UNIVERSITÄT DES SAARLANDES

# On the geometry of $p$-origamis and beyond

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# Abstract

The main topic of this thesis is the study of a special class of translation surfaces called *normal origamis*. The theory of translation surfaces is an active research area with applications in various fields such as dynamical systems, algebraic geometry, and geometric group theory. Normal origamis are surfaces with a maximal symmetry group and induce normal covers of the torus $\mathbb{T}$. We focus on *$p$-origamis*, where the deck transformation groups of the torus covers are $p$-groups, and answer the questions: Which strata contain $p$-origamis? Does already the deck transformation group determine the stratum?

We then turn toward the study of *Veech groups* of certain normal origamis. These groups are the stabilizer groups of an origami under an $\mathrm{SL}(2, \mathbb{Z})$-action. We are especially interested in the question, whether the occurring Veech groups are congruence groups. The $\mathrm{SL}(2, \mathbb{Z})$-orbits on normal origamis are closely related to the group-theoretic concept of $T_2$-systems. We investigate this relationship and transfer group-theoretic results to the geometric setting.

Cylinder decompositions are an important concept occurring in different contexts within this thesis. *Geminal origamis* exhibit special cylinder decompositions. Apisa and Wright asked whether geminal origamis are cyclic covers of the surface $(2 \times 2)$-torus. We use methods from group theory to answer this question partially.

This thesis contains results of the author's research articles [FT20] and [The21].

# Zusammenfassung

Im Zentrum dieser Dissertation steht das Studium *normaler Origamis*, einer Familie von Translationsflächen. Seit 40 Jahren sind Translationsflächen Gegenstand aktiver mathematischer Forschung mit Anwendungen in diversen mathematischen Bereichen wie algebraischer Geometrie und geometrischer Gruppentheorie. Normale Origamis haben eine maximale Symmetriegruppe und definieren normale Überlagerungen des Torus. Zunächst untersuchen wir *$p$-Origamis*, d.h. normale Origamis mit einer $p$-Gruppe als Decktransformationsgruppe. Wir beantworten die Fragen, welche Strata $p$-Origamis enthalten und ob die Decktransformationsgruppe bereits das Stratum festlegt.

Des Weiteren betrachten wir *Veechgruppen* bestimmter normaler Origamis. Diese Gruppen sind Stabilisatoren eines Origamis unter einer $\mathrm{SL}(2,\mathbb{Z})$-Wirkung. Unter anderem untersuchen wir die Fragen, ob und wann die Veechgruppen normaler Origamis Kongruenzgruppen sind. Zudem diskutieren wir den Zusammenhang zwischen den $\mathrm{SL}(2,\mathbb{Z})$-Bahnen normaler Origamis und dem gruppentheoretischen Konzept der $T_2$-Systeme.

Zylinderzerlegungen sind ein wichtiges Konzept in der Theorie der Translationsflächen, welches wir an verschiedenen Stellen in dieser Arbeit verwenden. *Geminale Origamis* sind Origamis mit sehr speziellen Zylinderzerlegungen. Unter gewissen Voraussetzungen beantworten wir die Frage, ob geminale Origamis den $(2\times 2)$-Torus zyklisch überlagern.

Diese Dissertation enthält Ergebnisse aus folgenden Publikationen der Autorin [FT20] und [The21].

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# Contents

# List of symbols

### General mathematical symbols

| | |
|---|---|
| $p$ | a prime number |
| $\mathbb{Q}$ | rational numbers |
| $\mathbb{R}$ | real numbers |
| $\mathbb{C}$ | complex numbers |
| $\mathbb{F}_q$ | finite field of order $q$ |
| $\mathbb{Z}_{\geq 0}$ | non-negative integers |
| $\mathbb{Z}_+$ | positive integers |
| $\mathbb{Z}$ | integers |
| $\uplus, \sqcup$ | disjoint union |

### Group theory

| | |
|---|---|
| $\mathrm{Aut}(G)$ | automorphism group of $G$ |
| $\gamma_i$ | $i$th group in the lower central series |
| $E(G)$ | set of generating pairs of a group $G$ |
| $E_k(G)$ | set of $k$-tuples with entries in $G$ that generate the group $G$ |
| $\exp(G)$ | exponent of a group $G$ |
| $G$ | group |
| $G'$ | commutator subgroup of a group $G$ |
| $(\hat{G}, \pi_i)$ | profinite group |
| $\varprojlim(G_i, \psi_{i,j})$ | inverse limit of an inverse system |
| $I_m$ | identity matrix of rank $m$ |
| $N_G(S), \mathrm{Norm}_G(S)$ | normalizer of $S$ in a group $G$ |
| $\mathrm{ord}(g)$ | order of a group element $g$ |
| $\mathrm{Out}(G)$ | group of outer automorphisms of a group $G$ |
| $\Omega_i(G) := \langle g \mid g \in G, g^{p^i} = 1 \rangle$ | omega subgroup of a $p$-group $G$ |
| $\mho^i(G) := \langle g^{p^i} \mid g \in G \rangle$ | agemo subgroup of a $p$-group $G$ |
| $\Phi(G)$ | Frattini subgroup of a group $G$ |
| $S$ | matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ |
| $\mathrm{Stab}_G(x), G_x$ | stabilizer subgroup of $G$ with respect to $x$ |
| $T$ | matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ |
| $Z(G)$ | center of a group $G$ |
| $\mathbb{Z}_{(p)}$ | $p$-adic integers |
| $[x, y] := x^{-1}y^{-1}xy$ | commutator of the group elements $x, y$ |

| | |
|---|---|
| $[G, H]$ | group generated by commutators $[g, h]$ for group elements $g \in G, h \in H$ |
| $\|G\|$ | order of a group or set |
| ${}^g x := g^{-1}xg$ | conjugation of $x$ with $g$ |
| $\#_{\tilde{x}}\ v$ | number of $\tilde{x}$ minus the number of $\tilde{x}^{-1}$ appearing in a word $v \in F_2 = \langle \tilde{x}, \tilde{y} \rangle$ |
| $\equiv_m$ | equivalence modulo a natural number $m$ |

**Commonly used groups**

| | |
|---|---|
| $\mathrm{Alt}(m)$ | alternating group on $m$ letters |
| $C_m$ | cyclic group of order $m$ |
| $\Gamma(m)$ | principal congruence group of level $m$ |
| $D_m$ | dihedral group of order $m$ |
| $D_\infty$ | infinite dihedral group |
| $F_k$ | free group on $k$ generators |
| $\mathrm{GL}(2, R)$ | general linear group over a ring $R$ |
| $\mathrm{PSL}(2, R)$ | projective special linear group over a ring $R$ |
| $\mathrm{PSL}(2, p)$ | projective special linear group $\mathrm{PSL}(2, \mathbb{F}_p)$ |
| $Q_8$ | quaternion group of order 8 |
| $Q_{2^m}$ | generalized quaternion group of order $2^m$ |
| $SD_{2^m}$ | semidihedral group of order $2^m$ |
| $\mathrm{Sp}(2, \mathbb{R})$ | symplectic group |
| $\mathrm{Sym}(m)$ | symmetric group on $m$ letters |
| $\mathrm{SL}(2, R)$ | special linear group over a ring $R$ |
| $T_{(a,b,c)}$ | triangle group with parameters $(a, b, c)$ |
| $(\mathbb{Z}/m\mathbb{Z})^*$ | group of units of $\mathbb{Z}/m\mathbb{Z}$ |

**Definitions of specific groups**

| | |
|---|---|
| $A_m$ | $\langle r, s \mid s^2 = r^{2^m} = 1,\ s^{-1}rs = r^{2^{m-1}-1} \rangle, m > 2$ |
| $B_m$ | $\langle r, s \mid s^2 = r^{2^m} = 1,\ s^{-1}rs = r^{1-2^{m-1}} \rangle, m > 2$ |
| $G^2_{(n,k)} = G_{(n,k)}$ | $\langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1,\ s^{-1}rs = r^{-1} \rangle, 0 \leq k \leq n-2$ |
| $G^p_{(n,k)}$ | $\langle r, s \mid r^{p^{k+1}} = s^{p^{n-k-1}} = 1,\ s^{-1}rs = r^{p+1} \rangle, p > 2$ and $0 \leq k < \frac{n}{2}$ |
| $W_m$ | $\langle x, y \mid x^{2^{m+1}} = y^{2^{m+1}} = x^{2^m} y^{2^m} = x^{-1}yxy = 1 \rangle, m \geq 1$ |

**Geometry**

| | |
|---|---|
| $\mathcal{F}_t$ | Teichmüller flow |
| $(G, x, y)$ | normal origami with deck transformation group $G$ and defining deck tranformations $x, y$ |
| $\mathcal{H}_g(k_1 \times a_1, \ldots, k_m \times a_m)$ | stratum of all translation surfaces of genus $g$ with $k_i$ singularities of multiplicity $a_i + 1$ for $1 \leq i \leq m$ |
| $\mathcal{H}_g(a_1, \ldots, a_m)$ | stratum of all translation surfaces of genus $g$ and with $m$ singularities of multiplicity $a_1, \ldots, a_m$ |

| | |
|---|---|
| $\mathbb{H}$ | upper halfplane |
| $\lambda_i$ | $i$th positive Lyapunov exponent |
| $\mathcal{M}_g$ | moduli space of complex curves of genus $g$ |
| $\mu, \mu_i, \mu_G$ | monodromy map of an origami |
| $\mathrm{mult}(x)$ | multiplicity of a singularity $x$ |
| $\mathrm{ord}(x)$ | order of a singularity $x$ |
| $\mathcal{O}$ | origami |
| $\mathcal{O}(G)$ | set of normal origamis with deck group $G$ |
| $PS_n$ | Penrose stairs origami of degree $n$ |
| $\pi_1(Y)$ | fundamental group |
| $\Sigma$ | set of singularities (of a translation surface) |
| $\mathrm{SL}(X)$ | Veech group of a translation surface $X$ |
| $St_n$ | staircase origami of degree $n$ |
| $St_\infty$ | infinite staircase origami |
| $\mathbb{T}$ | torus obtained by gluing the parallel edges of a unit square |
| $\mathbb{T}^*$ | punctured torus |
| $\mathbb{T}[2]$ | $(2 \times 2)$-torus obtained by gluing the parallel edges of a square with side length 2 |
| $X, (X, \Sigma, \mathcal{A})$ | a translation surface with set of singularities $\Sigma$ and translation atlas $\mathcal{A}$ |

# Chapter 1.

# Introduction

This thesis connects ideas and methods from geometry and group theory. The guiding theme throughout this work is to investigate certain geometric objects with large symmetry groups by using methods from group theory. We briefly discuss the central mathematical objects studied in this thesis: translation surfaces, normal origamis, and Veech groups. After exhibiting the motivation of this thesis and illustrating the main results, we briefly outline the structure of this thesis.

## Central mathematical objects

This thesis chiefly studies a special class of translation surfaces called *normal origamis*. *Translation surfaces* are closed Riemann surfaces with an additional structure given by a translation atlas, a holomorphic 1-form, or certain gluing data. Such surfaces can be constructed from finitely many polygons embedded in the Euclidean plane. These polygons are glued together along pairs of parallel edges by translations. The study of translation surfaces dates back to the pioneering work of Masur, Smillie, and Veech in the 1980s (see [Mas82; Vee89; KMS86]). Over the past 40 years, this study has developed into a fruitful research area with connections to and applications in numerous mathematical areas, such as algebraic geometry, dynamical systems, geometric group theory, low-dimensional topology, and number theory.

The matrix group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on the set of translation surfaces as follows. A matrix acts on a given translation surface by applying the matrix to the edges of the polygons that make up this surface. Since matrices in $\mathrm{SL}(2,\mathbb{R})$ map parallel edges to parallel edges, one obtains a translation surface. We call the stabilizer of a translation surface $X$ under this $\mathrm{SL}(2,\mathbb{R})$-action the *Veech group* of $X$ and we denote this stabilizer $\mathrm{SL}(X)$. Veech groups link the theory of translation surfaces to geometric group theory. Furthermore, these groups are of particular interest because they detect whether the image of the $\mathrm{SL}(2,\mathbb{R})$-orbit of a translation surface induces an algebraic curve in the moduli space of complex curves $\mathcal{M}_g$. Such an algebraic curve is referred to as *Teichmüller curve*. The $\mathrm{SL}(2,\mathbb{R})$-orbit of a translation surface $X$ defines a Teichmüller curve if and only if the Veech group is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Then, the Teichmüller curve is birational to the quotient $\mathbb{H}/\mathrm{SL}(X)$. In other words, the Veech group recognizes the corresponding Teichmüller curve up to birationality. See, e.g., [HS06; Zor06; Wri15] for further information on Veech groups and Teichmüller curves.

This thesis focuses on *origamis*, which are translation surfaces that are constructed by gluing the edges of finitely many unit squares through translations. An origami acquires a combinatorial nature from the gluing data, which can be described by two permutations of the squares that compose it. Moreover, origamis are particularly interesting because the Veech group of each origami is a lattice and thus each origami defines a Teichmüller curve. If we introduce the mild assumption that the origami is reduced, its Veech group is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. As a consequence, the Veech groups of origamis are easier to compute than the Veech groups of general translation surfaces. For example, Schmithüsen presents an algorithm for computing Veech groups of origamis in [Sch04]. See, e.g., [Sch05; HS06; Zor06; Wri15] for further information about origamis.

Investigating a special class of origamis called *p-origamis* with methods from the theory of $p$-groups was the starting point for this thesis. For this, topological covers are a powerful tool because they link the geometry of origamis to group theory. Each origami $\mathcal{O}$ naturally defines a torus cover $\mathcal{O} \to \mathbb{T}$ by sending each of the unit squares to the torus $\mathbb{T}$ (obtained by gluing the parallel edges of a unit square). This cover can be ramified over one point. If the cover is ramified, the preimages of the ramification point can have a cone angle which is larger than $2\pi$. We call such points (with cone angle larger than $2\pi$) *singularities*. Further, call the degree of a cover $\mathcal{O} \to \mathbb{T}$ the *degree* of the origami $\mathcal{O}$. We specifically examine origamis that induce a normal cover. These origamis are called *normal* or *regular origamis* and have a maximal symmetry group. Here, we consider deck transformations as symmetries. If the deck transformation group of a normal origami is a $p$-group, we call the origami a *p-origami*. Prominent $p$-origamis that have been intensively studied are, for instance, the "eierlegende Wollmilchsau" surface (see [HS08; For06; AN20; Möl11]) and certain dihedral origamis, also known as escalator origamis, (see [Zmi11, Section 4.2.2] and [SW17, Example 7]).

# Motivation

The motivation for this thesis is the study of geometric objects with maximal symmetry group. A natural question asks how the symmetry group of a normal origami influences its geometric properties. This thesis considers certain geometric aspects such as the types of singularities and the Veech group of a normal origami.

We mainly focus on $p$-origamis, whose definition originates from origamis such as the "eierlegende Wollmilchsau" surface (see [HS08; For06] and Example 2.2.3), which is a $p$-origami for the prime number $p = 2$. This surface has the quaternion group as deck transformation group and $\mathrm{SL}(2,\mathbb{Z})$ as Veech group. Furthermore, it is one out of two translation surfaces whose $\mathrm{SL}(2,\mathbb{R})$-orbit induces not only a Teichmüller curve in the moduli space of complex curves but also a Shimura curve in the moduli space of abelian varieties. Consequently, its Teichmüller curve has an extraordinary dynamical behavior (see [AN20; Möl11]). The "eierlegende Wollmilchsau" has often served as an important counterexample in the study of translation surfaces. Recently, Apisa and Wright introduced a specific family of origamis generalizing the "eierlegende Wollmilchsau", called *geminal origamis*, and studied $\mathrm{GL}(2,\mathbb{R})$-orbit closures of these origamis (see [AW21a]). Investigating expedient examples of translation surfaces and origamis has played an essential role in the theory of translation surfaces. We discover origamis with interesting properties by studying $p$-origamis.

This thesis is inspired by results that have been achieved studying normal origamis and normal covers of translation surfaces. Normal origamis were studied in the dissertations of Zmiaikou and Kremer (see [Zmi11; Kre09]). Zmiaikou studied normal origamis from a similar perspective by identifying a normal origami with a pair of deck tranformations generating the deck transformation group. He focused on normal origamis whose deck transformation group is, for example, an alternating group. This thesis focuses on normal origamis whose deck transformation groups have prime power order and uses the theory of $p$-groups to study these surfaces. Zmiaikou also investigated the $\mathrm{GL}(2,\mathbb{Z})$-action of specific origamis and examined the relationship between normal origamis and certain origamis that are not normal. Kremer considered normal origamis from another perspective and concentrated on questions motivated from algebraic geometry. For instance, he studied whether there exist normal origamis with isomorphic deck transformation group that define different Teichmüller curves. Kremer answered this question for groups up to order 250. We use his results to derive examples at several points in this thesis.

Further inspiration originates from results that concern either families of normal origamis or generalizations of normal origamis. Herrlich introduced *characteristic origamis*, which are normal origamis whose Veech group equals $\mathrm{SL}(2,\mathbb{Z})$ (see [Her06]). He described a construction that, for each origami $\mathcal{O}$, gives a characteristic origami $\mathcal{O}'$ and a cover $\mathcal{O}' \to \mathcal{O}$. In addition, he studied Teichmüller curves defined by characteristic origamis. In [Bau05], a characteristic origami of degree 108 was examined intensively using similar methods from group theory and covering theory as the methods used in this thesis. A generalization of normal origamis, called *quasi-regular origamis*, was studied by Matheus, Yoccoz, and Zmiaikou in [MYZ14]. Given that normal origamis are also called *regular origamis*, normal origamis are quasi-regular origamis satisfying an additional property. They used elementary representation theory to examine the connection between the automorphism group of an origami and the Lyapunov exponents of a dynamical system related to the origami. In this work, we study the sum of non-negative Lyapunov exponents of normal origamis using a combinatorial formula which was introduced in [EKZ14].

Finally, we refer to the dissertations [Kar20] and [Fin13]. Karg investigated the coarse geometry of countable infinite normal covers of translation surfaces. Such an infinite cover is quasi-isomorphic to a certain Cayley graph of the corresponding deck transformation group. Finster studied finite covers of translation surfaces. She connected the Veech group $\mathrm{SL}(X)$ of a finite translation surface $X$ to the Veech group of finite covers of the surface $X$.

Veech groups have played an essential role in the theory of translation surfaces since the groundbreaking work of Veech [Vee89]. He related properties of the Veech group to properties of trajectories in the context of mathematical billiards in this article. Even though Veech groups have been intensively studied, fundamental questions remain unanswered:

(1) Which Fuchsian groups occur as Veech groups of translations surfaces? (2) When are the Veech groups of translation surfaces congruence subgroups, and when are they far from being congruence subgroups? Partial results have been achieved considering congruence subgroups and origamis. On the one hand, Schmithüsen showed that almost all congruence groups of prime level appear as Veech groups (see [Sch05]). Furthermore, Ellenberg and McReynolds showed in [EM12] that all finite index subgroups of the principal congruence group $\Gamma(2)$ containing the matrix $-I$ occur as a Veech group. On the other hand, Hubert and Lelièvre studied the situation in the stratum $\mathcal{H}(2)$ (see [HL05]). They proved that, except for the Veech group of one origami, all occurring Veech groups are not congruence groups. Weitze-Schmithüsen introduced the *deficiency* of finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$, which measures how far a subgroup is from being a congruence subgroup (see [Wei13]). If the deficiency of a group $\Gamma$ is as large as possible, i.e., $\Gamma$ surjects onto each $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for each natural number $n$, we call $\Gamma$ a *totally non-congruence group*. In [Wei13] and [SW18], properties of subgroups of $\mathrm{SL}(2,\mathbb{Z})$ are presented which imply that these groups are totally non-congruence groups. However, these results do not fully answer the aforementioned questions. In this thesis, we are particularly interested in the question when Veech groups of normal origamis are congruence groups and totally non-congruence groups, respectively.

# Main results

A central topic of this thesis investigates the number and types of singularities of $p$-origamis. For this, we define the *stratum* $\mathcal{H}_g(k_1 \times a_1, \ldots, k_m \times a_m)$ as the set of all translation surfaces of genus $g$ with $k_i$ singularities of multiplicity $a_i + 1$ for $1 \leq i \leq m$. Here, the *multiplicity* $a_i + 1$ of a singularity describes the cone angle at the singularity, which equals $2\pi \cdot (a_i + 1)$. The strata $\mathcal{H}_g(k_1 \times a_1, \ldots, k_m \times a_m)$, $a_i, k_i \in \mathbb{Z}_+$, stratify the space of translation surfaces of genus $g$. We often omit the index $g$. One computes as follows in which stratum a normal origami is contained. Each normal origami is determined by its deck transformation group $G$ and a particularly chosen pair of generators $(x, y)$ of $G$. This description allows us to examine such origamis using group theory. If $G$ has order $d$ and the commutator of $x$ and $y$ has order $a$, then the corresponding origami lies in the stratum $\mathcal{H}(\frac{d}{a} \times (a-1))$ (see Remark 2.2.10).

We prove a precise characterization in which strata $p$-origamis occur. As for many questions in the theory of $p$-groups, we obtain two fundamentally different situations for the even prime 2 and for all odd primes.

**Theorem A (Theorem 3.2.3, Theorem 3.2.7)** *Let $n \in \mathbb{Z}_{\geq 0}$. Then any $p$-origami of degree $p^n$ has either no singularity and genus 1, or lies in one of the following strata:*

- $\mathcal{H}\left(2^{n-k} \times \left(2^k - 1\right)\right)$, *for $1 \leq k \leq n-2$, if $p = 2$,*
- $\mathcal{H}\left(p^{n-k} \times \left(p^k - 1\right)\right)$, *for $1 \leq k < \frac{n}{2}$, if $p > 2$.*

*Moreover, all of these strata occur.*

If we consider an abelian 2-generated $p$-group, the commutator of any pair of generators is trivial. Consequently, all $p$-origamis with abelian deck transformation group are tori and lie in the same stratum. This raises the question which $p$-groups behave in the same manner, i.e., for which $p$-groups $G$ lie all $p$-origamis with deck transformation group $G$ in the same stratum. We prove that far beyond the abelian case, the deck transformation group determines a unique stratum – one which is independent of the choice of the generators $x, y$ – in many more situations.

**Theorem B (Theorem 3.2.11)** *Many deck transformation groups of prime-power order admit only one possible stratum for their p-origamis, including all p-groups G which are regular, of maximal class, powerful, or those whose commutator subgroup $G'$ is regular, powerful, or order-closed. This includes all p-groups of order up to $p^{p+2}$ or of nilpotency class up to p.*

These results on strata of $p$-origamis are achieved using various methods from group theory. First, we derive results on the possible exponent of the commutator subgroup, which contains the commutator $[x, y]$ for each pair of generators $(x, y)$. We then show that these exponents can always be realized as commutator orders of pairs of generators. This yields a complete characterization of the possible orders of the considered commutators for $p$-groups and forms the group-theoretic analogue of Theorem A:

**Theorem C (Proposition 3.1.2, Proposition 3.1.4, Proposition 3.1.5, Proposition 3.1.8)**

1. *For any finite* 2*-group* $G$, $\exp(G') = 1$ *if* $|G| \leq 2$, *or else* $\exp(G') \leq \frac{|G|}{4}$.

2. *For all integers* $n \geq 2$ *and* $0 \leq k \leq n-2$, *there exists a* 2*-generated* 2*-group* $G$ *of order* $2^n$ *with generators* $x, y$ *such that*
$$\mathrm{ord}([x, y]) = \exp(G') = 2^k.$$

3. *For any non-trivial finite p-group G with odd prime p,*
$$\exp(G')^2 < |G|.$$

4. *For any odd prime p and any* $n, k \in \mathbb{Z}_{\geq 0}$ *with* $k < \frac{n}{2}$, *there exists a* 2*-generated p-group G of order* $p^n$ *with generators* $x, y$ *such that*
$$\mathrm{ord}([x, y]) = \exp(G') = p^k.$$

We similarly translate the question, when the deck transformation group determines the stratum of a $p$-origami (as partially answered in Theorem B), into a group-theoretic problem. This geometric phenomenon corresponds to the property of a $p$-group, that the commutator order is a fixed number for all pairs of generators. We call this *property* $(\mathcal{C})$ and we prove with Theorem B that many, but not all $p$-groups have property $(\mathcal{C})$. For this, we consider various well-known classes of $p$-groups such as regular, powerful, order-closed, and power-closed $p$-groups as well as $p$-groups of maximal class. In Section 3.1.2, the precise definitions of these properties are introduced. The proven implications are summarized in the following diagram (Theorem 3.1.29). This diagram forms the group-theoretic basis for Theorem B.

$$\begin{tikzcd}
 & G \text{ maximal class} \arrow[d] & & G \text{ powerful} \arrow[dll] \arrow[d] \\
G \text{ regular} \arrow[r] \arrow[d] & G' \text{ regular} \arrow[d] & & G' \text{ powerful} \arrow[dd, thick] \\
G \text{ order-closed} \arrow[r] \arrow[dd] & G' \text{ order-closed} \arrow[r] \arrow[dd] & G' \text{ weakly order-closed} \arrow[dr, thick] & \\
 & & & \boxed{G \text{ has property } (\mathcal{C})} \\
G \text{ power-closed} \arrow[r] & G' \text{ power-closed} \arrow[r] & G' \text{ weakly power-closed} \arrow[ur, "/" marking] &
\end{tikzcd}$$

Here, the thick arrows on the right represent the new implications proved in this thesis. The other implications are established facts in the theory of $p$-groups.

A rather new branch in the field of translation surfaces studies *infinite translation surfaces*. These surfaces are constructed of countably infinitely many polygons that are glued along pairs of parallel edges by translations (see, e.g., [Ran16; DHV]). Infinite translation surfaces occur for instance as infinite covers of finite translation surfaces (see, e.g., [HS10; HW12; HW13; HHW13]). Karg studied in [Kar20] the geometry of infinite normal covers of finite translation surfaces. In this thesis, we study the singularities of *infinite normal origamis*, which are infinite normal torus covers ramified over one point. We focus on infinite normal origamis with dense subgroups of profinite groups as deck transformation groups. The infinite staircase origami is a well-known example and has the infinite dihedral group as deck transformation group (see [HS10; HHW13]). After generalizing the definition of property $(\mathcal{C})$ to profinite and pro-$p$ groups, we transfer results from the setting of finite $p$-groups to these new situations.

**Theorem D (Proposition 3.3.8)** *A topologically 2-generated pro-$p$ group $\hat{G}$ has property $(\mathcal{C}^{\mathrm{pro}})$ if $\hat{G}'$ is either weakly order-closed or powerful.*

As in the case of finite $p$-groups in Theorem B, the group-theoretic results have a geometric interpretation concerning the singularities of infinite normal origamis.

The second main topic of this thesis centers around Veech groups of normal origamis. We are particularly interested in the following questions: When are the Veech groups of normal origamis congruence subgroups, and when are they far from being congruence subgroups? On the one hand, we study the Veech groups of infinite families of 2-origamis. On the other hand, we prove that certain properties of the deck transformation group imply that the Veech group of the corresponding normal origami is a *totally non-congruence group*, i.e., the Veech group surjects onto $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ for each natural number $n \in \mathbb{Z}_+$.

Starting point for the example series of 2-origamis, that we study, is the class of Penrose stairs origamis with dihedral groups as deck transformation groups. Penrose stairs origamis have been studied in [Zmi11, Section 4.2.2] and [SW17, Example 7] under the names dihedral origamis and escalator origamis, respectively. We determine all 2-origamis with these deck transformation groups and compute their Veech groups. On the basis of dihedral groups, we then construct various infinite families of 2-groups of the form $H_{(m,k)} = C_{2^m} \rtimes C_{2^k}$, e.g., generalized dihedral groups and all semidirect products for $k = 1$. For each of the infinite families of groups $H_{(m,k)}$, we determine all 2-origamis with $H_{(m,k)}$ as deck transformation group and show that the group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on these 2-origamis. Given a 2-origami $\mathcal{O}$ with deck transformation group $H_{(m,k)}$, we therefore obtain the index of the Veech group $\mathrm{SL}(\mathcal{O})$ in $\mathrm{SL}(2,\mathbb{Z})$ as the number of all 2-origamis with deck transformation group $H_{(m,k)}$. Although the group structure of the considered deck transformation groups is very similar, the indices of the corresponding Veech groups differ much. Subsequently, we prove that the Veech groups of the considered origamis are all congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and we compute their congruence level. We conclude the construction of example series by considering an infinite family of 2-groups such that each group has the quaternion group as a quotient. As a consequence, the corresponding 2-origamis are covers of the “eierlegende Wollmilchsau”. The results obtained for this family of origamis resemble the ones for the other example series. We summarize the results on the Veech groups of the 2-origamis under consideration in Table 1.1.

| deck transformation group $G$ | $D_{2m}$ | $A_m$ | $B_m$ | $W_m$ | $G^2_{(n,k)}$ |
|---|---|---|---|---|---|
| range of parameters | $m \in \mathbb{Z}_+$ | $m > 2$ | $m > 2$ | $m > 1$ | $1 \leq k \leq n-3$ |
| order of $G$ | $2m$ | $2^{m+1}$ | $2^{m+1}$ | $2^{2m-1}$ | $2^n$ |
| order of $\mathcal{O}(G)$ | 3 | 6 | $3 \cdot 2^{m-2}$ | $3 \cdot 2^{m-2}$ | $3 \cdot 2^{n-k-3}$ |
| Is $\mathrm{SL}(2,\mathbb{Z})$-action transitive on $\mathcal{O}(G)$? | yes | yes | yes | yes | yes |
| index of Veech group in $\mathrm{SL}(2,\mathbb{Z})$ | 3 | 6 | $3 \cdot 2^{m-2}$ | $3 \cdot 2^{m-2}$ | $3 \cdot 2^{n-k-3}$ |
| Is Veech group a congruence group? | yes | yes | yes | yes | yes |
| Congruence level of Veech group | 2 | 2 | $2^{m-1}$ | $2^m$ | $2^{n-k-2}$ |

Table 1.1.: This table shows some of the results for the families of normal origami studied in Section 4.1.

A connection between translation surfaces and dynamical systems enables us to derive information on the Lyapunov exponents of certain dynamical systems from the results on the considered Veech groups. Lyapunov exponents of the Kontsevich-Zorich cocycle describe the behavior of the Teichmüller flow, which is a dynamical system related to translation surfaces, and have been intensively studied (see, e.g., [Möl11; EKZ11; FMZ11; EKZ14; MYZ14; Esk+18; Api21]). Eskin, Kontsevich, and Zorich introduced a combinatorial formula for the sum of non-negative Lyapunov exponents for origamis in [EKZ14]. We simplify this formula in the case of normal origamis and compute the sum for all families of 2-origamis that we considered previously. For this, we use the results regarding the $\mathrm{SL}(2,\mathbb{Z})$-orbits. Finally, we prove that the sum of non-negative Lyapunov exponents is always an integer for the constructed example series. However, this is an exceptional behavior and we provide an example of a 2-origami where this is not the case. We summarize the results on the sums of non-negative Lyapunov exponents in Table 1.2.

| Deck group of normal origami | range of parameters | sum of non-negative Lyapunov exponents |
|---|---|---|
| $A_m$ | $m > 2$ | $3 \cdot 2^{m-3}$ |
| $B_m$ | $m > 2$ | $2^{m-3} + 1$ |
| $W_m$ | $m \in \mathbb{Z}_+$ | $3^{-1} \cdot (2^{m-1} + 1)$ |
| $G^2_{(n,k)}$ | $\frac{n-3}{2} \leq k \leq n-3$ | $3^{-1} \cdot (2^{n-2} + 2^{2k+3-n})$ |
| $G^2_{(n,k)}$ | $1 \leq k < \frac{n-3}{2}$ | $3^{-1} \cdot (2^{n-2} + 2^{2k+3-n} - 2^{n-2k-2} + 2^2 + \sum_{j=1}^{n-2k-4} 2^{2k+4+j-n})$ |

Table 1.2.: This table shows the sums of non-negative Lyapunov exponents of the normal origamis studied in Section 4.1.2.

As shown in Table 1.1, the Veech groups of all considered 2-origamis are congruence groups. We present conditions on the deck transformation group of a normal origami such that the Veech group is a totally non-congruence group.

**Theorem E (Theorem 4.2.9)** *Let $a, b, c \in \mathbb{Z}_{\geq 0}$ be pairwise coprime and $G = \langle x, y\rangle$ be a finite group with $a = \mathrm{ord}(x), b = \mathrm{ord}(y)$, and $c = \mathrm{ord}(xy)$. The Veech group of the normal origami $(G, y, x)$ is a totally non-congruence group.*

Finally, we study two mathematical objects that are closely related to normal origamis. We first discuss the relationship between the concept of $T_2$-systems from group theory and the $\mathrm{SL}(2,\mathbb{Z})$-orbits on the set of normal origamis $\mathcal{O}(G)$ with fixed deck transformation group $G$. For a finite group $G$, we consider the set $E(G)$ that consists of all pairs of generators of $G$, i.e., $E(G) = \{(x, y) \mid \langle x, y\rangle = G\}$. The $T_2$-*systems* are the orbits on the set $E(G)$ under a group action. Neumann and Neumann introduced $T_2$-systems, and more generally $T_k$-systems for natural numbers $k$, in [NN51]. Recent interest in $T_k$-systems is caused by the connection to the *Product Replacement Algorithm* (PRA) (see, e.g., [Pak01] and [GS09]). The PRA is used to construct random elements in finite groups and was analyzed in [Cel+95].

Given a finite group $G$, denote the number of $T_2$-systems by $n$. We discuss that the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits on the set of normal origamis $\mathcal{O}(G)$ lies between $n$ and $2n$. Using results from literature about $T_2$-systems, we then derive conclusions about the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits on the set of normal origamis with fixed deck transformation group.

At last, we examine a class of translation surfaces that we call *geminal origamis*. These surfaces were introduced in [AW21a] and are origamis with exceptional cylinder decompositions. The study of geminal origamis is related to $\mathrm{GL}(2,\mathbb{R})$-orbit closures of translation surfaces and motivated by work of Mirzakhani and Wright (see [MW18]). A prominent example for a geminal origami is the previously mentioned surface called "eierlegende Wollmilchsau". Apisa and Wright define a family of geminal origamis which generalize the "eierlegende Wollmilchsau" and have higher dimensional $\mathrm{GL}(2,\mathbb{R})$-orbit closures. Geminal origamis define not only a cover of the torus $\mathbb{T}$ but also a cover of the $(2 \times 2)$-torus $\mathbb{T}[2]$ which is ramified over up to four points of $\mathbb{T}[2]$ (see [AW21a]). Given a geminal origami $\mathcal{O}$, Apisa and Wright asked the question whether the induced cover $\mathcal{O} \to \mathbb{T}[2]$ is normal with a cyclic deck transformation group (see [AW21a, Problem 8.16]). We discuss the connection between this geometric question and a group-theoretic question regarding stabilizer subgroups in a symmetric group. Subsequently, we use the group-theoretic framework to prove the following theorem.

**Theorem F (Theorem 6.3.1)** *Let $\mathcal{O}$ be a geminal origami which induces a normal cover $\mathcal{O} \to \mathbb{T}[2]$ then the deck transformation group of this cover is cyclic.*

## 1.1. Outline

This thesis is organized as follows. Chapter 2 provides the mathematical preliminaries for translation surfaces, normal origamis, and Veech groups, which are needed in the subsequent chapters.

In Chapter 3, we explore two questions: (1) In which strata do $p$-origamis occur? (2) Do all $p$-origamis with isomorphic deck transformation groups lie in the same stratum? To answer these questions, we first introduce results on $p$-groups in Section 3.1. Section 3.1.1 contains the results that are used to classify which strata contain $p$-origamis. In Section 3.1.2, we study a group-theoretic property, which we call property $(\mathcal{C})$. A group $G$ has property $(\mathcal{C})$ if and only if all $p$-origamis with deck transformation group $G$ lie in the same stratum. Applying the results from Section 3.1.1 and Section 3.1.2, we answer the aforementioned questions in Section 3.2.1 and Section 3.2.2, respectively. Finally, we discuss infinite normal origamis in Section 3.3. We provide a construction of infinite normal origamis using inverse systems of finite groups and profinite groups. By generalizing property $(\mathcal{C})$ to profinite groups, we transfer some results from Section 3.1.2 and Section 3.2.2 into the setting of infinite normal origamis.

Chapter 4 addresses the study of Veech groups of normal origamis. In Section 4.1, we examine infinite families of 2-origamis with certain deck transformation groups and the Veech groups of these 2-origamis. Section 4.1.1 focuses on the study of the Veech groups of the considered 2-origamis. We then apply these results to compute the sum of non-negative Lyapunov exponents for these normal origamis. Section 4.1.2 presents these results. All normal origamis considered in Section 4.1 have Veech groups that are congruence groups. In Section 4.2, we investigate which properties of deck transformation groups of normal origamis indicate that the corresponding Veech groups are totally non-congruence groups, i.e., they are as far from being a congruence subgroup as possible.

Chapter 5 considers the concept of $T_2$-systems from group theory. The number of $\mathrm{SL}(2,\mathbb{Z})$-

orbits on the set of normal origamis with fixed deck transformation group is closely related to the number of $T_2$-systems. In Section 5.1, we discuss this relationship. In Section 5.2, we use results from literature about $T_2$-systems to derive conclusions about the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits on the set of normal origamis with a fixed deck transformation group.

In Chapter 6, we study geminal origamis. These surfaces exhibit exceptional cylinder decompositions. Furthermore, geminal origamis define not only a cover of the torus $\mathbb{T}$ but also a cover of the $(2\times 2)$-torus $\mathbb{T}[2]$. This cover of $\mathbb{T}[2]$ is ramified over up to four points. In Section 6.1, we explain the connection between a geometric question regarding geminal origamis and a group-theoretic question regarding stabilizer subgroups in a symmetric group. We use the group-theoretic framework to study the case when a geminal origami is a normal cover of $\mathbb{T}[2]$ in Section 6.3.

Chapter 7 identifies an outlook on open questions that arose during the process of writing this thesis and that require additional research.

In the last chapter, we describe code that was used to obtain the results presented in this thesis. The code has been published in [FT20].

## 1.2. Previously published content

Parts of this thesis have been published in the articles [FT20] and [The21].

- Chapter 1 is partially based on the article [FT20].
- Chapter 2 is partially based on the articles [FT20] and [The21].
- Chapter 3 is based on the article [FT20].
- Section 4.2 is based on the article [The21].
- Appendix A is based on the article [FT20].

The article [FT20] has been accepted for publication in Mathematische Nachrichten. This article as well as the content of Chapter 6 result from a collaboration with Johannes Flake. Both, Johannes Flake and I, contributed to these projects likewise.

# Chapter 2.

# Preliminaries

This chapter provides the preliminaries needed in the subsequent chapters of this thesis. We focus on the main concepts regarding translations surfaces, normal origamis, and Veech groups. All results presented in this chapter are well-known facts. Proofs are included, whenever it serves the exposition of this thesis.

We use [HS06], [Zor06], [Wri15], and [Kar20] as main references for well-known facts about translation surfaces. The dissertations [Zmi11] and [Kar20] serve as references on normal origamis. Furthermore, we refer the interested reader to [For91] for background knowledge about covers.

## 2.1. Translation surfaces and origamis

In this section, we introduce fundamental concepts regarding translation surfaces and origamis. There are three equivalent ways to define a translation surface. We discuss two possible definitions in detail because both points of view are relevant for this thesis. We follow the notation from [Kar20] in this section.

Let $\mathcal{P}$ denote a finite set of polygons in the Euclidean plane and let $\mathcal{E}(\mathcal{P})$ denote the edges of the polygons. It is important to fix an orientation of the plane and view the polygons and their edges as embedded objects in the Euclidean plane. Given a bijective map $g\ell : \mathcal{E}(\mathcal{P}) \to \mathcal{E}(\mathcal{P})$ with the property that for each edge $e$ the edges $e$ and $g\ell(e)$ are parallel, of equal length, oppositely oriented, and not equal. We call the map $g\ell$ **gluing map** and two edges $e$ and $g\ell(e)$ **partner edges**. The requirement that partner edges $e$ and $g\ell(e)$ are parallel and of equal length implies that the edges differ by a translation. The requirement that the edges have opposite orientation is necessary to ensure that identifying edges $e$ and $g\ell(e)$ via the corresponding translation in the disjoint union $\biguplus_{P\in\mathcal{P}} P$ defines an oriented surface. More precisely, the gluing map defines a relation on $\mathcal{E}(\mathcal{P}) \times \mathcal{E}(\mathcal{P})$ and the surface $X$ is defined as the quotient space

$$\biguplus_{P\in\mathcal{P}} P / \sim_{g\ell} .$$

We have now introduced the necessary objects to define a translation surfaces.

**Definition 2.1.1**

- Let $\mathcal{P}, \mathcal{E}(\mathcal{P}), g\ell$, and $X$ be as above. If the surface $X$ is connected, we call it a **(finite) translation surface**.

- The cone angle at a point $x \in X$ is $2\pi \cdot a$ for some natural number $a$. The **multiplicity** of $x$ is defined as $a$ and denoted by $\operatorname{mult}(x)$. A **singularity** is a point of $X$ which originates from a vertex of a polygon in $\mathcal{P}$ and has multiplicity larger than 1. Let $\Sigma$ denote the set of singularities. Define the **order** of a singularity $x$ as $\operatorname{mult}(x) - 1$ and denote it by $\operatorname{ord}(x)$.

**Example 2.1.2** The following figure shows a translation surface which is constructed out of one polygon by gluing opposite edges. All vertices of the polygon are identified in the translation surface.

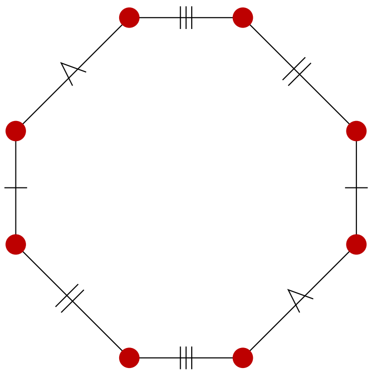

Figure 2.1.: This translation surface has one singularity of order 2 and lies in the stratum $\mathcal{H}(2)$. Edges with the same labels are identified by translations.

Given a translation surface, we consider the surface as unchanged if we cut a polygon in $\mathcal{P}$ along a straight line and glue it back along a partner edge. We call this a cut and glue operation. This operation induces an equivalence relation on the set of translation surfaces. From now on, we consider translation surfaces up to this equivalence relation. Accordingly, we call two translation surfaces $X$ and $X'$ equal if we obtain $X'$ from $X$ by finitely many cut and glue operations. In Figure 2.2, we give an example.

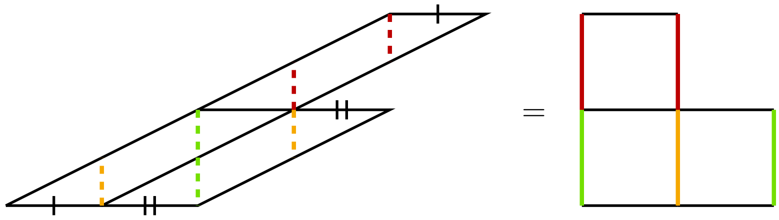

Figure 2.2.: This figure shows two translation surfaces. In both cases, the opposite edges are identified. The right surface is obtained from the left one by cutting along the red and green edges and gluing the parts along the partner edges.

**Definition 2.1.3** Let $a_i$ and $k_i$ be natural numbers for $1 \leq i \leq m$. Then, the **stratum** $\mathcal{H}(k_1 \times a_1, \ldots, k_m \times a_m)$ consists of all translation surfaces with $k_i$ singularities of multiplicity $a_i + 1$ for $1 \leq i \leq m$. We omit the parameter $k_i$ in this notation if it equals 1. Denote the set of translation surfaces without a singularity, i.e., the set of tori, by $\mathcal{H}(0)$.

**Remark 2.1.4** If we allow countable infinitely many polygons in the definition of a translation surface, we obtain **infinite translation surfaces**. In this case, the quotient space $\biguplus_{P \in \mathcal{P}} P/\sim_{g\ell}$ can have two further types of singularities which are called **infinite angle singularities** and **wild singularities**. For more details, see, e.g., [Kar20] or [Ran16]. We focus on the study of finite translation surfaces in this thesis and consider infinite origamis only in Section 3.3. Unless stated otherwise, we assume that a translation surface is finite.

The Euclidean metric on the plane $\mathbb{R}^2$ induces a metric on $X \setminus \Sigma$ which can be completed to a metric on $X$. By this, we obtain notions as directions, length, and geodesic on translation surfaces inherited from the ones on the Euclidean plane.

**Definition 2.1.5** Let $X$ be a translation surface. A **saddle connection** of $X$ is a geodesic with singularities as start and end point that does not contain a singularity in its interior.

Instead of defining the translation structure of a translation surface via gluing data of polygons, we can also define the translation structure using a translation atlas. We start with a connected surface $X$. Consider an atlas $\mathcal{A}$ of $X$ with charts $\varphi_i : U_i \to \mathbb{R}^2$. In particular, the sets $(U_i)_{i \in \mathbb{Z}_+}$ form an open cover of $X$ and the maps $\varphi_i : U_i \to \varphi_i(U_i)$ are homeomorphisms. We call the atlas a **translation atlas** if the transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are locally translations, i.e., of the form $x \mapsto x + b$ for some $b \in \mathbb{R}^2$. The following definition is equivalent to Definition 2.1.1.

**Definition 2.1.6** A **translation surface** is a tuple $(X, \Sigma, \mathcal{A})$, where $\Sigma$ is the smallest discrete subset of a compact surface $X$ such that $\mathcal{A}$ is a maximal translation atlas on $X \setminus \Sigma$.

Pulling back the Euclidean metric of $\mathbb{R}^2$, we obtain a metric on $X$ which coincides with the metric which was introduced in the polygon construction. The singularities are the elements of the set $\Sigma$. Two translation surfaces $(X, \Sigma, \mathcal{A})$ and $(X', \Sigma', \mathcal{A}')$ are **equivalent** if there exists a homeomorphism $\psi : X \to X'$ which induces translations in the charts and satisfies $\psi(\Sigma) = \Sigma'$. This induces an equivalence relation on the set of translation surfaces which coincides with the equivalence relation defined via cut and glue operations in the polygon construction. Recall that we consider translation surfaces only up to equivalence. From now on, we often denote the translation surface by $X$ and omit the tuple $(X, \Sigma, \mathcal{A})$.

Equivalently, one can define a translation surface as a closed Riemann surface together with a (non-trivial) holomorphic one-form. Here, the singularities are the zeros of the one-form and the order of a singularity coincides with the order of the corresponding zero. We refer the interested reader to [DHV, Chapter 1] for further details on this point of view and a proof of the fact that all three definitions are indeed equivalent.

We now turn toward certain maps between translation surfaces.

**Definition 2.1.7** An **affine homeomorphism** $\alpha : X \to X'$ between two translation surfaces is a homeomorphism respecting the sets of singularities, i.e., $\alpha(\Sigma) = \Sigma'$. In addition, it is locally of the form $x \mapsto Ax + b$ for $x \in X \setminus \Sigma$ and some $A \in \mathrm{GL}(2, \mathbb{R}), b \in \mathbb{R}^2$.

Let $\alpha$ be an affine homeomorphism as in the definition above. The matrix $A$ is independent of local coordinates. It is called the **derivative** of $\alpha$ and denoted by $D\alpha$. See, e.g., [Sch05, Section 1.3] for further information on the derivative map.

**Definition 2.1.8** A **translation** $\alpha : X \to X'$ between two translation surfaces is an affine homeomorphism whose derivative $D\alpha$ equals the identity matrix. If a topological cover of translation surfaces is locally a translation, i.e., of the form $x \mapsto x + b$ for $b \in \mathbb{R}^2$, we call it a **translation cover**.

This thesis focuses on the study of translation surfaces which can be constructed from finitely many unit squares. Such surfaces are called origamis or square-tiled surface.

**Definition 2.1.9** A translation surface that is constructed from finitely many unit squares is called an **origami** or a **square-tiled surface**. The **degree** of an origami is defined as the number of squares it is constructed of.

Each origami $\mathcal{O}$ defines naturally a translation cover $\mathcal{O} \to \mathbb{T}$ of the torus $\mathbb{T}$ by sending each unit square to $\mathbb{T}$. This cover is at most ramified over one point denoted by $\infty$ and the singularities are preimages of the point $\infty$. Note that the degree of an origami equals the degree of the corresponding torus cover.

