Monte-Carlo methods for backward stochastic differential equations: Segment-wise dynamic programming and fast rates for lower bounds

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Contents

Abstract

In this thesis, we study two different algorithms using Monte-Carlo methods for solving backward stochastic differential equations. In the first chapter, we present a new algorithm where the backward stochastic differential equation is discretized to a dynamic programming equation alternating between a multi-step forward approach on segments of the time grid and a one-step scheme between segments. Conditional expectations are computed via least squares regression on function spaces. We optimize the length of the segments in dependence on the dimension and smoothness of the backward stochastic differential equation and compute the complexity needed to achieve a desired accuracy in the limit as the number of time points in the discretization goes to infinity.
In the second chapter, we consider a discretized backward stochastic differential equation in form of a dynamic programming equation and study an algorithm for constructing lower bounds for its value at time zero. The algorithm uses a pre-computed approximate solution of this equation to sample a control process which is used to derive the lower bound for the solution. We derive asymptotic error bounds and compute the complexity required to achieve a desired accuracy in dependence on the input approximation.
The results of both algorithms are illustrated by numerical examples.

Zusammenfassung

Diese Arbeit beschäftigt sich mit zwei Algorithmen zur Lösung von rückwärtsstochastischen Differentialgleichungen mit Hilfe von Monte-Carlo Methoden. Im ersten Kapitel wird ein neuer Algorithmus vorgestellt, der auf einem Diskretisierungsverfahren beruht, welches zwischen einer Mehr-Schritt Darstellung auf Zeitsegmenten und einem Ein-Schritt Verfahren zwischen den Segmenten alterniert. Auftretende bedingte Erwartungswerte werden dabei als Projektionen auf endlich dimensionale Funktionenräume berechnet. Die Wahl der Segmentlänge wird in Abhängigkeit der Glattheit und der Dimension der Differentialgleichung optimiert und es wird der asymptotische Rechenaufwand ermittelt, welcher notwendig ist um eine vorgegebene Genauigkeit zu erzielen.
Im zweiten Kapitel wird ein Algorithmus zur Konstruktion von unteren Schranken der Lösung rückwärtsstochastischer Differentialgleichungen zum Startzeitpunkt untersucht. Hierfür wird mit Hilfe einer vorab berechneten Approximation der Lösung ein Kontrollprozess simuliert mit dessen Hilfe schließlich die Schranke berechnet wird. Es werden asymptotische Fehlerschranken sowie der erforderliche Rechenaufwand zur Erzielung einer vorgegebenen Genauigkeit hergeleitet.
Die theoretischen Ergebnisse bezüglich der beiden Algorithmen werden mit numerischen Beispielen illustriert.

Introduction

Linear backward stochastic differential equations (short BSDEs) were first introduced by Bismut (1973) as an instrument for stochastic control problems and were later generalized by Pardoux and Peng (1990) into a non-linear setting. They have numerous applications with the largest field of interest being applications in financial mathematics where option pricing problems using replication arguments can be expressed as backward stochastic differential equation, see e.g. El Karoui et al. (1997) or the early examples in Bergman (1995) and Avellaneda et al. (1995).
Due to this application BSDEs especially became a huge focus of research after the financial crisis which lead to peak interest in pricing problems in more sophisticated market models and exotic options, which could be expressed as non-linear BSDEs. Those include for example option pricing with credit valuation adjustment (see e.g. Brigo and Pallavicini, 2007, Crépey et al., 2013, Crépey et al., 2014), funding costs (see e.g. Crépey et al., 2013, Crépey et al., 2014, Laurent et al., 2014), model uncertainty (see e.g. Guyon and Henry-Labordere, 2010), collateralization (see e.g. Nie and Rutkowski, 2016) and transaction costs (see e.g. Guyon and Henry-Labordere, 2010).
Questions regarding the existence and uniqueness of solutions in the standard setting were addressed early on (see Pardoux and Peng, 1990); however, finding a closed-form is not possible in general. Therefore solutions have to be approximated by numerical methods. Early attempts for this were mainly based on techniques for partial differential equations, which are directly linked to BSDEs by the non-linear Feynman-Kac formula, like the methods in Ma et al. (1994) and Douglas Jr et al. (1996). However, those methods are only feasible for low-dimensional problems as they rely on a space discretization. More recent methods, therefore, tackled the stochastic problem directly. For this purpose, the usual first step is a discretization of the BSDE on a finite time grid. The resulting discretization error is well-studied. For example, we refer for a typical backward Euler-type discretization under standard assumptions to Bouchard and Touzi (2004) and Zhang et al. (2004), where it is shown that the time discretization error converges like $\Delta^{\frac{1}{2}}$ or to Gobet and Labart (2007) where it is shown that the rate of convergence even improves to Δ^1 under stronger regularity assumptions in the case of forward-backward stochastic differential equations, where Δ denotes the step length of the time partition of the discretization.

While some algorithms, outgoing from such a time-discretized BSDE, work forward in time, like for example those in Bender and Denk (2007) and Bender and Moseler (2010) using Picard iterations, most algorithms approximate the solution backward in time. The main difficulty then is the nesting of conditional expectations, as the solution at any given time depends on the conditional expectation of the solution at the next time step, which again is given in terms of a conditional expectation and so on. Hence, one needs a way to approximate these conditional expectations such that they can be evaluated multiple times without exploding costs. For this purpose, numerous methods were studied, including Malliavin Monte-Carlo (see e.g. Bouchard and Touzi, 2004), quantization methods (Bally et al., 2003), cubature on Wiener space (see e.g. Crisan and Manolarakis, 2012), sparse grid methods (see e.g. Zhang et al., 2013) and least squares Monte-Carlo (see e.g. Lemor et al., 2006, Gobet and Turkedjiev, 2016 or Bender and Denk, 2007), on which we will focus in the first part of this thesis. Those methods are not limited to low dimensions like PDE methods but still suffer from exploding costs as the dimension of the problem grows large. Those methods are best suited for intermediate dimensions, while for very high-dimensional problems, the recently developed deep learning methods (see e.g. Han et al., 2017, Germain et al., 2022, Huré et al., 2020, Han et al., 2018) or multi-level Picard iterations (see e.g. E et al., 2021, E et al., 2019, Hutzenthaler and Kruse, 2020) are better suited.

In this thesis, we study two different algorithms for solving BSDEs using Monte-Carlo methods. In the first chapter, we build on the main ideas of the work of Lemor et al. (2006) and Gobet and Turkedjiev (2016) and derive a new algorithm with the goal of a faster convergence speed in relation to the computation costs. In Lemor et al. (2006), the authors start with a discrete-time BSDE in the form of a dynamic programming equation, where the solution (Y, Z) is expressed via conditional expectations of the solution one step ahead. They assume that the pair (Y, Z) is given by deterministic but unknown functions of a Markovian explanatory process X, such that the conditional expectations in the dynamic programming equation can be viewed as a solution to a least squares problem. An approximation of (Y, Z) is then constructed backward in time by replacing the conditional expectations in the dynamic programming equation with empirical least square projections on finite-dimensional function spaces using simulations of X. When optimizing the parameters of the algorithm to achieve a given theoretical accuracy with minimal complexity in the limit, as the number of time points N tends to infinity, the dominating costs come from the approximation of the Y part of the solution, as an error propagation occurs between the time steps.

The algorithm presented by Gobet and Turkedjiev (2016) works similarly with the difference that they start with a dynamic programming equation where the solution (Y, Z) is at any time point expressed through the conditional expectations of all their future values up to the terminal time. This, for once, leads to higher simulation costs, as one needs to simulate whole paths of the process X but on the other hand, it improves convergence properties since the approximation error of the Y part averages out over the time steps. Through this effect, when optimizing the algorithm again to achieve a given theoretical accuracy, the costs for the approximation of the Z part become dominating. The results of Gobet and Turkedjiev (2016) show a faster convergence of their algorithm than derived in Lemor et al. (2006); however, both algorithms differ in the simulation scheme as Gobet and Turkedjiev (2016) re-simulate the process X in each time step of the algorithm. This is not done in Lemor et al. (2006) with the consequence of a dominating dependency error between the time steps.

In the first chapter of this thesis, we aim to further improve the two algorithms by using a multi-step scheme. Rather than always simulating to the terminal condition we simulate in segments and optimize the length of these segments to balance the costs for the approximation of Y and Z. This is in a similar spirit to the work of Egloff (2005), who interpolated between the one step Tsitsiklis–Van Roy algorithm (see Tsitsiklis and Van Roy, 1999) and the multi-step Longstaff–Schwartz algorithm (see Longstaff and Schwartz, 2001) for pricing Bermudan options. More precisely, we intend to achieve a reduction in complexity by using the better error propagation in the Y part of the solution through a multi-step scheme on the one hand and, on the other hand, avoid additional simulation costs of the process X by capping the used path length. This is achieved by separating the time interval into segments of a fixed length which can be controlled by a parameter of the algorithm. Then, in each segment, only simulations of X until the end of the corresponding segment are used. The algorithms from Lemor et al. (2006) and Gobet and Turkedjiev (2016) themselves can be reconstructed by choosing segments of length one or only one segment covering the whole time horizon. In this view, our algorithm interpolates between the two mentioned above. In comparison to Gobet and Turkedjiev (2016), we include the discretization error in our analysis by starting with a decoupled forward-backward stochastic differential equation in continuous time, i.e., we assume that the explanatory process X is given by the solution of a stochastic differential equation. We assume most components to be bounded; however, those assumptions could be relaxed with minor changes to the usual Lipschitz growth conditions. We provide a complete error analysis and optimize the algorithm parameters, including the segment length, to achieve a given convergence rate of the mean squared error in relation to the used time steps in the discretization with minimal costs. Results will show that the complexity can always be reduced by choosing the optimal segment length in comparison to using the algorithm of Gobet and Turkedjiev (2016), where the reduction is proportional to the lower simulation costs of the process X. To this end, we assume that X can be sampled on the discrete time grid. Besides this constraint, our algorithm is model-free just like the one in Gobet and Turkedjiev (2016). An outline of this part of the thesis is at the beginning of Chapter 1.

In the second chapter of this thesis, we study an algorithm for obtaining lower bounds for BSDEs with convex driver, which allows the construction of confidence intervals when paired with a second algorithm for calculating upper bounds. This is the well-known primal-dual methodology that was first proposed by Rogers (2002), Haugh and Kogan (2004) and Andersen and Broadie (2004) for pricing Bermudan options. There, to find the fair price of an option, one has to stop a stochastic process such that its value is maximized over the finite set of possible stopping times. Hence, any (possible non-optimal) stopping strategy results in a lower bound for the fair price. For the construction of an upper bound, a dual method was proposed by Haugh and Kogan (2004) and Rogers (2002), where the stopping problem can be considered pathwise. This approach uses information about the future, which has to be compensated by subtracting a martingale increment to obtain a tight bound. Here taking the minimum over the set of martingales leads again to the true price. In practice, an input approximation of an associated dynamic programming equation is used to replace conditional expectations and derive a stopping strategy for the primal lower bound and the martingale increments for the dual upper bound.
This methodology was generalized to general stochastic control problems in discrete time by Rogers (2007) and Brown et al. (2010) and later by Bender et al. (2017b) to backward dynamic programming equations associated with time discretization schemes of BSDEs with convex driver. Also, algorithms that allow the iterative improvements of constructed confidence intervals are available, like those proposed in Kolodko and Schoenmakers (2006) for optimal stopping problems or Bender et al. (2017a) for monotone and convex dynamic programming equations.
In the second chapter of this thesis, we introduce a version of the primal algorithm much like in Bender et al. (2017b) for the construction of lower bounds for a discretized BSDE in the form of a dynamic programming equation, which is equivalent to an optimal control problem. Instead of pairing the algorithm with a dual algorithm and constructing confidence intervals, we solely focus on a detailed error analysis of the primal algorithm. Such results are in the literature available in the special case of optimal stopping for Bermudan option pricing (see e.g. Belomestny, 2011 and Belomestny et al., 2015). Chapter 2 hence generalizes these results from an optimal stopping problem to the case of convex dynamic programming equations. We assume that input approximations of the conditional expectations can be constructed, which allows the simulation of samples of the optimal control process and by that a Monte-Carlo approximation of the solution of the dynamic programming equation. To obtain the (asymptotic) behavior of the mean squared error of the approximation, we pose slightly stronger integrability assumptions on the components compared to Bender et al. (2017b) and assume that the input approximations converge in a L^p sense with a specific rate. Under these assumptions, we can show that the mean squared error of the lower bound converges at least with the same speed as the input approximation, and a faster convergence is possible in some settings. We then optimize

the parameters of the algorithm and analyze how to balance the costs between the input approximations and the algorithm itself in order to achieve a given accuracy with minimal computation costs like in Belomestny et al. (2015) for Bermudan option pricing. A short outline of this part can be found at the beginning of the second chapter.

Chapter 1

Segment-wise dynamic programming algorithm for BSDEs

In this chapter, we introduce a new algorithm for solving backward stochastic differential equations which we will call segment-wise dynamic programming algorithm. The algorithm is based on those presented in Lemor et al. (2006) and Gobet and Turkedjiev (2016) and interpolates between these two intending to reduce the complexity of its predecessors. The motivating idea of how to achieve the reduction in complexity is explained in Section 1.1 where we also present the framework in which we analyze the segment-wise dynamic programming algorithm. The algorithm is then introduced in Section 1.2 in detail, such that the grade of interpolation between the algorithms of Lemor et al. (2006) and Gobet and Turkedjiev (2016) can be controlled by a parameter $\alpha \in [0, 1]$, and those algorithms themselves can be reconstructed by the choices $\alpha = 0$ and $\alpha = 1$ respectively. In Section 1.3 we state bounds for the quadratic error of the approximation with the segment-wise dynamic programming algorithm in dependency of the parameter α. This allows us to analyze how to optimally choose this parameter to minimize the complexity, which is described in the same section. The results will show that it is always optimal to choose α between zero and one in typical situations and hence our algorithm in fact improves its predecessors. Those results are illustrated in a numerical example in Section 1.4 while Section 1.5 is devoted to a complete and detailed error analysis of the algorithm, where the bounds stated in Section 1.3 are derived.

1.1 Setup and motivation

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability where the filtration is generated by a $\mathcal{D}$-dimensional Brownian motion W. Throughout this chapter, we consider a system of a decoupled stochastic differential equation (short SDE) and a backward stochastic differential equation (short BSDE) of the following form

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t, \\ X_0 &= x_0, \\ -dY_t &= f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T &= \xi(X_T). \end{aligned}$$

Such systems are called forward-backward stochastic differential equations (short FBSDEs). We suppose that the initial value $x_0 \in \mathbb{R}^D$ and the terminal time $T > 0$ are deterministic and impose the following standing assumptions on the coefficients of the system throughout the chapter.

Assumptions 1.1.1.

(A_ξ) The function $\xi : \mathbb{R}^D \to \mathbb{R}$ is bounded by some constant C_ξ.

