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Non-hyperoctahedral Categories of Two-Colored Partitions Part II: All Possible Parameter Values

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Abstract

This article is part of a series with the aim of classifying all non-hyperoctahedral categories of two-colored partitions. Those constitute by some Tannaka-Krein type result the representation categories of a specific class of quantum groups. In Part I we introduced a class of parameters which gave rise to many new non-hyperoctahedral categories of partitions. In the present article we show that this class actually contains all possible parameter values of all non-hyperoctahedral categories of partitions. This is an important step towards the classification of all non-hyperoctahedral categories.

Keywords Quantum group · Unitary easy quantum group · Unitary group · Half-liberation · Tensor category · Two-colored partition · Partition of a set · Category of partitions · Brauer algebra

Mathematics Subject Classification 05A18 (Primary) · 20G42 (Secondary)

1 Introduction

In [8], Woronowicz provided a Tannaka duality for the (today so-called) *compact matrix quantum groups* he defined in [7] and which can be seen as a certain class of complex Hopf-$*$-algebras (compare also [9] for general compact quantum groups). More precisely, Woronowicz's theorem establishes a 2-equivalence between, on the one hand, the opposite of the (2, 1)-category **CMQG** of compact matrix quantum groups and, on the other hand, the slice 2-category $\mathbf{gm}C^*\mathbf{Cat}_{\mathrm{scfr}}/Hilb_{\mathrm{f}}$ of small Cauchy-complete finite-dimensional rigid monoidal C^*-categories with a fixed a single-object generator (as 0-cells and with unitary monoidal functors as 1-cells and unitary monoidal natural transformations as 2-cells) over the category of finite-dimensional complex Hilbert spaces.

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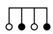
Banica and Speicher showed in [1] how to construct 0-cells of the 2-category $\mathbf{gm}C^*\mathbf{Cat}_{\mathrm{scfr}}/\mathit{Hilb}_{\mathrm{f}}$ from so-called *categories of partitions*, importantly utilizing combinatorics to produce heretofore scarce examples. Tarrago and the second author extended their construction in [6] to produce even more examples, now from *categories of two-colored partitions*. Moreover, they initiated a program to classify all such categories. The present article aims to further this effort.

Categories of two-colored partitions are explained in reference to a certain category $\mathcal{P}^{\circ\bullet}$. The latter is defined to have as objects all words $c_1 \ldots c_k$ over the alphabet $\{\circ, \bullet\}$. The morphism set from any word $c_1 \ldots c_k$ to any word $d_1 \ldots d_\ell$ consists of all set-theoretical partitions p of $\{{}^{\blacksquare}1, \ldots, {}^{\blacksquare}k, {}_{\blacksquare}1, \ldots, {}_{\blacksquare}\ell\}$. Morphisms are composed by "vertical concatenation", which importantly involves the (associative operation of) forming the join of two set-theoretical partitions. The identity of $c_1 \ldots c_k$ is the set containing exactly the sets $\{{}^{\blacksquare}i, {}_{\blacksquare}i\}$ for all $i \in \{1, \ldots, k\}$. Morphisms are frequently depicted graphically. E.g., the identity morphism of $\circ \circ \bullet$ is addressed as .

Moreover, $\mathcal{P}^{\circ\bullet}$ is equipped with the strict monoidal structure given on objects by the ("horizontal") concatenation of words, $c_1 \ldots c_k \otimes c'_1 \ldots c'_{k'} = c_1 \ldots c_k c'_1 \ldots c'_{k'}$, and on morphisms by an operation in the same spirit. The monoidal unit object is the empty word $\varnothing$. The dagger functor acts by "reflection", i.e., exchanging ${}^{\blacksquare}i \leftrightarrow {}_{\blacksquare}i$. With respect to this monoidal structure, $\mathcal{P}^{\circ\bullet}$ is rigid. Moreover, it is generated as a rigid monoidal category by the single object given by the one-letter word $\circ$, whose dual object is $\bullet$. The (left) evaluation and co-evaluation morphisms of $\circ$ correspond to  and , respectively. (That is enough to know because $\mathcal{P}^{\circ\bullet}$ can further be equipped with a symmetry, e.g.,  from $\circ\bullet$ to $\bullet\circ$.)

By definition, a category of two-colored partitions is now any wide (necessarily rigid) monoidal dagger-subcategory of $\mathcal{P}^{\circ\bullet}$ containing the evaluation and co-evaluation morphisms of $\circ$. Routinely, the symbol $\mathcal{P}^{\circ\bullet}$ is also used for the set of all morphisms of $\mathcal{P}^{\circ\bullet}$. And since categories of two-colored partitions are in particular supposed to be wide they are usually framed as subsets $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ of this set $\mathcal{P}^{\circ\bullet}$, subject to corresponding closure conditions. (See [6] or [5] for an unabridged version of all those defini.)

Since the classification program for categories of two-colored partitions was begun, different subclasses have been indexed by various contributors (see [2–4,6]). The present article is the second part of a series aiming to determine and describe all so-called non-hyperoctahedral categories, i.e., all categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ with  $\in \mathcal{C}$ or  $\notin \mathcal{C}$.

In this regard the first article [5] and the present one pursue complementary approaches to detecting whether a given set of partitions is a non-hyperoctahedral category: Part I gave sufficient conditions for being a non-hyperoctahedral category, Part II now provides necessary ones.

Let us take a closer look at the findings of Part I, [5]. Every two-colored partition can be equipped with two natural structures on its set of points: a measure-like one, the *color sum*, and a metric-like one, the *color distance*. Both [5] and the present article study tuples of six properties of any given partition:

(1) the set of block sizes,
(2) the set of block color sums,
(3) the color sum of the set of all points,
(4) the set of color distances between subsequent legs of the same block with identical (normalized) colors,
(5) the set of color distances between subsequent legs of the same block with different (normalized) colors and
(6) the set of color distances between legs belonging to crossing blocks.

By forming unions, one can aggregate these data over a given set of partitions. This information extracted from a set $\mathcal{S} \subseteq \mathcal{P}^{\circ\bullet}$ of partitions was called $Z(\mathcal{S})$ in [5].

There it was shown that one can give constraints on the above six properties which are preserved under category operations: A partially ordered set $(\mathcal{Q}, \leq)$ of parameters was introduced to prove that the sets of the form

$$\mathcal{R}_Q{:=}\{p \in \mathcal{P}^{\circ\bullet} \mid Z(\{p\}) \leq Q\} \quad \text{for } Q \in \mathcal{Q}$$

form non-hyperoctahedral categories.

The current article now shows that these constraints encoded in Z and $(\mathcal{Q}, \leq)$ are natural in the following sense. (See also Section 2 for the definitions.)

Main Theorem [Theorem 9.1] *Given any non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ of two-colored partitions, we have $Z(\mathcal{C}) \in \mathcal{Q}$.*

The importance of this result comes from its role in the overall program of the article series. On the one hand, it will be crucial to proving the main assertions of the ensuing articles. On the other hand, once those have been established, it will combine with them to show the final result of the entire series, roughly:

Main Theorem of the Series (Excerpt). *Z restricts to a one-to-one correspondence between the set $PCat^{\circ\bullet}_{\mathrm{NHO}}$ of non-hyperoctahedral categories of two-colored partitions and the parameter set $\mathcal{Q}$.*

The proof will go as follows: By Part I of the series, $\mathcal{R}_Q \subseteq PCat^{\circ\bullet}_{\mathrm{NHO}}$ for every $Q \in \mathcal{Q}$. Conversely, by the above Main Theorem of Part II, $Z(\mathcal{C}) \in \mathcal{Q}$ for any $\mathcal{C} \in PCat^{\circ\bullet}_{\mathrm{NHO}}$. In the subsequent articles we will define a set $\mathcal{G}_{Z(\mathcal{C})} \subseteq \mathcal{P}^{\circ\bullet}$ and show

$$\mathcal{G}_{Z(\mathcal{C})} \subseteq \mathcal{C} \subseteq \langle \mathcal{G}_{Z(\mathcal{C})} \rangle \quad \text{and} \quad \mathcal{G}_{Z(\mathcal{R}_{Z(\mathcal{C})})} \subseteq \mathcal{R}_{Z(\mathcal{C})} \subseteq \langle \mathcal{G}_{Z(\mathcal{R}_{Z(\mathcal{C})})} \rangle.$$

Proving $Z(\mathcal{R}_{Z(\mathcal{C})}) = Z(\mathcal{C})$ will then let us conclude $\mathcal{C} = \langle \mathcal{G}_{Z(\mathcal{C})} \rangle = \langle \mathcal{G}_{Z(\mathcal{R}_{Z(\mathcal{C})})} \rangle = \mathcal{R}_{Z(\mathcal{C})}$.

2 Reminder on Definitions from Part I

For the convenience of the reader we briefly repeat those definitions from [5, Sections 3–5] which are relevant to the current article. For definitions of partitions and categories of partitions see [5, Sections 3.1 and 4.2]. Throughout this article we will use the notations and definitions from [5, Sections 3–5].

Notation 2.1 For every set S denote its power set by $\mathfrak{P}(S)$.

Definition 2.2 [5, Definition 5.2] The *parameter domain* L is the sixfold Cartesian product of $\mathfrak{P}(\mathbb{Z})$.

Definition 2.3 [5, Definition 5.3] Using the notation from [5, Sections 3–5], we define the *analyzer* $Z: \mathfrak{P}(\mathcal{P}^{\circ\bullet}) \to L$ by

$$Z{:=}(\, F,\; V,\; \Sigma,\; L,\; K,\; X\,)$$

where, for all $\mathcal{S} \subseteq \mathcal{P}^{\circ\bullet}$,

(a) $F(\mathcal{S}){:=}\{\, |B| \mid p \in \mathcal{S},\; B \text{ block of } p\}$ is the set of block sizes,
(b) $V(\mathcal{S}){:=}\{\, \sigma_p(B) \mid p \in \mathcal{S},\; B \text{ block of } p\}$ is the set of block color sums,

(c) $\Sigma(\mathcal{S}){:=}\{\, \Sigma(p) \mid p \in \mathcal{S}\}$ is the set of total color sums,

(d) $L(\mathcal{S}){:=}\{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2,$

$$]\alpha_1, \alpha_2[_p \cap B = \varnothing,\ \sigma_p(\{\alpha_1, \alpha_2\}) \neq 0\}$$

is the set of color distances between any two subsequent legs of the *same* block having the *same* normalized color,

(e) $K(\mathcal{S}){:=}\{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2,$

$$]\alpha_1, \alpha_2[_p \cap B = \varnothing,\ \sigma_p(\{\alpha_1, \alpha_2\}) = 0\}$$

is the set of color distances between any two subsequent legs of the *same* block having *different* normalized colors and

(f) $X(\mathcal{S}){:=}\{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B_1, B_2 \text{ blocks of } p,\ B_1 \text{ crosses } B_2,$

$$\alpha_1 \in B_1,\ \alpha_2 \in B_2\}$$

is the set of color distances between any two legs belonging to two *crossing* blocks.

Notation 2.4 (a) For all $x, y \in \mathbb{Z}$ and $A, B \subseteq \mathbb{Z}$ write

$$xA + yB{:=}\{xa + yb \mid a \in A,\ b \in B\}.$$

Moreover, put $xA - yB{:=}xA + (-y)B$. Per $A = \{1\}$ expressions like $x + yB$ are defined as well, and per $x = 1$ so are such like $A + yB$.

(b) Let $\pm S{:=}S \cup (-S)$ for all sets $S \subseteq \mathbb{Z}$.

(c) For all $m \in \mathbb{Z}$ and $D \subseteq \mathbb{Z}$ define

$$D_m{:=}(D \cup (m - D)) + m\mathbb{Z} \quad \text{and} \quad D'_m{:=}(D \cup (m - D) \cup \{0\}) + m\mathbb{Z}.$$

(d) Use the abbreviations $[\![0]\!]{:=}\varnothing$ and $[\![k]\!]{:=}\{1, \ldots, k\}$ for all $k \in \mathbb{N}$.

Definition 2.5 ([5, Definition 5.7]). Define the *parameter range* Q as the subset of L comprising all tuples (f, v, s, l, k, x) listed below, where $u \in \{0\} \cup \mathbb{N}$, where $m \in \mathbb{N}$, where $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$, where $E \subseteq \{0\} \cup \mathbb{N}$ and where N is a subsemigroup of $(\mathbb{N}, +)$:

f	v	s	l	k	x
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N'_0$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{1, 2\}$	$\pm\{0, 1\}$	$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\{1, 2\}$	$\pm\{0, 1\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$

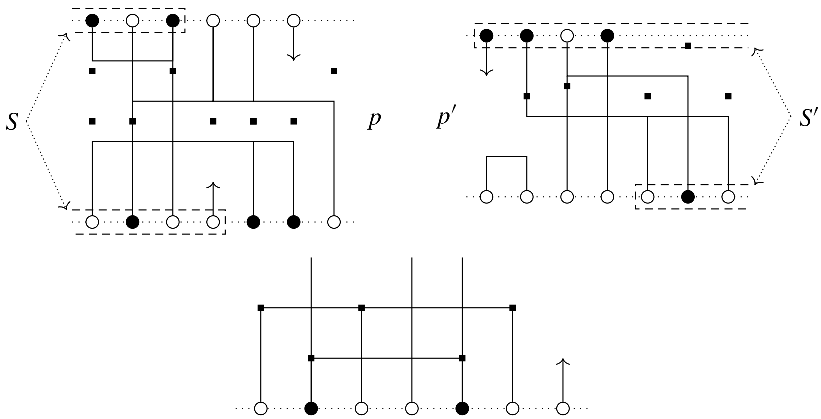
The goal of this article, as sketched in the introduction, is to prove that Z restricts to a map $PCat^{\circ\bullet}_{\text{NHO}} \to Q$ (see Theorem 9.1). Evidently, Q is not a Cartesian product; the six entries of the tuples cannot vary independently. Rather, only very special tuples of sets are allowed. Hence, if the claim $Z : PCat^{\circ\bullet}_{\text{NHO}} \to Q$ is to be true, then it is not enough to study the components of Z individually. We must also investigate the relations between them. In consequence, the argument follows a winding path, taking components into and out of consideration underway as required or convenient.

3 Tools: Equivalence and Projection

We introduce an equivalence relation on pairs of partitions and consecutive sets therein by which to compare partitions locally (cf. [3, Definition 6.2]).

Definition 3.1 For all $i \in \{1, 2\}$, let P_{p_i} denote the set of all points of $p_i \in \mathcal{P}^{\circ\bullet}$ and let $S_i \subseteq P_{p_i}$ be consecutive. We call (p_1, S_1) and (p_2, S_2) *equivalent* if $S_1 = S_2 = \varnothing$ or if the following is true: There exist $n \in \mathbb{N}$ and for each $i \in \{1, 2\}$ pairwise distinct points $\gamma_{i,1}, \ldots, \gamma_{i,n}$ in p_i such that $(\gamma_{i,1}, \ldots, \gamma_{i,n})$ is ordered in p_i and $S_i = \{\gamma_{i,1}, \ldots, \gamma_{i,n}\}$ and such that for all $j, j' \in \{1, \ldots, n\}$ (possibly $j = j'$) the following are true:

(1) The normalized colors of $\gamma_{1,j}$ in p_1 and $\gamma_{2,j}$ in p_2 agree.
(2) The points $\gamma_{1,j}$ and $\gamma_{1,j'}$ both belong to a block B_1 of p_1 with $B_1 \subseteq S_1$ if and only if $\gamma_{2,j}$ and $\gamma_{2,j'}$ both belong to a block B_2 of p_2 with $B_2 \subseteq S_2$.
(3) The points $\gamma_{1,j}$ and $\gamma_{1,j'}$ both belong to a block B_1 of p_1 with $B_1 \nsubseteq S_1$ if and only if $\gamma_{2,j}$ and $\gamma_{2,j'}$ both belong to a block B_2 of p_2 with $B_2 \nsubseteq S_2$.

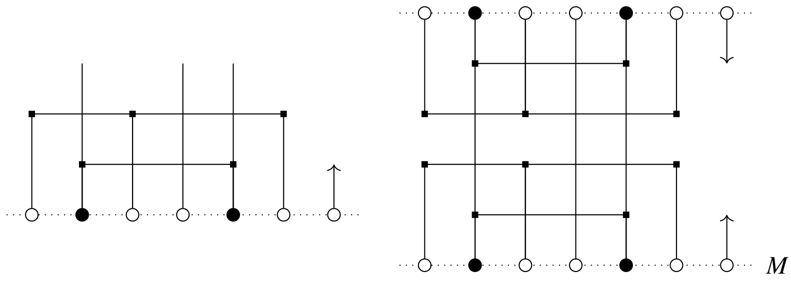


class of $(p, S) \cong (p', S')$

If (p_1, S_1) and (p_2, S_2) are equivalent, then S_1 and S_2 agree in size and normalized coloring up to a rotation ϱ and the induced partitions $\{B_1 \cap S_1 \mid B_1 \text{ block of } p_1\}$ of S_1 and $\{B_2 \cap S_2 \mid B_2 \text{ block of } p_2\}$ of S_2 concur up to ϱ. However, this is only a necessary condition. Equivalence further requires that a block $B_1 \cap S_1$ of the restriction of p_1 stems from a block B_1 of p_1 which has legs outside S_1 if and only if the corresponding statement $B_2 \nsubseteq S_2$ is true for the block B_2 of p_2 which B_1 is mapped to under ϱ.