**Example 2.1.10** The following origami induces a torus cover of degree 4. Edges with the same labels are identified.

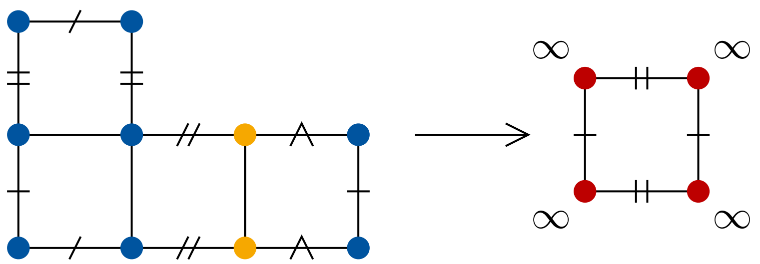


Figure 2.3.: An origami with one singularity of order 2, i.e., it lies in the stratum $\mathcal{H}(2)$.

Consider two origamis $\mathcal{O}_i$ and the induced covers $c_i : \mathcal{O}_i \to \mathbb{T}$ for $1 \leq i \leq 2$. In topology, these covers are called **equivalent** if there exists a homeomorphism $\alpha : \mathcal{O}_1 \to \mathcal{O}_2$ such that $c_1 = c_2 \circ \alpha$. As $c_1$ and $c_2$ are translation covers, the homeomorphism $\alpha$ is a translation and the origamis are equal.

## 2.2. Normal origamis and $p$-origamis

The main goal of this thesis is to consider origamis with maximal symmetry group and to examine the connection between properties of the symmetry group and geometric properties of origamis in this case. In our context, we consider deck transformations as symmetries.

Let $c : \mathcal{O} \to \mathbb{T}$ be the cover induced by an origami $\mathcal{O}$ of degree $d$. The **deck transformation group** of the origami consists of all homeomorphisms $\alpha : \mathcal{O} \to \mathcal{O}$ such that $c \circ \alpha = c$.

**Definition 2.2.1**

- An origami $\mathcal{O}$ is called **normal** (or **regular**) if the cover $c : \mathcal{O} \to \mathbb{T}$ is normal, i.e., the deck transformation group acts transitively on the squares the origami $\mathcal{O}$ consists of.
- Let $p$ be a prime number. An origami is called $p$**-origami** if it is normal and the deck transformation group of the corresponding cover is a finite $p$-group.

### 2.2.1. Monodromy maps

The concept of monodromy maps will be essential in the course of this thesis. It relates properties of a normal origami to group-theoretic properties of its deck transformation group. We consider the corresponding unramified cover of the once punctured torus $c^* : \mathcal{O}^* \to \mathbb{T}^*$, where $\mathcal{O}^* = \mathcal{O} \setminus c^{-1}(\infty)$ and $\mathbb{T}^* = \mathbb{T} \setminus \{\infty\}$. Recall that the fundamental group $\pi_1(\mathbb{T}^*)$ is the free group $F_2$ on two generators. We choose a base point $q$ on $\mathbb{T}^*$ which lies in the interior of the unit square from which $\mathbb{T}^*$ is constructed and not on an edge of this unit square. Label the preimages of $q$ (under $c$) by $q_1, \ldots, q_d$. In particular, in each square lies exactly one point $q_i$ for $1 \leq i \leq d$. Label this square by $s_i$. Denote the simple closed horizontal and vertical curve on $\mathbb{T}^*$ passing through $q$ by $a$ and $b$, respectively. These two curves generate the fundamental group $F_2$. The **monodromy map** $\mu : F_2 \to \mathrm{Sym}(d),\ w \mapsto \sigma_w^{-1}$ is a homomorphism defined as follows: A word $w \in F_2$ describes a path $f$ on $\mathbb{T}^*$. For $1 \leq i \leq d$, one obtains a lifted path $\tilde{f}_i$ of $f$ on $\mathcal{O}^*$ starting at the point $q_i$. Set $\sigma_w(i) := j$, if the path $\tilde{f}_i$ ends at the point $q_j$. This defines a permutation $\sigma_w \in \mathrm{Sym}(d)$. To obtain a homomorphism, one needs to invert this permutation in the definition of the monodromy map $\mu$. The origami $\mathcal{O}$ is determined by the permutations $\sigma_a$ and $\sigma_b$. These permutations describe the horizontal and vertical gluing of the $d$ squares. Note that $\sigma_a$ and $\sigma_b$ depend on the numbering of the $d$ squares. Renumbering the squares corresponds to conjugating $\sigma_a$ and $\sigma_b$ simultaneously with a permutation in $\mathrm{Sym}(d)$.

Let $c : \mathcal{O} \to \mathbb{T}$ be the cover induced by a normal origami and denote the deck transformation group of $\mathcal{O}$ by $G$. We can modify the codomain of the monodromy map in this case. For this, we fix a preimage of $q$ under the covering map, say $q_1$. Note that the degree of the cover is the order of the deck transformation group and for each point $q_i$ there is a unique deck transformation sending the point $q_1$ to $q_i$. We define the map $\mu_G : F_2 \to G, w \mapsto g_w$. Here, $g_w$ is the unique deck transformation mapping the point $q_1$ to the end point of the lift of $w$ to $\mathcal{O}$ starting at $q_1$. We do not invert $g_w$ in the definition of $\mu_G$ as in the definition of $\mu$ because the deck transformation group $G$ of the cover $\mathcal{O} \to \mathbb{T}$ is anti-isomorphic to the quotient of the corresponding fundamental groups. Choosing, instead of $q_1$, a different preimage of $q$ as reference point corresponds to simultaneous conjugation with a deck transformation in the definition of $\mu_G$. For further details, see, e.g., [Kar20, Sections 2.6 and 3.2].

**Lemma 2.2.2 (see, e.g., [Zmi11, Section 4.1] or [Kar20, Sections 3.2 and 3.3])** *The following holds:*

*(i) A finite 2-generator group $G$ together with an (ordered) pair of generators of the group $G$ determines a normal origami with deck transformation group $G$.*

*(ii) A normal origami is uniquely determined by its deck transformation group $G$ and the two deck transformations $\mu_G(a)$ and $\mu_G(b)$.*

**Proof** (i) Given a 2-generator group $G$ of order $d$ together with generators $x$ and $y$, we can construct a normal origami of degree $d$ as follows. Take $d$ squares labeled by the group elements. The right and upper neighbor of a square labeled by $g$ in $G$ is the one with label $gx$ and $gy$, respectively. This construction data defines an origami of degree $d$ with deck transformation group $G$. Since $x$ and $y$ generate $G$, the group acts transitively on the squares and thus the cover is normal. For further details on this construction, see [Kar20, Section 3.2].

(ii) Given a normal origami $\mathcal{O}$ with deck transformation group $G$ and the induced cover $\mathcal{O} \to \mathbb{T}$. Consider the deck transformations $x := \mu_G(a)$ and $y := \mu_G(b)$. These deck transformations represent passing from the square labeled by the identity element of $G$ to its right and upper neighbor, respectively. The deck transformation group $G$ is generated by $x$ and $y$. The procedure described in (i) reconstructs the origami $\mathcal{O}$ from the data $(G, x, y)$ (up to equivalence). ■

**Example 2.2.3** We consider the quaternion group

$$Q_8 := \left\langle i, j, k \mid i^2 = j^2 = k^2 = ijk,\ i^4 = 1 \right\rangle$$

with $(i, j)$ as the pair of generators. The group $Q_8$ can be viewed as a group of units in the quaternion division algebra, where $ijk = -1$. Following the construction described in Lemma 2.2.2 (i), we obtain the 2-origami $\mathcal{W}$ with deck transformation group $Q_8$ shown in Figure 2.4. This origami is called **“eierlegende Wollmilchsau”**. It is a well-known and extensively studied example (see [HS08; For06]).

Recall that origamis $c_1 : \mathcal{O}_1 \to \mathbb{T}$ and $c_2 : \mathcal{O}_2 \to \mathbb{T}$ are equivalent if there exists a homeomorphism $\alpha : \mathcal{O}_1 \to \mathcal{O}_2$ such that $c_1 = c_2 \circ \alpha$. Further, we consider equivalent origamis as equal. It is natural to ask when different pairs of generators of a given group describe the same origami. The following lemma answers this question.

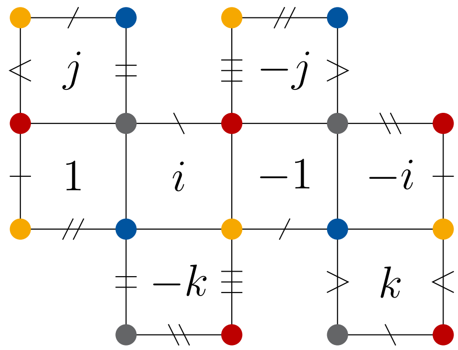


Figure 2.4.: The “eierlegende Wollmilchsau” is a 2-origami of degree 8 with 4 singularities of cone angle $4\pi$. It lies in the stratum $\mathcal{H}(4 \times 1)$.

**Lemma 2.2.4 (see, e.g., [Zmi11, Lemma 4.2] or [Kar20, Lemma 2.4])** *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be normal origamis with deck transformation group $G$ defined by pairs of generators $(x_1, y_1)$ and $(x_2, y_2)$, respectively. Let $\mu_i : F_2 \to G$ with $\mu_i(a) = x_i$ and $\mu_i(b) = y_i$ denote the monodromy maps of $\mathcal{O}_i$ for $i = 1, 2$. Then the following are equivalent*

*(i) the origamis $\mathcal{O}_1$ and $\mathcal{O}_2$ are equal,*

*(ii) the kernels of the monodromy maps $\mu_1$ and $\mu_2$ are equal,*

*(iii) there exists a group automorphism $\varphi : G \to G$ such that $(\varphi(x_1), \varphi(y_1)) = (x_2, y_2)$.*

**Proof** We begin by showing that (i) implies (ii). Let $\alpha : \mathcal{O}_1 \to \mathcal{O}_2$ be a homeomorphism such that $c_1 = c_2 \circ \alpha$. Recall that an element $w \in F_2$ defines a path $f$ on $\mathbb{T}^*$ passing through a chosen base point $q_1$. If $w$ lies in the kernel of $\mu_1$, each lift $\tilde{f}_i$ on $\mathcal{O}_1^*$ of $f$ starts and ends at the same preimage of $q_1$. Further, each of the paths $\tilde{f}_i$ on $\mathcal{O}_1^*$ induces a path $\alpha(\tilde{f}_i)$ which also has the same start and end point. This implies that $w$ lies in the kernel of $\mu_2$. Hence, we obtain the inclusion $\ker(\mu_1) \subseteq \ker(\mu_2)$. Using the inverse $\alpha^{-1}$, a similar argument shows the inclusion $\ker(\mu_2) \subseteq \ker(\mu_1)$.

Now we show that (ii) implies (iii). Suppose that the kernels of $\mu_1$ and $\mu_2$ are equal. We want to define a group isomorphism $\varphi : G \to G$ such that the following diagram commutes

$$\begin{tikzcd} F_2 \arrow[r, "\mu_1"] \arrow[d, "\mathrm{id}"'] & G \arrow[d, dashed, "\varphi"] \\ F_2 \arrow[r, "\mu_2"] & G. \end{tikzcd}$$

Let $K$ denote the kernel $\ker(\mu_1)$. Since the kernels of $\mu_1$ and $\mu_2$ coincide, we obtain isomorphisms $\overline{\mu}_i : F_2/K \to G$ with $\overline{\mu}_i(\overline{a}) = x_i$ and $\overline{\mu}_i(\overline{b}) = y_i$ for $1 \leq i \leq 2$. Define $\varphi$ as the composition $\overline{\mu}_2 \circ (\overline{\mu}_1)^{-1}$. Then $\varphi$ is an isomorphism such that $(\varphi(x_1), \varphi(y_1)) = (x_2, y_2)$.

Finally, we assume statement (iii). Let $\varphi : G \to G$ be an automorphism such that $(\varphi(x_1), \varphi(y_1)) = (x_2, y_2)$. Sending a square in the tiling of $\mathcal{O}_1$ labeled by the deck transformation $g$ to the square in the tiling of $\mathcal{O}_2$ labeled by $\varphi(g)$ defines a map $\alpha : \mathcal{O}_1 \to \mathcal{O}_2$.

Observe that $\alpha$ maps neighboring squares in $\mathcal{O}_1$ to neighboring squares in $\mathcal{O}_2$ and thus $\alpha$ is a well-defined homeomorphism. The equation $c_1 = c_2 \circ \alpha$ implies the equality of the origamis. ∎

**Remark 2.2.5** For a finite 2-generated group $G$, we consider the set

$$E(G) := \{(x, y) \mid \langle x, y\rangle = G\} \subseteq G \times G.$$

The automorphism group of $G$ acts on $E(G)$ via $\varphi \circ (x, y) := (\varphi(x), \varphi(y))$ for $\varphi \in \operatorname{Aut}(G)$ and $(x, y) \in E(G)$. Note that the stabilizer of each pair $(x, y) \in E(G)$ under this action is trivial. By Lemma 2.2.4, two normal origamis $(G, x, y)$ and $(G, x', y')$ are equal if and only if there exists an automorphism $\varphi \in \operatorname{Aut}(G)$ with $\varphi \circ (x, y) = (x', y')$. Hence, the set of cosets $E(G)/\operatorname{Aut}(G)$ coincides with the set of all normal origamis with deck transformation group $G$. We denote the set of normal origamis with deck transformation group $G$ by $\mathcal{O}(G)$.

**Example 2.2.6** Note that the automorphism group of the quaternion group $Q_8$ is isomorphic to the symmetric group $\operatorname{Sym}(4)$ and thus has order 24. By the following argument, we further know that there are exactly 24 pairs of generators $(x, y)$ in $E(Q_8)$. The restriction on $x, y$ is that $\{x, y\} \subseteq \{\pm i, \pm j, \pm k\}$ is a subset of size two not containing $\pm a$ for $a$ in $Q_8$. It is easy to check that these sets $\{x, y\}$ generate $Q_8$ and that $\pm 1$ cannot be contained in a generating set of size 2.

Since the stabilizer of each pair $(x, y) \in E(Q_8)$ under the action of the automorphism group $\operatorname{Aut}(Q_8)$ is trivial by Remark 2.2.5, the group $\operatorname{Aut}(Q_8)$ acts transitively on $E(Q_8)$. Hence, for any two pairs of generators $(x_1, y_1)$ and $(x_2, y_2)$, we find an automorphism $\varphi$ of $Q_8$ such that $(\varphi(x_1), \varphi(y_1))$ is equal to $(x_2, y_2)$. By Lemma 2.2.4, we obtain that the only 2-origami with deck transformation group $Q_8$ is the "eierlegende Wollmilchsau" (see Example 2.2.3).

From now on, we denote a normal origami with deck transformation group $G$ defined by the pair of generators $(x, y)$ by $(G, x, y)$.

**Example 2.2.7** For an even number $n$, we define the origami $PS_n$ of degree $n$ by the permutations

$$\sigma_a = (1, 2)(3, 4) \ldots (n - 1, n) \text{ and } \sigma_b = (2, 3)(4, 5) \ldots (n - 2, n - 1)(n, 1).$$

Here, the permutations $\sigma_a$ and $\sigma_b$ describe the horizontal and vertical gluing of the $n$ squares, respectively. We call such an origami a **Penrose stairs origami** as it resembles the infinite staircase known from work of Penrose [PP58] and a lithograph of Escher called *Klimmen en dalen*. The origami $PS_n$ is a normal origami with the dihedral group

$$D_n = \langle r, s \mid r^{n/2} = s^2 = 1, s^{-1}rs = r^{-1}\rangle$$

of order $n$ as deck transformation group and the pair of generators $(s, sr)$. Penrose stairs origamis have been studied in [Zmi11, Section 4.2.2] and [SW17, Example 7] under the

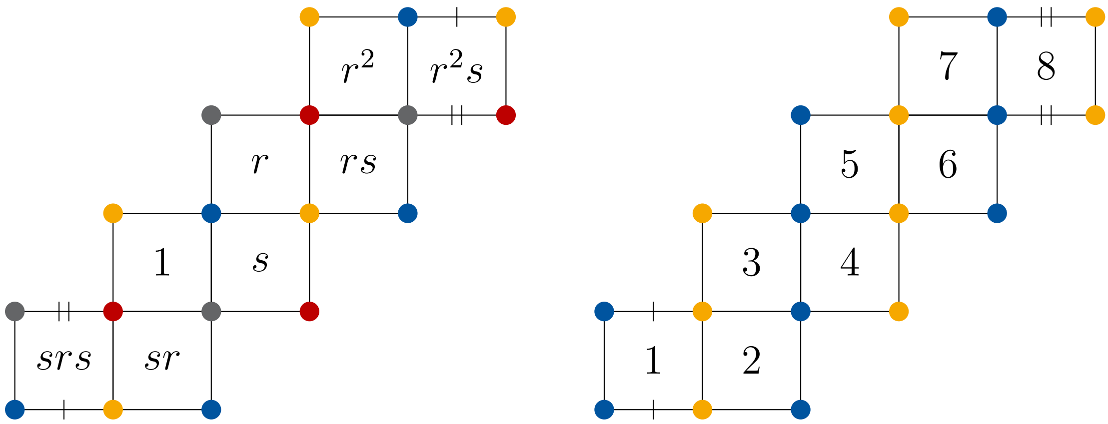


Figure 2.5.: While the Penrose stairs origami $PS_8 = (D_8, s, sr)$ has four singularities with cone angle $4\pi$, the origami $St_8$ has two singularities with cone angle $8\pi$. Opposite sides are identified, unless marked otherwise.

name dihedral origamis and escalator origamis, respectively. In Figure 2.5, the Penrose stairs origami $PS_8$ is shown.

Note that the family of Penrose stairs origamis is similar to the well-studied family of stair origamis (see, e.g., [Sch05, Section 5.3]). For an even number $n$, define the permutation $\sigma_b'$ as

$$(1)(2,3)(4,5)\dots(n-2,n-1)(n).$$

The stair origami $St_n$ defined by the permutations $\sigma_a$ and $\sigma_b'$ is not normal. We will consider in Figure 3.5 and Example 3.3.1 an infinite version of the Penrose stairs origamis called infinite staircase origami. This infinite origami can be viewed as a limit of certain Penrose stairs origamis $PS_n$ and has been intensively studied, for instance, in [HHW13] and [HS10].

### 2.2.2. Orders of singularities

The fact that normal origamis have maximal symmetry groups leads to restrictions on the number of singularities of a normal origami and the orders of these singularities. These restrictions can be phrased in terms of properties of the deck transformation group. We begin with a remark showing that all singularities of a normal origami have the same multiplicity. That gives a first restriction on the strata which contain normal origamis.

**Remark 2.2.8** The deck transformation group of a normal origami acts transitively on the squares of the origami. Hence, all singularities have the same multiplicity, and lie in a stratum of the form $\mathcal{H}(0)$ or $\mathcal{H}(k \times a)$ for $a, k \in \mathbb{Z}_+$. Further, a normal origami with deck transformation group $G$ and set of singularities $\Sigma$ with $s \in \Sigma$ satisfies the following equation

$$\sum_{s' \in \Sigma} \text{mult}(s') = |\Sigma| \cdot \text{mult}(s) = |G|.$$

In particular, we note that either all corners of the squares of a normal origami are singularities or all corners of the squares are regular points.

With the help of the next lemma we connect the multiplicity of singularities to a statement phrased in the language of group theory. For elements $x, y$ of a group, we denote their commutator $x^{-1}y^{-1}xy$ by $[x, y]$.

**Lemma 2.2.9 (see, e.g., [Zmi11, Section 4.1])** *Let $\mathcal{O} = (G, x, y)$ be a normal origami. The cover of the torus induced by the origami is unramified if and only if $[x, y] = 1$. In particular, all normal origamis with abelian deck transformation group lie in the stratum $\mathcal{H}(0)$. If the cover is ramified, the multiplicity of each singularity of $\mathcal{O}$ coincides with the order of $[x, y]$ in $G$.*

**Proof** Let $S$ denote the square labeled by the group element 1. Then the deck transformation $[x, y] = x^{-1}y^{-1}xy$ sends the square $S$ to the one lying $2\pi$ (with respect to the lower left corner of $S$) above $S$.

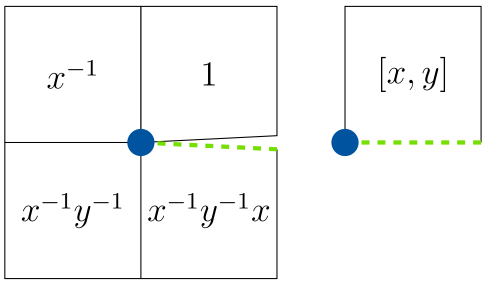


Figure 2.6.: The deck transformation $[x, y]$ maps the square labeled by 1 to the square labeled by the commutator $[x, y]$.

Hence, the deck transformation $[x, y]^m$ sends the square $S$ to the one lying $2\pi m$ above $S$ for $m \in \mathbb{Z}_+$. We conclude with Remark 2.2.8 that the cone angle at each corner is $2\pi\cdot$ ord$([x, y])$. ∎

**Remark 2.2.10** By the lemma above and Lemma 2.2.2, finding a normal origami of degree $d = (a+1)\cdot k$ in the stratum $\mathcal{H}(k \times a)$ is equivalent to finding a 2-generated group of order $d$ and a generating set of size two such that the commutator of the generators has order $a + 1$.

In Chapter 3, we will study in which strata $p$-origamis occur. For this, we will use the following connection between the deck transformation group and the type of singularities of normal $p$-origamis to derive conclusions about the stratum they lie in.

**Remark 2.2.11** By Remark 2.2.8, all singularities of a $p$-origami have the same multiplicity. Each $p$-origami outside the stratum $\mathcal{H}(0)$ satisfies the equation $d = (a + 1) \cdot k$, where $d$ is the degree, $a + 1$ is the multiplicity of each singularity, and $k$ is the number of singularities. This reduces the possible strata significantly. All $p$-origamis of degree $p^n$ (outside the stratum $\mathcal{H}(0)$) have $p^{n-k}$ singularities of multiplicity $p^k$ for $1 \leq k \leq n$, i.e., they lie in strata of the form $\mathcal{H}\left(p^{n-k} \times \left(p^k - 1\right)\right)$.

**Example 2.2.12** The Penrose stairs origami $PS_{2^m} = (D_{2^m}, s, sr)$ is a 2-origami with four singularities of order $2^{m-2}$ because the commutator $[s, sr] = r^2$ has order $2^{m-2}$. In Figure 2.5, we considered the origami $PS_8$.

We end this section by stating the famous Gauß-Bonnet formula in the context of translation surfaces. It enables us to compute the genus of a translation surface given the number of singularities and their orders. For a geometric proof, see, e.g., [Ran16, Proposition 1.14].

**Proposition 2.2.13 (Gauß-Bonnet formula)** *For a translation surface of genus $g$ that lies in the stratum $\mathcal{H}(k_1 \times a_1, \ldots, k_m \times a_m)$, the following equation holds*

$$g = 1 + \frac{1}{2}\sum_{i=1}^{m} k_i \cdot a_i.$$

## 2.3. Veech groups

The matrix group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on the Euclidean plane by matrix multiplication, i.e., a matrix $A$ sends a vector $v$ to the vector $A \cdot v$. Note that this action preserves parallelism. Geometrically, the matrices act, for instance, by shearing and rotating vectors. This $\mathrm{SL}(2,\mathbb{R})$-action induces an action on the translation surfaces in a given stratum $\mathcal{H}(k_1 \times a_1, \ldots, k_m \times a_m)$ by applying a matrix to the edges of the polygons the respective surface is constructed of. One can also use the description as a Riemann surface together with a translation atlas to define this group action: Given a translation surface $(X, \Sigma, \mathcal{A})$ and a matrix $M \in \mathrm{SL}(2,\mathbb{R})$, we obtain the translation surface $(X, \Sigma, \mathcal{A}_M)$. Here, the translation atlas $\mathcal{A}_M$ is defined by composing each chart in the atlas $\mathcal{A}$ with the linear map $z \mapsto M \cdot z$.

**Example 2.3.1** In Figure 2.7, we consider a matrix that stabilizes a translation surface. Using cut and glue operations, we see that both surfaces on the right side are equal (see Figure 2.2).

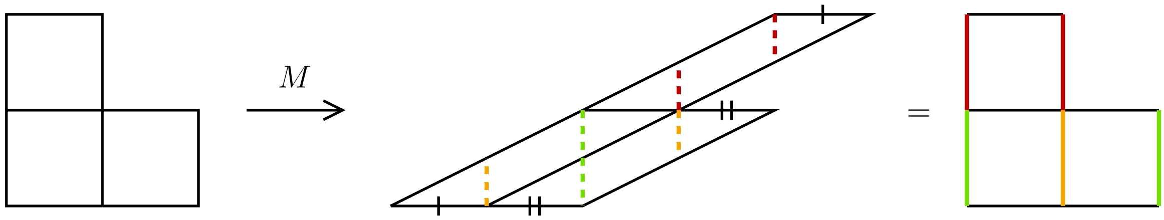


Figure 2.7.: The action of the matrix $M = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ on an origami. Opposite sides are identified unless marked otherwise.

**Definition 2.3.2** The **Veech group** of a translation surface $X$ is the stabilizer under the $\mathrm{SL}(2,\mathbb{R})$-action described above and is denoted by $\mathrm{SL}(X)$.

Equivalently, the Veech group can be defined as the image of the orientation preserving affine homeomorphisms on the translation surface under the derivative map (see, e.g., [Sch05, Section 1.3]). A main goal when studying translation surfaces is the description of their $\mathrm{SL}(2,\mathbb{R})$-orbits and -stabilizers. The $\mathrm{SL}(2,\mathbb{R})$-orbits of certain translation surfaces, e.g., origamis, define complex curves in a moduli space of curves $\mathcal{M}_g$. These curves are called Teichmüller curves and are birationally equivalent to the quotient of the upper halfplane by the Veech groups. For a detailed introduction about Veech groups we refer the reader to [HS04], [Sch05], and [Zor06].

### 2.3.1. Veech groups of origamis

In this section, we discuss special properties that hold for Veech groups of origamis. We begin by introducing the notion of reduced origamis. For this, we need the following way of assigning a vector in $\mathbb{R}^2 \cap \mathbb{Z}^2$ to a saddle connection of an origami. Given an origami $\mathcal{O}$ and a saddle connection starting at a singularity $s$, we place the origin of $\mathbb{R}^2$ in the point $s$. The direction and the length of the geodesic define a vector $v$ in $\mathbb{R}^2$ starting at the origin. Since the origami consists of unit squares and all singularities are corners of these squares, the vector $v$ lies in $\mathbb{Z}^2$. We say that the **saddle connections** of an origami **span** $\mathbb{Z}^2$ if the vectors corresponding to the saddle connections span $\mathbb{Z}^2$.

**Definition 2.3.3** An origami is **reduced** if the saddle connections span $\mathbb{Z}^2$.

Note that reduced origamis cannot be tiled by larger squares. Veech groups of reduced origamis are finite index subgroups of the group $\mathrm{SL}(2,\mathbb{Z})$ (see [GJ00, Theorem 5.5]). Thus, it is sufficient to consider the action of the matrix group $\mathrm{SL}(2,\mathbb{Z})$ instead of the $\mathrm{SL}(2,\mathbb{R})$-action. Normal origamis of genus $g > 1$ are always reduced. Therefore, we assume from now on that all origamis under consideration are reduced unless stated otherwise.

The special linear group $\mathrm{SL}(2,\mathbb{Z})$ is 2-generated, e.g., by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Geometrically, the action of $S$ corresponds to a rotation by $\frac{\pi}{2}$, while the action of $T$ corresponds to shearing the squares in the tiling of an origami by $T$. Alternatively, the action of these matrices can be also described with the help of the monodromy map. See [Wei13] for further information. Given an origami $\mathcal{O}$ defined by permutations $\sigma_a$ and $\sigma_b$, the permutations $\sigma_b^{-1}$ and $\sigma_a$ define the origami $S \cdot \mathcal{O}$. The permutations $\sigma_a$ and $\sigma_b\sigma_a^{-1}$ define the origami $T \cdot \mathcal{O}$. For a normal origami $(G,x,y)$, the actions of the matrices $S$, $S^{-1}$, $T$, and $T^{-1}$ are given by

$$\begin{aligned} S \cdot (G,x,y) &= (G,y^{-1},x), & S^{-1} \cdot (G,x,y) &= (G,y,x^{-1}), \\ T \cdot (G,x,y) &= (G,x,yx^{-1}), & T^{-1} \cdot (G,x,y) &= (G,x,yx). \end{aligned}$$

In particular, the $\mathrm{SL}(2,\mathbb{Z})$-action restricts to an action on the set of normal origamis with a fixed deck transformation group, i.e., on the cosets $E(G)/\operatorname{Aut}(G)$.

**Example 2.3.4** Example 2.2.6 shows that there is only one normal origami with deck transformation group $Q_8$. Hence, each matrix in $\mathrm{SL}(2,\mathbb{Z})$ stabilizes the "eierlegende Wollmilchsau" $\mathcal{W}$ and the Veech group $\mathrm{SL}(\mathcal{W})$ is $\mathrm{SL}(2,\mathbb{Z})$ (see [HS10, Proposition 2]).

## 2.3.2. Cylinder decompositions and parabolic matrices

The concept of cylinder decompositions is crucial at various points in this thesis. For instance, to construct parabolic elements in Veech groups and to study geminal origamis. In the subsequent definition, we follow the notation from [Ran16].

**Definition 2.3.5**

- A **cylinder** on a translation surface is an open subsurface that is isometric to a Euclidean cylinder $\mathbb{R}/w\mathbb{Z} \times (0,h)$ for $w,h > 0$. One calls $w$ the **circumference**, $h$ the **height**, and the quotient $\frac{h}{w}$ the **modulus** of the cylinder.
- If the genus of the translation surface is larger than one, a **maximal cylinder** (with respect to inclusion) is bounded by saddle connections. The direction of a saddle connection bounding a cylinder is called the **direction of the cylinder**.
- A **cylinder decomposition** is a collection of pairwise disjoint cylinders such that the union of their closures covers the whole surface. The directions of the cylinders in a cylinder decomposition coincide and this direction is called the **direction of the cylinder decomposition**.

**Example 2.3.6** Let $D_8 = \langle r,s \mid r^4 = s^2 = 1,\ s^{-1}rs = r^{-1}\rangle$ denote the dihedral group of order 8. Figure 2.8 shows the cylinder decompositions of the origami $(D_8,r,s)$ in horizontal and vertical direction. In horizontal direction, the decomposition consists of two cylinders of circumference 4 and height 1. The cylinder decomposition in vertical direction consists of four cylinders of circumference 2 and height 1.

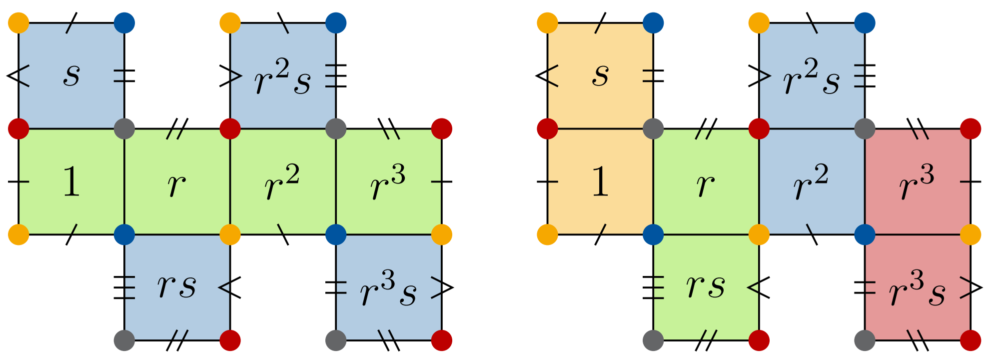


Figure 2.8.: The cylinder decomposition of the 2-origami $\mathcal{O} = (D_8,r,s)$ in horizontal direction consists of two cylinders shaded in green and blue, respectively. In vertical direction, we obtain four cylinders shaded in orange, green, blue, and red.

**Remark 2.3.7** A normal origami $\mathcal{O} = (G, x, y)$ of degree $n$ and genus $g \geq 2$ decomposes into $\frac{n}{\mathrm{ord}(x)}$ horizontal cylinders of circumference $\mathrm{ord}(x)$ and height 1 as the following argument shows. As $\mathcal{O}$ is normal, the set of singularities consists of all square corners and thus the height of the cylinders is 1. We use the bijection between the deck transformations and the squares of $\mathcal{O}$ introduced in the proof of Lemma 2.2.2 and obtain that the horizontal cylinder containing the trivial deck transformation consists of $\mathrm{ord}(x)$ squares. Since the deck transformation group acts transitively on the set of cylinders, the cylinders in the horizontal cylinder decomposition have equal circumference and height. Analogously, one shows that the vertical cylinder decomposition consists of $\frac{n}{\mathrm{ord}(y)}$ cylinders of circumference $\mathrm{ord}(y)$ and height 1.

Cylinder decompositions and parabolic elements of Veech groups are closely connected. This connection was first introduced by Veech in [Vee89] and is often used in the study of translation surfaces. In the following lemma, we use a version of this statement which is due to Vorobets (see [Vor96, Lemma 3.9]).

**Lemma 2.3.8 ([Vor96, Lemma 3.9])** *Let $\mathcal{O}$ be an origami, $v \in \mathbb{Z}^2$ be a rational direction, and $M \in \mathrm{SL}(2, \mathbb{Z})$ be a matrix mapping $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $v$. If $\mathcal{O}$ decomposes in direction $v$ into cylinders $C_1, \ldots, C_k$ with inverse moduli $m_1, \ldots, m_k$ and $m$ is the smallest common integer multiple of all the $m_i$, then the matrix $M \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot M^{-1}$ is contained in the Veech group $\mathrm{SL}(\mathcal{O})$.*

**Example 2.3.9** The left picture in Figure 2.8 shows the cylinder decomposition of the 2-origami $\mathcal{O} = (D_8, r, s)$ in horizontal direction that consists of two cylinders shaded in green and blue, respectively. The inverse modulus of both cylinders is 4. Choosing $A$ as the identity element, we obtain that the parabolic matrix $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ lies in the Veech group $\mathrm{SL}(\mathcal{O})$.

# Chapter 3.

# Strata of $p$-origamis

This chapter deals with the orders of singularities of $p$-origamis. The stratum, in which a given $p$-origami lies, is determined by the order of a certain deck transformation. In Section 3.1, we focus on the group-theoretic aspect. We find bounds for the orders of certain commutators in $p$-groups. Further, we study when the isomorphism type of the $p$-group determines the order of all commutators under consideration. In Section 3.2, we translate these results into the geometric setting of $p$-origamis and deduce implications of the orders of the singularities.

## 3.1. Results on $p$-groups

### 3.1.1. Bounds for the exponent of commutator subgroups of $p$-groups

The geometric setting in Section 2.2.2 motivates the study of the following problem. Given a finite 2-generated $p$-group $G$ of order $p^n$, find a bound for the order of commutators $[x, y]$ with $\langle x, y\rangle = G$. In this section, we answer a more general question. We give a sharp bound for the exponent of the commutator subgroup. This bound is different for the prime 2 and odd primes.

We recall some basic definitions and facts from the theory of $p$-groups. Let $G$ be a finite $p$-group. The **order** $|G|$ of $G$ is the number of its elements. For any element $x \in G$, the **order** $\operatorname{ord}(x)$ of $x$ is the smallest positive integer $n$ such that $x^n = 1$. The **exponent** $\exp(G)$ of $G$ is the greatest order of any element in $G$. The **commutator subgroup** $G'$ of $G$ is the subgroup generated by all commutators

$$[x, y] = x^{-1}y^{-1}xy \quad \text{for } x, y \in G.$$

The **center** $Z(G)$ of $G$ is the subgroup $\{x \in G \mid xy = yx \text{ for all } y \in G\}$. The **Frattini subgroup** $\Phi(G)$ of $G$ is the intersection of all maximal subgroups of $G$. For each $i \in \mathbb{Z}_+$, we define the **omega** and **agemo** subgroup,

$$\Omega_i(G) := \langle g \mid g \in G, g^{p^i} = 1 \rangle,$$
$$\mho^i(G) := \langle g^{p^i} \mid g \in G \rangle.$$

**Lemma 3.1.1 ([LM02, Proposition 1.2.4])** *Let $G$ be a finite $p$-group.*

- *The Frattini subgroup $\Phi(G)$ equals $G'\mho^1(G)$, the group generated by all commutators and $p$-th powers. In particular, $G/\Phi(G)$ is elementary abelian (that is, abelian and of exponent $p$).*
- *Burnside's basis theorem: A set of elements of $G$ is a (minimal) generating set if and only if the images in $G/\Phi(G)$ form a (minimal) generating set of $G/\Phi(G)$. In particular, every generating set for $G$ contains a generating set with exactly $d(G)$ elements, where $d(G)$ is the rank of the elementary abelian quotient $G/\Phi(G)$.*

This can be used to establish first bounds for the exponent of a $p$-group which holds for all prime numbers.

**Proposition 3.1.2** *For a finite $p$-group $G$ of order $p^n$, $\exp(G') = 1$ if $n \leq 2$ or otherwise*

$$\exp(G') \leq p^{n-2}.$$

**Proof** Any cyclic $p$-group is in particular abelian. Hence, $\exp(G') = 1$, in this case.

Suppose $G$ is a non-cyclic $p$-group of order $p^n$ with a minimal generating set of length $d \geq 2$. By Burnside's basis theorem, we obtain $|G/\Phi(G)| = p^d$ and thus $|\Phi(G)| = p^{n-d}$. The inclusion $G' \subseteq \Phi(G)$ implies the inequality $|G'| \leq p^{n-d}$. In particular, the inequality $\exp(G') \leq p^{n-d} \leq p^{n-2}$ holds. ∎

### 2-groups

In this section, we show that the bound in Proposition 3.1.2 is sharp for the prime 2. What is more, we construct 2-generated 2-groups with certain generators whose commutator has the desired order. These groups will be used to construct 2-origamis in Section 3.2.1. We need the following lemma.

**Lemma 3.1.3 ([Hup67, Kapitel III, Hilfssatz 1.11 a)])** *For a group $G$ generated by a subset $S$, the commutator subgroup $G'$ is generated by the set $\{g^{-1}[x, y]g \mid x, y \in S, g \in G\}$.*

**Proposition 3.1.4** *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $n > 2$ and $k \leq n - 2$. There exists a 2-generated 2-group $G$ of order $2^n$ with generators $x, y \in G$ such that*

$$\operatorname{ord}([x, y]) = \exp(G') = 2^k.$$

**Proof** Let $k$ be a natural number with $0 \leq k \leq n-2$. We construct a group $G^2_{(n,k)}$ of order $2^n$ and a pair of generators $r, s$ whose commutator is of order $2^k$. The group $G^2_{(n,k)}$ is a semidirect product of two cyclic groups $C_{2^{k+1}} = \langle r \rangle$ and $C_{2^{n-k-1}} = \langle s \rangle$ of order $2^{k+1}$ and $2^{n-k-1}$, respectively. First, define the group automorphism $\alpha\colon C_{2^{k+1}} \to C_{2^{k+1}},\ r^m \mapsto r^{-m}$. Since $\alpha^{2^{n-k-1}}$ is the identity map on $C_{2^{k+1}}$, the map

$$\varphi\colon C_{2^{n-k-1}} \to \mathrm{Aut}(C_{2^{k+1}}),\ \ s^m \mapsto \alpha^m$$

is a group homomorphism. Let $G^2_{(n,k)}$ be the semidirect product

$$C_{2^{k+1}} \rtimes_\varphi C_{2^{n-k-1}} = \langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1,\ \ s^{-1}rs = r^{-1} \rangle.$$

Then $G^2_{(n,k)}$ has order $2^n$. Using the defining relations of $G^2_{(n,k)}$, we conclude

$$[r, s] = r^{-1}s^{-1}rs = r^{-2}.$$

Hence, the commutator $[r, s]$ has order $2^k$.

Finally, we show that the commutator subgroup $(G^2_{(n,k)})'$ has exponent $2^k$. The group $G^2_{(n,k)}$ is generated by $\{r, s\}$. By Lemma 3.1.3, the commutator subgroup is generated by elements of the form $g^{-1}[r, s]g$ for $g \in G^2_{(n,k)}$. As $[r, s] = r^{-2}$ and $s^{-1}[r, s]s = r^2$, each of the elements $g^{-1}[r, s]g$ is contained in $\langle r^2 \rangle$. Hence $(G^2_{(n,k)})'$ is cyclic of order $2^k$. ■

Note that all the 2-groups constructed in the proof of Proposition 3.1.4 are semidirect products of two cyclic groups.

### $p$-groups for odd primes $p$

Throughout this section, let $p$ denote an odd prime. In Proposition 3.1.5, we introduce a much stronger bound on the exponent of the commutator subgroup which holds for odd primes. This is a generalization of a theorem by van der Waall (see [Waa73, Theorem 1]). There, the order of the commutator subgroup of finite $p$-groups is bounded under the condition that the commutator subgroup is cyclic.

Subsequently, we show in Proposition 3.1.8 that the bound introduced in Proposition 3.1.5 is sharp. To this end, we construct 2-generated $p$-groups and generators whose commutators have the desired orders. These groups are used to construct certain $p$-origamis in Section 3.2.1.

**Proposition 3.1.5** *For a non-trivial finite $p$-group $G$, $p$ odd, the following inequality holds*

$$\exp(G')^2 < |G|. \tag{3.1}$$

**Proof** As the inequality holds for all cyclic $p$-groups, we may use an induction and consider a $p$-group $G$ such that the inequality holds for all $p$-groups of smaller order.

By Lemma III 7.5 in [Hup67], a finite $p$-group $G$ for $p$ odd is either cyclic, or it has a normal subgroup $N \trianglelefteq G$ isomorphic to $C_p \times C_p$. In the first case, again the inequality (3.1) holds. So without loss of generality, we may assume that there exists a normal subgroup $N \trianglelefteq G$ isomorphic to $C_p \times C_p$. Define $H$ as the quotient $G/N$. By the induction hypothesis we have $\exp(H')^2 < |H|$. Consider the canonical epimorphism $\varphi : G \to H$. It maps commutators of $G$ to commutators of $H$, so for $g \in G'$, the image $\varphi(g)$ lies in $H'$. Thus, $\varphi(g)$ has order at most $\exp(H')$, and $g$ has order at most $\exp(N) \cdot \exp(H') = p \cdot \exp(H')$ in $G$. Hence, we obtain the desired inequality

$$\exp(G')^2 \leq p^2 \cdot \exp(H')^2 < p^2 \cdot |H| = |N| \cdot |H| = |G|. \qquad \blacksquare$$

**Corollary 3.1.6** *Let $G$ be a 2-generated $p$-group of order $p^n$ for an odd prime $p$. For generators $x, y$ of $G$, the order of their commutator obeys the inequality*

$$\operatorname{ord}([x, y]) < p^{\frac{n}{2}}.$$

**Proof** Let $\exp(G') = p^m, |G| = p^n$, and $\operatorname{ord}([x, y]) = p^k$. Since $k \leq \exp(G')$, it is sufficient to show that $m < \frac{n}{2}$. This is equivalent to $2m < n$. By Proposition 3.1.5, we have $\exp(G')^2 < |G|$ and thus the inequality $2m < n$ holds. $\blacksquare$

As in the case of 2-groups, we construct for natural numbers $n, k$ with $k < \frac{n}{2}$ a $p$-group of order $p^n$ which is a semidirect product of two cyclic groups and generators $x, y$ such that the order of $[x, y]$ equals $p^k$. This implies that the proven bound is sharp. The construction given in Proposition 3.1.8 works similarly as the one for 2-groups in the proof of Proposition 3.1.4. However, the group homomorphism defining the semidirect products needs to be chosen more carefully for odd primes.

We begin with a purely number-theoretic observation which will be useful when constructing the semidirect products. The following lemma can be proved directly using the binomial expansion, which we include for completeness. See, for instance, [Rot95, Proof of Theorem 6.7, p. 129] for an alternative proof.

**Lemma 3.1.7** *Let $p$ be an odd prime and let $k$ be a positive natural number. Then $p+1$ has order $p^k$ in $\left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^*$.*

**Proof** First, we prove by induction on $m$ that for each $m \geq 0$, the following congruence holds

$$(1 + p)^{p^m} \equiv 1 + p^{m+1} \mod p^{m+2}. \tag{3.2}$$

This is clear for $m = 0$. We assume that the congruence holds for some $m$, i.e., we have

$$(1 + p)^{p^m} = 1 + p^{m+1}(1 + pq)$$

for some $q \in \mathbb{Z}_{\geq 0}$. Using this, we compute

$$\begin{aligned}
(1+p)^{p^{m+1}} &= (1+p^{m+1}(1+pq))^p \\
&= 1 + \underbrace{p \cdot p^{m+1}(1+pq)}_{\equiv\, p^{m+2} \mod p^{m+3}} + \sum_{i=2}^{p} \underbrace{\binom{p}{i} p^{(m+1)\cdot i}}_{\equiv\, 0 \mod p^{m+3}} (1+pq)^i \\
&\equiv 1 + p^{m+2} \mod p^{m+3}.
\end{aligned}$$

This shows that the congruence relation (3.2) is true for $m+1$. By induction, it holds for all $m \in \mathbb{Z}_{\geq 0}$.

Choosing $m = k$ we get

$$\begin{aligned}
(1+p)^{p^k} &\equiv 1 + p^{k+1} \mod p^{k+2} \\
&\equiv 1 \mod p^{k+1}.
\end{aligned}$$

In particular, the order of $1+p$ in $\left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^*$ divides $p^k$. For $1 \leq m < k$, we have

$$\begin{aligned}
(1+p)^{p^m} &\equiv 1 + p^{m+1} \mod p^{m+2} \\
&\not\equiv 1 \mod p^{m+2}.
\end{aligned}$$

We conclude that

$$(1+p)^{p^m} \not\equiv 1 \mod p^{k+1}$$

for $1 \leq m < k$. Thus, the order of $1+p$ is $p^k$. ∎

**Proposition 3.1.8** *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k < \frac{n}{2}$. There exists a 2-generated p-group $G$ of order $p^n$ with generators $x, y \in G$ such that*

$$\operatorname{ord}([x,y]) = \exp(G') = p^k.$$

**Proof** Fix a positive natural number $n$ and let $k$ be an integer with $0 \leq k < \frac{n}{2}$. The group $G^p_{(n,k)}$ is constructed as a semidirect product of two cyclic groups $C_{p^{k+1}} = \langle r \rangle$ and $C_{p^{n-k-1}} = \langle s \rangle$ of order $p^{k+1}$ and $p^{n-k-1}$, respectively. First, consider the automorphism group of $C_{p^{k+1}}$. From elementary group theory, we know that

$$\operatorname{Aut}(C_{p^{k+1}}) \cong \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^* \cong C_{\varphi(p^{k+1})} = C_{p^k(p-1)}.$$

The map

$$\alpha\colon C_{p^{k+1}} \to C_{p^{k+1}}, r^m \mapsto r^{m\cdot(p+1)}$$

defines a group automorphism, since $p$ and $p+1$ are coprime.

Now, we consider the map

$$\varphi\colon C_{p^{n-k-1}} \to \operatorname{Aut}(C_{p^{k+1}}), s^m \mapsto \alpha^m.$$

We claim that $\varphi$ is a well-defined group homomorphism. To see this, we need to show that $\alpha^{p^{n-k-1}}$ is the identity map on $C_{p^{k+1}}$. We prove this as follows. The inequality $k \leq \frac{n-1}{2}$

implies $n-k-1 \geq k$. Hence, we obtain the congruence $(p+1)^{p^{n-k-1}} \equiv 1 \bmod p^{k+1}$ because $p+1$ has order $p^k$ in $\left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^*$ by Lemma 3.1.7. Write

$$(p+1)^{p^{n-k-1}} = j \cdot p^{k+1} + 1$$

for some natural number $j$. Then we have

$$\begin{aligned}
\alpha^{p^{n-k-1}}(r^i) &= r^{i\cdot(p+1)^{p^{n-k-1}}} \\
&= r^{i\cdot j\cdot p^{k+1}} \cdot r^i \\
&= r^i
\end{aligned}$$

for each $1 \leq i \leq p^{k+1}$. Since $r$ has order $p^{k+1}$ in $C_{p^{k+1}}$, the last equality follows. We conclude that $\alpha^{p^{n-k-1}}$ is the identity map on $C_{p^{k+1}}$.

Let $G^p_{(n,k)}$ be the semidirect product

$$C_{p^{k+1}} \rtimes_\varphi C_{p^{n-k-1}} = \langle r, s \mid r^{p^{k+1}} = s^{p^{n-k-1}} = 1,\ s^{-1}rs = r^{p+1} \rangle.$$

Then $G^p_{(n,k)}$ has order $p^n$. We claim that $G^p_{(n,k)}$ together with the pair of generators $(r, s)$ has the desired properties. Using the defining relations of $G^p_{(n,k)}$, we conclude

$$[r, s] = r^{-1}s^{-1}rs = r^p.$$

In particular, the commutator $[r, s]$ has order $p^k$.