(A_L) The functions $b : [0, T] \times \mathbb{R}^D \to \mathbb{R}^D$, $\sigma : [0, T] \times \mathbb{R}^D \to \mathbb{R}^{D\times\mathfrak{D}}$ and $f : [0, T] \times \mathbb{R}^D \times \mathbb{R} \times \mathbb{R}^{\mathfrak{D}} \to \mathbb{R}$ are measurable, $\frac{1}{2}$-Hölder-continuous in the first variable and Lipschitz-continuous in the other variables, i.e., there exist constants L_X and L_f such that

$$\begin{aligned} |b(t, x) - b(t', x')| + |\sigma(t, x) - \sigma(t', x')| &\le L_X \left(|t - t'|^{\frac{1}{2}} + |x - x'|\right) \\ |f(t, x, y, z) - f(t', x', y', z')| &\le L_f \left(|t - t'|^{\frac{1}{2}} + |x - x'| + |y - y'| + |z - z'|\right) \end{aligned}$$

for all $x, x' \in \mathbb{R}^D$, $t, t' \in [0, T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^{\mathfrak{D}}$.

(A_f) The function f is uniformly bounded by a constant C_f, i.e.,

$$f(t, x, y, z) \le C_f$$

for all $t \in [0, T]$, $x \in \mathbb{R}^D$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{\mathfrak{D}}$.

Assumption (A_L) is standard for FBSDEs and yields important characteristics of the processes X, Y and Z. For once, the assumption on b and σ ensures the existence of a unique strong solution X of the SDE and that this solution satisfies $E[\sup_{t\in[0,T]} |X_t|^2] < \infty$ (see e.g. Karatzas and Shreve, 2012). Then, paired with the assumptions on f and ξ the solution of the BSDE can be expressed by deterministic functions of the SDE solution X, i.e., there exist deterministic functions $\overline{y} : [0, T] \times \mathbb{R}^D \to \mathbb{R}$ and $\overline{z} : [0, T] \times \mathbb{R}^D \to \mathbb{R}^{\mathfrak{D}}$ such that $\overline{y}(t, X_t) = Y_t$ and $\overline{z}(t, X_t) = Z_t$ (see e.g. El Karoui et al., 1997). The boundedness conditions on f and ξ are posed for convenience only and could be relaxed with minor changes in the error analysis.

We now define the equidistant time grid

$$\pi := \{t_i = i\Delta;\ i = 0, \ldots, N\}$$

with step width $\Delta = T/N$ for a fixed $N \in \mathbb{N}$. We denote the increments of the Brownian motion W on this time grid with ΔW_i, i.e., $\Delta W_i := W_{t_i} - W_{t_{i-1}}$, and define the functions

$$\begin{aligned} \overline{q}_i^N(x) &:= E\left[Y_{t_{i+1}} | X_{t_i} = x\right], \\ \overline{z}_i^N(x) &:= E\left[\frac{\Delta W_{i+1}}{\Delta} Y_{t_{i+1}} \middle| X_{t_i} = x\right] \end{aligned}$$

for all $i \in \{0, \ldots, N-1\}$. Then, under the standing assumptions, it holds that

$$\lim_{N\to\infty} \left(\max_{i=0,\ldots,N-1} E\left[|\overline{q}_i^N(X_{t_i}) - Y_{t_i}|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} |\overline{z}_i^N(X_{t_i}) - Z_s|^2 ds\right]\right) = 0,$$

where the rate of convergence depends on the regularity of Z, see e.g. Zhang (2001). The functions $\overline{q}_i^N$ and $\overline{z}_i^N$ can, therefore, be interpreted as time-discretized versions of the BSDE solution (Y, Z). To obtain an implementable approximation scheme for $\overline{q}_i^N$ and $\overline{z}_i^N$, fix any function

$$\tau : \{0, \ldots, N-2\} \to \{1, \ldots, N-1\}$$

satisfying $\tau(i) \geq i+1$ for all $i \in \{0, \ldots, N-2\}$. Then the tower property of the conditional expectation and the Markov property of X yield for any $i \in \{0, \ldots, N-2\}$

$$\begin{aligned} \overline{q}_i^N(x) &= E[Y_{t_{i+1}} | X_{t_i} = x] \\ &= E\left[Y_{t_{\tau(i)+1}} + \int_{t_{i+1}}^{t_{\tau(i)+1}} f(t, X_t, Y_t, Z_t)dt - \int_{t_{i+1}}^{t_{\tau(i)+1}} Z_t dW_t \middle| X_{t_i} = x\right] \\ &= E\left[\overline{q}_{\tau(i)}^N(X_{\tau(i)}) + \sum_{j=i+1}^{\tau(i)} \int_{t_j}^{t_{j+1}} f(t, X_t, Y_t, Z_t)dt \middle| X_{t_i} = x\right] \\ &\approx E\left[\overline{q}_{\tau(i)}^N(X_{\tau(i)}) + \sum_{j=i+1}^{\tau(i)} \Delta f\left(t_j, X_{t_j}, \overline{q}_j^N(X_{t_j}), \overline{z}_j^N(X_{t_j})\right) \middle| X_{t_i} = x\right] \end{aligned}$$

and similarly

$$\begin{aligned}
\overline{z}_i^N(x) &= E\left[\frac{\Delta W_{i+1}}{\Delta} Y_{t_{i+1}} \middle| X_{t_i} = x\right] \\
&= E\left[\frac{\Delta W_{i+1}}{\Delta}\left(Y_{t_{\tau(i)+1}} + \int_{t_{i+1}}^{t_{\tau(i)+1}} f(t, X_t, Y_t, Z_t)dt - \int_{t_{i+1}}^{t_{\tau(i)+1}} Z_t dW_t\right) \middle| X_{t_i} = x\right] \\
&= E\left[\frac{\Delta W_{i+1}}{\Delta}\left(\overline{q}_{\tau(i)}^N(X_{\tau(i)}) + \sum_{j=i+1}^{\tau(i)} \int_{t_j}^{t_{j+1}} f(t, X_t, Y_t, Z_t)dt\right) \middle| X_{t_i} = x\right] \\
&\approx E\left[\frac{\Delta W_{i+1}}{\Delta}\left(\overline{q}_{\tau(i)}^N(X_{\tau(i)}) + \sum_{j=i+1}^{\tau(i)} \Delta f\left(t_j, X_{t_j}, \overline{q}_j^N(X_{t_j}), \overline{z}_j^N(X_{t_j})\right)\right) \middle| X_{t_i} = x\right].
\end{aligned}$$

This motivates the time discretization scheme

$$\begin{aligned}
Q_{N-1}^N &:= E\left[\xi(X_{t_N}) \middle| \mathcal{F}_{t_{N-1}}\right] \\
Z_{N-1}^N &:= E\left[\frac{\Delta W_N}{\Delta}\xi(X_{t_N}) \middle| \mathcal{F}_{t_{N-1}}\right] \\
Q_i^N &:= E\left[Q_{\tau(i)}^N + \sum_{j=i+1}^{\tau(i)} \Delta f\left(t_j, X_{t_j}, Q_j^N, Z_j^N\right) \middle| \mathcal{F}_{t_i}\right], \quad i = N-2, \dots, 0 \qquad (1.1) \\
Z_i^N &:= E\left[\frac{\Delta W_{i+1}}{\Delta}\left(Q_{\tau(i)}^N + \sum_{j=i+1}^{\tau(i)} \Delta f\left(t_j, X_{t_j}, Q_j^N, Z_j^N\right)\right) \middle| \mathcal{F}_{t_i}\right], \quad i = N-2, \dots, 0.
\end{aligned}$$

By the tower property of the conditional expectation, this definition of Q_i^N and Z_i^N does not depend on the choice of τ. However, the appearing conditional expectations can not be calculated in closed form in general. Hence, when attempting to solve the BSDE, one has to replace the conditional expectations with some approximation operator resulting in different schemes depending on the choice of τ. As our results will show, the choice of τ then influences both, the computational costs as well as the convergence properties.

The most natural choices for τ would be for once setting $\tau(i) = i+1$ or $\tau(i) = N-1$ for all $i \in \{0, \dots, N-2\}$. The first results in the classical one-step scheme of Lemor et al. (2006) (to which we further refer to as ODP), the latter in the multi-step forward scheme (MDP for short) by Gobet and Turkedjiev (2016). To understand the idea of the segment-wise dynamic programming algorithm that will be introduced in the next section, it is worth reviewing these two schemes and comparing the resulting algorithms.

Both algorithms work recursively backward in time by constructing estimates of the functions $\overline{q}_i^N$ and $\overline{z}_i^N$ through approximating the conditional expectations in the corresponding time discretization scheme via empirical orthogonal projections on finite-dimensional function spaces, where the components Q_j^N, Z_j^N with $j > i$ on the right-hand side of the

discretization scheme are replaced by the approximations found in the previous step of the recursion. As a result of the different schemes, the approximation of the ODP algorithm depends at each time point t_i only on the approximations at the time t_{i+1} while in the MDP scheme, the approximation at each step t_i depends on all the previously constructed ones, i.e., those at the time points from t_{i+1} up to t_{N-1}. Since the approximations of $\overline{q}_i^N$ and $\overline{z}_i^N$ have to be evaluated, one has to simulate in each step of the MDP algorithm segments of the form $(X_{t_i}, X_{t_{i+1}}, \dots, X_{t_N})$ while it suffices in the ODP algorithm to simulate values of X at just the current and the following time point. This obviously leads to higher simulation costs in the MDP scheme. However, since the algorithms recursively reuse the obtained approximations of $\overline{q}_i^N$ and $\overline{z}_i^N$ an error propagation between the time steps occurs. Results show that from this perspective, the MDP scheme is advantageous as errors average out over time, leading to better convergence properties (see Gobet and Turkedjiev, 2016).
The idea of the segment-wise approach, which will be introduced in the next chapter, is to interpolate between the two extreme cases of the ODP and the MDP scheme in order to balance these two aspects, the computation costs and convergence properties.

1.2 Segment-wise dynamic programming algorithm

In this section, we present the segment-wise dynamic programming algorithm (short SDP) in detail. First, a specific family of functions τ_α is introduced to obtain the time discretization scheme that interpolates between the ones from the ODP scheme and the MDP scheme via (1.1). Then the algorithm is described in detail based on this discretization scheme.
For any $\alpha \in [0, 1]$, consider the time grid

$$\overline{\pi}_\alpha := \{(\Delta n \lceil N^\alpha \rceil) \wedge (T - \Delta); n \in \mathbb{N}\}$$

with step width $\lceil N^\alpha \rceil$ (up to a possibly smaller size in the last step), that consists of $\lceil N^{1-\alpha} \rceil$ time points at most. Based on these time grids define the functions

$$\tau_\alpha : \{0, 1, \dots, N-2\} \to \{1, \dots, N-1\}$$

as

$$\tau_\alpha(i) := \min\{j > i : j\Delta \in \overline{\pi}_\alpha\}.$$

For a fixed α, the choice $\tau = \tau_\alpha$ in (1.1) then defines a discretization scheme where the time grid π is separated in segments consisting of $\lceil N^\alpha \rceil$ points by the coarser time grid $\overline{\pi}_\alpha$. The resulting discretization scheme corresponds to an MDP scheme on each of these segments paired with a single step of an ODP scheme between consecutive segments connecting them. Moreover choosing $\alpha = 0$ or $\alpha = 1$ results in the classical ODP or MDP scheme

respectively.
Now for a fixed $\alpha \in [0,1]$ the SDP algorithm works as follows.

Algorithm 1.2.1.

- Choose basis functions

$$\begin{aligned} p_{q,i}^k &: \mathbb{R}^D \to \mathbb{R}, \qquad k = 1, \dots, K_{q,i} \\ p_{z,i}^k &: \mathbb{R}^D \to \mathbb{R}^{\mathcal{D}}, \quad k = 1, \dots, K_{z,i} \end{aligned}$$

  for each $i \in \{0, \dots, N-1\}$ such that

$$\sum_{i=0}^{N-1} \sum_{k=1}^{K_{q,i}} E\left[|p_{q,i}^k(X_{t_i})|^2 \right] + \sum_{i=0}^{N-1} \sum_{k=1}^{K_{z,i}} E\left[|p_{z,i}^k(X_{t_i})|^2 \right] < \infty.$$

  Here the number of basis functions $K_{q,i}$, $K_{z,i} \in \mathbb{N}$ may depend on the time point t_i. We denote the function spaces spanned by these basis functions with $\mathcal{K}_{q,i}$ and $\mathcal{K}_{z,i}$ respectively, i.e.,

$$\begin{aligned} \mathcal{K}_{q,i} &:= span\left(p_{q,i}^1, \dots, p_{q,i}^{K_{q,i}}\right) \\ \mathcal{K}_{z,i} &:= span\left(p_{z,i}^1, \dots, p_{z,i}^{K_{z,i}}\right). \end{aligned}$$

  The algorithm will approximate $\overline{q}_i^N$ by empirical orthogonal projections on the subspaces $\mathcal{K}_{q,i}$ and $\overline{z}_i^N$ by projections on $\mathcal{K}_{z,i}$.

- Initialize the algorithm by setting

$$\Xi_{N-1}^{N,M}(x_N) := \xi(x_N)$$

  for all $(x_N) \in \mathbb{R}^D$. Then, assuming $\Xi_i^{N,M}$ is already constructed, perform the following backward incursion for $i = N-1, N-2, \dots, 0$:

  1*) If $i = N-1$: Choose $M_{N-1} \in \mathbb{N}$, then simulate M_{N-1} independent copies

$$\left(X_{t_{N-1}}^{[N-1,m,N]}, X_{t_N}^{[N-1,m,N]}, \Delta W_N^{[N-1,m,N]}\right)_{m=1,\dots,M_{N-1}}$$

  of the segment $(X_{t_{N-1}}, X_{t_N}, \Delta W_N)$ and set

$$X^{[N-1,m,N]} := X_{t_N}^{[N-1,m,N]}.$$

1) If $i < N-1$: Choose a $M_i \in \mathbb{N}$, then simulate M_i independent copies

$$\left(X_{t_i}^{[i,m,N]}, \dots, X_{t_{\tau_\alpha(i)}}^{[i,m,N]}, \Delta W_{i+1}^{[i,m,N]}\right)_{m=1,\dots,M_i}$$

of the segment $(X_{t_i}, \dots, X_{\tau_\alpha(i)}, \Delta W_{i+1})$ and set

$$X^{[i,m,N]} := \left(X_{t_{i+1}}^{[i,m,N]}, \dots, X_{t_{\tau_\alpha(i)}}^{[i,m,N]}\right).$$

2) Find solutions to the linear least-squares regression problems

$$\varphi_i^{q^{N,M}} = \underset{\psi \in \mathcal{K}_{q,i}}{\operatorname{argmin}} \left(\frac{1}{M} \sum_{m=1}^{M_i} \left| \psi\left(X_{t_i}^{[i,m,N]}\right) - \Xi_i^{N,M}\left(X^{[i,m,N]}\right) \right|^2 \right)$$

and

$$\varphi_i^{z^{N,M}} = \underset{\psi \in \mathcal{K}_{z,i}}{\operatorname{argmin}} \left(\frac{1}{M} \sum_{m=1}^{M_i} \left| \psi\left(X_{t_i}^{[i,m,N]}\right) - \frac{\Delta W_{i+1}^{[i,m,N]}}{\Delta} \Xi_i^{N,M}\left(X^{[i,m,N]}\right) \right|^2 \right).$$

3) Define approximations $q_i^{N,M}$ and $z_i^{N,M}$ of the functions $\overline{q}_i^N$ and $\overline{z}_i^N$ via

$$q_i^{N,M} := \mathcal{T}_{C_{q,i}} \circ \varphi_i^{q^{N,M}}, \quad z_i^{N,M} := \mathcal{T}_{C_{z,i}} \circ \varphi_i^{z^{N,M}}$$

where $C_{q,i} := C_\xi + (T - t_{i+1})C_f$ and $C_{z,i} := \frac{C_{q,i}}{\Delta}$ are positive constants and $\mathcal{T}_c$ is the truncation function defined as

$$\mathcal{T}_c(x) := \operatorname{sign}(x) \min\{|x|, c\}$$

for any constant $c > 0$ (acting componentwise on $\varphi^{z^{N,M}}$).