We define and construct special representatives of the classes of this equivalence relation. Recall that a partition $p \in \mathcal{P}^{\circ\bullet}$ is called *projective* if p is self-adjoint, i.e., $p = p^*$, and idempotent, i.e., the pair (p, p) is composable and $pp = p$.

Definition 3.2 For every consecutive set S in $p \in \mathcal{P}^{\circ\bullet}$ we call the unique projective partition q with lower row M such that (q, M) and (p, S) are equivalent the *projection* $P(p, S)$ of (p, S).



class of $(p, S) \cong (q, M)$ $\qquad$ $P(p, S) = q$

In truth, of course, for any consecutive set S in $p \in \mathcal{P}^{\circ\bullet}$ the projection $P(p, S)$ depends only on the equivalence class of (p, S). The following lemma constitutes a generalization of [3, Lemma 6.4].

Lemma 3.3 *$P(p, S) \in \langle p \rangle$ for any consecutive set S in any $p \in \mathcal{P}^{\circ\bullet}$.*

Proof As $S = \varnothing$ implies $P(p, S) = \varnothing \in \langle p \rangle$, let $S \neq \varnothing$. By rotation we can assume that S is the lower row of p. Then S has the same size and coloring in p as in $q{:=}pp^*$. We show $q = P(p, S)$. By the nature of composition the blocks of p which are contained in S are blocks of q as well. We only need to care about the other blocks of q. If we identify the upper row of p and the lower row of p^*, the same partition s is induced there by p and p^*. Consequently, the meet of the two induced partitions is identical with s as well. That means that every block D of s intersects exactly one block B of p and exactly one block of p^*, namely the mirror image of B. The block of q resulting from D therefore contains exactly the restriction of B to the lower row and the mirror image of that set on the upper row. That means $q = P(p, S)$, which proves the claim. □

4 Step 1: Component *F* in Isolation

We now take our first step towards proving the main result that the analyzer Z from Definition 2.3 restricts to a map $PCat^{\circ\bullet}_{\mathrm{NHO}} \to Q$ (see Theorem 9.1). Namely, we verify (see Proposition 4.3) that, for every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, the set

$$F(\mathcal{C}){:=}\{|B| \mid p \in \mathcal{C},\ B \text{ block of } p\}$$

of block sizes appearing in $\mathcal{C}$ can only be one of the three sets of integers admissible as a first component for tuples in Q by Definition 2.5.

Lemma 4.1 [6, Lemmata 1.3 (b), 2.1 (a)] *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a category.*

(a) $\langle \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle = \langle \uparrow_{\bullet} \otimes \uparrow_{\circ} \rangle = \langle \overset{\circ}{\downarrow}\underset{\circ}{\uparrow} \rangle = \langle \overset{\bullet}{\downarrow}\underset{\bullet}{\uparrow} \rangle$.

(b) The following statements are equivalent:

- *(1) There exists in $\mathcal{C}$ a partition with a singleton block.*
- *(2)* $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$.

(c) If $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$, then $\mathcal{C}$ is closed under disconnecting points from their blocks.

Proof (a) All transformations can be achieved by basic and cyclic rotations.

(b) Projecting to a singleton block produces $\uparrow^{\circ}_{\circ}$ or $\uparrow^{\bullet}_{\bullet}$. Hence, Part (a) and Lemma 3.3 prove the claim.
(c) Rotate a given partition such that the leg to disconnect from its block is the only lower point. Composing from below with $\uparrow^{\circ}_{\circ}$ or $\uparrow^{\bullet}_{\bullet}$, depending on the color of the leg, and reversing the rotation achieves what is claimed. Hence, Part (a) concludes the proof. □

Lemma 4.2 [6, Lemmata 1.3 (d), 2.1 (b)] *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a category.*

(a) $\langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle = \langle \sqcap\!\sqcap\!\sqcap_{\bullet\circ\bullet\circ} \rangle = \left\langle \mathsf{H}^{\circ\bullet}_{\circ\bullet} \right\rangle = \left\langle \mathsf{H}^{\bullet\circ}_{\bullet\circ} \right\rangle$.
(b) The following statements are equivalent:

(1) There exists in $\mathcal{C}$ a partition with a block with at least three legs.
(2) $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$.

(c) If $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$*, then $\mathcal{C}$ is closed under connecting the two points in any turn.*

Proof (a) Once again, by basic and cyclic rotations we can transform the partitions into each other.
(b) Suppose B is a block in $p \in \mathcal{C}$ with at least three legs, $\alpha, \beta \in B$, $\alpha \neq \beta$ and $]\alpha, \beta[_p \cap B = \emptyset$. Let T be the set of the first lower and the first upper point of $P(p, [\alpha, \beta]_p)$. The partition $P(P(p, [\alpha, \beta]_p), T)$ is either $\mathsf{H}^{\circ\bullet}_{\circ\bullet}$ or $\mathsf{H}^{\bullet\circ}_{\bullet\circ}$. Thus follows the claim by Part (a) and Lemma 3.3.
(c) Let T be the turn in $p \in \mathcal{C}$ whose points we want to connect. By rotation we can assume that T is the upper row of p. By composing p from above with $\mathsf{H}^{\circ\bullet}_{\circ\bullet}$ or $\mathsf{H}^{\bullet\circ}_{\bullet\circ}$, depending on the sequence of colors in T, and reversing the initial rotation we achieve exactly what is claimed. So, Part (a) implies the assertion. □

Recall the cases $\mathcal{O}$, $\mathcal{B}$, $\mathcal{S}$ from [5, Definition 4.1].

Proposition 4.3 *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a non-hyperoctahedral category.*

(a) The set $F(\mathcal{C})$ is given by $\{2\}$, $\{1, 2\}$ or $\mathbb{N}$.
(b) If $\mathcal{C}$ is case $\mathcal{O}$, then $F(\mathcal{C}) = \{2\}$.
(c) If $\mathcal{C}$ is case $\mathcal{B}$, then $F(\mathcal{C}) = \{1, 2\}$.
(d) If $\mathcal{C}$ is case $\mathcal{S}$, then $F(\mathcal{C}) = \mathbb{N}$.

Proof By definition of a category, $\sqcap_{\circ\bullet} \in \mathcal{C}$ and thus $\{2\} \subseteq F(\mathcal{C})$.

(a) The first claim follows from the other three.
(b) Because $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$, Lemmata 4.1 (b) and 4.2 (b) show that every block in every partition of $\mathcal{C}$ has exactly two legs, i.e., $F(\mathcal{C}) = \{2\}$.
(c) The assumption $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$ implies by Lemma 4.2 (c) that no partition of $\mathcal{C}$ has blocks with more than two legs: $F(\mathcal{C}) \subseteq \{1, 2\}$. Because $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$, it is clear that $\{1\} \subseteq F(\mathcal{C})$. Thus, $F(\mathcal{C}) = \{1, 2\}$ has been proven.
(d) It suffices to show $\mathbb{N} \subseteq F(\mathcal{C})$. Let $n \in \mathbb{N}$ be arbitrary. Then,

$$p := (\uparrow_{\circ} \otimes \uparrow_{\bullet})^{\otimes \lceil \frac{n}{2} \rceil} \in \mathcal{C}.$$

Thanks to $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ we can, by Lemma 4.2 (c), connect the first n points in p to produce a partition in $\mathcal{C}$ containing a block with n points, proving $\{n\} \subseteq F(\mathcal{C})$. □

5 Step 2: Component *V* and its Relation to *F* and *L*

The next objective is to narrow down the range of the component V of Z over $PCat^{\circ\bullet}_{\text{NHO}}$. Given a non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, we show that the set

$$V(\mathcal{C}) := \{\sigma_p(B) \mid p \in \mathcal{C},\ B \text{ block of } p\}$$

of block color sums occurring in $\mathcal{C}$ can only be one of the five sets allowed as second components for tuples of Q by Definition 2.5. Beyond that, we can use Proposition 4.3 to show a result about the three parameters $V(\mathcal{C})$, $F(\mathcal{C})$ and

$$L(\mathcal{C}) := \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{C},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2,\\ ]\alpha_1, \alpha_2[_p \cap B = \varnothing,\ \sigma_p(\{\alpha_1, \alpha_2\}) \neq 0\},$$

the set of color distances between legs of the same block with identical normalized colors appearing in $\mathcal{C}$: Viewed together as $(F, V, L)(\mathcal{C})$, they satisfy the conditions necessary for $Z(\mathcal{C})$ to be element of Q by Definition 2.5.

Proposition 5.1 *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a non-hyperoctahedral category.*

(a) The set $V(\mathcal{C})$ is given by $\{0\}$, $\pm\{0, 2\}$, $\pm\{0, 1\}$, $\pm\{0, 1, 2\}$ or $\mathbb{Z}$.

(b) If $\mathcal{C}$ is case $\mathcal{O}$, then

$$V(\mathcal{C}) = \begin{cases} \pm\{0, 2\} & \text{if } L(\mathcal{C}) \neq \varnothing, \\ \{0\} & \text{otherwise.} \end{cases}$$

(c) If $\mathcal{C}$ is case $\mathcal{B}$, then

$$V(\mathcal{C}) = \begin{cases} \pm\{0, 1, 2\} & \text{if } L(\mathcal{C}) \neq \varnothing, \\ \pm\{0, 1\} & \text{otherwise.} \end{cases}$$

(d) If $\mathcal{C}$ is case $\mathcal{S}$, then $L(\mathcal{C}) \neq \varnothing$ and $V(\mathcal{C}) = \mathbb{Z}$.

Proof Two general facts about $V(\mathcal{C})$ in advance: In any case, $0 \in V(\mathcal{C})$ since $V\left(\{\sqcap_{\circ\bullet}\}\right) = \{0\}$. And [5, Lemma 6.4], using the fact that $p \in \mathcal{C}$ implies $\tilde{p} \in \mathcal{C}$, showed $V(\mathcal{C}) = -V(\mathcal{C})$.

(a) Claim (a) follows from the other three.

(b) A pair block B in $p \in \mathcal{C}$ satisfies $\sigma_p(B) = 0$ if and only if that block has no two (necessarily subsequent) legs of the same normalized colors. Otherwise it has color sum -2 or 2.

(c) And a singleton block always has color sums -1 or 1. The rest follows from the proof of Part (b).

(d) If $\mathcal{C}$ is case $\mathcal{S}$, then $\sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ and $\uparrow_{\circ}\otimes\uparrow_{\bullet} \in \mathcal{C}$. Hence, we can use $\uparrow_{\circ}\otimes\uparrow_{\bullet}$ to disconnect the left black point in $\sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet}$ by Lemma 4.1 (c) to obtain $p := \sqcap_{\circ\uparrow\circ\bullet} \in \mathcal{C}$ with $V(\{p\}) = \{-1, 1\}$. Given any $n \in \mathbb{N}$, we use $\sqcap\sqcap\sqcap_{\circ\bullet\circ\bullet}$ to connect in $p^{\otimes n} \in \mathcal{C}$ all the n many three-leg blocks together (leaving the disconnected singletons alone) in accordance with Lemma 4.2 (c). That procedure results in the partition $q \in \mathcal{C}$ with $V(\{q\}) = \{-1, n\}$. By $V(\mathcal{C}) = -V(\mathcal{C})$ it then follows $V(\mathcal{C}) = \mathbb{Z}$ as claimed. □

6 Step 3: Component Σ in Isolation

Easily, we can confirm that for all non-hyperoctahedral categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ the set

$$\Sigma(\mathcal{C}) := \{\Sigma(p) \mid p \in \mathcal{C}\}$$

of all total color sums appearing in $\mathcal{C}$ is within the range of allowed third entries of tuples in Q by Definition 2.5. The following proposition contains a generalization of [6, Lemma 2.6] and [6, Proposition 2.7].

Proposition 6.1 *For every category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ the set $\Sigma(\mathcal{C})$ is a subgroup of $\mathbb{Z}$.*

Proof [5, Lemma 6.5 (c)] implies $\Sigma(\mathcal{C}) + \Sigma(\mathcal{C}) \subseteq \Sigma(\mathcal{C})$. And $-\Sigma(\mathcal{C}) \subseteq \Sigma(\mathcal{C})$ was shown in [5, Lemma 6.4]. As also $\Sigma($$) = 0$ and  $\in \mathcal{C}$ by definition, the set $\Sigma(\mathcal{C})$ is indeed a subgroup of $\mathbb{Z}$. □

7 Step 4: General Relations between Σ, *L*, *K* and *X*

The goal remains proving that Z (see Definition 2.3) maps the set $PCat^{\circ\bullet}_{\mathrm{NHO}}$ of non-hyperoctahedral categories to Q (see Definition 2.5). So far, we have tackled this problem, more or less, one component of Z at a time. In that way, what we have managed to show is, mostly, that the values over $PCat^{\circ\bullet}_{\mathrm{NHO}}$ of each of the three maps F, V and Σ, viewed individually, are confined to the range of parameters allowed by Q as corresponding entries of its elements. To complete this picture, we would also like to see that for any non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ the three sets $L(\mathcal{C})$,

$$\begin{aligned} K(\mathcal{C}) &:= \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{C},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2, \\ &\qquad]\alpha_1, \alpha_2[_p \cap B = \varnothing,\ \sigma_p(\{\alpha_1, \alpha_2\}) = 0\} \\ \text{and} \quad X(\mathcal{C}) &:= \{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{C},\ B_1, B_2 \text{ blocks of } p,\ B_1 \text{ crosses } B_2, \\ &\qquad \alpha_1 \in B_1,\ \alpha_2 \in B_2\}, \end{aligned}$$

too, can only be of the kinds allowed as fourth, fifth and sixth components of tuples in Q, respectively, by Definition 2.5. However, due to the strong interdependences between these three components of Z, it is not even possible to prove this basic claim about the ranges of the individual maps by studying them one at a time. Instead, now, the reasonable thing to do is to consider the tuple (Σ, L, K, X) and make inferences about its range over $PCat^{\circ\bullet}_{\mathrm{NHO}}$. That will give us (see Proposition 7.23) the claim about the individual ranges of L, K and X but also many more of the relations between them (and Σ), which we need to verify the main result.

7.1 Abstract Arithmetic Lemma

As a first step, it is best to study the relationship between the Σ-, L-, K- and X-components of Z in an abstract context, merely talking about arbitrary subsets of $\mathbb{Z}$ subject to certain axioms. Our goal for this subsection is to prove the Arithmetic Lemma (7.13): Assuming certain axioms (7.1), we may deduce a certain parameter range. We will show in Subsection 7.3 that for non-hyperoctahedral categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ our sets $\Sigma(\mathcal{C})$, $L(\mathcal{C})$, $K(\mathcal{C})$ and $X(\mathcal{C})$ satisfy these axioms. Recall $\overline{\bullet} := \circ$ and $\overline{\circ} := \bullet$.

Axioms 7.1 Let σ as well as κ_{c_1,c_2} and ξ_{c_1,c_2} for all $c_1, c_2 \in \{\circ, \bullet\}$ be subsets of $\mathbb{Z}$. Throughout this subsection, make the following assumptions:

(i) σ is a subgroup of $\mathbb{Z}$.

For all $(\omega_{c_1,c_2})_{c_1,c_2 \in \{\circ,\bullet\}} \in \{(\kappa_{c_1,c_2})_{c_1,c_2 \in \{\circ,\bullet\}}, (\xi_{c_1,c_2})_{c_1,c_2 \in \{\circ,\bullet\}}\}$ and for all $c_1, c_2 \in \{\circ, \bullet\}$:

(ii) $\omega_{c_1,c_2} + \sigma \subseteq \omega_{c_1,c_2}$.
(iii) $\omega_{c_1,c_2} \subseteq -\omega_{\overline{c_2},\overline{c_1}}$.
(iv) $\omega_{c_1,c_2} \subseteq -\omega_{c_2,c_1} + \sigma$.

For all $c_1, c_2, c_3 \in \{\circ, \bullet\}$:

(v) $\xi_{c_1,c_2} \subseteq \xi_{c_1,\overline{c_2}} \cup \big(-\xi_{c_2,\overline{c_1}} + \sigma\big)$.
(vi) $0 \in \kappa_{\circ\bullet} \cap \kappa_{\bullet\circ}$.
(vii) $\kappa_{c_1,c_2} + \kappa_{\overline{c_2},c_3} \subseteq \kappa_{c_1,c_3}$.
(viii) $\kappa_{c_1,c_2} + \xi_{\overline{c_2},c_3} \subseteq \xi_{c_1,c_3}$.

Let us first study how much κ_{c_1,c_2} and ξ_{c_1,c_2} depend on $c_1, c_2 \in \{\circ, \bullet\}$.

Lemma 7.2 *For any* $(\omega_{c_1,c_2})_{c_1,c_2\in\{\circ,\bullet\}} \in \{(\kappa_{c_1,c_2})_{c_1,c_2\in\{\circ,\bullet\}}, (\xi_{c_1,c_2})_{c_1,c_2\in\{\circ,\bullet\}}\}$*:*

(a) $\omega_{\circ\circ} = \omega_{\bullet\bullet}$ *and* $\omega_{\circ\circ} = -\omega_{\circ\circ} = \omega_{\circ\circ} + \sigma$.
(b) $\omega_{\circ\bullet} = \omega_{\bullet\circ}$ *and* $\omega_{\circ\bullet} = -\omega_{\circ\bullet} = \omega_{\circ\bullet} + \sigma$.