Finally, we show that the commutator subgroup of $G^p_{(n,k)}$ has exponent $p^k$. Since $G^p_{(n,k)}$ is generated by $\{r, s\}$, the commutator subgroup is generated by elements of the form $g^{-1}[r, s]g$ for $g \in G^p_{(n,k)}$ by Lemma 3.1.3. Each of these elements is contained in $\langle r^p \rangle$ because

$$s^{-1}[r, s]s = s^{-1}r^p s = r^{p\cdot(p+1)}.$$

Hence, $(G^p_{(n,k)})'$ is cyclic of order $p^k$. ■

### 3.1.2. On the order of certain commutators in 2-generated $p$-groups

In this section, we study a second question that arises from the geometric setting in Section 2.2.2. Recall that the group of deck transformations of a normal origami is always a finite 2-generated group. Further, recall that two normal origamis with isomorphic deck transformation groups $(G, x_1, y_1)$ and $(G, x_2, y_2)$ lie in the same stratum if and only if the orders of the commutators $[x_1, y_1]$ and $[x_2, y_2]$ agree.

We first note that the sets of possible strata for normal origamis with a fixed deck transformation group depend only on its isoclinism class and the order of the group. We recall that isoclinism is an equivalence relation for groups generalizing isomorphism. For its definition, we use the observation that the commutator in any group $G$ induces a well-defined map

$$[.,.] : G/Z(G) \times G/Z(G) \to G', (\overline{x}, \overline{y}) \mapsto [x, y].$$

**Definition 3.1.9 ([Hal40])** Two groups $G_1, G_2$ are **isoclinic** if there are isomorphisms $\phi\colon G_1/Z(G_1) \to G_2/Z(G_2)$ and $\psi\colon G_1' \to G_2'$ which are compatible with the commutator maps in the sense that the following diagram is commutative

$$\begin{tikzcd} G_1/Z(G_1) \times G_1/Z(G_1) \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "\phi\times\phi"'] & G_1' \arrow[d, "\psi"'] \\ G_2/Z(G_2) \times G_2/Z(G_2) \arrow[r, "{[\cdot,\cdot]}"] & G_2'. \end{tikzcd}$$

In particular, all abelian groups are isoclinic.

**Lemma 3.1.10** *The set of possible commutator orders* $\mathrm{ord}([x,y])$ *for generators* $x, y$ *of a* 2*-generated group* $G$ *depends only on the isoclinism class of* $G$*.*

**Proof** Assume $G_1$ and $G_2$ are isoclinic groups and $[x,y]$ has order $n$ for generators $x, y$ of $G_1$. Then $n$ is also the order of $\psi([x,y]) = [\phi(\overline{x}), \phi(\overline{y})]$, where $\overline{x}, \overline{y}$ are the images in $G_1/Z(G_1)$. Thus, $\mathrm{ord}([x', y']) = \mathrm{ord}([x,y])$ for any $x', y'$ in $G_2$ such that $x' \equiv \phi(x)$ and $y' \equiv \phi(y)$ modulo $Z(G_2)$.

If $Z(G_2)$ does not lie in the Frattini subgroup $\Phi(G_2)$, then a central element can be chosen as one of two generators of $G_2$ (see Lemma 3.1.1). So, $G_2$ is abelian and $G_1$ is abelian, and the only possible commutator order is 1 in each of these groups.

Otherwise, $Z(G_2)$ lies in $\Phi(G_2)$. Let $x'$ and $y'$ be preimages of the elements $\phi(\overline{x})$ and $\phi(\overline{y})$ in $G_2$, respectively. The images of $x', y'$ in $G_2/\Phi(G_2)$ generate the quotient, so $x', y'$ are generators of $G_2$ and the order of $[x', y']$ equals the order of $[x, y]$. Since isoclinism is a reflexive relation, the argument is symmetric in $G_1$ and $G_2$. Thus, we have proved the assertion. ∎

**Corollary 3.1.11** *For each* $n \geq 1$*, the dihedral group, the generalized quaternion group, and the semidihedral group with* $2^n$ *elements have the same set of possible commutator orders* $\mathrm{ord}([x,y])$ *for generators* $x, y$*.*

**Proof** It is well-known that these groups are isoclinic (see, for instance, [Ber08, §29 Exercise 4]). ∎

We will return to these groups, recall their definition, and compute the set of possible commutator orders in the section on $p$-groups of maximals class (see Proposition 3.1.18 and Lemma 3.1.19).

We have seen that the possible strata of $p$-origami with a given deck transformation group depends on the possible commutator orders for pairs of generators of this group (see Remark 2.2.10). We will see that, in fact, for many groups there is only one stratum possible (see Theorem 3.1.29 and Theorem 3.2.11). To study such groups, we first translate this property into the language of group theory.

**Definition 3.1.12** We say that a finite 2-generated group $G$ has **property** $(\mathcal{C})$, if there exists a natural number $n$ such that for each 2-generating set $\{x, y\}$ of $G$ the order of $[x, y]$ equals $n$.

We pose the following question:

**Question 3.1.13** Which finite 2-generated $p$-groups have property $(\mathcal{C})$?

For a large class of $p$-groups, we prove property $(\mathcal{C})$. However, in Lemma 3.1.34 we give a counterexample for each prime $p$, i.e., we construct a finite $p$-group with generating sets $\{x, y\}$ and $\{x', y'\}$ such that $\operatorname{ord}([x, y]) \neq \operatorname{ord}([x', y'])$.

In a first example, we show that most alternating groups – which are not $p$-groups – do not have property $(\mathcal{C})$. We use this example to construct origamis in Section 3.2.

**Example 3.1.14** For $n \in \mathbb{N}_{\geq 5}$ odd, we consider the alternating group $\operatorname{Alt}(n)$ with two pairs of generators: $((1, 2, \ldots, n-1, n), (1, 2, 3))$ and $((3, 4, \ldots, n-1, n), (1, 3)(2, 4))$. The orders of the commutators

$$[(1, 2, \ldots, n-1, n), (1, 2, 3)] = (1, 2, 4)$$
$$[((3, 4, \ldots, n-1, n), (1, 3)(2, 4))] = (1, 2, 5, 4, 3)$$

are 3 and 5, respectively. Hence, there are two pairs of generators such that the order of their commutator is different and thus $\operatorname{Alt}(n)$ does not have property $(\mathcal{C})$ for $n \geq 5$. Notice that those alternating groups $\operatorname{Alt}(n)$ are not $p$-groups, since their order is $\frac{n!}{2}$.

Further, note that we multiply permutations from the left because we label the squares of a normal origami by multiplying generators of the deck transformation group from the right.

In the following, we will prove property $(\mathcal{C})$ for certain families of $p$-groups.

### Regular and order-closed $p$-groups

We begin by stating some basic properties of regular $p$-groups. See, e.g., [Hup67; LM02; HEO05] for further information on $p$-groups. Recall that a $p$-group $G$ is **regular** if for each $g, h \in G$ and $i \in \mathbb{Z}_+$, there exists some $c \in \langle g, h\rangle'$ such that

$$(gh)^{p^i} = g^{p^i} h^{p^i} c.$$

Note that the commutator subgroup of a regular $p$-group is regular.

We call a $p$-group $G$ **weakly order-closed** if the product of elements of order at most $p^k$ has order at most $p^k$ for any $k \geq 0$. In the literature, $p$-groups for which all sections (i.e., subquotients) are weakly order-closed according to our definition have been studied and are called **order-closed** $p$-groups (see [Man76]). Clearly, order-closed $p$-groups are weakly order-closed and all subgroups of a weakly order-closed group are so, as well. Hence, a $p$-group is order-closed if and only if all its quotients are weakly order-closed. The class of $p$-groups we call weakly order-closed has been called $\mathcal{O}_p$ in [Wil02].

**Lemma 3.1.15 (see [LM02], Lemma 1.2.13)**

*(i) Any regular $p$-group is order-closed, and hence also weakly order-closed.*

*(ii) A $2$-group is regular if and only if it is abelian.*

**Lemma 3.1.16** *Let $G$ be a weakly order-closed $p$-group. If $x_1, \ldots, x_r \in G$ generate $G$, then the exponent of $G$ is equal to the maximum of the orders $\operatorname{ord}(x_i)$, $1 \leq i \leq r$.*

**Proof** Note that the orders of $x_i$ and $x_i^{-1}$ are equal. As every group element can be written as a word in $\{x_i,\ x_i^{-1} \mid 1 \leq i \leq r\}$, the claim follows. ∎

**Proposition 3.1.17** *Any finite $2$-generated $p$-group with weakly order-closed commutator subgroup has property $(\mathcal{C})$. In particular, any finite $2$-generated $p$-group with regular commutator subgroup has property $(\mathcal{C})$.*

**Proof** Let $G$ be a finite 2-generated $p$-group with regular commutator subgroup. Further, let $x, y$ and $x', y'$ be two pairs of generators of $G$. Hence, by Lemma 3.1.3, the commutator subgroup $G'$ is generated by each of the sets

$$\left\{g^{-1}[x,y]g \mid g \in G\right\},$$
$$\left\{g^{-1}[x',y']g \mid g \in G\right\}.$$

We have

$$\operatorname{ord}(g^{-1}[x,y]g) = \operatorname{ord}([x,y]),$$
$$\operatorname{ord}(g^{-1}[x',y']g) = \operatorname{ord}([x',y']).$$

By the previous lemma, $G'$ being weakly order-closed implies that

$$\operatorname{ord}([x,y]) = \exp(G') = \operatorname{ord}([x',y']).$$

Hence, the orders of the commutators $[x,y]$ and $[x',y']$ coincide. ∎

As we will discuss in Section 3.3, a version of this result holds for pro-$p$ groups (see Proposition 3.3.8).

### $p$-groups of maximal class

In this section, we prove that property $(\mathcal{C})$ holds for $p$-groups of maximal class.

We recall that the **lower central series** of a group $G$ is the series of subgroups

$$\gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \dots,$$

where $\gamma_1(G) := G$ and $\gamma_i(G) := [\gamma_{i-1}(G), G]$ for $i > 1$. The **nilpotency class** of $G$ is $c$ if

$$\gamma_c(G) \gneq \gamma_{c+1}(G) = \langle 1 \rangle .$$

A $p$-group of order $p^n$ has nilpotency class at most $n-1$ and is called of **maximal class** in that case.

As 2-groups of maximal class are completely classified by the following proposition, we treat these groups separately from the $p$-groups for odd primes $p$.

**Proposition 3.1.18 ([Hup67, Kapitel III, Satz 11.9])** *Each 2-group of maximal class and order $2^n$ is isomorphic to one of the following groups*

- *a dihedral group, i.e., a group given by the presentation*

$$D_{2^n} = \left\langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ s^{-1}rs = r^{-1} \right\rangle,$$

- *a generalized quaternion group, i.e., a group given by the presentation*

$$Q_{2^n} = \left\langle r, s \mid r^{2^{n-1}} = 1,\ s^2 = r^{2^{n-2}},\ s^{-1}rs = r^{-1} \right\rangle,$$

- *a semidihedral group, i.e., a group given by the presentation*

$$SD_{2^n} = \left\langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ s^{-1}rs = r^{2^{n-2}-1} \right\rangle.$$

We can now strengthen the result in Corollary 3.1.11 for 2-groups of maximal class. Note that according to the classification in Proposition 3.1.18, each such group is 2-generated.

**Lemma 3.1.19** *Any 2-generated 2-group of maximal class and of order $2^n$ has property $(\mathcal{C})$ and for each pair of generators $x, y$, the order of the commutator $[x, y]$ is $2^{n-2}$.*

**Proof** Let $G$ be a 2-group of maximal class and of order $2^n$. By Proposition 3.1.18, it is isomorphic to a dihedral group, quaternion group or semidihedral group. In each case, we show that the commutator subgroup is regular and that for a pair of generators $r, s$ the commutator $[r, s]$ has order $2^{n-2}$. By Proposition 3.1.17, this proves the claim.

For the semidihedral group $SD_{2^n}$, the commutator subgroup is generated by the set $\{g^{-1}[r,s]g \mid g \in SD_{2^n}\}$. Since the commutator $[r,s]$ equals $r^{2^{n-2}-2}$, the commutator subgroup is cyclic. In particular, it is abelian and thus regular. The order of the commutator $[r,s]$ is $2^{n-2}$.

For the dihedral group $D_{2^n}$ and the generalized quaternion group $Q_{2^n}$, the commutator subgroup is generated by $[r,s] = r^{-2}$. Here, we use the relation $rs = sr^{-1}$. Hence, the commutator subgroup is abelian and thus regular. As in the previous case, the commutator $[r,s]$ has order $2^{n-2}$. ∎

We turn our attention to $p$-groups of maximal class for odd primes $p$.

**Lemma 3.1.20** *Let $G$ be a 2-generated $p$-group of maximal class and order $p^n$ for a natural number $n$ with $5 \leq n \leq p+1$ and an odd prime $p$. Then $G$ has property $(\mathcal{C})$ and for each pair of generators $x, y$, the order of the commutator $[x,y]$ equals $p$.*

**Proof** Assume $G$ is a group of order $p^n$ of maximal class for $5 \leq n \leq p+1$. Then the commutator subgroup $G'$ has exponent $p$ by [Hup67, Kapitel III, Hilfssatz 14.14]. Hence, all elements of $G'$ have either order $p$ or order 1. Let $(x,y)$ be a pair of generators of $G$. Then $G'$ is generated by elements of the form $g^{-1}[x,y]g$ for $g \in G$. Since $G$ is of maximal class, it is non abelian. We conclude that $[x,y] \neq 1$ has order $p$. ∎

**Lemma 3.1.21** *Any finite 2-generated $p$-group of maximal class for $p$ odd has property $(\mathcal{C})$.*

**Proof** Let $G$ be a group of order $p^n$. If $n \leq 4$, then the order of the commutator subgroup $G'$ is smaller than $p^4$. Hence, we have $|G'| \leq p^p$. By [Hup67, Kapitel III, Satz 10.2 b)], the commutator subgroup is regular. Using Lemma 3.1.15, the claim follows from Proposition 3.1.17. In Lemma 3.1.20, we showed the claim for groups of order $p^n$ of maximal class with $5 \leq n \leq p+1$.

Now assume $G$ has order $p^n$ and is of maximal class for $n > p+1$. Then there exists a maximal subgroup $H$ of $G$ which is regular (see [Hup67, Kapitel III, Satz 14.22]). Recall that the Frattini subgroup $\Phi(G)$ is the intersection of the set of maximal subgroups. We have the following inclusions

$$G' \subseteq \Phi(G) \subseteq H.$$

Since subgroups of regular groups are regular as well, the commutator subgroup $G'$ is regular. By Proposition 3.1.17, the claim follows. ∎

We obtain the following theorem from Lemma 3.1.19 and Lemma 3.1.21.

**Theorem 3.1.22** *Any finite 2-generated $p$-group of maximal class has property $(\mathcal{C})$.*

### Powerful $p$-groups

We introduce some further basics from the theory of $p$-groups to show that property $(\mathcal{C})$ holds for the class of powerful $p$-groups.

A $p$-group $G$ is called **powerful** if either $p$ is odd and $G' \subseteq \mho^1(G)$, or $p = 2$ and $G' \subseteq \mho^2(G) = \langle g^4 \mid g \in G \rangle$. A normal subgroup $N \trianglelefteq G$ is **powerfully embedded** in $G$ if either $p$ is odd and $[N, G] \subseteq \mho^1(N)$, or $p = 2$ and $[N, G] \subseteq \mho^2(N)$. In particular, any powerfully embedded $p$-group is powerful.

**Corollary 3.1.23** *Any finite 2-generated powerful $p$-group $G$ has a cyclic commutator subgroup $G'$.*

**Proof** Let $G$ be a powerful $p$-group with 2-generating set $\{x, y\}$. Then the commutator subgroup $G'$ is powerfully embedded in $G$ by [LM87, Theorem 1.1. and Theorem 4.1.1.]. Further, $G'$ is generated by all elements of the form $g^{-1}[x, y]g$ for $g \in G$ by Lemma 3.1.3. Now [LM87, Theorem 1.10. and Theorem 4.1.10.] state that if a powerfully embedded subgroup of a powerful $p$-group is the normal closure of a subset, then it is generated by this subset. Thus, we conclude that $G'$ is generated by $[x, y]$, i.e., $G'$ is cyclic. ■

**Corollary 3.1.24** *Any finite 2-generated powerful $p$-group has property $(\mathcal{C})$.*

**Proof** $G'$ is cyclic, so in particular, $G'$ is regular. Using Proposition 3.1.17, the claim follows. ■

**Proposition 3.1.25** *Let $G$ be a finite 2-generated $p$-group such that $G'$ is powerful. Then $G$ has property $(\mathcal{C})$.*

**Proof** Let $G$ be a finite 2-generated $p$-group such that $G'$ is powerful. Let $\{x, y\}$ be a generating set. Denote the order of $[x, y]$ by $p^m$. Recall that $G'$ is generated by the set $\{g^{-1}[x, y]g \mid g \in G\}$ whose elements are all of order $p^m$ (see Lemma 3.1.3). Since $G'$ is powerful, we apply [LM87, Theorem 1.9. and Theorem 4.1.9.] stating that in a powerful $p$-group, any agemo subgroup is generated by the corresponding powers of a given set of generators, and deduce

$$\begin{aligned} \mho^m(G') &= \langle g^{p^m} \mid g \in G' \rangle \\ &= \langle \left(g^{-1}[x, y]g\right)^{p^m} \mid g \in G \rangle \\ &= \{1\}. \end{aligned}$$

In particular, the exponent of $G'$ equals $p^m$, and hence equals the order of $[x, y]$. As $\{x, y\}$ is an arbitrary pair of generators of $G$, this proves the claim. ■

Since the commutator subgroup of a powerful $p$-group is powerful by [LM87, Theorem 1.1], Corollary 3.1.24 is also a consequence of Proposition 3.1.25.

We will see in Section 3.3, that the result of the proposition can be extended to pro-$p$ groups (see Proposition 3.3.8).

### Power-closed $p$-groups

Recall from Section 3.1.2 that we call a $p$-group weakly order-closed, if products of elements of order at most $p^k$ have order at most $p^k$ for all $k \geq 0$. We call a $p$-group **weakly power-closed**, if products of $p^k$-th powers are $p^k$-th powers for all $k \geq 0$. Such groups generalize **power-closed** $p$-groups ([Man76]), for which all sections (i.e., subquotients) have to be weakly power-closed in our sense. As quotients of weakly power-closed groups are automatically weakly power-closed, a $p$-group is power-closed if and only if all its subgroups are weakly power-closed. In [Wil02], the class of $p$-groups we call weakly power-closed has been called $\mathcal{P}_p$.

It is known that order-closed $p$-groups generalize regular $p$-groups, and that power-closed $p$-groups generalize order-closed $p$-groups (see [Man76]).

Recall that in Proposition 3.1.17, we have shown that 2-generated $p$-groups $G$ for which $G'$ is weakly order-closed have property ($\mathcal{C}$).

**Example 3.1.26** Using the `GAP` code in Listing A.3 (in Appendix A), we find instances of a 2-group $G$ such that $G'$ is weakly power-closed, but which does have generators $x, y$ such that $\operatorname{ord}([x, y]) \neq \operatorname{ord}([x, y^3])$. As $G$ is a 2-group, $x, y^3$ is a pair of generators, as well, so $G$ does not have property ($\mathcal{C}$).

We obtain, for example, the permutations $x, y \in \operatorname{Sym}(16)$ given as

$$\begin{aligned} x &:= (1, 13, 2, 14)(3, 16, 4, 15)(5, 9, 7, 11, 6, 10, 8, 12) \ , \\ y &:= (1, 16, 6, 11, 4, 14, 7, 9, 2, 15, 5, 12, 3, 13, 8, 10) \ , \end{aligned}$$

and $G = \langle x, y \rangle$. Then, the commutators

$$\begin{aligned} [x, y] &= (1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 16, 12, 15) \ , \\ [x, y^3] &= (1, 6)(2, 5)(3, 7)(4, 8)(9, 15)(10, 16)(11, 14)(12, 13) \end{aligned}$$

have order 4 and 2, respectively. The group $G$ has order $2^{12}$ and nilpotency class 7.

However, this counterexample is not a power-closed $p$-group, since it has subgroups which are not weakly power-closed. We have found such subgroups using the computer algebra system `GAP`. This example yields the following corollary.

**Corollary 3.1.27** *There are 2-groups with weakly power-closed commutator subgroup which do not have property ($\mathcal{C}$).*

However, let us contrast this with the case of minimal non-power-closed or minimal non-order-closed $p$-groups, where minimal means that all proper sections are power-closed or order-closed, respectively.

**Lemma 3.1.28** *Let $G$ be a minimal non-power-closed $p$-group or a minimal non-order-closed $p$-group. Then $G$ has property ($\mathcal{C}$) and for each pair of generators $x, y$, the order of the commutator $[x, y]$ equals $p$.*

**Proof** Let $G$ be a minimal non-power-closed or a minimal non-order-closed $p$-group. By [Man76, Theorem 3 and Theorem 6], the Frattini group $\Phi(G)$ has exponent $p$ and the center satisfies $Z(G) \neq G$. As $G'$ is contained in $\Phi(G)$, we conclude that $G'$ has exponent $p$. Hence, $G'$ is regular. Proposition 3.1.17 implies that $G$ has property $(\mathcal{C})$. For a pair of generators $x, y$ of $G$, the commutator subgroup $G'$ is generated by all conjugates of $[x, y]$. Since $G'$ is not trivial, the commutator $[x, y]$ has exponent $p$. ∎

Let us summarize our results on groups with property $(\mathcal{C})$.

**Theorem 3.1.29** *The implications and the non-implication visualized in the following diagram are true for any finite 2-generated $p$-group $G$. In particular, all $p$-groups falling into the classes above the gray line have property $(\mathcal{C})$.*

$$\begin{tikzcd}
 & G \text{ maximal class} \arrow[d] & & G \text{ powerful} \arrow[d] \arrow[dll] \\
G \text{ regular} \arrow[r] \arrow[d] & G' \text{ regular} \arrow[d] & & G' \text{ powerful} \arrow[d] \\
G \text{ order-closed} \arrow[r] \arrow[d] & G' \text{ order-closed} \arrow[r] \arrow[d] & G' \text{ weakly order-closed} \arrow[r] & \boxed{G \text{ has property } (\mathcal{C})} \\
G \text{ power-closed} \arrow[r] & G' \text{ power-closed} \arrow[r] & G' \text{ weakly power-closed} \arrow[ur, "/" marking] &
\end{tikzcd}$$

**Proof** The implications regular $\Rightarrow$ order-closed $\Rightarrow$ power-closed are due to [Man76]. Each of these properties is inherited by subgroups, so by $G'$ from $G$. By our definition, the properties weakly order-closed and power-closed generalize order-closed and power-closed, respectively. Powerful $p$-groups and $p$-groups of maximal class have regular commutator subgroups as explained in the proof of Corollary 3.1.24 and Lemma 3.1.21, respectively. The commutator subgroup of a powerful $p$-group is powerful by [LM87, Theorem 1.1, Theorem 4.1.1].

The results that $G'$ powerful or weakly order-closed imply property $(\mathcal{C})$, are given in Proposition 3.1.25 and Proposition 3.1.17, respectively. Corollary 3.1.27 shows that property $(\mathcal{C})$ does not follow from the property that $G'$ is weakly power-closed. ∎

**Remark 3.1.30** We note that, in particular, all 2-generated $p$-groups of order at most $p^{p+2}$ or of nilpotency class at most $p$ have property $(\mathcal{C})$. In the first case, $G'$ has order at most $p^p$, because it lies in the Frattini subgroup which has index $p^2$ in $G$, so $G'$ is regular by [Hup67, Kapitel III, Satz 10.2 b)]. Similarly, in the second case, $G$ is regular by [Hup67, Kapitel III, Satz 10.2 a)].

It is an open questions, whether $G$ or $G'$ being power-closed implies property $(\mathcal{C})$ for a $p$-group $G$. To study this and similar questions, we offer a reduction argument.

**Lemma 3.1.31** *Let $\mathcal{F}$ be a family of finite 2-generated p-groups which is closed under quotients and such that property $(\mathcal{C})$ holds for all members of $\mathcal{F}$ with cyclic center. Then property $(\mathcal{C})$ holds for all members of $\mathcal{F}$.*

**Proof** Observe that property $(\mathcal{C})$ holds for all abelian groups and assume that property $(\mathcal{C})$ holds all members of $\mathcal{F}$ with cyclic center. Consider a non-abelian group $G \in \mathcal{F}$ whose center is not cyclic such that property $(\mathcal{C})$ holds for any member of $\mathcal{F}$ of smaller cardinality.

Since the center of $G$ has rank at least two, i.e., the center is not cyclic, we can choose two trivially intersecting central cyclic subgroups $N_1, N_2$ of order $p$. We consider pairs of generators $x, y$ and $x', y'$ of $G$, and we define

$$\begin{aligned} c &:= [x, y], \quad m := \operatorname{ord}([x, y]), \\ c' &:= [x', y'],\ m' := \operatorname{ord}([x', y']). \end{aligned}$$

Without loss of generality, we may assume $m \geq m' \geq p$. Now $c^{\frac{m}{p}}$ has order $p$. Since $N_1$ and $N_2$ intersect trivially, $c^{\frac{m}{p}}$ lies in at most one of these subgroups. We may assume that $c^{\frac{m}{p}} \notin N_1$. Consider the canonical epimorphism $\overline{\phantom{x}} : G \to G/N_1$. The order of $\bar{c}$ is equal to $m$ because $c^{\frac{m}{p}} \notin N_1$. Now $\overline{x}, \overline{y}$ and $\overline{x'}, \overline{y'}$ are pairs of generators of the $p$-group $G/N_1$. By our assumptions, $G/N_1$ has property $(\mathcal{C})$, so

$$m = \operatorname{ord}(\overline{c}) = \operatorname{ord}(\overline{c'}).$$

Thus, we conclude that the $m'$ is either $m$ or $m \cdot p$. Since $m \geq m'$ holds, we obtain that $m$ and $m'$ coincide. This shows that $G$ has property $(\mathcal{C})$.

An induction using this argument proves the assertion. ∎

**Corollary 3.1.32** *$G$ being power-closed or $G'$ being power-closed implies property $(\mathcal{C})$ if it implies property $(\mathcal{C})$ together with the assumption that $G$ has cyclic center.*

**Proof** The family of power-closed $p$ groups is closed under taking arbitrary quotients. The same is true for the family of $p$-groups with power-closed commutator subgroup, since for any group, the commutator subgroup of a quotient is a quotient of the commutator subgroup. ∎

### A family of counterexamples

We conclude our discussion of property $(\mathcal{C})$ by constructing 2-generated $p$-groups for all primes $p$ which do not have property $(\mathcal{C})$: for each such group, we exhibit two pairs of generators with different commutator order.

As these will be realized as certain subgroups of a $p$-Sylow subgroup of the symmetric group on $p^r$ letters, for some $r \geq 0$, let us first recall a description of one of these Sylow subgroups which we will denote $P_{p,r}$. Recall that we multiply permutations from the left.

**Definition 3.1.33** For any prime $p$ and any $r \geq 0$, consider the perfect $p$-ary tree with $p^r$ leaves (as in Figure 3.1 below for $p = r = 3$). Given $1 \leq i \leq r, 1 \leq j \leq p^{i-1}$, let $N$ be the $j$-th node of this tree at level $i-1$ (where the root node is at level 0), then we define a tree automorphism by fixing all nodes which are not descendants of $N$, while rotating the subtrees starting at the $p$ direct descendants of $N$ cyclically to the right. We denote the permutation of the $p^r$ leaves of the tree induced by this graph automorphism by $\mathbf{e_{i,j}}$. Let $\mathbf{P_{p,r}}$ be the subgroup of $S_{p^r}$ generated by all $e_{i,j}$ for $1 \leq i \leq r, 1 \leq j \leq p^{i-1}$.

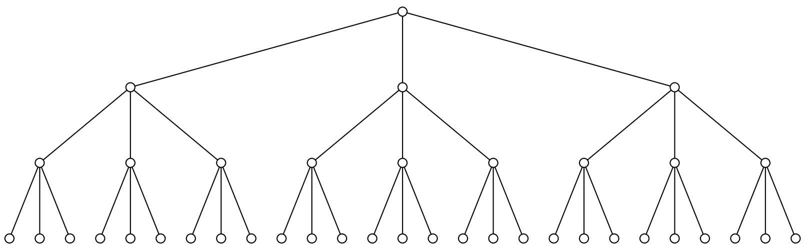


Figure 3.1.: A perfect tertiary tree with $3^3$ leaves. Its automorphism group contains a group isomorphic to $P_{3,3}$.

In cycle notation, $e_{i,j}$ is a product of disjoint $p$-cycles defined by

$$e_{i,j} := \prod_{k=p^{r+1-i}(j-1)+1}^{p^{r+1-i}(j-1)+p^{r-i}} (k, k+p^{r-i}, \ldots, k+p^{r-i}(p-1)) \ . \tag{3.3}$$

From Equation (3.3) we get the conjugation relation

$$e_{i,j}^{-1} e_{k,l} e_{i,j} = e_{k,l'} \tag{3.4}$$

for $i \leq k$, where $l'$ is defined by

$$(j-1)p^{k-i} < l' \leq jp^{k-i} \quad \text{and} \quad l' - l \equiv p^{k-i-1} \mod p^{k-i}$$

if $i < k$ and $(j-1)p^{k-i} < l \leq jp^{k-i}$, and $l = l'$ otherwise.

This implies that $P_{p,r}$ is, in fact, generated by $(e_{i,1})_{1 \leq i \leq r}$. We can check that $P_{p,r}$ is isomorphic to the $r$-fold wreath product of $C_p$ by identifying the respective generators. So, $P_{p,r}$ is indeed a Sylow $p$-subgroup of $S_{p^r}$. For further information on iterated wreath products of cyclic groups $C_p$ see, e.g., [LM02, Section 2.4].

**Lemma 3.1.34** *For any prime $p$, there exist elements $x, x', y \in P_{p,4}$ such that*

$$H_p := \langle x, y \rangle = \langle x', y \rangle \quad \text{and} \quad \operatorname{ord}([y, x']) = p \neq p^2 = \operatorname{ord}([y, x]) \ .$$

*In particular, the $p$-group $H_p$ does not have property $(\mathcal{C})$.*

**Proof** We define

$$x := e_{1,1}e_{2,1} \ , \quad y := e_{1,1}e_{2,1}e_{3,1}e_{4,1+p^2} \quad \in P_{p,4} \ .$$

Assume $p > 3$, then by Equation (3.4),

$$\begin{aligned} x^{-1}yx &= e_{2,1}^{-1}(e_{1,1}e_{2,2}e_{3,1+p}e_{4,1+2p^2})e_{2,1} \\ &= e_{1,1}e_{2,1}e_{3,1+p}e_{4,1+2p^2} \ , \\ x^{-2}yx^2 &= e_{2,1}^{-1}(e_{1,1}e_{2,2}e_{3,1+2p}e_{4,1+3p^2})e_{2,1} \\ &= e_{1,1}e_{2,1}e_{3,1+2p}e_{4,1+3p^2} \ , \\ \text{so} \quad [y,x] &= y^{-1}x^{-1}yx = e_{4,1+p^2}^{-1}e_{3,1}^{-1}e_{3,1+p}e_{4,1+2p^2} \ , \\ [y,x^2] &= y^{-1}x^{-2}yx^2 = e_{4,1+p^2}^{-1}e_{3,1}^{-1}e_{3,1+2p}e_{4,1+3p^2} \ . \end{aligned}$$

We compute the restriction of the permutation $[y,x]$ to the set $S := \{p^3+1,\dots,p^3+p^2\}$:

$$\begin{aligned} [y,x]|_S &= (e_{4,1+p^2}^{-1}e_{3,1}^{-1}e_{3,1+p}e_{4,1+2p^2})|_S \\ &= (e_{4,1+p^2}^{-1}e_{3,1+p})|_S \\ &= (p^3+1,p^3+2,\dots,p^3+p)^{-1}(p^3+1,p^3+p+1,\dots,p^3+p(p-1)+1) \\ &\quad \dots(p^3+p,p^3+2p,\dots,p^3+p^2) \\ &= (p^3+p,p^3+2p-1,\dots,p^3+p^2-1, \\ &\quad p^3+p-1,p^3+2p-2,\dots,p^3+p^2-2, \\ &\quad \dots, \\ &\quad p^3+1,p^3+2p,\dots,p^3+p^2) \ . \end{aligned}$$

Since the permutation $[y,x]|_S$ is a $p^2$-cycle, we obtain the inequality $\operatorname{ord}([y,x]) \geq p^2$. As all $p$-cycles occurring in $e_{3,1}$ or $e_{4,1+2p^2}$ are disjoint from each other and from the set $S$, indeed, $\operatorname{ord}([y,x]) = p^2$. At the same time, our formula for $[y,x^2]$ exhibits it as a product of disjoint $p$-cycles, hence $\operatorname{ord}([y,x^2]) = p$. So, we have shown

$$\operatorname{ord}([y,x^2]) = p \neq p^2 = \operatorname{ord}([y,x]) \ .$$

Since $p$ is an odd prime, we deduce the equality $\langle x,y\rangle = \langle x^2,y\rangle$. This proves the assertion with $x' := x^2$.

It can be verified that the same is true for $p = 3$, even though the formulae for $x^{-2}yx^2$ and $[y,x^2]$ become slightly different:

$$\begin{aligned} x^{-2}yx^2 &= e_{2,1}^{-1}(e_{1,1}e_{2,2}e_{3,1+2p}e_{4,1})e_{2,1} = e_{1,1}e_{2,1}e_{3,1+2p}e_{4,1+p} \ , \\ [y,x^2] &= e_{4,1+p^2}^{-1}e_{3,1}^{-1}e_{3,1+2p}e_{4,1+p} \ . \end{aligned}$$

All conclusions, however, remain valid.

For $p = 2$, take

$$\begin{aligned} x &:= e_{1,1}e_{2,1}e_{3,1}e_{4,1} = (1,9,5,13,3,11,7,15,2,10,6,14,4,12,8,16)\ , \\ x' &:= x^3 = (1,13,7,10,4,16,5,11,2,14,8,9,3,15,6,12)\ , \\ y &:= e_{1,1}e_{3,4}e_{4,1} = (1,9,2,10)(3,11)(4,12)(5,15,7,13)(6,16,8,14)\ . \end{aligned}$$

Since 3 does not divide the order of $x$, we obtain the equality $\langle x, y\rangle = \langle x', y\rangle$. At the same time, we compute

$$\mathrm{ord}([y,x]) = 2 \neq 4 = \mathrm{ord}([y,x']).$$

This proves the assertion for the prime $p = 2$. ■

## 3.2. Results on strata of $p$-origamis

The purpose of this section is to derive results about $p$-origamis from the results about $p$-groups in Section 3.1. In Section 3.2.1, we answer the question in which strata $p$-origamis occur. Subsequently, we study in Section 3.2.2 under which conditions $p$-origamis with isomorphic deck transformation groups lie in the same stratum.

### 3.2.1. Strata of $p$-origamis

The answer to the question in which strata $p$-origamis occur depends on whether the considered prime is 2 or not. We begin this section with some facts that hold for all primes. In the following two subsections, we consider 2-origamis and $p$-origamis for odd primes $p$, respectively.

Recall from Lemma 2.2.9 that all normal origamis with abelian deck transformation group lie in the stratum $\mathcal{H}(0)$. Since all abelian groups are isoclinic, the following can be viewed as a generalization of the previous observation:

**Corollary 3.2.1** *The set of possible types of singularities of normal origamis with a given deck transformation group depends only on the isoclinism class of this deck transformation group.*

**Proof** By Lemma 3.1.10, the set of possible commutator orders for pairs of generators coincides for isoclinic 2-generated groups. The commutator order determines the multiplicity of the singularity. ∎

**Remark 3.2.2** As an example, we have seen that for each $n \geq 1$, there is a stratum containing all 2-origamis with the dihedral group, the generalized quaternion group, or the semidihedral group of order $2^n$ as the group of deck transformations (see Corollary 3.1.11). It was computed to be $\mathcal{H}(4 \times (2^{n-2} - 1))$ in Lemma 3.1.19.

#### Strata of 2-origamis

In this subsection, we classify the strata of 2-origamis. We will see in the next subsection that the occurring strata differ from the ones of $p$-origamis for odd primes $p$.

**Theorem 3.2.3** *Let $n \in \mathbb{Z}_{\geq 0}$. For 2-origamis of degree $2^n$, exactly the following strata appear*

- $\mathcal{H}(0)$,
- $\mathcal{H}\left(2^{n-k} \times \left(2^k - 1\right)\right)$, *where* $1 \leq k \leq n-2$.

**Proof** For $n \leq 2$, all groups of order $2^n$ are abelian, so the corresponding 2-origamis lie in the trivial stratum by Lemma 2.2.9.

Let $n, k \in \mathbb{Z}_{\geq 0}$ with $n > 2$ and $k \leq n-2$. By Proposition 3.1.4, there exists a 2-generated group $G$ of order $2^n$ and generators $x, y$ such that $\operatorname{ord}([x,y]) = 2^k$. Hence, the origami $(G, x, y)$ lies in the stratum $\mathcal{H}(0)$ for $k = 0$ and in $\mathcal{H}\left(2^{n-k} \times \left(2^k - 1\right)\right)$ for $k > 0$.

It remains to prove that other strata cannot occur. Let $\mathcal{O} = (G, x, y)$ be a 2-origami of degree $2^n$. By Remark 2.2.11, the only possible strata are of the form $\mathcal{H}(k \times (a-1))$ where $a$ is the multiplicity of each singularity and $k$ is the number of singularities. Using Proposition 3.1.2, we deduce that the inequality $\exp(G') \leq 2^{n-2}$ holds. By Lemma 2.2.9, the multiplicity of each singularity equals $\operatorname{ord}([x,y])$. Since $\operatorname{ord}([x,y]) \leq 2^{n-2}$, the claim follows. ∎

**Remark 3.2.4** We recall the definition of the series of 2-groups in the proof of Proposition 3.1.4. For $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n-2$, we define the semidirect product

$$G^2_{(n,k)} := C_{2^{k+1}} \rtimes_\varphi C_{2^{n-k-1}} = \langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1,\ s^{-1}rs = r^{-1} \rangle.$$

The commutator $[r,s]$ has order $2^k$ and thus the 2-origami $(G^2_{(n,k)}, r, s)$ lies in the stratum $\mathcal{H}\left(2^{n-k} \times \left(2^k - 1\right)\right)$.

In particular, this shows that in each of the occurring strata there exists a 2-origami with a semidirect product of two cyclic groups as deck transformation group.

**Example 3.2.5** For $n = 3$ and $k = 1$, we obtain the group

$$G^2_{(3,1)} = C_4 \rtimes_\varphi C_2 = \langle r, s \mid r^4 = s^2 = 1,\ s^{-1}rs = r^{-1} \rangle.$$

This group is isomorphic to the dihedral group $D_8$ of order 8. In Figure 3.2, we consider the 2-origami $\mathcal{O} = (G^2_{(3,1)}, r, s)$. The commutator $[r,s] = r^{-2}$ has order 2 and thus $\mathcal{O}$ lies in $\mathcal{H}(4 \times 1)$.

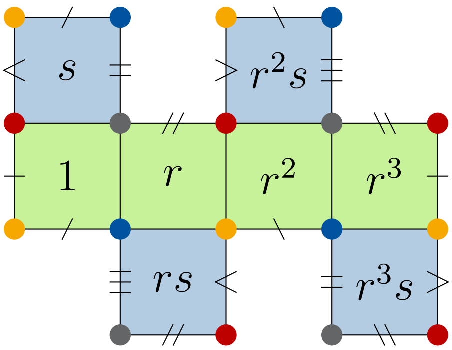


Figure 3.2.: The 2-origami $\mathcal{O} = (G^2_{(3,1)}, r, s)$ with marked horizontal cylinders.

Choosing $(s, rs)$ as the pair of generators of the group $G^2_{(3,1)}$, we obtain the normal origami $\mathcal{O}' = (G^2_{(3,1)}, s, rs)$ shown in Figure 3.3. Note that this is the Penrose stairs origami $PS_8$

(see Figure 2.5). Since $[s, rs] = r^{-2}$ has order 2, the origamis $\mathcal{O}$ and $\mathcal{O}'$ lie in the same stratum. In the following, we show that the origamis are different. Recall that a maximal cylinder is a maximal open subsurface that is isometric to an Euclidean cylinder. For a normal origami $(H, x, y)$, the length of each cylinder in horizontal and vertical direction equals the order of $x$ and $y$, respectively. We conclude that the horizontal cylinders of $\mathcal{O}'$ have length 2, whereas the horizontal cylinders of $\mathcal{O}$ have length 4. Hence, the origamis are different. (Alternatively, one can use Lemma 2.2.4 and the fact that group isomorphisms preserve orders.)

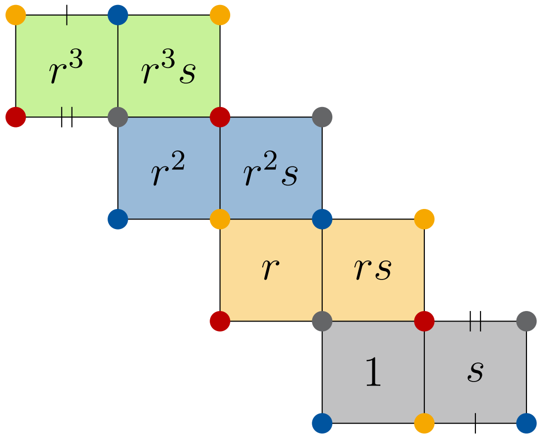


Figure 3.3.: The 2-origami $\mathcal{O}' = (G^2_{(3,1)}, s, rs)$ with marked horizontal cylinders. It coincides with the Penrose stairs origami $PS_8$. Opposite sides are identified unless marked otherwise.

The example above gives two different 2-origamis with isomorphic deck transformation group. In Section 3.2.2, we address the question whether the deck transformation group determines the stratum of a normal origami.

**Remark 3.2.6** We recall from Section 2.3 that the matrix group $\mathrm{SL}(2,\mathbb{Z})$ is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and acts on the set of normal origamis with a fixed deck transformation group. Note that Example 3.2.5 shows an instance of this $\mathrm{SL}(2,\mathbb{Z})$-action because of the equation

$$T \cdot \mathcal{O}' = \left(G^2_{(3,1)}, s, rss^{-1}\right) = (G, s, r) = \mathcal{O}.$$

In particular, $\mathcal{O}$ and $\mathcal{O}'$ lie in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit. Recall that the dihedral group $D_8$ is isomorphic to the group $G^2_{(3,1)}$. In Proposition 4.1.4, we will see that there are three normal origamis with deck transformation group $D_8$ and that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on these origamis.

### Strata of $p$-origamis for odd primes $p$

Throughout this section, let $p$ denote an odd prime. We study the question in which strata $p$-origamis lie. Compared to the case of 2-origamis fewer strata occur. This is shown in the following theorem.

**Theorem 3.2.7** *Let $n \in \mathbb{Z}_{\geq 0}$. For $p$-origamis of degree $p^n$, exactly the following strata appear*

- $\mathcal{H}(0)$
- $\mathcal{H}\left(p^{n-k} \times \left(p^k - 1\right)\right)$, *where* $1 \leq k < \frac{n}{2}$.

**Proof** For $n \leq 2$, all groups of order $p^n$ are abelian, so the corresponding $p$-origamis lie in the trivial stratum by Lemma 2.2.9.

Let $n, k \in \mathbb{Z}_{\geq 0}$ with $n > 2$ and $k < \frac{n}{2}$. By Proposition 3.1.8, there exist a 2-generated group $G$ of order $p^n$ and generators $x, y$ such that $\operatorname{ord}([x,y]) = p^k$. Hence, the origami $(G, x, y)$ lies in the stratum $\mathcal{H}(0)$ for $k = 0$ and in the stratum $\mathcal{H}\left(p^{n-k} \times \left(p^k - 1\right)\right)$ for $k > 0$.

It remains to prove that other strata cannot occur. Let $\mathcal{O} = (G, x, y)$ be a $p$-origami of degree $p^n$. By Remark 2.2.11, the only possible strata are of the form $\mathcal{H}(k \times (a-1))$ where $a$ is the multiplicity of each singularity and $k$ is the number of singularities. By Lemma 2.2.9, the multiplicity of each singularity equals $\operatorname{ord}([x,y])$. Using Corollary 3.1.6, we deduce that the inequality $\operatorname{ord}([x,y]) \leq p^k$ holds, where $k < \frac{n}{2}$. ∎

**Remark 3.2.8** We recall the definition of the series of $p$-groups in the proof of Proposition 3.1.8. For $n, k \in \mathbb{Z}_{\geq 0}$ with $k < \frac{n}{2}$, we define the semidirect product

$$G^p_{(n,k)} := C_{p^{k+1}} \rtimes_\varphi C_{p^{n-k-1}} = \langle r, s \mid r^{p^{k+1}} = s^{p^{n-k-1}} = 1,\ s^{-1}rs = r^{p+1} \rangle.$$

The commutator $[r,s]$ has order $p^k$ and thus the $p$-origami $(G^p_{(n,k)}, r, s)$ lies in the stratum $\mathcal{H}\left(p^{n-k} \times \left(p^k - 1\right)\right)$. As in Remark 3.2.4, this shows that in each of the occurring strata there exists a $p$-origami with a semidirect product of two cyclic groups as deck transformation group.

**Example 3.2.9** For $p = 3$, $n = 3$ and $k = 1$, we obtain the group

$$G^3_{(3,1)} = C_9 \rtimes_\varphi C_3 = \langle r, s \mid r^9 = s^3 = 1,\ s^{-1}rs = r^4 \rangle.$$

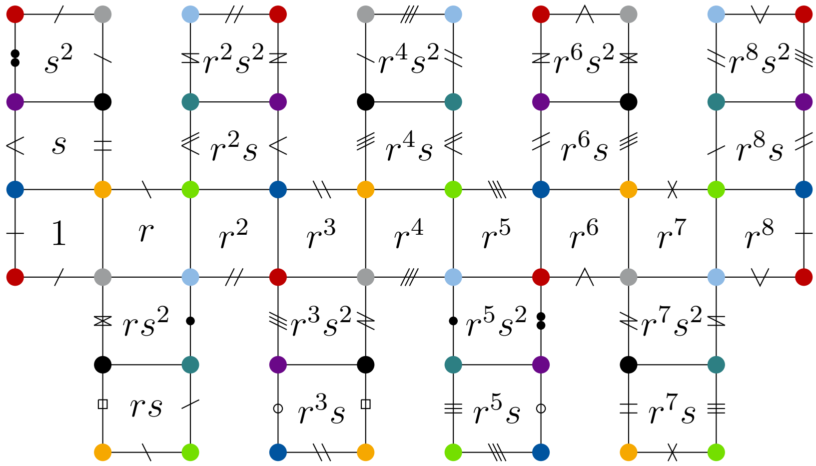


Figure 3.4.: The 3-origami $(G^3_{(3,1)}, r, s)$. All horizontal cylinders are of length 9 and all vertical cylinders are of length 3.

We consider the origami $\mathcal{O} = (G^3_{(3,1)}, r, s)$. The commutator $[r, s]$ has order 3 and thus the origami $\mathcal{O}$ lies in $\mathcal{H}(9 \times 2)$. Hence, the origami has nine singularities of angle $3 \cdot 2\pi$. Using the `GAP`-package [Ert+21], we compute that the set of normal origamis with deck transformation group $G^3_{(3,1)}$ consists of 8 origamis.

### 3.2.2. $p$-origamis with isomorphic deck transformation groups

In this section, we study the question whether the deck transformation group determines the stratum of a normal origami. This question was motivated by computer experiments. For $p$-origamis, computer experiments suggested that the stratum depends only on the isomorphism class of the deck transformation group. Using Example 3.1.14, we show that this does not hold for all finite groups.

**Example 3.2.10** For $n \in \mathbb{N}_{\geq 5}$ odd, we consider the alternating group $\mathrm{Alt}(n)$ with two pairs of generators $((1, 2, \dots, n-1, n), (1, 2, 3))$ and $((3, 4, \dots, n-1, n), (1, 3)(2, 4))$. Recall from Example 3.1.14 that the orders of the commutators

$$[(1, 2, \dots, n-1, n), (1, 2, 3)] \text{ and } [((3, 4, \dots, n-1, n), (1, 3)(2, 4))]$$

are 3 and 5, respectively. Hence, the normal origami

$$\mathcal{O}_n := (\mathrm{Alt}(n), (1, 2, \dots, n-1, n), (1, 2, 3))$$

has $\frac{n!}{6}$ singularities of multiplicity 3, whereas the normal origami

$$\mathcal{O}'_n := (\mathrm{Alt}(n), (3, 4, \dots, n-1, n), (1, 3)(2, 4))$$

has $\frac{n!}{10}$ singularities of multiplicity 5. It follows that there are two pairs of generators of $\mathrm{Alt}(n)$ defining normal origamis lying in different strata.

