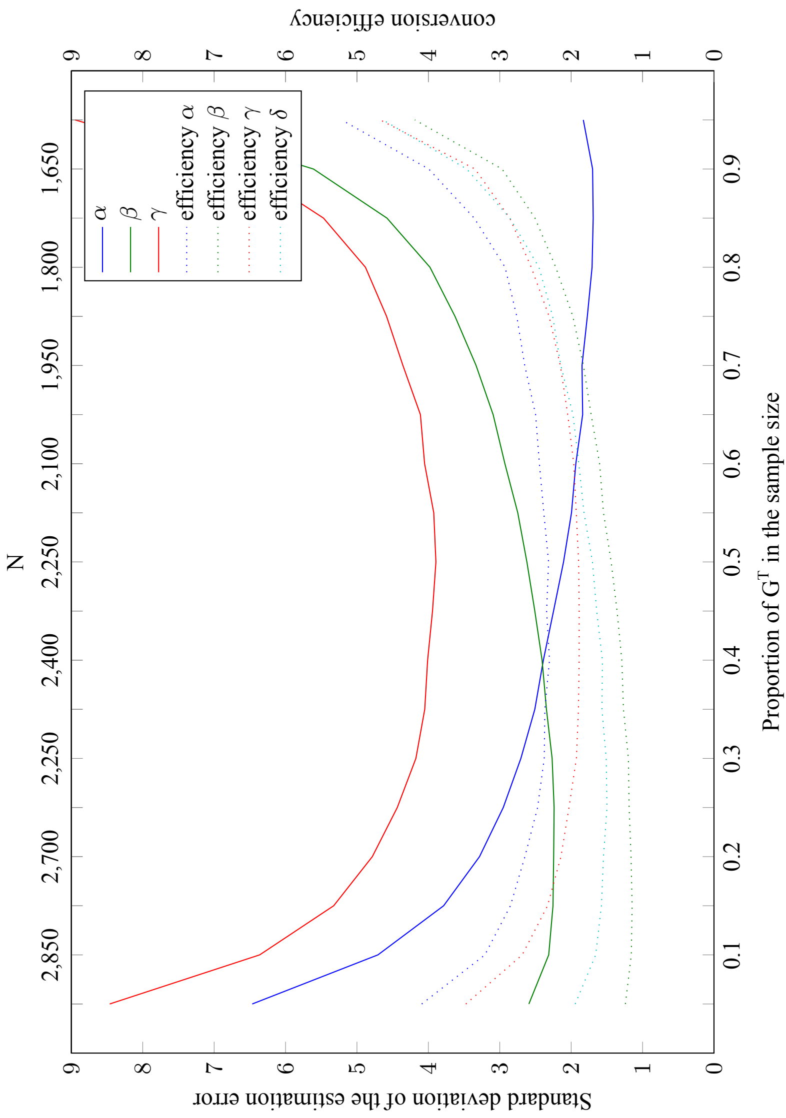

Figure 6.3: Efficiency for dynamic control.

In the $0.4 \leq |G^T_{NCD}| \leq 0.8$ area the conversion efficiency is nearly constant whereas the estimation of the NCD does not lose a lot of quality with regard to γ. The estimator for α in the NCD is even better with a growing $|G^T_{NCD}|$. The results apply analogously to the YCD.

There are sure to be a number of organisational problems when conducting a survey of this type. Depending on the nature of the survey there may however be other serious problems. If, for example, the time it takes for a person to decide whether to participate in a survey correlates with their response behaviour (cheater or honest survey participant) or the existence of the characteristic, there is the possibility that the population groups are not evenly distributed across the RRT subsamples. As will be shown in 6.3, this results in biased estimators.

Figure 6.4 shows the course of a simulated survey that was controlled dynamically. The population parameters unknown to us are $\alpha = 0.3$, $\beta = 0.1$, $\gamma = 0.55$ and $\delta = 0.05$. In the planning phase, the decision was taken to conduct an NCD as there were good reasons for assuming $\delta = 0$. We expect that around 2,000 participants are theoretically sufficient to be able to conduct a TCD without problems. We want to keep the option open to switch as quickly as possible to a TCD and therefore use $|G^T_{NCD}| = 0.8$.

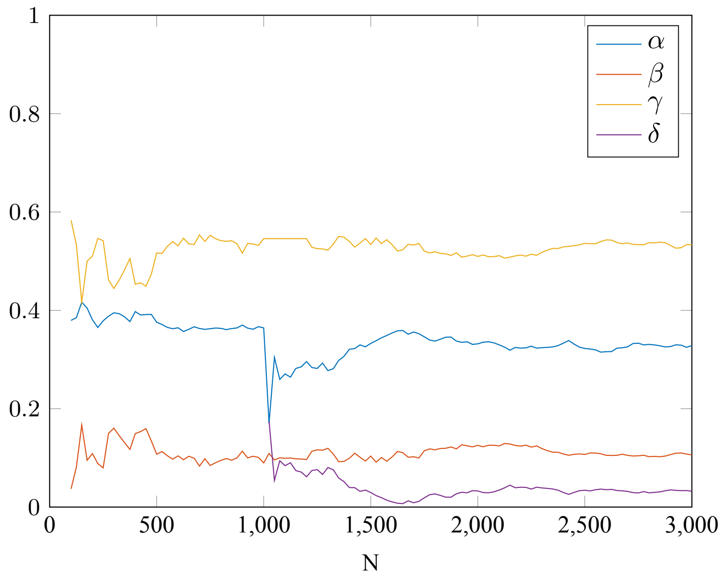

Figure 6.4: Example for estimations in a dynamically controlled survey.

