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# Horrocks splitting on Segre–Veronese varieties

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**Abstract**
We prove an analogue of Horrocks' splitting theorem for Segre–Veronese varieties building upon the theory of Tate resolutions on products of projective spaces.

**Keywords** Horrocks splitting · Segre–Veronese varieties · Tate resolutions

**Mathematics Subject Classification** 14F05 · 14M99

## Introduction

Horrocks' famous splitting theorem [3] on $\mathbb{P}^n$ says that a vector bundle $\mathcal{F}$ on $\mathbb{P}^n$ splits into a direct sum of line bundles

$$\mathcal{F} \cong \bigoplus_j \mathcal{O}(k_j)$$

if and only if $\mathcal{F}$ has no intermediate cohomology, i.e., if

$$H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0 \quad \forall k \in \mathbb{Z} \text{ and } \forall i \text{ with } 0 < i < n.$$

In this note we prove a similar criterion for Segre–Veronese varieties

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \hookrightarrow \mathbb{P}^N$$

embedded by the complete linear system of a very ample line bundle $\mathcal{O}(H) = \mathcal{O}(d_1, \ldots, d_t)$, so $N = (\prod_{j=1}^t \binom{n_j+d_j}{n_j}) - 1$.

**Theorem 0.1** *Let $\mathcal{O}(H) = \mathcal{O}(d_1, \ldots, d_t)$ be a very ample line bundle on a product of projective spaces $\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ of dimension $m = n_1 + \cdots + n_t$ with $t \geq 2$ factors. A torsion free sheaf $\mathcal{F}$ on $\mathbb{P}$ splits into a direct sum $\mathcal{F} \cong \bigoplus_j \mathcal{O}(k_j H)$ if and only if*

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$$\forall \quad i \in \{1, \ldots, m-1\} \quad H^i(\mathbb{P}, \mathcal{F}(a_1, \ldots, a_t)) = 0$$

*for all twists with* $\mathcal{O}(a_1, \ldots, a_t)$ *such that the cohomology groups* $H^i(\mathbb{P}, \mathcal{O}(kH) \otimes \mathcal{O}(a_1, \ldots, a_t))$ *vanish for all* $i \in \{1, \ldots, m-1\}$ *and all* $k \in \mathbb{Z}$.

We can rephrase the theorem as follows: *If a torsion free sheaf* $\mathcal{F}$ *on a product* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ *has no intermediate cohomology in the range where the sheaves* $\mathcal{O}(kH)$ *have no intermediate cohomology, then it is a direct sum of these sheaves.*

***Example 0.2*** For $\mathbb{P} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, the line bundle $\mathcal{O}(a_1, a_2)$ has a nonzero cohomology group

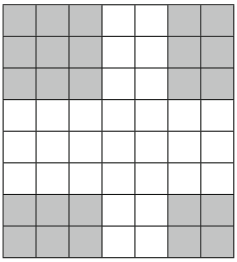
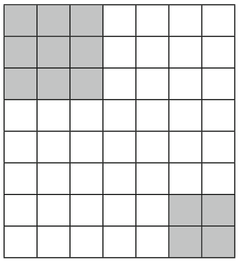
$$\begin{array}{ll} H^{n_1}(\mathcal{O}(a_1, a_2)) \neq 0, & H^0(\mathcal{O}(a_1, a_2)) \neq 0, \\ H^m(\mathcal{O}(a_1, a_2)) \neq 0, & H^{n_2}(\mathcal{O}(a_1, a_2)) \neq 0, \end{array}$$

for $a = (a_1, a_2)$ in the range

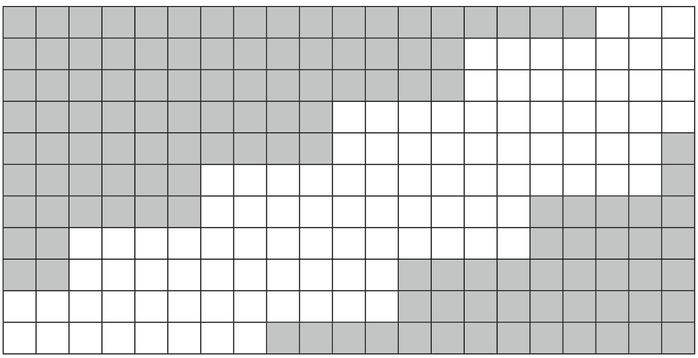
$$\begin{array}{ll} \{a_1 < -n_1, a_2 \geq 0\}, & \{a_1 \geq 0, a_2 \geq 0\} \\ \{a_1 < -n_1, a_2 < -n_2\}, & \{a_1 \geq 0, a_2 < -n_2\} \end{array}$$

respectively and is zero otherwise.