The origami constructed in [Ath+20, Example 7.3] is a normal origami with deck transformation group $\mathrm{Alt}(5)$. It lies in the same stratum as origami $\mathcal{O}'_5$ and thus could replace $\mathcal{O}'_5$ in the example above for $n = 5$.

Recall that two normal origamis $(G, x_1, y_1)$ and $(G, x_2, y_2)$ with isomorphic group of deck transformation group lie in the same stratum if and only if the orders of the commutators $[x_1, y_1]$ and $[x_2, y_2]$ agree (see Remark 2.2.10). This is the case for all possible pairs of generators of a group $G$ if and only if the deck transformation group has property $(\mathcal{C})$ (see Definition 3.1.12). Using Theorem 3.1.29 and Remark 3.1.30, we obtain Theorem B from the introduction.

**Theorem 3.2.11** *Let $G$ be a finite 2-generated $p$-group. If $G$ satisfies one of the properties (1) to (8), then all $p$-origamis with deck transformation group $G$ lie in the same stratum.*

*(i) $G$ is regular.*
*(ii) $G$ has maximal class.*
*(iii) $G$ is powerful.*
*(iv) $G'$ is regular.*
*(v) $G'$ is powerful.*
*(vi) $G'$ is order-closed.*
*(vii) $G$ has order at most $p^{p+2}$.*
*(viii) $G$ has nilpotency class at most $p$.*

**Proof** By definition, a finite 2-generated group $G$ has property $(\mathcal{C})$, if there exists a natural number $n$ such that for each 2-generating set $\{x, y\}$ of $G$ the order of $[x, y]$ equals $n$. Hence, it is sufficient to show that property $(\mathcal{C})$ holds for all groups satisfying one of the properties (i) to (viii). This follows from Theorem 3.1.29. The connection to groups up to a certain order or nilpotency class is made in Remark 3.1.30. ∎

For certain $p$-groups with property $(\mathcal{C})$, we studied the constant given by the order of the commutator of a pair of generators in Section 3.1.2 (see Lemma 3.1.19, Lemma 3.1.20, and Lemma 3.1.28). We deduce the corresponding results for the strata of the respective $p$-origamis.

**Corollary 3.2.12** *Any 2-origami of degree $2^n$ whose deck transformation group has maximal class lies in the stratum $\mathcal{H}\left(4 \times (2^{n-2} - 1)\right)$.*

**Corollary 3.2.13** *For $5 \leq n \leq p+1$ and an odd prime $p$, any $p$-origami of degree $p^n$ whose deck transformation group has maximal class lies in the stratum $\mathcal{H}\left(p^{n-1} \times (p-1)\right)$.*

**Corollary 3.2.14** *Any $p$-origami of degree $p^n$ whose deck transformation group is a minimal non-power-closed $p$-group or a minimal non-order-closed $p$-group lies in the stratum $\mathcal{H}\left(p^{n-1} \times (p-1)\right)$.*

In Lemma 3.1.34, we constructed for each prime $p$ a 2-generated $p$-group that does not have property $(\mathcal{C})$. Hence, we obtain the following proposition.

**Proposition 3.2.15** *For each prime $p$, there exist $p$-origamis with isomorphic deck transformation group that lie in different strata.*

**Proof** In Lemma 3.1.34, we proved for each prime $p$ the existence of a 2-generated $p$-group $H_p$ which is contained in the Sylow $p$-subgroup of the symmetric group $\mathrm{Sym}(p^4)$ and does not have property $(\mathcal{C})$. Hence, there exist $p$-origamis with deck transformation group isomorphic to $H_p$ that lie in different strata. ∎

**Remark 3.2.16** The 2-group $G$ with pairs of generators $(x, y)$ and $(x, y^3)$ defined in Example 3.1.26 is weakly power-closed, but not power-closed. Recall that the orders of the commutators $\mathrm{ord}([x, y])$ and $\mathrm{ord}([x, y^3])$ are 4 and 2, respectively. We obtain that the 2-origamis $(G, x, y)$ and $(G, x, y^3)$ lie in different strata, namely $\mathcal{H}(2^{10} \times 3)$ and $\mathcal{H}(2^{11} \times 1)$. Here we use that the group $G$ has order $2^{12}$.

## 3.3. Infinite normal origamis

So far, we have considered surfaces that are also called finite translation surfaces, i.e., the surface can be described as finitely many polygons with edge identifications via translations. As a generalization, **infinite translation surfaces** have been studied during the past 10 years (see, e.g., [BV13]). In contrast to finite translation surfaces, one allows countably many polygons glued by translations. For a detailed introduction to infinite translation surfaces see [Ran16] and [DHV].

In this section, we consider a well-known infinite translation surface called staircase origami. Moreover, we generalize the notion of property $(\mathcal{C})$ to pro-$p$ groups, certain infinite analogs of finite $p$-groups. We then transfer some results from Section 3.1.2 to pro-$p$ groups and deduce conclusions about a class of translation surfaces which we call infinite normal origamis.

Let $\mathcal{O} \to \mathbb{T}$ be a countably infinite, normal cover of the torus $\mathbb{T}$ ramified over at most one point. Then the corresponding surface $\mathcal{O}$ is called an **infinite normal origami**. Amongst others, these surfaces have been studied by [Kar20], where they are called regular origamis. As in the finite case, they correspond to a special class of infinite translation surfaces where all polygons are squares of the same size. The concepts introduced in Section 2.2 carry over to infinite origamis. Given a countably infinite group $G$ with 2-generating set $(x, y)$, one constructs an infinite normal origami $(G, x, y)$ as in Lemma 2.2.2. One has a natural bijection between the squares in the tiling and the elements of the deck transformation group. Singularities of infinite normal origamis can have finite cone angle, i.e., $2\pi n$ for $n \in \mathbb{Z}_+$ as in the case of finite origamis, or infinite angle, i.e., a neighborhood of the singularity is isometric to a neighborhood of the branching point of the infinite cyclic branched cover of $\mathbb{R}^2$. As in the case of finite normal origamis, the cone angle of all singularities of the origami $(G, x, y)$ coincide and are equal to $2\pi \cdot a$, where $a$ is the order of the commutator $[x, y]$. If the order of $[x, y]$ is infinite, then the cone angle is infinite as well.

**Example 3.3.1** An example for an infinite normal origami is the staircase origami $St_\infty$ in Figure 3.5. It has been studied both from the geometric (see, e.g., [HS10]) and from the dynamical point of view (see, e.g., [HHW13]). The deck transformation group of the origami $St_\infty$ is the infinite dihedral group

$$D_\infty := \langle r, s \mid s^2 = 1, srs = r^{-1} \rangle.$$

For the origami $St_\infty$, the elements $s, sr$ are chosen as generators. The commutator subgroup of $D_\infty$ is the infinite cyclic group generated by $[r, s] = r^{-2}$. Hence, for any pair of generators $x, y$, the commutator $[x, y]$ has order infinity. We conclude that each infinite normal origami with deck transformation group $D_\infty$ has 4 singularities of infinite cone angle.

Choosing the generators $r, s$ we obtain a surface different from $St_\infty$. In contrast to the latter, it has 2 infinite horizontal cylinders, as shown in Figure 3.6. Both normal origamis lie in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit because of the following equation

$$ST \cdot (D_\infty, s, sr) = S \cdot (D_\infty, s, r^{-1}) = (D_\infty, r, s).$$

We see that the horizontal cylinders of origamis in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit can be finite and infinite.

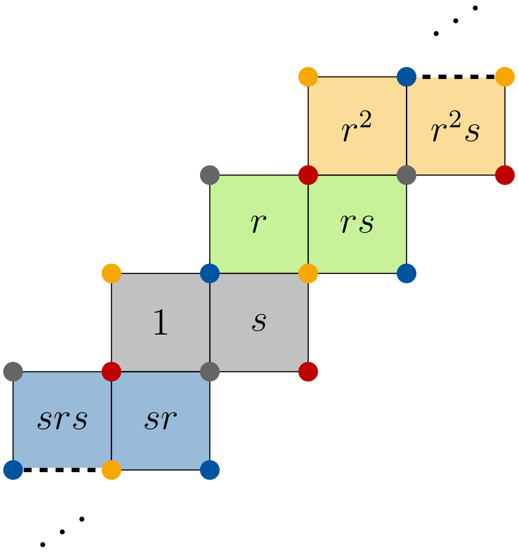


Figure 3.5.: The infinite staircase origami $St_\infty = (D_\infty, s, sr)$ has four singularities of infinite cone angle. All vertical and horizontal cylinders have length 2. Here, the horizontal cylinders are shaded in different colors. Opposite sides are identified unless marked otherwise.

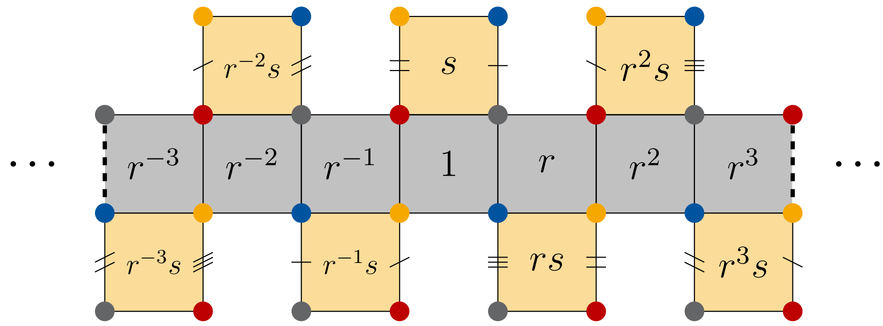


Figure 3.6.: The origami $(D_\infty, r, s)$ has two infinite horizontal cylinders and infinitely many vertical cylinders of height 2. Here, the horizontal cylinders are shaded in different colors. Opposite sides are identified unless marked otherwise.

The infinite dihedral group is a dense subgroup of the pro-2 group $\mathbb{Z}_{(2)} \ltimes C_2$. Pro-$p$ groups have played an essential role in the study of finite $p$-groups (see, e.g., [LM00]). It is a natural question, whether results of Section 3.1.2 and Section 3.2.2 can be transferred to certain infinite groups, in particular profinite and pro-$p$ groups, and infinite normal origamis. To this end, we extend the definition of property $(\mathcal{C})$ to possibly infinite groups which are 2-generated. Note that, as we do not consider topological groups yet, the groups under consideration are still countable.

**Definition 3.3.2** A (possibly countably infinite) 2-generated group $G$ has **property** $(\mathcal{C})$ if there is an element $k \in \mathbb{Z}_+ \cup \{\infty\}$ such that the order of $[x, y]$ equals $k$ for any pair $x, y$ of generators of $G$.

**Example 3.3.3** Recall that the infinite dihedral group $D_\infty$ has an infinite cyclic commutator subgroup, i.e., $D_\infty$ has property $(\mathcal{C})$ (see Example 3.3.1).

In the following, we consider 2-generated profinite groups. For this purpose, recall that an **inverse system** of groups is a collection of groups $(G_i)_{i \in I}$ indexed by a directed poset $I$ and homomorphisms $\psi_{i,j} \colon G_j \to G_i$ for all $i, j \in I, i \leq j$, such that $\psi_{i,i}$ is the identity and $\psi_{i,j} \circ \psi_{j,k} = \psi_{i,k}$ for all $i \leq j \leq k$. The **inverse limit** of an inverse system $(G_i)_i, (\psi_{i,j})_{i \leq j}$ is the group

$$\hat{G} := \{(g_i)_i \in \prod_i G_i : \psi_{i,j}(g_j) = g_i \quad \text{for all } i \leq j\}$$

together with the projection maps $\pi_i \colon \hat{G} \to G_i$ onto the components $G_i$. It is the categorical limit for the diagram described by the inverse system in the category of groups, and is distinguished by the corresponding universal property (see Diagram (3.5)).

Recall that a **profinite group** is a Hausdorff, compact, totally disconnected topological group. For any profinite group $\hat{G}$, the quotients $\hat{G}/N$ for all normal open subgroups $N$ of $\hat{G}$ form an inverse system of finite groups whose inverse limit is isomorphic to $\hat{G}$. Equivalently, one can define profinite groups as those which are isomorphic to the inverse limit of an inverse system of finite groups. We refer the interested reader to [Dix+99] for background knowledge about profinite groups.

For profinite, or more generally, topological groups, we can consider **topological generating sets**, i.e., subsets which generate a dense subgroup. A profinite group is topologically 2-generated if and only if it is isomorphic to the inverse limit of an inverse system of 2-generated finite groups ([Dix+99, Prop. 1.5]). This motivates the following definition.

Recall that a **Steinitz number** $k$ is a formal product $\prod_p p^{\nu_p(k)}$, where $p$ ranges over all prime numbers and $\nu_p(k)$ is a non-negative integer or $\infty$ for each prime $p$. Steinitz numbers were first considered in [Ste10] and the order of each element in a profinite group can be viewed as a Steinitz number (see, e.g., [Wil98, Chapter 2]).

**Definition 3.3.4** We say that a topologically 2-generated profinite group has **property** $(\mathcal{C}^{\mathrm{pro}})$ if there exists a Steinitz number $k$ such that for all topological 2-generating sets $x, y$ the commutator order $\mathrm{ord}([x, y])$ equals $k$.

Using the connection between commutator orders of 2-generating sets and cone angles of singularities for infinite normal origamis, we obtain the following lemma directly from the definition of property $(\mathcal{C}^{\mathrm{pro}})$.

**Lemma 3.3.5** *Let $\hat{G}$ be a topologically 2-generated profinite group with property $(\mathcal{C}^{\mathrm{pro}})$. The singularities of all infinite normal origamis whose deck transformation group is a (countable) dense subgroup of $\hat{G}$ have the same cone angle.*

One way of verifying that a given profinite group has property ($\mathcal{C}^{\mathrm{pro}}$) is given by the following lemma.

**Lemma 3.3.6** *A topologically 2-generated profinite group has property ($\mathcal{C}^{\mathrm{pro}}$) if it is isomorphic to the inverse limit of an inverse system of finite groups which have property ($\mathcal{C}$).*

**Proof** Consider the profinite group $\hat{G} \cong \varprojlim(G_i, \psi_{i,j})$ for an inverse system $((G_i)_i, (\psi_{i,j})_{i\leq j})$ of finite groups with property ($\mathcal{C}$). Denote the structural projections of the inverse limit by $\pi_i \colon \hat{G} \to G_i$.

From the inverse limit construction it is clear that the order of any element $g \in \hat{G}$ is the least common multiple (as a Steinitz number) of the element orders $(\operatorname{ord}(\pi_i(g)))_i$ in the respective groups $G_i$. If $g$ is the commutator of a pair of (topological) generators $x, y$ of the group $\hat{G}$, then $\pi_i(g)$ is the commutator of a pair of generators in $G_i$ for every $i$, since $\pi_i$ is an epimorphism. By assumption, the group $G_i$ has property ($\mathcal{C}$) for each $i$. Hence, the number $\operatorname{ord}(\pi_i(g))$ is independent of the choice of the generators $x, y$ for each $i$. We conclude that the least common multiple of the element orders $(\operatorname{ord}(\pi_i(g)))_i$ is independent of the choice of the generators $x, y$ as well. This proves that $\hat{G}$ has property ($\mathcal{C}^{\mathrm{pro}}$). ■

**Remark 3.3.7** Let $(G_i)_i, (\psi_{i,j})_{i\leq j}$ be an inverse system of 2-generated finite groups. We call 2-generating sets $(x_i, y_i)_i$ **compatible** if $\psi_{i,j}(x_j) = x_i$ and $\psi_{i,j}(y_j) = y_i$ for all $i$ and $j$. Denote the inverse limit by $(\hat{G}, \pi_i)$. Applying the universal mapping property of the inverse limit to the free group $F_2 = \langle a, b\rangle$ and the monodromy maps $m_i : F_2 \to G_i$ of the normal origamis $(G_i, x_i, y_i)$, one obtains a unique group homomorphism $\alpha : F_2 \to \hat{G}$ and the following commutative diagram

$$\begin{tikzcd}
 & F_2 \arrow[d, dashed, "\alpha"'] \arrow[ddl, bend right, "m_i"'] \arrow[ddr, bend left, "m_j"] & \\
 & \hat{G} \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
G_j \arrow[rr, "\psi_{i,j}"] & & G_i.
\end{tikzcd} \tag{3.5}$$

The image $\alpha(F_2)$ is a (countable) dense subgroup of $\hat{G}$ with the following 2-generating set $x := \alpha(a)$ and $y := \alpha(b)$ (in the sense of classical group theory). Note that $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ for all $i \in I$. We obtain an infinite normal origami $(\alpha(F_2), x, y)$ associated with the infinite sequence of finite normal origamis $(G_i, x_i, y_i)_i$.

Similarly, any given infinite normal origami $(H, x', y')$ for a (countable) dense subgroup $H$ of $\hat{G}$ yields infinitely many finite normal origamis $(G_i, \pi_i(x_i'), \pi_i(y_i'))_i$. Here, we use that the image $\pi_i(H)$ equals $G_i$ for each $i \in I$ (see [Dix+99, Proposition 1.5]).

Recall that for any prime $p$, a **pro-$p$ group** is a profinite group $\hat{G}$ such that $\hat{G}/N$ is a finite $p$-group for any open normal subgroup $N$ of $\hat{G}$. Equivalently, it is a group which is isomorphic to the inverse limit of an inverse system of finite $p$-groups. Pro-$p$ groups play a central role in the coclass conjectures by Leedham-Green and Newman ([LN80]), proved by Leedham-Green ([Lee94]) and Shalev ([Sha94]), concerning a way of classifying all finite $p$-groups.

We can now extend some of our results from finite $p$-groups to pro-$p$ groups. Let us call a pro-$p$ group **weakly order-closed** if products of elements of order $p^k$ have order at most $p^k$, for any $k \geq 0$. Let us also recall that a pro-$p$ group is called **powerful** if $G/\operatorname{cl}(\{g^{p^k} : g \in G\})$ is abelian for $k = 1$ if $p > 2$ and for $k = 2$ if $p = 2$, where $\operatorname{cl}(S)$ denotes the minimal closed subgroup generated by a set $S$ (see [Dix+99, Definition 3.1]).

We observe that key results for finite $p$-groups, e.g., Proposition 3.1.17 and Corollary 3.1.24, can be generalized to pro-$p$ groups.

**Proposition 3.3.8** *A topologically* 2*-generated pro-*$p$ *group* $\hat{G}$ *has property* $(\mathcal{C}^{\text{pro}})$ *if*

*(i)* $\hat{G}'$ *is weakly order-closed, or*

*(ii)* $\hat{G}'$ *is powerful.*

**Proof** If the order of the commutator $[x, y]$ is infinite for all (topological) generators $x, y$ of $\hat{G}$, the claim follows. Assume that $x$ and $y$ are (topological) generators of $\hat{G}$ such that the order of the commutator $[x, y]$ is finite. The commutator subgroup $\hat{G}'$ is generated by conjugates of $[x, y]$, all having the same order, say $p^k$ for $k \geq 1$. We will show that assuming (i) or (ii) implies that this order is the exponent of $\hat{G}'$ and thus is independent of the choice of $x, y$. This proves the assertion that $\hat{G}$ has property $(\mathcal{C}^{\text{pro}})$.

If we assume (i), then the countable subgroup of $\hat{G}'$ generated (without topological closure) by all conjugates of $[x, y]$ consists of elements of order at most $p^k$. Hence indeed, $p^k$ is the exponent of the countable subgroup, and of $\hat{G}'$ being its closure.

If we assume (ii), then $\hat{G}'$ is a powerful pro-$p$ group topologically generated by conjugates of $[x, y]$. Hence, by [Dix+99, Prop. 3.6 (iii)], the set of all $p^k$-th powers in $\hat{G}'$ equals the closed subgroup generated by the $p^k$-th powers of the conjugates of $[x, y]$, which is the trivial subgroup. Thus, all $p^k$-th powers in $\hat{G}'$ have to be trivial, so $p^k$ is the exponent of the commutator subgroup $\hat{G}'$. ∎

As for finite $p$-groups this has implications for families of normal origamis:

**Corollary 3.3.9** *Let* $\hat{G}$ *be a topologically* 2*-generated pro-*$p$ *group which is either powerful or has a weakly order-closed commutator subgroup. The singularities of all infinite normal origamis whose deck transformation group is a (countable) dense subgroup of* $\hat{G}$ *have the same cone angle.*

We conclude our exploration into the world of infinite deck transformation groups with some examples.

**Example 3.3.10** In this example, we introduce a setup to construct inverse systems from semidirect products of cyclic groups. Such groups appeared several times in Section 3.1.1 and Section 3.2 (e.g. Proposition 3.1.4, Proposition 3.1.8, Remark 3.2.4, and Remark 3.2.8). For a prime $p$, let $C_{p^m} = \langle x \rangle$ and $C_{p^\ell} = \langle y \rangle$ be cyclic groups of order $p^m$ and $p^\ell$, respectively. If $a \in \mathbb{Z}$ is coprime to $p$ and $a^{p^m} \equiv 1 \bmod p^\ell$, then the map $\varphi_a : C_{p^m} \to \operatorname{Aut}(C_{p^\ell})$, $x \mapsto (y \mapsto y^a)$ is a group homomorphism and defines a semidirect product

$$H_{(\ell,m,a)} := C_{p^\ell} \rtimes_{\varphi_a} C_{p^m} = \langle x, y \mid x^{p^m} = y^{p^\ell} = 1, x^{-1}yx = y^a \rangle.$$

Now for any $m' \geq m$ and $\ell' \geq \ell$ with $m' - \ell' \geq m - \ell$, we show that the congruences $a^{p^{m'}} \equiv 1 \bmod p^{\ell'}$ and $a^{p^{m'}} \equiv 1 \bmod p^\ell$ hold as well. For this, it suffices to prove the congruence $a^{p^{m+1}} \equiv 1 \bmod p^{\ell+1}$ and apply an induction argument. By assumption, we have $a^{p^m} = d \cdot p^\ell + 1$ for some integer $d$. Using binomial expansion, we compute

$$\begin{aligned}
a^{p^{m+1}} &\equiv (dp^\ell + 1)^p \\
&\equiv \sum_{k=1}^{p} \binom{p}{k} \cdot \left(dp^\ell\right)^{p-k} \cdot 1^k \\
&\equiv 1 + \binom{p}{p-1} \cdot dp^\ell \\
&\equiv 1 \bmod p^{\ell+1}.
\end{aligned}$$

Hence, the groups $H_{(\ell',m',a)}$ and $H_{(\ell,m',a)}$ are still well-defined and we have epimorphisms $H_{(\ell',m',a)} \to H_{(\ell,m,a)}$ and $H_{(\ell,m',a)} \to H_{(\ell,m,a)}$ sending $x \mapsto x$ and $y \mapsto y$ in the respective groups. Using these epimorphisms, we get an inverse system

$$\left(H_{(\ell,m',a)}\right)_{m' \geq m}$$

with inverse limit $C_{p^\ell} \rtimes \mathbb{Z}_{(p)}$, or an inverse system

$$\left(H_{(\ell',m',a)}\right)_{m' \geq m, \ell' \geq \ell, m'-\ell' \geq m-\ell}$$

with inverse limit $\mathbb{Z}_{(p)} \rtimes \mathbb{Z}_{(p)}$.

Choosing compatible 2-generating sets for the groups forming an inverse system, one obtains an infinite sequence of finite normal origamis and an infinite normal origami which has a (countable) dense subgroup of the inverse limit as its deck transformation group (see Remark 3.3.7).

Note, that all groups of the form $H_{(\ell,m,a)}$ have property $(\mathcal{C})$, since the commutator subgroup is always cyclic (generated by $y^{a-1}$). Hence, both constructions of inverse systems yield pro-$p$ groups with property $(\mathcal{C}^{\text{pro}})$ (see Lemma 3.3.6). The constructions of inverse systems given above can be applied as well to the inverse systems formed by the groups $G^p_{(n,k)}$ considered in Remark 3.2.4 and Remark 3.2.8 for $p = 2$ and $p > 2$, respectively.

The dihedral groups $D_{2^n} = \langle r_n, s_n \mid r_n^{2^{n-1}} = s_n^2 = 1, s_n r_n s_n = r_n^{-1} \rangle$, considered in Example 3.3.1, form an inverse system constructed in a similar, but slightly different way.

Here, we fix $m = 1$ as well as $a = -1$ and let $\ell$ vary. The infinite dihedral group $D_\infty$ is a 2-generated countable dense subgroup of the inverse limit $\hat{D}_2 = \mathbb{Z}_{(2)} \rtimes C_2$ of the 2-generated 2-groups $(D_{2^n})_{n\geq 0}$. By Lemma 3.1.21, the dihedral groups have property $(\mathcal{C})$ and thus the pro-2 group has property $(\mathcal{C}^{\text{pro}})$ (see Lemma 3.3.6). For $n \in \mathbb{Z}_+$, the tuples $(r_n, s_n)_n$ form compatible 2-generating sets. Using the construction in Remark 3.3.7, we obtain the normal infinite origami $(D_\infty, r, s)$ in Figure 3.6. Choosing $(s_n, s_n r_n)_n$ as compatible 2-generators, the construction yields the infinite staircase origami in Figure 3.5. For each natural number $n$, the finite normal origami $(D_{2^n}, s_n, s_n r_n)$ is the Penrose stairs origami $PS_{2^n}$ (see Example 2.2.7). We observe that the infinite staircase origami can be viewed as the limit of the finite Penrose stairs origamis $PS_{2^n}$.

In the following example, we construct an inverse system taking the quaternion group as a starting point. This yields an infinite series of normal origamis covering the “eierlegende Wollmilchsau” (see Example 2.2.3).

**Example 3.3.11** The groups

$$W_n = \langle x, y \mid x^{2^{n+1}} = y^{2^{n+1}} = x^{2^n} y^{2^n} = 1, x^{-1}yx = y^{-1} \rangle$$

are 2-generated 2-groups of order $2^{2n+1}$. For any $n \geq 1$, consider the group element $z := y^2 = [x, y]$. It lies in the commutator subgroup $W_n'$, commutes with $y$, and obeys the equality

$$x^{-1}zx = x^{-1}y^2x = y^{-2} = z^{-1} \; .$$

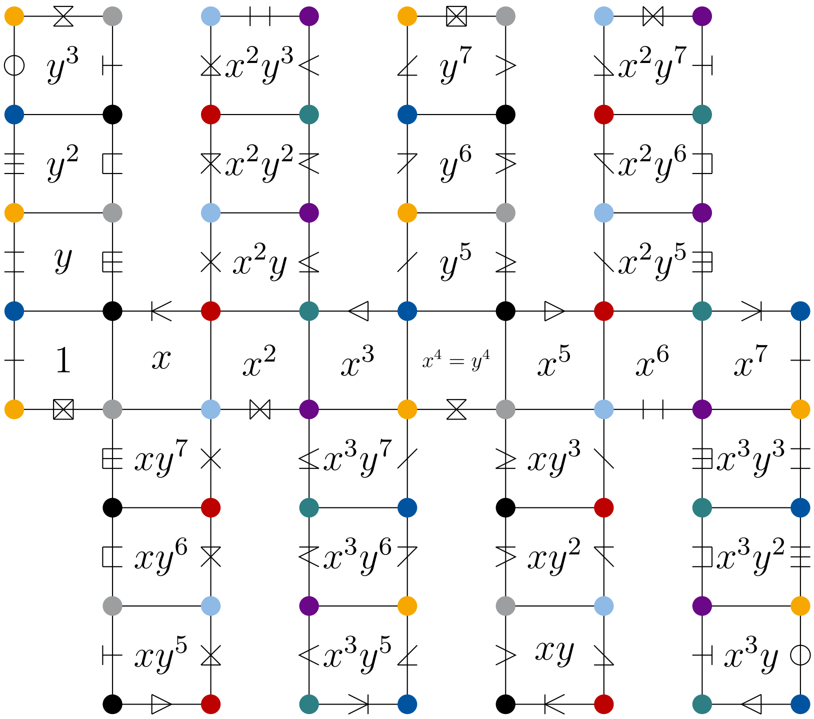


Figure 3.7.: The origami $(W_2, x, y)$ as in Example 3.3.11 is a cover of the “eierlegende Wollmilchsau” of degree 4.

Hence, by Lemma 3.1.3, the commutator subgroup $W_n' = \langle z \rangle$ is cyclic of order $2^n$. The defining relations of $W_n$ imply that its elements are of the form $y^i x^j$ for $0 \leq i < 2^{n+1}$ and $0 \leq j < 2^n$. The images of $x$ and $y$ in the abelianization $W_n/W_n' = W_n/\langle y^2 \rangle$ have order $2^n$ and 2, respectively, and the abelianization is isomorphic to $C_{2^n} \times C_2$.

Note that we have epimorphisms from $W_{n+1}$ to $W_n$ for any $n \geq 1$ sending $x \mapsto x$, $y \mapsto y$ (and $z \mapsto z$) in the respective groups. We obtain an inverse system whose inverse limit has property $(\mathcal{C}^{\text{pro}})$ by Lemma 3.3.6. The group $W_1$ is isomorphic to the quaternion group with 8 elements and the generators $(x, y)$ (viewed as elements in the groups $W_n$) form a set of compatible generators. Hence, we obtain an infinite sequence of normal origamis $(W_n, x, y)$ covering the "eierlegende Wollmilchsau". We give an example of these origamis for $n = 2$ in Figure 3.7.

# Chapter 4.

# Veech groups of normal origamis

In this chapter, we study the $\mathrm{SL}(2,\mathbb{Z})$-action on and the Veech groups of certain normal origamis. We are especially interested in the following questions: When are the Veech groups of origamis congruence subgroups, and when are they far from being congruence subgroups? On the one hand, [Sch05] demonstrated that almost all congruence groups occur as Veech groups of origamis. On the other hand, Hubert and Lelièvre studied the situation in the stratum $\mathcal{H}(2)$ in [HL05]. They proved that, except for the Veech group of one origami, all occurring Veech groups are not congruence groups. However, these results do not fully answer the aforementioned questions.

In Section 4.1, we mostly consider families of 2-groups that are semidirect products of cyclic groups. We construct and examine all normal origamis with these groups as deck transformation groups and study properties of their Veech groups as well as their $\mathrm{SL}(2,\mathbb{Z})$-orbits in Section 4.1.1. For instance, we prove that the Veech groups of the 2-origamis under consideration are a congruence subgroup and that the group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the set of normal origamis $\mathcal{O}(G)$ with deck transformation group $G$ for all considered groups $G$.

Translation surfaces are closely related to the study of dynamical systems, e.g., the Teichmüller flow. Lyapunov exponents capture the dynamical behavior of the homology along the orbits of the Teichmüller flow. In Section 4.1.2, we use a combinatorial formula introduced by Eskin, Kontsevich, and Zorich (see Theorem 4.1.24) together with the results obtained in Section 4.1.1 to compute the sum of non-negative Lyapunov exponents for the normal origamis under consideration.

The results in Section 4.1.1 raise the question whether there exist normal origamis whose Veech groups are not congruence groups. In Section 4.2, conditions for the deck transformation group of a normal origami $\mathcal{O}$ are introduced which imply that its Veech group is a totally non-congruence group, i.e., $\mathrm{SL}(\mathcal{O})$ surjects onto $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{Z}_+$. The notion of totally non-congruence groups was established by Weitze-Schmithüsen (see [Wei13]). In [SW18], for each stratum an infinite family of origamis with totally non-congruence groups as Veech groups is constructed. These origamis have only few symmetries and thus are far from being normal origamis. In Section 4.2, we give examples of normal origamis whose Veech groups are totally non-congruence groups. For instance, we construct such a normal origami for each Hurwitz group.

## 4.1. Veech groups of $2$-origamis

In this section, we study families of normal origamis whose deck transformation groups are isomorphic and have a particular group structure. Starting from dihedral groups and the family of groups $G^2_{(n,k)}$ constructed in Section 3.1, we consider more general families of 2-groups. After investigating the set of normal origamis $\mathcal{O}(G)$ with these groups as deck transformation groups, the $\mathrm{SL}(2,\mathbb{Z})$-action on the set $\mathcal{O}(G)$, and the occurring Veech groups in Section 4.1.1, we compute the sum of non-negative Lyapunov exponents for the normal origamis in $\mathcal{O}(G)$.

We begin by recalling the definition of congruence subgroups.

**Definition 4.1.1**

- For $n \in \mathbb{Z}_+$, the **principal congruence subgroup** $\Gamma(n)$ is defined as the kernel of the natural projection $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$.

- A subgroup $G$ of $\mathrm{SL}(2,\mathbb{Z})$ is a **congruence subgroup** if it contains a principal congruence subgroup $\Gamma(n)$ for some natural number $n$. The minimal $n$ such that $\Gamma(n)$ is contained in $G$ is called the **level** of $G$.

The following lemma and corollary are useful to decide whether the Veech group of a normal origami is a congruence subgroup of level 2.

**Lemma 4.1.2** *For a normal origami $\mathcal{O} = (G, x, y)$, its Veech group contains the parabolic matrices*

$$\begin{pmatrix} 1 & \mathrm{ord}(x) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \mathrm{ord}(y) & 1 \end{pmatrix}.$$

**Proof** Consider the cylinder decomposition in horizontal direction. Each horizontal cylinder has height 1, circumference $\mathrm{ord}(x)$ and thus inverse modulus $\mathrm{ord}(x)$ (see Remark 2.3.7). Hence, the first matrix lies in the Veech group $\mathrm{SL}(\mathcal{O})$ (see Lemma 2.3.8). For the second matrix, consider the cylinder decomposition of $\mathcal{O}$ in vertical direction. These cylinders have height 1, circumference $\mathrm{ord}(y)$ and hence inverse modulus $\mathrm{ord}(y)$. The matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ maps the vertical direction to the horizontal one and is inverse to itself. We compute

$$M \cdot \begin{pmatrix} 1 & \operatorname{ord}(y) \\ 0 & 1 \end{pmatrix} \cdot M = M \cdot \begin{pmatrix} \operatorname{ord}(y) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \operatorname{ord}(y) & 1 \end{pmatrix}.$$

Thus, the second matrix lies in $\mathrm{SL}(\mathcal{O})$. ■

**Corollary 4.1.3** *The Veech group of a normal origami $\mathcal{O} = (G, x, y)$ is a congruence group of level at most 2 if the orders of $x$ and $y$ are 2.*

**Proof** Recall that the principal congruence subgroup $\Gamma(2)$ is generated by the matrices

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \text{ and } ST^{-2}S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

By Lemma 4.1.2, the matrices $T^2$ and $ST^{-2}S^{-1}$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Applying the matrix $S^2$ to the origami $\mathcal{O}$ yields the origami $(G, x^{-1}, y^{-1})$. This equals $\mathcal{O}$ and thus the claim follows. ■

In the rest of this section, we study the Veech groups of a first family of normal origamis. Recall that $D_{2m} = \langle r, s \mid r^m = s^2 = 1, s^{-1}rs = r^{-1} \rangle$ denotes the dihedral group with $2m$ elements for $m \in \mathbb{N}_{>1}$. Further, recall that the set of normal origamis with $D_{2m}$ as deck transformation group is denoted by $\mathcal{O}(D_{2m})$. Note that we call the normal origamis $(D_{2m}, s, sr)$ Penrose stairs origamis $PS_{2m}$ (see Example 2.2.7 and Example 3.2.5). However, we will see in Proposition 4.1.4 that for each $m$ there are two other origamis with the dihedral group of order $2m$ as deck transformation group.

Recall that two normal origamis $\mathcal{O} = (G, x, y)$ and $\mathcal{O}' = (G, x', y')$ are equal if and only if there exists an automorphism $\varphi$ of $G$ sending the pair of generators $(x, y)$ to the pair $(\varphi(x), \varphi(y)) = (x', y')$. Further, the $\mathrm{SL}(2, \mathbb{Z})$-action on the set of normal origamis $\mathcal{O}(G)$ can be described in terms of the pairs of generators (see Remark 2.2.5 and Section 2.3.1 for further information). In the following proposition, we use this to describe the $\mathrm{SL}(2, \mathbb{Z})$-action on all origamis in $\mathcal{O}(D_{2m})$ and study the Veech groups of these origamis. The Veech group of the Penrose stairs origamis $PS_{2m}$ has been computed in a different way by Zmiaikou in [Zmi11, Proposition 4.10 and Section 2.6].

**Proposition 4.1.4** *The set $\mathcal{O}(D_{2m})$ has cardinality 3 and $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on $\mathcal{O}(D_{2m})$. Veech groups of origamis in $\mathcal{O}(D_{2m})$ have index 3 and are congruence subgroups of level 2.*

**Proof** Recall that each element in the dihedral group $D_{2m}$ can be written as $r^j s^k$ with $0 \leq j < m$, $0 \leq k < 2$. Further, note that elements of the form $r^j s$ have order 2. This implies that 2-generating sets of $D_{2m}$ are of the form

(i) $\{r^j, r^k s\}$ with $j \in (\mathbb{Z}/m\mathbb{Z})^*$, $0 \leq k < m$, or

(ii) $\{r^j s, r^k s\}$ with $j - k \in (\mathbb{Z}/m\mathbb{Z})^*$.

Define $\alpha_{(j,k)} : D_{2m} \to D_{2m}$ by $\alpha_{(j,k)}(r) = r^j$ and $\alpha_{(j,k)}(s) = r^k s$ for $j \in (\mathbb{Z}/m\mathbb{Z})^*$ and $0 \leq k < m$. One easily checks that $\alpha_{(j,k)}$ defines an automorphism of $D_{2m}$. Since group automorphisms are order preserving, no further automorphism exists. Each automorphism sends $r$ to a power of $r$ and thus applying an automorphism cannot change the form of a 2-generating set. Pairs of generators of form (i) yield the following two orbits under the action of $\mathrm{Aut}(D_{2m})$

(1) $o_1 = [(r,s)] = \{(r^j, r^k s) \mid j \in (\mathbb{Z}/m\mathbb{Z})^*,\ 0 \leq k < m\}$,

(2) $o_2 = [(s,r)] = \{(r^k s, r^j) \mid j \in (\mathbb{Z}/m\mathbb{Z})^*,\ 0 \leq k < m\}$.

Consider a pair of generators $(r^j s, r^k s)$ with $j - k \in (\mathbb{Z}/m\mathbb{Z})^*$. Applying $\alpha_{(1,-j)}$ yields $(s, r^{k-j}s)$. The automorphism $\alpha_{((k-j)^{-1},0)}$ sends this pair of generators to $(s, rs)$. We conclude that all tuples of generators arising from generating sets of form (ii) lie in the same orbit

(3) $o_3 = [(s, rs)] = \{(r^j s, r^k s) \mid j - k \in (\mathbb{Z}/m\mathbb{Z})^*\}$.

Each of the orbits $o_i = [(g,h)]$ defines a normal origami $\mathcal{O}_i = (D_{2m}, g, h)$ with deck transformation group $D_{2m}$ for $1 \leq i \leq 3$ and each origami in $\mathcal{O}(D_{2m})$ arises in this way. This proves the statement about the cardinality of $\mathcal{O}(D_{2m})$.

To prove the statement about the Veech group, we study the $\mathrm{SL}(2,\mathbb{Z})$-action on the set $\mathcal{O}(D_{2m})$. We first consider the action of the matrix $S$:

$$\begin{aligned} S \cdot \mathcal{O}_1 &= (D_{2m}, s^{-1}, r) = \mathcal{O}_2, \\ S \cdot \mathcal{O}_2 &= (D_{2m}, r^{-1}, s) = \mathcal{O}_1, \\ S \cdot \mathcal{O}_3 &= (D_{2m}, s^{-1}r^{-1}, s) = \mathcal{O}_3. \end{aligned}$$

Hence, the permutation $\sigma_S = (1,2)$ describes the action of $S$. The action of the matrix $T$ on the set $\mathcal{O}(D_{2m})$ is given by:

$$\begin{aligned} T \cdot \mathcal{O}_1 &= (D_{2m}, r, sr^{-1}) = (D_{2m}, r, rs) = \mathcal{O}_1, \\ T \cdot \mathcal{O}_2 &= (D_{2m}, s, rs^{-1}) = \mathcal{O}_3, \\ T \cdot \mathcal{O}_3 &= (D_{2m}, s, rss^{-1}) = \mathcal{O}_2. \end{aligned}$$

This corresponds to the permutation $\sigma_T = (2,3)$. We conclude that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathcal{O}(D_{2m})$. Hence, the Veech groups of the origamis $\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_3$ have index 3 in $\mathrm{SL}(2,\mathbb{Z})$ and are conjugated in $\mathrm{SL}(2,\mathbb{Z})$.

Finally, we show that the Veech groups $\mathrm{SL}(\mathcal{O}_i)$ is a congruence group of level 2 for $1 \leq i \leq 3$. Since principal congruence subgroups are normal in $\mathrm{SL}(2,\mathbb{Z})$ and the Veech groups under consideration are conjugated in $\mathrm{SL}(2,\mathbb{Z})$, it is sufficient to prove the claim for the origami $\mathcal{O}_3$. Note that the orders of the group elements $s$ and $rs$ are 2. By Corollary 4.1.3, the Veech group of the origami $O_3 = (D_{2m}, s, rs)$ is a congruence subgroup of level at most 2. We proved that the index of $\mathrm{SL}(\mathcal{O}_3)$ is 3 and thus the congruence level equals 2. ∎

The family of normal origamis $\mathcal{O}(D_{2m})$ studied in the previous proposition illustrates how one can use the concepts introduced in Chapter 2 to study the $\mathrm{SL}(2,\mathbb{Z})$-action of normal origamis with a fixed deck transformation group and the ocurring Veech groups.

### 4.1.1. 2-origamis with congruence groups as Veech groups

In this section, we construct interesting families of finite groups and use similar techniques as in Proposition 4.1.4 to deduce results on the Veech groups of the corresponding normal origamis and the $\mathrm{SL}(2,\mathbb{Z})$-action. We will see that even if the group structure of the deck transformation groups of normal origamis are very similar, the corresponding normal origamis may behave very differently. To this end, we mainly focus on studying 2-groups that are semidirect products of cyclic groups. However, at the end of this section we consider the family of 2-groups introduced in Example 3.3.11 that are extensions of the quaternion group. We begin by defining generalized dihedral groups.

**Definition 4.1.5** A **generalized dihedral group** is a semidirect product $H \rtimes_\varphi C_2$, where $H$ is an abelian group, $C_2 = \langle g \rangle$ is a cyclic group of size 2, and $\varphi(g) : H \to H,\ h \mapsto h^{-1}$.

Given a generalized dihedral group $G = H \rtimes_\varphi C_2$ that is also a 2-generated 2-group, the group $H$ is cyclic and of order $2^m$ for some $m$. Hence, $G$ is of the form $C_{2^m} \rtimes_\varphi C_2$ with $\varphi$ as above and $m \in \mathbb{Z}_{\geq 0}$. Notice that for $m \geq 1$ the group $C_{2^m} \rtimes_\varphi C_2$ is isomorphic to the group $D_{2^{m+1}} \cong G^2_{(m+1,m-1)}$ (we recall the definition of the groups $G^2_{(n,k)}$ below). As we have studied this case in Proposition 4.1.4, we will not consider these groups in the following.

In Chapter 3, we classified the strata in which $p$-origamis occur. To construct origamis in the respective strata, we considered the following family of 2-groups (see Remark 3.2.4). We recall the definition of the groups $G^2_{(n,k)}$:

$$C_{2^{k+1}} \rtimes_\varphi C_{2^{n-k-1}} = \langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1,\ s^{-1}rs = r^{-1} \rangle,$$

where $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n-2$. In this chapter, we set $G_{(n,k)} := G^2_{(n,k)}$ to simplify the notation. These groups generalize dihedral groups by other means. We use Remark 2.2.5 as well as the definition of the $\mathrm{SL}(2,\mathbb{Z})$-action on the generating pairs to compute all

2-origamis with a group $G_{(n,k)}$ for $1 \leq k \leq n-3$. Note that we do not consider the case $k = n-2$ because the corresponding groups $G_{(n,n-2)}$ are dihedral groups.

We will use frequently several relations and automorphisms of the group $G_{(n,k)}$ when studying the normal origamis with deck transformation group $G_{(n,k)}$. Observe that the relation $s^{-1}rs = r^{-1}$ implies the following relations for an even integer $i$ and an odd integer $j$:

$$\begin{aligned} r^j s &= s r^{-j}, & r s^i &= s^i r, \\ (rs)^j &= r s^j, & (rs)^i &= s^i. \end{aligned}$$

Further, we obtain the following automorphisms for $1 \leq i, \ell \leq 2^{k+1}, 1 \leq j \leq 2^{n-k-1}$ with $i, j$ odd:

$$\begin{aligned} \alpha_{(i,j)} : G_{(n,k)} \to G_{(n,k)} &\text{ defined by } \alpha_{(i,j)}(r) = r^i \text{ and } \alpha_{(i,j)}(s) = s^j, \\ \beta_\ell : G_{(n,k)} \to G_{(n,k)} &\text{ defined by } \beta_\ell(r) = r \text{ and } \beta_\ell(s) = r^\ell s, \\ \gamma : G_{(n,k)} \to G_{(n,k)} &\text{ defined by } \gamma(r) = r s^{2^{n-k-2}} \text{ and } \gamma(s) = s. \end{aligned}$$

To show that these maps are indeed automorphisms, one checks the defining relations for the images of $r$ and $s$ and observes that the images of $r$ and $s$ generate the group $G_{(n,k)}$.

**Proposition 4.1.6** *Let $n, k \in \mathbb{Z}_+$ such that $1 \leq k \leq n-3$. The set $\mathcal{O}(G_{(n,k)})$ of normal origamis with deck transformation group $G_{(n,k)}$ has cardinality $3 \cdot 2^{n-k-3}$ and is contained in the stratum $\mathcal{H}\left(2^{n-k} \times \left(2^k - 1\right)\right)$. Representatives of these origamis are given as the union $R = R_1 \,\dot\cup\, R_2 \,\dot\cup\, R_3$, where*

$$\begin{aligned} R_1 &:= \left\{(G_{(n,k)},\ s,\ rs^m) \mid 1 \leq m \leq 2^{n-k-2},\ 2 \nmid m\right\}, \\ R_2 &:= \left\{(G_{(n,k)},\ s,\ rs^{2\cdot m}) \mid 1 \leq m \leq 2^{n-k-3}\right\}, \\ R_3 &:= \left\{(G_{(n,k)},\ rs^{2\cdot m},\ s) \mid 1 \leq m \leq 2^{n-k-3}\right\}. \end{aligned}$$

**Proof** Fix natural numbers $n, k$ such that $1 \leq k \leq n-3$. Consider the normal origami $(G_{(n,k)}, s, r)$. The commutator $[s,r] = r^2$ has order $2^k$ and thus this origami lies in the claimed stratum. Note that the conjugates $r^{-1}[s,r]r = r^2$ and $s^{-1}[s,r]s = r^{-2}$ are contained in the group generated by $r^2$. We conclude that the commutator subgroup $G'_{(n,k)}$ is generated by the commutator $[s,r]$ and thus it is regular (see Lemma 3.1.3). By Theorem 3.2.11, all normal origamis with deck transformation group $G_{(n,k)}$ lie in the same stratum. Hence, we have $\mathcal{O}(G_{(n,k)}) \subseteq \mathcal{H}\left(2^{n-k} \times \left(2^k - 1\right)\right)$.

The relation $rs = sr^{-1}$ implies that each group element $g \in G_{(n,k)}$ can be written as $r^\ell s^m$ for $1 \leq \ell \leq 2^{k+1}$ and $1 \leq m \leq 2^{n-k-1}$. Further, two group elements $r^{\ell_1}s^{m_1}$ and $r^{\ell_2}s^{m_2}$ with $1 \leq \ell_i \leq 2^{k+1}$ and $1 \leq m_i \leq 2^{n-k-1}$ are equal if and only if $\ell_1 = \ell_2$ and $m_1 = m_2$.

Let $\mathcal{T} = (r^{\ell_1}s^{m_1}, r^{\ell_2}s^{m_2})$ be a tuple of generators of $G_{(n,k)}$. We first show that there is an automorphism $\delta \in \operatorname{Aut}(G_{(n,k)})$ mapping $\mathcal{T}$ to a representative in $R$. Suppose both $m_1$ and $m_2$ are even integers. Then the relation $rs = sr^{-1}$ implies that the group generated by $r^{\ell_1}s^{m_1}$ and $r^{\ell_2}s^{m_2}$ does not contain the group element $s$. However, the tuple $\mathcal{T}$ generates $G_{(n,k)}$ by assumption and this yields a contradiction. We conclude that one of the exponents $m_1$ and $m_2$ is odd.