Proof Because $0 \in \sigma$ by Assumption (i), the Assumption (ii) actually means

$$\omega_{c_1,c_2} = \omega_{c_1,c_2} + \sigma \tag{ii'}$$

for all $c_1, c_2 \in \{\circ, \bullet\}$. And with this new identity we can, for all $c_1, c_2 \in \{\circ, \bullet\}$, refine Assumption (iv) to

$$\omega_{c_1,c_2} \subseteq -\omega_{c_2,c_1} \tag{iv'}$$

as $-\omega_{c_2,c_1} + \sigma = -(\omega_{c_2,c_1} - \sigma) = -(\omega_{c_2,c_1} + \sigma) = -\omega_{c_2,c_1}$ due to $\sigma = -\sigma$.

(a) Version (ii') of Assumption (ii) yields $\omega_{\circ\circ} = \omega_{\circ\circ}+\sigma$ as claimed. And Assumption (iv) in the form of (iv') proves

$$\omega_{\circ\circ} \overset{(iv)}{\subseteq} -\omega_{\circ\circ} \overset{(iv)}{\subseteq} \omega_{\circ\circ} \quad \text{and} \quad \omega_{\bullet\bullet} \overset{(iv)}{\subseteq} -\omega_{\bullet\bullet} \overset{(iv)}{\subseteq} \omega_{\bullet\bullet},$$

thus verifying $\omega_{\circ\circ} = -\omega_{\circ\circ}$ and $\omega_{\bullet\bullet} = -\omega_{\bullet\bullet}$. Now, if we apply Assumption (iii) to conclude

$$\omega_{\circ\circ} \overset{(iii)}{\subseteq} -\omega_{\bullet\bullet} \overset{(iii)}{\subseteq} \omega_{\circ\circ},$$

we can infer $\omega_{\circ\circ} = \omega_{\bullet\bullet}$. That proves the remainder of the claims about $\omega_{\circ\circ}$ and $\omega_{\bullet\bullet}$.

(b) Here also, Version (ii') of Assumption (ii) implies $\omega_{\circ\bullet} = \omega_{\circ\bullet} + \sigma$. Now, though, for $\omega_{\circ\bullet}$ and $\omega_{\bullet\circ}$ the roles of Assumptions (iii) and (iv) reverse. First, we apply the former to conclude

$$\omega_{\circ\bullet} \overset{(iii)}{\subseteq} -\omega_{\circ\bullet} \overset{(iii)}{\subseteq} \omega_{\circ\bullet} \quad \text{and} \quad \omega_{\bullet\circ} \overset{(iii)}{\subseteq} -\omega_{\bullet\circ} \overset{(iii)}{\subseteq} \omega_{\bullet\circ},$$

which shows the claims $\omega_{\circ\bullet} = -\omega_{\circ\bullet}$ and $\omega_{\bullet\circ} = -\omega_{\bullet\circ}$. Then, it is the refined version (iv') of Assumption (iv) that yields

$$\omega_{\circ\bullet} \overset{(iv)}{\subseteq} -\omega_{\bullet\circ} \overset{(iv)}{\subseteq} \omega_{\circ\bullet},$$

implying $\omega_{\circ\bullet} = \omega_{\bullet\circ}$ and thus completing the proof. □

In the case of $(\omega_{c_1,c_2})_{c_1,c_2\in\{\circ,\bullet\}} = (\xi_{c_1,c_2})_{c_1,c_2\in\{\circ,\bullet\}}$ of Lemma 7.2 we can go even further and combine the objects of Parts (a) and (b).

Lemma 7.3 $\xi_{\circ\circ} = \xi_{\circ\bullet}$.

Proof Since $\xi_{c_2,\overline{c_1}} = \xi_{c_2,\overline{c_1}} + \sigma$ for all $c_1, c_2 \in \{\circ, \bullet\}$ by Version (ii') of Axiom (ii), our Assumption (v) actually spells

$$\xi_{c_1,c_2} \subseteq \xi_{c_1,\overline{c_2}} \cup (-\xi_{c_2,\overline{c_1}}) \tag{v'}$$

for all $c_1, c_2 \in \{\circ, \bullet\}$ as $\sigma = -\sigma$. Using this version of the assumption twice, we conclude

$$\xi_{\circ\circ} \overset{(v)}{\subseteq} \xi_{\circ\bullet} \cup (-\xi_{\circ\bullet}) = \xi_{\circ\bullet} \overset{(v)}{\subseteq} \xi_{\circ\circ} \cup (-\xi_{\bullet\bullet}) = \xi_{\circ\circ},$$

where we have used the results $\xi_{\circ\bullet} = -\xi_{\circ\bullet}$ and $\xi_{\circ\circ} = -\xi_{\bullet\bullet}$ of Lemma 7.2. It follows that indeed $\xi_{\circ\circ} = \xi_{\circ\bullet}$. □

Definition 7.4 Write $\lambda := \kappa_{\circ\circ} = \kappa_{\bullet\bullet}$ and $\kappa := \kappa_{\circ\bullet} = \kappa_{\bullet\circ}$ and $\xi := \xi_{\circ\circ} = \xi_{\bullet\bullet} = \xi_{\circ\bullet} = \xi_{\bullet\circ}$.

Our next step is to show that the pair (λ, κ) is of a very simple form (Lemma 7.7).

Definition 7.5 Define the non-negative integers

$$d := \begin{cases} \min(\kappa \cap \mathbb{N}) & \text{if } \kappa \cap \mathbb{N} \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad l := \begin{cases} \min(\lambda \cap \mathbb{N}) & \text{if } \lambda \cap \mathbb{N} \neq \varnothing, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 7.6 *(a)* $\kappa = d\mathbb{Z}$.
(b) If $\lambda \neq \varnothing$*, then* $l \in \lambda$ *and* $\lambda - l \supseteq \kappa$.
(c) $\lambda - l \subseteq \kappa$.
(d) If $\lambda \neq \varnothing$ *and* $d \neq 0$*, then* $l \leq d$.
(e) If $\lambda \neq \varnothing$ *and* $d \neq 0$*, then* $l \neq 0$.
(f) If $\lambda \neq \varnothing$*, then* $2l\mathbb{Z} \subseteq d\mathbb{Z}$.
(g) If $\lambda \neq \varnothing$*, then* $d = l$ *or* $d = 2l$.

Proof (a) Of course, $0 \in \kappa$ by Assumption (vi). And $-\kappa = \kappa$ was established in Lemma 7.2 (b). And with the choices $c_1 = \circ$, $c_2 = c_3 = \bullet$, Assumption (vii) implies that

$$\kappa + \kappa = \kappa_{\circ\bullet} + \kappa_{\circ\bullet} \overset{(vii)}{\subseteq} \kappa_{\circ\bullet} = \kappa.$$

Hence, κ is indeed a subgroup of $\mathbb{Z}$. The definition of d makes d a generator of κ, implying $\kappa = d\mathbb{Z}$.
(b) As $\lambda = -\lambda$ by Lemma 7.2 (a), assuming $\lambda \neq \varnothing$ ensures $\lambda \cap (\{0\} \cup \mathbb{N}) \neq \varnothing$. Hence, under this assumption, $l \in \lambda$ by definition of l. If we choose $c_1 = c_3 = \circ$ and $c_2 = \bullet$ in Assumption (vii), it follows that

$$\kappa + \lambda = \kappa_{\circ\bullet} + \kappa_{\circ\circ} \overset{(vii)}{\subseteq} \kappa_{\circ\circ} = \lambda.$$

Since $l \in \lambda$, we can specialize the λ on the left hand side of that inclusion to l and then subtract l on both sides. We obtain $\kappa \subseteq \lambda - l$.
(c) If $\lambda = \varnothing$, there is nothing to prove. Hence, let $\lambda \neq \varnothing$, implying $l \in \lambda$ by Part (b). Using Assumption (vii) once more, this time with the choices $c_1 = c_2 = \circ$ and $c_3 = \bullet$, yields

$$\lambda - \lambda = \lambda + \lambda = \kappa_{\circ\circ} + \kappa_{\bullet\bullet} \overset{(vii)}{\subseteq} \kappa_{\circ\bullet} = \kappa,$$

where we have used $\lambda = -\lambda$ (Lemma 7.2 (a)) in the first step. Specializing on the left hand side the second instance of λ to l yields $\lambda - l \subseteq \kappa$.

(d) Actually, we show the contraposition. Hence, suppose $\lambda \neq \varnothing$ and $l > d$. Since $\lambda = l + d\mathbb{Z}$ by Parts (a)–(c), it then follows that $l - d \in \lambda \cap \mathbb{N}$. The definition of l consequently requires $l \leq l - d$, i.e. $d \leq 0$. As $d \geq 0$ by definition, $d = 0$ is the only possibility.
(e) We prove the contraposition indirectly. As $\lambda = l + d\mathbb{Z}$ by Parts (a)–(c), supposing $l = 0$ entails $\lambda = d\mathbb{Z}$. Thus, if $d \neq 0$ were true, then $\varnothing \neq d\mathbb{Z} \cap \mathbb{N} = \lambda \cap \mathbb{N}$ would yield the contradiction $0 < \min(\lambda \cap \mathbb{N}) = l = 0$ by definition of l.
(f) In the proof of Part (c) we saw $\lambda + \lambda \subseteq \kappa$. Specializing therein both instances of λ on the left hand side to l (which we can do due to $\lambda \neq \varnothing$ by Part (b)) yields $2l \in \kappa = d\mathbb{Z}$. It follows $2l\mathbb{Z} \subseteq d\mathbb{Z}$ as asserted.
(g) From $2l\mathbb{Z} \subseteq d\mathbb{Z}$, as shown in Part (f), it is immediate that, if $d = 0$, then $l = 0 = d$ as claimed. If $d \neq 0$, we know, firstly, $l \leq d$ by Part (d), secondly, $l \neq 0$ by Part (e) and, thirdly, $2l\mathbb{Z} \subseteq d\mathbb{Z}$ by Part (f). That is only possible if $d = l$ or $d = 2l$: Indeed, if $c \in \mathbb{Z}$ is such that $2l = cd$, then $l > 0$ and $d \geq 0$ ensure $c > 0$. Moreover, $l \leq d$ implies $2l \leq 2d$, i.e., $cd \leq 2d$. We infer $c \leq 2$ by $d > 0$. Hence, $c \in \{1, 2\}$ by $c > 0$. □

Lemma 7.7 *(a) If $\lambda = \varnothing$, then $(\lambda, \kappa) = (\varnothing, d\mathbb{Z})$.*
(b) If $\lambda \neq \varnothing$, then (λ, κ) is equal to $(l + 2l\mathbb{Z}, 2l\mathbb{Z})$ or $(l\mathbb{Z}, l\mathbb{Z})$.

Proof In Lemma 7.6 we established that $\kappa = d\mathbb{Z}$ (Part (a)) and that $\lambda = \varnothing$ or $\lambda = l + d\mathbb{Z}$ (Parts (b) and (c)), where $d = l$ or $d = 2l$ (Part (g)). In other words, we have proven that (λ, κ) is of the asserted form. □

We can immediately relate σ to κ.

Definition 7.8 Define

$$k := \begin{cases} \min(\sigma \cap \mathbb{N}) & \text{if } \sigma \cap \mathbb{N} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.9 $\sigma = k\mathbb{Z} \subseteq d\mathbb{Z} = \kappa$.

Proof Because σ is a subgroup of $\mathbb{Z}$, the definition of k implies $\sigma = k\mathbb{Z}$. Moreover, we know $\kappa = \kappa + \sigma$ by Lemma 7.2 (b). Hence Assumption (vi), namely $0 \in \kappa$, implies $k\mathbb{Z} = \sigma \subseteq \kappa + \sigma \subseteq \kappa = d\mathbb{Z}$. □

Let us now turn to the description of ξ.

Lemma 7.10 *(a) $\xi = \xi + d\mathbb{Z}$.*
(b) If $\lambda \neq \varnothing$, then $\xi = \xi + l\mathbb{Z}$.

Proof (a) Picking $c_1 = \circ$, $c_2 = c_3 = \bullet$, Assumption (viii) implies the inclusion

$$\kappa + \xi = \kappa_{\circ\bullet} + \xi_{\circ\bullet} \overset{(viii)}{\subseteq} \xi_{\circ\bullet} = \xi.$$

As the reverse inclusion is trivially true by $0 \in \kappa$ (Assumption (vi)), we have thus verified our claim $\xi = \xi + d\mathbb{Z}$ by Lemma 7.6 (a).
(b) Assumption (viii), applied a second time, now with $c_1 = c_2 = c_3 = \circ$, allows us to conclude

$$\lambda + \xi = \kappa_{\circ\circ} + \xi_{\bullet\circ} \overset{(viii)}{\subseteq} \xi_{\circ\circ} = \xi.$$

If $\lambda \neq \emptyset$, then $l \in \lambda$ by Lemma 7.6 (b). Hence, the above inclusion shows in particular $\xi + l \subseteq \xi$. Using this, induction proves $\xi + l\mathbb{N} \subseteq \xi$. Lemma 7.6 (g) established that $d = l$ or $d = 2l$. Either way, $\xi = \xi + d\mathbb{Z}$, as seen in Part (a), then ensures $\xi - 2l \subseteq \xi$. Combining this conclusion with $\xi + l \subseteq \xi$ lets us infer $\xi - l = (\xi + l) - 2l \subseteq \xi$. Again, it follows $\xi - l\mathbb{N} \subseteq \xi$ by induction. Hence, altogether we have shown $\xi + l\mathbb{Z} = (\xi - l\mathbb{N}) \cup \xi \cup (\xi + l\mathbb{N}) \subseteq \xi$. Of course, the converse inclusion is true as well because $0 \in \mathbb{Z}$, proving $\xi = \xi + l\mathbb{Z}$ as claimed. □

In order to obtain a refined understanding of ξ we need the following preparatory lemma.

Lemma 7.11 *Let $\chi \subseteq \mathbb{Z}$ and $m \in \mathbb{N}$ satisfy $\chi = -\chi = \chi + m\mathbb{Z}$.*

(a) $\chi = (\chi \cap (\{0\} \cup [\![m-1]\!]))_m$.
(b) $\chi \cap [\![m-1]\!] = m - (\chi \cap [\![m-1]\!])$.
(c) $\chi = (\chi \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]))_m$.
(d) $\chi = \mathbb{Z} \backslash D_m$ *for* $D = (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]) \backslash \chi$.

Proof The mapping $S \mapsto S_m := (S \cup (m - S)) + m\mathbb{Z}$ of subsets $S \subseteq \mathbb{Z}$ is a closure operator with respect to $\subseteq$, i.e., for all $S, T \subseteq \mathbb{Z}$ with $S \subseteq T$ we have $S \subseteq S_m$ and $S_m \subseteq T_m$ and $(S_m)_m = S_m$. In particular $S = S_m$ if and only if $S = -S = S + m\mathbb{Z}$.

(a) The assumption $\chi = -\chi = \chi + m\mathbb{Z}$ implies $\chi = \chi_m$. Hence, $\chi = \chi_m \supseteq (\chi \cap (\{0\} \cup [\![m-1]\!]))_m$ is clear by monotonicity of $S \mapsto S_m$. We show the converse: If $x \in \chi$, we find $x' \in \{0\} \cup [\![m-1]\!]$ such that $x' - x \in m\mathbb{Z}$. Consequently, $x' \in x + m\mathbb{Z} \subseteq \chi + m\mathbb{Z} \subseteq \chi$ by assumption. We conclude $x \in x' + m\mathbb{Z} \subseteq (\chi \cap (\{0\} \cup [\![m-1]\!])) + m\mathbb{Z} \subseteq (\chi \cap (\{0\} \cup [\![m-1]\!]))_m$, which is what we needed to show.

(b) We further deduce from $\chi = -\chi = \chi + m\mathbb{Z}$ that $m - \chi \subseteq \chi$. Naturally, $m - (\chi \cap [\![m-1]\!]) \subseteq m - [\![m-1]\!] = [\![m-1]\!]$. Combining this with $m - (\chi \cap [\![m-1]\!]) \subseteq m - \chi \subseteq \chi$ yields $m - (\chi \cap [\![m-1]\!]) \subseteq \chi \cap [\![m-1]\!]$. We conclude $\chi \cap [\![m-1]\!] = m - (m - (\chi \cap [\![m-1]\!])) \subseteq m - (\chi \cap [\![m-1]\!])$, which proves one inclusion.
Now, the converse. From $\chi = -\chi = \chi + m\mathbb{Z}$ we can infer $m - \chi = -(m - \chi) = (m - \chi) + m\mathbb{Z}$. In consequence we can apply the inclusion we just proved to the set $m - \chi$ in the role of χ. Since $m - [\![m-1]\!] = [\![m-1]\!]$, the resulting inclusion $(m - \chi) \cap [\![m-1]\!] \subseteq m - ((m - \chi) \cap [\![m-1]\!])$ actually spells $m - (\chi \cap [\![m-1]\!]) \subseteq \chi \cap [\![m-1]\!]$. That is just what we had to show.