After the first 1,000 answers we can assume that the returns are much higher than expected so that we can handle a total of around 3,000 answers instead of the planned 2,000. This causes us to decide to utilise the next 200 answers to expand to a TCD. There are now three reasonable options for what could happen.

1. The assumption $\delta = 0$ is confirmed but we stay with the TCD as the unexpectedly high number of responses roughly delivers the originally expected quality.

2. The assumption $\delta = 0$ is confirmed. We go back to the NCD and work towards $|\mathrm{G}^{\mathrm{T}}_{\mathrm{NCD}}| = |\mathrm{G}^{\mathrm{N}}_{\mathrm{NCD}}|$. We would lose the 200 responses that we cannot appropriately include in the NCD calculation because the Forced Answer probabilities are absolutely inappropriate. Nevertheless the NCD with 2,800 answers delivers a more reliable result than the TCD with 3,000 answers.

   Choosing an alternative in order to preserve the 200 records would typically lead to worse estimations:

   - If we calculate the NCD over three RRT subsamples there is no real ML estimator and the relatively small size of this third subsample offers a high potential for statistical outliers.
   - If we work the 200 answers into the two subsamples G^{T} or G^{N} we have to use the weighted averages of the Forced Answer probabilities in the process matrix. Considered asymptomatically the determinant of the process matrix worsens linearly with the additional answers whereas the rising sample size only contributes with exponent $\frac{1}{2}$ to the improvement of the estimation.

3. Our assumption $\delta = 0$ is not confirmed. So we remain with the TCD and work towards improving the quality of the estimation by trying to reach equal RRT subsample sizes.

Although parameter δ (unknown to us up to this point) is only very low, we can see quickly that our assumption $\delta = 0$ was incorrect. We take the approach in option 3.

6.3 Different Response Behaviour in the RRT Subsamples

One assumption that was made when deriving the method is that a person's response behaviour does not depend on the assignment to a particular RRT subsample. This is important with regard to cheating which may occur for different reasons. It may be the case that a person deliberately attempts to sabotage the survey or simply does not understand the instruction. However these reasons for cheating do not affect the assumption made, provided that these people are on average assigned equally to the subsamples. Furthermore, the impression of not being secure may result in cheating. This may of course be different in each RRT subsample as the forced answer probabilities are different. In general, the experienced protection of anonymity depends on the probability of the alternative instruction for the embarrassing property. If the answer yes is embarrassing the protection is estimated even higher when more participants are forced to answer yes as a result of the instruction. If one considers a TCD with

$$P^{\mathrm{Y}} = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.7 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{N}} = \begin{pmatrix} 0.1 \\ 0.7 \\ 0.1 \end{pmatrix},$$

the yes response is better protected in the third subsample than in the first two, where this protection is equally weak. Nevertheless, if we understand cheating as a result of a decision which respondents take at the moment they are surveyed, the first two RRT subsamples also differ in the share of those who decide how to answer: although the yes response is equally weakly protected in these subsamples, this affects a very different share of participants. In the first subsample, the share of $0.9 \cdot \alpha$ people with the embarrassing property will be tempted to ignore the instructions to answer yes to or provide an honest answer. This share significantly exceeds the share in the second RRT subsample which is only $0.3 \cdot \alpha$.

Different response behaviours in the RRT subsamples mean that the estimators are (in general) biased. We will show this numerically with an example using the NCD with the forced answer probabilities of the first two subsamples shown above: for an NCD with a total size $N = 1{,}000$, $N_1 = N_2 = 500$ survey participants are simulated from two survey populations which are assigned to G^{T} and G^{N}. In the RRT subsample G^{T} the share of potential cheaters is significantly higher than in G^{N}. Therefore the population parameters

$$\alpha_1 = 0.2, \quad \beta_1 = 0.2 \quad \text{and} \quad \gamma_1 = 0.6$$

for the first population and

$$\alpha_2 = 0.3, \quad \beta_2 = 0.1 \quad \text{and} \quad \gamma_2 = 0.6$$

for the second one, reflecting the different willingness towards "no" cheating due to a different number of tempted participants with the embarrassing property. With an unbiased estimator one would expect in this situation precisely the average of the parameter in both sub-populations, i.e.

$$\hat{\alpha} = 0.25, \quad \hat{\beta} = 0.15 \quad \text{and} \quad \hat{\gamma} = 0.6.$$

Under these conditions the simulation was conducted 10,000 times and the average estimations were

$$\hat{\alpha} = 0.1832, \quad \hat{\beta} = 0.0670 \quad \text{and} \quad \hat{\gamma} = 0.7498.$$

The estimators therefore vary significantly from the true scores.

If it is possible to set up a functional link between the forced answer probabilities and the feeling of protection or the willingness to cheat empirically, this can be taken into account when deriving the method and maybe also be compensated for accordingly in the calculation. Until then, the only way to counteract this is to find instructions which hide the fact that the probability of the sensitive question is relatively high or that affirming the embarrassing characteristic is poorly protected.