4) If $i \geq 1$, set

$$\Xi_{i-1}^{N,M}(x_i, \dots, x_{\tau_\alpha(i-1)}) := q_{\tau_\alpha(i-1)}^{N,M}(x_{\tau_\alpha(i-1)}) + \sum_{j=i}^{\tau_\alpha(i-1)} \Delta f\left(t_j, x_j, q_j^{N,M}(x_j), z_j^{N,M}(x_j)\right)$$

as preparation for the next iteration.

The solutions to the empirical least squares problems can be computed numerically using a singular value decomposition. Hence, the algorithm is fully implementable as long as the segments $(X_{t_i}, \dots, X_{t_{\tau_\alpha(i)}})$ can be simulated. Then, in the typical situation, for example $X = W$, the average costs for the simulation of one segment $(X_{t_i}, \dots, X_{t_{\tau_\alpha(i)}})$ are of order $\mathcal{O}(N^\alpha)$. For $\alpha < 1$ this leads to smaller computation costs through simulations as the MDP scheme, where the average costs for simulating one set $(X_{t_i}, \dots, X_{t_N})$ are of order $\mathcal{O}(N)$.

When X can not be sampled perfectly, it would, in principle, be possible to replace X with some approximation scheme with minor changes in the error analysis. The problem is, however, to approximate X in a way that sustains the gain in computation costs compared to the MDP scheme, which would not be the case in the simplest approach when approximating X with a naive Euler scheme starting at the time 0. In theory, one could approximate X_{t_i} with some high-order approximation scheme and use an Euler scheme inside the segment $(X_{t_i}, \ldots, X_{t_{\tau_\alpha(i)}})$ and preserve at least some gain in computation costs. However, we restrict the theoretical analysis to the assumption that the values of X can be sampled directly on the time grid π.

1.3 Convergence rates and complexity

In this section, we state error bounds for the quadratic error of the SDP algorithm in dependency of the number of time steps N in the time discretization. We argue how these bounds show the (asymptotic) convergence rate of our approximations for $N \to \infty$, i.e., when making the time discretization finer. We then analyze how to optimally choose the parameters of the algorithm and state the complexity in terms of the used time steps N and in dependency of the discretization scheme through the parameter α. The results will show that the optimal α is always in the open interval $(0, 1)$, and hence the SDP algorithm presented in Section 1.2 is advantageous when compared to the MDP and ODP schemes. We do this once under the standing assumptions and once under additional regularity assumptions resulting in an even faster rate of convergence.
Under both sets of assumptions, a first bound for the total quadratic error of the approximation is given in the following theorem:

Theorem 1.3.1. *Under the standing assumptions, it holds*

$$
\begin{aligned}
&\max_{0\le i\le N-1} E\left[|q_i^{N,M}(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + \sum_{i=0}^{N-1} \Delta E\left[|z_i^{N,M}(X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2\right] \\
&\le c \max_{i\in\mathcal{J}} \left(N^{1-\alpha} \inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + N^{2-2\alpha}\frac{K_{q,i}}{M_i} + N^{2-2\alpha}\frac{K_{q,i}\log(M_i)}{M_i}\right) \\
&\quad + c \max_{0\le i\le N-1} \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[|\psi(X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2\right] \right.\\
&\quad \left. + \frac{K_{q,i}}{M_i} + N\frac{K_{z,i}}{M_i} + \frac{K_{q,i}\log(M_i)}{M_i} + N\frac{K_{z,i}\log(M_i)}{M_i}\right) \\
&\quad + cN\mathcal{R}^N
\end{aligned}
$$

where $\mathcal{I} := \{i : t_i \in \overline{\pi}_\alpha\}$, *c is a positive constant not depending on* N *and*

$$\mathcal{R}^N := \sum_{i=1}^{N-1} E\left[\left(\int_{t_i}^{t_{i+1}} E\left[f\left(s, X_s, Y_s, Z_s\right) - f\left(t_i, X_{t_i}, \overline{q}_i^N(X_{t_i}), \overline{z}_i^N(X_{t_i})\right)\middle| X_{t_{i-1}}\right] ds\right)^2\right].$$

We can think of this bound as a composition of terms due to three different error sources. The appearing expectations occur due to the projection on finite-dimensional subspaces and we will therefore refer to those as projection errors. They can be controlled through the choice of basis functions where more basis functions result in a lower projection error. The term $\mathcal{R}^N$ only depends on the true BSDE solution and the chosen time grid (through the discretized functions $\overline{q}_i^N$ and $\overline{z}_i^N$), but not on the approximation obtained with the algorithm. It can be interpreted as part of the time discretization error. The remaining terms are statistical error terms that occur due to the use of simulations of the process X. Those can be controlled by the number of used simulations. However, more simulations are required when more basis functions are used.

The bound shows the influence of the parameter α, as the projection and statistical error terms regarding $\overline{q}^N$ appear once at all time steps and once on the time steps of the coarser time grid $\overline{\pi}$ with different factors that are decreasing in α. Although bigger values for α seem favorable from this perspective, increasing α also results in higher computation costs through the required simulation of X on larger segments. Hence, when choosing α a trade-off between convergence properties and computation costs has to be considered. We shall argue in the optimization that the optimal value of α in typical situations is in the open interval $(0, 1)$.

For $\alpha = 1$ the terms in the first bracket are dominated by the remaining terms, and we essentially reproduce the error analysis of the MDP scheme in Gobet and Turkedjiev (2016) with some slight differences in representation: Here, the projection error is formulated in terms of the true continuous time solution (Y, Z) of the BSDE via the functions $\overline{q}_i^N$ and $\overline{z}_i^N$ while it is stated in terms of the backward Euler discretization scheme for BSDEs in Gobet and Turkedjiev (2016). Secondly, the factor $\mathcal{R}^N$ is absent in the error analysis of the MDP scheme since the discretization error is not included in the analysis there.

On the other hand, when choosing $\alpha = 0$, one ends up with an error analysis for the ODP scheme with independent re-simulation at every time point. This differs from the error analysis in Lemor et al. (2006), as there only one cloud of simulations is used at all time points which results in a dominating interdependency error. Hence, the bound obtained with Theorem 1.3.1 for $\alpha = 0$ also allows a better comparison of the ODP and MDP algorithms.

Possible bounds for the term $\mathcal{R}^N$ in dependence of N are of order N^{-2} or N^{-3}, depending on the assumptions on ξ, X and the regularity of the BSDE as illustrated by the following theorems.

Theorem 1.3.2. *Under the standing assumptions, the total quadratic error of the approximation is bounded by*

$$\begin{aligned}
&\max_{0\le i\le N-1} E\left[|q_i^{N,M}(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} |z_i^{N,M}(X_{t_i}) - \overline{z}(s, X_s)|^2 ds\right] \\
&\le c \max_{i\in\mathfrak{I}} \left(N^{1-\alpha} \inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + N^{2-2\alpha}\frac{K_{q,i}}{M_i} + N^{2-2\alpha}\frac{K_{q,i}\log(M_i)}{M_i}\right) \\
&\quad + c \max_{0\le i\le N-1} \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[|\psi(X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2\right]\right. \\
&\left. + \frac{K_{q,i}}{M_i} + N\frac{K_{z,i}}{M_i} + \frac{K_{q,i}\log(M_i)}{M_i} + N\frac{K_{z,i}\log(M_i)}{M_i}\right) \\
&+ cN^{-1}
\end{aligned}$$

where $\mathfrak{I} := \{i : t_i \in \overline{\pi}_\alpha\}$ *and* c *is a positive constant not depending on* N. *Furthermore, assuming that* $\overline{z}$ *is* $\frac{1}{2}$*-Hölder continuous in* t *and Lipschitz continuous in* x*, it holds*

$$\begin{aligned}
&\max_{0\le i\le N-1} E\left[|q_i^{N,M}(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} |z_i^{N,M}(X_{t_i}) - \overline{z}(s, X_s)|^2 ds\right] \\
&\le c \max_{i\in\mathfrak{I}} \left(N^{1-\alpha} \inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] + N^{2-2\alpha}\frac{K_{q,i}}{M_i} + N^{2-2\alpha}\frac{K_{q,i}\log(M_i)}{M_i}\right) \\
&\quad + c \max_{0\le i\le N-1} \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[|\psi(X_{t_i}) - \overline{z}(t_i, X_{t_i})|^2\right]\right. \\
&\left. + \frac{K_{q,i}}{M_i} + N\frac{K_{z,i}}{M_i} + \frac{K_{q,i}\log(M_i)}{M_i} + N\frac{K_{z,i}\log(M_i)}{M_i}\right) \\
&+ cN^{-1},
\end{aligned}$$

i.e., we can express the projection error in terms of the functions $\overline{y}$ *and* $\overline{z}$.

Theorem 1.3.2 shows that the convergence rate under the standing assumptions is limited by N^{-1} at best due to the discretization error. Under additional assumptions, this limitation can be lifted to N^{-2} as follows.

Theorem 1.3.3. *Additionally to the standing assumptions, suppose that* $X = W$*, and that* f *and* $\overline{y}$ *are respectively twice and* $s+1$ *times continuous differentiable with bounded deriva-*

tives each for an $s \geq 2$. Then the total error of our approximation is bounded by

$$\max_{0\leq i\leq N-1} E\left[|q_i^{N,M}(W_{t_i}) - \overline{y}(t_i, W_{t_i})|^2\right] + \sum_{i=0}^{N-1} \Delta E\left[|z_i^{N,M}(W_{t_i}) - \overline{z}(t_i, W_{t_i})|^2\right]$$
$$\leq c \max_{i\in\mathcal{I}} \left(N^{1-\alpha} \inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(W_{t_i}) - \overline{q}_i^N(W_{t_i})|^2\right] + N^{2-2\alpha}\frac{K_{q,i}}{M_i} + N^{2-2\alpha}\frac{K_{q,i}\log(M_i)}{M_i}\right)$$
$$+ c \max_{0\leq i\leq N-1} \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(W_{t_i}) - \overline{q}_i^N(W_{t_i})|^2\right] + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[|\psi(W_{t_i}) - \overline{z}_i^N(W_{t_i})|^2\right]\right.$$
$$\left.+ \frac{K_{q,i}}{M_i} + N\frac{K_{z,i}}{M_i} + \frac{K_{q,i}\log(M_i)}{M_i} + N\frac{K_{z,i}\log(M_i)}{M_i}\right)$$
$$+ cN^{-2}$$

where $\mathcal{I} := \{i : t_i \in \overline{\pi}_\alpha\}$ and c is a positive constant not depending on N. Furthermore, q^N and z^N are bounded and respectively $s+1$ times and s times continuous differentiable with bounded derivatives.

Compared to the more general assumptions in Theorem 1.3.2, it is unfortunately not possible to express the approximation error in terms of the continuous-time functions $\overline{y}$ and $\overline{z}$ without worsening the limiting error term N^{-2}. Note however that the second statement of Theorem 1.3.3 shows that smoothness properties of $\overline{y}$ and $\overline{z}$ carry over to the discretized versions $\overline{q}^N$ and $\overline{z}^N$, which will suffice for the analysis of the projection error.
Before we prove the theorems presented above in Section 1.5, we analyze how to optimal choose α and derive the resulting complexity of the algorithm.
As Theorems 1.3.2 and 1.3.3 show, the achievable convergence rate is limited by $N^{-2\theta}$ for $\theta = \frac{1}{2}$ under the general setting of Theorem 1.3.2 or $\theta = 1$ in the setting of Theorem 1.3.3, where a higher regularity of the BSDE is required. We therefore analyse how to optimally choose the parameters of the algorithm to achieve this convergence rate with the smallest computation costs possible.
As argued before, the projection error can be controlled by the choice of approximation spaces $\mathcal{K}_{q,i}$, $\mathcal{K}_{z,i}$. Using more basis functions and hence a higher dimensional approximation space reduces the approximation error but, in return increases the statistical error terms leading to a higher number of required simulations M_i needed to bound these error terms.
For simplicity we assume that we use the same approximation space $\mathcal{K}_z$ for the approximation of $\overline{z}^N$ in each time step. Due to the additional projection error on the coarser time grid $\overline{\pi}_\alpha$ we differ between the time points in and outside of $\overline{\pi}_\alpha$ for the approximation of $\overline{q}^N$. We assume that the same approximation space $\overline{\mathcal{K}}_q$ is used for all time points $t_i \in \overline{\pi}_\alpha$ while we use a possibly different approximation space $\mathcal{K}_q$ on all other time points. Analogously we assume that we use $M_i = M$ simulations at each time point t_i not in the coarser time grid $\overline{\pi}$ and $M_i = \overline{M}$ simulations otherwise. As already mentioned, we assume that one

segment $(X_i, \dots, X_{\tau_\alpha(i)})$ of a path of X can be simulated at the cost of N^α. Under these assumptions, assuming we can evaluate the driver and the basis functions at cost 1, the performed simulations and evaluations during the algorithm lead to costs of order

$$NN^\alpha M + N^\alpha N^{1-\alpha}\overline{M}.$$

As basis functions we take local polynomials on hypercubes which we choose disjoint such that their union contains the set $\{x \in \mathbb{R}^D : |x| < C_b\}$ for a constant $C_b > 0$. We suppose that the edge length of the hypercube is $\overline{\delta}_q$ for the approximation of $\overline{q}^N$ on the time grid $\overline{\pi}_\alpha$, δ_q for the approximation of $\overline{q}^N$ on all other time points and δ_z for the approximation of $\overline{z}^N$. Assuming that $\overline{y}$ and $\overline{z}$ are respectively $s+1$ and s times continuous differentiable with bounded derivatives we set the degree of the polynomials as s for the approximation of $\overline{q}^N$ and $s-1$ for $\overline{z}^N$. Note that this allows us to express the approximation error in terms of $\overline{y}$ and $\overline{z}$ in the setting of Theorem 1.3.2. In the setting of Theorem 1.3.3 the following argumentation holds true analogously when replacing $\overline{y}$ and $\overline{z}$ with their discretized versions $\overline{q}^N$ and $\overline{z}^N$. We denote the set of polynomials of degree less than or equal to l with P_l. Then a Taylor expansion on each hypercube gives us

$$\begin{aligned}\inf_{\psi \in \mathcal{K}_q} E\left[|\psi(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] &\leq E\left[|\overline{y}(t_i, X_{t_i})|^2 \mathbb{1}_{|X_{t_i}|>C_b}\right] \\ &\quad + \sum_{H \in \mathcal{H}_q} \min_{\psi \in P_s} E\left[|\psi(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2 \mathbb{1}_{X_{t_i} \in H}\right] \\ &\leq \|\overline{y}(t_i, .)\|_\infty^2 P\left(|X_{t_i}| > C_b\right) + c\|\overline{y}(t_i, .)^{(s+1)}\|_\infty^2 (\delta_q^{s+1})^2 \end{aligned} \quad (1.2)$$

for the approximation of $\overline{q}^N$ at all time points not in $\overline{\pi}$, where we denote the set of used hypercubes with $\mathcal{H}_q$ and take the polynomials to be equal to the first $(s+1)$ terms of the Taylor expansion. Under the assumption that $\sup_{0\leq i \leq N} E[e^{\varpi |X_{t_i}|}] < \infty$ for some $\varpi > 0$, it follows by the Markov inequality that the choice $C_b = 2\theta\varpi^{-1}\log(N+1)$ ensures that the first term in (1.2) is of order $N^{-2\theta}$. The same holds for the second term when choosing the edge length of the hypercubes as $\delta_q = cN^{-\frac{\theta}{s+1}}$. Therefore we can assume that it suffices to choose K_q of the order $N^{D\frac{\theta}{s+1}} \log^D(N+1)$ to ensure that

$$\inf_{\psi \in \mathcal{K}_q} E[|\psi(X_i) - \overline{y}(t_i, X_{t_i})|^2] \in O(N^{-2\theta}).$$