(c) Due to the monotonicity and idempotency of the mapping $S \mapsto S_m$, it suffices by Part (a) to prove $\chi \cap (\{0\} \cup [\![m-1]\!]) \subseteq (\chi \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]))_m$. Let $x \in \chi \cap (\{0\} \cup [\![m-1]\!])$ be arbitrary. If $x \leq \lfloor \frac{m}{2} \rfloor$, then, naturally, $x \in \chi \cup (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]) \subseteq (\chi \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]))_m$. Hence, we can assume $x > \lfloor \frac{m}{2} \rfloor$. By Part (b) we know $m - x \in \chi$. By assumption, $m - x < m - \lfloor \frac{m}{2} \rfloor$. If m is even, then this inequality says $m - x < m - \frac{m}{2} = \frac{m}{2} = \lfloor \frac{m}{2} \rfloor$. Should m be odd instead, it means $m - x < m - \frac{m-1}{2} = \frac{m+1}{2}$, which implies $m - x \leq \frac{m+1}{2} - 1 = \frac{m-1}{2} = \lfloor \frac{m}{2} \rfloor$. Thus, $m - x \leq \lfloor \frac{m}{2} \rfloor$ in all cases. Hence we have shown $m - x \in \chi \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!])$. It follows $x = m - (m - x) \in m - (\chi \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!])) \subseteq (\chi (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]))_m$. That is what we needed to see.

(d) The assumption $\chi = -\chi = \chi + m\mathbb{Z}$ implies $\mathbb{Z} \backslash \chi = -(\mathbb{Z}\underline{a}ckslash\chi) = (\mathbb{Z} \backslash \chi) + m\mathbb{Z}$. Hence, we can apply Part (c) to the set $\mathbb{Z} \backslash \chi$ in the role of χ and obtain $\mathbb{Z} \backslash \chi = ((\mathbb{Z} \backslash \chi) \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]))_m$. Since $(\mathbb{Z} \backslash \chi) \cap (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]) = (\{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]) \backslash \chi = D$ we have shown $\mathbb{Z} \backslash \chi = D_m$. It follows $\chi = \mathbb{Z} \backslash D_m$ as claimed. □

Lemma 7.12 *(a) If $d=0$, then $\xi=\mathbb{Z}\backslash E_0$ for $E=(\{0\}\cup\mathbb{N})\backslash\xi$.*
(b) If $d\geq 1$ and $\lambda\neq\varnothing$, then $\xi=\mathbb{Z}\backslash D_l$ for $D=(\{0\}\cup[\![\lfloor\frac{l}{2}\rfloor]\!])\backslash\xi$.
(c) If $d\geq 1$ and $\lambda=\varnothing$, then $\xi=\mathbb{Z}\backslash D_d$ for $D=(\{0\}\cup[\![\lfloor\frac{d}{2}\rfloor]\!])\backslash\xi$.

Proof (a) The defining equations $E=(\{0\}\cup\mathbb{N})\backslash\xi$ and $E_0=E\cup(-E)$ imply $E_0=((\{0\}\cup\mathbb{N})\backslash\xi)\cup((-(\{0\}\cup\mathbb{N}))\backslash(-\xi))$. Hence, $\xi=-\xi$ (by Lemma 7.2) shows $E_0=\mathbb{Z}\backslash\xi$ and thus the claim $\xi=\mathbb{Z}\backslash E_0$.
(b) Because $\lambda\neq\varnothing$, Lemma 7.6 (g) guarantees $d=l$ or $d=2l$. Hence, the assumption $d\geq 1$ implies $l\geq 1$. Moreover, Lemma 7.10 (b) assures us that $\xi=\xi+l\mathbb{Z}$. And, we already know $\xi=-\xi$ by Lemma 7.2. Hence, Lemma 7.11 (d) yields the claim.
(c) Still, $\xi=-\xi$, of course. And $\xi=\xi+d\mathbb{Z}$ by Lemma 7.10 (a) as $d\geq 1$. Thus, once more, Lemma 7.11 (d) proves the claim. □

In conclusion we have shown the following auxiliary result.

Lemma 7.13 (Arithmetic Lemma). *If the nine sets of integers σ and κ_{c_1,c_2}, ξ_{c_1,c_2} for $c_1,c_2\in\{\circ,\bullet\}$ satisfy Axioms 7.1, then*

$$\kappa_{\circ\circ}=\kappa_{\bullet\bullet}=:\lambda,\quad \kappa_{\circ\bullet}=\kappa_{\bullet\circ}=:\kappa\quad\textit{and}\quad \xi_{\circ\circ}=\xi_{\bullet\bullet}=\xi_{\circ\bullet}=\xi_{\bullet\circ}=:\xi$$

and there exist $u\in\{0\}\cup\mathbb{N}$, $m\in\mathbb{N}$, $D\subseteq\{0\}\cup[\![\lfloor\frac{m}{2}\rfloor]\!]$ and $E\subseteq\{0\}\cup\mathbb{N}$ such that the tuple $(\sigma,\lambda,\kappa,\xi)$ is given by one of the following:

σ	λ	κ	ξ
$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash E_0$

Proof That λ, κ and ξ are well-defined was shown in Lemmata 7.2 and 7.3. Hence, we can let k, d and l be as in Definitions 7.8 and 7.5. We distinguish five cases in total.

Case 1: First, suppose that $\lambda=\varnothing$. Then, $\kappa=d\mathbb{Z}$. By Lemma 7.7 (a). There are now two possibilities depending on the value of $d\in\{0\}\cup\mathbb{N}$.

Case 1.1: If $d=0$, which is to say $\kappa=\{0\}$, then Lemma 7.12 (a) yields $\xi=\mathbb{Z}\backslash E_0$ for $E:=(\{0\}\cup\mathbb{N})\backslash\xi$. And Lemma 7.9 proves $\sigma=k\mathbb{Z}\subseteq d\mathbb{Z}=\{0\}$, implying $k=0$ and thus $\sigma=\{0\}$. As, naturally, $E\subseteq\{0\}\cup\mathbb{N}$, the tuple $(\sigma,\lambda,\kappa,\xi)$ is indeed as claimed in the fifth row of the table.

Case 1.2: Should $d\geq 1$ on the other hand, then by Lemma 7.12 (c) we infer $\xi=\mathbb{Z}\backslash D_d$ for $D:=(\{0\}\cup[\![\lfloor\frac{d}{2}\rfloor]\!])\backslash\xi$. Since $\sigma=k\mathbb{Z}\subseteq d\mathbb{Z}$ by Lemma 7.9, if we put $u:=\frac{k}{d}$, then $\sigma=ud\mathbb{Z}$. Recognizing $D\subseteq\{0\}\cup[\![\lfloor\frac{d}{2}\rfloor]\!]$ and defining $m:=d$ thus proves that $(\sigma,\lambda,\kappa,\xi)$ is as asserted by the third row of the table.

Case 2: Now, let $\lambda\neq\varnothing$ instead. Then, $(\lambda,\kappa)=(l+2l\mathbb{Z},2l\mathbb{Z})$ or $(\lambda,\kappa)=(l\mathbb{Z},l\mathbb{Z})$ by Lemma 7.7 (a). Respectively, $d=2l$ or $d=l$. We now distinguish two cases based on the value of $l\in\{0\}\cup\mathbb{N}$.

Case 2.1: Assuming $l=0$ lets us conclude $l\mathbb{Z}=2l\mathbb{Z}=l+2l\mathbb{Z}=\{0\}$, which implies $(\lambda,\kappa)=(\{0\},\{0\})$. Lemma 7.9 gives $\sigma=k\mathbb{Z}\subseteq\kappa=\{0\}$ and thus $k=0$ and $\sigma=\{0\}$. Because $d=l=2l=0$ we can infer $\xi=\mathbb{Z}\backslash E_0$ for $E:=(\{0\}\cup\mathbb{N})\backslash\xi$ by Lemma 7.12 (a). As $E\subseteq\{0\}\cup\mathbb{N}$, the tuple $(\sigma,\lambda,\kappa,\xi)$ is hence given by the fourth row of the table.

Case 2.2: Finally, let $l\geq 0$. Then, also $d\geq 0$, no matter whether $d=l$ or $d=2l$. In conclusion, $\xi=\mathbb{Z}\backslash D_l$ for $D:=(\{0\}\cup[\![\lfloor\frac{l}{2}\rfloor]\!])\backslash\xi$ by Lemma 7.12 (c).

Case 2.2.1: If $(\lambda, \kappa) = (l + 2l\mathbb{Z}, 2l\mathbb{Z})$, i.e., $d = 2l$, then the implication $\sigma = k\mathbb{Z} \subseteq d\mathbb{Z} = 2l\mathbb{Z}$ of Lemma 7.9 lets us define $u \in \{0\} \cup \mathbb{N}$ by $u{:=}\frac{k}{2l}$ and obtain $\sigma = 2ul\mathbb{Z}$. Hence, choosing $m{:=}l$ proves that $(\sigma, \lambda, \kappa, \xi)$ fits the second row of the table.

Case 2.2.2: If instead, $(\lambda, \kappa) = (l\mathbb{Z}, l\mathbb{Z})$, i.e., $d = l$, then Lemma 7.9 yields $\sigma = k\mathbb{Z} \subseteq d\mathbb{Z} = l\mathbb{Z}$, thus permitting us to define $u \in \{0\} \cup \mathbb{N}$ by $u{:=}\frac{k}{l}$ and obtain $\sigma = ul\mathbb{Z}$. The choice $m{:=}l$ hence shows $(\sigma, \lambda, \kappa, \xi)$ to be given by the first row. □

As mentioned before, our goal will be to show (Section 7.3) that for every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ the tuple $(\Sigma, L, K, X)(\mathcal{C})$ is of the form given in the table of the Arithmetic Lemma.

7.2 Reduction to Singleton and Pair Blocks

Let us return to categories of partitions. To elucidate the ranges of K, L and X over $PCat^{\circ\bullet}_{\mathrm{NHO}}$ and central relations between $\Sigma(\mathcal{C})$, $K(\mathcal{C})$, $L(\mathcal{C})$ and $X(\mathcal{C})$ for non-hyperoctahedral categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, we must consider certain decompositions of K, L and X according to leg colors.

Definition 7.14 Let $\mathcal{S} \subseteq \mathcal{P}^{\circ\bullet}$ and $c_1, c_2 \in \{\circ, \bullet\}$ be abitrary. Then, define

$$K_{c_1,c_2}(\mathcal{S}){:=}\{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B \text{ block of } p,\ \alpha_1, \alpha_2 \in B,\ \alpha_1 \neq \alpha_2,\ ]\alpha_1, \alpha_2[_p \cap B = \varnothing,\ \forall i = 1,2 : \alpha_i \text{ of normalized color } c_i \}, \tag{a}$$

$$X_{c_1,c_2}(\mathcal{S}){:=}\{\, \delta_p(\alpha_1, \alpha_2) \mid p \in \mathcal{S},\ B_1, B_2 \text{ blocks of } p,\ B_1 \text{ and } B_2 \text{ cross},\ \alpha_1 \in B_1,\ \alpha_2 \in B_2,\ \forall i = 1,2 : \alpha_i \text{ of normalized color } c_i \}. \tag{b}$$

L, K and X can then be written as, where the union occurs pointwise,

$$L = \bigcup_{\substack{c_1,c_2 \in \{\circ,\bullet\} \\ c_1 = c_2}} K_{c_1,c_2}, \quad K = \bigcup_{\substack{c_1,c_2 \in \{\circ,\bullet\} \\ c_1 \neq c_2}} K_{c_1,c_2}, \quad \text{and} \quad X = \bigcup_{c_1,c_2 \in \{\circ,\bullet\}} X_{c_1,c_2}.$$

Recall that $\mathcal{P}^{\circ\bullet}_{\leq 2}$ denotes the set of all partitions with block sizes one or two and that it is a category (see [5, Lemma 4.4 (a)]). By the next lemma we may always restrict to partitions in $\mathcal{P}^{\circ\bullet}_{\leq 2}$ when studying K_{c_1,c_2} and X_{c_1,c_2}. This is trivial in cases $\mathcal{O}$ and $\mathcal{B}$, while for case $\mathcal{S}$ this basically follows from Lemma 4.1(c).

Lemma 7.15 *For all non-hyperoctahedral categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ and $c_1, c_2 \in \{\circ, \bullet\}$:*

(a) $K_{c_1,c_2}(\mathcal{C}) = K_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2})$.
(b) $X_{c_1,c_2}(\mathcal{C}) = X_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2})$.

Proof (a) If $\mathcal{C}$ is case $\mathcal{O}$ or case $\mathcal{B}$, i.e., if $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}_{\leq 2}$ by Proposition 4.3, there is nothing to show. Hence, suppose that $\mathcal{C}$ is case $\mathcal{S}$ and let $c_1, c_2 \in \{\circ, \bullet\}$. We only need to prove $K_{c_1,c_2}(\mathcal{C}) \subseteq K_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2})$. Let α_1 and α_2 with $\alpha_1 \neq \alpha_2$ be points in $p \in \mathcal{C}$ such that α_i is of normalized color c_i for every $i \in \{1, 2\}$ and such that $\alpha_1, \alpha_2 \in B$ and $]\alpha_1, \alpha_2[_p \cap B = \varnothing$ for some block B in p. Because $\mathcal{C}$ is case $\mathcal{S}$, by Lemma 4.1 (c) we do not violate the assumption $p \in \mathcal{C}$ by assuming that every block other than B is a singleton. In the same way we can assume that α_1 and α_2 are the only legs of B. None of these assumptions affect $\delta_p(\alpha_1, \alpha_2)$ or the normalized colors of α_1 or α_2. As they ensure $p \in \mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2}$ though, we have shown $\delta_p(\alpha_1, \alpha_2) \in K_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2})$, which is what we needed to see.

(b) Again, all that we need to prove is that $X_{c_1,c_2}(\mathcal{C}) \subseteq X_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2})$ if $\mathcal{C}$ is case $\mathcal{S}$ and if $c_1, c_2 \in \{\circ, \bullet\}$. Let the points α_1 of normalized color c_1 and α_2 of normalized color c_2 in $p \in \mathcal{C}$ belong to the blocks B_1 and B_2, respectively, and suppose that B_1 and B_2 cross. Because $\mathcal{C}$ is case $\mathcal{S}$ we can, by Lemma 4.1 (c), assume that all other blocks of p besides B_1 and B_2 are singletons. Now the only thing standing in the way of $p \in \mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2}$ is the possibility of at least one of B_1 and B_2 having more than two legs. We would like to assume that B_1 and B_2 have only two legs each and still maintain all the other assumptions including $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$ and, of course, not alter $\delta_p(\alpha_1, \alpha_2)$. By Lemma 4.1 (c), we can always remove surplus legs of B_1 and B_2. But it is not immediately clear that we can remove legs without affecting the other assumptions. A priori, the crossing between B_1 and B_2 only implies that we can find points $\beta_1, \gamma_1 \in B_1$ and $\beta_2, \gamma_2 \in B_2$ such that $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ is ordered in p. If now $\alpha_1 \in \{\beta_1, \gamma_1\}$ and $\alpha_2 \in \{\beta_2, \gamma_2\}$, then we can certainly remove all legs except $\{\beta_i, \gamma_i\}$ from B_i for all $i \in \{1, 2\}$ and still maintain the other assumptions. In fact, we can do so in general as well:
Let us only consider the "worst case" that $\alpha_1 \notin \{\beta_1, \gamma_1\}$ and $\alpha_2 \notin \{\beta_2, \gamma_2\}$. There are 20 possible arrangements of the points $\{\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2\}$ relative to each other with respect to the cyclic order respecting that $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ is ordered.

$\overset{\downarrow}{\alpha}_1\beta_1\beta_2\gamma_1\gamma_2$	$\beta_1\overset{\downarrow}{\alpha}_1\beta_2\gamma_1\gamma_2$	$\beta_1\beta_2\overset{\downarrow}{\alpha}_1\gamma_1\gamma_2$	$\beta_1\beta_2\gamma_1\overset{\downarrow}{\alpha}_1\gamma_2$
$\underline{\alpha_2\alpha_1}\beta_1\underline{\beta_2\gamma_1}\gamma_2$	$\underline{\alpha_2}\beta_1\underline{\alpha_1\beta_2\gamma_1}\gamma_2$	$\underline{\alpha_2\beta_1\beta_2\alpha_1}\gamma_1\gamma_2$	$\underline{\alpha_2\beta_1\beta_2}\gamma_1\underline{\alpha_1}\gamma_2$
$\underline{\alpha_1\alpha_2\beta_1\beta_2}\gamma_1\gamma_2$	$\beta_1\underline{\alpha_2\alpha_1\beta_2\gamma_1}\gamma_2$	$\underline{\beta_1\alpha_2}\beta_2\underline{\alpha_1}\gamma_1\underline{\gamma_2}$	$\underline{\beta_1\alpha_2}\beta_2\gamma_1\underline{\alpha_1\gamma_2}$
$\underline{\alpha_1}\beta_1\underline{\alpha_2}\beta_2\underline{\gamma_1\gamma_2}$	$\beta_1\underline{\alpha_1\alpha_2}\beta_2\underline{\gamma_1\gamma_2}$	$\underline{\beta_1}\beta_2\underline{\alpha_2\alpha_1}\gamma_1\underline{\gamma_2}$	$\underline{\beta_1}\beta_2\underline{\alpha_2}\gamma_1\underline{\alpha_1\gamma_2}$
$\underline{\alpha_1}\beta_1\beta_2\underline{\alpha_2\gamma_1\gamma_2}$	$\beta_1\underline{\alpha_1}\beta_2\underline{\alpha_2\gamma_1\gamma_2}$	$\beta_1\beta_2\underline{\alpha_1\alpha_2\gamma_1\gamma_2}$	$\underline{\beta_1}\beta_2\gamma_1\underline{\alpha_2\alpha_1\gamma_2}$
$\underline{\alpha_1}\beta_1\underline{\beta_2\gamma_1\alpha_2}\gamma_2$	$\beta_1\underline{\alpha_1\beta_2\gamma_1\alpha_2}\gamma_2$	$\underline{\beta_1\beta_2\alpha_1}\gamma_1\underline{\alpha_2}\gamma_2$	$\underline{\beta_1\beta_2}\gamma_1\underline{\alpha_1\alpha_2}\gamma_2$

We remove all legs of B_1 and B_2 except for the underlined ones. Then the above table shows that we can always turn B_1 and B_2 into crossing pair blocks containing α_1 and α_2, respectively. That concludes the proof. □

7.3 Verifying the Axioms

We want to apply the Arithmetic Lemma 7.13 to the sets $\sigma := \Sigma(\mathcal{C})$, $\kappa_{c_1,c_2} := K_{c_1,c_2}(\mathcal{C})$ and $\xi_{c_1,c_2} := X_{c_1,c_2}(\mathcal{C})$ for $c_1, c_2 \in \{\circ, \bullet\}$ and non-hyperoctahedral categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$. In order to be able to do so, we, of course, need to show that these sets actually satisfy the prerequisite Axioms 7.1. Proving that will crucially utilize the reduction to singleton and pair blocks from Lemma 7.15.