In particular for $\mathbb{P}^2 \times \mathbb{P}^3$ and the area $\{-5 \leq a_1 \leq 1, -5 \leq a_2 \leq 2\}$, nonzero cohomology and nonzero intermediate cohomology occur in the shaded regions

 and  respectively.

Thus for $\mathcal{O}(H) = \mathcal{O}(4, 2)$, the assumption of the theorem in this case is that the intermediate cohomology occurs only in a range as indicated in the area $\{-8 \leq a_1 \leq 12, -4 \leq a_2 \leq 6\}$ by the shaded region below:



***Remark 0.3*** For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ the condition

$$H^i(\mathbb{P}, \mathcal{F}(kH)) = 0 \text{forall} i \in \{1, \dots, m-1\} \text{andall} k \in \mathbb{Z}$$

implies that $\mathcal{F}$ is locally free unless $\mathcal{F}$ has a zero dimensional subsheaf. The condition $\mathcal{F}$ torsion free in Theorem 0.1 is only used to exclude such torsion subsheaves.

## 1 Preliminaries and notation

The Tate resolutions of a sheaf on products of projective spaces is a generalization of the Tate resolution on $\mathbb{P}^n$ [2]. We recall from [1] the basic notation.

Let $\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} = \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_t)$ be a product of $t$ projective spaces over an arbitrary field $K$. Set $V_i = W_i^*$ and $V = \bigoplus_i V_i$. Let $E$ be the $\mathbb{Z}^t$-graded exterior algebra on $V$, where elements of $V_i \subset E$ have degree $(0, \dots, 0, -1, 0, \dots, 0)$ with $-1$ in the $i$-th place.

For a sheaf $\mathcal{F}$ on $\mathbb{P}$ the Tate resolution $\mathbf{T}(\mathcal{F})$ is a minimal exact complex of graded $E$-modules with terms

$$\mathbf{T}(\mathcal{F})^d = \bigoplus_{a \in \mathbb{Z}^t} \mathrm{Hom}_K(E, H^{d-|a|}(\mathbb{P}, \mathcal{F}(a))),$$

where the cohomology group $H^{d-|a|}(\mathbb{P}, \mathcal{F}(a))$ is regarded as a vector space concentrated in degree $a$, and $|a| = \sum_{j=1}^t a_j$ denotes the total degree.

Since $\omega_E = \mathrm{Hom}_K(E, K)$ is the free $E$-module of rank 1 with socle in degree 0 and hence generator in degree $(n_1 + 1, \dots, n_t + 1)$, the differential of the complex $\mathbf{T}(\mathcal{F})$ is given by a matrix with entries in $E$. More precisely, the component $\mathrm{Hom}_K(E, H^{d-|a|}(\mathbb{P}, \mathcal{F}(a))) \to \mathrm{Hom}_K(E, H^{d+1-|b|}(\mathbb{P}, \mathcal{F}(b)))$ is given by a $h^{d+1-|b|}(\mathbb{P}, \mathcal{F}(b)) \times h^{d-|a|}(\mathbb{P}, \mathcal{F}(a))$-matrix with entries in

$$\Lambda^{b-a} V := \Lambda^{b_1 - a_1} V_1 \otimes \cdots \otimes \Lambda^{b_t - a_t} V_t.$$

In particular, if $b_j < a_j$ for some $j$, then the corresponding block is zero. Moreover, all blocks corresponding to cases with $a = b$ are also zero, since $\mathbf{T}(\mathcal{F})$ is a minimal complex.