7 Outlook

The derivation of the method applies to any number of RRT subsamples. It is therefore possible to develop other generalisations for the methods derived here that can estimate more parameters. For example, one could add two more population groups to the four introduced here:

- ε - People who have the embarrassing characteristic who behave correctly with the Forced Answer instruction and answer no to the sensitive question.
- φ - People who do not have the embarrassing characteristic who behave correctly with the Forced Answer instruction and answer yes to the sensitive question.

As with the DCD, one could also derive a method which only estimates some of the population parameters if there is a well-founded assumption that the remaining parameters equal zero.

In Chapter 5.2, we saw that estimating more parameters requires a higher survey size if the estimation is not to lose quality. If one transfers these observations to a method with five RRT subsamples that estimates six population parameters, this requires 2.5 times as many survey participants to reach the quality of a method that was designed for just three parameters.

There are other practical difficulties for generalisation of the method. Most notably, the forced yes and forced no probabilities in the RRT subsamples must be selected such that on the one hand the respondents feel protected and on the other hand that the determinant of the process matrix is neither singular nor that the absolute value of the determinant becomes very small. The more RRT subsamples are included in the method the more complex this variation of the Forced Answer probabilities is because for each additional parameter the process matrix is extended by one additional row which must differ from the other rows only in the Forced Answer probabilities. If one row equals another the matrix is already singular. If two rows differ only slightly the absolute value of the determinant becomes very small. Nevertheless, for practical reasons, if the instructions are to be kept simple, rows that are almost identical are nearly unavoidable. Additionally, the derivation and implementation of the method becomes exponentially more complex: If one uses R RRT subsamples to calculate $R+1$ population parameters $2^R - 1$ marginal cases must be considered.

By and large, besides the possibility of deriving more complex methods based on the ideas presented in this book, there are strong practical limitations to this idea. However, social-scientific theories dealing with embarrassing characteristics do not necessarily afford more complex methods than those methods developed until now. Therefore, RRT with cheater detection may support social scientific theory development in spite of these practical limitations as they have already done in recent years.

A Derivation of the Triangular Representation

Let $\delta \in [0, 1)$ (the transformation cannot be applied for $\delta = 1$ but in this case we obviously don't need the transformation). Let further

$$T := \left\{ \Theta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \alpha + \beta + \gamma = 1 - \delta \wedge \alpha, \beta, \gamma \in [0, 1] \right\}$$

be the set of admissible population parameters for the given δ and let $\Theta \in T$. We now choose

$$B := \left\{ B_a = \begin{pmatrix} 1-\delta \\ 0 \\ 0 \end{pmatrix}, B_b = \begin{pmatrix} 0 \\ 1-\delta \\ 0 \end{pmatrix}, B_c = \begin{pmatrix} 0 \\ 0 \\ 1-\delta \end{pmatrix} \right\}$$

as basis of $\mathbb{R}^3$ and rewrite $\Theta = \lambda_a B_a + \lambda_b B_b + \lambda_c B_c$. This yields

$$\lambda_a (1-\delta) + \lambda_b (1-\delta) + \lambda_c (1-\delta) = 1 - \delta \Leftrightarrow \lambda_a + \lambda_b + \lambda_c = 1$$

due to $\alpha + \beta + \gamma = 1 - \delta$. But we also know that $\alpha, \beta, \gamma \in [0, 1]$, hence it is $\lambda_a, \lambda_b, \lambda_c \in [0, 1]$. Therefore every $\Theta \in T$ can uniquely be represented as a convex combination of the basis vectors in B, i.e. Θ lies on the convex hull of B. By choice of B, its convex hull is an equilateral triangle with edge length $(1-\delta)\sqrt{2}$ (cf. Figure A.1). The coefficients $\lambda_a, \lambda_b, \lambda_c$ of above convex combination can be interpreted as barycentric coordinates in this triangle T. Let $T' \subset \mathbb{R}^3$ be another equilateral triangle with the same edge length. If $\Theta' \in T'$ is a point with barycentric coordinates $\lambda'_a, \lambda'_b, \lambda'_c$ and if $\lambda_a = \lambda'_a, \lambda_b = \lambda'_b, \lambda_c = \lambda'_c$ holds, then Θ and Θ' have the same relative positions in their respective triangles.

Choose

$$B' := \left\{ B'_a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, B'_b = \begin{pmatrix} (1-\delta)\sqrt{2} \\ 0 \\ 1 \end{pmatrix}, B'_c = \begin{pmatrix} (1-\delta)\frac{\sqrt{2}}{2} \\ (1-\delta)\sqrt{\frac{3}{2}} \\ 1 \end{pmatrix} \right\}.$$