Following the same argumentation, we set

$$\overline{K}_q = cN^{D\frac{\theta + \frac{1-\alpha}{2}}{s+1}} \log^D(N+1), \quad K_z = cN^{D\frac{\theta}{s}} \log^D(N+1)$$

for a positive constant c to ensure

$$N^{1-\alpha} \inf_{\psi \in \overline{\mathcal{K}}_q} E\left[|\psi(X_i) - \overline{y}(t_i, X_{t_i})|^2\right] \in O(N^{-2\theta}), \quad \inf_{\psi \in \mathcal{K}_z} E\left[|\psi(X_i) - \overline{z}(t_i, X_{t_i})|^2\right] \in O(N^{-2\theta})$$

where the change from $s+1$ to s in the number of basis functions in the approximation of $\overline{z}^N$ occurs due to the lower smoothness of $\overline{z}^N$ and $\overline{K}_q$ and K_z denote the number of basis functions of the space $\overline{\mathcal{K}}_q$ and $\mathcal{K}_z$ respectively.
Given the size of the approximation spaces we hence have to choose M of the order

$$N^{2\theta} \max\{K_q, NK_z\} = N^{1+2\theta} K_z$$

and $\overline{M}$ of the order

$$N^{2\theta} \max\{N^{2-2\alpha} \overline{K}_q, NK_z\}$$

in order to bound the statistical error terms asymptotically by a multiple of $N^{-2\theta}$ as long as the driver is not independent of Z. Then, in dependence of α, the total order of computation costs of the algorithm is given by

$$\mathcal{C} = \max\{N^{1+\alpha} M, N\overline{M}\} =: \max\{\mathcal{C}_\pi, \mathcal{C}_{\overline{\pi}}\}.$$

Here $\mathcal{C}_\pi$ is increasing in α and $\mathcal{C}_{\overline{\pi}}$ is non-increasing in α and both coincide for

$$\alpha = \alpha_{opt} = \frac{\frac{1}{2} - \frac{\theta}{s} + \frac{s+1}{D}}{\frac{1}{2} + \frac{3(s+1)}{D}},$$

which therefore is optimal and always in the open interval $(0,1)$ for $\vartheta = \frac{1}{2}$, $s \geq 1$ and $\vartheta = 1$, $s \geq 2$. The resulting computation costs are of the order

$$\begin{aligned} \mathcal{C} = NN^{\alpha_{opt}} M &= N^{2+2\theta+D\frac{\theta}{s} + \frac{\frac{1}{2} - \frac{\theta}{s} + \frac{s+1}{D}}{\frac{1}{2} + \frac{3(s+1)}{D}}} \\ &= N^{3+2\theta+D\frac{\theta}{s} - \frac{\frac{\theta}{s} + \frac{2(s+1)}{D}}{\frac{1}{2} + \frac{3(s+1)}{D}}}. \end{aligned}$$

In comparison, if we would choose $\alpha = 1$ in our optimization, we would get computation costs of order

$$\mathcal{C} = cN^{3+2\theta+D\frac{\theta}{s}},$$

as stated analogously in Gobet and Turkedjiev (2016). Hence, by using the presented SDP algorithm with the optimal choice of α we can improve the calculation costs by a factor

$$N^{1-\alpha_{opt}} = N^{\frac{\theta/s+2(s+1)/D}{1/2+3(s+1)/D}} = N^{\frac{2}{3}+\frac{\theta/s-1/3}{1/2+3(s+1)/D}}$$

which corresponds to the lower simulation costs of a segment of length $N^{\alpha_{opt}}$ compared to a segment of length N. If $\overline{q}^N$ only fulfils the minimal requirement needed for our analysis, i.e., $\overline{q}^N \in C^2$ in the case $\theta = 1/2$ and $\overline{q}^N \in C^3$ in the case $\theta = 1$ the gain in complexity is of the factor $N^{27/39}$ and $N^{39/57}$ respectively in dimension 1 and further increasing in higher dimensions. Depending on the dimension, the gain in complexity is decreasing with the smoothness s up to some point and then increases again. In the limit, assuming that $\overline{q}^N, \overline{z}^N$ are smooth functions, i.e., $\overline{q}^N, \overline{z}^N \in \mathcal{C}^\infty$, this would lead to a gain in computation time of $N^{\frac{2}{3}}$ for any dimension. However the SDP algorithm suffers, just like the ODP and MDP, from the typical "curse of dimensionality" through exploding costs in the limit $D \to \infty$.

1.4 Numerical example

In this section, we run both the MDP and the SDP algorithm for the same BSDE and compare the approximations in order to illustrate our theoretical results.
For this purpose, we define for each $x \in \mathbb{R}^D$ and all $t \in [0, 0.2]$ the function

$$\varphi(t,x) := \exp\left(-\sum_{d=1}^{D} |x^{(d)} - t|^{0.3}\right) \sum_{d=1}^{D} \left(x^{(d)} - t\right)^2$$

and for $d = 1, \ldots, D$

$$\phi_d(t,x) := \exp\left(-\sum_{d=1}^{D} |x^{(d)} - T|^{0.3}\right) \left(x^{(d)} - t\right) \left(2 - 0.3|x^{(d)} - t|^{0.3}\right) \sum_{e=1, e\neq d}^{D} \left(x^{(d)} - t\right)^2.$$

We then consider the BSDE driven by a Brownian motion $X = W$ with terminal time $T = 0.2$, the terminal condition

$$\xi(W_T) = \varphi(T, W_T)$$

and driver

$$f(t,x,y,z) := |z| - |\nabla\varphi(t,x)| - (\partial_t + \frac{1}{2}\triangle)\varphi(t,x).$$

It can be easily checked by the Ito-formula that the analytic solution to this BSDE is given by

$$\begin{aligned} Y_t &= \varphi(t, W_t) \\ Z_t^{(d)} &= \frac{\partial}{\partial x^{(d)}}\varphi(t, W_t) = \phi_d(t, W_t) \quad d = 1, \ldots, D. \end{aligned}$$

First, note that φ is bounded and twice continuously differentiable with bounded derivatives. Furthermore, the terminal condition is bounded and the driver is Lipschitz continuous in all variables but unbounded in z. However, assumption (A_f) holds true if we restrict the definition space of the driver to the image of $\nabla\varphi$ since $\nabla\varphi$ is bounded. Since the approximations will be truncated during the algorithm, this is possible without additional error. Hence, all assumptions of the theoretical analysis are fulfilled.
For a comparison between the MDP algorithm and the SDP algorithm, we want to compare the asymptotic rates, in which the squared error of the approximation decreases, and the computation time in relation to the used time steps N increases in both algorithms. For this purpose, we ran both algorithms for an increasing number of time points N, 50 times each. We do this in the dimensions $D = 1$ and $D = 2$ and measure the computation time and quadratic errors. For each value of N, we calibrated the algorithms according to the theoretical results such that the quadratic error should decrease like N^{-1}, but deviated slightly from the theoretical optimization in the following way:
First, instead of using the same approximation spaces $\mathcal{K}_z$ and $\mathcal{K}_q$ at all time points, we utilize that the Brownian motion $(W_t)_{t\in[0,T]}$ is less likely to take bigger values for small t and reduce the outer bound of the hypercubes in each step towards the starting time 0. In accordance with this, we also use different numbers of simulations at each time point. Additionally, we not only re-simulate in each time step but also for the approximation of Z_t and Y_t and only used the required number for each instead of the maximum of both at that time step. Note that this may influence the total computation costs but does not change the asymptotic rate in which the computation costs increase in relation to the used time steps N.
Measuring of the errors:
We measured the quadratic error of both algorithms for the approximation of $\overline{q}_i^N$ and $\overline{z}_i^N$ separately and considered the average error over all the time steps. Additionally, we measured the approximation error of Y at time zero since we are most interested in the approximation there in typical applications. We measured those errors in the form of the

terms

$$\mathcal{C}_{y,av} := \frac{1}{N}\sum_{i=1}^{N-1}\sum_{H\in\mathcal{H}_{q,i}} \left|q_i^{M,N}(\Theta_H) - \overline{y}(t_i,\Theta_H)\right|^2 P\big(X_{t_i}\in H\big)$$
$$\mathcal{C}_{y,0} := \left|q_0^{M,N}(0) - \overline{y}(0,0)\right|^2$$
$$\mathcal{C}_{z,av} := \frac{1}{N}\sum_{i=0}^{N-1}\sum_{H\in\mathcal{H}_{z,i}} \left|z_i^{M,N}(\Theta_H) - \overline{z}(t_i,\Theta_H)\right|^2 P\big(X_{t_i}\in H\big)$$

where Θ_H is the componentwise middle point of the hypercube H, and $\mathcal{H}_{q,i}$ and $\mathcal{H}_{z,i}$ are the sets of hypercubes used at time t_i for the approximation of $\overline{q}_i^N$ and $\overline{z}_i^N$ respectively. We further refer to the values $\mathcal{C}_{y,av}$ and $\mathcal{C}_{z,av}$ as average quadratic error over the time steps for the approximation and analog to $\mathcal{C}_{y,0}$ as quadratic error at time 0. Those approximately correspond to the error terms in Theorem 1.3.3 as the cube length and the time steps in the discretization get smaller but allow us to avoid the additional numerical solution of multidimensional integrals for each time point and hypercube.

Calibration in dimension 1:

We use piecewise linear functions for the approximation of $\overline{q}_i^N$ and piecewise constant functions for the one of $\overline{z}_i^N$ in both algorithms. At each time, we set the outer bound of the hypercubes as a multiple of the standard deviation of the Brownian motion at that time. More precisely, we chose $C_{b,i} := (2\log(N)+2)\sqrt{t_i}$ at the time t_i. The edge lengths of the hypercubes were set to $\delta_z = \sqrt{T}/N^{\frac{1}{2}}$ and $\delta_Q = \sqrt{T}/N^{\frac{1}{4}}$ in the MDP algorithm. This choice leads to $K_{z,i} = \lceil C_{b,i}/\delta_z\rceil = cN^{\frac{1}{2}}$ Basis functions for the approximation of $\overline{z}^N$ at time t_i and $K_{Q,i} = \lceil 2C_{b,i}/\delta_y\rceil = cN^{\frac{1}{4}}$ for $\overline{q}^N$ as derived as optimal under the choice $\alpha = 1$ in the theory. We re-simulate the sample paths at each time step as well as for the approximation of Q and Z where we use $M_{q,i} = 10NK_{q,i} = cN^{1+\frac{1}{4}}$ simulations for Q at time t_i and $M_{z,i} = 10N^2K_{z,i} = cN^{2+\frac{1}{2}}$ simulations for Z.

Note that only the required basis functions and simulations used for the approximation of $\overline{q}_i^N$ at time points $t_i \in \overline{\pi}$ depend on the choice of α. Hence, following the notation used in the optimization, we chose the parameters δ_z, $K_{z,i}$,$M_{z,i}$, δ_q, $K_{q,i}$ and $M_{q,i}$ for $t_i \notin \overline{\pi}$ in the SDP algorithm equal to those in the MDP algorithm. In this situation, according to our calculations in Section 1.3, the optimal alpha is given by $\alpha = 4/13$. For the time points in $\overline{\pi}$ we hence have to choose $\overline{\delta_q} = \sqrt{T}/N^{-\frac{1}{2}+\frac{\alpha}{4}}$ what leads to $\overline{K}_{q,i} = cN^{\frac{1}{2}+\frac{\alpha}{4}} = cN^{\frac{11}{26}}$. As number of simulations for the approximation of Q in these time points we choose $M_{q,i} = 10N^{3-2\alpha}\overline{K}_{q,i} = cN^{3-\frac{5}{26}}$.