The complex $\mathbf{T}(\mathcal{F})$ has various exact free subquotient complexes: For $c \in \mathbb{Z}^t$ a degree and $I, J, K \subset \{1, \dots, t\}$ disjoint subsets we have the subquotient complex $T_c(I, J, K)$ with terms

$$T_c(I, J, K)^d = \sum_{\substack{a \in \mathbb{Z}^t \\ a_i < c_i \text{for} i \in I \\ a_i = c_i \text{for} i \in J \\ a_i \geq c_i \text{for} i \in K}} \mathrm{Hom}_K(E, H^{d-|a|}(\mathbb{P}, \mathcal{F}(a)))$$

By [1, Theorem 3.3 and Corollary 3.5] these complexes are exact as long as $I \cup J \cup K \subsetneq \{1, \dots, t\}$. The complexes $T_c(\emptyset, J, \emptyset)$ can be used to compute the direct image complex of $\mathcal{F}(c)$ along a partial projection $\pi_J : \mathbb{P} \to \prod_{j \notin J} \mathbb{P}^{n_j}$ [1, Corollary 0.3 and Proposition 3.6].

**Lemma 1.1** *Let $\mathcal{F}$ be a coherent sheaf on a product of projective spaces $\mathbb{P} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_t}$ and let $a = (a_1, a_2, \dots, a_t) = (a', a_t) \in \mathbb{Z}^t = \mathbb{Z}^{t-1} \times \mathbb{Z}$ and $n \in \mathbb{Z}$. If*

$$H^n(\mathcal{F}(a', a_t)) = H^{n-1}(\mathcal{F}(a', a_t + 1)) = \ldots = H^{n-n_t}(\mathcal{F}(a', a_t + n_t)) = 0$$

*then $H^n(\mathcal{F}(a', a_t - 1)) = 0$ as well. A similar statement holds for the cohomology along the $j$-th strand $T_a(\emptyset, \{1, \dots, t\} \backslash \{j\}, \emptyset)$.*

***Proof*** We consider the strand $T_a(\emptyset, \{1, \ldots, t-1\}, \emptyset)$ of $\mathbf{T}(\mathcal{F})$. The differential starting at the summand $\mathrm{Hom}_K(E, H^n(\mathcal{F}(a', a_t - 1)) \subset T_a(\emptyset, \{1, \ldots, t-1\}, \emptyset)$ maps in the strand to the summands

$$\mathrm{Hom}_K(E, H^n(\mathcal{F}(a', a_t))) \oplus \ldots \oplus \mathrm{Hom}_K(E, H^{n-n_t}(\mathcal{F}(a', a_t + n_t))).$$

By assumption the target is zero. Since of $T_a(\emptyset, \{1, \ldots, t-1\}, \emptyset)$ is minimal and exact, the source is zero as well.

The proof of the our main theorem uses the corner complexes $T_{\ulcorner c}(\mathcal{F})$ which are defined as the cone of a map of complexes

$$\varphi_c : T_c(\{1, \ldots, t\}, \emptyset, \emptyset)[-t] \to T_c(\emptyset, \emptyset, \{1, \ldots, t\})$$

from the last quadrant complex to the first quadrant complex. The map $\varphi_c$ is the composition of $t$ maps

$$T_c(\{1, \ldots, k\}, \emptyset, \{k+1, \ldots, t\})[-k] \to T_c(\{1, \ldots, k-1\}, \emptyset, \{k, \ldots, t\})[-k+1]$$

each of which is obtained from the differential of $\mathbf{T}(\mathcal{F})$ by taking the terms with source in one quadrant and target in the next quadrant. The corner complexes are exact as well by [1, Theorem 4.3 and Corollary 4.5].

If we follow a path from the last quadrant to the first quadrant using a different order of the elements in the set $\{1, \ldots, t\}$, we obtain an isomorphic complex. Indeed, all of these corner complexes are exact and their differentials

$$T_{\ulcorner c}(\mathcal{F})^d \to T_{\ulcorner c}(\mathcal{F})^{d+1}$$

coincide for sufficiently large cohomological degree $d$, since those differentials involve only terms from the first quadrant $T_c(\emptyset, \emptyset, \{1, \ldots, t\})$.