Lemma 7.16 *For every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, the set $\sigma := \Sigma(\mathcal{C})$ satisfies Axiom (i) of 7.1: σ is a subgroup of $\mathbb{Z}$.*

Proof That was shown in Proposition 6.1. □

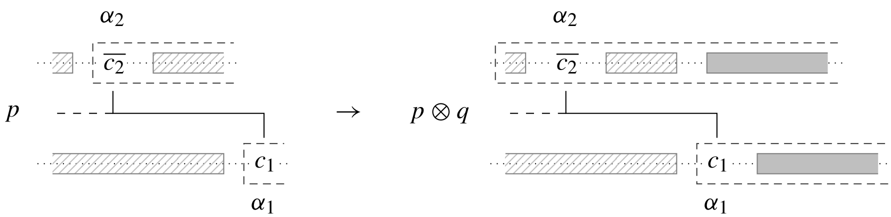
Lemma 7.17 *For every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, the sets $\sigma := \Sigma(\mathcal{C})$ and $\kappa_{c_1,c_2} := K_{c_1,c_2}(\mathcal{C})$ for $c_1, c_2 \in \{\circ, \bullet\}$ satisfy Axioms (ii)–(iv) of 7.1:*

$$\text{(ii) } \kappa_{c_1,c_2} + \sigma \subseteq \kappa_{c_1,c_2}, \quad \text{(iii) } \kappa_{c_1,c_2} \subseteq -\kappa_{\overline{c_2},\overline{c_1}}, \quad \text{(iv) } \kappa_{c_1,c_2} \subseteq -\kappa_{c_2,c_1} + \sigma$$

for all $c_1, c_2 \in \{\circ, \bullet\}$.

Proof Let $c_1, c_2 \in \{\circ, \bullet\}$ be arbitrary and let α_1 and α_2 be distinct points of the same block B in $p \in \mathcal{C}$ such that $]\alpha_1, \alpha_2[_p \cap B = \varnothing$ and such that α_i has normalized color c_i for every $i \in \{1, 2\}$. In other words, let $\delta_p(\alpha_1, \alpha_2)$ be a generic element of $K_{c_1,c_2}(\mathcal{C}) = \kappa_{c_1,c_2}$.

Axiom (ii): Let $q \in \mathcal{C}$ be arbitrary. None of the assumptions about p, α_1, α_2 and $\delta_p(\alpha_2, \alpha_2)$ are impacted by assuming that p is rotated in such a way that α_1 is the rightmost lower point of p. Then, B is a block of $p \otimes q \in \mathcal{C}$ as well and $]\alpha_1, \alpha_2[_{p\otimes q} \cap B = \varnothing$.

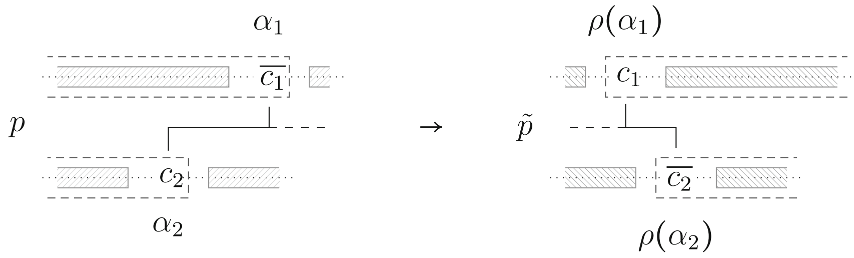


Now, because all points stemming from q lie within $]\alpha_1, \alpha_2[_{p\otimes q}$,

$$\delta_{p\otimes q}(\alpha_1, \alpha_2) = \delta_p(\alpha_1, \alpha_2) + \Sigma(q).$$

That proves $\delta_p(\alpha_1, \alpha_2) + \Sigma(q) \in K_{c_1,c_2}(\mathcal{C}) = \kappa_{c_1,c_2}$, which is what we needed to see.

Axiom (iii): The verticolor reflection $\tilde{p}$ of p belongs to $\mathcal{C}$. The set $]\alpha_1, \alpha_2]_p$ in p is mapped by the reflection ρ to the set $[\rho(\alpha_2), \rho(\alpha_1)[_{\tilde{p}}$ in $\tilde{p}$. As the operation of verticolor reflection inverts normalized colors, $\sigma_p(S) = -\sigma_{\tilde{p}}(\rho(S))$ for any set S of points in p.

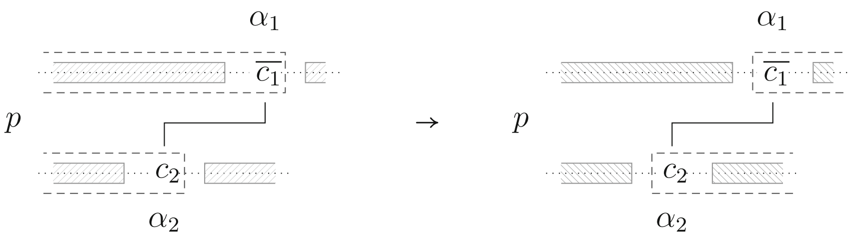


Using the case distinction free formula for $\delta_p(\alpha_1, \alpha_2)$ given in the proof of [5, Lemma 3.1 (b)], we thus compute

$$\begin{aligned}
\delta_p(\alpha_1, \alpha_2) &= \sigma_p(]\alpha_1, \alpha_2]_p) + \tfrac{1}{2}(\sigma_p(\alpha_1) - \sigma_p(\alpha_2)) \\
&= -\sigma_{\tilde{p}}([\rho(\alpha_2), \rho(\alpha_1)[_{\tilde{p}}) - \tfrac{1}{2}(\sigma_{\tilde{p}}(\rho(\alpha_1)) - \sigma_{\tilde{p}}(\rho(\alpha_2))) \\
&= -\sigma_{\tilde{p}}(]\rho(\alpha_2), \rho(\alpha_1)]_{\tilde{p}}) - \sigma_{\tilde{p}}(\rho(\alpha_2)) + \sigma_{\tilde{p}}(\rho(\alpha_1)) \\
&\quad - \tfrac{1}{2}(\sigma_{\tilde{p}}(\rho(\alpha_1)) - \sigma_{\tilde{p}}(\rho(\alpha_2))) \\
&= -\sigma_{\tilde{p}}(]\rho(\alpha_2), \rho(\alpha_1)]_{\tilde{p}}) - \tfrac{1}{2}(\sigma_{\tilde{p}}(\rho(\alpha_2)) - \sigma_{\tilde{p}}(\rho(\alpha_1))) \\
&= -\delta_{\tilde{p}}(\rho(\alpha_2), \rho(\alpha_1)).
\end{aligned}$$

Because, for every $i \in \{1, 2\}$, the point $\rho(\alpha_i)$ has normalized color $\overline{c_i}$ in $\tilde{p}$ and because $\rho(B)$ is a block of $\tilde{p}$ with $]\rho(\alpha_2), \rho(\alpha_1)[_{\tilde{p}} \cap \rho(B) = \varnothing$, we conclude $\delta_p(\alpha_1, \alpha_2) \in -K_{\overline{c_2},\overline{c_1}}(\mathcal{C}) = -\kappa_{\overline{c_2},\overline{c_1}}$. And that is what we had to show.

Axiom (iv): So far, we have not made use of Lemma 7.15. Now, though, we employ it to additionally assume $p \in \mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2}$. In particular, then, $B = \{\alpha_1, \alpha_2\}$ is a pair block. Consequently, not only $]\alpha_1, \alpha_2[_p \cap B = \varnothing$ but also $]\alpha_2, \alpha_1[_p \cap B = \varnothing$.



By [5, Lemma 2.1 (b)] we infer

$$\delta_p(\alpha_1, \alpha_2) = -\delta_p(\alpha_2, \alpha_1) \mod \Sigma(p).$$

As $\Sigma(p) \in \Sigma(\mathcal{C})$, it follows $\delta_p(\alpha_1, \alpha_2) \in -K_{c_2,c_1}(\mathcal{C}) + \Sigma(\mathcal{C}) = -\kappa_{c_2,c_1} + \sigma$, which is what we wanted to see. □

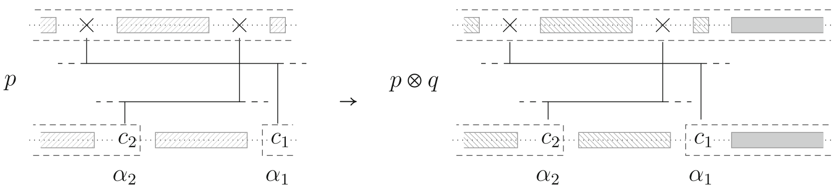
Lemma 7.18 *For every non-hyperoctahedral category* $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$*, the sets* $\sigma:=\Sigma(\mathcal{C})$ *and* $\xi_{c_1,c_2}:=X_{c_1,c_2}(\mathcal{C})$ *for* $c_1, c_2 \in \{\circ, \bullet\}$ *satisfy Axioms (ii)–(iv) of 7.1:*

$$\text{(ii) } \xi_{c_1,c_2} + \sigma \subseteq \xi_{c_1,c_2}, \quad \text{(iii) } \xi_{c_1,c_2} \subseteq -\xi_{\overline{c_2},\overline{c_1}}, \quad \text{(iv) } \xi_{c_1,c_2} \subseteq -\xi_{c_2,c_1} + \sigma$$

for all $c_1, c_2 \in \{\circ, \bullet\}$.

Proof The proof is similar to that of Lemma 7.17. Let $c_1, c_2 \in \{\circ, \bullet\}$, let B_1 and B_2 be crossing blocks of $p \in \mathcal{C}$ and let $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$ have normalized colors c_1 and c_2, respectively. That makes $\delta_p(\alpha_1, \alpha_2)$ a generic element of $X_{c_1,c_2}(\mathcal{C}) = \xi_{c_1,c_2}$.

Axiom (ii): Just like in the proof of Lemma 7.17, we can assume that α_1 is the rightmost lower point. Given arbitrary $q \in \mathcal{C}$, the sets B_1 and B_2 are crossing blocks of $p \otimes q$ as well,

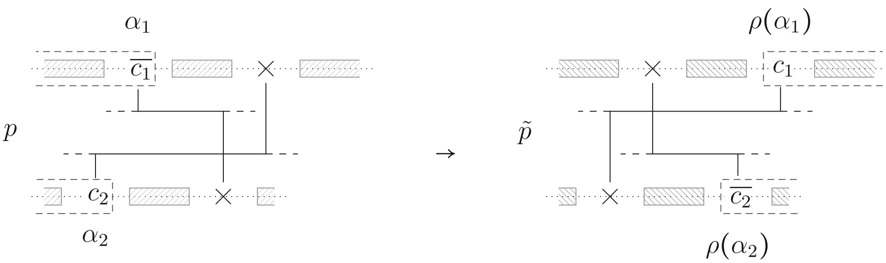


which proves

$$\delta_p(\alpha_1, \alpha_2) + \Sigma(q) = \delta_{p\otimes q}(\alpha_1, \alpha_2) \in X_{c_1,c_2}(\mathcal{C}) = \xi_{c_1,c_2}.$$

Thus, $\xi_{c_1,c_2} + \sigma \subseteq \xi_{c_1,c_2}$ as claimed.

Axiom (iii): Likewise, the sets $\rho(B_1)$ and $\rho(B_2)$ are still crossing blocks in $\tilde{p} \in \mathcal{C}$. There, α_i has normalized color $\overline{c_i}$ for every $i \in \{1, 2\}$.



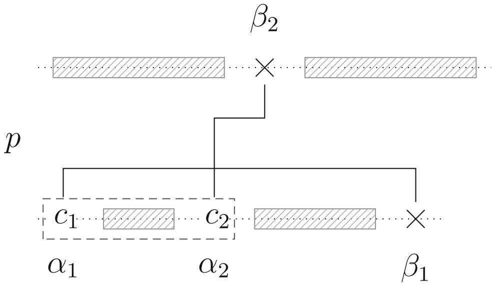
and because, $\delta_p(\alpha_1, \alpha_2) \equiv -\delta_p(\alpha_2, \alpha_1) \mod \Sigma(p)$ (by [5, Lemma 2.1 (b)]), we can immediately conclude $\delta_p(\alpha_1, \alpha_2) \in -X_{c_2,c_1}(\mathcal{C}) + \Sigma(\mathcal{C})$. Thus, $\xi_{c_1,c_2} \subseteq -\xi_{c_2,c_1} + \sigma$. Differently from Lemma 7.17, we did not need Lemma 7.15 to see this. □

Lemma 7.19 *For every non-hyperoctahedral category* $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$*, the sets* $\sigma := \Sigma(\mathcal{C})$ *and* $\xi_{c_1,c_2} := X_{c_1,c_2}(\mathcal{C})$ *for* $c_1, c_2 \in \{\circ, \bullet\}$ *satisfy Axiom (v) of 7.1: For all* $c_1, c_2 \in \{\circ, \bullet\}$,

$$\xi_{c_1,c_2} \subseteq \xi_{c_1,\overline{c_2}} \cup \left(-\xi_{c_2,\overline{c_1}} + \sigma\right).$$

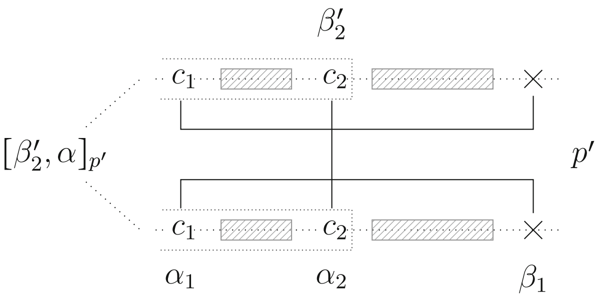
Proof Let B_1 and B_2 be crossing blocks in $p \in \mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2}$ and let $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$ have normalized colors $c_1 \in \{\circ, \bullet\}$ and $c_2 \in \{\circ, \bullet\}$, respectively. According to Lemma 7.15 then, every element of $\xi_{c_1,c_2} = X_{c_1,c_2}(\mathcal{C}) = X_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}^{\circ\bullet}_{\leq 2})$ is of the form $\delta_p(\alpha_1, \alpha_2)$. Because $p \in \mathcal{P}^{\circ\bullet}_{\leq 2}$, the blocks B_1 and B_2 are pairs. Hence, the crossing between these blocks means that we find points $\beta_1 \in B_1$ and $\beta_2 \in B_2$ with $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$ such that either $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ or $(\alpha_2, \alpha_1, \beta_2, \beta_1)$ is ordered.

Case 1: First, we suppose that $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is ordered and show $\delta_p(\alpha_1, \alpha_2) \in X_{c_1,\overline{c_2}}(\mathcal{C})$. We can assume that α_1 is the leftmost and β_1 the rightmost lower point.



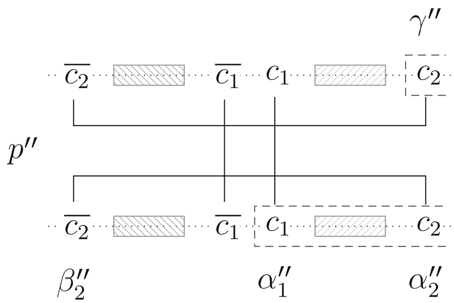
By Lemma 3.3, the partition $p' := P(p, [\alpha_1, \beta_1]_p)$ belongs to $\mathcal{C}$. The definition of the projection operation has the following consequences: The three lower points α_1, α_2 and β_1 of p, also points of p', all retain their normalized colors in p'; the set $B_1 = \{\alpha_1, \beta_1\}$ is still a block of p'; the point α_2 is now connected to its counterpart β_2' on the upper row of p', implying in particular that the blocks of α_1 and α_2 still cross in p'; and it holds

$$\delta_{p'}(\alpha_1, \alpha_2) = \delta_p(\alpha_1, \alpha_2).$$



We apply Lemma 3.3 a second time to infer $p'' := P(p', [\beta_2', \alpha_2]) \in \mathcal{C}$. Denote the images of the points β_2', α_1 and α_2 of p' in p'' by β_2'', α_1'' and α_2'', respectively. Now, β_2'' is the leftmost lower point and α_2'' the rightmost lower point of p'' and the two form a block; the point $\alpha_1'' \in [\beta_2'', \alpha_2'']_{p''}$ is connected to its counterpart on the upper row; and

$$\delta_{p''}(\alpha_1'', \alpha_2'') = \delta_{p'}(\alpha_1, \alpha_2) = \delta_p(\alpha_1, \alpha_2).$$



There are two crucial observations to make about the successor γ'' of α''_2 in p'', the rightmost upper point of p''. Firstly, γ'' has the inverse normalized color $\overline{c_2}$ of α''_2, which in particular implies that $\delta_{p''}(\alpha''_2, \gamma'') = 0$. Secondly, γ'' forms a block of p'' together with the leftmost upper point of p'', which entails that its block crosses the block of α''_1 in p''. Hence, $\delta_{p''}(\alpha''_1, \gamma'') \in X_{c_1,\overline{c_2}}(\mathcal{C})$ and

$$\delta_{p''}(\alpha''_1, \gamma'') = \delta_{p''}(\alpha''_1, \alpha''_2) + \delta_{p''}(\alpha''_2, \gamma'') = \delta_p(\alpha_1, \alpha_2)$$

together show $\delta_p(\alpha_1, \alpha_2) \in X_{c_1,\overline{c_2}}(\mathcal{C}) = \xi_{c_1,\overline{c_2}}$, which is what we set out to prove.