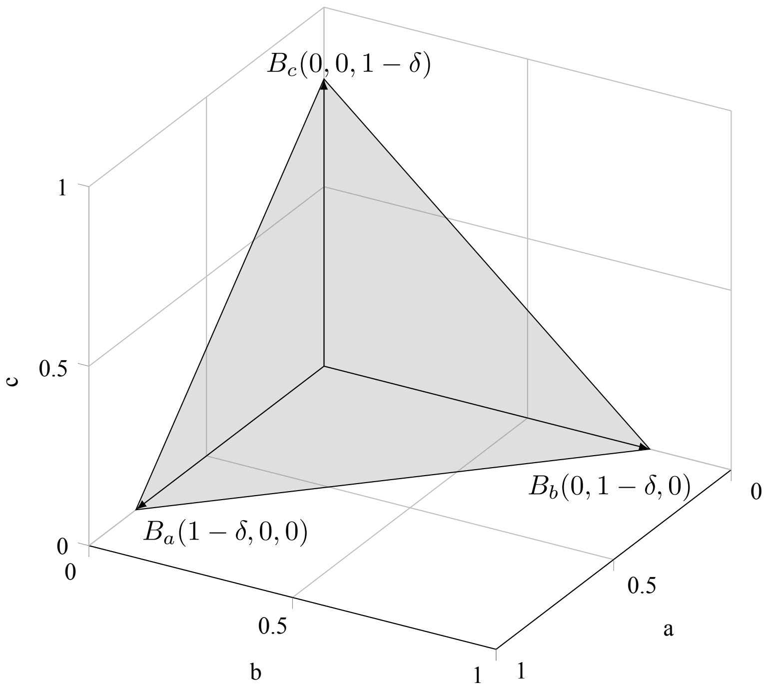

Figure A.1: The convex hull of B_a, B_b and B_c (grey area) is an equilateral triangle with edge length $(1-\delta)\sqrt{2}$.

One can easily see that B' is a basis of $\mathbb{R}^3$ and that its basis vectors also span an equilateral triangle with side length $(1-\delta)\sqrt{2}$. This triangle T' spanned by B' naturally embeds into $\mathbb{R}^2$ by simply dropping the third coordinate from all of its points. However, we need to keep the information $\lambda'_a + \lambda'_b + \lambda'_c = 1$ from the dropped coordinate. The embedded triangle has the vertices

$$\begin{aligned} T_A &= (T_{A,x}, T_{A,y}) = (0,0) \\ T_B &= (T_{B,x}, T_{B,y}) = \Big((1-\delta)\sqrt{2}, 0\Big) \\ T_C &= (T_{C,x}, T_{C,y}) = \left((1-\delta)\frac{\sqrt{2}}{2}, (1-\delta)\sqrt{\frac{3}{2}}\right). \end{aligned}$$

For any point $\Theta_2 = (x,y)$ in this embedded triangle, we can compute its barycentric coordinates with respect to the original triangle T' via

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = M \begin{pmatrix} \lambda'_a \\ \lambda'_b \\ \lambda'_c \end{pmatrix} \text{ mit } M = \begin{pmatrix} 1 & 1 & 1 \\ T_{A,x} & T_{B,x} & T_{C,x} \\ T_{A,y} & T_{B,y} & T_{C,y} \end{pmatrix}.$$

Now that we have $\lambda'_a, \lambda'_b, \lambda'_c$, we can go back to our original set T of admissible population parameters by identifying

$$\Theta_2 \mathrel{\widehat{=}} \lambda'_a B_a + \lambda'_b B_b + \lambda'_c B_c,$$

i.e. we know exactly which population parameters are represented by a point $(x,y) \in \mathbb{R}^2$.

B Figures and Tables

B.1 Figures and Tables for TCD

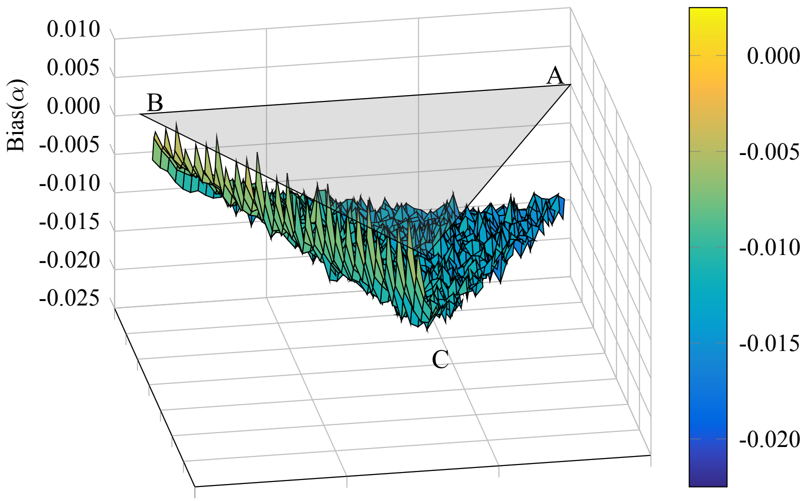

Figure B.1: Estimation errors for α in the TCD ($N = 1{,}500$, $\delta = 0.00$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

Table B.1: Average TCD estimation errors for the simulated data scaled by a factor of 10^6.