## 2 Proof of the main result

We use the partial order $a \geq b$ on $\mathbb{Z}^t$ defined by $a_j \geq b_j$ for $j = 1, \ldots, t$ and write $a > b$ if $a \geq b$ and $a \neq b$.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. If $H^m(\mathbb{P}, \mathcal{F}(a)) \neq 0$ then $H^m(\mathbb{P}, \mathcal{F}(b)) \neq 0$ for all $b \leq a$ as we see from applying $H^m$ to the surjection

$$H^0(\mathbb{P}, \mathcal{O}(a-b)) \otimes \mathcal{F}(b) \to \mathcal{F}(a).$$

An *extremal $H^m$-position* of $\mathcal{F}$ is a degree $a \in \mathbb{Z}^t$ such that $H^m(\mathbb{P}, \mathcal{F}(a)) \neq 0$ but $H^m(\mathbb{P}, \mathcal{F}(c)) = 0$ for all $c > a$.

**Proposition 2.1** *Let $\mathcal{F}$ be a torsion free sheaf on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ satisfying the assumption of Theorem 0.1 with respect to $\mathcal{O}(H) = \mathcal{O}(d_1, \ldots, d_t)$. There exists an extremal $H^m$-position for $\mathcal{F}$ of the form*

$$(a_1, \ldots, a_t) = (kd_1 - n_1 - 1, \ldots, kd_t - n_t - 1)$$

*for some $k \in \mathbb{Z}$.*

Note that $\mathcal{O}(-n_1 - 1, \ldots, -n_t - 1) \cong \omega_{\mathbb{P}}$ is the canonical sheaf on $\mathbb{P}$.

***Proof*** Since $\mathcal{F}$ is nonzero and torsion free, we have $H^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}})\neq 0$ for $k\ll 0$ and $H^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}})=0$ for $k\gg 0$. Let $k$ be the maximum such that $H^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}})\neq 0$. We claim that $(kd_1-n_1-1,\ldots,kd_t-n_t-1)$ is an extremal $H^m$-position. Suppose it is not. Then there exists a maximal $a$ in the range

$$(kd_1-n_1-1,\ldots,kd_t-n_t-1)<a\leq((k+1)d_1-n_1-1,\ldots,(k+1)d_t-n_t-1)$$

such that $H^m(\mathcal{F}(a))\neq 0$. At least for one $i$ we have $kd_i-n_i-1<a_i$. Then for any $j\neq i$ we consider $J=\{1,\ldots,t\}\backslash\{j\}$ and look at the $j$-th strand $T_a(\emptyset,J,\emptyset)$ through $a$. Lemma 1.1 implies $a_j=(k+1)d_j-n_j-1$: If $a_j<(k+1)d_j-n_j-1$, then we cannot reach the intermediate cohomology range of $\mathcal{F}$ after at most $n_j+1$ steps along this strand, contradicting the maximality of $a$. Starting with $kd_j-n_j-1<a_j=(k+1)d_j-n_j-1$ and interchanging the role of $i$ and $j$ in the argument above, we deduce $a_i=(k+1)d_i-n_i-1$ for all $i$. This is a contradiction to the maximality of $k$.

**Proposition 2.2** *Let $\mathcal{F}$ be a torsion free sheaf on $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_t}$ satisfying the assumption of Theorem 0.1 with respect to $\mathcal{O}(H)=\mathcal{O}(d_1,\ldots,d_t)$. If*

$$(kd_1-n_1-1,\ldots,kd_t-n_t-1)$$

*is an extremal $H^m$-position for $\mathcal{F}$, then*

$$\mathcal{F}\cong\mathcal{O}(kH)\oplus\mathcal{F}'.$$

***Proof*** We consider the corner complex $T_{\ulcorner c}(\mathcal{F})$ for $c=(kd_1-n_1,\ldots,kd_t-n_t)$. The first part of the corner map

$$T_c(\{1,\ldots,t\},\emptyset,\emptyset)[-t]\to T_c(\{1,\ldots,t-1\},\emptyset,\{t\})[-t+1]$$

with source $\mathrm{Hom}_K(E,H^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}})$ is a map

$$\mathrm{Hom}_K(E,H^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}}))\to\mathrm{Hom}_K(E,H^{m-n_t}(\mathcal{F}\otimes\omega_{\mathbb{P}}\otimes\mathcal{O}(0,\ldots,0,n_t+1)))$$