Results in dimension 1:

The results for the averaged error terms are visualized in Figure 1.1, where we plotted the log-log rates of the quadratic error terms in relation to the number of time points N along with the best-fitted linear regression line. All error terms approximately achieve the expected rate of N^{-1} in both algorithms or outperform them. While the results for the

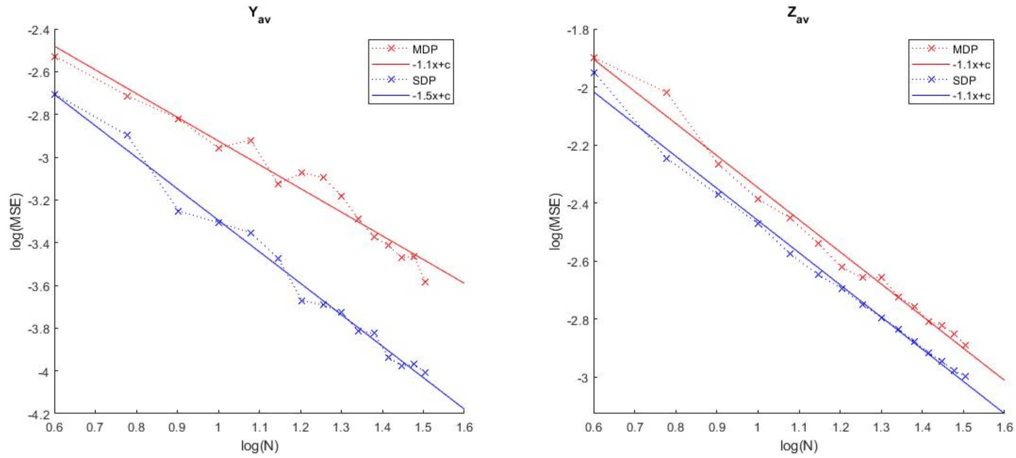

Figure 1.1: Averaged quadratic error in relation to N in dimension $d = 1$

approximation of Z averaged over the time steps are very similar in both algorithms, the results for the averaged error in the approximation of Y differ more.
First, we can notice that the results of the MDP algorithm have a higher variance, which probably occurs since we use the whole sample path of the Brownian motion X in each time step, such that the approximations are influenced by the higher variance of the Brownian motion when sampled over a longer time interval when compared to the simulations of the SDP algorithm.
The second thing to notice here is that the SDP algorithm over-performs in the approximation of Y with a rate of $N^{-1.5}$ instead of the expected N^{-1}. This effect even increases in the quadratic error term of Y at the time $t = 0$ (illustrated in Figure 1.2), where the error decreases like N^{-2}. A possible explanation for this effect at time 0 could be that the approximations used for constructing q_0^M already had a small bias, such that the approximation error of q_0^M mostly depends on the variance. Then q_0^M will have a smaller variance in the SDP algorithm than in the MDP algorithm (due to the dependency on fewer previously constructed approximations) such that more samples as needed are used.

The average computation time required for one run of the algorithms is also plotted in Figure 1.2 in relation to the used time steps N, again in log-log rates. Along with the measured computation time, we plotted the expected rate at which the computation time should increase with N and attached it at the last measured point. First, we can see that, for larger N, the SDP algorithm is faster than the MDP algorithm. In both algorithms the computation costs increase a bit faster than expected for small values of N, but also seem to approach the expected rate more and more for bigger N. This is no surprise, as the computation costs are influenced by several log-factors that were neglected in the optimization:
First, we neglected a cost factor of $\log(N+1)$ from the increasing number of basis functions used. Additionally, at each time, the sample paths have to be sorted into the hypercubes.

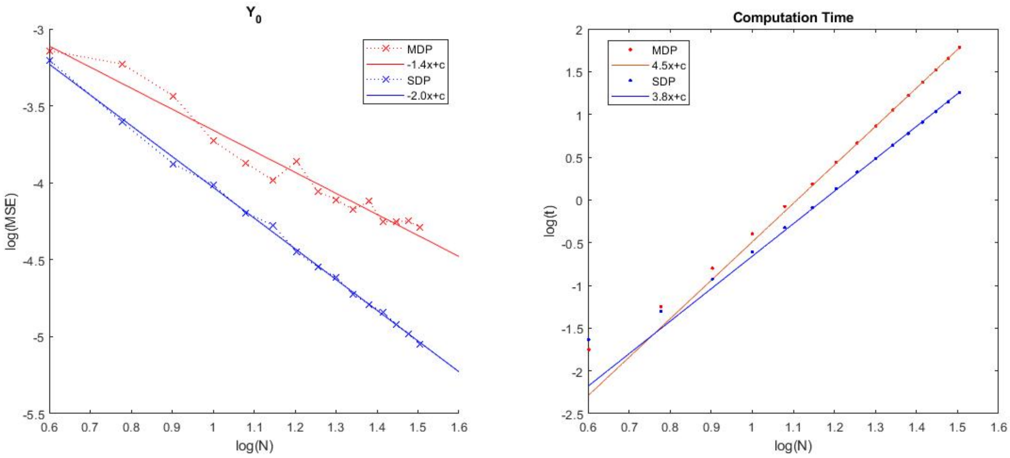

Figure 1.2: Quadratic Error of the Y-approximation at time 0 and computation time in relation to N in dimension $d = 1$

This results in additional costs of order $log(N)^D$ that were neglected in the optimization. These factors still weigh in for the relatively small number of time points $N = 18$ that we used at most. This effect is seen in both algorithms.

Calibration in dimension 2:

We did the same simulations also in dimension 2 to illustrate the results. While we bounded the hypercubes in each dimension as before and chose again piecewise linear functions for the approximation of Y and piecewise constant functions for the one of Z, the optimal choice of the parameters slightly differs.

First, we chose $\delta_z = \sqrt{T}/N^{\frac{1}{2}}$ and $\delta_q = \sqrt{T}/N^{\frac{1}{4}}$ for the MDP algorithm what leads to $K_{z,i} \in \mathcal{O}(\lceil R_i/\delta_z \rceil^D) = \mathcal{O}(N^1)$ and $K_{q,i} = \lceil R_i/\delta_q \in \mathcal{O}(\rceil^D) = \mathcal{O}(N^{\frac{1}{2}})$ at time t_i. Again we re-simulate the sample paths for each approximation and use $M_{q,i} = 10NK_{q,i} = cN^{1+\frac{1}{2}}$ simulations for Q at time t_i and $M_{z,i} = 10N^2K_{z,i} = cN^3$ simulations for Z.

The optimal parameters δ_z, $K_{z,i}$ and $M_{z,i}$ for the SDP algorithm are the same as in the MDP algorithm and so are δ_q and $K_{q,i}$ and $M_{q,i}$ for $t_i \notin \overline{\pi}$. The time grid $\overline{\pi}$ is now defined for $\alpha = \frac{2}{7}$. For the time points in $\overline{\pi}$ we choose $\overline{\delta_q} = 1.5\sqrt{T}/N^{-\frac{1}{2}+\frac{\alpha}{4}}$ what leads to $\overline{K}_{q,i} \in \mathcal{O}(N^{1+\frac{\alpha}{2}})$. As number of simulations for the approximation of Q in these time points we choose $M_{q,i} = 10N^{3-2\alpha}\overline{K}_{q,i} = cN^{4-\frac{5}{2}\alpha}$.

Results in dimension 2:

The resulting averaged error terms are plotted in Figure 1.3 in relation to the used time steps N along with the best fit linear regression line, again in log-log rates. Most observations are similar to those in dimension 1. Firstly, the rate of N^{-1} is achieved in all error terms, and both algorithms perform similarly in the approximation of Z

Also, analog to the results in dimension one, we can observe the results of the algorithms differ more in the approximation of Y. Here the SDP algorithm again performs much

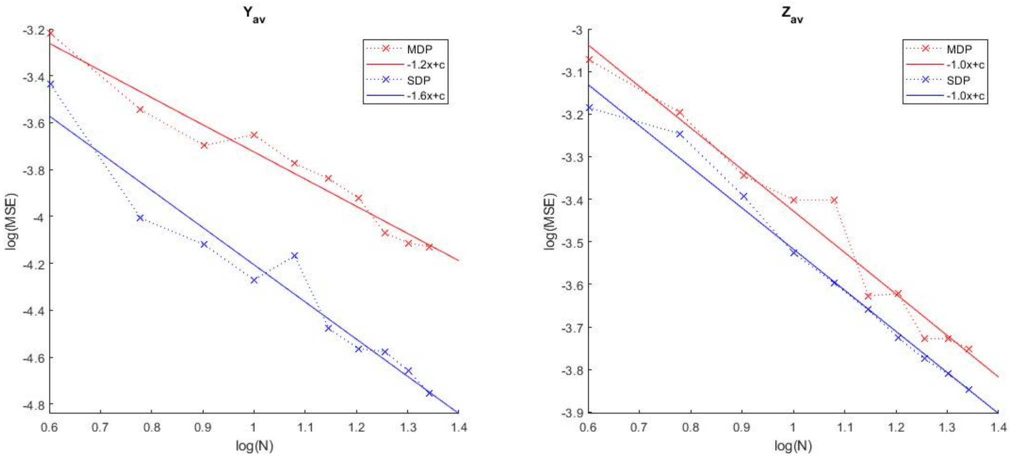

Figure 1.3: Averaged quadratic error in relation to N in dimension $d = 2$

better than the MDP algorithm and outperforms the theoretically expected rate of N^{-1} in the approximation of Y, both at time 0 and on average over the time points. At time 0, this difference is even bigger than in dimension 1.
Also, the results regarding the computation time (plotted in Figure 1.4) are similar to those in dimension 1. Again, the required time in both algorithms increases slightly faster than expected for small N, which can be explained, as mentioned in the one-dimensional case, due to log-factors that were neglected in the optimization and weigh in even more heavily in higher dimensions. Nonetheless, with a higher number of time points N, the rates approach the expected rate more and more such that we can assume convergence at the expected rate for larger numbers of time points N. Again we can observe that the computation time increases in the SDP algorithm, as suggested by the theoretical results, with a slower rate than in the MDP algorithm, such that the SDP algorithm hence becomes faster than the MDP for larger numbers of time points.
Summarizing, we can say that the observations confirm the ones from dimension $d = 1$. Both algorithms achieve a similar convergence rate in the error terms, while the approximations of the SDP algorithm have a smaller variance. Also, when increasing the number of time steps N to achieve more accurate approximations, the SDP algorithm becomes faster when compared to the MDP algorithm. Hence we can say that the SDP algorithm in fact outperforms the MDP algorithm when using a higher number of time points N. Therefore, the SDP algorithm has clear advantages, especially when we are interested in highly accurate approximations which require a large number of steps in the time discretization.

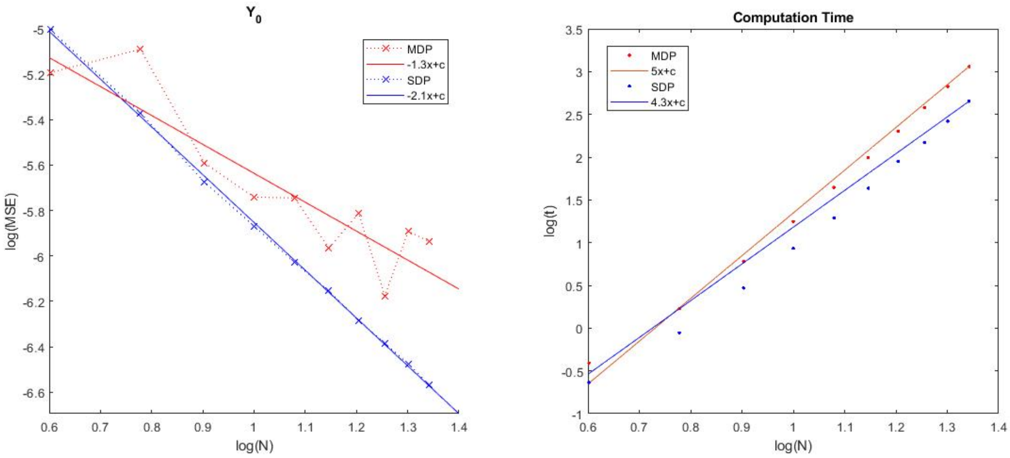

Figure 1.4: Quadratic Error of the Y-approximation at time 0 and computation time in relation to N in dimension $d = 2$

1.5 Error analysis

In this section, we present a complete and detailed error analysis of the SDP algorithm. In the first subsection, we introduce additional notation used in the proofs and present some key tools for the error analysis. The following subsections are then dedicated to the derivation of error bounds for the approximation of $\overline{q}^N$ and $\overline{z}^N$ respectively. We will establish a sort of recursion formula for both parts, allowing us to bound the quadratic error at the time t_i by the one at time t_{i+1} plus an additional driver-dependent term. This illustrates the error propagation between the time steps. We will then derive global bounds for the quadratic error for both approximations before we analyze the driver-dependent terms appearing in both recursions further in Subsection 1.5.4. Finally, in the last subsection, we combine the obtained bounds to derive the result presented in Theorem 1.3.1 and proof the two Theorems 1.3.2 and 1.3.3 by bounding the discretization error under the different assumptions.

1.5.1 Preliminaries and key-tools

In this section, we lay the groundwork for the error analysis by introducing additional objects and presenting key tools for the analysis. The proof of the lemmas in this section is postponed to the appendix.
First, recall the definition of the functions $\overline{q}_i^N$ and $\overline{z}_i^N$. By setting

$$\Xi_{N-1}^N(\underline{x}_{N-1}) := \xi(x_N)$$

$$\Xi_i^N(\underline{x}_i) := \overline{q}_{\tau_\alpha(i)}^N(x_{\tau_\alpha(i)}) + \int_{t_{i+1}}^{t_{\tau_\alpha(i)+1}} f(s, x_s, Y_s, Z_s), ds \qquad i \in \{0, \ldots, N-2\}$$

for any $\underline{x}_i := (x_s)_{t_i \le s \le T} \in \mathbb{R}^{D[t_i,T]}$ it holds

$$\overline{q}_i^N(x) = E\left[\Xi_i^N\left((X_s)_{t_i \le s \le T}\right)\middle| X_{t_i} = x\right]$$

and

$$\overline{z}_i^N(x) = E\left[\frac{\Delta W_{i+1}}{\Delta}\Xi_i^N\left((X_s)_{t_i \le s \le T}\right)\middle| X_{t_i} = x\right].$$

Note that Ξ_i^N differs from $\Xi_i^{N,M}$ in two ways: in the functions $\Xi_i^{N,M}$, the true solution (Y, Z) of the BSDE is replaced by approximations of the algorithm and the integral is discretized. In order to make use of properties of the least squares projection, we need the analogs of the least squares solutions $\varphi^{q^{N,M}}$ and $\varphi^{z^{N,M}}$ based on the functions Ξ^N. Since those depend on the whole path of X rather than just the values on the time grid π additional fictional simulations are required for the theoretical error analysis which motivates the following definition:

Definition 1.5.1. For $i \in \{0, \ldots, N-1\}$, let $\mathcal{S}_i := \{\Delta W_{i+1}^{[i,m]}, X^{[i,m]} : m = 1, \ldots, M\}$ be a cloud of independent random variables defined on the probability space $(\Omega^M, \mathcal{F}^M, P^M)$ with $X^{[i,m]} = (X_s^{[i,m]})_{t_i \le s \le T}$ such that $X^{[i,m]}$ is distributed like a segment of the SDE solution X. Furthermore, we assume that these simulations match the ones used in the SDP algorithm on the time grid π, i.e., $\Delta W_{i+1}^{[i,m]} = \Delta W_{i+1}^{[i,m,N]}$ and $X_{t_j}^{[i,m]} = X_{t_j}^{[i,m,N]}$ for all $i \in \{0, \ldots, N-1\}$, $j \ge i$ and $m \in \{1, \ldots, M_i\}$. Then, for every $\omega \in \Omega^M$, let $\nu_i^M(\omega, .)$ be the measure on $\left((\mathbb{R}^D)^{[t_i,T]} \times \mathbb{R}^{\mathcal{D}}, \mathcal{B}\left((\mathbb{R}^D)^{[t_i,T]} \times \mathbb{R}^{\mathcal{D}}\right)\right)$ defined by

$$\nu_i^M(\omega, x) := \frac{1}{M_i}\sum_{m=1}^{M_i} \delta_{\left(\Delta W_{i+1}^{[i,m]}(\omega), X^{[i,m]}(\omega)\right)}(x)$$

where $\delta_c(.)$ is the Dirac-measure on c.