δ	χ	Bias $(\hat{\chi})$ with marginal cases	Bias $(\hat{\chi})$ without marginal cases
0.00	β	8,445	-
	δ	83,258	-
0.05	β	-3,824	-2,061
	δ	-13,548	-2,511
0.10	β	-11,509	-5,602
	δ	-23,762	-8,702
0.15	β	-19,614	-3,147
	δ	-24,743	-5,078
0.20	β	-27,075	-6,299
	δ	-16,145	-2,990
0.35	β	-28,682	-5,926
	δ	23,781	6,502
0.50	β	-19,141	-8,912
	δ	18,264	17,942
0.65	β	-35,129	-5,673
	δ	21,189	9,243
0.80	β	-17,175	-2,810
	δ	9,673	11,556
0.95	β	-388	-
	δ	-10,765	-

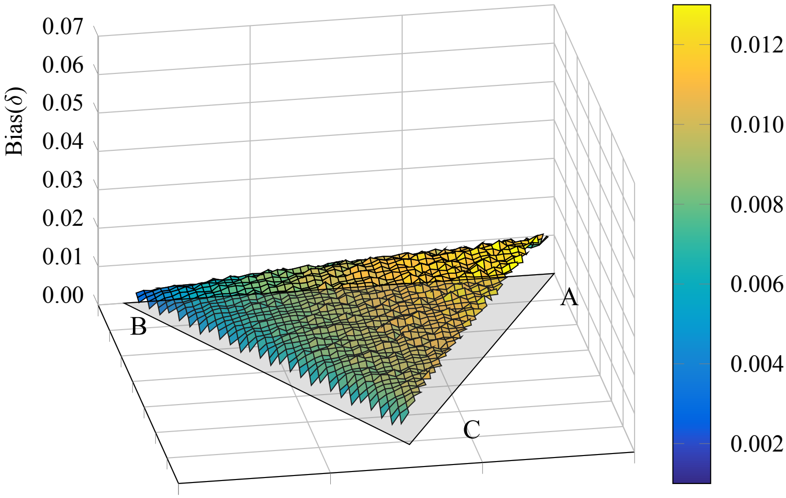

Figure B.2: Estimation errors for δ in the TCD ($N = 1{,}500$, $\delta = 0.00$, $1{,}000$ repetitions). Depiction in triangle coordinates as per Chapter 4.4.

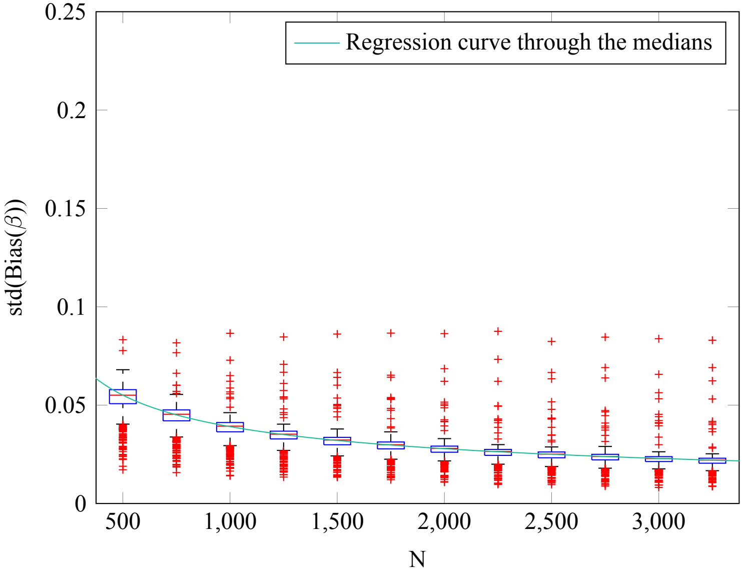

Figure B.3: Development of the standard deviation of the estimation error of β in the TCD as per Chapter 4.4.2. For notes on the depiction refer to Figure 4.4.

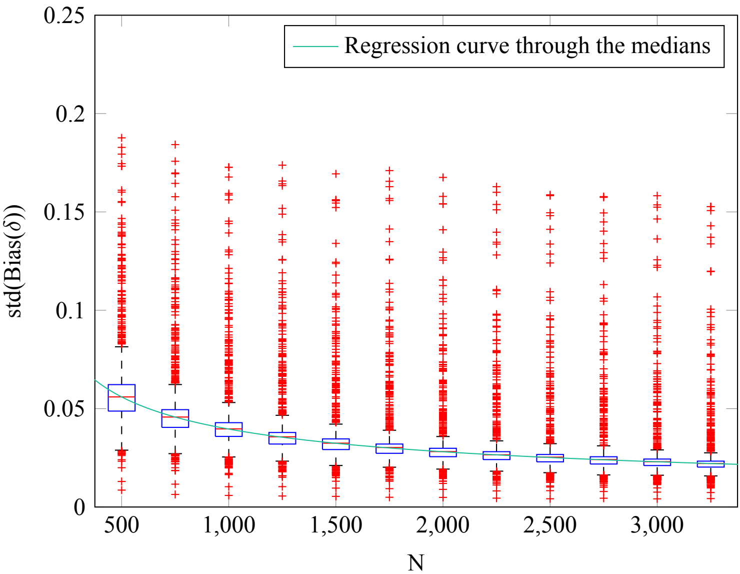

Figure B.4: Development of the standard deviation of the estimation error of δ in the TCD as per Chapter 4.4.2. For notes on the depiction refer to Figure 4.4.