given by a matrix with entries in $\Lambda^{n_t+1}V_t$. Indeed, $H^m((\mathcal{F}\otimes\omega_{\mathbb{P}}\otimes\mathcal{O}(0,\ldots,0,1))=0$ holds since $(kd_1-n_1-1,\ldots,kd_t-n_t-1)$ is extremal. Since the map follows the strand $T_c(\emptyset,\{1,\ldots,t-1\},\emptyset)$, the group $H^{m-n_t}(\mathcal{F}\otimes\omega_{\mathbb{P}}\otimes\mathcal{O}(0,\ldots,0,n_t+1)$ is the first possible non-zero intermediate cohomology group by assumption. Composed with the second part of the corner complex

$$T_c(\{1,\ldots,t-1\},\emptyset,\{t\})[-t+1]\to T_c(\{1,\ldots,t-2\},\emptyset,\{t-1,t\})[-t+2]$$

the image is in

$$\mathrm{Hom}_K(E,H^{m-n_t-n_{t-1}}(\mathcal{F}\otimes\omega_{\mathbb{P}}\otimes\mathcal{O}(0,\ldots,0,n_{t-1}+1,n_t+1)))$$

since $\Lambda^{n_t+2}V_t=0$ and other possible intermediate cohomology groups in the strand $T_c(\emptyset,\{1,\ldots,t-2\},\emptyset)$ vanish by assumption. Repeating these arguments, we conclude that the corner map with source $\mathrm{Hom}_K(E,H^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}})$ has an image only in $\mathrm{Hom}_K(E,H^0(\mathcal{F}(kH))$. It is given by an

$$h^0(\mathcal{F}(kH))\times h^m(\mathcal{F}(kH)\otimes\omega_{\mathbb{P}})\text{−matrix}$$

with entries in the one-dimensional space

$$\Lambda^{m+t}V=\Lambda^{n_1+1}V_1\otimes\cdots\otimes\Lambda^{n_t+1}V_t.$$

Consider the submatrix of the differential in the corner complex with target equal to the summand $\mathrm{Hom}_K(E, H^0(\mathcal{F}(kH))$. The only other subspaces in the source which have this target come from $H^0$-groups:

$$\mathrm{Hom}_K(E, H^0(\mathcal{F}(kH)(-1, 0, \dots, 0)), \dots, \mathrm{Hom}_K(E, H^0(\mathcal{F}(kH)(0, \dots, 0, -1)).$$

Thus this differential is given by an

$$h^0(\mathcal{F}(kH)) \times [(h^m(\mathcal{F}(kH) \otimes \omega_{\mathbb{P}}) + h^0(\mathcal{F}(kH) \otimes \mathcal{B}))]\text{−matrix}$$

with $\mathcal{B} = \mathcal{O}(-1, 0, \dots, 0) \oplus \dots \oplus \mathcal{O}(0, \dots, 0, -1)$. Note that $h^0(\mathcal{F}(kH)) \geq h^m(\mathcal{F}(kH) \otimes \omega_{\mathbb{P}})$, because otherwise a generator of $\mathrm{Hom}_K(E, H^m(\mathcal{F}(kH) \otimes \omega_{\mathbb{P}})$ would map to zero which is impossible because $T_{\vec{c}}(\mathcal{F})$ is exact and minimal. Thus in a suitable basis the matrix has shape

$$\varphi = \left(\begin{array}{ccc|ccc} v & & 0 & \ell_{1j} & \dots & \ell_{1n} \\ & \ddots & & \vdots & & \vdots \\ 0 & & v & \ell_{rj} & \dots & \ell_{rn} \\ \hline & & & \ell_{r+1j} & \dots & \ell_{r+1n} \\ & 0 & & \vdots & & \vdots \\ & & & \ell_{sj} & \dots & \ell_{sn} \end{array}\right)$$

with $v \in \Lambda^{m+t} V$ a fixed basis element and $\ell_{ij} \in V_1 \cup \dots \cup V_t$.

We claim now that $\ell_{1j}$ is a $K$-linear combination of $\ell_{r+1j}, \dots, \ell_{sj}$. Indeed if not, we could multiply the $j$-th column by a decomposable element $w \in \Lambda^{m+t-1} V$ which annihilates $\ell_{r+1j}, \dots, \ell_{sj}$ but does not annihilate $\ell_{1j}$, so that $\ell_{1j} w = v$. This would give us a column

$$\left(\begin{array}{ccc|c} v & & 0 & v \\ & \ddots & & \vdots \\ 0 & & v & \lambda_r v \\ \hline & & & 0 \\ & 0 & & \vdots \\ & & & 0 \end{array}\right)$$

for possibly zero scalars $\lambda_2, \dots, \lambda_r$, and the first column would be an $E$-linear combination of columns 2 to $j$. This is impossible since no generator can map to zero in $T_{\vec{c}}(\mathcal{F})$.