Assume $m_1$ is odd. For $m_1^{-1} \in (\mathbb{Z}/2^{n-k-1}\mathbb{Z})$, the automorphism $\alpha_{(1,m_1^{-1})}$ maps the tuple $\mathcal{T}$ to

$$\mathcal{T}_1 = (r^{\ell_1}s, r^{\ell_2}s^{m_2m_1^{-1}}).$$

Next, consider the automorphism $\beta_{-\ell_1}$ which maps $\mathcal{T}_1$ to

$$\mathcal{T}_2 = (s, r^{\ell_2-\ell_1 m}s^{m_2m_1^{-1}}),$$

where $m = 1$ if $m_2m_1^{-1}$ is odd, and $m = 0$ else. The tuple $\mathcal{T}_2$ generates $G_{(n,k)}$. Hence, the exponent $\ell := \ell_2 - \ell_1 m$ of $r$ is odd. Applying the automorphism $\alpha_{(\ell^{-1},1)}$ to the tuple $\mathcal{T}_2$, we obtain the tuple of generators $\mathcal{T}_3 = (s, rs^{m_2m_1^{-1}})$.

Since $\gamma : G_{(n,k)} \to G_{(n,k)}$ given by $\gamma(r) = rs^{2^{n-k-2}}$, $\gamma(s) = s$ defines an automorphism, we may assume that $m_2m_1^{-1} \leq 2^{n-k-2}$. Hence, the normal origami $(G_{(n,k)}, s, rs^{m_2m_1^{-1}})$ lies in $R_1$ or $R_2$ and thus is a representative in $R$.

Assume $m_2$ is odd and $m_1$ is even. Analog computations as above yield an automorphism $\delta \in \mathrm{Aut}(G_{(n,k)})$ mapping $\mathcal{T}$ to a tuple of the form $(rs^{2\cdot m}, s)$ for $1 \leq m \leq 2^{n-k-3}$. The normal origami $(G_{(n,k)}, rs^{2\cdot m}, s)$ lies in $R_3$.

It remains to show that each representative given in $R$ defines a different origami. To prove this, one shows that $x_i \mapsto a_j,\ y_i \mapsto b_j$ does not define an automorphism of $G_{(n,k)}$. Here, the normal origamis $(G_{(n,k)}, x_i, y_i)$ and $(G_{(n,k)}, a_j, b_j)$ are elements of $R_i$ and $R_j$, respectively, and $1 \leq i, j \leq 3$.

We begin with the case $i = 2$ and $j = 3$. Let $(G_{(n,k)},\ s,\ rs^{2\cdot\ell})$ and $(G_{(n,k)},\ rs^{2\cdot m},\ s)$ be origamis in $R_2, R_3$, respectively. Suppose that

$$\delta(rs^{2\cdot m}) := s \text{ and } \delta(s) := rs^{2\cdot\ell} \tag{4.1}$$

define an automorphism $\delta$ of $G_{(n,k)}$. We deduce the following equation for the image of $r$

$$\delta(r) = s \cdot (rs^{2\cdot\ell})^{-2m} = r^{2m}s^{1-4\ell m}.$$

Since the relation $s^{-1}rs = r^{-1}$ holds in $G_{(n,k)}$, the images of $r$ and $s$ under $\delta$ have to satisfy the same relation. For the left hand side, we obtain the equality

$$\begin{aligned}\delta(s)^{-1} \cdot \delta(r) \cdot \delta(s) &= \left(rs^{2\cdot\ell}\right)^{-1} \cdot \left(r^{2m}s^{1-4\ell m}\right) \cdot \left(rs^{2\cdot\ell}\right) \\ &= r^{2m-2}s^{1-4\ell m}.\end{aligned} \tag{4.2}$$

For the right hand side, we obtain the equality

$$\delta(r)^{-1} = \left(r^{2m}s^{1-4\ell m}\right)^{-1} = r^{2m}s^{4\ell m-1}. \tag{4.3}$$

Since $2m$ and $2m-2$ are not equivalent modulo $2^{k+1}$, the equations (4.2) and (4.3) are not equal. Hence, (4.1) does not define an automorphism. This implies that the considered normal origamis are different.

We check the other cases in an analogous manner using the relation $rs = sr^{-1}$ in $G_{(n,k)}$. Consider the case $i = 1$ and $j = 2$. The origamis are of the form $\mathcal{O} = (G_{(n,k)}, s, rs^m)$ and $\mathcal{O}' = (G_{(n,k)}, s, rs^\ell)$ with $m$ even and $\ell$ odd. Then there is an automorphism $\delta$ of $G_{(n,k)}$ with $\delta(s) = s$ and $\delta(r) = rs^{m-\ell}$. Note $m-\ell$ is odd. As $\delta$ is an automorphism, we obtain the equality $\delta(r)\delta(s) = \delta(s)\delta(r^{-1})$. For the left side, we obtain

$$rs^{m-\ell}s = rs^{m-\ell+1} = s^{m-\ell+1}r.$$

For the right side, we obtain

$$ss^{\ell-m}r^{-1} = s^{\ell-m+1}r^{-1} =: x.$$

The equality of both sides yields

$$\begin{aligned} 1 &= s^{m-\ell+1}r \cdot x^{-1} \\ &= s^{m-\ell+1}rrs^{m-\ell-1} \\ &= r^2 s^{2m-2\ell+2}. \end{aligned}$$

However, this does not hold because $k \neq 0$.

Consider the case $i = 1$ and $j = 3$. The origamis are of the form $\mathcal{O} = (G_{(n,k)}, s, rs^m)$ and $\mathcal{O}' = (G_{(n,k)}, rs^\ell, s)$ with $m$ odd and $\ell$ even. Then there is an automorphism $\delta$ of $G_{(n,k)}$ with $\delta(s) = rs^\ell$ and $\delta(rs^m) = s$. Then we have

$$\begin{aligned} \delta(r) &= s \cdot \delta(s^m)^{-1} \\ &= s(rs^\ell)^{-m} \\ &= r^m s^{1-m\ell}. \end{aligned}$$

Here, we use that $\ell$ is an even integer. As $\delta$ is an automorphism, we obtain the equality $\delta(r)\delta(s) = \delta(s)\delta(r^{-1})$. For the left side, we obtain

$$r^m s^{1-m\ell} \cdot rs^\ell = r^{m-1}s^{1-m\ell+\ell}.$$

For the right side, we obtain

$$rs^\ell \cdot s^{m\ell-1}r^{-m} = r^{1+m}s^{\ell+m\ell-1} =: x.$$

The equality of both sides yields

$$\begin{aligned} 1 &= r^{m-1}s^{1-m\ell+\ell} \cdot x^{-1} \\ &= r^{m-1}s^{1-m\ell+\ell} \cdot s^{1-\ell-m\ell}r^{-1-m} \\ &= r^{m-1}s^{2-2m\ell}r^{-1-m} \\ &= r^{-2}s^{2-2m\ell}. \end{aligned}$$

This does not hold because $k \neq 0$.

Finally, consider the case $1 \leq i, j \leq 3$ with $i = j$. Without loss of generality, we may assume that the origamis are of the form $\mathcal{O} = (G_{(n,k)}, s, rs^m)$ and $\mathcal{O}' = (G_{(n,k)}, s, rs^\ell)$ such that $m$ and $\ell$ have the same parity and $m \neq \ell$. Then there is an automorphism $\delta$ of $G_{(n,k)}$ with $\delta(s) = s$ and $\delta(rs^m) = rs^\ell$. Then we have

$$\begin{aligned}\delta(r) &= rs^\ell \cdot \delta(s^m)^{-1} \\ &= rs^\ell s^{-m} \\ &= rs^{\ell-m}.\end{aligned}$$

Note $m-\ell$ is even. As $\delta$ is an automorphism, we obtain the equality $\delta(r)\delta(s) = \delta(s)\delta(r^{-1})$. For the left side, we obtain

$$rs^{\ell-m}s = rs^{\ell-m+1}.$$

For the right side, we obtain

$$ss^{m-\ell}r^{-1} = s^{m-\ell+1}r^{-1} =: x.$$

The equality of both sides yields

$$\begin{aligned}1 &= rs^{\ell-m+1} \cdot x^{-1} \\ &= rs^{\ell-m+1}rs^{\ell-m-1} \\ &= s^{2\ell-2m}.\end{aligned}$$

Hence, $2\ell - 2m$ is a multiple of $2^{n-k-2}$ and thus $\ell - m$ is a multiple of $2^{n-k-3}$. This is a contradiction because $0 \neq \ell - m < 2^{n-k-3}$. ∎

**Proposition 4.1.7** *The group* $\mathrm{SL}(2,\mathbb{Z})$ *acts transitively on the set of normal origamis* $\mathcal{O}(G_{(n,k)})$. *The Veech groups of origamis in* $\mathcal{O}(G_{(n,k)})$ *are conjugated in* $\mathrm{SL}(2,\mathbb{Z})$ *and have index* $3 \cdot 2^{n-k-3}$.

**Proof** We use the notation from Proposition 4.1.6. Computing the action of $S$ on origamis in $R_2$ and $R_3$, one obtains

$$\begin{aligned}S \cdot (G_{(n,k)}, s, rs^{2m}) &= (G_{(n,k)}, s^{-2m}r^{-1}, s) \\ &= (G_{(n,k)}, r^{-1}s^{-2m}, s) \\ &= (G_{(n,k)}, rs^{-2m}, s), \\ S \cdot (G_{(n,k)}, rs^{-2m}, s) &= (G_{(n,k)}, s^{-1}, rs^{-2m}) \\ &= (G_{(n,k)}, s, rs^{2m}).\end{aligned}$$

Here, we use that $r \mapsto r^{-1}, s \mapsto s$ and $r \mapsto r, s \mapsto s^{-1}$ define automorphisms of $G_{(n,k)}$. Using the automorphism $\gamma$ with $\gamma(r) = rs^{2^{n-k-2}}$ and $\gamma(s) = s$, we observe that the normal origami $(G_{(n,k)}, rs^{-2m}, s) = (G_{(n,k)}, rs^{2^{n-k-2}-2m}, s)$ lies in $R_3$. We consider the generating tuple $(rs^{-2m}, s)$ to simplify the calculation above. The permutation describing the action of $S$ on the origamis in $R_2$ and $R_3$ consists of disjoint 2-cycles. Each 2-cycle connects an origami in $R_2$ with one in $R_3$. The matrix $T^{-1}$ acts on origamis in $R_1$ and $R_2$ as follows

$$T^{-1} \cdot (G_{(n,k)}, s, rs^i) = (G_{(n,k)}, s, rs^{i+1}).$$

We obtain a $2^{n-k-2}$-cycle describing the action of $T$ on the origamis in $R_1$ and $R_2$. This permutation acts transitively on all origamis in $R_1$ and $R_2$.

Combining these two results, we obtain the following diagram

$$
\begin{tikzcd}
\mathcal{O}_1 \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{n-k-3}+1} \arrow[r, "T^{-1}"'] \arrow[d, bend right, "S"'] & \mathcal{O}_2 \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{n-k-3}+2} \arrow[r, "T^{-1}"'] \arrow[d, bend right, "S"'] & \cdots \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{n-k-3}} \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{n-k-2}} \arrow[d, bend right, "S"'] \arrow[llllll, bend right, "T^{-1}"'] \\
 & \mathcal{O}_{2^{n-k-2}+1} \arrow[u, bend right, "S"'] & & \mathcal{O}_{2^{n-k-2}+2} \arrow[u, bend right, "S"'] & & & \mathcal{O}_{3\cdot 2^{n-k-3}} \arrow[u, bend right, "S"']
\end{tikzcd}
\tag{4.4}
$$

Here, the origamis in $R_{i+1}$ are labeled by $\mathcal{O}_{i\cdot 2^{n-k-3}+1}, \ldots, \mathcal{O}_{(i+1)\cdot 2^{n-k-3}}$ in a suitable way for $0 \leq i \leq 2$. Proposition 4.1.6 implies that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the set of normal origamis $\mathcal{O}(G_{(n,k)})$. This shows that the Veech groups of such origamis have index $3 \cdot 2^{n-k-2}$ in $\mathrm{SL}(2,\mathbb{Z})$ and are conjugated in $\mathrm{SL}(2,\mathbb{Z})$. ∎

In the following corollaries, we use the notation from Proposition 4.1.6. Note that we will use the notation for the sets of representatives $R_1, R_2$, and $R_3$ in later parts of this thesis as well. We will then refer to Proposition 4.1.6. Further, note that we will use both corollaries in the proof of Proposition 4.1.11. There, we show that the Veech group of each normal origami with deck transformation group $G_{(n,k)}$ is a congruence group and we compute the congruence level.

**Corollary 4.1.8** *Let $\mathcal{O}$ be an origami in $R_1 \cup R_2$. The smallest natural number $m \in \mathbb{Z}_+$ with $T^m \in \mathrm{SL}(\mathcal{O})$ is $m = 2^{n-k-2}$.*

We have seen in Diagram (4.4) that the difference $n-k$ is essential when describing the $\mathrm{SL}(2,\mathbb{Z})$-action on the set $\mathcal{O}(G_{(n,k)})$. Note that the representatives in the sets $R_1, R_2$, and $R_3$ also depend on the difference $n-k$ (see Proposition 4.1.6). The subsequent corollary shows that the occurring Veech groups depend only on the difference $n-k$ as well. To see this, we set $d := n-k$. We then use that there is a natural bijection between the sets of representatives if we consider integers $n_1, k_1, n_2, k_2$ with $1 \leq k_i \leq n_i - 3$ and $d = n_i - k_i$ for $i = 1, 2$. Further, we show that the $\mathrm{SL}(2,\mathbb{Z})$-action on these sets of representatives coincide.

**Corollary 4.1.9** *Let $(n_1, k_1)$ and $(n_2, k_2)$ be pairs of natural numbers with $1 \leq k_i \leq n_i - 3$ and $n_1 - k_1 = n_2 - k_2$. Further, let $F_2 = \langle \tilde{r}, \tilde{s} \rangle$ be the free group on two generators and $\mu_i : F_2 \to G_{(n_i,k_i)}$ be the group homomorphism defined by $\mu_i(\tilde{r}) = r$ and $\mu_i(\tilde{s}) = s$ with $i = 1, 2$. Given words $v, w \in F_2$ such that $(G_{(n_1,k_1)}, \mu_1(v), \mu_1(w))$ defines a normal origami, the tuple $(G_{(n_2,k_2)}, \mu_2(v), \mu_2(w))$ defines a normal origami as well. The Veech groups of both origamis coincide.*

**Proof** Without loss of generality, we may assume that $\mathcal{O}_i := (G_{(n_i,k_i)}, \mu_i(v), \mu_i(w))$ is one of the representatives given in Proposition 4.1.6 for $i = 1, 2$. The equality $n_1 - k_1 = n_2 - k_2$

implies that the representatives of the normal origamis with deck transformation group $G_{(n_1,k_1)}$ and deck transformation group $G_{(n_2,k_2)}$ are equal (as formal words in the generators $r$ and $s$). Denote $d = n_1 - k_1$.

We claim that the $\mathrm{SL}(2,\mathbb{Z})$-action on the representatives coincides for $G_{(n_1,k_1)}$ and $G_{(n_2,k_2)}$. For this, note that $\mu_i(v) = s$ or $\mu_i(w) = s$ for $i = 1, 2$. We have

$$T \cdot \mathcal{O}_i = (G_{(n_i,k_i)}, \mu_i(v), \mu_i(w)\mu_i(v)^{-1}).$$

If $\mu_i(v) = s$, then $\mu_i(w) = rs^\ell$ holds for some natural number $\ell$ with $1 \le \ell \le 2^{n-k-2}$. Hence, we compute $T \cdot \mathcal{O}_i = (G_{(n_i,k_i)}, s, rs^{\ell-1})$, i.e., the action of the matrix $T$ on these representatives coincides. If $\mu_i(w) = s$, then $\mu_i(v) = rs^{2\cdot\ell}$ holds for some natural number $\ell$ with $1 \le m \le 2^{n-k-3}$. Hence, we obtain

$$\begin{aligned} T \cdot \mathcal{O}_i &= (G_{(n_i,k_i)}, rs^{2\cdot\ell}, s(rs^{2\cdot\ell})^{-1}) \\ &= (G_{(n_i,k_i)}, rs^{2\cdot\ell}, s^{1-2\cdot\ell}r^{-1}) \\ &= (G_{(n_i,k_i)}, rs^{2\cdot\ell\cdot a}, rs) \\ &= (G_{(n_i,k_i)}, r(r^{-1}s)^{2\cdot\ell\cdot a}, s) \\ &= (G_{(n_i,k_i)}, rs^{2\cdot\ell\cdot a}, s). \end{aligned}$$

Here, $a$ denotes $(1 - 2\ell)^{-1} \in \mathbb{Z}/2^{d-1}$. Recall $d = n_1 - k_1$. If $2 \cdot \ell \cdot a > 2^{d-3}$ holds, apply the automorphism defined by $s \mapsto s,\ r \mapsto rs^{2^{d-2}}$. This proves the claim for the action of the matrix $T$ on the sets $\mathcal{O}(G_{(n_i,k_i)})$ for $i = 1, 2$.

The action of $S$ is given by

$$S \cdot \mathcal{O}_i = (G_{(n_i,k_i)}, \mu_i(w)^{-1}, \mu_i(v)).$$

For $\{\mu_i(v), \mu_i(w)\} = \{s, rs^{2\cdot\ell}\}$, we compute

$$\begin{aligned} S \cdot (G_{(n_i,k_i)}, s, rs^{2\cdot\ell}) &= (G_{(n_i,k_i)}, s^{-2\cdot\ell}r^{-1}, s) \\ &= (G_{(n_i,k_i)}, rs^{-2\cdot\ell}, s), \\ S \cdot (G_{(n_i,k_i)}, rs^{2\cdot\ell}, s) &= (G_{(n_i,k_i)}, s^{-1}, rs^{2\cdot\ell}) \\ &= (G_{(n_i,k_i)}, s, rs^{-2\cdot\ell}). \end{aligned}$$

Since $s \mapsto s,\ r \mapsto rs^{2^{d-2}}$ defines an automorphism of $G_{(n_i,k_i)}$ for $i = 1, 2$, we deduce that the action of the matrix $S$ on these representatives coincides. For an odd natural number $\ell$ with $1 \le \ell \le 2^{d-2}$, we obtain

$$\begin{aligned} S \cdot (G_{(n_i,k_i)}, s, rs^{\ell}) &= (G_{(n_i,k_i)}, s^{-\ell}r^{-1}, s) \\ &= (G_{(n_i,k_i)}, rs^{-\ell}, s), \\ &= (G_{(n_i,k_i)}, r(r^{-1}s)^{-\ell}, r^{-1}s), \\ &= (G_{(n_i,k_i)}, s^{-\ell}, r^{-1}s), \\ &= (G_{(n_i,k_i)}, s, rs^{(-\ell)^{-1}}). \end{aligned}$$

Hence, the action of $S$ on the representatives coincides. We have shown that the $\mathrm{SL}(2,\mathbb{Z})$-action on the sets $\mathcal{O}(G_{(n_i,k_i)})$ coincide for $i = 1, 2$. This implies that the Veech groups of the origamis $(G_{(n_1,k_1)}, \mu_1(v), \mu_1(w))$ and $(G_{(n_2,k_2)}, \mu_2(v), \mu_2(w))$ coincide as well. ■

Using [Sch05, Proposition 3.5.] and the natural projection $\mathbb{Z}^2 \to (\mathbb{Z}/m\mathbb{Z})^2$, one obtains the following lemma.

**Lemma 4.1.10** *Let $m \in \mathbb{Z}_+$, $M \in \Gamma(m)$, and $(G, x, y)$ be a normal origami. There exist $v, w \in F_2 = \langle \tilde{x}, \tilde{y} \rangle$ with*

$$\begin{aligned} \#_{\tilde{x}}\ v &\equiv 1 \mod m, & \#_{\tilde{x}}\ w &\equiv 0 \mod m,\\ \#_{\tilde{y}}\ v &\equiv 0 \mod m, & \#_{\tilde{y}}\ w &\equiv 1 \mod m. \end{aligned}$$

*and $M \cdot (G, x, y) = (G, \mu(v), \mu(w))$. Here, $\mu$ denotes the monodromy map $F_2 \to G$ with $\mu(\tilde{x}) = x$ and $\mu(\tilde{y}) = y$. Further, $\#_{\tilde{x}}\ z$ denotes the number of $\tilde{x}$ minus the number of $\tilde{x}^{-1}$ appearing in a word $z \in F_2$. One defines $\#_{\tilde{y}}$ analogously.*

**Proposition 4.1.11** *The Veech groups of 2-origamis with deck transformation group $G_{(n,k)}$ are congruence subgroups of level $2^{n-k-2}$ for $1 \le k \le n-3$.*

**Proof** By Corollary 4.1.9, it is sufficient to consider the groups $G_{(n,1)}$ for $n \ge 3$. Principal congruence groups are normal subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the 2-origamis with deck transformation group $G_{(n,1)}$ (see Proposition 4.1.7). Hence, it suffices to consider the origami $\mathcal{O} := (G_{(n,1)}, r, s)$. Consider a matrix $M \in \Gamma(2^{n-3})$. By Lemma 4.1.10, we have $M \cdot \mathcal{O} = (G_{(n,1)}, \mu(v(\tilde{r}, \tilde{s})), \mu(w(\tilde{r}, \tilde{s})))$ for words $v, w \in F_2 = \langle \tilde{r}, \tilde{s} \rangle$ with

$$\begin{aligned} \#_{\tilde{r}}\ v &\equiv 1 \mod 2^{n-3}, & \#_{\tilde{r}}\ w &\equiv 0 \mod 2^{n-3},\\ \#_{\tilde{s}}\ v &\equiv 0 \mod 2^{n-3}, & \#_{\tilde{s}}\ w &\equiv 1 \mod 2^{n-3}. \end{aligned}$$

We write $\mu(v)$ as $r^i s^j$ for some $i, j$ with $1 \le i \le 4$ and $1 \le j \le 2^{n-2}$. The relation $rs = sr^{-1}$ implies that $j \in \{2^{n-2}, 2^{n-3}\}$ and $i$ is odd. Moreover, $\mu(w) = r^\ell s^m$ for some $\ell, m$ with $m \in \{1, 1+2^{n-3}\}$. Applying the automorphisms $\alpha_{(1,m^{-1})}, \beta_{-\ell}, \alpha_{(i^{-1},1)}$ and $\gamma$, we obtain

$$\begin{aligned} M \cdot \mathcal{O} &= (G_{(n,1)}, r^i s^j, r^\ell s^m)\\ &= (G_{(n,1)}, r^i s^{jm^{-1}}, r^\ell s)\\ &= (G_{(n,1)}, r^i s^{jm^{-1}}, s)\\ &= (G_{(n,1)}, r s^{jm^{-1}}, s)\\ &= (G_{(n,1)}, r, s)\\ &= \mathcal{O}. \end{aligned}$$

Hence, the Veech group $\mathrm{SL}(\mathcal{O})$ contains $M$ and is thus a congruence subgroup of level at most $2^{n-3}$.

Finally, we show that the Veech group $\mathrm{SL}(\mathcal{O})$ contains no principal congruence subgroup $\Gamma(m)$ for $m < 2^{n-3}$. To this end, consider the matrices $T^m = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ for $m \in \mathbb{Z}_+$. By Corollary 4.1.8, the matrix $T^m$ does not lie in $\mathrm{SL}((G_{(n,1)}, s, r))$ for $m < 2^{n-3}$. Using $\mathcal{O} \in \mathrm{SL}(2,\mathbb{Z}) \cdot (G_{(n,1)}, s, r)$ and $\Gamma(m) \trianglelefteq \mathrm{SL}(2,\mathbb{Z})$, we deduce that the congruence level is $2^{n-3}$. The claim follows from Corollary 4.1.9 for all $k, n \in \mathbb{Z}_+$ with $1 \le k \le n-3$. ∎

In a next step, we allow that the group homomorphism defining the semidirect product structure differs. More precisely, we study normal origamis with semidirect products of the form $C_{2^m} \rtimes_\varphi C_2$ as deck transformation groups. Although the group structures are very similar, the corresponding normal origamis and Veech groups behave differently (see Proposition 4.1.13, Proposition 4.1.16, and Proposition 4.1.18).

**Lemma 4.1.12** *Let $C_2$ and $C_{2^m}$ be cyclic groups generated by $s$ and $r$, respectively. For $m > 2$, the nontrivial group homomorphisms $C_2 \to \mathrm{Aut}(C_{2^m})$ are given by*

$$\begin{aligned}
\varphi_1(s) &= \left(C_{2^m} \to C_{2^m},\ r \mapsto r^{-1}\right),\\
\varphi_2(s) &= \left(C_{2^m} \to C_{2^m},\ r \mapsto r^{2^{m-1}-1}\right),\\
\varphi_3(s) &= \left(C_{2^m} \to C_{2^m},\ r \mapsto r^{1-2^{m-1}}\right).
\end{aligned}$$

*The nontrivial semidirect products of the form $C_{2^m} \rtimes C_2$ are given by $C_{2^m} \rtimes_{\varphi_i} C_2$ for $1 \leq i \leq 3$.*

**Proof** Each nontrivial group homomorphism $C_2 \to \mathrm{Aut}(C_{2^m})$ is uniquely defined by an element of order 2 in the automorphism group $\mathrm{Aut}(C_{2^m})$ (the image of the generator $s$). Note that we have the following isomorphisms

$$\mathrm{Aut}(C_{2^m}) \cong (\mathbb{Z}/2^m\mathbb{Z})^* \cong C_2 \times C_{2^{m-2}}$$

(see, e.g., [Hup67, Kapitel I, Satz 4.6]). Hence, $\mathrm{Aut}(C_{2^m})$ has thus three elements of order 2. These are given by $\varphi_1, \varphi_2$, and $\varphi_3$. Thus, the nontrivial semidirect products are of the form $C_{2^m} \rtimes_{\varphi_i} C_2$ for $1 \leq i \leq 3$. ■

Recall that $C_{2^m} \rtimes_{\varphi_1} C_2$ is equal to the dihedral group $D_{2^{m+1}}$. The corresponding normal origamis have been studied in Proposition 4.1.4. For $m \geq 2$, define

$$\begin{aligned}
A_m &:= C_{2^m} \rtimes_{\varphi_2} C_2 = \langle r, s \mid s^2 = r^{2^m} = 1,\ s^{-1}rs = r^{2^{m-1}-1}\rangle,\\
B_m &:= C_{2^m} \rtimes_{\varphi_3} C_2 = \langle r, s \mid s^2 = r^{2^m} = 1,\ s^{-1}rs = r^{1-2^{m-1}}\rangle.
\end{aligned}$$

Note that the group $A_2$ is isomorphic to the direct product $C_4 \times C_2$ and the group $B_2$ is isomorphic to the dihedral group $D_8$. Hence, we only consider the groups $A_m$ and $B_m$ for $m > 2$. Further, note that because of the last relation each element in $A_m$ and $B_m$ can be written in the form $s^j r^\ell$ with $0 \leq j \leq 1$ and $0 \leq \ell < 2^m$. This will be used extensively throughout the rest of this section. In the following, we study the sets of normal origamis $\mathcal{O}(A_m)$ and $\mathcal{O}(B_m)$.

**Proposition 4.1.13** *For $m > 2$, the set $\mathcal{O}(A_m)$ is contained in the stratum $\mathcal{H}\left(4 \times (2^{m-1}-1)\right)$ and has cardinality 6. The group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathcal{O}(A_m)$ and the Veech group of each origami in $\mathcal{O}(A_m)$ is the principal congruence group $\Gamma(2)$.*

**Proof** Fix a natural number $m > 2$. Consider the normal origami $\mathcal{O} = (A_m, r, s)$. The commutator $[r, s] = r^{2^{m-1}-2}$ has order $2^{m-1}$ and thus $\mathcal{O}$ lies in the claimed stratum. Using Lemma 3.1.3 and the relation $s^{-1}rs = r^{2^{m-1}-1}$, we conclude that the commutator subgroup $A_m'$ is generated by the commutator $[r, s]$ and thus $A_m'$ is regular. By

Theorem 3.2.11, all normal origamis with deck transformation group $A_m$ lie in the same stratum. Hence, we have the inclusion $\mathcal{O}(A_m) \subseteq \mathcal{H}\left(4 \times (2^{m-1}-1)\right)$.

In a next step, we compute the set $\mathcal{O}(A_m)$ using Remark 2.2.5. For this, we determine the automorphism group $\mathrm{Aut}(A_m)$. Since automorphisms preserve the order of elements, $r$ and $s$ are mapped to elements of order $2^m$ and 2, respectively. The elements of order $2^m$ and 2 are

$$\mathcal{R} := \{r^\ell \mid \ell \equiv_2 1\} \text{ and } \mathcal{S} := \{r^{2^{m-1}}, sr^\ell \mid \ell \equiv_2 0\},$$

respectively. If $s$ is mapped to $r^{2^{m-1}}$, the images of $r$ and $s$ do not generate $A_m$. Hence, this cannot happen. For $\ell \equiv_2 1$ and $j \equiv_2 0$, one easily checks that $\alpha_{(\ell,j)} : A_m \to A_m$ with

$$\alpha_{(\ell,j)}(r) = r^\ell, \ \alpha_{(\ell,j)}(s) = sr^j$$

defines an automorphism. Hence, $\mathrm{Aut}(A_m)$ contains only automorphisms of this form.

The 2-generating sets of $A_m$ are of the form

(i) $\{sr^\ell, r^j\}$ with $j \equiv_2 1$ or

(ii) $\{sr^\ell, sr^j\}$ with $\ell + j \equiv_2 1$.

Applying an automorphism of $A_m$ cannot change the form of the 2-generating set because each automorphism sends $r$ to a power of $r$. The generating sets of form (i) yield the following four orbits under the action of $\mathrm{Aut}(A_m)$

(1) $o_1 = [(s,r)] = \{(sr^\ell, r^j) \mid \ell \equiv_2 0, j \equiv_2 1\}$,

(2) $o_2 = [(r,s)] = \{(r^j, sr^\ell) \mid \ell \equiv_2 0, j \equiv_2 1\}$,

(3) $o_3 = [(sr,r)] = \{(sr^\ell, r^j) \mid \ell \equiv_2 1, j \equiv_2 1\}$,

(4) $o_4 = [(r,sr)] = \{(r^j, sr^\ell) \mid \ell \equiv_2 1, j \equiv_2 1\}$.

To see this, observe first that all elements inside a set $o_i$ differ by an automorphism of $A_m$ for $1 \leq i \leq 4$. For instance, applying the automorphism $\alpha_{(\ell,j)}$ to the pair of generators $(s,r)$ yields the pair $(sr^\ell, r^j)$ for $\ell \equiv_2 0$ and $j \equiv_2 1$. Further, check that two elements $(x,y)$ and $(x',y')$ of sets $o_i$ and $o_j$, respectively, do not lie in the same $\mathrm{Aut}(A_m)$-orbit for $i \neq j$ because $\gamma(x) := x'$ and $\gamma(y) := y'$ does not define an automorphism of $A_m$. Recall that we have computed all automorphisms of $A_m$ above. For instance, assume that the pairs of generators $(s,r)$ and $(r,s)$ lie in the same $\mathrm{Aut}(A_m)$-orbit. Then $\gamma(r) := s$ and $\gamma(s) := r$ define an automorphism. This yields a contradiction because all automorphism of $A_m$ are of the form $\alpha_{(\ell,j)}$ for $\ell \equiv_2 0$ and $j \equiv_2 1$. Hence, the pairs of generators $(s,r)$ and $(r,s)$ lie in the different $\mathrm{Aut}(A_m)$-orbits.

The generating sets of form (ii) yield two orbits under the action of $\mathrm{Aut}(A_m)$

(5) $o_5 = [(sr, s)] = \{(sr^\ell, sr^j) \mid \ell \equiv_2 1, j \equiv_2 0\}$,

(6) $o_6 = [(s, sr)] = \{(sr^\ell, sr^j) \mid \ell \equiv_2 0, j \equiv_2 1\}$.

Each of the orbits $o_i = [(g, h)]$ defines a normal origami $\mathcal{O}_i = (A_m, g, h)$ with deck transformation group $A_m$ for $1 \leq i \leq 6$. This shows the statement about the cardinality of $\mathcal{O}(A_m)$.

To prove the statement about the Veech groups, we study the $\mathrm{SL}(2, \mathbb{Z})$-action on $\mathcal{O}(A_m)$. We first consider the action of the matrix $S$.

$$\begin{aligned}
S \cdot \mathcal{O}_1 &= (A_m, r^{-1}, s) = (A_m, r, s) = \mathcal{O}_2,\\
S \cdot \mathcal{O}_2 &= (A_m, s^{-1}, r) = (A_m, s, r) = \mathcal{O}_1,\\
S \cdot \mathcal{O}_3 &= (A_m, r^{-1}, sr) = (A_m, r, sr) = \mathcal{O}_4,\\
S \cdot \mathcal{O}_4 &= (A_m, r^{-1}s^{-1}, r) = (A_m, sr^{-(2^{m-1}-1)}, r) = \mathcal{O}_3,\\
S \cdot \mathcal{O}_5 &= (A_m, s^{-1}, sr) = (A_m, s, sr) = \mathcal{O}_6,\\
S \cdot \mathcal{O}_6 &= (A_m, r^{-1}s^{-1}, s) = (A_m, sr^{-(2^{m-1}-1)}, s) = \mathcal{O}_5.
\end{aligned}$$

Hence, the permutation $\sigma_S = (1, 2)(3, 4)(5, 6)$ describes the action of $S$. The action of the matrix $T$ is given by

$$\begin{aligned}
T \cdot \mathcal{O}_1 &= (A_m, s, rs^{-1}) = (A_m, s, sr^{2^{m-1}-1}) = \mathcal{O}_6,\\
T \cdot \mathcal{O}_2 &= (A_m, r, sr^{-1}) = (A_m, r, sr) = \mathcal{O}_4,\\
T \cdot \mathcal{O}_3 &= (A_m, sr, rr^{-1}s^{-1}) = (A_m, sr, s) = \mathcal{O}_5,\\
T \cdot \mathcal{O}_4 &= (A_m, r, srr^{-1}) = (A_m, r, s) = \mathcal{O}_2,\\
T \cdot \mathcal{O}_5 &= (A_m, sr, sr^{-1}s^{-1}) = (A_m, sr, r^{1-2^{m-1}}) = \mathcal{O}_3,\\
T \cdot \mathcal{O}_6 &= (A_m, s, srs^{-1}) = (A_m, s, r^{2^{m-1}-1}) = \mathcal{O}_1.
\end{aligned}$$

This corresponds to the permutation $\sigma_T = (1, 6)(2, 4)(3, 5)$. We conclude that $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on $\mathcal{O}(A_m)$. Hence, the Veech groups of the origamis have index 6 in $\mathrm{SL}(2, \mathbb{Z})$ and are conjugated in $\mathrm{SL}(2, \mathbb{Z})$.

Finally, we show that the Veech group $\Gamma(\mathcal{O}_1)$ is the principal congruence group $\Gamma(2)$. Recall that $\Gamma(2)$ is generated by the matrices

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \text{ and } ST^{-2}S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Since $\sigma_S$ and $\sigma_T$ consist of disjoint transpositions, these matrices fix all origamis in $\mathcal{O}(A_m)$. Hence, we have the inclusion $\Gamma(2) \subseteq \Gamma(\mathcal{O}_i)$ for $1 \leq i \leq 6$. As $\Gamma(2)$ has index 6 in $\mathrm{SL}(2, \mathbb{Z})$, equality follows. ■

**Remark 4.1.14** Observe that the 6 normal origamis with $A_m$ as deck transformation are defined by the following pairs of generators $(s, r), (s, sr), (sr, r), (sr, s), (r, s)$, and $(r, sr)$

for each $m > 2$. The first generators of these pairs have order $2, 2, 4, 4, 2^m$, and $2^m$. We will use this fact in the proof of Proposition 4.1.26.

**Example 4.1.15** For $m = 3$, we obtain the group

$$A_3 = C_8 \rtimes_\varphi C_2 = \langle r, s \mid r^8 = s^2 = 1,\ s^{-1}rs = r^3 \rangle.$$

We consider the origami $\mathcal{O} = (A_3, r, s)$. The commutator $[r, s] = r^2$ has order 4 and thus $\mathcal{O}$ lies in $\mathcal{H}(4 \times 3)$.

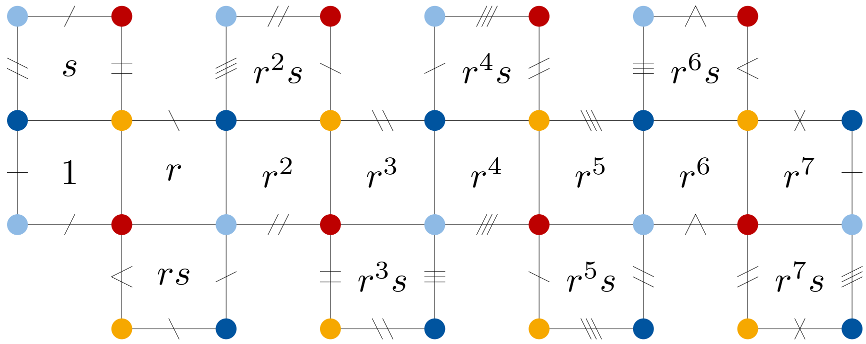


Figure 4.1.: The $\mathrm{SL}(2,\mathbb{Z})$-orbit of the origami $\mathcal{O} = (A_3, r, s)$ contains 6 origamis. The Veech group $\mathrm{SL}(\mathcal{O})$ is the principal congruence group $\Gamma(2)$.

We now turn toward the second family of 2-groups defined as a semidirect products. In the following, we study for $m > 2$ the set of normal origamis with deck transformation group

$$B_m = \langle r, s \mid s^2 = r^{2^m} = 1,\ s^{-1}rs = r^{1-2^{m-1}} \rangle,$$

denoted by $\mathcal{O}(B_m)$.

**Proposition 4.1.16** *For $m > 2$, the set $\mathcal{O}(B_m)$ is contained in the stratum $\mathcal{H}(2^m \times 1)$ and has cardinality $3 \cdot 2^{m-2}$. The group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathcal{O}(B_m)$ and the Veech group of each origami in $\mathcal{O}(B_m)$ has index $3 \cdot 2^{m-2}$.*

**Proof** Fix a natural number $m > 2$. Consider the normal origami $\mathcal{O} = (B_m, r, s)$. The commutator $[r, s] = r^{-2^{m-1}}$ has order 2 and thus $\mathcal{O}$ lies in the claimed stratum. Since the commutator subgroup $B'_m$ is generated by the commutator $[r, s]$, it is regular. By Theorem 3.2.11, all normal origamis with deck transformation group $B_m$ lie in the same stratum. Hence, we have $\mathcal{O}(B_m) \subseteq \mathcal{H}(2^{m-1} \times 1)$.

In a next step, we compute the set $\mathcal{O}(B_m)$ as described in Remark 2.2.5. For this, we determine the automorphism group $\mathrm{Aut}(B_m)$. Since automorphisms preserve the order of elements, $r$ and $s$ are mapped to elements of order $2^m$ and 2, respectively. The elements of order $2^m$ and 2 are

$$\mathcal{R} := \{r^\ell, sr^\ell \mid \ell \equiv_2 1\} \text{ and } \mathcal{S} := \{s, sr^{2^{m-1}}, r^{2^{m-1}}\},$$

respectively.

The images of $s$ and $r$ need to generate $B_m$. Hence, if $s$ is mapped to $r^{2^{m-1}}$ then $r$ has to be mapped to $sr^\ell$ for $\ell \equiv_2 1$. Suppose $\gamma(s) = r^{2^{m-1}}$ and $\gamma(r) = sr^\ell$ with $\ell$ odd defines an automorphism of $B_m$. We compute

$$\begin{aligned}
\gamma(s)^{-1}\gamma(r)\gamma(s) &= r^{-2^{m-1}} sr^\ell r^{2^{m-1}} \\
&= sr^{-2^{m-1}\cdot(1-2^{m-1})+2^{m-1}+\ell} \\
&= sr^\ell \\
&\neq \left(sr^\ell\right)^{1-2^{m-1}} \\
&= \gamma(r)^{1-2^{m-1}}.
\end{aligned}$$

Here we use that $sr^\ell$ has order $2^m$. We conclude that there is no automorphism mapping $s$ to $r^{2^{m-1}}$. For $\ell \equiv_2 1$ and $j \in \{0,1\}$, we claim that $\alpha_{(\ell,j)},\ \beta_{(\ell,j)} : B_m \to B_m$ with

$$\begin{aligned}
&\alpha_{(\ell,j)}(r) = r^\ell,\ \alpha_{(\ell,j)}(s) = sr^{2^{m-1}\cdot j}, \\
&\beta_{(\ell,j)}(r) = sr^\ell,\ \beta_{(\ell,j)}(s) = sr^{2^{m-1}\cdot j}
\end{aligned} \tag{4.5}$$

define automorphisms. We prove that $\beta_{(\ell,j)}$ defines an automorphism. Note that $\beta_{(\ell,j)}(s)$ and $\beta_{(\ell,j)}(r)$ generate the group $B_m$.

First, we check that $\beta_{(\ell,j)}(s)^2 = 1$. We compute

$$\begin{aligned}
\beta_{(\ell,j)}(s)^2 &= s^2 r^{(1-2^{m-1})\cdot 2^{m-1}j + 2^{m-1}\cdot j} \\
&= r^{(2-2^{m-1})\cdot 2^{m-1}j} \\
&= r^{(1-2^{m-2})\cdot 2^m j} \\
&= 1.
\end{aligned}$$

Now, we check that $\beta_{(\ell,j)}(r)^{2^m} = 1$. We have

$$\begin{aligned}
\beta_{(\ell,j)}(r)^{2^m} &= (sr^\ell \cdot sr^\ell)^{2^{m-1}} \\
&= \left(s^2 r^{\ell+(1-2^{m-1})\cdot\ell}\right)^{2^{m-1}} \\
&= r^{2^m\ell\cdot(1-2^{m-2})} \\
&= 1.
\end{aligned}$$

Finally, we check the relation

$$\beta_{(\ell,j)}(r)\beta_{(\ell,j)}(s) = \beta_{(\ell,j)}(s)\beta_{(\ell,j)}(r)^{1-2^{m-1}}.$$

For the left side, we obtain

$$\begin{aligned}
sr^\ell \cdot sr^{2^{m-1}j} &= s^2 r^{\ell(1-2^{m-1})+2^{m-1}j} \\
&= r^{\ell(1-2^{m-1})+2^{m-1}j}.
\end{aligned}$$

For the right side, we compute

$$\begin{aligned} sr^{2^{m-1}j} \cdot (sr^\ell)^{1-2^{m-1}} &= (sr^{2^{m-1}j} sr^\ell)(sr^\ell)^{-2^{m-1}} \\ &= r^{(1-2^{m-1})2^{m-1}j+\ell} \cdot r^{\ell\cdot(2-2^{m-1})\cdot(-2^{m-2})} \\ &= r^{2^{m-1}j+\ell+\ell 2^{m-1}} \\ &= r^{\ell(1+2^{m-1})+2^{m-1}j}. \end{aligned}$$

Hence, the equality holds and $\beta_{(\ell,j)}$ defines an automorphism. Analogously, one checks that $\alpha_{(\ell,j)}$ is a homomorphism. The computations are less tedious and we leave this to the reader. We conclude that the automorphism group $\mathrm{Aut}(B_m)$ consists of the automorphisms $\alpha_{(\ell,j)}$ and $\beta_{(\ell,j)}$ for $\ell \equiv_2 1$ and $j \in \{0,1\}$.

The 2-generating sets of $B_m$ are of the form

(i) $\{sr^\ell, r^j\}$ with $\ell \equiv_2 1$ and $j \equiv_2 1$,

(ii) $\{sr^\ell, r^j\}$ with $\ell \equiv_2 0$ and $j \equiv_2 1$,

(iii) $\{sr^\ell, sr^j\}$ with $\ell \equiv_2 1$ and $j \equiv_2 0$.

Recall that we denote the pairs of generators by $E(B_m) = \{(x,y) \mid \langle x,y\rangle = B_m\}$. We claim that there are three types of orbits under the action of $\mathrm{Aut}(B_m)$ on the set $E(B_m)$ given by

(1) $[(r, sr^i)]$ for $1 \le i \le 2^{m-1}, i \equiv_2 1$,

(2) $[(r, sr^i)]$ for $1 \le i \le 2^{m-1}, i \equiv_2 0$,

(3) $[(sr^i, r)]$ for $1 \le i \le 2^{m-1}, i \equiv_2 0$.

Observe that an automorphism fixing $r$ is either the identity or sends $s$ to $sr^{2^{m-1}}$. This implies that the above orbits are all disjoint.

It remains to show that each pair of generators lies in one of the orbits. Let $(sr^k, r^i)$ be a pair of generators with $k \equiv_2 1$ and $i \equiv_2 1$. Applying $\beta_{(1,0)}$ yields $(r^{k_1}, sr^{i_1})$ with $k_1 \equiv_2 1$ and $i_1 \equiv_2 1$. We apply $\alpha_{(k_1^{-1},0)}$ and obtain $(r, sr^{i_1 k_1^{-1}})$. If $i_1 k_1^{-1} > 2^{m-1}$, then apply $\alpha_{(1,1)}$ to conclude that this pair of generators lies in an orbit of the form (1). Given a pair of generators $(r^i, sr^k)$ with $i \equiv_2 1$, apply $\alpha_{(i^{-1},0)}$. One obtains $(r, sr^{ki^{-1}})$. By the above argument, this lies in an orbit of the form (1) or (2).

Given a pair of generators $(sr^k, r^i)$ with $k \equiv_2 0$ and $i \equiv_2 1$, the automorphism $\alpha_{(i^{-1},0)}$ sends this pair to $(sr^{ki^{-1}}, r)$. Since $ki^{-1} \equiv_2 0$, such a pair of generators lies in an orbit of the form (3).

Finally, we consider pair of generators of form (iii). Let $(sr^k, sr^i)$ with $k \equiv_2 1$ and $i \equiv_2 0$. Applying the automorphism $\beta_{(1,0)}$ yields $(r^{k_1}, sr^{i_1})$ with $k_1 \equiv_2 1$ and $i_1 \equiv_2 0$. Hence, this pair of generators lies in an orbit of the form (2). Given a pair $(sr^k, sr^i)$ with $k \equiv_2 0$ and $i \equiv_2 1$. Again, we apply $\beta_{(1,0)}$ and obtain $(sr^{k_1}, r^{i_1})$ with $k_1 \equiv_2 0$ and $i_1 \equiv_2 1$. Thus, this pair of generators lies in an orbit of the form (3). We conclude that $E(B_m)/\operatorname{Aut}(B_m)$ consists of the orbits $[(r, sr^i)], [(r, sr^j)]$, and $[(sr^j, r)]$ with $1 \leq i, j \leq 2^{m-1}, i \equiv_2 1, j \equiv_2 0$. Thus, the set $\mathcal{O}(B_m)$ has cardinality $3 \cdot 2^{m-2}$.

Lastly, we prove that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathcal{O}(B_m)$. Number the origamis in $\mathcal{O}(B_m)$ as follows: Let $\mathcal{O}_1, \ldots, \mathcal{O}_{2^{m-2}}$ denote the origamis defined by pairs of generators of type (1). For pairs of generators of type (2), denote the corresponding origamis by $\mathcal{O}_{2^{m-2}+1}, \ldots, \mathcal{O}_{2^{m-1}}$. Finally, let $\mathcal{O}_{2^{m-1}+1}, \ldots, \mathcal{O}_{3\cdot 2^{m-2}}$ denote the origamis with generators of type (3). Computing the action of $S$ one obtains

$$\begin{aligned} S \cdot (B_m, r, sr^i) &= (B_m, r^{-i}s^{-1}, r) \\ &= (B_m, sr^{-i\cdot(1-2^{m-1})}, r) \\ &= (B_m, sr^{-i}, r), \\ S \cdot (B_m, sr^{-i}, r) &= (B_m, r^{-1}, sr^{-i}) \\ &= (B_m, r, sr^i), \end{aligned}$$

for an even integer $i$ with $1 \leq i \leq 2^{m-1}$. We conclude that the permutation describing the action of the matrix $S$ on the origamis of type (2) and (3) consists of 2-cycles. Each 2-cycle connects an origami of type (2) with one of type (3).

The matrix $T^{-1}$ acts on origamis of type (1) and (2) as follows

$$T^{-1} \cdot (B_m, r, sr^i) = (B_m, r, sr^{i+1}),$$

$1 \leq i \leq 2^{m-1}$. We obtain a $2^{m-1}$-cycle describing the action of $T^{-1}$ on the origamis of type (1) and (2). This permutation acts transitively on all origamis of type (1) and (2).