B.2 Figures for NCD

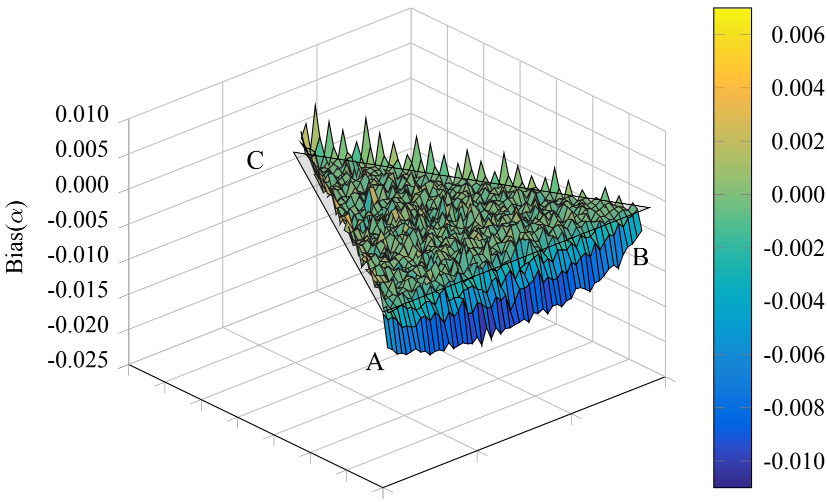

Figure B.5: Estimation errors for α in the NCD ($N = 1{,}000$, $\delta = 0.00$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

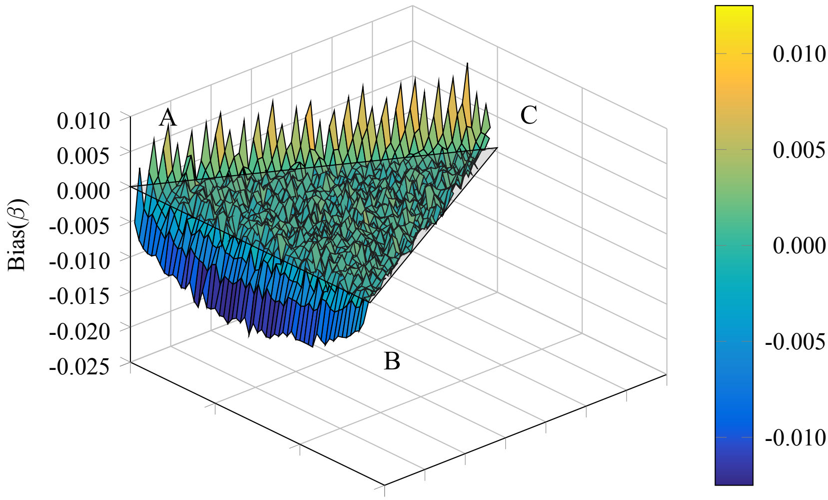

Figure B.6: Estimation errors for β in the NCD ($N = 1{,}000$, $\delta = 0.00$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

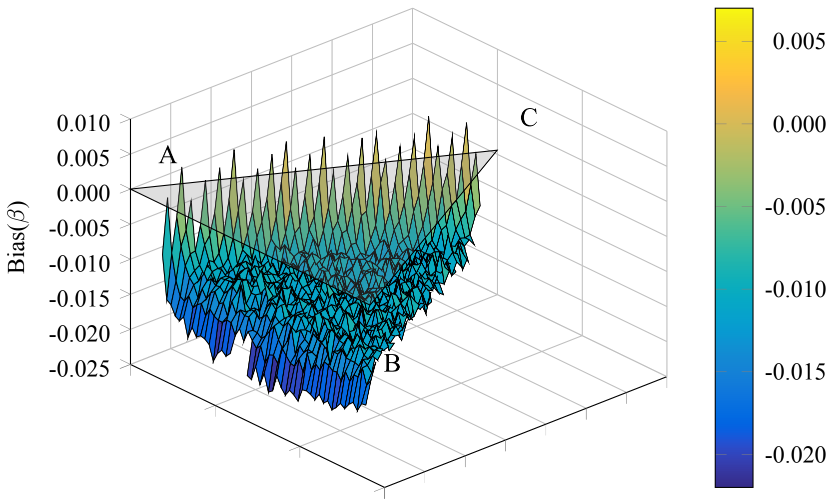

Figure B.7: Estimation errors for β in the NCD ($N = 1{,}000$, $\delta = 0.10$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

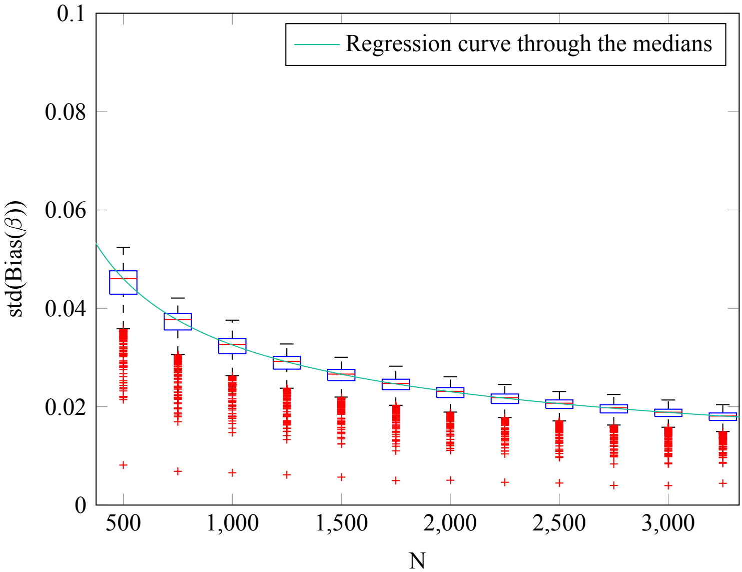

Figure B.8: Development of the standard deviation of the estimation error of β in the NCD as per Chapter 4.5.2. For notes on the depiction refer to Figure 4.4. The scaling of the ordinate differs from Figures 4.4 and 4.5.