The calculations in this chapter are done on the probability space $(\Omega^M, \mathcal{F}^M, P^M)$, where we suppose that there exists a $\mathcal{D}$-dimensional Brownian motion W on $(\Omega^M, \mathcal{F}^M, P^M)$, which is independent of all simulations and hence also a copy X of the SDE solution independent of the simulations.
Given these additional random variables, we denote with $\varphi_i^{\overline{q}^N}$ and $\varphi_i^{\overline{z}^N}$ the solutions of the least-squares problems

$$\varphi_i^{\overline{q}^N} := \underset{\psi \in \mathcal{K}_{q,i}}{\operatorname{argmin}}\left(\frac{1}{M}\sum_{m=1}^{M}\left|\psi\left(X_{t_i}^{[i,m]}\right) - \Xi_i^N\left(X^{[i,m]}\right)\right|^2\right)$$

and

$$\varphi_i^{\overline{z}^N} := \underset{\psi \in \mathcal{K}_{z,i}}{\operatorname{argmin}} \left(\frac{1}{M} \sum_{m=1}^{M_i} \left| \psi\left(X_{t_i}^{[i,m]}\right) - \frac{\Delta W_{i+1}^{[i,m]}}{\Delta} \Xi_i^N \left(X^{[i,m]}\right) \right|^2 \right).$$

Remark 1.5.2. Recall the functions $\varphi_i^{q^{N,M}}$ and $\varphi_i^{z^{N,M}}$ from the SDP algorithm. Since it holds by definition of the ghost sample that $X_\pi^{[i,m]} := (X_{t_j}^{[i,m]})_{t_i < t_j \in \pi} = X^{[i,m,N]}$, we have

$$\varphi_i^{q^{N,M}} = \underset{\psi \in \mathcal{K}_{q,i}}{\operatorname{argmin}} \left(\frac{1}{M} \sum_{m=1}^{M} \left| \psi\left(X_{t_i}^{[i,m]}\right) - \Xi_i^{N,M} \left(X_\pi^{[i,m]}\right) \right|^2 \right)$$

and

$$\varphi_i^{z^{N,M}} = \underset{\psi \in \mathcal{K}_{z,i}}{\operatorname{argmin}} \left(\frac{1}{M} \sum_{m=1}^{M_i} \left| \psi\left(X_{t_i}^{[i,m]}\right) - \frac{\Delta W_{i+1}^{[i,m]}}{\Delta} \Xi_i^{N,M} \left(X_\pi^{[i,m]}\right) \right|^2 \right),$$

i.e., for any fixed outcomes of the simulations $X^{[i,m]}$, both pairs of functions $\varphi_i^{q^{N,M}}$ and $\varphi_i^{z^{N,M}}$, and $\varphi_i^{\overline{q}^N}$ and $\varphi_i^{\overline{z}^N}$ solve a least squares problem with respect to the same measure $\nu_i^M(\omega, .)$.

This allows us to utilize the following lemma, which is a key tool in the error analysis. It matches essentially Proposition 4.12 in Gobet and Turkedjiev (2016) where the domain of the function Ξ is generalized in order to cover our setting.

Lemma 1.5.3. *For each $\omega \in \Omega^M$, let $(A, \mathcal{A}, \nu(\omega, .))$ be a measurable space with*

$$\nu(\omega, .) = \frac{1}{M} \sum_{m=1}^{M} \delta_{\chi^{[m]}(\omega)}$$

for i.i.d random variables $\chi^{[1]} \ldots, \chi^{[M]} : \Omega^M \to A$. Furthermore, let $\mathcal{K}$ be a linear function space spanned by $\mathbb{R}^l$-valued basis functions $\{p^k(.), 1 \le k \le K\}$ with $\sum_{k=1}^K E\left[|p^k(\chi^{[m]})|^2\right] < \infty$ for all m. For any $\mathcal{F}^M \otimes \mathcal{A}$-measurable, $\mathbb{R}^l$-valued random variable Ξ with $\Xi(\omega, .) \in L^2(\mathcal{A}, \nu(\omega, .))$ for P^M-a.e. ω, set

$$\varphi(\omega, .) := \underset{\psi \in \mathcal{K}}{\operatorname{arginf}} \frac{1}{M} \sum_{m=1}^{M} \left| \psi\left(\chi^{[m]}(\omega)\right) - \Xi\left(\omega, \chi^{[m]}(\omega)\right) \right|^2.$$

Then:

(i) The mapping $\Xi \mapsto \varphi$ is linear.

(ii) It holds

$$\|\varphi\|_{L^2(\mathcal{A},\nu(\omega,.))} \leq \|\Xi\|_{L^2(\mathcal{A},\nu(\omega,.))}$$

where we denote with $\|.\|_{L^2((\mathcal{A},\nu(\omega,.))}$ *the* L^2*-norm with respect to the measure* $\nu(\omega,.)$*.*

(iii) Suppose there is a sub-σ*-fields* $\mathcal{G}$ *of* $\mathcal{F}^M$ *such that* $\left(p^k(\chi^{[1]}),\ldots,p^k(\chi^{[M]})\right)$ *is* $\mathcal{G}$*-measurable for each* $k=1,\ldots,K$*. Then*

$$E\left[\varphi|\mathcal{G}\right](\omega,.) = \operatorname*{argmin}_{\psi\in\mathcal{K}} \frac{1}{M}\sum_{i=1}^{M}\left|\psi\left(\chi^{[m]}(\omega)\right) - \Xi_{\mathcal{G}}\left(\chi^{[m]}(\omega)\right)\right|^2$$

where $\Xi_{\mathcal{G}}(x) := E\left[\Xi(x)|\mathcal{G}\right]$*.*

(iv) In the situation of (iii), suppose that $\mathcal{G}$ *is given by* $\sigma(g(\chi^{[m]})_{m=1,\ldots,M})$ *for a* $\mathcal{A}$*-measurable function*

$$g : A \to \mathbb{R}^{l'}.$$

Furthermore, assume that there is a sub-σ*-field* $\mathcal{H}$ *independent of* $\sigma((\chi^{[m]})_{m=1\ldots,M})$ *such that* Ξ *is* $\mathcal{H}\otimes\mathcal{A}$*-measurable and that the conditional variance*

$$E\left[\left|\Xi\left(\chi^{[m]}\right) - E\left[\Xi\left(\chi^{[m]}\right)|\mathcal{G}\vee\mathcal{H}\right]\right|^2\Big|\mathcal{G}\vee\mathcal{H}\right]$$

is P^M*-almost surely uniformly bounded by some constant* σ^2 *for all* $m\in\{1,\ldots,M\}$*. Then*

$$E\left[\|\varphi - E\left[\varphi|\mathcal{G}\vee\mathcal{H}\right]\|^2_{L^2(\mathcal{A},\nu(\omega,.))}\Big|\mathcal{G}\vee\mathcal{H}\right] \leq \sigma^2\frac{K}{M}.$$

While Lemma 1.5.3 and the objects defined so far help us to utilize projection properties, we still need tools to handle the dependency on the different sets of simulations used in the algorithm. For this purpose, we first consider the following norms allowing us to distinguish between the dependence on simulations or the actual law of the true SDE solution X more clearly:

Definition 1.5.4. Let $\varphi : \Omega^M \times \mathbb{R}^D \to \mathbb{R}^l$ be $\mathcal{F}^M \times \mathcal{B}(\mathbb{R}^D)$-measurable. For each $i = 0,\ldots,N-1$, define the random norms $\|.\|_{i,\infty}$ and $\|.\|_{i,M}$ via

$$\|\varphi\|^2_{i,\infty} := \int_{\mathbb{R}^D}|\varphi(x)|^2 P_{X_{t_i}}(dx), \qquad \|\varphi\|^2_{i,M} := \frac{1}{M}\sum_{m=1}^{M}\left|\varphi\left(X^{[i,m]}_{t_i}\right)\right|^2$$

where $P_{X_{t_i}}$ denotes the distribution of X_{t_i}.

Note that we are interested in the error with respect to the law of the SDE solution X, i.e., in the difference between the approximations $q_i^{N,M}$ and $z_i^{N,M}$, and the functions $\overline{q}_i^N$ and $\overline{z}_i^N$ respectively in the $\|.\|_{i,\infty}$-norm. The following lemma allows us to lead this difference back to the one in the $\|.\|_{i,M}$-norm that appears naturally in the error analysis due to the use of simulations. It is an adapted version of Proposition 4.10 in Gobet and Turkedjiev (2016) where the analogue result is shown for $\epsilon = 1$ instead of $\epsilon \in (0, 1]$.

Lemma 1.5.5. *It holds for all $i = 0, \ldots, N-1$ and any $\epsilon \in (0, 1]$ that*

$$E\left[\|q_i^{N,M} - \overline{q}_i^N\|_{i,\infty}^2\right] \le (1+\epsilon)E\left[\|q_i^{N,M} - \overline{q}_i^N\|_{i,M}^2\right] + \frac{C_1 K_{q,i}\log(C_2 M_i)}{M_i \epsilon}$$
$$E\left[\|z_i^{N,M} - \overline{z}_i^N\|_{i,\infty}^2\right] \le (1+\epsilon)E\left[\|z_i^{N,M} - \overline{z}_i^N\|_{i,M}^2\right] + \frac{\mathcal{D} C_1 K_{z,i}\log(C_2 M_i)}{\Delta M_i \epsilon}$$

for positive constants C_1, C_2 independent of ϵ, Δ, and M_i.

Finally, the following lemma allows us to reduce the dependency on a sampled path of X to only the value of the sample at one time point t_i. A proof can be found in Chapter 5 of Kallenberg (1997).

Lemma 1.5.6. *Let $\mathcal{G}$ and $\mathcal{H}$ be independent sub-σ-fields of $\mathcal{F}^M$. For $l \ge 1$ let $F : \Omega^M \times \mathbb{R}^D \to \mathbb{R}^l$ be bounded and $\mathcal{G} \times \mathcal{B}(\mathbb{R}^D)$-measurable and $U : \Omega^M \to \mathbb{R}^D$ be $\mathcal{H}$-measurable. Then $E[F(U)|\mathcal{H}] = j(U)$ where $j(x) = E[F(x)]$ for all $x \in \mathbb{R}^D$.*

To see how we can utilize this lemma, note that each sample $X^{[i,m]}$ satisfies

$$X_{t_i+s}^{[i,m]} = X_{t_i}^{[i,m]} + \int_{t_i}^{t_i+s} b(l, X_l)dl + \int_{t_i}^{t_i+s} \sigma(l, X_l)dW_l^{[i,m]}$$

for a Brownian motion $W^{[i,m]}$. Substituting the time variable and setting $\tilde{W}_u = W_{u+t_i}^{[i,m]} - W_{t_i}^{[i,m]}$ leads to

$$X_{t_i+s}^{[i,m]} = X_{t_i}^{[i,m]} + \int_0^s b(u+t_i, X_{u+t_i})du + \int_0^{t-s} \sigma(u+t_i, X_{u+t_i})d\tilde{W}_u$$

which shows that the sample $X^{[i,m]}$ is the solution to a forward SDE starting in t_i with initial value $X_{t_i}^{[i,m]}$ with respect to the Brownian motion $\tilde{W}_u$. Hence, we can express the path $X^{[i,m]}$ as a deterministic function h of $X_{t_i}^{[i,m]}$ and $(\tilde{W}_u)_{t_i \le u \le T}$ only, i.e., we can write $X_s^{[i,m]} = h(s, X_{t_i}^{[i,m]}, (\tilde{W}_u)_{0 \le u \le s-t_i})$ for any $s \ge t_i$. Since the Brownian motion $\tilde{W}$ is independent of

$\sigma(X_{t_i}^{[i,m]})$, we can then use Lemma 1.5.6 on the function

$$
\begin{aligned}
F(x_{t_i}) = &\int_{t_{i+1}}^{t_{\tau(i)+1}} f\left(s, h(s, x_{t_i}, (\tilde{W}_u)_{0\le u\le s-t_i}), Y_s, Z_s\right) ds \\
&- \Delta \sum_{j=i+1}^{\tau(i)} f\left(t_j, h(t_j, x_{t_i}, (\tilde{W}_u)_{0\le u\le t_j-t_i}), q_j^{N,M}\left(h(t_j, x_{t_i}, (\tilde{W}_u)_{0\le u\le t_j-t_i})\right),\right. \\
&\left. z_j^{N,M}\left(h(t_j, x_{t_i}, (\tilde{W}_u)_{0\le u\le t_j-t_i}))\right)\right)
\end{aligned}
$$

and get

$$
\begin{aligned}
&E\left[\Xi_i^N\left(X^{[i,m]}\right) - \Xi_i^{N,M}\left(X^{[i,m]}\right)\middle|\sigma(\mathcal{S}_j : j > i) \vee \sigma\left(X_{t_i}^{[i,m]}\right)\right] \\
&\quad = E\left[\Xi_i^N\left(\underline{X}_{t_i}\right) - \Xi_i^{N,M}\left(\underline{X}_{t_i}\right)\middle|\sigma(\mathcal{S}_j : j > i), X_{t_i} = X_{t_i}^{[i,m]}\right] \qquad (1.3)
\end{aligned}
$$

where we set $\underline{X}_{t_i} = (X_s)_{s\in[t_i,T]}$. Completely analogously, it holds

$$
\begin{aligned}
&E\left[\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\left(\Xi_i^N(X^{[i,m]}) - \Xi_i^{N,M}(X^{[i,m]})\right)\middle|\sigma(\mathcal{S}_j : j > i) \vee \sigma\left(X_{t_i}^{[i,m]}\right)\right] \\
&= E\left[\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\left(\Xi_i^N(\underline{X}_{t_i}) - \Xi_i^{N,M}(\underline{X}_{t_i})\right)\middle|\sigma(\mathcal{S}_j : j > i), X_{t_i} = X_{t_i}^{[i,m]}\right]. \qquad (1.4)
\end{aligned}
$$

Hence, Lemma 1.5.6 allows us to reduce the dependency on a sample path to a conditional expectation according to the actual law of X given the value of the sample path at the current time.