Case 2: Now, let $(\alpha_2, \alpha_1, \beta_2, \beta_1)$ be ordered instead. By Case 1 then, $\delta_p(\alpha_2, \alpha_1) \in X_{c_2,\overline{c_1}}(\mathcal{C})$. [5, Lemma 2.1 (b)] shows $\delta_p(\alpha_2, \alpha_1) \equiv -\delta_p(\alpha_1, \alpha_2) \mod \Sigma(p)$. That implies $\delta_p(\alpha_1, \alpha_2) \in -X_{c_2,\overline{c_1}}(\mathcal{C}) + \Sigma(\mathcal{C}) = -\xi_{c_2,\overline{c_1}} + \sigma$, which is what we needed to see. □

Lemma 7.20 *For every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, the sets $\sigma{:=}\Sigma(\mathcal{C})$ and $\kappa_{c_1,c_2}{:=}K_{c_1,c_2}(\mathcal{C})$ for $c_1, c_2 \in \{\circ, \bullet\}$ satisfy Axiom (vi) of 7.1:* $0 \in \kappa_{\circ\bullet} \cap \kappa_{\bullet\circ}$.

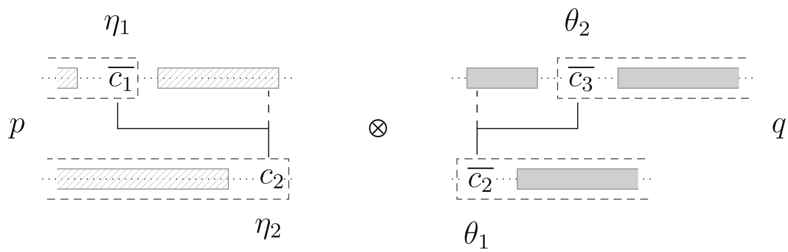
Proof Since $\sqcap_{\circ\bullet} \in \mathcal{C}$ and $K_{\circ\bullet}(\{\sqcap_{\circ\bullet}\}) = K_{\bullet\circ}(\{\sqcap_{\circ\bullet}\}) = \{0\}$, this is clear. □

Lemma 7.21 *For every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, the sets $\sigma{:=}\Sigma(\mathcal{C})$ and $\kappa_{c_1,c_2}{:=}K_{c_1,c_2}(\mathcal{C})$ for $c_1, c_2 \in \{\circ, \bullet\}$ satisfy Axiom (vii) of 7.1:*

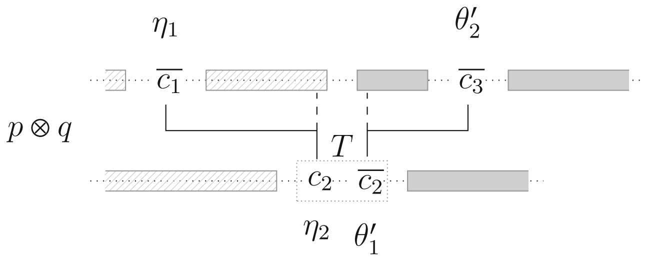
$$\kappa_{c_1,c_2} + \kappa_{\overline{c_2},c_3} \subseteq \kappa_{c_1,c_3}$$

for all $c_1, c_2, c_3 \in \{\circ, \bullet\}$.

Proof Let $c_1, c_2, c_3 \in \{\circ, \bullet\}$ be arbitrary and let η_1 and η_2 be distinct points of the same block B of $p \in \mathcal{C}$ such that $]\eta_1, \eta_2[_p \cap B = \emptyset$ and such that η_i has normalized color c_i in p for every $i \in \{1, 2\}$. Furthermore, let θ_1 and θ_2 be distinct points of the same block C of $q \in \mathcal{C}$ with $]\theta_1, \theta_2[_q \cap C = \emptyset$ such that θ_1 has normalized color $\overline{c_2}$ in q and θ_2 normalized color c_3. None of these assumptions are impacted and neither $\delta_p(\eta_1, \eta_2)$ nor $\delta_q(\theta_1, \theta_2)$ altered by assuming that η_2 is the rightmost lower point of p and θ_1 the leftmost lower point of q.

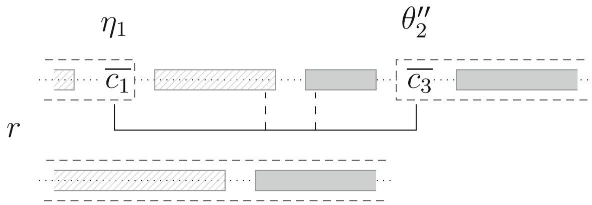


Denote the images of the points θ_1 and θ_2 of q in $p \otimes q \in \mathcal{C}$ by θ'_1 and θ'_2, respectively. The assumptions about the normalized colors of η_2 and θ_1 imply that $T{:=}\{\eta_2, \theta'_1\}$ is a turn in $p \otimes q$, meaning in particular $\delta_{p\otimes q}(\eta_2, \theta'_1) = 0$.



Moreover, $\delta_{p\otimes q}(\eta_1,\eta_2) = \delta_p(\eta_1,\eta_2)$ and $\delta_{p\otimes q}(\theta'_1,\theta'_2) = \delta_q(\theta_1,\theta_2)$ by nature of the tensor product.

Let θ''_2 denote the image of θ'_2 in $r := E(p\otimes q, T) \in \mathcal{C}$, the partition obtained from $p\otimes q$ by erasing the turn T (see [5, Section 4.3]). By definition of the erasing operation, η_1 and θ''_2 belong to the same block D in r with $]\eta_1,\theta''_2[_r \cap D = \emptyset$.



Hence, from $\delta_r(\eta_1,\theta''_2) \in K_{c_1,c_3}(\mathcal{C}) = \kappa_{c_1,c_3}$ and from

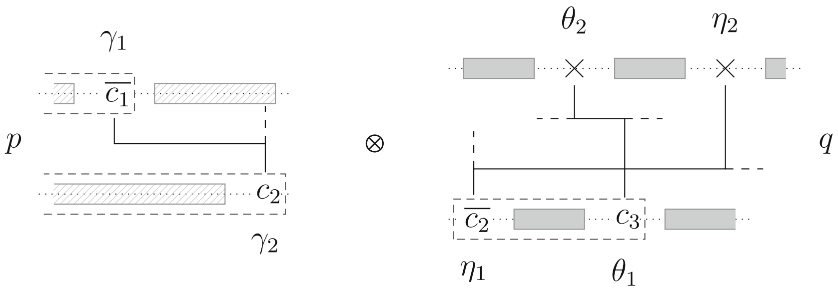
$$\begin{aligned}\delta_r(\eta_1,\theta''_2) &= \delta_{p\otimes q}(\eta_1,\theta_2) - \sigma_{p\otimes q}(T)\\ &= \delta_{p\otimes q}(\eta_1,\theta_2)\\ &= \delta_{p\otimes q}(\eta_1,\eta_2) + \delta_{p\otimes q}(\eta_2,\theta'_1) + \delta_{p\otimes q}(\theta'_1,\theta'_2)\\ &= \delta_{p\otimes q}(\eta_1,\eta_2) + \delta_{p\otimes q}(\theta'_1,\theta'_2)\\ &= \delta_p(\eta_1,\eta_2) + \delta_q(\theta_1,\theta_2)\end{aligned}$$

it follows $\delta_p(\eta_1,\eta_2) + \delta_q(\theta_1,\theta_2) \in \kappa_{c_1,c_3}$. And that is what we needed to show. □

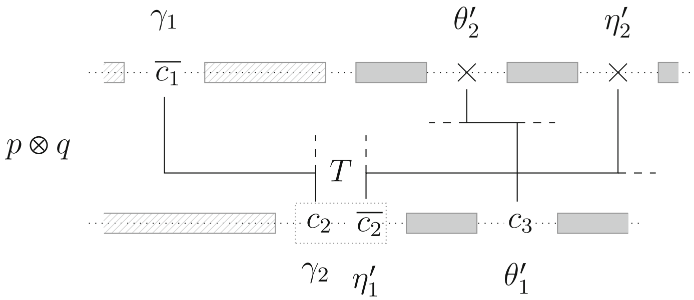
Lemma 7.22 *For every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, the sets $\sigma := \Sigma(\mathcal{C})$ and $\kappa_{c_1,c_2} := K_{c_1,c_2}(\mathcal{C})$ and $\xi_{c_1,c_2} := X_{c_1,c_2}(\mathcal{C})$ for $c_1, c_2 \in \{\circ,\bullet\}$ satisfy Axiom (viii) of 7.1: For all $c_1, c_2, c_3 \in \{\circ,\bullet\}$,*

$$\kappa_{c_1,c_2} + \xi_{\overline{c_2},c_3} \subseteq \xi_{c_1,c_3}.$$

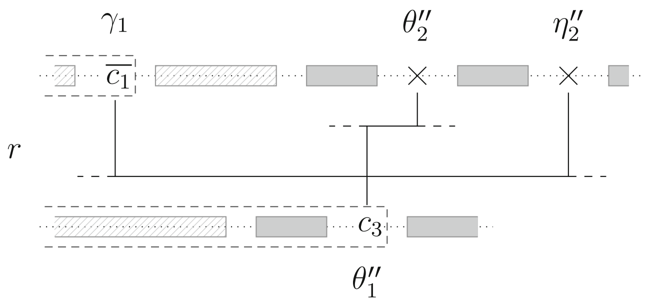
Proof We adapt the proof of Lemma 7.21. Let $c_1, c_2 \in \{\circ,\bullet\}$ be arbitrary. Let $p, q \in \mathcal{C}$, let B be a block in p, and let C and D be two crossing blocks in q. Let γ_1 and γ_2 be two distinct points of B of normalized colors c_1 respectively c_2 in p with $]\gamma_1,\gamma_2[_p \cap B = \emptyset$. In q, let $\eta_1 \in C$ have normalized color $\overline{c_2}$ and $\theta_1 \in D$ normalized color c_3. Then, $\delta_p(\gamma_1,\gamma_2)$ is a generic element of $K_{c_1,c_2}(\mathcal{C}) = \kappa_{c_1,c_2}$ and $\delta_q(\eta_1,\theta_1)$ one of $X_{\overline{c_2},c_3}(\mathcal{C}) = \xi_{\overline{c_2},c_3}$. No generality is lost assuming that γ_2 is the rightmost lower point of p and η_1 the leftmost lower point of q. We find $\eta_2 \in C$ and $\theta_2 \in D$ such that $\eta_1 \neq \eta_2$ and $\theta_1 \neq \theta_2$ and such that $(\eta_1,\theta_1,\eta_2,\theta_2)$ or $(\eta_1,\theta_2,\eta_2,\theta_1)$ is ordered in q.



Let η_1', η_2', θ_1' and θ_2' denote the images of, respectively, η_1, η_2, θ_1 and θ_2 in $p \otimes q \in \mathcal{C}$. By nature of the tensor product, B is a block of $p \otimes q$. Likewise, η_1' and η_2' belong to the same block in $p \otimes q$ and so do θ_1' and θ_2'. And each involved point has the same normalized color in $p \otimes q$ as the corresponding preimage in p or q. The set $T{:=}\{\gamma_2, \eta_1'\}$ is a turn in $p \otimes q$.



If we denote by η_2'', θ_1'' and θ_2'' the images of η_1', θ_1' and θ_2' in $r{:=}E(p \otimes q, T) \in \mathcal{C}$, then γ_1 and η_2'' belong to the same block in r and so do θ_1' and θ_2'. Because $(\gamma_1, \gamma_2, \eta_1', \theta_i', \eta_2', \theta_{\neg i}')$ is ordered in $p \otimes q$ for some $i, \neg i \in \{1, 2\}$ with $\{i, \neg i\} = \{1, 2\}$, the tuple $(\gamma_1, \theta_i'', \eta_2'', \theta_{\neg i}'')$ is then ordered in r. Thus, the blocks of γ_1 and η_2'' and of θ_1'' and θ_2'' cross in r.



Consequently, from $\delta_r(\gamma_1, \theta_1'') \in X_{c_1,c_3}(\mathcal{C})$ and from

$$\begin{aligned}
\delta_r(\gamma_1, \theta_1'') &= \delta_{p\otimes q}(\gamma_1, \theta_1') - \sigma_{p\otimes q}(T) \\
&= \delta_{p\otimes q}(\gamma_1, \theta_1') \\
&= \delta_{p\otimes q}(\gamma_1, \gamma_2) + \delta_{p\otimes q}(\gamma_2, \eta_1') + \delta_{p\otimes q}(\eta_1', \theta_1') \\
&= \delta_{p\otimes q}(\gamma_1, \gamma_2) + \delta_{p\otimes q}(\eta_1', \theta_1') \\
&= \delta_p(\gamma_1, \gamma_2) + \delta_q(\eta_1, \theta_1)
\end{aligned}$$

it follows $\delta_p(\gamma_1, \gamma_2) + \delta_q(\eta_1, \theta_1) \in X_{c_1,c_3}(\mathcal{C}) = \xi_{c_1,c_3}$. And that is what we needed to see. □

Finally, we can give the final result of this section.

Proposition 7.23 *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a non-hyperoctahedral category. Then,*

$$L(\mathcal{C}) = K_{\circ\circ}(\mathcal{C}) = K_{\bullet\bullet}(\mathcal{C}), \qquad K(\mathcal{C}) = K_{\circ\bullet}(\mathcal{C}) = K_{\bullet\circ}(\mathcal{C})$$

and

$$X(\mathcal{C}) = X_{\circ\circ}(\mathcal{C}) = X_{\bullet\bullet}(\mathcal{C}) = X_{\circ\bullet}(\mathcal{C}) = X_{\bullet\circ}(\mathcal{C})$$

and there exist $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$ and $E \subseteq \{0\} \cup \mathbb{N}$ such that the tuple $(\Sigma, L, K, X)(\mathcal{C})$ is one of the following:

$\Sigma(\mathcal{C})$	$L(\mathcal{C})$	$K(\mathcal{C})$	$X(\mathcal{C})$
$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$2um\mathbb{Z}$	$m + 2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash E_0$

Proof Follows from Lemmata 7.16–7.22 and the Arithmetic Lemma 7.13. □

8 Step 5: Special Relations between Σ, *L*, *K* and *X* depending on *F* and *V*

Our objective remains proving $Z(\mathcal{C}) \in \mathcal{Q}$ for any non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$. After studying components F (Section 4) and Σ (Section 6) in isolation and after investigating the images of the mappings (F, V, L) (Section 5) and (Σ, L, K, X) (Section 7), we have arrived at the point where we must take all six components of $Z = (F, V, \Sigma, L, K, X)$ into account simultaneously. Fortunately, we can capitalize on the results of Sections 4–7 in this endeavor. In consequence, it largely suffices to understand better the behavior of (Σ, L, K, X) as dependent on (F, V) or, roughly, on F.

Recall from [5, Definition 4.1] that a category is non-hyperoctahedral if and only if it is case $\mathcal{O}$, $\mathcal{B}$ or $\mathcal{S}$ and that these cases are mutually exclusive.

8.1 Special Relations in Case $\mathcal{S}$

For case $\mathcal{S}$ categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, i.e., by Proposition 4.3 assuming $F(\mathcal{C}) = \mathbb{N}$, there is just a single fact about $(\Sigma, L, K, X)(\mathcal{C})$ we have to note, one about $L(\mathcal{C})$.

Proposition 8.1 $0 \in L(\mathcal{C})$ *for every case $\mathcal{S}$ category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$.*

Proof As $\uparrow \otimes \uparrow \in \mathcal{C}$, we can, by Lemma 4.1 (c), disconnect the black points in $\sqcap\sqcap \in \mathcal{C}$ and obtain $\sqcap\uparrow\uparrow \in \mathcal{C}$. It follows $\{0\} = L(\{\sqcap\uparrow\uparrow\}) \subseteq L(\mathcal{C})$. □

8.2 Special Relations in Case $\mathcal{O}$

For case $\mathcal{O}$ categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, i.e., assuming $F(\mathcal{C}) = \{2\}$, more than what Proposition 7.23 is able to discern can be said about $\Sigma(\mathcal{C})$ and $X(\mathcal{C})$.