B.3 Figures and Tables for YCD

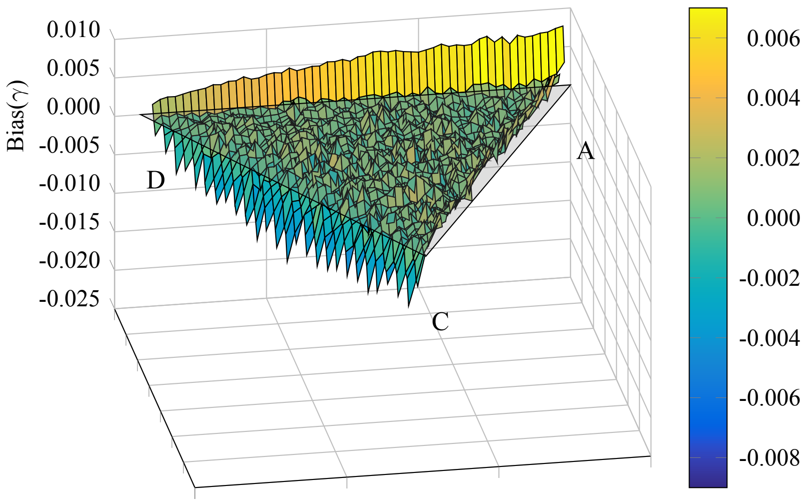

Figure B.9: Estimation errors for γ in the YCD ($N = 1{,}000$, $\beta = 0.00$, $1{,}000$ repetitions). Depiction in triangle coordinates as per Chapter 4.4.

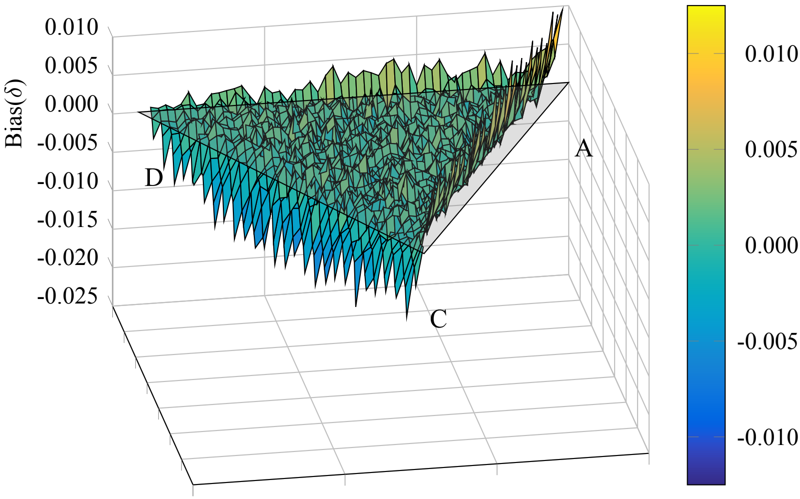

Figure B.10: Estimation errors for δ in the YCD ($N = 1{,}000$, $\beta = 0.00$, 1,000 repetitions). Depiction in triangle coordinates as per Chapter 4.4.

Table B.2: Interaction of the standard variations of the estimation errors for YCD with the determinant of the process matrix. The determinant is exact; the quantiles are scaled by a factor of 1,000 and rounded.

χ	$\lvert\det M\rvert$	Std (Bias ($\hat{\chi}$))			Std (Bias ($\hat{\chi}$)) $\cdot$ $\lvert\det M\rvert$		
		10%	50%	90%	10%	50%	90%
α	0.54	32.72	40.07	42.96	17.67	21.64	23.20
	0.40	43.59	54.69	58.30	16.56	20.78	22.15
	0.22	52.47	67.90	72.83	15.74	20.37	21.85
	0.16	68.07	91.90	100.30	12.93	17.46	19.06
γ	0.54	15.23	22.13	24.06	8.22	11.95	12.99
	0.40	21.72	28.42	30.55	8.25	10.80	11.61
	0.22	26.38	35.20	38.22	7.91	10.56	11.47
	0.16	33.77	45.63	50.46	6.42	8.67	9.59
δ	0.54	22.96	26.64	28.32	12.40	14.38	15.30
	0.40	28.04	33.60	35.51	10.66	12.77	13.50
	0.22	31.32	38.48	40.97	9.40	11.54	12.29
	0.16	39.87	51.66	55.76	7.58	9.82	10.59