We close this section with some abbreviations for the notation: Throughout the rest of this chapter, we denote with $\mathcal{F}_i^M := \sigma(\mathcal{S}_k : k > i) \vee \sigma(X_{t_i}^{[i,m]} : m = 1, \dots, M_i)$ the σ-field generated by the simulations used up to the time k (backwards starting from N) for a fixed $N \in \mathbb{N}$. With $\mathcal{F}_i := \sigma((W_s)_{0\le s\le t_i})$ we denote the σ-field generated by the Brownian motion which is independent of the simulations. The conditional expectations given those σ-fields we denote with $E_i^M[.] = E[.|\mathcal{F}_i^M]$ and $E_i[.] := E[.|\mathcal{F}_i]$. Additionally, we shorten the notation of the driver f by dropping clearly indicated function arguments through the notation $f(t, x, g, g') := f(t, x, g(x), g'(x))$ for any functions $g, g' : \mathbb{R}^D \to \mathbb{R}^l$ and $f(t, x, g_t, g'_t) := f(t, x, (g(t, x), g'(t, x))$ for any functions $g, g' : [0, T] \times \mathbb{R}^D \to \mathbb{R}^l$.

With these considerations we are ready for the derivation of the error bounds.

1.5.2 Error of the Q approximation

In this section, we analyze the expected quadratic error of the approximation of $\overline{q}^N$ in the form of the terms

$$E\left[\|q_i^{N,M} - \overline{q}_i^N\|_{i,M}^2\right].$$

We first establish a bound on the error propagation between the time steps, allowing us to express the expected difference of this term by the one in the next time step. Afterward, we derive a local error bound for the term as well as a bound for the global error by bounding the maximum of the quadratic error terms over all time points $t_i \in \pi$. The first part of the proof, in which we establish the error propagation, is inspired by the error analysis of the MDP scheme in Gobet and Turkedjiev (2016).

Error propagation: First, note that

$$\begin{aligned} E_i^M\left[\Xi_i^N\left(X^{[i,m]}\right)\right] &= E_i^M\left[\overline{q}^N_{\tau_\alpha(i)}\left(X^{[i,m]}_{t_{\tau_\alpha(i)}}\right) + \int_{t_{i+1}}^{t_{\tau_\alpha(i)+1}} f\left(s, X_s^{[i,m]}, Y_s, Z_s\right) ds\right] \\ &= E\left[\overline{q}^N_{\tau_\alpha(i)}\left(X_{t_{\tau_\alpha(i)}}\right) + \int_{t_{i+1}}^{t_{\tau_\alpha(i)+1}} f\left(s, X_s, Y_s, Z_s\right) ds \middle| X_{t_i} = X^{[i,m]}_{t_i}\right] \\ &= \overline{q}_i^N\left(X^{[i,m]}_{t_i}\right). \end{aligned}$$

Hence, it holds by Lemma 1.5.3 (iii) that $E_i^M[\varphi_i^{\overline{q}^N}]$ is the least squares projection of $\overline{q}_i^N$ on the subspace $\mathcal{K}_{q,i}$ with respect to the measure ν_i^M, i.e., it holds

$$E_i^M\left[\varphi_i^{\overline{q}^N}\right] = \operatorname*{arginf}_{\psi \in \mathcal{K}_{q,i}} \left(\frac{1}{M}\sum_{i=1}^M \left|\psi\left(X^{[i,m]}_{t_i}\right) - \overline{q}_i^N\left(X^{[i,m]}_{t_i}\right)\right|^2\right).$$

Therefore, by the properties of least square projections, $\overline{q}_i^N - E_i^M[\varphi_i^{\overline{q}^N}]$ is orthogonal on $E_i^M[\varphi_i^{\overline{q}^N}] - \psi_i^{q^{N,M}}$ with respect to ν_i^M. Additionally, note that, since ξ and f are bounded due to the assumptions (A_ξ) and (A_f), it follows by a simple backward recursion that

$$|\overline{q}_i^N(x)| \le C_{q,i} = C_\xi + (T - t_{i+1})C_f \le C_q := C_\xi + TC_f$$

for all $x \in \mathbb{R}^D$. We conclude that $\mathcal{T}_{C_{q,i}}(\overline{q}_i^N(x)) = \overline{q}_i^N(x)$ and obtain:

$$\begin{aligned}
E\left[\|\overline{q}_i^N(.) - q_i^{N,M}(.)\|_{i,M}^2\right] &= E\left[\left\|\mathcal{T}_{C_{q,i}}\left(\overline{q}_i^N(.)\right) - \mathcal{T}_{C_{q,i}}\left(\varphi_i^{q^{N,M}}(.)\right)\right\|_{i,M}^2\right] \\
&\leq E\left[\left\|\overline{q}_i^N(.) - E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right] + E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right] - \varphi_i^{q^{N,M}}(.)\right\|_{i,M}^2\right] \\
&= E\left[\left\|\overline{q}_i^N(.) - E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right]\right\|_{i,M}^2\right] + E\left[\left\|E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right] - \varphi_i^{q^{N,M}}(.)\right\|_{i,M}^2\right] \\
&\leq E\left[\left\|\overline{q}_i^N(.) - E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right]\right\|_{i,M}^2\right] \\
&\quad + (1+\kappa)E\left[\left\|E_i^M\left[\varphi_i^{q^{N,M}}(.) - \varphi_i^{\overline{q}^N}(.)\right]\right\|_{i,M}^2\right] \\
&\quad + (1+\kappa^{-1})E\left[\left\|E_i^M\left[\varphi_i^{q^{N,M}}(.)\right] - \varphi_i^{q^{N,M}}(.)\right\|_{i,M}^2\right]
\end{aligned}$$

where κ is an arbitrary positive constant. We handle the appearing summands separately: As argued before, it holds

$$E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right] = \operatorname*{arginf}_{\psi \in \mathcal{K}_{q,i}} \|\overline{q}_i^N(.) - \psi(.)\|_{i,M}^2.$$

Hence,

$$\begin{aligned}
E\left[\left\|\overline{q}_i^N(.) - E_i^M\left[\varphi_i^{\overline{q}^N}(.)\right]\right\|_{i,M}^2\right] &= E\left[\inf_{\psi \in \mathcal{K}_{q,i}} \left\|\overline{q}_i^N(.) - \psi(.)\right\|_{i,M}^2\right] \\
&\leq \inf_{\psi \in \mathcal{K}_{q,i}} \frac{1}{M}\sum_{i=1}^M E\left[\left|\overline{q}_i^N\left(X_{t_i}^{[i,m]}\right) - \psi\left(X_{t_i}^{[i,m]}\right)\right|^2\right] \\
&= \inf_{\psi \in \mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right].
\end{aligned}$$

This term describes the best approximation error due to the projection on the subspace $\mathcal{K}_{q,i}$ and is part of the final error representation.
For the next term, note that $\Xi_i^{N,M}$ is bounded by $C_{q,i}$ under assumptions (A_ξ) and (A_f), since the approximations $q_i^{N,M}$ are (due to the truncation in the algorithm). Additionally, for each i, the function $\Xi_i^{N,M}$ is built using only the simulations in the sets $\mathcal{S}_k$ for $k > i$. Hence, it follows directly by Lemma 1.5.3 (iv) with $\mathcal{H} = \sigma(\mathcal{S}_k, k > i)$ and $g(X^{[i,m]}) = X_{t_i}^{[i,m]}$ that

$$E\left[\left\|E_i^M\left[\varphi_i^{q^{N,M}}(.)\right] - \varphi_i^{q^{N,M}}(.)\right\|_{i,M}^2\right] \leq C_{q,i}^2 \frac{K_{q,i}}{M_i}.$$

Since, as argued before, the functions $\overline{q}_i^N$ are bounded by $C_{q,i}$ as well, we can apply a similar argument to the functions Ξ_i^N. Those are again built using only the simulations in $\mathcal{S}_k$ for $k > i$. Setting $\xi_i^q(x) := E[\Xi_i^{N,M}(\underline{X}_{t_i}) - \Xi_i^N(\underline{X}_{t_i})|X_{t_i} = x, \mathcal{F}_0^M]$, we have by (1.3) that

$$E_i^M \left[\Xi_i^{N,M}(X^{[i,m]}) - \Xi_i^N(X^{[i,m]})\right] = \xi_i^q(X_i^{[i,m]}).$$

Then Lemma 1.5.3 (i) and (iii) imply that

$$E_i^M \left[\varphi_i^{q^{N,M}}(.) - \varphi_i^{\overline{q}^N}(.)\right] = \operatorname*{arginf}_{\psi \in \mathcal{K}_{q,i}} \left(\frac{1}{M} \sum_{i=1}^M \left| \psi\left(X_{t_i}^{[i,m]}\right) - \xi_i^q\left(X_{t_i}^{[i,m]}\right) \right|^2 \right).$$

Hence, by Lemma 1.5.3 (ii) it holds

$$E\left[\left\| E_i^M \left[\varphi_i^{q^{N,M}}(.) - \varphi_i^{\overline{q}^N}(.)\right] \right\|_{i,M}^2\right] \leq E\left[\left\|\xi_i^q(.)\right\|_{i,M}^2\right] = E\left[\xi_i^q\left(X_{t_i}\right)^2\right].$$

Plugging in the estimates derived so far we have:

$$\begin{aligned} E\left[\left\|\overline{q}_i^N(.) - q_i^{N,M}(.)\right\|_{i,M}^2\right] &\leq \inf_{\psi \in \mathcal{K}_{q,i}} E\left[\left(\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right)^2\right] \\ &\quad + (1+\kappa^{-1})C_{q,i}^2 \frac{K_{q,i}}{M_i} + (1+\kappa)E\left[\xi_i^q(X_{t_i})^2\right]. \end{aligned} \tag{1.5}$$

To further estimate $E[\xi_i^q(X_{t_i})^2]$, we have to distinguish between the time points. If the time points t_i and t_{i+1} are in the same segment defined by $\overline{\pi}_\alpha$, i.e., at time points t_i such that $t_{i+1} \notin \overline{\pi}_\alpha$, it holds $\tau_\alpha(i) = \tau_\alpha(i+1)$ by our choice of τ_α and hence:

$$\begin{aligned} \Xi_i^{N,M}\left(\underline{X}_{t_i}\right) &= q_{\tau_\alpha(i)}^{N,M}\left(X_{t_{\tau_\alpha(i)}}\right) + \sum_{j=i+1}^{\tau_\alpha(i)} \Delta f\left(t_j, X_{t_j}, q_j^{N,M}, z_j^{N,M}\right) \\ &= \Xi_{i+1}^{N,M}\left(\underline{X}_{t_{i+1}}\right) + \Delta f\left(t_{i+1}, X_{t_{i+1}}, q_{i+1}^{N,M}, z_{i+1}^{N,M}\right) \end{aligned}$$

and

$$\begin{aligned} \Xi_i^N\left(\underline{X}_{t_i}\right) &= \overline{q}_{\tau_\alpha(i)}^N\left(X_{t_{\tau_\alpha(i)}}\right) + \int_{t_{i+1}}^{t_{\tau_\alpha(i)+1}} f\left(t, X_t, Y_t, Z_t\right) dt \\ &= \Xi_{i+1}^N\left(\underline{X}_{t_{i+1}}\right) + \int_{t_{i+1}}^{t_{i+2}} f\left(t, X_t, Y_t, Z_t\right) dt. \end{aligned}$$

This allows us to estimate $E[\xi_i^q(X_{t_i})^2]$ as:

$$\begin{aligned}
&E\left[\xi_i^q(X_{t_i})^2\right] \\
&= E\Bigg[E\Bigg[\overline{q}^N_{\tau_\alpha(i)}\left(X_{t_{\tau_\alpha(i)}}\right) - q^{N,M}_{\tau_\alpha(i)}\left(X_{t_{\tau_\alpha(i)}}\right) \\
&\quad + \sum_{j=i+1}^{\tau_\alpha(i)} \int_{t_j}^{t_{j+1}} f\left(s, X_s, Y_s, Z_s\right) - f\left(t_j, X_{t_j}, q_j^{N,M}, z_j^{N,M}\right) ds \Big| \mathfrak{F}_0^M, X_{t_i}\Bigg]^2\Bigg] \\
&\leq (1+\Gamma\Delta) E\Bigg[E\Big[E\Big[\Xi^N_{i+1}(\underline{X}_{t_{i+1}}) - \Xi^{N,M}_{i+1}(\underline{X}_{t_{i+1}})\Big| \mathfrak{F}_0^M, X_{t_i}, X_{t_{i+1}}\Big]\Big|\mathfrak{F}_0^M, X_{t_i}\Big]^2\Bigg] \\
&\quad + (1+\frac{1}{\Gamma\Delta}) E\Bigg[E\Big[\int_{t_{i+1}}^{t_{i+2}} f\left(s, X_s, Y_s, Z_s\right) - f\left(t_{i+1}, X_{t_{i+1}}, q_{i+1}^{N,M}, z_{i+1}^{N,M}\right) ds\Big|\mathfrak{F}_0^M, X_{t_i}\Big]^2\Bigg] \\
&\leq (1+\Gamma\Delta) E\left[\left(\xi^q_{i+1}(X_{t_{i+1}})\right)^2\right] \\
&\quad + (1+\frac{1}{\Gamma\Delta}) E\Bigg[E\Big[\int_{t_{i+1}}^{t_{i+2}} f\left(s, X_s, Y_s, Z_s\right) - f\left(t_{i+1}, X_{t_{i+1}}, q_{i+1}^{N,M}, z_{i+1}^{N,M}\right) ds\Big|\mathfrak{F}_0^M, X_{t_i}\Big]^2\Bigg].
\end{aligned}$$

Here we first used Hölder's inequality with some constant $\Gamma > 0$ that will be specified later and the tower property of the conditional expectation in the first inequality. Then in the second step, we used Jensen's inequality, the Markov property of X and once more the tower property of the conditional expectation. The calculation shows that the expected error term $E[\xi_i^q(X_i)^2]$ is bounded by the one in the next time step and a driver-dependent term.