Let $r_1 - r$ denote the dimension of the linear span of $\ell_{r+1j}, \dots, \ell_{sj}$. Then after row operations we may assume that $\varphi$ has the shape

$$\varphi = \left(\begin{array}{ccc|c|cc} v & & 0 & 0 & \dots & \ell_{1n} \\ \hline & \ddots & & \vdots & & \vdots \\ 0 & & v & \ell_{rj} & \dots & \ell_{rn} \\ \hline & & & \ell_{r+1j} & \dots & \ell_{r+1n} \\ & 0 & & \vdots & & \vdots \\ & & & \ell_{r_1 j} & \dots & \ell_{r_1 n} \\ \hline & & & \vdots & & \vdots \\ & & & 0 & \dots & \ell_{sn} \end{array}\right)$$

with $\ell_{r+1j}, \dots \ell_{r_1 j}$ $K$-linearly independent.

Next we note that the columns of the matrix

$$\left(\begin{array}{ccc|ccc|c}
v & & 0 & & & & \ell_{1j+1} \\
 & \ddots & & & 0 & & \vdots \\
0 & & v & & & & \vdots \\
\hline
 & & & v & & 0 & \vdots \\
 & 0 & & & \ddots & & \vdots \\
 & & & 0 & & v & \ell_{r_1 j+1} \\
\hline
 & & & & & & \ell_{r_1+1j+1} \\
 & 0 & & & 0 & & \vdots \\
 & & & & & & \ell_{sj+1}
\end{array}\right)$$

are in the $E$-column span of $\varphi$. Arguing as before, we see that $\ell_{1j+1}$ is a linear combination of $\ell_{r_1+1j+1}, \dots, \ell_{sj+1}$, and repeating the arguments, we find that $\varphi$ can be transformed by row operations into a matrix of type

$$\left(\begin{array}{c|cc|ccc}
v & & 0 & 0 & \dots & 0 \\
\hline
 & \ddots & & \ell_{2j} & \dots & \ell_{2n} \\
0 & & v & \vdots & & \vdots \\
\hline
 & & & \vdots & & \vdots \\
 & & 0 & \ell_{sj} & \dots & \ell_{sn}
\end{array}\right) = \left(\begin{array}{c|c} v & 0 \\ \hline 0 & \varphi' \end{array}\right)$$

We conclude that $T_{\ulcorner c}(\mathcal{O}(kH))$ is a direct summand of the complex $T_{\ulcorner c}(\mathcal{F})$, and

$$\mathcal{F} \cong \mathcal{O}(kH) \oplus \mathcal{F}',$$

since we can recover $\mathcal{F}$ from its corner complex with the Beilinson functor $\mathbf{U}$ applied to $T_{\ulcorner c}(\mathcal{F})(a)[|a|]$ for a suitable $a \in \mathbb{Z}^t$ by [1, Theorem 0.1]. Indeed $\mathbf{U}(T_{\ulcorner c}(\mathcal{F})(a)[|a|])$ and $\mathbf{U}(\mathbf{T}(\mathcal{F})(a)[|a|])$ coincide for $a \gg 0$.

*Proof of Theorem.* Let $\mathcal{F}$ be a torsion free sheaf on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ with no intermediate cohomology where the sheaves $\mathcal{O}(kH)$ for $\mathcal{O}(H) = \mathcal{O}(d_1, \dots, d_t)$ have no intermediate cohomology. By Proposition 2.1 there is an extremal $H^m$-position of $\mathcal{F}$ of the form

$$(k_1 d_1 - n_1 - 1, \dots, k_1 d_t - n_t - 1)$$

and by Proposition 2.2 we get a summand

$$\mathcal{F} \cong \mathcal{O}(k_1 H) \oplus \mathcal{F}'.$$

If $\operatorname{rank} \mathcal{F} = 1$, we are done: $\mathcal{F}' = 0$ since $\mathcal{F}$ is torsion free. Otherwise we can argue by induction on the rank since $\mathcal{F}'$ satisfies the assumption of the Theorem again. $\square$

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