8.2.1 Relation of Σ to L and K in Case $\mathcal{O}$

First, we treat the total color sums of case $\mathcal{O}$ categories.

Proposition 8.2 *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a case $\mathcal{O}$ category and let $m \in \mathbb{N}$.*

(a) If $(L, K)(\mathcal{C}) = (\varnothing, m\mathbb{Z})$, then $\Sigma(\mathcal{C}) = \{0\}$.
(b) If $(L, K)(\mathcal{C}) = (m\mathbb{Z}, m\mathbb{Z})$ or $(L, K)(\mathcal{C}) = (m + 2m\mathbb{Z}, 2m\mathbb{Z})$, then

$$\Sigma(\mathcal{C}) = 2um\mathbb{Z}$$

for some $u \in \{0\} \cup \mathbb{N}$.

Proof (a) By Proposition 7.23 there exists $\tilde{u} \in \{0\} \cup \mathbb{N}$ such that $\Sigma(\mathcal{C}) = \tilde{u}m\mathbb{Z}$. We suppose $\tilde{u} \neq 0$ and derive a contradiction. As $\mathcal{C}$ is closed under erasing turns and as erasing turns does not affect total color sum, we find $p \in \mathcal{C}$ with no turns such that $\Sigma(p) = \tilde{u}m$. Because $\tilde{u}m > 0$, the partition p has at least one block. As all blocks of p are pairs by Proposition 4.3, there is a block B of p with (necessarily subsequent) legs $\alpha, \beta \in B$ and $\alpha \neq \beta$. Since p has no turns, all points of p have normalized color $\circ$. In particular, α and β do. That proves $L(\mathcal{C}) \neq \varnothing$, contradicting the assumption.
(b) Proposition 7.23 guarantees that $\Sigma(\mathcal{C}) = \tilde{u}m\mathbb{Z}$ for some $\tilde{u} \in \{0\} \cup \mathbb{N}$ and that $\tilde{u}$ is even if $(L, K)(\mathcal{C}) = (m + 2m\mathbb{Z}, 2m\mathbb{Z})$. We want to show that $\tilde{u}$ is even also if $(L, K)(\mathcal{C}) = (m\mathbb{Z}, m\mathbb{Z})$. If $\tilde{u} = 0$, this claim is true. Hence, suppose $\tilde{u} > 0$. As in Part (a), we utilize $p \in \mathcal{C}$ with no turns such that $\Sigma(p) = \tilde{u}m > 0$ and, this time, also with no upper points.
For every $i \in \mathbb{N}$ with $i \leq m$ consider the set

$$S_i = \{\blacksquare j \mid j \in (i + m\mathbb{N}_0),\ j \leq \tilde{u}m\}$$

comprising the i-th lower point and all its m-th neighbors to the right. Then, $\bigcup_{i=1}^{m} S_i$ comprises all points of p and $|S_i| = \tilde{u}$ for every $i \in \mathbb{N}$ with $i \leq m$.
The sets $S_1, \ldots, S_m$ must all be subpartitions of p: Otherwise, we find $j, j' \in \mathbb{N}$ with $j < j' \leq \tilde{u}m$ and $j' - j \notin m\mathbb{Z}$ such that $\blacksquare j$ and $\blacksquare j'$ belong to the same block. As all of $]\blacksquare j, \blacksquare j']_p$ has normalized color $\circ$,

$$\delta_p(\blacksquare j, \blacksquare j') = \sigma_p(]\blacksquare j, \blacksquare j']_p) = |]\blacksquare j, \blacksquare j']_p| = j' - j \notin m\mathbb{Z}.$$

That contradicts the assumption $L(\mathcal{C}) = m\mathbb{Z}$.
Because all blocks of p are pairs by Proposition 4.3, subpartitions of p have even cardinality. We conclude $\tilde{u} = |S_1| \in 2\mathbb{Z}$, which then proves the claim. □

8.2.2 Relation of X to L and K in Case $\mathcal{O}$

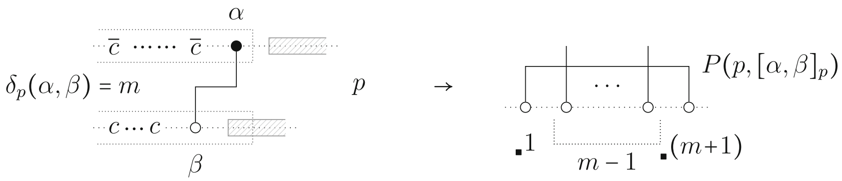
When studying $X(\mathcal{C})$ further for case $\mathcal{O}$ categories $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$, it is best to distinguish whether $(L \cup K)(\mathcal{C})$ contains non-zero elements or not.

Proposition 8.3 *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a case $\mathcal{O}$ category and let $m \in \mathbb{N}$.*

(a) If $(L, K)(\mathcal{C}) = (m + 2m\mathbb{Z}, 2m\mathbb{Z})$, then $X(\mathcal{C}) = \mathbb{Z}$ or $X(\mathcal{C}) = \mathbb{Z} \backslash m\mathbb{Z}$.
(b) If $(L, K)(\mathcal{C}) = (m\mathbb{Z}, m\mathbb{Z})$ or $(L, K)(\mathcal{C}) = (\varnothing, m\mathbb{Z})$, then $X(\mathcal{C}) = \mathbb{Z}$.

Proof No matter which of the three values $(L, K)(\mathcal{C})$ takes, by Proposition 7.23 the set $X(\mathcal{C})$ is m-periodic. Therefore, showing $[\![m]\!] \subseteq X(\mathcal{C})$ already implies $X(\mathcal{C}) = \mathbb{Z}$. Likewise, provided $m \geq 2$, establishing $[\![m-1]\!] \subseteq X(\mathcal{C})$ forces the conclusion that $X(\mathcal{C}) = \mathbb{Z} \backslash m\mathbb{Z}$ or $X(\mathcal{C}) = \mathbb{Z}$.

(a) First, let $(L, K)(\mathcal{C}) = (m + 2m\mathbb{Z}, 2m\mathbb{Z})$. If $m = 1$, the 1-periodicity of $X(\mathcal{C})$ immediately implies $X(\mathcal{C}) = \varnothing$ or $X(\mathcal{C}) = \mathbb{Z}$. Hence, we can suppose $m \geq 2$ and only need to prove $[\![m-1]\!] \subseteq X(\mathcal{C})$ by the initial remark.
Proposition 7.23 lets us infer $K_{\circ\circ}(\mathcal{C}) = m + 2m\mathbb{Z}$. Hence, we find a partition $p \in \mathcal{C} \subseteq \mathcal{P}_2^{\circ\bullet}$, therein a block $\{\alpha, \beta\}$ with α and β both of normalized color $\circ$, with $\alpha \neq \beta$ and with $\delta_p(\alpha, \beta) = m$. Without infringing on any of these assumptions we can additionally suppose that there are no turns T in p such that $T \subseteq]\alpha, \beta[_p$ (otherwise we erase them). Then, all of $]\alpha, \beta[_p$ has the same normalized color $c \in \{\circ, \bullet\}$.



Because α and β also identically have normalized color $\circ$,

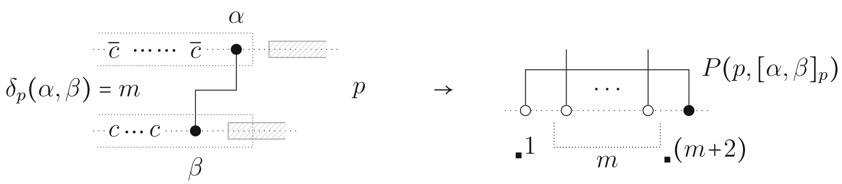
$$m = \delta_p(\alpha, \beta) = \sigma_p(]\alpha, \beta]_p) = \begin{cases} |]\alpha, \beta]_p| & \text{if } c = \circ, \\ -|]\alpha, \beta]_p| & \text{otherwise.} \end{cases}$$

As $m > 0$, the only option is $c = \circ$. That means $[\alpha, \beta]_p$ consists of $m + 1$ points of normalized color $\circ$.
By definition of the projection operation and by Lemma 3.3, it is possible to further add the premise $p = P(p, [\alpha, \beta]_p)$ without impacting any of the previous assumptions. Now, p is also projective and $[\alpha, \beta]_p = [_\blacksquare 1, _\blacksquare (m+1)]_p$ is its lower row.
For every $j \in \mathbb{N}$ with $1 < j < m+1$ the point $_\blacksquare j$ belongs to a through block: Assuming otherwise, forces us to accept the existence of $j, j' \in \mathbb{N}$ with $1 < j < j' < m+1$ such that $_\blacksquare j$ and $_\blacksquare j'$ belong to the same block. But then, the uniform color $\circ$ of $[\alpha, \beta]_p$ implies

$$1 \leq \delta_p(_\blacksquare j, _\blacksquare j') = j' - j \leq m - 2 \leq m - 1$$

and thus $L(\mathcal{C}) \cap \{1, \ldots, m-1\} \neq \varnothing$, contradicting $L(\mathcal{C}) \subseteq m\mathbb{Z}$.
Thus we have shown that $\alpha = _\blacksquare 1$ and $_\blacksquare j$ belong to crossing blocks for every $j \in \mathbb{N}$ with $1 < j < m + 1$. Because $\delta_p(\alpha, _\blacksquare j) = j - 1$ for every such j, this proves $[\![m-1]\!] \subseteq X(\mathcal{C})$. And that is what we needed to show.

(b) Let $(L, K)(\mathcal{C})$ be given by $(m\mathbb{Z}, m\mathbb{Z})$ or $(\varnothing, m\mathbb{Z})$. We adapt the proof of Part (a). However, this time, we do *not* yet impose any restriction on m.
Proposition 7.23 assures us that $K_{\circ\bullet}(\mathcal{C}) = K(\mathcal{C}) = m\mathbb{Z}$. Hence, we again find $p \in \mathcal{C}$, a block B of p and legs $\alpha, \beta \in B$ with $\alpha \neq \beta$, with $]\alpha, \beta[_p \cap B = \varnothing$ and with $\delta_p(\alpha, \beta) = m$, but this time, such that α is of normalized color $\circ$ and β of normalized color $\bullet$. By the same argument as before we can assume that all points of $]\alpha, \beta[_p$ share the same normalized color. Then, the deviating assumption on the colors of α and β implies $m = \delta_p(\alpha, \beta) = \sigma_p(]\alpha, \beta[_p) = |]\alpha, \beta[_p|$, which forces $[\alpha, \beta]_p$ to consist of exactly $m + 2$ points (rather than $m + 1$ as in Part (a)), the first $m + 1$ of which have normalized color $\circ$. Once more, we can assume $p = P(p, [\alpha, \beta]_p)$.



If $m = 1$, then $F(\{p\}) = \{2\}$ requires the unique point $_{\blacksquare}2 \in]_{\blacksquare}1, _{\blacksquare}3[_p$ to belong to a through block, proving $1 \in X(\mathcal{C})$ and thus $X(\mathcal{C}) = \mathbb{Z}$ as claimed. Hence, suppose $m \geq 2$ in the following.

We prove that only through blocks intersect $]_{\blacksquare}1, _{\blacksquare}(m+2)[_p$: Supposing that $_{\blacksquare}j$ and $_{\blacksquare}j'$, where $j, j' \in \mathbb{N}$ and $1 < j < j' < m+2$, belong to the same block requires us to believe, as both $_{\blacksquare}j$ and $_{\blacksquare}j'$ are ◦-colored, that

$$1 \leq \delta_p(_{\blacksquare}j, _{\blacksquare}j') = j' - j \leq (m+1) - 2 = m - 1$$

and thus $L(\mathcal{C}) \cap \{1, \ldots, m-1\} \neq \emptyset$. As this would contradict the assumption $L(\mathcal{C}) \subseteq m\mathbb{Z}$, this cannot be the case.

Now, the conclusion that the blocks of $_{\blacksquare}1$ and of $_{\blacksquare}j$ cross for every $j \in \mathbb{N}$ with $1 < j < m+2$ and the fact $\delta_p(_{\blacksquare}1, _{\blacksquare}j) = j - 1$ let us deduce $[\![m]\!] \subseteq X(\mathcal{C})$, which is what needed to see. □

Proposition 8.4 *Let $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$ be a case $\mathcal{O}$ category.*

(a) If $(L, K)(\mathcal{C}) = (\{0\}, \{0\})$, then $X(\mathcal{C}) = \mathbb{Z}\backslash N_0$ for a subsemigroup N of $(\mathbb{N}, +)$.
(b) If $(L, K)(\mathcal{C}) = (\emptyset, \{0\})$, then there exists a subsemigroup N of $(\mathbb{N}, +)$ such that $X(\mathcal{C}) = \mathbb{Z}\backslash N_0$ or $X(\mathcal{C}) = \mathbb{Z}\backslash N_0'$.

Proof Let $(L, K)(\mathcal{C})$ be given by $(\{0\}, \{0\})$ or $(\emptyset, \{0\})$. We show the two claims jointly in two steps:

Step 1: First, we prove that there exists a subsemigroup N of $(\mathbb{N}, +)$ such that $X(\mathcal{C}) = \mathbb{Z}\backslash N_0$ or $X(\mathcal{C}) = \mathbb{Z}\backslash N_0'$. That in itself requires two steps as well.

Step 1.1: Recall from [3, Definition 4.1] that by $\mathcal{S}_0$ we denote the set of all $p \in \mathcal{P}_2^{\circ\bullet}$ with $\sigma_p(B) = 0$ and $\delta_p(\alpha, \beta) = 0$ for all blocks B of p and all $\alpha, \beta \in B$. We justify that it suffices to prove

$$\{|z| \mid z \in X(\mathcal{C})\}\backslash\{0\} \overset{!}{\subseteq} \{|z| \mid z \in X(\mathcal{C} \cap \mathcal{S}_0)\} \qquad (*)$$

in order to verify the assertion of Step 1.

Indeed, in [4, Theorem 8.3, Lemmata 8.1 (b) and 7.16 (c)] it was shown that for every category $\mathcal{I} \subseteq \mathcal{S}_0$ there exists a subsemigroup N of $(\mathbb{N}, +)$ such that

$$\{|z| \mid z \in X(\mathcal{I})\}\backslash\{0\} = \mathbb{N}\backslash N.$$

The set $\mathcal{S}_0$ is a category by [3, Proposition 5.3], which means that so is $\mathcal{C} \cap \mathcal{S}_0$. Thus, we find a corresponding subsemigroup N for the special case $\mathcal{I} = \mathcal{C} \cap \mathcal{S}_0$. If we now suppose $(*)$, which can immediately be sharpened to

$$\{|z| \mid z \in X(\mathcal{C})\}\backslash\{0\} = \{|z| \mid z \in X(\mathcal{C} \cap \mathcal{S}_0)\}\backslash\{0\},$$

that implies

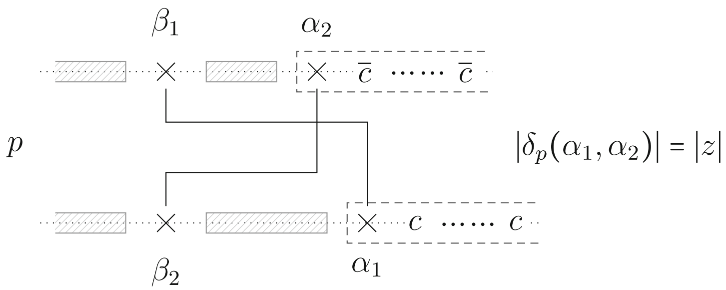
$$\{|z| \mid z \in X(\mathcal{C})\}\backslash\{0\} = \mathbb{N}\backslash N.$$

As we know $X(\mathcal{C}) = -X(\mathcal{C})$ by Proposition 7.23, this is equivalent to

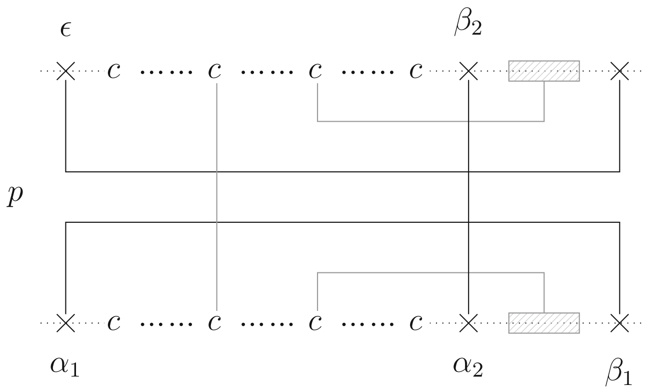
$$X(\mathcal{C})\backslash\{0\} = \mathbb{Z}\backslash N_0'$$

and thus the claim of Step 1. Hence, it is indeed sufficient to show $(*)$.