Renumbering the origamis within the types (1) to (3) appropriately and combining these two results, we obtain the following diagram describing the action:

$$\begin{tikzcd}
\mathcal{O}_1 \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{m-2}+1} \arrow[r, "T^{-1}"'] \arrow[d, bend right, "S"'] & \mathcal{O}_2 \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{m-2}+2} \arrow[r, "T^{-1}"'] \arrow[d, bend right, "S"'] & \cdots \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{m-2}} \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{m-1}} \arrow[d, bend right, "S"'] \arrow[llllll, bend right, "T^{-1}"'] \\
& \mathcal{O}_{2^{m-1}+1} \arrow[u, bend right, "S"'] & & \mathcal{O}_{2^{m-1}+2} \arrow[u, bend right, "S"'] & & & \mathcal{O}_{3\cdot 2^{m-2}} \arrow[u, bend right, "S"']
\end{tikzcd}$$

Hence, $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathcal{O}(B_m)$. This shows that the Veech groups of origamis in $\mathcal{O}(B_m)$ have index $3 \cdot 2^{m-2}$ in $\mathrm{SL}(2,\mathbb{Z})$ and are conjugated in $\mathrm{SL}(2,\mathbb{Z})$. ■

**Corollary 4.1.17** *For a 2-origami $\mathcal{O} = (B_m, r, sr^i)$ with $0 \leq i \leq 2^{m-1}$, the smallest natural number $k \in \mathbb{Z}_+$ with $T^k \in \mathrm{SL}(\mathcal{O})$ is $k = 2^{m-1}$.*

**Proposition 4.1.18** *The Veech groups of 2-origamis with deck transformation group $B_m$ are congruence subgroups of level $2^{m-1}$ for $m > 2$.*

**Proof** Principal congruence groups are normal subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the 2-origamis with deck transformation group $B_m$ by Proposition 4.1.16. Hence, it suffices to consider the Veech group of the origami $\mathcal{O} := (B_m, r, s)$. Consider a matrix $M$ in the principal congruence group $\Gamma(2^{m-1})$. By Lemma 4.1.10, we obtain $M \cdot \mathcal{O} = (B_m, \mu(v(\tilde{r},\tilde{s})), \mu(w(\tilde{r},\tilde{s})))$ for words $v, w \in F_2 = \langle \tilde{r}, \tilde{s}\rangle$ with

$$\begin{aligned} \#_{\tilde{r}}\ v &\equiv 1 \mod 2^{m-1}, & \#_{\tilde{r}}\ w &\equiv 0 \mod 2^{m-1},\\ \#_{\tilde{s}}\ v &\equiv 0 \mod 2^{m-1}, & \#_{\tilde{s}}\ w &\equiv 1 \mod 2^{m-1}. \end{aligned} \tag{4.6}$$

Recall that $\mu : F_2 \to B_m$ denotes the monodromy map. We have $\mu(v) = s^j r^i$ with $0 \le j \le 1$ and $0 \le i < 2^m$. The relations $rs = sr^{1-2^{m-1}}$ and $s = s^{-1}$ imply that $j = 0$ and $i$ is odd. Since $s^2 = 1$, there exist $\ell$ even and $k_1, \ldots, k_\ell \in \mathbb{Z}_{\ge 0}$ such that

$$\begin{aligned} \mu(w) &= r^{k_1} s r^{k_2} s r^{k_3} \cdots s r^{k_\ell}\\ &= sr^k, \end{aligned}$$

where

$$\begin{aligned} k &= \sum_{\substack{i\in\{1,\ldots,\ell\}\\ i \text{ odd}}} k_i(1-2^{m-1}) + \sum_{\substack{i\in\{1,\ldots,\ell\}\\ i \text{ even}}} k_i\\ &= -\sum_{\substack{i\in\{1,\ldots,\ell\}\\ i \text{ odd}}} k_i \cdot 2^{m-1} + \sum_{i\in\{1,\ldots,\ell\}} k_i \end{aligned}$$

Here, we use $(1-2^{m-1})^2 \equiv_{2^m} 1$. Using the equivalence given in (4.6), we obtain the equation $\sum_{i=1}^{\ell} k_i \equiv 0$ modulo $2^{m-1}$ and therefore the exponent $k$ is equivalent to 0 modulo $2^{m-1}$. We conclude $M \cdot \mathcal{O} = (B_m, r^i, sr^{j\cdot 2^{m-1}})$ for $i$ odd and $j \in \{0,1\}$. We obtain $M \cdot \mathcal{O} = (B_m, r, s)$ using the automorphisms $\alpha_{(i^{-1},0)}$ and $\alpha_{(1,1)}$ defined in (4.5) in the proof of Proposition 4.1.16. Hence, $M$ lies in the Veech group $\mathrm{SL}(\mathcal{O})$. This shows that $\mathrm{SL}(\mathcal{O})$ is a congruence group of level at most $2^{m-1}$.

Finally, we show that the Veech group $\mathrm{SL}(\mathcal{O})$ contains no principal congruence subgroup $\Gamma(k)$ for $k < 2^{m-1}$. To this end, observe that $T^k = \left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ for $k \in \mathbb{Z}_{\ge 0}$. By Corollary 4.1.17, the matrix $T^k$ does not lie in $\mathrm{SL}(\mathcal{O})$ for $k < 2^{m-1}$. Thus, the congruence level is $2^{m-1}$. ∎

**Example 4.1.19** For $m = 3$ we obtain the group

$$B_3 = C_8 \rtimes_\varphi C_2 = \langle r, s \mid r^8 = s^s = e,\ s^{-1}rs = r^{-3}\rangle.$$

We consider the origami $\mathcal{O} = (B_3, r, s)$. The commutator $[r,s] = r^{-4}$ has order 2 and thus $\mathcal{O}$ lies in $\mathcal{H}(8 \times 1)$.

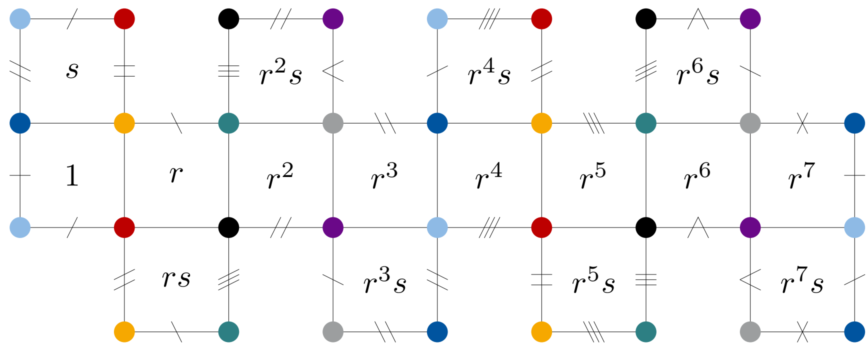


Figure 4.2.: The $\mathrm{SL}(2,\mathbb{Z})$-orbit of the origami $\mathcal{O} = (B_3, r, s)$ contains 6 origamis. The Veech group $\mathrm{SL}(\mathcal{O})$ has index 6 in $\mathrm{SL}(2,\mathbb{Z})$ and is a congruence subgroup of level 4. In particular, the Veech group $\mathrm{SL}(\mathcal{O})$ is different from the Veech group of the origami $(A_3, r, s)$ (see Figure 4.1).

Finally, we consider the example series of normal origamis introduced in Example 3.3.11 covering the "eierlegende Wollmilchsau". The deck transformation groups of origamis in this example series are extensions of the quaternion group. We recall the definition of the deck transformation groups $W_m$. For $m \in \mathbb{Z}_+$, define the group

$$W_m = \langle x, y \mid x^{2^{m+1}} = y^{2^{m+1}} = x^{2^m} y^{2^m} = 1, x^{-1}yx = y^{-1} \rangle.$$

Note that the groups $W_m$ are not semidirect products of cyclic groups as the families of groups studied previously. Further, recall that every element in $W_m$ can be written as $y^i x^j$ for $0 \leq i < 2^{m+1}, 0 \leq j < 2^m$ and thus $W_m$ has order $2^{2m+1}$ (see Example 3.3.11).

Let $\mathcal{O}(W_m)$ denote the set of normal origamis with deck transformation group $W_m$. Since the "eierlegende Wollmilchsau" is the only normal origami with deck transformation group $Q_8$, the set $\mathcal{O}(W_1)$ consists of this origami. In the remaining part of this section, we study $\mathcal{O}(W_m)$ for $m > 1$.

**Proposition 4.1.20** *For $m \geq 2$, the set $\mathcal{O}(W_m)$ is contained in the stratum $\mathcal{H}(2^{m+1} \times (2^m - 1))$ and has cardinality $3 \cdot 2^{m-1}$. The group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathcal{O}(W_m)$ and the Veech group of each origami in $\mathcal{O}(W_m)$ has index $3 \cdot 2^{m-1}$.*

**Proof** Fix a natural number $m \geq 2$. Consider the normal origami $\mathcal{O} = (W_m, x, y)$. The commutator $[x, y] = y^2$ has order $2^m$. Since $W_m$ has order $2^{2m+1}$, the origami $\mathcal{O}$ lies in the claimed stratum. The commutator subgroup $W_m'$ is generated by the commutator $[x, y]$ and thus it is regular. By Theorem 3.2.11, all normal origamis with deck transformation group $W_m$ lie in the same stratum. Hence, we obtain the inclusion $\mathcal{O}(W_m) \subseteq \mathcal{H}(2^{m+1} \times (2^m - 1))$.

In a next step, we compute the set $\mathcal{O}(W_m)$ as described in Remark 2.2.5 and claim that it is the disjoint union of the sets

(1) $R_1 := \{(W_m, x, yx^i) \mid 0 \leq i < 2^m, i \equiv_2 1\}$,

(2) $R_2 := \{(W_m, x, yx^i) \mid 0 \leq i < 2^m, i \equiv_2 0\}$,

(3) $R_3 := \{(W_m, yx^i, x) \mid 0 \le i < 2^m, i \equiv_2 0\}$.

Let $\mathcal{T} = (y^i x^j, y^k x^\ell)$ with $0 \le i,k < 2^{m+1}$ and $0 \le j,\ell < 2^m$ be an arbitrary tuple generating the group $W_m$. The relation $x^{-1}yx = y^{-1}$ implies that $j$ is odd or $\ell$ is odd.

If $j$ is odd, then the map

$$\alpha_1 : W_m \to W_m \text{ given by } \alpha_1(x) = x^{j^{-1}} \text{ and } \alpha_1(y) = y$$

defines an automorphism with $j^{-1} \in (\mathbb{Z}/2^{m+1}\mathbb{Z})^*$. It maps the tuple $\mathcal{T}$ to $\mathcal{T}_1 = (y^i x, y^k x^{\tilde{\ell}})$ for some $0 \le \tilde{\ell} < 2^{m+1}$. Without loss of generality, we assume $\tilde{\ell} < 2^m$. Otherwise, replace $k$ and $\tilde{\ell}$ by $k + 2^m$ and $\tilde{\ell} - 2^m$, respectively. Applying the automorphism

$$\alpha_2 : W_m \to W_m \text{ defined by } \alpha_2(x) = y^{-i}x \text{ and } \alpha_2(y) = y$$

to $\mathcal{T}_1$, yields $\mathcal{T}_2 = (x, y^{\tilde{k}} x^{\tilde{\ell}})$ for some $0 \le \tilde{k} < 2^{m+1}$. As the tuple $\mathcal{T}_2$ generates $W_m$, the exponent $\tilde{k}$ is odd and thus invertible in $(\mathbb{Z}/2^{m+1}\mathbb{Z})$. We apply the automorphism

$$\alpha_3 : W_m \to W_m \text{ given by } \alpha_3(x) = x \text{ and } \alpha_3(y) = y^{\tilde{k}^{-1}}$$

to $\mathcal{T}_2$ and obtain $\mathcal{T}_3 = (x, yx^{\tilde{\ell}})$. Hence, the normal origami $(W_m, y^i x^j, y^k x^\ell)$ equals $(W_m, x, yx^{\tilde{\ell}})$ and lies in $R_1$ or $R_2$.

If $j$ is even and $\ell$ is odd, then analog computations as above yield an automorphism $\alpha \in \mathrm{Aut}(W_m)$ mapping $\mathcal{T}$ to a tuple of the form $(yx^i, x)$ for an even number $i \le 2^m$. The origami $(W_m, yx^i, x)$ lies in $R_3$.

It remains to show that all origamis in $R_1, R_2$, and $R_3$ are different. To prove this, one shows that $v_i \mapsto a_j,\ w_i \mapsto b_j$ does not define an automorphism of $W_m$. Here, $(W_m, v_i, w_i)$ and $(W_m, a_j, b_j)$ are elements of $R_i$ and $R_j$, respectively, and $1 \le i, j \le 3$.

We begin with the case $i = 1$ and $j = 3$. Let $\mathcal{O} = (W_m,\ x,\ yx^k)$ and $\mathcal{O}' = (W_m,\ yx^\ell,\ x)$ be origamis in $R_1$ and $R_3$, respectively. Hence, the indices obey $0 \le k, \ell < 2^m, k \equiv_2 1$, and $\ell \equiv_2 0$. Suppose that the origamis $\mathcal{O}$ and $\mathcal{O}'$ are equal. Then the equalities

$$\delta(x) := yx^k \text{ and } \delta(yx^\ell) := x$$

define an automorphism $\delta$ of $W_m$. We deduce the following equation for the image of $y$

$$\delta(y) = x \cdot (yx^k)^{-\ell} = x^{1-k\ell}.$$

Here, we use that $k$ is odd and $\ell$ is even. Since the relation $x^{-1}yx = y^{-1}$ holds in $W_m$, the images of $x$ and $y$ under $\delta$ have to satisfy the same relation. For the left hand side, we obtain the equality

$$\begin{aligned} \delta(x)^{-1} \cdot \delta(y) \cdot \delta(x) &= \left(yx^k\right)^{-1} \cdot \left(x^{1-k\ell}\right) \cdot \left(yx^k\right) \\ &= y^2 x^{1-k\ell}. \end{aligned} \tag{4.7}$$

For the right hand side, we obtain the equality

$$\delta(y)^{-1} = \left(x^{1-k\ell}\right)^{-1} = x^{k\ell-1}. \tag{4.8}$$

The equations (4.7) and (4.8) are not equal and thus the origamis $\mathcal{O}$ and $\mathcal{O}'$ are different.

We check the other cases in an anologous manner using the same relation. For this, consider first the case $i = 1$ and $j = 2$. The origamis are of the form $\mathcal{O} = (W_m, x, yx^i)$ and $\mathcal{O}' = (W_m, x, yx^j)$ with $i$ odd and $j$ even. Then there is an automorphism $\delta$ of $W_m$ with $\delta(x) = x$ and $\delta(y) = yx^{j-i}$. Note $j - i$ is odd. As $\delta$ is an automorphism, we obtain the equality $\delta(x^{-1})\delta(y)\delta(x) = \delta(y^{-1})$. For the left side, we obtain

$$x^{-1}yx^{j-i}x = y^{-1}x^{j-i}.$$

For the right side, we obtain

$$x^{i-j}y^{-1} = yx^{i-j}.$$

The equality of both sides implies

$$\begin{aligned} 1 &= y^{-1}x^{j-i} \cdot (yx^{i-j})^{-1} \\ &= y^{-1}x^{2j-2i}y^{-1} \\ &= y^{-2}x^{2j-2i}. \end{aligned}$$

However, this does not hold.

Consider the case $i = 2$ and $j = 3$. The origamis are of the form $\mathcal{O} = (W_m, x, yx^i)$ and $\mathcal{O}' = (W_m, yx^j, x)$ with $i$ and $j$ even. Then, there is an automorphism $\delta$ of $W_m$ with $\delta(x) = yx^j$ and $\delta(yx^i) = x$. Thus, we have

$$\begin{aligned} \delta(y) &= x \cdot (yx^j)^{-i} \\ &= xx^{-ij}y^{-i} \\ &= y^i x^{1-ij}. \end{aligned}$$

Here, we use that $i$ and $j$ are even integers. As $\delta$ is an automorphism, we obtain the equality $\delta(x^{-1})\delta(y)\delta(x) = \delta(y^{-1})$. For the left side, we obtain

$$x^{-j}y^{-1} \cdot y^i x^{1-ij} \cdot yx^j = y^{i-2}x^{1-ij}.$$

For the right side, we obtain

$$x^{ij-1}y^{-i} = y^i x^{ij-1}.$$

The equality of both sides yields

$$\begin{aligned} 1 &= y^{i-2}x^{1-ij} \cdot (y^i x^{ij-1})^{-1} \\ &= y^{i-2}x^{2-2ij}y^{-i} \\ &= y^{-2}x^{2-2ij}. \end{aligned}$$

This does not hold.

Finally, consider the case $1 \leq i, j \leq 3$ with $i = j$. We assume that $i = j = 3$. The other two cases $i = j = 1$ and $i = j = 2$ follow analogously. So, let $\mathcal{O}$ and $\mathcal{O}'$ be origamis of the form $\mathcal{O} = (W_m, x, yx^i)$ and $\mathcal{O}' = (W_m, x, yx^j)$ such that $i$ and $j$ have the same parity and $i \neq j$. Then there is an automorphism $\delta$ of $W_m$ with $\delta(x) = x$ and $\delta(yx^i) = yx^j$. Then we have $\delta(y) = yx^{j-i}$. Note $j - i$ is even. As $\delta$ is an automorphism, we obtain the equality $\delta(x^{-1})\delta(y)\delta(x) = \delta(y^{-1})$. For the left side, we obtain

$$x^{-1}yx^{j-i}x = y^{-1}x^{j-i}.$$

For the right side, we obtain

$$x^{i-j}y^{-1} = y^{-1}x^{i-j}.$$

Both equalities are equal if and only if $2^m$ divides $i - j$. However, we have $0 \neq |i-j| < 2^m$ which yields a contradiction. We conclude that the set $\mathcal{O}(W_m)$ has cardinality $3 \cdot 2^{m-1}$.

Lastly, we prove that $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on $\mathcal{O}(W_m)$. Number the origamis in $\mathcal{O}(W_m)$ as follows: Let $\mathcal{O}_1, \ldots, \mathcal{O}_{2^{m-1}}$ denote the origamis in $R_1$, let $\mathcal{O}_{2^{m-1}+1}, \ldots, \mathcal{O}_{2^m}$ denote the origamis in $R_2$ and let $\mathcal{O}_{2^m+1}, \ldots, \mathcal{O}_{3 \cdot 2^{m-1}}$ denote the origamis in $R_3$. Computing the action of $S$, one obtains for an even number $i$

$$\begin{aligned} S \cdot (W_m, x, yx^i) &= (W_m, x^{-i}y^{-1}, x) \\ &= (W_m, yx^{-i}, x), \\ S \cdot (W_m, yx^{-i}, x) &= (W_m, x^{-1}, yx^{-i}) \\ &= (W_m, x, yx^i). \end{aligned}$$

Here, we use that $x \mapsto x, y \mapsto y^{-1}$ as well as $x \mapsto x^{-1}, y \mapsto y$ define automorphisms of $W_m$. We conclude that the permutation describing the $S$-action on the set $R_2 \cup R_3$ consists of 2-cycles. Each 2-cycle connects an origami in $R_2$ with one in $R_3$.

The matrix $T^{-1}$ acts on the set $R_1 \cup R_2$ as follows

$$T^{-1} \cdot (W_m, x, yx^i) = (W_m, x, yx^{i+1}),$$

$1 \leq i \leq 2^m$. We obtain a $2^m$-cycle describing the action of $S$ on the set $R_1 \cup R_2$. This permutation acts transitively on the set $R_1 \cup R_2$.

Renumbering the origamis within the sets $R_1, R_2$, and $R_3$ appropriately and combining these two results, we obtain the following diagram:

$$\begin{tikzcd}
\mathcal{O}_1 \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{m-1}+1} \arrow[r, "T^{-1}"'] \arrow[d, bend right, "S"'] & \mathcal{O}_2 \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^{m-1}+2} \arrow[r, "T^{-1}"'] \arrow[d, bend right, "S"'] & \cdots \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^m-1} \arrow[r, "T^{-1}"'] & \mathcal{O}_{2^m} \arrow[llllll, bend right, "T^{-1}"'] \arrow[d, bend right, "S"'] \\
 & \mathcal{O}_{2^m+1} \arrow[u, bend right, "S"'] & & \mathcal{O}_{2^m+2} \arrow[u, bend right, "S"'] & & & \mathcal{O}_{3 \cdot 2^{m-1}} \arrow[u, bend right, "S"']
\end{tikzcd}$$

Hence, $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on $\mathcal{O}(W_m)$. This shows that the Veech groups of origamis in $\mathcal{O}(W_m)$ have index $3 \cdot 2^{m-1}$ in $\mathrm{SL}(2, \mathbb{Z})$ and are conjugated in $\mathrm{SL}(2, \mathbb{Z})$. ∎

**Corollary 4.1.21** *For a 2-origami $\mathcal{O} = (W_m, x, yx^i)$ with $0 \leq i < 2^m$, $m \geq 2$, the smallest natural number $k \in \mathbb{Z}_+$ with $T^k \in \mathrm{SL}(\mathcal{O})$ is $k = 2^m$.*

**Proposition 4.1.22** *The Veech groups of 2-origamis with deck transformation group $W_m$ are congruence subgroups of level $2^m$ for $m \geq 2$.*

**Proof** Principal congruence groups are normal subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the set of 2-origamis $\mathcal{O}(W_m)$ by Proposition 4.1.20. Hence, it suffices to consider the Veech group of the origami $\mathcal{O} := (W_m, x, y)$. Consider a matrix $M \in \Gamma(2^m)$. By Lemma 4.1.10, we obtain $M \cdot \mathcal{O} = (W_m, \mu(v(\tilde{x},\tilde{y})), \mu(w(\tilde{x},\tilde{y})))$ for words $v, w$ in the free group $F_2 = \langle \tilde{x}, \tilde{y} \rangle$ with

$$\begin{aligned} \#_{\tilde{x}}\, v &\equiv 1 \mod 2^m, & \#_{\tilde{x}}\, w &\equiv 0 \mod 2^m, \\ \#_{\tilde{y}}\, v &\equiv 0 \mod 2^m, & \#_{\tilde{y}}\, w &\equiv 1 \mod 2^m. \end{aligned} \tag{4.9}$$

Recall that $\mu$ denotes the monodromy map. We have $\mu(v) = y^i x^j$ with $0 \leq i < 2^{m+1}$ and $0 \leq j < 2^m$. The relations $yx = xy^{-1}$ and $x^{2^m} = y^{2^m}$ imply that $\mu(v) = y^i x$. Using these relations for $\mu(w)$, we obtain $\mu(w) = y^k$ with $0 \leq k < 2^{m+1}$. As $\mu(v)$ and $\mu(w)$ generate $W_m$ we conclude that $k$ is odd. Applying suitable automorphisms of $W_m$, we obtain

$$\begin{aligned} M \cdot \mathcal{O} &= (W_m, y^i x, y^k) \\ &= (W_m, y^i x, y) \\ &= (W_m, x, y) \\ &= \mathcal{O}. \end{aligned}$$

Hence, the Veech group $\mathrm{SL}(\mathcal{O})$ contains $M$ and thus is a congruence subgroup of level at most $2^m$. Consider the matrices $T^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ for $k \in \mathbb{Z}_{\geq 0}$. By Corollary 4.1.21, the matrix $T^k$ does not lie in $\mathrm{SL}(\mathcal{O})$ for $k < 2^m$. Thus, the congruence level of $\mathrm{SL}(\mathcal{O})$ is $2^m$. ∎

**Remark 4.1.23** So far, all groups considered in this section have the property that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the corresponding normal origamis (see Proposition 4.1.4, Proposition 4.1.7, Proposition 4.1.16, Proposition 4.1.13, and Proposition 4.1.20). This depends on the structure of the deck transformation groups and is not true in general. For each prime number $p$, two $p$-origamis with isomorphic deck transformation groups exist that lie in different strata (see Proposition 3.2.15). Since the $\mathrm{SL}(2,\mathbb{Z})$-action fixes the stratum, the action is not transitive in this case.

Moreover, the Veech groups of all 2-origamis studied in this section are congruence groups of level $2^k$ for some $k \in \mathbb{Z}_{\geq 0}$. However, this does not hold in general. In Proposition 4.2.4 and Theorem 4.2.9, conditions for the deck transformation group of a normal origami $\mathcal{O}$ are introduced which imply that the Veech group is a totally non-congruence group, i.e., $\mathrm{SL}(\mathcal{O})$ surjects onto $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{Z}_+$. Furthermore, examples of normal origamis with this property are given (see Example 4.2.10 and Corollary 4.2.5).

### 4.1.2. Application to sums of Lyapunov exponents

This section aims at applying the results on the normal origamis studied in Section 4.1.1 to deduce information about dynamical systems related to these normal origamis. We give a brief and informal description of these dynamical systems. For more details, see, e.g., [Wil17] and [Kap11].

The **Teichmüller flow** $\mathcal{F}_t$ acts on the moduli space of translation surfaces. For a translation surface $X$, it is defined as $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right) \cdot X$, i.e., it stretches the East-West direction and contracts the North-South direction. The Kontsevich-Zorich cocycle captures the change of the homology along the orbits of the Teichmüller flow and takes values $A(t)$ in the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$. Finally, the **Lyapunov exponents** of the Kontsevich-Zorich cocycle $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2g}$ are defined as the logarithm of the eigenvalues of the matrix

$$\Lambda := \lim_{t \to \infty} \left( A(t)^T A(t) \right)^{-2t}.$$

Here, $g$ denotes the genus of the surface $X$. As $A(t)$ is a symmetric matrix, the Lyapunov exponents satisfy the equality $\lambda_i = -\lambda_{2g+1-i}$ for $1 \leq i \leq g$. In particular, the top $g$ Lyapunov exponents are non-negative and the bottom $g$ ones are non-positive.

In this section, we compute the sum of the top $g$ Lyapunov exponents of the Kontsevich-Zorich cocycle defined by certain normal origamis. For this, we use the following theorem of Eskin, Kontsevich, and Zorich and adapt it to the case of normal origamis.

**Theorem 4.1.24 ([EKZ14, Corollary 8])** *Let $\mathcal{O}$ be an origami of genus $g$ in some stratum $\mathcal{H}(a_1, \ldots, a_m)$. Then the sum of the top $g$ Lyapunov exponents of the Kontsevich-Zorich cocycle defined by $\mathcal{O}$ satisfies the following equation*

$$\sum_{i=1}^{g} \lambda_i = \frac{1}{12} \cdot \sum_{i=1}^{m} \frac{a_i \cdot (a_i + 2)}{a_i + 1} + \frac{1}{|\operatorname{SL}(2, \mathbb{Z}) \cdot \mathcal{O}|} \cdot \sum_{\mathcal{O}_i \in \operatorname{SL}(2,\mathbb{Z}) \cdot \mathcal{O}} \sum_{\substack{\textit{horizontal} \\ \textit{cylinders } c_{ij} \\ \textit{such that} \\ \mathcal{O}_i = \sqcup c_{ij}}} \frac{h_{ij}}{w_{ij}}.$$

*Here, $h_{ij}$ and $w_{ij}$ denote the height and circumference of the horizontal cylinder $c_{ij}$, respectively.*

We deduce the following simplified formula for normal origamis.

**Corollary 4.1.25** *For a normal origami $\mathcal{O} = (G, x, y)$ in some stratum $\mathcal{H}_g(a_1, \ldots, a_m)$, the sum of the top $g$ Lyapunov exponents satisfies the following equation*

$$\sum_{i=1}^{g} \lambda_i = \frac{m}{12} \cdot \frac{a_1 \cdot (a_1 + 2)}{a_1 + 1} + \frac{1}{|\operatorname{SL}(2, \mathbb{Z}) \cdot \mathcal{O}|} \cdot \sum_{\substack{\mathcal{O}_i = (G, x_i, y_i) \\ \in \operatorname{SL}(2,\mathbb{Z}) \cdot \mathcal{O}}} \frac{|G|}{\operatorname{ord}(x_i)^2}. \tag{4.10}$$

**Proof** Note that all singularities of a normal origami have the same order. This explains the simplification of the first sum. Given a normal origami $\mathcal{O}_i = (G, x_i, y_i)$ of genus $g \geq 2$, the cylinder decomposition in horizontal direction consists of cylinders of height 1 and circumference $\mathrm{ord}(x_i)$. The origami $\mathcal{O}_i$ has degree $|G|$ and thus the number of horizontal cylinders is equal to $\frac{|G|}{\mathrm{ord}(x_i)}$ (see Remark 2.3.7). ■

In the remaining part of this section, we compute the sum of the positive Lyapunov exponents for the families of normal origamis studied in Section 4.1.1. For this, we apply Corollary 4.1.25 and use the results obtained in Section 4.1.1. It will be especially useful that Equation (4.10) is invariant under the $\mathrm{SL}(2, \mathbb{Z})$-action.

**Proposition 4.1.26** *The sum of the top $2^m - 1$ Lyapunov exponents for a 2-origami in $\mathcal{O}(A_m)$ is $3 \cdot 2^{m-3}$ for $m > 2$.*

**Proof** Let $m > 2$. By Proposition 4.1.13, the group $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on the set $\mathcal{O}(A_m)$ of all 2-origamis with deck transformation group $A_m$. Each origami in $\mathcal{O}(A_m)$ lies in the stratum $\mathcal{H}(4 \times (2^{m-1} - 1))$ and has genus $2^m - 1$ (see Proposition 2.2.13). Corollary 4.1.25 implies that the sum of the top $2^m - 1$ Lyapunov exponents coincides for all origamis in $\mathcal{O}(A_m)$. We obtain the equation

$$\sum_{i=1}^{2^m-1} \lambda_i = \frac{2^{2m-1} - 2}{3 \cdot 2^m} + \frac{1}{6} \sum_{\mathcal{O}_i = (A_m, x_i, y_i) \in \mathcal{O}(A_m)} \frac{|A_m|}{\mathrm{ord}(x_i)^2}.$$

Using the characterization of the origamis in $\mathcal{O}(A_m)$ from the proof of Proposition 4.1.13 and the orders of the possible generators $x_i$ noted in Remark 4.1.14, yields

$$\begin{aligned}
\sum_{i=1}^{2^m-1} \lambda_i &= \frac{2^{2m-1} - 2}{3 \cdot 2^m} + \frac{1}{6} \sum_{\mathcal{O}_i \,\in\, \mathcal{O}(A_m)} \frac{2^{m+1}}{\mathrm{ord}(x_i)^2} \\
&= \frac{2^{2m-1} - 2}{3 \cdot 2^m} + \frac{1}{6} \cdot 2 \cdot \left( \frac{2^{m+1}}{2^2} + \frac{2^{m+1}}{4^2} + \frac{2^{m+1}}{2^{m \cdot 2}} \right) \\
&= \frac{2^{2m-1} - 2}{3 \cdot 2^m} + \frac{2^{2m-1} + 2^{2m-3} + 2}{3 \cdot 2^m} \\
&= \frac{2^{2m} + 2^{2m-3}}{3 \cdot 2^m} \\
&= \frac{2^m + 2^{m-3}}{3} \\
&= 2^{m-3} \cdot \frac{2^3 + 1}{3} \\
&= 3 \cdot 2^{m-3}.
\end{aligned}$$

■

**Proposition 4.1.27** *The sum of the top $2^{m-1} + 1$ Lyapunov exponents for a 2-origami in $\mathcal{O}(B_m)$ is $2^{m-3} + 1$ for $m > 2$.*

**Proof** Let $m > 2$. By Proposition 4.1.16, the group $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on the set $\mathcal{O}(B_m)$ of 2-origamis with deck transformation group $B_m$. Each origami in $\mathcal{O}(B_m)$ lies in the stratum $\mathcal{H}(2^m \times 1)$ and has genus $2^{m-1}+1$ (see Proposition 2.2.13). By Corollary 4.1.25, the sum of the top $2^{m-1}+1$ Lyapunov exponents coincides for all origamis in $\mathcal{O}(B_m)$. For the summand depending on the stratum, we obtain

$$\frac{2^m}{12} \cdot \frac{3}{2} = 2^{m-3}.$$

In a next step, we compute

$$\frac{1}{3 \cdot 2^{m-2}} \cdot \sum_{(B_m, x_i, y_i) \, \in \, \mathcal{O}(B_m)} \frac{|B_m|}{\operatorname{ord}(x_i)^2}$$

Recall that for $2^{m-1}$ origamis $x_i$ can be chosen as $r$ (see the proof of Proposition 4.1.16). Call the set of these origamis $\mathfrak{O}_1$. We compute

$$\begin{aligned} & \frac{1}{3 \cdot 2^{m-2}} \cdot \sum_{(B_m, r, y_i) \, \in \, \mathfrak{O}_1} \frac{2^{m+1}}{2^{2m}} \\ = & \frac{2^{m-1}}{3 \cdot 2^{m-2}} \cdot \frac{1}{2^{m-1}} \\ = & \frac{1}{3 \cdot 2^{m-2}}. \end{aligned}$$

Recall that for the remaining $2^{m-2}$ origamis $x_i$ can be chosen as $sr^{2i}$ with $1 \leq i \leq 2^{m-2}$ (see thee proof of Proposition 4.1.16). Call the set of these origamis $\mathfrak{O}_2$. For $2i = 2^k \cdot \ell$ with $\ell$ odd, the order of $sr^{2i}$ is $2^{m-k}$. This follows from the equation

$$\left(sr^{2i}\right)^2 = r^{2i \cdot (1-2^{m-1})+2i} = r^{4i}.$$

We conclude

$$\begin{aligned} & \frac{1}{3 \cdot 2^{m-2}} \cdot \sum_{(B_m, x_i, y_i) \, \in \, \mathfrak{O}_2} \frac{2^{m+1}}{\operatorname{ord}(x_i)^2} \\ = & \frac{1}{3 \cdot 2^{m-2}} \cdot \sum_{i=1}^{2^{m-2}} \frac{2^{m+1}}{\operatorname{ord}(sr^{2i})^2} \\ = & \frac{1}{3 \cdot 2^{m-2}} \cdot \left( \frac{2^{m+1}}{2^2} + \sum_{j=2}^{m-1} 2^{j-2} \cdot \frac{2^{m+1}}{2^{2j}} \right) \\ = & \frac{1}{3 \cdot 2^{m-2}} \cdot \left( 2^{m-1} + 2^{m-3} \cdot \sum_{j=0}^{m-3} 2^{-j} \right) \\ = & \frac{1}{3 \cdot 2^{m-2}} \cdot \left( 2^{m-1} + 2^{m-3} \cdot \left( -2 \cdot \left( 2^{2-m} - 1 \right) \right) \right) \\ = & \frac{1}{3 \cdot 2^{m-2}} \cdot \left( 2^{m-1} - 1 + 2^{m-2} \right) \\ = & \frac{3 \cdot 2^{m-2} - 1}{3 \cdot 2^{m-2}}. \end{aligned}$$

Since $\mathcal{O}(B_m) = \mathfrak{O}_1 \cup \mathfrak{O}_2$, we obtain

$$\begin{aligned}\sum_{i=1}^{2^{m-1}+1} \lambda_i &= 2^{m-3} + \frac{1}{3 \cdot 2^{m-2}} \cdot \sum_{\mathcal{O}_i \,\in\, \mathcal{O}(B_m)} \frac{|B_m|}{\operatorname{ord}(x_i)^2} \\ &= 2^{m-3} + \frac{1}{3 \cdot 2^{m-2}} + \frac{3 \cdot 2^{m-2} - 1}{3 \cdot 2^{m-2}} \\ &= 2^{m-3} + 1.\end{aligned}$$

∎

**Proposition 4.1.28** *The sum of the top $2^{2m} - 2^m + 1$ Lyapunov exponents for a 2-origami in $\mathcal{O}(W_m)$ is an integer and equals $\frac{2^{2m-1}+1}{3}$ for $m \geq 1$.*

**Proof** Let $m \geq 1$. By Proposition 4.1.20, the group $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on the set $\mathcal{O}(W_m)$ of normal origamis with deck transformation group $W_m$. Using Corollary 4.1.25, we deduce that the sum of the top $2^{2m} - 2^m + 1$ Lyapunov exponents coincides for all origamis in $\mathcal{O}(W_m)$. Denote this sum by $\mathcal{L}_m$. Each origami in $\mathcal{O}(W_m)$ lies in the stratum $\mathcal{H}\left(2^{m+1} \times (2^m - 1)\right)$ and has genus $2^{2m} - 2^m + 1$ (see Proposition 4.1.20 and Proposition 2.2.13). For the summand depending on the stratum, we obtain

$$\frac{2^{m+1}}{12} \cdot \frac{(2^m - 1) \cdot (2^m + 1)}{2^m} = \frac{1}{6} \cdot \left(2^{2m} - 1\right).$$

In a next step, we compute

$$\frac{1}{3 \cdot 2^{m-1}} \cdot \sum_{(W_m, v, w) \,\in\, \mathcal{O}(W_m)} \frac{|W_m|}{\operatorname{ord}(v)^2}.$$

In the following, we use the notation of the proof of Proposition 4.1.20. Recall that for the $2^m$ origamis in $R_1 \cup R_2$ the group element $v$ can be chosen as $x$ (see the proof of Proposition 4.1.20). For the $2^{m-1}$ origamis in $R_3$, we choose $v$ as an element of the form $yx^\ell$ with an even integer $\ell$. Using that $x$ as well as elements of the form $yx^\ell$ have order $2^{m+1}$, we compute

$$\begin{aligned}&\frac{1}{3 \cdot 2^{m-1}} \cdot \sum_{(W_m, v, w) \,\in\, \mathcal{O}(W_m)} \frac{|W_m|}{\operatorname{ord}(v)^2} \\ &= \frac{1}{3 \cdot 2^{m-1}} \cdot \sum_{i=1}^{3 \cdot 2^{m-1}} \frac{2^{2m+1}}{2^{(m+1) \cdot 2}} \\ &= \frac{3 \cdot 2^{m-1}}{3 \cdot 2^{m-1}} \cdot \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Here, we use that the set of origamis $\mathcal{O}(W_m)$ has cardinality $3 \cdot 2^{m-1}$. Consequently, the sum of Lyapunov exponents obeys the following equality

$$\begin{aligned}\mathcal{L}_m &= \frac{2^{2m}-1}{6}+\frac{1}{2}\\ &= \frac{2^{2m}+2}{6}\\ &= \frac{2^{2m-1}+1}{3}.\end{aligned}$$

It remains to show that $\mathcal{L}_m$ is an integer. Note that $2^k \equiv_3 (-1)^k$ for $k \in \mathbb{Z}_{\geq 0}$. Hence, we obtain $2^{2m-1} \equiv_3 -1$ and thus $2^{2m-1}+1$ is divisible by 3. ∎

**Proposition 4.1.29** *Let $n, k \in \mathbb{Z}_+$ such that $1 \leq k \leq n-3$. The sum of the top $2^{n-1}-2^{n-k-1}+1$ Lyapunov exponents for a 2-origami with deck transformation group $G_{(n,k)}$ is an integer and equals*

$$\begin{aligned}&\frac{1}{3}\left(2^{n-2}+2^{2k+3-n}\right), \qquad \text{if } k \geq \frac{n-3}{2}\\ &\frac{1}{3}\left(2^{n-2}+2^{2k+3-n}-2^{n-2k-2}+2^2+\sum_{j=1}^{n-2k-4}2^{2k+4+j-n}\right), \ \text{else.}\end{aligned}$$

**Proof** Let $n, k \in \mathbb{Z}_+$ such that $1 \leq k \leq n-3$. By Proposition 4.1.7, the matrix group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the set $\mathcal{O}(G_{(n,k)})$ of 2-origamis with deck transformation group $G_{(n,k)}$. Each origami in $\mathcal{O}(G_{(n,k)})$ lies in the stratum $\mathcal{H}\left(2^{n-k}\times\left(2^k-1\right)\right)$ and has genus $2^{n-1}-2^{n-k-1}+1$ (see Proposition 4.1.6 and Proposition 2.2.13). Using Corollary 4.1.25, we deduce that the sum of the top $2^{n-1}-2^{n-k-1}+1$ Lyapunov exponents coincides for all origamis in $\mathcal{O}(G_{(n,k)})$. Denote this sum by $\mathcal{L}_{(n,k)}$. For the summand depending on the stratum, we obtain

$$\frac{2^{n-k}}{12}\cdot\frac{\left(2^k-1\right)\cdot\left(2^k+1\right)}{2^k}=\frac{1}{3}\cdot\left(2^{n-2}-2^{n-2k-2}\right).$$

In a next step, we compute

$$\frac{1}{3\cdot 2^{n-k-3}}\cdot\sum_{(G_{(n,k)},x_i,y_i)\ \in\ \mathcal{O}(G_{(n,k)})}\frac{|G_{(n,k)}|}{\mathrm{ord}(x_i)^2}.$$

For this, we use the notation of the proof of Proposition 4.1.6. Recall that $\mathcal{O}(G_{(n,k)})$ is the disjoint union of the sets $R_1, R_2$, and $R_3$ (see Proposition 4.1.6). Further, note that for the $2^{n-k-2}$ origamis in $R_1 \cup R_2$ the group element $x_i$ can be chosen as $s$, which has order $2^{n-k-1}$. We compute

$$\begin{aligned}&\frac{1}{3\cdot 2^{n-k-3}}\cdot\sum_{(G_{(n,k)},s,y_i)\ \in\ R_1\cup R_2}\frac{2^n}{2^{(n-k-1)\cdot 2}}\\ =\ &\frac{2^{n-k-2}}{3\cdot 2^{n-k-3}}\cdot\frac{1}{2^{n-2k-2}}\\ =\ &\frac{2^{2k+3-n}}{3}.\end{aligned}$$

We recall the definitions of $G_{(n,k)}$ and $R_3$ (see Proposition 4.1.6)

$$\begin{aligned} G_{(n,k)} &= C_{2^{k+1}} \rtimes_\varphi C_{2^{n-k-1}} = \langle r, s \mid r^{2^{k+1}} = s^{2^{n-k-1}} = 1,\ s^{-1}rs = r^{-1}\rangle, \\ R_3 &= \left\{(G_{(n,k)},\ rs^{2\cdot m},\ s) \mid 1 \leq m \leq 2^{n-k-3}\right\}. \end{aligned}$$

Since $rs = sr^{-1}$ and $r^{-1}s = sr$, the order of $rs^{2m}$ equals $\max\{2^{k+1},\ 2^{n-k-j-2}\}$, where $2^j$ is the maximal power of 2 dividing $m$. In the following, we consider two cases.

If $k \geq \frac{n-3}{2}$, then the inequality $k+1 \geq n-k-j-2$ for all $1 \leq m \leq 2^{n-k-2}$ and thus we obtain $\operatorname{ord}(rs^{2m}) = 2^{k+1}$. Hence, we compute

$$\begin{aligned} &\frac{1}{3\cdot 2^{n-k-3}} \cdot \sum_{j=1}^{2^{n-k-3}} \frac{2^n}{2^{(k+1)\cdot 2}} \\ =& \frac{2^{n-k-3}}{3\cdot 2^{n-k-3}} \cdot \frac{2^n}{2^{2k+2}} \\ =& \frac{2^{n-2k-2}}{3}. \end{aligned}$$

In this case, the sum of Lyapunov exponents obeys the following equality

$$\begin{aligned} \mathcal{L}_{(n,k)} &= \frac{1}{3}\cdot\left(2^{n-2} - 2^{n-2k-2} + 2^{2k+3-n} + 2^{n-2k-2}\right) \\ &= \frac{1}{3}\cdot\left(2^{n-2} + 2^{2k+3-n}\right). \end{aligned}$$

To prove the claim for $k \geq \frac{n-3}{2}$, it remains to show that $\mathcal{L}_{(n,k)}$ is an integer. Note that $2^m \equiv_3 (-1)^m$ for $m \in \mathbb{Z}_{\geq 0}$. The equality

$$n-2 \not\equiv_2 n-3 \equiv_2 2k+3-n$$

implies that

$$2^{n-2} \equiv_3 -2^{2k+3-n}.$$

This proves the claim for $k \geq \frac{n-3}{2}$.

If $k < \frac{n-3}{2}$, then we have $0 < n-2k-3$. For $1 \leq j \leq n-2k-4$, there are $2^{n-k-3-j}$ many group elements of the form $rs^{2m}$ and order $2^{n-k-j-2}$. Further, there are $2^{n-k-3-(n-2k-4)} = 2^{k+1}$ many group elements of the form $rs^{2m}$ and order $2^{k+1}$. We obtain the following equation

$$\begin{aligned} &\frac{1}{3\cdot 2^{n-k-3}} \cdot \left(\sum_{j=1}^{n-2k-4}\left(2^{n-k-3-j}\cdot\frac{2^n}{2^{(n-k-j-2)\cdot 2}}\right) + 2^{k+1}\cdot\frac{2^n}{2^{(k+1)\cdot 2}}\right) \\ =& \frac{2}{3}\cdot\left(\sum_{j=1}^{n-2k-4}\left(2^{2k+j+3-n}\right) + 2\right). \end{aligned}$$

In this case, the sum of Lyapunov exponents equals

$$\mathcal{L}_{(n,k)} = \frac{1}{3} \cdot \left( 2^{n-2} - 2^{n-2k-2} + 2^{2k+3-n} + 2^2 + \sum_{j=1}^{n-2k-4} \left( 2^{2k+j+4-n} \right) \right).$$

It remains to show that $\mathcal{L}_{(n,k)}$ is an integer for the case $k < \frac{n-3}{2}$. Recall that $2^m \equiv_3 (-1)^m$ for $m \in \mathbb{Z}_{\geq 0}$. Then $n - 2k - 2 \equiv_2 n - 2$ implies $2^{n-2} - 2^{n-2k-2} \equiv_3 0$. For $n \in \mathbb{Z}_{\geq 0}$ odd, the sum $\sum_{j=1}^{n-2k-4} \left(2^{2k+j+4-n}\right)$ has an odd number of summands. The inequality

$$2k + j + 4 - n \not\equiv_2 2k + j + 1 + 4 - n$$

implies that

$$2^{2k+j+4-n} \equiv_3 -2^{2k+j+1+4-n}.$$

Thus, the sum $\sum_{j=1}^{n-2k-4} \left(2^{2k+j+4-n}\right)$ is equivalent to the first summand $2^{2k+5-n}$ modulo 3. Moreover, we have

$$2k + 3 - n \equiv_2 2 \equiv_2 2k + 5 - n$$

and thus we conclude

$$2^{2k+3-n} + 2^2 + 2^{2k+5-n} \equiv_3 3 \cdot 1 \equiv_3 0.$$

If $n$ is even, then the sum $\sum_{j=1}^{n-2k-4} \left(2^{2k+j+4-n}\right)$ has an even number of summands and is equivalent to 0 modulo 3. As $2k + 3 - n$ is odd in this case, the remaining terms satisfy the equation

$$2^{2k+3-n} + 2^2 \equiv_3 -1 + 1 \equiv_3 0.$$

This shows that $\mathcal{L}_{(n,k)}$ is an integer. ■

The example series studied in Proposition 4.1.26, Proposition 4.1.27, Proposition 4.1.28, and Proposition 4.1.29 might suggest that the sum of the positive Lyapunov exponents of the Kontsevich-Zorich cocycle defined by a normal origami is always an integer. However, this is not the case. Consider the normal origami $\mathcal{O} := (G, x, y)$ with $G = \langle x, y \rangle$ and

$$\begin{aligned} x &:= (1, 9, 5, 13, 3, 11, 7, 15, 2, 10, 6, 14, 4, 12, 8, 16)\ , \\ y &:= (1, 9, 2, 10)(3, 11)(4, 12)(5, 15, 7, 13)(6, 16, 8, 14) \end{aligned}$$

(see Lemma 3.1.34). This origami has degree $2^{12}$, lies in the stratum $\mathcal{H}\left(2^{11} \times 1\right)$, and has genus 1025. We use the `SageMath` package [DFL] to compute that the sum of the top 1025 Lyapunov exponents equals $\frac{25}{16}$.

## 4.2. Normal origamis with totally non-congruence groups as Veech groups

In the previous section, we studied families of normal origamis whose Veech groups are congruence groups. This motivates the question of interest in this section: Are there normal origamis with Veech groups that are far away from being a congruence subgroup? In [Wei13], Weitze-Schmithüsen introduced the deficiency of finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$. It measures how far the group is from being a congruence subgroup. She also established the notion of totally non-congruence groups, namely, groups that are as far from being a congruence group as possible. Such a group projects surjectively onto $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for each $n \in \mathbb{Z}_+$, i.e., no information about the group itself can be recovered from the images under these natural projections. In [SW18], an infinite family of origamis with totally non-congruence subgroups as Veech groups are constructed for each stratum. These origamis have only few symmetries. In this section, we present sufficient conditions for normal origamis to have totally non-congruence subgroups as Veech groups and introduce a class of normal origamis satisfying these conditions.

**Definition 4.2.1** Consider the natural projection $\pi_n : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for a natural number $n \in \mathbb{Z}_+$. A finite index subgroup $G$ of $\mathrm{SL}(2,\mathbb{Z})$ is a **totally non-congruence subgroup** if the image of $G$ under $\pi_n$ equals $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ for all $n \in \mathbb{Z}_+$.