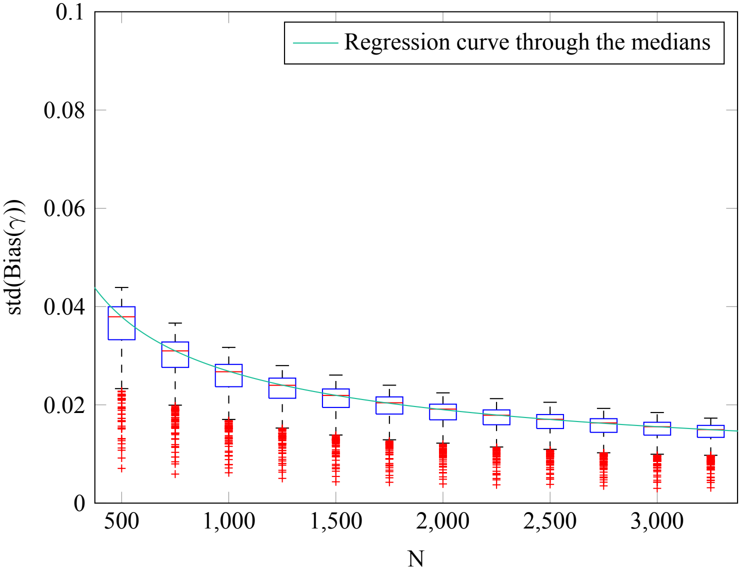

Figure B.11: Development of the standard deviation of the estimation error of γ in the YCD as per Chapter 4.6.2. For notes on the depiction refer to Figure 4.4. The scaling of the ordinate differs from Figures 4.4 and 4.5.

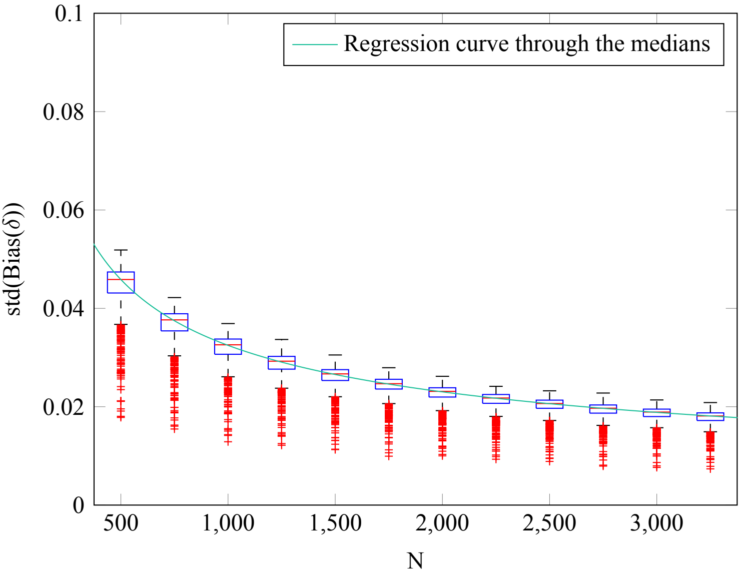

Figure B.12: Development of the standard deviation of the estimation error of δ in the YCD as per Chapter 4.6.2. For notes on the depiction refer to Figure 4.4. The scaling of the ordinate differs from Figures 4.4 and 4.5.

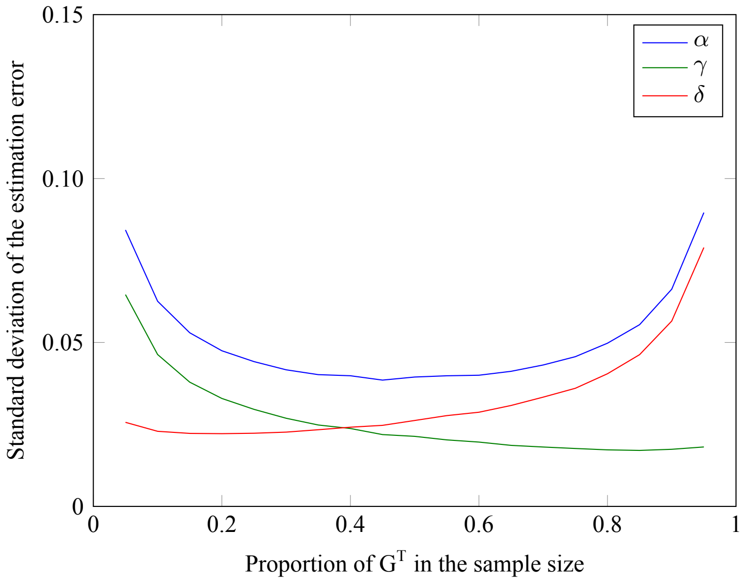

Figure B.13: Effect of differently sized RRT subsamples in the YCD (N = 1,500).

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Finding out how often an embarrassing characteristic exists in a population is accompanied by numerous problems that are due in particular to false answers as a consequence of social undesirability. The oldest of the techniques developed to compensate for these distortions is the Randomized Response Technique which is currently still the most widely used, evaluated and researched technique in the field of sensitive subjects. Although this technique guarantees complete security for the person being questioned and therefore opens up options for answering even threatening questions honestly, even here "cheating" occurs in terms of not complying with the instructions. This volume describes the mathematical derivation of techniques to detect the extent to which this "cheating" occurs. It also provides analyses on the process characteristics and extends these analyses to include practical advice on how to use this process flexibly. A general solution is developed for various forms of "cheating" for the first time.