Step 1.2: We prove $(*)$. As $\mathcal{C} \subseteq \mathcal{P}_2^{\circ\bullet}$ by Proposition 4.3, we are assured by Lemma 7.15 and Proposition 7.23 that $X(\mathcal{C}) = X_{c_1,c_2}(\mathcal{C} \cap \mathcal{P}_2^{\circ\bullet})$ for all $c_1, c_2 \in \{\circ, \bullet\}$. Now, let $z \in X(\mathcal{C})\backslash\{0\}$ be arbitrary. By definition we find $p \in \mathcal{C} \cap \mathcal{P}_2^{\circ\bullet}$ and therein crossing blocks B_1 and B_2 as well as points $\alpha_1 \in B_1$ and $\alpha_2 \in B_2$ such that $\delta_p(\alpha_1, \alpha_2) = z$. Then, there exist points $\beta_1 \in B_1$ and $\beta_2 \in B_2$ such that $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$ and such that either $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ or $(\alpha_2, \alpha_1, \beta_2, \beta_1)$ is ordered in p. As $\Sigma(\mathcal{C}) = \{0\}$ by Proposition 7.23 and thus $\Sigma(p) = 0$, we know $\delta_p(\alpha_2, \alpha_1) = -\delta_p(\alpha_1, \alpha_2)$ by [5, Lemma 2.1]. Hence, by renaming B_1 and B_2 if necessary we can, at the cost of weakening $\delta_p(\alpha_1, \alpha_2) = z$ to $|\delta_p(\alpha_1, \alpha_2)| = |z|$, assume that $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is ordered. As $\mathcal{C} \cap \mathcal{P}_2^{\circ\bullet}$ is closed under erasing turns and as $(B_1 \cup B_2) \cap]\alpha_1, \alpha_2[_p = \varnothing$ we can further suppose that no turns T exist in p with $T \subseteq]\alpha_1, \alpha_2[_p$. In other words, there is $c \in \{\circ, \bullet\}$ such that every point in $]\alpha_1, \alpha_2[_p$ has normalized color c.

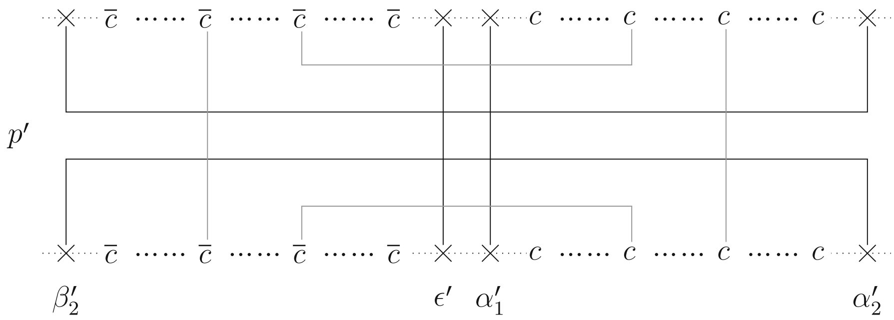


Even further, by Lemma 3.3 none of the previous assumptions are violated by assuming that $p = P(p, [\alpha_1, \beta_1]_p)$. Then, β_2 is the counterpart of α_2 on the upper row, $\alpha_1 \in [\beta_2, \alpha_2]_p$ and $\beta_1 \notin [\beta_2, \alpha_2]_p$. If we let ϵ be the predecessor of α_1, i.e., if ϵ is the leftmost upper point of p, then $(\beta_2, \epsilon, \alpha_1, \alpha_2, \beta_1)$ is ordered.



Recall that there are no turns T in p with $T \subseteq]\alpha_1, \alpha_2[_p$. As $p = p^*$, there are none with $T \subseteq]\beta_2, \epsilon[_p$ either. That means every point in $]\alpha_1, \alpha_2[_p$ has normalized color c and every point in $]\beta_2, \epsilon[_p$ normalized color $\overline{c}$. We can also say a lot about the blocks of p which intersect $[\beta_2, \alpha_2]_p$: If a point $\theta_1 \in]\alpha_1, \alpha_2[_p$ belongs to a through block it must be connected to its counterpart on the upper row because $p \in \mathcal{P}_2^{\circ\bullet}$ is projective. If θ_1 belongs to a non-through block instead, then the partner θ_2 of θ_1 must lie outside $[\beta_2, \alpha_2]_p$: Supposing otherwise, i.e., $\theta_2 \in]\alpha_1, \alpha_2[_p$, produces a contradiction: If $(\alpha_1, \theta_i, \theta_{\neg i}, \beta_2)$ with $i, \neg i \in \{1, 2\}$ and $\{i, \neg i\} = \{1, 2\}$ is ordered, then, as all points in $[\theta_i, \theta_{\neg i}]_p$ are c-colored, the consequence $|\delta_p(\theta_i, \theta_{\neg i})| = |]\theta_i, \theta_{\neg i}]_p| > 0$ violates $L(\mathcal{C}) \subseteq \{0\}$, which follows from $K(\mathcal{C}) = \{0\}$ by Proposition 7.23.

Define $p' := P(p, [\beta_2, \alpha_2]_p) \in \mathcal{C} \cap \mathcal{P}_2^{\circ\bullet}$ and denote by $\beta_2', \epsilon', \alpha_1'$ and α_2' the images in p' of $\beta_2, \epsilon, \alpha_1$ and α_2, respectively. In p' the leftmost lower point β_2' and the rightmost lower point α_2' form a block. The points $\epsilon', \alpha_1' \in [\beta_2', \alpha_2']$ are each paired with their respective counterpart on the upper row. In particular the blocks of α_1' and α_2' cross in p'.

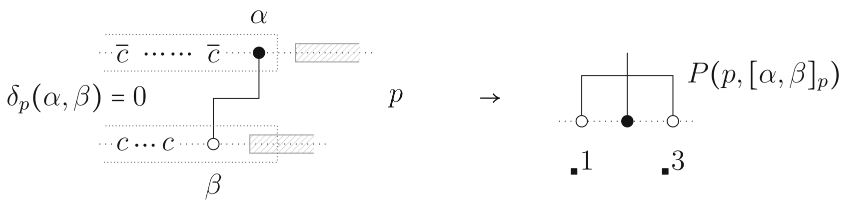


Our knowledge about the blocks of p intersecting $[\beta_2, \alpha_2]_p$ lets us draw the following conclusions about the blocks of p': A point in $]\alpha_1', \alpha_2'[_p$ is either partnered with its reflection at the center $[\epsilon', \alpha_1']_{p'}$ of the lower row of p' in $]\beta_2', \epsilon'[_{p'}$ or, as p' is projective, it is partnered with its counterpart on the opposite row. As $]\beta_2', \epsilon'[_{p'}$ is uniformly $\overline{c}$-colored and $]\alpha_1', \alpha_2'[_{p'}$ uniformly c-colored, that means that all blocks emanating from $]\beta_2', \epsilon'[_{p'} \cup]\alpha_1', \alpha_2'[_p$ are neutral. But then, all blocks of p' are neutral. Due to $L(\mathcal{C}) \subseteq \{0\}$ and $K(\mathcal{C}) = \{0\}$, this is already enough to know $p' \in \mathcal{S}_0$. Because $\delta_{p'}(\alpha_1', \alpha_2') = \delta_p(\alpha_1, \alpha_2)$, that proves $|z| = |\delta_p(\alpha_1, \alpha_2)| = |\delta_{p'}(\alpha_1', \alpha_2)| \in \{|z| \mid z \in X(\mathcal{C} \cap \mathcal{S}_0)\}$. As z was arbitrary, $(*)$ holds true and Part (b) has been proven.

Step 2: In order to prove Part (a) it remains to show $0 \in X(\mathcal{C})$ provided $L(\mathcal{C}) = \{0\}$. Under this latter assumption, by Proposition 7.23 we infer $K_{\circ\circ}(\mathcal{C}) = \{0\}$. Hence, we find $p \in \mathcal{C}$, therein a block B and legs $\alpha, \beta \in B$ of normalized color $\circ$ with $\alpha \neq \beta$, with $]\alpha, \beta[_p \cap B = \varnothing$ and with $\delta_p(\alpha, \beta) = 0$. As in the proof of Proposition 8.3 we can assume that there are no turns T in p such that $T \subseteq]\alpha, \beta[_p$, i.e. that all points in $]\alpha, \beta[_p$ have the same normalized color $c \in \{\circ, \bullet\}$. From

$$0 = \delta_p(\alpha, \beta) = \sigma_p(]\alpha, \beta]_p) = \sigma_p(]\alpha, \beta[_p) + \sigma_p(\{\beta\}) = \begin{cases} |]\alpha, \beta[_p| + 1 & \text{if } c = \circ, \\ -|]\alpha, \beta[_p| + 1 & \text{otherwise} \end{cases}$$

and from $|]\alpha, \beta[_p| \geq 0$ it follows that $c = \bullet$ and that $]\alpha, \beta[_p$ is a singleton set. Emulating the proof of Proposition 8.3 further, we can assume $p = P(p, [\alpha, \beta]_p)$.



Then, the lower row $[\alpha, \beta]_p = [{}_\blacksquare 1, {}_\blacksquare 3]_p$ of p has coloration $\circ \bullet \circ$. As $p \in \mathcal{P}_2^{\circ\bullet}$ and as p is projective, the block of ${}_\blacksquare 2$ is the pair $\{{}_\blacksquare 2, {}^\blacksquare 2\}$. That means the blocks of $\alpha = {}_\blacksquare 1$ and ${}_\blacksquare 2$ cross, implying $0 = \delta_p({}_\blacksquare 1, {}_\blacksquare 2) \in X(\mathcal{C})$. That concludes the proof. □

9 Step 6: Synthesis

Combining the results from Sects. 4–8, we are able to show the main theorem.

Theorem 9.1 *$Z(\mathcal{C}) \in \mathcal{Q}$ for every non-hyperoctahedral category $\mathcal{C} \subseteq \mathcal{P}^{\circ\bullet}$.*

Proof By Lemma 7.23 there exist $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$ and $E \subseteq \{0\} \cup \mathbb{N}$ such that the tuple $(\Sigma, L, K, X)(\mathcal{C})$ is given by one of the following:

Σ	L	K	X
$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$2um\mathbb{Z}$	$m + 2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash E_0$

(∗)

We treat the three cases $\mathcal{O}$, $\mathcal{B}$ and $\mathcal{S}$ individually. The formulaic presentation will mirror that of Definition 2.5 exactly, to facilitate cross-checking.

Case $\mathcal{B}$: First, let $\mathcal{C}$ be case $\mathcal{B}$. Proposition 4.3 (c) implies $F(\mathcal{C}) = \{1, 2\}$. So, we can immediately add the column for $F(\mathcal{C})$ to table (∗). Further, Proposition 5.1 (c) shows $V(\mathcal{C}) = \pm\{0, 1, 2\}$ if and only if $L(\mathcal{C}) \neq \varnothing$ and $V(\mathcal{C}) = \pm\{0, 1\}$ otherwise. That allows us to fill in the column for $V(\mathcal{C})$ as well. The result is that $Z(\mathcal{C})$ concurs with a row of the table

F	V	Σ	L	K	X
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$2um\mathbb{Z}$	$m{+}2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{1, 2\}$	$\pm\{0, 1\}$	$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{1, 2\}$	$\pm\{0, 1, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\{1, 2\}$	$\pm\{0, 1\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash E_0$

for some $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$ and $E \subseteq \{0\} \cup \mathbb{N}$. Hence, by Definition 2.5, we have shown $Z(\mathcal{C}) \in \mathcal{Q}$ if $\mathcal{C}$ is case $\mathcal{B}$.

Case $\mathcal{S}$: Next, let $\mathcal{C}$ be case $\mathcal{S}$. Propositions 4.3 (d) and 5.1 (d) guarantee $F(\mathcal{C}) = \mathbb{N}$ and $V(\mathcal{C}) = \mathbb{Z}$. Hence, we can fill in the columns for F and V in (∗) once more. Moreover, $0 \in L(\mathcal{C})$ by Proposition 8.1. Thus, we can exclude that $(\Sigma, L, K, X)(\mathcal{C})$ is given by the second, third or fifth rows of (∗). In other words, there are $u \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, $D \subseteq \{0\} \cup [\![\lfloor \frac{m}{2} \rfloor]\!]$ and $E \subseteq \{0\} \cup \mathbb{N}$ such that $Z(\mathcal{C})$ is given by one of the rows of the following table:

F	V	Σ	L	K	X
$\mathbb{N}$	$\mathbb{Z}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\mathbb{N}$	$\mathbb{Z}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$

And, by Definition 2.5, this means $Z(\mathcal{C}) \in \mathcal{Q}$ for $\mathcal{C}$ in case $\mathcal{S}$.

Case $\mathcal{O}$: Lastly, let $\mathcal{C}$ be case $\mathcal{O}$. Once more, Propositions 4.3 (b) and 5.1 (b) give, on the one hand, $F(\mathcal{C}) = \{2\}$ and, on the other hand, $V(\mathcal{C}) = \pm\{0, 2\}$ if $L(\mathcal{C}) \neq \varnothing$ and $V(\mathcal{C}) = \{0\}$

otherwise. That enables us to fill in the columns for $F(\mathcal{C})$ and $V(\mathcal{C})$ in (∗):

F	V	Σ	L	K	X
$\{2\}$	$\pm\{0,2\}$	$um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{2\}$	$\{0\}$	$um\mathbb{Z}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}\backslash D_m$
$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash E_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash E_0$

(∗∗)

This is not yet what we claim as this range is not contained in Q. We need to exclude certain values for u, D and E by taking into account the results of Section 8.2. This we shall do on a row-by-row basis.

Case $\mathcal{O}$.1: First, suppose $(L, K)(\mathcal{C}) = (m\mathbb{Z}, m\mathbb{Z})$ for some $m \in \mathbb{N}$, as in the first row of Table (∗∗). Then $\Sigma(\mathcal{C}) \subseteq 2m\mathbb{Z}$ (corresponding to parameters $u \in 2\mathbb{Z}$) according to Proposition 8.2 (b). Moreover, $X(\mathcal{C}) = \mathbb{Z}$ (corresponding to $D = \varnothing$) as seen in Proposition 8.3 (b). Hence, we can replace the first row of Table (∗∗) by

F	V	Σ	L	K	X
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$

still for parameters $u \in \{0\} \cup \mathbb{N}$ and $m \in \mathbb{N}$ exactly as before.

Case $\mathcal{O}$.2: Now, proceeding to the second row of Table (∗∗), let $(L, K)(\mathcal{C}) = (m + 2m\mathbb{Z}, 2m\mathbb{Z})$ for some $m \in \mathbb{N}$. By Proposition 8.3 (a) the only two values $X(\mathcal{C})$ can possibly take are $\mathbb{Z}$ and $\mathbb{Z}\backslash m\mathbb{Z}$ (corresponding to $D = \varnothing$ and $D = \{0\}$, respectively). Thus, we can delete the second row of Table (∗∗) and insert the two new rows

F	V	Σ	L	K	X
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0,2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$

in its stead, still for parameters $m \in \mathbb{N}$ and $u \in \{0\} \cup \mathbb{N}$.

Case $\mathcal{O}$.3: Next, assume $(L, K)(\mathcal{C}) = (\varnothing, m\mathbb{Z})$ for some $m \in \mathbb{N}$ as in row three of Table (∗∗). Then, in fact, $\Sigma(\mathcal{C}) = \{0\}$ as seen in Proposition 8.2 (a). Furthermore, $X(\mathcal{C}) = \mathbb{Z}$ by Proposition 8.3 (b). Hence, we rewrite the third row of (∗∗) as

F	V	Σ	L	K	X
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}$

depending only on the parameter $m \in \mathbb{N}$.

Case $\mathcal{O}$.4: Let $(L, K)(\mathcal{C}) = (\{0\}, \{0\})$, i.e., consider the fourth row of Table (∗∗). Then, $X(\mathcal{C}) = \mathbb{Z}\backslash N_0$ for some subsemigroup of $(\mathbb{N}, +)$ by Proposition 8.4 (a) (corresponding to $E = N$ being a subsemigroup). Accordingly, we can replace the fourth row of Table (∗∗) by

F	V	Σ	L	K	X
$\{2\}$	$\pm\{0,2\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N_0$

for a new table parameter N, running through all subsemigroups of $(\mathbb{N}, +)$.

Case $\mathcal{O}$.5: Lastly, suppose $(L, K)(\mathcal{C}) = (\varnothing, \{0\})$ as in the fifth row of Table (∗∗). In Proposition 8.4 (a) we showed $X(\mathcal{C})$ is of the form $\mathbb{Z}\backslash N_0$ or $\mathbb{Z}\backslash N_0'$ for some subsemigroup N of $(\mathbb{N}, +)$ (corresponding to $E = N$ and $E = \{0\} \cup N$, respectively). Thus, strike the last row of Table (∗∗) and append the two rows

F	V	Σ	L	K	X
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N_0'$

to the table, with N being a subsemigroup of $(\mathbb{N}, +)$.

Synthesis in case $\mathcal{O}$: If we combine the results of Cases 1–5, then we can say that there exist $m \in \mathbb{N}$, $u \in \{0\} \cup \mathbb{N}$ and a subsemigroup N of $(\mathbb{N}, +)$ such that $Z(\mathcal{C})$ is given by one of the rows of the following table:

F	V	S	L	K	X
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m\mathbb{Z}$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$2um\mathbb{Z}$	$m+2m\mathbb{Z}$	$2m\mathbb{Z}$	$\mathbb{Z}\backslash m\mathbb{Z}$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$m\mathbb{Z}$	$\mathbb{Z}$
$\{2\}$	$\pm\{0, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\mathbb{Z}\backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N_0$
$\{2\}$	$\{0\}$	$\{0\}$	$\varnothing$	$\{0\}$	$\mathbb{Z}\backslash N_0'$

Definition 2.5 thus yields $Z(\mathcal{C}) \in \mathcal{Q}$ if $\mathcal{C}$ is case $\mathcal{O}$. Hence, the overall claim is true. □

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Declarations

Conflict of interests The authors declare that they have no conflict of interest.

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