The results in this section are based upon the following theorem that gives a sufficient condition when finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ are totally non-congruence groups. It is used in the proofs of Proposition 4.2.4 and Theorem 4.2.9.

**Theorem 4.2.2 ([SW18, Theorem 1])** *Let $\Gamma$ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Suppose that for each prime $p$ there exist matrices $A_1, A_2 \in \mathrm{SL}(2,\mathbb{Z})$ with the following properties:*

*(i) For all $j \in \mathbb{Z}_+$, the inequality $A_1 e_1 \not\equiv_p j \cdot A_2 e_1$ holds.*

*(ii) There exist $m_1, m_2 \in \mathbb{Z}_+$ with $A_1 T^{m_1} A_1^{-1},\ A_2 T^{m_2} A_2^{-1} \in \Gamma$ such that $p$ divides neither $m_1$ nor $m_2$.*

*Then $\Gamma$ is a totally non-congruence group.*

In this section, we use cylinder decompositions in different directions to construct the parabolic matrices occurring in Theorem 4.2.2 (see Lemma 2.3.8). The moduli of the considered cylinders coincide with the order of certain deck transformations (see Remark 2.3.7). We apply Theorem 4.2.2 to normal origamis with deck transformation groups that admit for each prime number a cylinder decomposition with a suitable modulus. The following lemma computes the inverse moduli of the cylinders in the directions of interest.

**Lemma 4.2.3** *Let $\mathcal{O} = (G, x, y)$ be a normal origami. For $m \in \mathbb{Z}_{\geq 0}$, the inverse modulus of all cylinders in direction $\binom{1}{-m}$ coincides with the order of $xy^m$.*

**Proof** Denote $\binom{1}{-m}$ by $v$. Acting with the matrix $A = \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} = S^{-1}T^mS \in \mathrm{SL}(2,\mathbb{Z})$ maps the horizontal direction to the direction $v$, i.e., $A \cdot e_1 = v$. The inverse modulus of all horizontal cylinders of the origami $A \cdot \mathcal{O} = (G, xy^m, y)$ coincides with the order of $xy^m$ (see Remark 2.3.7). Note that acting by matrices in $\mathrm{SL}(2,\mathbb{Z})$ does not change the modulus of a cylinder. Hence, the inverse modulus of the cylinder in direction $v$ of the origami $\mathcal{O}$ equals the order of $xy^m$. ∎

Using Theorem 4.2.2 and Lemma 4.2.3, we deduce a sufficient condition for normal origamis to have a totally non-congruence group as Veech group.

**Proposition 4.2.4** *Let $\mathcal{O} = (G, x, y)$ be a normal origami. If for each prime $p$ one of the following holds*

*(i) $p$ does not divide $\mathrm{ord}(y) \cdot \mathrm{ord}(yx)$ or*

*(ii) there exist natural numbers $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ with $m_1 \not\equiv_p m_2$ and $p$ does not divide $\mathrm{ord}(xy^{-m_1}) \cdot \mathrm{ord}(xy^{-m_2})$.*

*Then the Veech group $\mathrm{SL}(\mathcal{O})$ is a totally non-congruence group.*

**Proof** Fix a prime $p$. If condition (i) holds, consider the matrices $S^{-1}T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and $TS^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. We obtain

$$\begin{aligned} S^{-1}T^{-1} \cdot \mathcal{O} &= (G, yx, x^{-1}), \\ TS^{-1} \cdot \mathcal{O} &= (G, y, x^{-1}y^{-1}). \end{aligned}$$

The inverse moduli of the horizontal cylinders of the normal origamis $(G, yx, x^{-1})$ and $(G, y, (yx)^{-1})$ are $\mathrm{ord}(yx) =: a$ and $\mathrm{ord}(y) =: b$, respectively (see Remark 2.3.7). By Lemma 2.3.8, the Veech group $\mathrm{SL}(\mathcal{O})$ contains the matrices

$$\begin{aligned} S^{-1}T^{-1} \cdot T^a \cdot TS &= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}, \\ TS^{-1} \cdot T^b \cdot ST^{-1} &= \begin{pmatrix} 1-b & b \\ -b & 1+b \end{pmatrix}. \end{aligned}$$

Moreover, we obtain for each $j \in \mathbb{Z}_+$ the inequality

$$\begin{aligned} S^{-1}T^{-1} \cdot e_1 &\equiv_p \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &\not\equiv_p j \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ &\equiv_p j \cdot TS^{-1} \cdot e_1. \end{aligned}$$

Using Theorem 4.2.2, the claim follows in this case.

If condition (ii) holds, then let $m_1, m_2$ be natural numbers satisfying condition (2). Define the matrices

$$A_1 = S^{-1}T^{m_1}S = \begin{pmatrix} 1 & 0 \\ m_1 & 1 \end{pmatrix},$$
$$A_2 = S^{-1}T^{m_2}S = \begin{pmatrix} 1 & 0 \\ m_2 & 1 \end{pmatrix}.$$

Since $m_1 \not\equiv_p m_2$, we have $A_1e_1 \not\equiv j \cdot A_2e_1$ modulo $p$ for each $j \in \mathbb{Z}_+$.

Note that $p$ does not divide $\mathrm{ord}(xy^{-m_1}) \cdot \mathrm{ord}(xy^{-m_2})$. Further, set $k_1 := \mathrm{ord}(xy^{-m_1})$ and $k_2 := \mathrm{ord}(xy^{-m_2})$. Using Lemma 4.2.3 and Lemma 2.3.8, we conclude that the matrices $A_iT^{k_i}A_i^{-1}$ are contained in the Veech group of the origami $\mathcal{O}$. Again, the claim follows by Theorem 4.2.2. ∎

In the following corollary, we construct generating sets $\{x, y\}$ of alternating groups $\mathrm{Alt}(n)$ satisfying the conditions given in Proposition 4.2.4. Consequently, the infinite family of normal origamis $(\mathrm{Alt}(n), x, y)$ have totally non-congruence groups as Veech groups.

**Corollary 4.2.5** *For each prime number $n \geq 5$, the normal origami $(\mathrm{Alt}(n), (1,2,3), (1,2,3,\ldots,n))$ has a totally non-congruence group as Veech group.*

**Proof** Set $x := (1,2,3)$ and $y := (1,2,3,\ldots,n)$. For each prime $p \neq n$, we consider the group elements $y$ and $yx$. Since the orders of $y$ and $yx$ are equal to $n$, the prime $p$ does not divide $\mathrm{ord}(y) \cdot \mathrm{ord}(yx)$.

For the prime $n$, we consider the group elements $xy^{n-1}$ and $x$, i.e., $m_1 = 1-n$ and $m_2 = 0$. Note that $1 - n \not\equiv_n 0$. The permutation $xy^{n-1}$ has the fixed point 2. Since $n$ is a prime, this implies that the permutation $xy^{n-1}$ is not a cycle of length $n$ and its order is not divisible by $n$. Since $\mathrm{ord}(x) = 3 < n$, the prime $n$ does not divide the order of $x$ either. By Proposition 4.2.4, the claim follows. ∎

**Remark 4.2.6** Denote the origami $(\mathrm{Alt}(n), (1,2,3), (1,2,3,\ldots,n))$ by $\mathcal{O}_n$ for $n \in \mathbb{Z}_+$. We determine the strata in which the normal origamis $\mathcal{O}_n$ for $n$ prime lie. For this, we compute the commutator $[(1,2,3), (1,2,3,\ldots,n)] = (1,4,2)$. Hence, the origami $\mathcal{O}_n$ lies in the stratum $\mathcal{H}(k \times 2)$ with $k = \frac{n!}{6}$ and has genus $g = \frac{n!}{6} + 1$.

**Example 4.2.7** We consider the normal origami $\mathcal{O} := (\mathrm{Alt}(5), (1,2,3), (1,2,3,4,5))$ given in Corollary 4.2.5 for the prime $n = 5$. Using the `GAP` package [Ert+21], we compute that the Veech group $\mathrm{SL}(\mathcal{O})$ is generated by the matrices

$$S^2, TST^{-1}, T^3, T^{-1}STS^{-1}, STST^{-3}S^{-1}$$

and has index 9 in $\mathrm{SL}(2,\mathbb{Z})$. Representatives of the $\mathrm{SL}(2,\mathbb{Z})$-orbit are given by the following origamis

$\mathcal{O} = (\mathrm{Alt}(5), (1,2,3), (1,2,3,4,5))$, $\mathcal{O}_2 := (\mathrm{Alt}(5), (2,4)(3,5), (1,2,3,4,5))$,
$\mathcal{O}_3 := (\mathrm{Alt}(5), (1,2,4,5,3), (1,2,3,5,4))$, $\mathcal{O}_4 := (\mathrm{Alt}(5), (3,5,4), (1,2,3,4,5))$,
$\mathcal{O}_5 := (\mathrm{Alt}(5), (1,3,2,5,4), (1,2)(3,4))$, $\mathcal{O}_6 := (\mathrm{Alt}(5), (1,2,3,4,5), (1,2,3))$,
$\mathcal{O}_7 := (\mathrm{Alt}(5), (1,3,5,4,2), (1,2,3))$, $\mathcal{O}_8 := (\mathrm{Alt}(5), (1,2,3,4,5), (1,2,3,5,4))$,
$\mathcal{O}_9 := (\mathrm{Alt}(5), (3,4,5), (1,2,3))$.

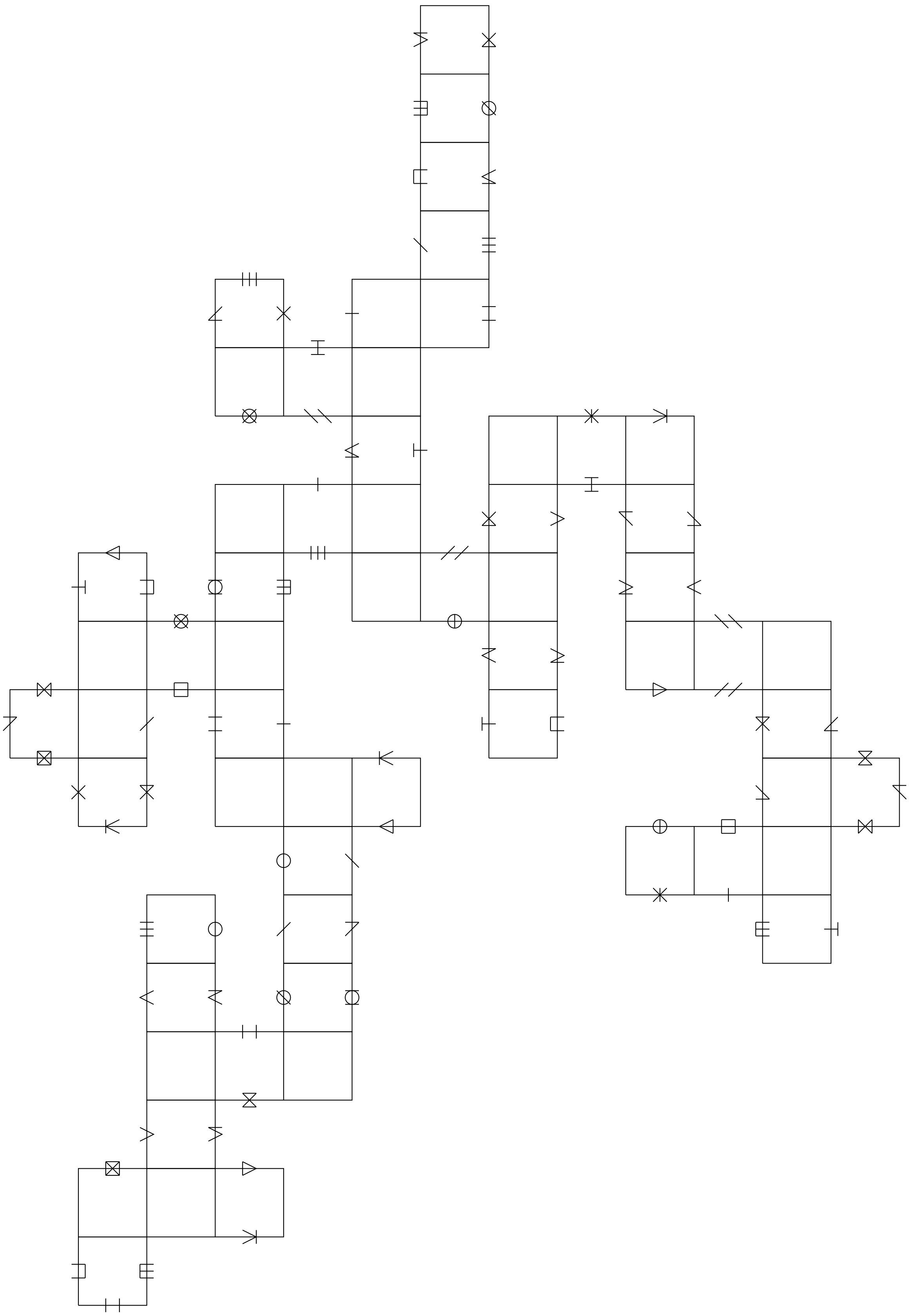

Figure 4.3.: This figure shows the origami $\mathcal{O}$. Each unlabeled edge is glued to the opposite edge. For clarity, these edges are not labeled.

Corollary 4.2.5 motivates to examine finite simple groups more generally. Simple groups form an interesting class of 2-generated groups. The natural question, how the orders of generators for a fixed group can be chosen, has been studied intensively (see e.g. [JLM18] for further information). In this context it is natural to consider $(a, b, c)$-groups.

**Definition 4.2.8** A finite group generated by two elements $x$ and $y$ with $\mathrm{ord}(x) = a$, $\mathrm{ord}(y) = b$, and $\mathrm{ord}(xy) = c$ is called an $(a, b, c)$**-group**. We call such generators $(a, b, c)$**-generators**.

Each $(a, b, c)$-group is a finite quotient of the **triangle group**

$$T_{(a,b,c)} = \langle x, y, z \mid x^a = y^b = z^c = xyz = 1\rangle.$$

The following theorem shows that $(a, b, c)$-groups where $a, b$, and $c$ are chosen pairwise coprime produce normal origamis with a totally non-congruence group as Veech group.

**Theorem 4.2.9** *Let $a, b, c \in \mathbb{Z}_{\geq 0}$ be pairwise coprime and $G$ be an $(a, b, c)$-group with $(a, b, c)$-generators $x, y$. The Veech group of the normal origami $(G, y, x)$ is a totally non-congruence group.*

**Proof** We prove that the assumptions of Theorem 4.2.2 are satisfied for the Veech group of the normal origami $\mathcal{O} = (G, y, x)$. Let $p$ be a prime. Since $a, b$, and $c$ are pairwise coprime, $p$ divides at most one of the numbers $a, b$, and $c$. We consider each of the three cases separately.

If $p$ is coprime to $b \cdot c$, then consider the matrices $I_2$ and $S^{-1}T^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. We obtain $I_2 \cdot \mathcal{O} = \mathcal{O}$ and $S^{-1}T^{-1} \cdot \mathcal{O} = (G, xy, y^{-1})$. The inverse moduli of the horizontal cylinders of the normal origamis $\mathcal{O}$ and $(G, xy, y^{-1})$ are $\mathrm{ord}(y) = b$ and $\mathrm{ord}(xy) = c$, respectively. Hence, $T^b$ and $S^{-1}T^{-1} \cdot T^c \cdot TS$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we obtain for each integer $j \in \mathbb{Z}_+$

$$S^{-1}T^{-1} \cdot e_1 \equiv_p \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \not\equiv_p j \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

If $p$ is coprime to $a \cdot c$, then consider the matrices $S^{-1}T^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $TS^{-1} = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We obtain the normal origamis

$$\begin{aligned} S^{-1}T^{-1} \cdot \mathcal{O} &= (G, xy, y^{-1}), \\ TS^{-1} \cdot \mathcal{O} &= (G, x, y^{-1}x^{-1}). \end{aligned}$$

The inverse moduli of the horizontal cylinders of the normal origamis $(G, xy, y^{-1})$ and $(G, x, (xy)^{-1})$ are $\mathrm{ord}(xy) = c$ and $\mathrm{ord}(x) = a$, respectively. Hence, the matrices $S^{-1}T^{-1} \cdot T^c \cdot TS$ and $TS^{-1} \cdot T^a \cdot ST^{-1}$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we obtain for each $j \in \mathbb{Z}_+$ the inequality

$$\begin{aligned} S^{-1}T^{-1} \cdot e_1 &\equiv_p \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \\ &\not\equiv_p j \cdot \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right) \\ &\equiv_p j \cdot TS^{-1} \cdot e_1. \end{aligned}$$

If $p$ is coprime to $a \cdot b$, then consider the matrices $I_2$ and $S^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We obtain the normal origamis $I_2 \cdot \mathcal{O} = \mathcal{O}$ and $S^{-1} \cdot \mathcal{O} = (G, x, y^{-1})$. The inverse moduli of the horizontal cylinders of the normal origamis $\mathcal{O}$ and $(G, x, y^{-1})$ are $\mathrm{ord}(y) = b$ and $\mathrm{ord}(x) = a$, respectively. Hence, $T^b$ and $S^{-1} \cdot T^a \cdot S$ lie in the Veech group $\mathrm{SL}(\mathcal{O})$. Moreover, we have for each integer $j \in \mathbb{Z}_+$

$$S^{-1} \cdot e_1 \equiv_p \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \not\equiv_p j \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right). \qquad \blacksquare$$

**Example 4.2.10** A well-studied family of groups satisfying the assumption in Theorem 4.2.9 are $(2, 3, 7)$-groups, which are also called Hurwitz groups. Hurwitz groups are of interest from a geometric point of view because they arise as automorphism groups of compact Riemann surfaces of genus $g > 1$ with maximal order, i.e., of order $84(g-1)$. The smallest Hurwitz group is the projective linear group $\mathrm{PSL}(2, 7)$ and has order 168. For further information about Hurwitz, see, e.g., [Con90] and [Con10].

# Chapter 5.

# $T_2$-systems and normal origamis

In this chapter, we investigate the connection between the group theoretic concept of $T_2$-systems and the $\mathrm{SL}(2,\mathbb{Z})$-orbits on sets of normal origamis. Subsequently, we deduce results for such $\mathrm{SL}(2,\mathbb{Z})$-orbits from the results known for $T_2$-systems.

## 5.1. Connection between $T_2$-systems and normal origamis

First, we introduce the notion of $T_k$-systems that are orbits under a special group action. We then describe the connection between $T_2$-systems and the $\mathrm{SL}(2,\mathbb{Z})$-action on normal origamis with a fixed deck transformation group. The $T_2$-systems turn out to be unions of up to two orbits under the $\mathrm{SL}(2,\mathbb{Z})$-action. Zmiaikou discussed in [Zmi11, Chapter 4] the analog connection for a $\mathrm{GL}(2,\mathbb{Z})$-action which generalizes the $\mathrm{SL}(2,\mathbb{Z})$-action. In this case, the $T_2$-systems coincide with the $\mathrm{GL}(2,\mathbb{Z})$-orbits.

Let $G$ be a finite group and $k \in \mathbb{Z}_+$. Define the set

$$E_k(G) := \{(g_1, \dots, g_k) \mid \langle g_1, \dots, g_k \rangle = G\}.$$

That is, $E_k(G)$ contains $k$-tuples with generators of $G$ as entries. Such a $k$-tuple can be viewed as an epimorphism from the free group $F_k$ on $k$ letters to $G$ sending the standard generators $x_1, \dots, x_k$ of $F_k$ to the entries of the tuple. The two automorphism groups $\mathrm{Aut}(F_k)$ and $\mathrm{Aut}(G)$ act on $E_k(G)$ via precomposing and postcomposing the homomorphisms, respectively. More precisely, for $g = (g_1, \dots, g_k)$ in $E_k(G)$ define $\alpha_g : F_k \to G$ via $\alpha_g(x_i) = g_i$ for $1 \leq i \leq k$. The $\mathrm{Aut}(F_k) \times \mathrm{Aut}(G)$-action is defined by

$$(\psi, \varphi) \cdot \alpha_g := \varphi \circ \alpha_g \circ \psi^{-1}$$

for $(\psi, \varphi) \in \mathrm{Aut}(F_k) \times \mathrm{Aut}(G)$ and $g \in E_k(G)$. Again, the images of $x_1, \dots, x_k$ under the epimorphism $\varphi \circ \alpha_g \circ \psi^{-1}$ define an element in $E_k(G)$.

**Definition 5.1.1** For a finite group $G$ and $k \in \mathbb{Z}_+$, the orbits of the $\mathrm{Aut}(F_k) \times \mathrm{Aut}(G)$-action on the set $E_k(G)$ are called $T_k$**-systems**.

The concept of $T_k$-systems was introduced in the context of ultracharacteristic groups by B. H. Neumann and H. Neumann in [NN51]. Since then $T_k$-systems have been studied by various mathematicians such as Dunwoody, Evans, Gilman, Guralnick, and Pak (see, e.g., [Dun63], [Eva93], [Gil77], and [GP03]). Renewed interest in $T_k$-systems was caused by the connection between $T_k$-systems and the *Product Replacement Algorithm* (PRA) (see, e.g., [Pak01] and [GS09]). The PRA was introduced and analyzed in [Cel+95]. It is used to construct random elements in finite groups and implemented as a standard routine in the computer algebra systems `GAP` and `MAGMA`.

In the following, we briefly describe the connection between $T_k$-systems and the PRA. For this, we introduce a particular generating set of the automorphism group of $F_k$. This generating set consists of the following automorphisms which are called **Nielsen transformations**

$$\begin{aligned}
R_{i,j} &: (x_1, \ldots, x_i, \ldots, x_k) && \mapsto (x_1, \ldots, x_i \cdot x_j, \ldots, x_k),\\
L_{i,j} &: (x_1, \ldots, x_i, \ldots, x_k) && \mapsto (x_1, \ldots, x_j \cdot x_i, \ldots, x_k),\\
P_{i,j} &: (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) && \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k),\\
I_i &: (x_1, \ldots, x_i, \ldots, x_k) && \mapsto (x_1, \ldots, x_i^{-1}, \ldots, x_k),
\end{aligned}$$

for $1 \leq i \neq j \leq k$. Further, consider for $1 \leq i \neq j \leq k$ the automorphisms

$$\begin{aligned}
R^-_{i,j} &= I_j \circ R_{i,j} \circ I_j,\\
L^-_{i,j} &= I_j \circ L_{i,j} \circ I_j.
\end{aligned}$$

The PRA constructs random group elements by conducting a random walk on the **product replacement graph** $\Gamma_k(G)$, where the vertices of $\Gamma_k(G)$ are the elements of $E_k(G)$. An edge connects two vertices of $\Gamma_k(G)$ if one vertex is mapped to the other vertex by an automorphism $R^-_{i,j}, R_{i,j}, L^-_{i,j}$, or $L_{i,j}$ for $1 \leq i \neq j \leq k$. One is interested in studying the connected components of the graph $\Gamma_k(G)$.

Similarly, one can construct the **extended product replacement graph** $\tilde{\Gamma}_k(G)$. Again, the vertices are the elements of $E_k(G)$. An edge corresponds to the application of an automorphism $P_{i,j}, I_i, R^-_{i,j}, R_{i,j}, L^-_{i,j}$, or $L_{i,j}$ for $1 \leq i \neq j \leq k$. Thus, the number of $T_k$-systems is less or equal to the number of connected components of the graph $\tilde{\Gamma}_k(G)$. Further, the number of connected components of the graph $\tilde{\Gamma}_k(G)$ is smaller or equal to the number of connected components of the graph $\Gamma_k(G)$.

In the remaining part of this chapter, we focus on the case $k = 2$.

**Remark 5.1.2** Recall that we identify normal origamis with deck transformation group $G$ with pairs of generators of $G$. More precisely, the set of normal origamis $\mathcal{O}(G)$ coincides with the quotient $E_2(G)/\operatorname{Aut}(G)$, where $\operatorname{Aut}(G)$ acts componentwise on tuples in $E_2(G)$ as in the definition of $T_2$-systems (see Remark 2.2.5). Note that inner automorphisms of $F_2$ act as automorphism of $G$ and thus it is possible to consider the orbits under the action of $\operatorname{Out}(F_2) \times \operatorname{Aut}(G)$ instead of $\operatorname{Aut}(F_2) \times \operatorname{Aut}(G)$ in the definition of $T_2$-systems. Further, recall that for the free group on two generators the outer automorphism group

$\mathrm{Out}(F_2)$ is isomorphic to the group $\mathrm{GL}(2,\mathbb{Z})$. Since $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup of $\mathrm{GL}(2,\mathbb{Z})$, we can assign an automorphism in $\mathrm{Out}(F_2)$ to each element of $\mathrm{SL}(2,\mathbb{Z})$. The $\mathrm{SL}(2,\mathbb{Z})$-action introduced in Section 2.3 coincides with the $\mathrm{Out}(F_2)$-action considered for $T_2$-systems. For instance, the action of the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to the automorphisms (viewed as elements in the outer automorphism group)

$$\begin{aligned} \psi_S &= P_{1,2} \circ I_1, x \mapsto y, y \mapsto x^{-1} \text{ and} \\ \psi_T &= R_{1,2}, x \mapsto x, y \mapsto yx, \end{aligned}$$

respectively.

The following lemma connects the number of $T_2$-systems with the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in the set of normal origamis with fixed deck transformation group.

**Lemma 5.1.3** *For a finite group $G$, denote the number of $T_2$-systems by $n$. Then the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in $\mathcal{O}(G)$ lies between $n$ and $2n$.*

**Proof** Let $G$ be a group. Recall that the $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathcal{O}(G)$ coincides with the group action of $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{Aut}(G)$ on $E_2(G)$ occurring in the definition of $T_2$-systems. The group $\mathrm{SL}(2,\mathbb{Z})$ is the kernel of the determinant map $\mathrm{GL}(2,\mathbb{Z}) \to \mathbb{Z}^*$ and thus it has index 2 in $\mathrm{GL}(2,\mathbb{Z})$. Hence, we deduce the claim: If $n$ denotes the number of $T_2$-systems, the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in the set $\mathcal{O}(G)$ lies between $n$ and $2n$. ∎

We obtain the following corollary and apply it to the families of groups considered in Section 4.1.1.

**Corollary 5.1.4** *If the group $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on the set $\mathcal{O}(G)$ for a group $G$, then $G$ admits only one $T_2$-system.*

**Example 5.1.5** In Section 4.1.1, we showed that the $\mathrm{SL}(2,\mathbb{Z})$-action is transitive on each of the sets of normal origamis $\mathcal{O}(D_{2m}), \mathcal{O}(G^2_{n,k}), \mathcal{O}(A_m), \mathcal{O}(B_m)$, and $\mathcal{O}(W_m)$ (see Proposition 4.1.4, Proposition 4.1.7, Proposition 4.1.13, Proposition 4.1.16, and Proposition 4.1.20). By Corollary 5.1.4, each of these groups admits only one $T_2$-system.

Given a $T_2$-system $\mathcal{T}$ of a group $G$, we examine whether $\mathcal{T}$ induces one or two $\mathrm{SL}(2,\mathbb{Z})$-orbits. Consider a matrix $M \in \mathrm{GL}(2,\mathbb{Z}) \setminus \mathrm{SL}(2,\mathbb{Z})$ and a normal origami $\mathcal{O} = (G,x,y)$ defined by a pair of generators $(x,y)$ in $\mathcal{T}$. Then $M \cdot \mathcal{O}$ is either contained in the orbit $\mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$ or not. We first consider the case that $M \cdot \mathcal{O}$ lies in $\mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$. There exists a matrix $M' \in \mathrm{SL}(2,\mathbb{Z})$ such that $M \cdot \mathcal{O} = M' \cdot \mathcal{O}$. We obtain the equality $\mathcal{O} = M^{-1}M' \cdot \mathcal{O}$ and thus $M^{-1}M' \in \mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O})$. This implies that the stabilizer $\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O})$ is not contained in $\mathrm{SL}(2,\mathbb{Z})$.

We now consider the case $M \cdot \mathcal{O} \notin \mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$. Recall that the general linear group $\mathrm{GL}(2,\mathbb{Z})$ is the disjoint union $\mathrm{SL}(2,\mathbb{Z}) \uplus \mathrm{SL}(2,\mathbb{Z}) \cdot M$. We conclude that each matrix in

$\mathrm{GL}(2,\mathbb{Z}) \setminus \mathrm{SL}(2,\mathbb{Z})$ is not contained in the stabilizer $\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O})$. As a consequence, we obtain

$$\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O}) = \mathrm{SL}(\mathcal{O}) \subseteq \mathrm{SL}(2,\mathbb{Z}).$$

Since $\mathrm{GL}(2,\mathbb{Z})$ equals $\mathrm{SL}(2,\mathbb{Z}) \uplus \mathrm{SL}(2,\mathbb{Z}) \cdot M$, the argument above is independent of the matrix $M$. Furthermore, it does not depend on the choice of the pair of generators $(x,y) \in \mathcal{T}$. We obtain the following lemma:

**Lemma 5.1.6** *The* $\mathrm{GL}(2,\mathbb{Z})$*-orbit of a normal origami* $\mathcal{O}$ *splits into two* $\mathrm{SL}(2,\mathbb{Z})$*-orbits if the inclusion* $\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O}) \subseteq \mathrm{SL}(2,\mathbb{Z})$ *holds. If this inclusion is not satisfied, then the orbits* $\mathrm{GL}(2,\mathbb{Z}) \cdot \mathcal{O}$ *and* $\mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$ *coincide.*

**Remark 5.1.7** We consider the action of the matrix $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on a normal origami $\mathcal{O} = (G,x,y)$. This action corresponds to the automorphism $\psi$ of $F_2$ with $\psi(x_1) = x_1$ and $\psi(x_2) = x_2^{-1}$ in the setting of $T_2$-systems. The origami $\mathcal{O}$ is stabilized by the matrix $M$ if $x \mapsto x$ and $y \mapsto y^{-1}$ defines an automorphism of $G$. If this is the case, we have $\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O}) \not\subseteq \mathrm{SL}(2,\mathbb{Z})$ and derive that the $T_2$-system containing the pair $(x,y)$ induces only one $\mathrm{SL}(2,\mathbb{Z})$-orbit.

**Example 5.1.8** The considered groups in Example 5.1.5 have the property that the $T_2$-systems and $\mathrm{SL}(2,\mathbb{Z})$-orbits coincide. We now search for small groups $G$ such that a $T_2$-system of $G$ splits into two $\mathrm{SL}(2,\mathbb{Z})$-orbits. For all groups $G$ of order smaller than 81, the number of $T_2$-systems of $G$ and the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in the set of normal origamis $\mathcal{O}(G)$ coincide. To see this, one computes for each such group $G$ the $\mathrm{SL}(2,\mathbb{Z})$-orbits in the set of normal origamis $\mathcal{O}(G)$ using the `GAP`-package [Ert+21]. For most groups $G$, this $\mathrm{SL}(2,\mathbb{Z})$-action is transitive and thus the number of $T_2$-systems and the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits are equal. This is consistent with the results of Kremer who examined which groups $G$ of order up to 250 admit more than one $\mathrm{SL}(2,\mathbb{Z})$-orbit (see [Kre09, Appendix A.4]). The possible orders of such groups are $60, 81, 120, 160, 162, 168, 180, 189, 192, 200, 216, 240,$ and $243$.

If the $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathcal{O}(G)$ is not transitive, one checks for a normal origami $\mathcal{O}$ in $\mathcal{O}(G)$ whether $M \cdot \mathcal{O}$ lies in the orbit $\mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$. Here, $M$ denotes the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The smallest group $G$ such that $M \cdot \mathcal{O}$ does not lie in the orbit $\mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$ for $\mathcal{O} \in \mathcal{O}(G)$ has order 81 and was assigned the identification number $(81,10)$ in the Small Groups library in `GAP`. It is generated by the two permutations

$$\begin{aligned} x = {} & (1,2,4,8,16,30,15,23,37)(3,6,12,17,32,52,29,46,65) \\ & (5,10,20,31,50,68,43,62,77)(7,14,27,33,54,21,39,55,69) \\ & (9,18,34,49,67,78,59,76,81)(11,22,41,51,24,35,56,70,79) \\ & (13,25,38,53,71,80,66,19,36)(26,45,63,72,47,57,75,40,61) \\ & (28,48,60,73,42,64,74,44,58), \end{aligned}$$

$$
\begin{aligned}
y = {} & (1,3,7,15,29,39,8,17,33)(2,5,11,23,43,56,16,31,51)\\
& (4,9,19,37,59,71,30,49,25)(6,13,26,46,66,75,32,53,72)\\
& (10,21,40,62,27,47,50,69,45)(12,24,44,65,22,42,52,70,48)\\
& (14,28,34,55,74,81,54,73,78)(18,35,57,76,41,63,67,79,61)\\
& (20,38,60,77,36,58,68,80,64).
\end{aligned}
$$

The set $\mathcal{O}(G)$ consists of 8 normal origamis and splits into two $\mathrm{SL}(2,\mathbb{Z})$-orbits of length 4. For the origami $\mathcal{O} = (G, x, y)$, the origami $M \cdot \mathcal{O} = (G, x, y^{-1})$ does not lie in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit as the origami $\mathcal{O}$. Thus, we obtain the inclusion $\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{Z})}(\mathcal{O}) \subseteq \mathrm{SL}(2,\mathbb{Z})$. We conclude that there is one $T_2$-system of $G$ which induces two $\mathrm{SL}(2,\mathbb{Z})$-orbits.

We consider yet another example. The next larger group with the desired property has order 162 and was assigned the identification number $(162, 31)$ in the Small Groups library. It is generated by the two permutations

$$
\begin{aligned}
x = {} & (1,2,4,8,16,30,52,81,113)(3,6,12,17,32,56,82,115,143)\\
& (5,10,20,31,54,84,114,142,158)(7,14,27,33,58,21,39,60,85)\\
& (9,18,34,53,83,116,141,157,140)(11,22,41,55,24,35,61,87,117)\\
& (13,25,38,57,88,118,144,19,36)(15,23,37,59,86,45,66,93,120)\\
& (26,46,69,89,48,62,94,40,67)(28,49,65,90,42,70,92,44,63)\\
& (29,47,73,91,121,78,99,126,71)(43,68,98,119,75,103,125,77,96)\\
& (50,79,102,122,64,95,124,74,100)(51,80,109,123,147,101,130,148,107)\\
& (72,104,133,145,105,127,149,110,131)(76,106,129,146,111,134,150,97,128)\\
& (108,137,155,159,138,151,162,132,154)(112,139,153,160,135,156,161,136,152),\\
y = {} & (1,3,7,15,29,51,8,17,33,59,91,123,52,82,39,66,99,130)\\
& (2,5,11,23,43,72,16,31,55,86,119,145,81,114,61,93,125,149)\\
& (4,9,19,37,64,97,30,53,25,45,74,106,113,141,88,120,79,111)\\
& (6,13,26,47,76,108,32,57,89,121,146,159,115,144,94,126,150,162)\\
& (10,21,40,68,101,132,54,85,46,75,107,137,142,27,48,77,109,138)\\
& (12,24,44,73,105,136,56,87,49,78,110,139,143,22,42,71,104,135)\\
& (14,28,50,80,112,140,58,90,122,147,160,34,60,92,124,148,161,116)\\
& (18,35,62,95,127,151,83,117,67,100,131,154,157,41,69,102,133,155)\\
& (20,38,65,98,129,153,84,118,70,103,134,156,158,36,63,96,128,152).
\end{aligned}
$$

The set $\mathcal{O}(G)$ consists of 24 normal origamis and splits into two $\mathrm{SL}(2,\mathbb{Z})$-orbits of length 12. For the origami $\mathcal{O} = (G, x, y)$, the origami $M \cdot \mathcal{O} = (G, x, y^{-1})$ does not lie in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit as the origami $\mathcal{O}$. Again, we conclude that there is one $T_2$-system of $G$ which induces two $\mathrm{SL}(2,\mathbb{Z})$-orbits.

Finally, we present one example showing the opposite behaviour. The matrix group $\mathrm{PSL}(3,2)$ has order 168 and was assigned the identification number $(168, 42)$ in the Small Groups library. Note that this group is isomorphic to the group $\mathrm{PSL}(2,7)$ and is a Hurwitz group (see Example 4.2.10). It is generated by the two permutations

$$\begin{aligned} x = {} & (1,2)(3,5)(4,7)(6,10)(8,12)(9,13)(11,16)(14,20)(15,21)(17,24) \\ & (18,25)(19,27)(22,31)(23,32)(26,36)(28,38)(29,39)(30,41)(33,45)(34,46) \\ & (35,48)(37,51)(40,55)(42,57)(43,58)(44,59)(47,63)(49,65)(50,66)(52,67) \\ & (53,68)(54,70)(56,73)(60,77)(61,78)(62,80)(64,83)(69,89)(71,81)(72,91) \\ & (74,92)(75,85)(76,94)(79,98)(82,100)(84,101)(86,103)(87,105)(88,107) \\ & (90,110)(93,114)(95,108)(96,116)(97,118)(99,121)(102,125)(104,119) \\ & (106,128)(109,130)(111,131)(112,132)(113,134)(115,137)(117,140)(120,142) \\ & (122,143)(123,144)(124,146)(126,149)(127,151)(129,153)(133,152)(135,158) \\ & (136,159)(138,160)(139,154)(141,157)(145,161)(147,164)(148,165)(150,166) \\ & (155,162)(156,167)(163,168), \\ y = {} & (1,3,6)(2,4,8)(5,9,14)(7,11,17)(10,15,22)(12,18,26)(13,19,28)(16,23,33) \\ & (20,29,40)(21,30,42)(24,34,47)(25,35,49)(27,37,52)(31,43,59)(32,44,60) \\ & (36,50,51)(38,53,69)(39,54,71)(41,56,74)(45,61,79)(46,62,81)(48,64,84) \\ & (55,72,73)(57,75,93)(58,76,95)(63,82,83)(65,85,102)(66,86,104)(67,87,106) \\ & (68,88,108)(70,90,111)(77,96,117)(78,97,119)(80,99,122)(89,109,110) \\ & (91,112,133)(92,113,135)(94,115,138)(98,120,121)(100,123,145)(101,124,147) \\ & (103,126,150)(105,127,152)(107,129,154)(114,136,137)(116,139,161) \\ & (118,141,151)(125,148,149)(128,146,153)(130,155,166)(131,156,144) \\ & (132,143,163)(134,157,140)(142,162,160)(158,167,165)(159,164,168). \end{aligned}$$

Using `GAP`, one computes that the Veech group of the origami $(\mathrm{PSL}(3,2),x,y)$ has index 16 in $\mathrm{SL}(2,\mathbb{Z})$. Furthermore, one sees $x \mapsto x$ and $y \mapsto y^{-1}$ defines an automorphism of the group $\mathrm{PSL}(3,2)$. Hence, we obtain the equality $(\mathrm{PSL}(3,2),x,y) = M \cdot (\mathrm{PSL}(3,2),x,y)$ and conclude that the $T_2$-system containing the tuple $(x,y)$ induces only one $\mathrm{SL}(2,\mathbb{Z})$-orbit (see Remark 5.1.7). The set $\mathcal{O}(\mathrm{PSL}(3,2))$ splits into four $\mathrm{SL}(2,\mathbb{Z})$-orbits of length $7, 16, 16$, and $18$. Note that the following is a consequence of the orbit-stabilizer theorem: If a $T_2$-system splits into two $\mathrm{SL}(2,\mathbb{Z})$-orbits, then both $\mathrm{SL}(2,\mathbb{Z})$-orbits have the same length. The normal origami $(\mathrm{PSL}(3,2),x,y)$ lies in an $\mathrm{SL}(2,\mathbb{Z})$-orbit of length 16 and thus we conclude that each $T_2$-system induces only one $\mathrm{SL}(2,\mathbb{Z})$-orbit.

In addition, we compute with the help of `GAP` that all 32 normal origamis with deck transformation group $\mathrm{PSL}(3,2)$ that lie in an $\mathrm{SL}(2,\mathbb{Z})$-orbit of length 16 are contained in the stratum $\mathcal{H}(42 \times 3)$. The normal origamis in the $\mathrm{SL}(2,\mathbb{Z})$-orbits of length 7 and 18 lie in the stratum $\mathcal{H}(24 \times 6)$ and $\mathcal{H}(56 \times 2)$, respectively.

## 5.2. Transfer of results to $\mathrm{SL}(2,\mathbb{Z})$-orbits of normal origamis

In this section, we collect known results about $T_2$-systems and use Lemma 5.1.3 to deduce results on $\mathrm{SL}(2,\mathbb{Z})$-orbits of normal origamis with isomorphic deck transformation groups. More precisely, we show that the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in the set of normal origamis $\mathcal{O}(G)$ is unbounded for a finite group $G$ that is either simple, a $p$-group, or equals $\mathrm{PSL}(2,p)$ for $p$ prime.

We begin with a result on the number of $T_2$-systems for the groups $\mathrm{PSL}(2,p)$ with $p$ prime.

**Theorem 5.2.1 ([GP03, Theorem 1.2])** *For each natural number $n$, there exists a prime number $p$ such that the number of $T_2$-systems for the group $\mathrm{PSL}(2,p)$ is at least $n$.*

**Corollary 5.2.2** *For each natural number $n$, there exists a prime number $p$ such that the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in the normal origamis $\mathcal{O}(\mathrm{PSL}(2,p))$ is at least $n$.*

Garion and Shalev showed that the number of $T_2$-systems for finite simple groups is also unbounded.

**Theorem 5.2.3 ([GS09, Theorem 1.8])** *Let $G$ be a finite simple group. Then the number of $T_2$-systems in $G$ tends to infinity as $|G| \to \infty$.*

**Corollary 5.2.4** *Let $G$ be a finite simple group. Then the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits in the normal origamis $\mathcal{O}(G)$ tends to infinity as $|G| \to \infty$.*

Let $\mathcal{T}$ denote a $T_2$-system of a group $G$. By Higman's Lemma (see [Neu56, Section 2]), there exists a natural number $k$ such that for each tuple $(x,y)$ in $\mathcal{T}$ the order of the commutator $[x,y]$ equals $k$. So, the commutator order is an invariant of a $T_2$-system. In [Neu56, Section 3], Neumann constructs a 2-group $G$ of order $2^{15}$ with more than one $T_2$-system. He considers group elements $x, y, y' \in G$ such that

$$\begin{aligned} G &= \langle x, y\rangle = \langle x, y'\rangle, \\ \mathrm{ord}([x,y]) &= 2, \\ \mathrm{ord}([x,y']) &= 4. \end{aligned}$$

As the order of the commutator $[a,b]$ is fixed for all pairs of generators $(a,b)$ in the $T_2$-system containing the pair of generators $(x,y)$, the pair $(x,y')$ lies in a different $T_2$-system than the pair $(x,y)$. Neumann computes that the nilpotency class of $G$ is 12. Considering a suitable quotient of $G$, he obtains a group which has more than one $T_2$-system, order $2^{13}$, nilpotency class 10, and derived length 3. Note that a group has derived length 3 if the derived series has length 3, i.e., $G^{(2)} \supsetneq G^{(3)} = \langle 1\rangle$. He poses the following question: What is the minimal natural number $n$ such that there exists a 2-group of nilpotency class $n$ with more than one $T_2$-system? Dunwoody answered this question in the following theorem.

**Theorem 5.2.5 ([Dun63, Theorem 1])** *For each natural number $n$ and each prime number $p$, there exists a finite $p$-group of nilpotency class 2 with at least $n$ $T_2$-systems.*

**Corollary 5.2.6** *For each natural number $n$ and each prime number $p$, there exists a finite $p$-group $G$ of nilpotency class 2 with at least $n$ $\mathrm{SL}(2,\mathbb{Z})$-orbits in the set of normal origamis $\mathcal{O}(G)$.*

However, the order of the groups constructed in the proof of Theorem 5.2.5 grows very quickly.

**Example 5.2.7** In Lemma 3.1.34, we considered the group $G = \langle x, y\rangle$, where

$$\begin{aligned} x &:= e_{1,1}e_{2,1}e_{3,1}e_{4,1} = (1,9,5,13,3,11,7,15,2,10,6,14,4,12,8,16), \\ x' &:= x^3 = (1,13,7,10,4,16,5,11,2,14,8,9,3,15,6,12), \\ y &:= e_{1,1}e_{3,4}e_{4,1} = (1,9,2,10)(3,11)(4,12)(5,15,7,13)(6,16,8,14). \end{aligned}$$

This group is a 2-group of order $2^{12}$, nilpotency class 7, and derived length 3. Since the order of $x$ is 16 and 3 is coprime to 16, one has $\langle x\rangle = \langle x'\rangle$ and thus $\langle x, y\rangle = \langle x', y\rangle$. At the same time, we compute

$$\mathrm{ord}([y,x]) = 2 \neq 4 = \mathrm{ord}([y,x']).$$

We used this example to construct two normal origamis $(G, y, x)$ and $(G, y, x')$ with isomorphic deck transformation group that lie in different strata. This implies that they also lie in different $\mathrm{SL}(2,\mathbb{Z})$-orbits. Applying Higman's Lemma, we see that the tuples $(y, x)$ and $(y, x')$ in $E_2(G)$ lie in different $T_2$-system of the group $G$ as well.

# Chapter 6.

# Geminal origamis

The content of this chapter is motivated by recent studies of Apisa and Wright on generalizations of the “eierlegende Wollmilchsau” origami (see [AW21a]). In their work, Apisa and Wright examine $\mathrm{GL}(2,\mathbb{R})$-orbit closures that consist of surfaces with extraordinary cylinder decompositions. They call such orbit closures *geminal* and provide a classification which is useful in different frameworks. Firstly, Apisa and Wright answer two questions from Mirzakhani and Wright regarding $\mathrm{GL}(2,\mathbb{R})$-invariant varieties (see [AW21a, Section 1.4] and [MW18]). Secondly, Apisa derives results on certain varieties with a degenerate Lyapunov spectrum (see [Api21]). Lastly, Apisa and Wright obtain results on $\mathrm{GL}(2,\mathbb{R})$-invariant varieties of high rank (see [AW21b]).

One case in the classification of geminal orbit closures involves origamis covering the $(2\times 2)$-torus $\mathbb{T}[2]$, which we call **geminal origamis**. In this chapter, we investigate the conjecture of Apisa and Wright that each geminal origami induces a cyclic cover of $\mathbb{T}[2]$ (see [AW21a, Section 8.4]).

## 6.1. Connection between the geometric and the group-theoretic setting

In this section, we discuss the connection between a geometric question regarding geminal origamis (see Question 6.1.2) and a group-theoretic question regarding stabilizer subgroups in a symmetric group (see Question 6.1.7). We begin by defining geminal origamis.

**Definition 6.1.1** Let $\mathcal{O}$ be an origami that allows a cover $c_1 : \mathcal{O} \to \mathbb{T}[2]$ with up to four ramification points. The possible ramification points are the corners of the four squares in the $(2\times 2)$-torus $\mathbb{T}[2]$. We further require that the cover $c_1$ is compatible with the cover $c : \mathcal{O} \to \mathbb{T}$, i.e., if $c_2$ denotes the natural cover $\mathbb{T}[2] \to \mathbb{T}$, then the equality $c = c_2 \circ c_1$ holds. Denote the degree of the cover $c_1$ by $d$. We call the origami $\mathcal{O}$ **geminal** if the monodromy map $\mu : F_2 \to \mathrm{Sym}(4d)$ corresponding to the cover $\mathcal{O} \to \mathbb{T}$ satisfies the following property: For each pair of generators $(a', b')$ of the group $F_2$, the images $\mu(a')$ and $\mu(b')$ consist of two disjoint $2d$-cycles each.

Each geminal origami $\mathcal{O}$ is described by two permutations $x$ and $y$ describing the horizontal and vertical gluings of the origami via the monodromy map (see Section 2.2.1). As $\mathcal{O}$ covers the surface $\mathbb{T}[2]$, the degree of the origami is $4d$. Hence, $x$ and $y$ are permutations in the group $\mathrm{Sym}(4d)$. We denote the group $\langle x, y\rangle$ by $G$ and identify the fundamental group of the punctured torus $\mathbb{T}^*$ with the free group $F_2$. The monodromy map $\mu : F_2 \to \mathrm{Sym}(4d)$ sends the standard generators $a$ and $b$ to $x$ and $y$, respectively. Thus, the permutations $x$ and $y$ consist of two disjoint $2d$-cycles each. We call such permutations a $(2d, 2d)$**-cycle**. Note that the cycle structure of the permutation $x$ implies that the cylinder decomposition in horizontal direction consists of two cylinders of circumference $2d$ and height 1.

A geometric property of cylinder decompositions motivates the definition of geminal origamis. The cylinder decomposition in each direction consists of two cylinders of equal height and circumference, respectively. We consider the projection of a cylinder from a geminal origami to the torus $\mathbb{T}$. Each core curve of a cylinder is mapped to a curve in the torus. For a geminal origami, the length of the core curve of each cylinder divided by the length of the corresponding curve in the torus equals $2d$. This is caused by the group-theoretic property in the definition of geminal origamis which states that generators of $F_2$ are mapped to $(2d, 2d)$-cycles. Different generators of the free group $F_2$ correspond to core curves of cylinder decompositions in different directions. This relates the group-theoretic property of generators of $F_2$ to the geometric property of cylinder decompositions. See [AW21a, Section 8.4] for further details.

Apisa and Wright asked the following question:

**Question 6.1.2 ([AW21a, Problem 8.16])** Let $\mathcal{O}$ be a geminal origami. Is the cover $c_1 : \mathcal{O} \to \mathbb{T}[2]$ normal with a cyclic deck transformation group?

**Example 6.1.3** The standard example for a geminal origami is the "eierlegende Wollmilchsau" which we have considered for instance in Example 2.2.3 and Example 3.3.11. Using the permutation notation, the "eierlegende Wollmilchsau" is defined by the permutations $x = (1, 2, 3, 4)(5, 6, 7, 8)$ and $y = (1, 5, 3, 7)(2, 8, 4, 6)$. The "eierlegende Wollmilchsau" is shown in Figure 6.1.

**Example 6.1.4** A further example for a geminal origami was introduced in [FM08]. See also [MW15] and [AW21a] for further information. This origami is often called "Ornithorynque". Using the permutation notation, the "Ornithorynque" is defined by the permutations $x = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)$ and $y = (1, 11, 5, 7, 3, 9)(2, 12, 4, 10, 6, 8)$. The "Ornithorynque" is shown in Figure 6.2.

Apisa and Wright formulated Question 6.1.2 in the language of group theory using covers and fundamental groups (see [AW21a, Section 8.4]). In the following, we explain the connection between the geometric and the group-theoretic statement. For this, we consider a second group homomorphism

$$\alpha : F_2 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \text{ defined by } \alpha(a) = (1, 0) \text{ and } \alpha(b) = (0, 1).$$

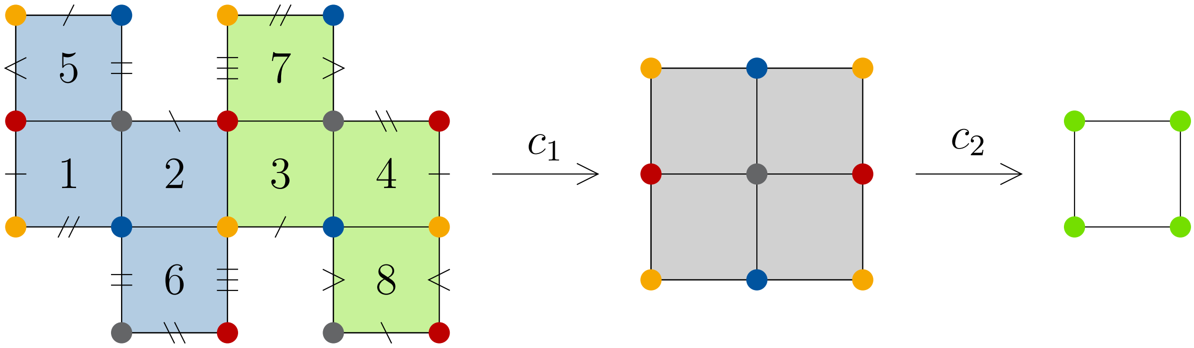


Figure 6.1.: This figure shows the cover $c_1$ from the "eierlegende Wollmilchsau" to $\mathbb{T}[2]$ and the cover $c_2$ from $\mathbb{T}[2]$ to $\mathbb{T}$. The cover $c_1$ has degree 2 and the sheets of the cover are shaded in blue and green. The deck transformation group is the cyclic group of order 2.

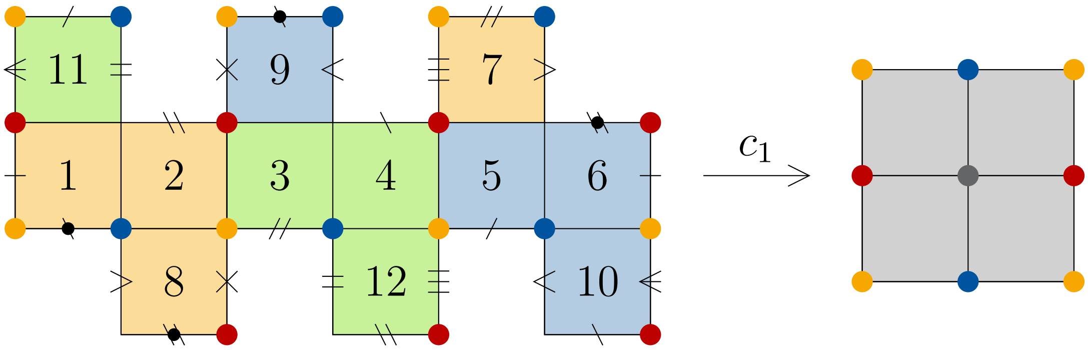


Figure 6.2.: This figure shows the cover $c_1$ from the "Ornithorynque" to $\mathbb{T}[2]$. The cover $c_1$ has degree 3 and the sheets of the cover are shaded in orange, green, and blue.

Note that the kernel $\ker(\alpha)$ is the smallest normal subgroup of $F_2$ containing $a^2, b^2$, and the commutator $[a, b]$. The fundamental groups of the surfaces $\mathbb{T}[2]$ and $\mathcal{O}$ are the kernel $\ker(\alpha)$ and the preimage $\mu^{-1}(\mathrm{Stab}_G(1))$, respectively. Here, we consider the permutation action of the symmetric group on the squares $1, \ldots, 4d$ the origami $\mathcal{O}$ consists of. Further, note that the monodromy group $\mu(F_2)$ is isomorphic to the group $G$. We identify the monodromy group with $G$ and thus we notate the intersection $\mathrm{Stab}_{\mathrm{Sym}(4d)}(1) \cap \mu(F_2)$ by $\mathrm{Stab}_G(1)$. As we have the covers

$$\mathcal{O} \xrightarrow{c_1} \mathbb{T}[2] \xrightarrow{c_2} \mathbb{T},$$

we obtain the inclusions

$$\mu^{-1}(\mathrm{Stab}_G(1)) \subseteq \ker(\alpha) \subseteq F_2.$$

The cover $c_1$ is normal if and only if the group $\mu^{-1}(\mathrm{Stab}_G(1))$ is normal in the kernel $\ker(\alpha)$. Furthermore, the corresponding deck transformation group is cyclic if and only if the quotient $\ker(\alpha)/\mu^{-1}(\mathrm{Stab}_G(1))$ is cyclic. We denote $\ker(\alpha)$ by $K$ in the remaining part of this chapter. In summary, Question 6.1.2 is equivalent to the question below.

**Question 6.1.5** Is the stabilizer $\mu^{-1}(\mathrm{Stab}_G(1))$ a normal subgroup of $K$? Is the quotient $K/\mu^{-1}(\mathrm{Stab}_G(1))$ cyclic in this case?

The following lemma transfers this question from the free group $F_2$ to the finite group $\mathrm{Sym}(4d)$ using basic facts from group theory. This setting will be used in the remaining part of this chapter.

**Lemma 6.1.6** *The following holds:*

*(i) The stabilizer $\mu^{-1}(\mathrm{Stab}_G(1))$ is a normal subgroup of $K$ if and only if $\mathrm{Stab}_G(1)$ is a normal subgroup of $\mu(K)$.*

*(ii) Let $\mu^{-1}(\mathrm{Stab}_G(1))$ be a normal subgroup of $K$. The quotient $K/\mu^{-1}(\mathrm{Stab}_G(1))$ is cyclic if and only if the quotient $\mu(K)/\mathrm{Stab}_G(1)$ is cyclic.*

**Proof** Statement (i) follows from [Fra14, Chapter 3, Theorem 5.16].

For statement (ii), note that $\mu$ induces a homomorphism

$$\overline{\mu} : K/\mu^{-1}(\mathrm{Stab}_G(1)) \to \mu(K)/\mathrm{Stab}_G(1).$$

On the one hand, a generator of the cyclic group $K/\mu^{-1}(\mathrm{Stab}_G(1))$ is mapped to a generator of the group $\mu(K)/\mathrm{Stab}_G(1)$ under $\overline{\mu}$. On the other hand, the preimage of a generator of the cyclic group $\mu(K)/\mathrm{Stab}_G(1)$ is a generator of the quotient $K/\mu^{-1}(\mathrm{Stab}_G(1))$. ∎

Using the lemma above, we deduce that Question 6.1.5 is equivalent to the following question:

**Question 6.1.7** Is the stabilizer $\mathrm{Stab}_G(1)$ a normal subgroup of $\mu(K)$? Is the quotient $\mu(K)/\mathrm{Stab}_G(1)$ cyclic in this case?

The following commutative diagram shows the inclusions between the groups under consideration:

$$\begin{tikzcd}
 & & \mathrm{Sym}(4d) \\
\mathbb{T} & \pi_1(\mathbb{T}) = F_2 = \langle a,b\rangle \arrow[r, "\mu"] & G = \langle x,y\rangle \arrow[u, hook] \\
\mathbb{T}[2] \arrow[u, "c_2"] & \pi_1(\mathbb{T}[2]) = K \arrow[r, "\mu"] \arrow[u, hook] & \mu(K) \arrow[u, hook] \\
\mathcal{O} \arrow[u, "c_1"] & \pi_1(\mathcal{O}) = \mu^{-1}(\mathrm{Stab}_G(1)) \arrow[r, "\mu"] \arrow[u, hook] & \mathrm{Stab}_G(1). \arrow[u, hook]
\end{tikzcd}$$

**Example 6.1.8** Consider again the “eierlegende Wollmilchsau” (see Example 2.2.3). We show that in this particular case the answer to Question 6.1.7 is yes. The group $\mu(K)$ is cyclic and generated by the permutation

$$x^2 = y^2 = [x,y] = (1,3)(2,4)(5,7)(6,8).$$

Further, the stabilizer $\mathrm{Stab}_G(1)$ is trivial for $G = \langle x,y\rangle$. In particular, the stabilizer $\mathrm{Stab}_G(1)$ is normal in $K$ and the quotient is cyclic.

**Remark 6.1.9** The stabilizer $\mathrm{Stab}_G(1)$ is trivial if the geminal origami $\mathcal{O}$ is a normal origami, i.e., the cover $\mathcal{O} \xrightarrow{c_2 \circ c_1} \mathbb{T}$ is normal. In this case, the stabilizer $\mathrm{Stab}_G(1)$ is a normal subgroup of $\mu(K)$ and Question 6.1.7 is equivalent to the question whether $\mu(K)$ is cyclic.

**Example 6.1.10** Next, consider the "Ornithorynque" origami from Example 6.1.4. In the following, we show that the answer to Question 6.1.7 is yes for this surface as well. Again, we compute the following permutations in $\mu(K)$

$$\begin{aligned} x^2 &= (1,3,5)(2,4,6)(7,9,11)(8,10,12), \\ y^2 &= (1,5,3)(2,4,6)(7,9,11)(8,12,10), \\ [x,y] &= (1,5,3)(2,6,4)(7,9,11). \end{aligned}$$

Using `GAP`, we check that the group $\langle x^2, y^2, [x,y] \rangle$ is a normal subgroup of the group $G = \langle x, y \rangle$. Hence, this group equals $\mu(K)$. Furthermore, note that the separate cycles occurring in the permutations $x^2, y^2$, and $[x,y]$ are all powers of the disjoint cycles $(1,3,5)$, $(2,4,6)$, $(7,9,11)$, or $(8,10,12)$. This implies that the permutations $x^2, y^2$, and $[x,y]$ commute with each other and thus the group $\mu(K)$ is abelian. More precisely, the group $\mu(K)$ is isomorphic to the elementary abelian group $C_3 \times C_3 \times C_3$. Since each subgroup of an abelian group is normal, the group $\mathrm{Stab}_G(1)$ is a normal subgroup of $\mu(K)$. In Theorem 6.3.1, we will show that this implies that the quotient $\mu(K)/\,\mathrm{Stab}_G(1)$ is a cyclic group.

## 6.2. Permutations describing geminal origamis

This section examines conditions which are satisfied by permutations defining a geminal origami. We use these conditions to obtain a bound for the number of geminal surfaces of fixed degree. Moreover, the results obtained in this section will be useful to partially answer Question 6.1.7 in Section 6.3.

In this section, we use the notation introduced in Section 6.1. Recall that the permutations $x$ and $y$ define the horizontal and vertical gluings of a geminal origami $\mathcal{O}$, respectively. Further, recall that both permutations consist of two disjoint $2d$-cycles each. The group $\langle x, y \rangle$ is denoted by $G$ and $\mu(K)$ is the smallest normal subgroup of $G$ containing the permutations $x^2, y^2$, and $[x,y]$. The following definition will be useful to study the permutations $x$ and $y$.

**Definition 6.2.1** We denote the entries of the permutations $x$ and $y$ as follows

$$\begin{aligned} x &= (x_1, \ldots, x_{2d})(x_{2d+1}, \ldots, x_{4d}), \\ y &= (y_1, \ldots, y_{2d})(y_{2d+1}, \ldots, y_{4d}). \end{aligned}$$

Further, we define the following sets

$$\begin{aligned} X_i &:= \{x_{2d\cdot i+k} \mid 1 \leq k \leq 2d\}, \\ X_i^j &:= \{x_{2d\cdot i+k} \mid 1 \leq k \leq 2d,\ k \equiv_2 j+1\}, \\ Y_i &:= \{y_{2d\cdot i+k} \mid 1 \leq k \leq 2d\}, \\ Y_i^j &:= \{y_{2d\cdot i+k} \mid 1 \leq k \leq 2d,\ k \equiv_2 j+1\}. \end{aligned}$$

for $i, j \in \{0, 1\}$.

The sets $X_i$ and $Y_i$ consist of the entries of the cycles of the permutations $x$ and $y$, respectively. Furthermore, the sets $X_i^j$ with $j \in \{0, 1\}$ consist of the entries of the cycles of the permutation $x$ which have even and odd indices, respectively. The sets $Y_i^j$ with $j \in \{0, 1\}$ are defined analogously for the permutation $y$.

**Definition 6.2.2** We say that the permutation $x$ **is alternating in the blocks of** $y$ if the following holds: if $k$ and $\ell$ are natural numbers such that $\ell \equiv_{2d} k+1$ and $\{x_k, x_\ell\} \subseteq X_i$ for some $i \in \{0, 1\}$, then

$$|\{x_k, x_\ell\} \cap Y_j| = 1$$

for $j = 0$ and $j = 1$. Analogously, we define that the permutation $y$ **is alternating in the blocks of** $x$.

In other words, the permutation $x$ is alternating in the blocks of $y$ if two consecutive entries of a $2d$-cycle of $x$ do not lie in the same $2d$-cycle of $y$. The subsequent lemma shows that permutations describing a geminal surface are alternating in the blocks of each other.

**Lemma 6.2.3** *Let $x$ and $y$ be permutations in* $\mathrm{Sym}(4d)$ *describing a geminal origami. The following holds:*

*(i) The permutation $x$ is alternating in the blocks of $y$.*

*(ii) The permutation $y$ is alternating in the blocks of $x$.*

**Proof** To proof statement (i), consider natural numbers $k$ and $\ell$ such that $\ell \equiv_{2d} k+1$ and $\{x_k, x_\ell\} \subseteq X_i$ for some $i \in \{0, 1\}$. Assume $x$ is not alternating in the blocks of $y$. Then we obtain $|\{x_k, x_\ell\} \cap Y_j| = 2$ for some $j \in \{0, 1\}$, i.e., $\{x_k, x_\ell\} \subseteq Y_j$ for some $j \in \{0, 1\}$. Thus, there exist $k'$ and $\ell'$ such that $x_k = y_{k'}$, $x_\ell = y_{\ell'}$, and the numbers $y_{k'}$ and $y_{\ell'}$ lie in the same cycle of the permutation $y$. Set $r := k' - \ell'$ and compute

$$y^r x(x_k) = y^r(x_\ell) = y^r(y_{\ell'}) = y_{k'} = x_k.$$

This implies that the permutation $y^r x$ has the fixed point $x_k$ and thus is not a $(2d, 2d)$-cycle. However, $(y^r x, y)$ is a generating pair of the group $G = \langle x, y \rangle$. Since $x$ and $y$ define a geminal origami, both permutations have to be $(2d, 2d)$-cycles. This yields a contradiction. This proves statement (i).

Note that the above argument is symmetric in $x$ and $y$. Hence, statement (ii) follows analogously with the above argument. ∎

**Corollary 6.2.4** *The following equality holds*

$$\{X_i^j \mid 0 \le i, j \le 1\} = \{Y_i^j \mid 0 \le i, j \le 1\}.$$

**Proof** Assume that $\{x_k, x_\ell\} \subseteq X_i^j$ for some $1 \le k, \ell \le 4d$ and $0 \le i, j \le 1$. In particular, $k-\ell$ is even. Let $0 \le i', j' \le 1$ such that $x_k$ is contained in the set $Y_{i'}^{j'}$. By Lemma 6.2.3 (i), we conclude $x_\ell$ lies either in the set $Y_{i'}^{j'}$ or in the set $Y_{i'}^{1-j'}$. In the following, we show $x_\ell$ lies in $Y_{i'}^{j'}$. Suppose to the contrary that $x_\ell$ lies in $Y_{i'}^{1-j'}$. Then, there are natural numbers $k'$ and $\ell'$ such that $x_k = y_{k'}, x_\ell = y_{\ell'}$, and $k' - \ell' \equiv_2 1$. Using Lemma 6.2.3 (ii), we conclude that the numbers $y_{k'}$ and $y_{\ell'}$ lie in different cycles of the permutation $x$. This is a contradiction to the inclusion $\{x_k, x_\ell\} \subseteq X_i^j$. Since all the sets $X_i^j$ and $Y_i^j$ with $0 \le i, j \le 1$ have cardinality $d$, the claim follows. ∎

Without loss of generality, we may assume that the number 1 is contained in both sets $X_0^0$ and $Y_0^0$. Otherwise, we change the order of the $2d$-cycles of the permutations $x$ and $y$, respectively. We obtain the following result.

**Lemma 6.2.5** *Assuming that the number* 1 *is contained in both sets* $X_0^0$ *and* $Y_0^0$*, we obtain the equalities*

$$\begin{aligned} X_0^0 &= Y_0^0, & X_0^1 &= Y_1^{j_1} \\ X_1^0 &= Y_{i_2}^{j_2}, & X_1^1 &= Y_{1-i_2}^{j_3} \end{aligned}$$

*for some* $i_2, j_1, j_2,$ *and* $j_3 \in \{0, 1\}$.

**Proof** Lemma 6.2.3 (ii) implies that for all $i, j \in \{0, 1\}$, there is an $i' \in \{0, 1\}$ such that $Y_i^j \subset X_{i'}$. Since the number 1 is contained in both sets $X_0^0$ and $Y_0^0$, we have the inclusion $Y_0^0 \subset X_0$. We conclude

$$X_0 = Y_0^0 \sqcup Y_{i_1}^{j_1}, \quad X_1 = Y_{i_2}^{j_2} \sqcup Y_{i_3}^{j_3}$$

for some $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0, 1\}$. Hence, we obtain the equalities

$$\begin{aligned} X_0^0 \sqcup X_0^1 &= Y_0^0 \sqcup Y_{i_1}^{j_1}, \\ X_1^0 \sqcup X_1^1 &= Y_{i_2}^{j_2} \sqcup Y_{i_3}^{j_3}. \end{aligned}$$

Lemma 6.2.3 (i) implies that $i_1 \neq 0$ and $i_2 \neq i_3$. Thus, we have $i_1 = 1$. Finally, we use Corollary 6.2.4 to obtain

$$\begin{aligned} X_0^0 &= Y_0^0, & X_0^1 &= Y_1^{j_1}, \\ X_1^0 &= Y_{i_2}^{j_2}, & X_1^1 &= Y_{1-i_2}^{j_3}. \end{aligned}$$

∎

Note that we may assume without loss of generality that the permutation $x$ equals $(1, 2, \ldots, 2d)(2d + 1, \ldots, 4d)$. Otherwise, conjugate $x$ and $y$ simultaneously with an appropriate permutation in $\mathrm{Sym}(4d)$. This does not change the origami $\mathcal{O}$, but corresponds to renumbering the squares of the origami. Furthermore, we can change the entry of the $2d$-cycles of the permutation $y$ where the numbering of the entries starts without changing the permutation. In other words, the permutations $(y_1, y_2, \ldots, y_{2d})$ and $(y_{1+k \mod 2d}, y_{2+k \mod 2d}, \ldots, y_{2d+k \mod 2d})$ are equal for each natural number $k$. Here, we set $y_0 := y_{2d}$. Hence, we may assume $y_1 = 1$ and that $y_{2d+1}$ is the smallest number in the second cycle of $y$ without changing $y$.

**Proposition 6.2.6** *Let $x$ and $y$ be permutations in* $\mathrm{Sym}(4d)$ *describing a geminal origami* $\mathcal{O}$. *Assume* $x = (1, 2, \ldots, 2d)(2d+1, \ldots, 4d)$ *and* $y_1 = 1$. *Further, assume that* $y_{2d+1}$ *is the smallest number in the second cycle of the permutation* $y$. *Then, there exists a permutation* $y'$ *such that* $\mathcal{O} = (x, y')$ *and* $y'$ *has a cycle representation such that the following holds*

- *each of the two* $2d$*-cycles of* $y$ *is alternating with respect to being* $\leq 2d$ *or* $> 2d$,
- *each of the two* $2d$*-cycles of* $y$ *is constant with respect to the parity.*

**Proof** Since $x = (1, 2, \ldots, 2d)(2d + 1, \ldots, 4d)$, we obtain

$$\begin{aligned} X_0^0 &= \{1 \leq k \leq 2d \mid k \equiv_2 1\}, & X_0^1 &= \{1 \leq k \leq 2d \mid k \equiv_2 0\}, \\ X_1^0 &= \{2d + 1 \leq k \leq 4d \mid k \equiv_2 1\}, & X_1^1 &= \{2d + 1 \leq k \leq 4d \mid k \equiv_2 0\}. \end{aligned}$$

Using Lemma 6.2.5, we conclude that either

$$Y_0^0 = X_0^0, Y_0^1 = X_1^1, Y_1^0 = X_0^1, Y_1^1 = X_1^0$$

or

$$Y_0^0 = X_0^0, Y_0^1 = X_1^0, Y_1^0 = X_0^1, Y_1^1 = X_1^1.$$

Recall that simultaneous conjugation of two permutations defining an origami does not change the origami. Conjugating $x$ with the permutation $(2d+1, \ldots, 4d)$ does not change $x$ because this permutation lies in the stabilizer of $x$. If each of the two $2d$-cycles of $y$ is alternating with respect to the parity (as in the first case), then conjugating $y$ with the permutation $(2d + 1, \ldots, 4d)$ yields a permutation $y'$. This permutation consists of two $2d$-cycles that are both constant with respect to the parity (as in the second case). This proves the claim. ∎

**Example 6.2.7** Observe that the permutations defining the “eierlegende Wollmilchsau” and the “Ornithorynque” in Example 6.1.3 and Example 6.1.4, respectively, satisfy the properties given in Proposition 6.2.6.

Using Proposition 6.2.6, we obtain the following bound of the number of geminal origamis of fixed degree.

**Corollary 6.2.8** *The number of geminal origamis of degree* $4d$ *is bounded above by*

$$d! \cdot ((d - 1)!)^3.$$

**Proof** Proposition 6.2.6 implies that we may assume $y_1 = 1$ and $y_{2d+1} = 2$. Thus, there are $(d - 1)!$ choices for the entries with odd index of the first cycle and second cycle of $y$, respectively. Using that conjugating $y$ with even powers of the permutation $(2d+1, \ldots, 4d)$ does not change the origami and does preserve the properties given in Proposition 6.2.6, one can choose $y_2 = 2d + 1$. We obtain $(d - 1)!$ choices for the entries with even index of the first cycle of $y$. Furthermore, one has $d!$ choices for the entries with even index of the second cycle of $y$. Thus, we obtain the claimed bound. ∎

**Example 6.2.9** We consider the case $d = 2$. By Corollary 6.2.8, there exist at most two geminal origamis of degree 8. The permutations describing the gluings of the two possible origamis are $(x, y)$ and $(x, y')$, where

$$\begin{aligned} x &= (1,2,3,4)(5,6,7,8), \\ y &= (1,5,3,7)(2,8,4,6), \\ y' &= (1,5,3,7)(2,6,4,8). \end{aligned}$$

The origami defined by the permutations $x$ and $y$ is the "eierlegende Wollmilchsau" (see Example 6.1.3).

In the following, we show that the origami $\mathcal{O}$ defined by the permutations $x$ and $y'$ is not a geminal origami. To see this, compute

$$\begin{aligned} y'x^3 &= (1,5,3,7)(2,6,4,8) \cdot (4,3,2,1)(8,7,6,5) \\ &= (1,8)(2,5)(3,6)(4,7). \end{aligned}$$

Furthermore, define $G := \langle x, y' \rangle$ and observe that the two permutations $y'x^3$ and $x$ generate the group $G$. Hence, the group $G$ has the pair of generators $(y'x^3, x)$ such that one generator consists of 4 transpositions and not a $(4, 4)$-cycle. We conclude that the origami $\mathcal{O}$ is not a geminal origami.

**Remark 6.2.10** Let $x$ and $y$ be two permutations satisfying the conditions from Proposition 6.2.6. If $i$ or $j$ is coprime to $2d$, then permutations of the form $x^i y^j$ have no fixed points. However, Proposition 6.2.6 does not prevent that permutations of the form $x^i y^j$ consist of at least three cycles of length at least two each if $i$ or $j$ is coprime to $2d$. To introduce further conditions on $x$ and $y$ that prevent this behavior is work in progress. Thereby, one could also improve the bound given in Corollary 6.2.8.

## 6.3. Normality implies cyclicality

In this section, we work toward answering the Question 6.1.7 and give the following partial answer. We prove that if $\operatorname{Stab}_G(1)$ is a normal subgroup of $\mu(K)$ then the quotient $\mu(K)/\operatorname{Stab}_G(1)$ is cyclic. For this, we use the notation introduced in Section 6.1 and Section 6.2.

**Theorem 6.3.1** *If the stabilizer $\operatorname{Stab}_G(1)$ is a normal subgroup of $\mu(K)$, then the quotient $\mu(K)/\operatorname{Stab}_G(1)$ is cyclic.*

**Proof** Suppose that the stabilizer $\operatorname{Stab}_G(1)$ is a normal subgroup of $\mu(K)$. Recall that $K$ is the smallest normal subgroup of $F_2$ containing $a^2, b^2$, and the commutator $[a, b]$. Thus, the group $\mu(K)$ is the smallest normal subgroup of $G$ containing $x^2, y^2$, and $[x, y]$. Here, we use the fact that the image of $\mu$ equals $G$. To show that $\mu(K)/\operatorname{Stab}_G(1)$ is cyclic, we construct a set $M$ of elements in the stabilizer $\operatorname{Stab}_G(1)$ such that the quotient $\mu(K)/N$

is cyclic. Here, $N$ denotes the smallest normal subgroup of $\mu(K)$ that contains $M$. Then it follows that $\mu(K)/\operatorname{Stab}_G(1)$ is cyclic as well.

By Proposition 6.2.6, $m := y^2(1) \leq 2d$ is odd. We obtain the permutation $x^{1-m}y^2$, which is contained in the stabilizer $\operatorname{Stab}_G(1)$. Similarly, $m' := y^{-2}(1) \leq 2d$ is odd by Proposition 6.2.6. Therefore, the permutation $y^2x^{m'-1}$ lies in the stabilizer $\operatorname{Stab}_G(1)$. Consider the permutation $yxy^{-1}$. For $1 \leq i \leq 4d$, we claim that both $i$ and $yxy^{-1}(i)$ lie in the same cycle of the permutation $x$ and are of different parity. To see this, observe that for all $i$ the numbers $y^{-1}(i)$ and $y(i)$ both lie in the cycle of $x$ in which the number $i$ is not contained. However, the numbers $x(i)$ and $i$ lie in the same cycle of $x$ for all $i$. Further, note that applying $y$ to $i$ preserves the parity for each $i$ with $1 \leq i \leq 4d$. Hence, $n := yxy^{-1}(1)$ is even and satisfies $n \leq 2d$. Therefore, $x^{1-n}yxy^{-1}$ lies in the stabilizer $\operatorname{Stab}_G(1)$.

We conclude that the permutations $x^{1-m}y^2, y^2x^{m'-1}$, and $x^{1-n}yxy^{-1}$ lie in the stabilizer $\operatorname{Stab}_G(1)$ for some $m, m'$, and $n$. This means, each element in $\mu(K)/\operatorname{Stab}_G(1)$ can be written as $x^iy^k$ with $i \in \mathbb{Z}$ and $k \in \{0, 1\}$.

Finally, we show that each element $g$ in $\mu(K)$ is of even total degree in $y$, i.e., by considering $g$ as a word in $x$ and $y$ and by adding all the exponents occurring for $y$, one obtains an even number. Recall that $\mu(K)$ is the normal closure of $\langle S \rangle$ with $S = \{x^2, y^2, [x, y]\}$. Hence, we have

$$\mu(K) = \langle g^{-1}sg \mid g \in G, s \in S \rangle.$$

However, conjugating an element $s \in S$ with some $g \in G$ does not change its total degree in $y$. All elements in $S$ have even total degree in $y$. Consequently, we have a generating set of $\mu(K)$ that consists of elements with even total degree in $y$. It follows that all elements in $\mu(K)$ can be written as words in this generating set and all these words have even total degree in $y$. The elements $x^{1-m}y^2, y^2x^{m'-1}$, and $x^{1-n}yxy^{-1}$ lie in the stabilizer and have even total degree in $y$. This implies that we can write each element in $\mu(K)/\operatorname{Stab}_G(1)$ as $x^i$ for some $i$. In particular, we obtain that $\mu(K)/\operatorname{Stab}_G(1)$ is cyclic. ■

**Remark 6.3.2** If a geminal origami $\mathcal{O}$ is a normal origami, the stabilizer $\operatorname{Stab}_G(1)$ is trivial and thus a normal subgroup of $\mu(K)$ (see Remark 6.1.9). Furthermore, we obtain the equality $\mu(K) = \mu(K)/\operatorname{Stab}_G(1)$. By Theorem 6.3.1, the group $\mu(K)$ is cyclic.

**Remark 6.3.3** Let $\mathcal{O}$ be a geminal origami $\mathcal{O}$ such that $\mu(K)$ is an abelian group. Then, the stabilizer $\operatorname{Stab}_G(1)$ is a normal subgroup of $\mu(K)$. By Theorem 6.3.1, the quotient $\mu(K)/\operatorname{Stab}_G(1)$ is cyclic.

# Chapter 7.

# Open problems

During the work on this thesis several questions arose, which remain open for further research. A first interesting problem involves quasi-regular origamis studied in [MYZ14]. These translation surfaces generalize normal origamis, which are also called regular origamis. Matheus, Yoccoz, and Zmiaikou used methods from representation theory to investigate the relation between the automorphism group of a quasi-regular origami and the Lyapunov exponents of the Kontsevich-Zorich cocycle defined by the origami. A natural approach would consider quasi-regular origamis instead of regular ones and apply methods used in this thesis to study this more general class of origamis. Below, we discuss the remaining questions following the structure of this thesis.

## Strata of $p$-origamis

We begin with questions regarding the strata of normal origamis.

### $p$-origamis with isomorphic deck transformation groups

Recall that the deck transformation group $G$ of a normal origami determines its stratum if and only if the deck transformation group $G$ has property $(\mathcal{C})$, i.e., if and only if there exists a natural number $n$ such that for each 2-generating set $\{x, y\}$ of $G$ the order of $[x, y]$ equals $n$ (see Definition 3.1.12). In Section 3.2.2, we studied which $p$-groups have property $(\mathcal{C})$. For large classes of $p$-groups, for instance, groups of maximal class and groups whose commutator subgroups are powerful or weakly order-closed, we showed that they fulfill property $(\mathcal{C})$ (see Theorem 3.1.29). Moreover, we constructed a $p$-group whose commutator subgroup is weakly power-closed that does not have property $(\mathcal{C})$ (see Corollary 3.1.27). It would be interesting to study the following question, where we consider not only $p$-groups, but arbitrary finite groups.

**Problem 7.1** *Which finite groups have property* $(\mathcal{C})$*?*

In Proposition 4.1.4, we computed all normal origamis with dihedral groups as deck transformation group. When considering these origamis, we realize that all dihedral groups satisfy property $(\mathcal{C})$, whereas alternating groups $\mathrm{Alt}(n)$ do not satisfy property $(\mathcal{C})$ for $n \geq 5$ odd (see Example 3.2.10). Furthermore, symmetric groups $\mathrm{Sym}(n)$

do not satisfy property $(\mathcal{C})$ for $n \geq 5$. To see this, consider the normal origamis $\mathcal{O}_1 = ((1,2),(1,2,\ldots,n),\mathrm{Sym}(n))$ and $\mathcal{O}_2 = ((2,2+p),(1,2,\ldots,n),\mathrm{Sym}(n))$. Here, $p$ denotes a prime number which is smaller then $n-1$ and coprime to $n$. We compute the commutators

$$[(1,2),(1,2,\ldots,n)] = (1,n,2),$$
$$[(2,2+p),(1,2,\ldots,n)] = (1,1+p)(2,2+p).$$

Hence, the origamis $\mathcal{O}_1$ and $\mathcal{O}_2$ lie in the stratum $\mathcal{H}(\frac{n!}{3}\times 2)$ and $\mathcal{H}(\frac{n!}{2}\times 1)$. Consequently, the symmetric group does not satisfy property $(\mathcal{C})$.

In the following, we discuss the situation for small groups of order up to 250. Kremer computed in his dissertation that the $\mathrm{SL}(2,\mathbb{Z})$-action on the set of normal origamis $\mathcal{O}(G)$ is transitive for most such groups $G$ (see [Kre09, Appendix A.4]). He listed the 30 groups for which this is not the case. The $\mathrm{SL}(2,\mathbb{Z})$-action leaves the stratum invariant and thus all normal origamis in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit lie in the same stratum. From the 30 groups with a intransitive $\mathrm{SL}(2,\mathbb{Z})$-action, 18 groups admit normal origamis that lie in different strata. We see that almost all of these groups of small order satisfy property $(\mathcal{C})$. A natural approach would examine groups of larger order.

### Infinite normal origamis

In Section 3.3, we considered infinite normal origamis and generalized the concept property $(\mathcal{C})$ to these surfaces. Again, a deck transformation group $G$ of an infinite normal origami has property $(\mathcal{C}^{\mathrm{pro}})$ if and only if the deck transformation group $G$ determines the order of all singularities of the origami. Due to the correspondence between infinite families of finite normal origamis and infinite normal origamis described in Remark 3.3.7, we studied infinite normal origamis with dense subgroups of profinite groups as deck transformation groups. We were especially interested in pro-$p$ groups, i.e., profinite groups that are the inverse limit of an inverse system consisting of $p$-groups. In this case, we were able to generalize results from Section 3.1.2 (see Proposition 3.3.8). As in the finite case, the following problem is compelling:

**Problem 7.2** *Characterize which profinite groups have property* $(\mathcal{C}^{\mathrm{pro}})$.

Moreover, we are interested in constructing further intriguing infinite families of normal origamis that contain surfaces which have been studied in literature. We would like to use the setting described in Remark 3.3.7 to deduce new results for such surfaces.

# Veech groups of normal origamis

We next discuss some open questions regarding the study of Veech groups of normal origamis. In Section 4.1, we studied 2-origamis and their Veech groups for certain families of deck transformation groups. The considered groups were either semidirect groups or extensions of the quaternion group. In a next step, more general families of deck transformation groups should be considered. For example, it would be interesting to consider $p$-groups for odd primes $p$ and 2-groups with a more complex group structure. A next goal could be the study of normal origamis whose deck transformation group is a semidirect product of two cyclic groups as well as the study of the corresponding Veech groups. This could give further insights regarding the following question:

**Problem 7.3** *Which finite index subgroups of* $\mathrm{SL}(2,\mathbb{Z})$ *occur as Veech groups of normal origamis?*

Several results in Chapter 4 (see, e.g., Corollary 4.1.3 and Theorem 4.2.9) show that certain properties of the deck transformation group of a (finite) normal origami imply properties of its Veech group. This raises the question whether a similar behavior occurs in the case of infinite normal origamis.

**Problem 7.4** *Which properties of the Veech group of an infinite normal origami are influenced by certain properties of its deck transformation group?*

Two remaining questions regarding infinite normal origamis concern the construction of infinite normal origamis using inverse systems and profinite groups (see Remark 3.3.7).

**Problem 7.5 (Geometry)** *Are there infinite normal origamis constructed as in Remark 3.3.7 such that the corresponding deck transformation groups are not isomorphic and the closures of the deck transformation groups are isomorphic?*

The equivalent question using group theory is the following:

**Problem 7.5 (Group theory)** *Are there compatible generating sets* $(x_i, y_i)$ *and* $(a_i, b_i)$ *of finite groups* $G_i$ *and* $H_i$ *defining inverse systems* $(G_i)_i$ *and* $(H_i)_i$*, respectively, with the following property? The inverse limits* $\varprojlim G_i$ *and* $\varprojlim H_i$ *are isomorphic and the corresponding elements* $x, y, a, b$ *in the inverse limits define dense subgroups* $\langle x, y\rangle$ *and* $\langle a, b\rangle$ *which are not isomorphic?*

Several families of groups studied in Section 4.1.1 form inverse systems of groups and were considered in examples in Section 3.3 as well (see, e.g., Example 3.3.11). We observe that finite normal origamis whose deck transformation groups belong to the same family share several properties. For instance, see Proposition 4.1.20 and Proposition 4.1.22 for the family $W_m$. This raises the following question:

**Problem 7.6** *Is the Veech group of an infinite normal origami constructed as in Remark 3.3.7 related to the Veech groups of the corresponding finite normal origamis?*

## Application to sums of Lyapunov exponents

In Section 4.1.2, we computed the sum of non-negative Lyapunov exponents for infinite families of normal origamis. The sums of non-negative Lyapunov exponents were integers for the considered surfaces. However, we gave an example for a normal origami such that the sum of non-negative Lyapunov exponents is not an integer. We ask the following question:

**Problem 7.7** *When is the sum of non-negative Lyapunov exponents of a normal origami an integer?*

Instead of the sum of non-negative Lyapunov exponents, it would be interesting to compute individual Lyapunov exponents for the considered surfaces. The individual Lyapunov exponents capture the dynamics of the Teichmüller flow (see Section 4.1.2). In Table 7.1, the second largest Lyapunov exponent $\lambda_2$ of the normal origami in $(G^2_{(n,k)}, r, s)$ is listed for several values of $n$ and $k$. The software package [DFL] was used for these computations.

| $n$ | $k$ | $\frac{2^k-1}{2^k}$ | Lyapunov exponent $\lambda_2$ of the origami $(G^2_{(n,k)}, r, s)$ |
|---|---|---|---|
| 3 | 1 | 0.5 | 0.5 |
| 4 | 1 | 0.5 | 0.5 |
| 5 | 1 | 0.5 | 0.5 |
| 6 | 1 | 0.5 | 0.5 |
| 7 | 1 | 0.5 | 0.5 |
| 4 | 2 | 0.75 | 0.75 |
| 5 | 2 | 0.75 | 0.75 |
| 6 | 2 | 0.75 | 0.75 |
| 7 | 2 | 0.75 | 0.75 |
| 5 | 3 | 0.875 | 0.875 |
| 6 | 3 | 0.875 | 0.875 |
| 7 | 3 | 0.875 | 0.875 |
| 6 | 4 | 0.9375 | 0.938 |

Table 7.1.: The second largest Lyapunov exponent $\lambda_2$ of the normal origami in $(G^2_{(n,k)}, r, s)$ was computed for several values of $n$ and $k$ using the software package [DFL]. The values are rounded.

The values in Table 7.1 suggest that the Lyapunov exponent $\lambda_2$ of the normal origami $(G^2_{(n,k)}, r, s)$ does only depend on $k$. They support the conjecture that the Lyapunov exponent of the origami $(G^2_{(n,k)}, r, s)$ equals $\frac{2^k-1}{2^k}$ for $n \geq k+2$. This example motivates the following problem.

**Problem 7.8** *Compute and analyze the Lyapunov exponents of normal origamis whose deck transformation groups lie in certain families of groups. Under which circumstances are the Lyapunov exponents in such infinite families of normal origamis related?*

Again, it would be interesting to consider the case where the deck transformation groups form an inverse system.

**Problem 7.9** *Are the Lyapunov exponents of an infinite normal origami constructed as in Remark 3.3.7 related to the Lyapunov exponents of the corresponding finite normal origamis?*

## Normal origamis with totally non-congruence groups as Veech groups

In Section 4.2, we introduced properties of the deck transformation group of a normal origami implying that the Veech group is a totally non-congruence group. This was motivated by the normal origamis studied in Section 4.1.1 whose Veech groups were all congruence subgroups. The main results of Section 4.2 (Proposition 4.2.4 and Theorem 4.2.9) are not applicable to $p$-groups because all elements of a $p$-group have order $p^m$ for some $m$. This raises the following questions:

**Problem 7.10** *Are there $p$-origamis with totally non-congruence groups that are not equal to* $\mathrm{SL}(2,\mathbb{Z})$ *as Veech groups? Which properties of a $p$-group imply this behavior?*

More generally, we would like to understand the interaction between the properties of the deck transformation group of normal origamis and the properties of their Veech groups. We would be especially interested in the deficiency of Veech groups, a concept introduced in [Wei13] for measuring how far a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ is from being a congruence group. More precisely, the deficiency of a finite index subgroup $\Gamma$ of $\mathrm{SL}(2,\mathbb{Z})$ is defined as follows (see [Wei13, Section 3] for further details). For each natural number $m$, we consider the natural projections $\pi_m : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})$ and the exact sequences defining the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Gamma(m) \arrow[r, hook] & \mathrm{SL}(2,\mathbb{Z}) \arrow[r, "\pi_m"] & \mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z}) \arrow[r] & 1 \\
1 \arrow[r] & \Gamma \cap \Gamma(m) \arrow[r, hook] \arrow[u, hook] & \Gamma \arrow[r, "\pi_m"] \arrow[u, hook] & \pi_m(\Gamma) \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

Recall that the principal congruence group $\Gamma(m)$ is the kernel of the projection $\pi_m$. Define

$$\begin{aligned}
f_m &:= [\Gamma(m) : \Gamma \cap \Gamma(m)], \\
d &:= [\mathrm{SL}(2,\mathbb{Z}) : \Gamma], \\
e_m &:= [\mathrm{SL}(2\mathbb{Z}/m\mathbb{Z}) : \pi_m(\Gamma)].
\end{aligned}$$

Observe that the indices obey the equation $d = e_m \cdot f_m$ as the above diagram commutes. The **deficiency** of $\Gamma$ is defined as the minimal $f_m$ for $m \in \mathbb{Z}_+$. Note that the group $\Gamma$ is a congruence subgroup if and only if $f_m = 1$ and $e_m = d$. Further, the group $\Gamma$ is a totally non-congruence group if and only if $f_m = d$ and $e_m = 1$. In Chapter 4, we studied these two cases. An interesting question for further research is the general case.

**Problem 7.11** *Which properties of the deck transformation group are related to the deficiency of Veech groups of normal origamis?*

# $T_2$-systems and normal origamis

Next, we discuss some questions concerning the study of $T_2$-systems which is related to the study of $\mathrm{SL}(2,\mathbb{Z})$-orbits on the set of normal origamis with fixed deck transformation group. In Lemma 5.1.3, we saw that the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits lies between $n$ and $2n$, where $n$ equals the number of $T_2$-systems. In Example 5.1.8, we gave examples where $T_2$-systems split into two $\mathrm{SL}(2,\mathbb{Z})$-orbits as well as examples where $T_2$-systems induce only one $\mathrm{SL}(2,\mathbb{Z})$-orbit. This raises two obvious questions.

**Problem 7.12** *For which groups $G$ coincide the number of $T_2$-systems and the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits on $\mathcal{O}(G)$? When is the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits twice the number of $T_2$-systems?*

Another question is related to Example 5.2.7 where we defined a group of order $2^{12}$ which has at least two $T_2$-systems.

**Problem 7.13** *Which is the minimal exponent $m$ such that there exists a group of order $2^m$ with at least two $T_2$-systems?*

Finally, we pose a question which is related to infinite normal origamis. One can consider profinite groups instead of finite ones in the definition of $T_2$-systems. In this case, one considers topological generating sets instead of generating sets in the sense of classical group theory. Moreover, one considers the free profinite group on two generators $\hat{F}_2$ instead of the free group $F_2$.

Lubotzky showed that each profinite group has only one $T_2$-system. This is a consequence of the following theorem (see [Lub01, Proposition 2.2]).

**Theorem 7.14** *Let $\hat{G}$ be a (topologically) 2-generated profinite group with 2-generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Consider the two corresponding epimorphisms $\pi_1$ and $\pi_2$ from the free profinite group $\hat{F}_2$ on two generators to $\hat{G}$. Then there exists an automorphism $\alpha$ of $\hat{F}_2$ such that $\pi_1 \circ \alpha = \pi_2$.*

Since we consider the free profinite group here, we cannot simply proceed as in the finite case to deduce a connection between the number of $T_2$-systems and the number of $\mathrm{SL}(2,\mathbb{Z})$-orbits. We ask the question.

**Problem 7.15** *Does Theorem 7.14 imply a bound on the number of* $\mathrm{SL}(2,\mathbb{Z})$*-orbits of infinite origamis whose deck transformation groups are dense subgroups of a profinite group?*

# Geminal origamis

We showed in Chapter 6 that if a geminal origami induces a normal cover of the $(2 \times 2)$-torus $\mathbb{T}[2]$ then this cover is cyclic. The question whether each geminal origami induces such a normal cover remains open for further research. Another interesting project is the classification of geminal surfaces. For this purpose, one could continue to study the properties of two permutations defining a geminal surface using group theory.

Overall, this thesis demonstrates that the interaction between the geometric properties of normal origamis and the properties of the corresponding deck transformation groups yields a powerful tool for studying normal origamis. Moreover, this thesis provides a basis for further research on various interesting problems regarding normal origamis.

# Appendix A.

# GAP code

All of the following is GAP4 code (see [GAP19]).

Listing A.1: For natural numbers $n, k$ with $1 \leq k \leq n-2$, the following code defines the group $G^2_{(n,k)}$ as well as the corresponding 2-generating set constructed in Proposition 3.1.4. A very similar code can be used to define the groups $G^p_{(n,k)}$ and their generating sets as discussed in Proposition 3.1.8

```
G2 := function(n,k)
  local C1, C2, alpha, phi, G, x, y;
  C1 := CyclicGroup(2^(k+1)); C2 := CyclicGroup(2^(n-k-1));
  alpha := GroupHomomorphismByImages(C1, C1, [C1.1], [(C1.1)^(-1)]);
  phi := GroupHomomorphismByImages(C2, AutomorphismGroup(C1), [C2.1],
       [alpha]);
  G := SemidirectProduct(C2, phi,C1);
  x := Image(Embedding(G,2), C1.1); y := Image(Embedding(G,1), C2.1);
  return [G, x, y];
end;
```

Listing A.2: Variations of the following code were used to find $p$-groups which do not have property $(\mathcal{C})$.

```
p := 3; n := 4;
g := SylowSubgroup(SymmetricGroup(p^n), p);

repeat x := Random(g); y := Random(g);
until Order(Comm(x, y)) <> Order(Comm(x, y^2));
```

Listing A.3: The following code defines a function to test whether a given $p$-group is weakly power-closed (i.e., products of $p^k$-th powers are $p^k$-th powers for any $k \geq 0$), and uses it to find a 2-generated subgroup $G$ of the 2-Sylow subgroup of the symmetric group $S_{24}$ with generators $x, y$ such that $\mathrm{ord}([x, y]) \neq \mathrm{ord}([x, y^3])$ and $G'$ is weakly power-closed.

```
IsWeaklyPowerClosedPGroup := function(g)
  local powers, el;
  if IsTrivial(g) then return true; fi;
  powers := g;
  repeat
    powers := Set(powers, x -> x^PrimePGroup(g));
    if Size(Group(powers)) > Size(powers) then return false; fi;
  until IsTrivial(powers);
  return true;
end;

p := 2; n := 4;
gg := SylowSubgroup(SymmetricGroup(p^n), p);;
repeat x := Random(gg); y := Random(gg);
  g := Group(x, y); d := DerivedSubgroup(g);
until Order(Comm(x, y)) <> Order(Comm(x, y^(p+1)))
  and IsWeaklyPowerClosedPGroup(d);
```

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