At time points at the end of a segment, i.e., for time points t_i such that $t_{i+1} \in \overline{\pi}_\alpha$, the function $\Xi_i^{N,M}$ and $\Xi_{i+1}^{N,M}$ are defined on different segments and hence the first inequality in the calculations above does not hold true in this case. However, to get a similar bound, we can once more use Hölder's inequality and a zero addition to get

$$
\begin{aligned}
&E\left[(\xi_i^q(X_{t_i}))^2\right] \\
&= E\Bigg[E\Big[\overline{q}^N_{\tau_\alpha(i)}\left(X_{t_{\tau_\alpha(i)}}\right) - q^{N,M}_{\tau_\alpha(i)}\left(X_{t_{\tau_\alpha(i)}}\right) \\
&\quad + \int_{t_{i+1}}^{t_{i+2}} f\left(s, X_s, Y_s, Z_s\right) - f\left(t_{i+1}, X_{t_{i+1}}, q^{N,M}_{i+1}, z^{N,M}_{i+1}\right) ds\Big|\mathcal{F}_0^M, X_{t_i}\Big]^2\Bigg] \\
&\leq (1+\Gamma\Delta)E\Big[\|\overline{q}^N_{i+1} - q^{N,M}_{i+1}\|^2_{i+1,\infty}\Big] \\
&\quad + (1+\frac{1}{\Gamma\Delta})E\Bigg[E\Big[\int_{t_{i+1}}^{t_{i+2}} f\left(s, X_s, Y_s, Z_s\right) - f\left(t_{i+1}, X_{t_{i+1}}, q^{N,M}_{i+1}, z^{N,M}_{i+1}\right) ds\Big|\mathcal{F}_0^M, X_{t_i}\Big]^2\Bigg] \\
&\leq (1+\Gamma\Delta)(1+\kappa)(1+\epsilon)E\left[\left(\xi^q_{i+1}(X_{t_{i+1}})\right)^2\right] \\
&\quad + (1+\frac{1}{\Gamma\Delta})E\Bigg[E\Big[\int_{t_{i+1}}^{t_{i+2}} f\left(s, X_s, Y_s, Z_s\right) - f\left(t_{i+1}, X_{t_{i+1}}, q^{N,M}_{i+1}, z^{N,M}_{i+1}\right) ds\Big|\mathcal{F}_0^M, X_{t_i}\Big]^2\Bigg] \\
&\quad + (1+\Gamma\Delta)\left(E\left[\|\overline{q}^N_{i+1} - q^{N,M}_{i+1}\|^2_{i+1,\infty}\right] - (1+\kappa)(1+\epsilon)E\left[\left(\xi^q_{i+1}(X_{t_{i+1}})\right)^2\right]\right)_+
\end{aligned}
$$

with positive constants κ and ϵ, which we will specify later. Again we have bounded the error term $E[\xi_i^q(X_{t_i})^2]$ by the one at the next time point and a driver-dependent term, but now with an additional error term that depends on the approximation of $\overline{q}^N$ at the time point t_{i+1}. This additional error term could be expected since the SDP algorithm works on the segment containing t_i like the MDP scheme where the correct terminal condition of the BSDE restricted to the corresponding time segment has been replaced by the approximation $q^N_{\tau_\alpha(i)}$ at the time point $\tau_\alpha(i) = t_{i+1}$.
Local and global bounds:
Iterating this step leads to the following local and global bounds for the quadratic error of the approximation of $\overline{q}^N$, that are stated in terms of the expectations $E[\xi_i^q(X_i)^2]$ for later use. To obtain a corresponding bound for the terms $E[\|\overline{q}_i^N - q_i^{N,M}\|^2_{i,M}]$, one can simply follow the arguments that lead to (1.5) and apply the lemma afterwards.

Lemma 1.5.7. *For a positive constant* Γ, *set* $\lambda_i := (1+\Gamma\Delta)^i(1+N^{\alpha-1})^{2|\{j\leq i: t_j\in\overline{\pi}_\alpha\}|}$ *for*

$i \in \{0, \dots, N\}$. *It then holds under the standing assumptions that*

$$
\begin{aligned}
&E\left[(\xi_i^q(X_{t_i}))^2\right] \\
&\leq \lambda_i E\left[(\xi_i^q(X_{t_i}))^2\right] \\
&\leq (1+\frac{1}{\Delta\Gamma})\sum_{j=i}^{N-2}\lambda_j E\left[E\left[\int_{t_{j+1}}^{t_{j+2}} f(s, X_s, Y_s, Z_s) - f\left(t_{j+1}, X_{t_{j+1}}, q_{j+1}^{N,M}, z_{j+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_j}\right]^2\right] \\
&\quad + \lceil N^{1-\alpha}\rceil \lambda_N \sup_{j\in\mathcal{J}}\left(E\left[\|\overline{q}_j^N - q_j^{N,M}\|_{j,\infty}^2\right] - (1+N^{\alpha-1})^2 E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\right)_+
\end{aligned}
$$

for all $i \in \{0, \dots, N-2\}$ *and*

$$
\begin{aligned}
&\max_{0\leq i\leq N-1} E\left[(\xi_i^q(X_{t_i}))^2\right] \\
&\leq \sum_{j=0}^{N-2}(1+\frac{1}{\Delta\Gamma})\lambda_j E\left[E\left[\int_{t_{j+1}}^{t_{j+2}} f(s, X_s, Y_s, Z_s) - f\left(t_{j+1}, X_{t_{j+1}}, q_{j+1}^{N,M}, z_{j+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_j}\right]^2\right] \\
&\quad + \lceil N^{1-\alpha}\rceil \lambda_N \max_{j\in\mathcal{J}}\left((1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,j}} E\left[\left|\overline{q}_j^N(X_{t_j}) - \psi(X_{t_j})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,j}^2 K_{q,j}}{M_j}\right)\right. \\
&\quad \left. + \frac{N^{1-\alpha} C_1 K_{q,j}\log(C_2 M_j)}{M_j}\right).
\end{aligned}
$$

where $\mathcal{J} := \{i : t_i \in \overline{\pi}_\alpha\}$.

Proof. Iterating the previous calculations where we choose $\kappa = \epsilon = N^{\alpha-1}$ yields

$$
\begin{aligned}
&E\left[(\xi_i^q(X_{t_i}))^2\right] \leq \lambda_i E\left[(\xi_i^q(X_{t_i}))^2\right] \\
&\leq \lambda_{i+1} E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right] \\
&\quad + \lambda_i(1+\frac{1}{\Delta\Gamma})E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f(s, X_s, Y_s, Z_s) - f\left(t_{i+1}, X_{t_{i+1}}, q_{i+1}^{N,M}, z_{i+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\quad + \lambda_{i+1}\left(E\left[\|\overline{q}_{i+1}^N - q_{i+1}^{N,M}\|_{i+1,\infty}^2\right] - (1+N^{\alpha-1})^2 E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right]\right)_+ \mathbb{1}_{\mathcal{J}}(i+1) \\
&\leq \lambda_{N-1} E\left[\left(\xi_{N-1}^q(X_{t_{N-1}})\right)^2\right] \\
&\quad + (1+\frac{1}{\Delta\Gamma})\sum_{j=i}^{N-2}\lambda_j E\left[E\left[\int_{t_{j+1}}^{t_{j+2}} f(s, X_s, Y_s, Z_s) - f\left(t_{j+1}, X_{t_{j+1}}, q_{j+1}^{N,M}, z_{j+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_j}\right]^2\right] \\
&\quad + \lceil N^{1-\alpha}\rceil \lambda_N \max_{j\in\mathcal{J}}\left(E\left[\|\overline{q}_j^N - q_j^{N,M}\|_{j,\infty}^2\right] - (1+N^{\alpha-1})^2 E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\right)_+.
\end{aligned}
$$

Then we have by definition $E[\xi_{N-1}^q(X_{t_{N-1}})^2] = 0$ since $\Xi_{N-1}^{N,M} = \Xi_{N-1}^N$ and the recursion terminates. This already proofs the first statement of Lemma 1.5.7. Additionally, using Lemma 1.5.5 with $\epsilon = N^{\alpha-1}$ and the inequality (1.5) for $\kappa = N^{\alpha-1}$ we have for any j:

$$\begin{aligned}
&\left(E\left[\|\overline{q}_j^N - q_j^{N,M}\|_{j,\infty}^2\right] - (1+N^{\alpha-1})^2 E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\right)_+ \\
&\leq \Bigg((1+N^{\alpha-1})E\left[\|\overline{q}_j^N - q_j^{N,M}\|_{j,M}^2\right] + \frac{N^{1-\alpha}C_1K_{q,j}\log(C_2M_j)}{M_j} \\
&\quad - (1+N^{\alpha-1})^2 E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\Bigg)_+ \\
&\leq \Bigg((1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,j}} E\left[\left|\overline{q}_j^N(X_{t_j}) - \psi(X_{t_j})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,j}^2K_{q,j}}{M_j}\right. \\
&\quad \left. +(1+N^{\alpha-1})E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\right) + \frac{N^{1-\alpha}C_1K_{q,j}\log(C_2M_j)}{\epsilon M_j} - (1+N^{\alpha-1})^2 E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\Bigg)_+ \\
&\leq (1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,j}} E\left[\left|\overline{q}_j^N(X_{t_j}) - \psi(X_{t_j})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,j}^2K_{q,j}}{M_j}\right) + \frac{N^{1-\alpha}C_1K_{q,j}\log(C_2M_j)}{M_j}.
\end{aligned} \tag{1.6}$$

The second statement of Lemma 1.5.7 then follows by plugging in the estimate above in the first statement and taking the maximum over all time points. □

1.5.3 Error of the Z approximation

Analogously to the previous section, we now analyze the quadratic error of the approximation of $\overline{z}^N$ via the terms $E[\|\overline{z}_i^N - z_i^{N,M}\|_{i,M}^2]$. Again, we first establish a bound on the error propagation between the time steps before deriving global bounds.

Error propagation: While the later steps require changes, we can get an analog of the inequality (1.5) by applying the same arguments as before. It holds that

$$E_i^M\left[\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\Xi_i^N\left(X^{[i,m]}\right)\right] = \overline{z}_i^N\left(X_{t_i}^{[i,m]}\right)$$

and therefore, we have by by Lemma 1.5.3 (iii)

$$E_i^M\left[\varphi_i^{\overline{z}^N}\right] = \operatorname*{arginf}_{\psi\in\mathcal{K}_{z,i}}\left(\frac{1}{M}\sum_{i=1}^{M}\left|\psi\left(X_{t_i}^{[i,m]}\right) - \frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\overline{z}_i^N\left(X_{t_i}^{[i,m]}\right)\right|^2\right). \tag{1.7}$$

We conclude that $\overline{z}_i^N - E_i^M[\varphi_i^{\overline{z}^N}]$ is orthogonal on $E_i^M[\varphi_i^{\overline{z}^N}] - \varphi_i^{z^{N,M}}$ with respect to $\|.\|_{i,M}$. Additionally, since $|\Xi_i^N| \leq C_{q,i}$ due to assumptions (A_ξ) and (A_f), it holds

$$\left|\overline{z}_i^{N,(d)}(x)\right| \leq C_{z,i} = \frac{C_{q,i}}{\Delta}$$

for each component $\overline{z}_i^{N,(d)}$, $d = 1, \dots, \mathcal{D}$ of $\overline{z}_i^N$, $x \in \mathbb{R}^D$ and $i \in \{0, \dots, N-1\}$. With that, we obtain for an arbitrary $\kappa > 0$ that

$$\begin{aligned}
\Delta E\left[\|\overline{z}_i^N - z_i^{N,M}\|_{i,M}^2\right] &= \Delta E\left[\|\mathcal{T}_{C_{z,i}}\left(\overline{z}_i^N(.)\right) - \mathcal{T}_{C_{z,i}}\left(\varphi_i^{z^{N,M}}(.)\right)\|_{i,M}^2\right] \\
&\leq \Delta E\left[\|\overline{z}_i^N(.) - E_i^M\left[\varphi_i^{\overline{z}^N}(.)\right] + E_i^M\left[\varphi_i^{\overline{z}^N}(.)\right] - \varphi_i^{z^{N,M}}(.)\|_{i,M}^2\right] \\
&= \Delta\left(E\left[\|\overline{z}_i^N(.) - E_i^M\left[\varphi_i^{\overline{z}^N}(.)\right]\|_{i,M}^2\right] + E\left[\|E_i^M\left[\varphi_i^{\overline{z}^N}(.)\right] - \varphi_i^{z^{N,M}}(.)\|_{i,M}^2\right]\right) \\
&\leq \Delta\left(E\left[\|\overline{z}_i^N(.) - E_i^M\left[\varphi_i^{\overline{z}^N}(.)\right]\|_{i,M}^2\right]\right. \\
&\quad + (1+\kappa^{-1}) + E\left[\|E_i^M\left[\varphi_i^{z^{N,M}}(.)\right] - \varphi_i^{z^{N,M}}(.)\|_{i,M}^2\right] \\
&\quad \left. +(1+\kappa)E\left[\|E_i^M\left[\varphi_i^{z^{N,M}}(.) - \varphi_i^{\overline{z}^N}(.)\right]\|_{i,M}^2\right]\right).
\end{aligned}$$

Once more we handle the appearing terms separately:
First, analog as for the corresponding term in in the previous section, it follows due to equation (1.7) that

$$E[\|\overline{z}_i^N(.) - E_i^M[\varphi_i^{\overline{z}^N}(.)]\|_{i,M}^2] \leq \inf_{\psi \in \mathcal{K}_{z,i}} E[|\psi(X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2],$$

which describes the best approximation error of $\overline{z}^N$ using the basis functions and is part of the final error representation.
For the next term, note again that $\Xi_i^{N,M}$ is bounded by $C_{q,i}$ for all $i \in \{0, \dots, N-1\}$. We conclude that

$$\begin{aligned}
E_i^M\left[\left|\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\Xi_i^{N,M}(X^{[i,m]}) - E_i^M\left[\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\Xi_i^{N,M}(X^{[i,m]})\right]\right|^2\right] &\leq E_i^M\left[\left|\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\Xi_i^{N,M}(X^{[i,m]})\right|^2\right] \\
&\leq \frac{\mathcal{D}C_{q,i}^2}{\Delta}.
\end{aligned}$$

Then, since $\Xi_i^{N,M}$ is built using only simulations of the clouds $\mathcal{S}_k$ for $k > i$, it follows by Lemma 1.5.3 (iv) that $E[\|E_i^M[\varphi_i^{z^{N,M}}] - \varphi_i^{z^{N,M}}\|_{i,M}^2]$ is bounded by $C_{q,i}^2 \frac{\mathcal{D}K_{z,i}}{\Delta M_i}$.
For the last term, we have by (1.4) that

$$E_i^M\left[\frac{\Delta W_{i+1}^{[i,m]}}{\Delta}\left(\Xi_i^{N,M}(X^{[i,m]}) - \Xi_i^N(X^{[i,m]})\right)\right] = \xi_i^z(X_{t_i}^{[i,m]})$$

with

$$\xi_i^z(x) = E\left[\frac{\Delta W_{i+1}}{\Delta}\left(\Xi_i^N(\underline{X}_{t_i}) - \Xi_i^{N,M}(\underline{X}_{t_i})\right)\middle| X_{t_i} = x, \mathcal{F}_0^M\right].$$

Then, by Lemma 1.5.3 (i) and (iii), it follows that

$$E_i^M\left[\varphi_i^{z^{N,M}}(.)-\varphi_i^{\overline{z}^N}(.)\right]=\operatorname*{arginf}_{\psi\in\mathcal{K}_{z,i}}\left(\frac{1}{M}\sum_{i=1}^{M}\left|\psi\left(X_{t_i}^{[i,m]}\right)-\xi_i^z\left(X_{t_i}^{[i,m]}\right)\right|^2\right).$$

Hence, by Lemma 1.5.3 (ii) we have

$$E\left[\left\|E_i^M\left[\psi_i^{z^{N,M}}-\psi_i^{\overline{z}^N}\right]\right\|_{i,M}^2\right]\le E\left[\left\|\xi_i^z\right\|_{i,M}^2\right]=E\left[(\xi_i^z(X_{t_i}))^2\right].$$

Plugging in the estimates obtained so far we have

$$\begin{aligned}\Delta E\left[\left\|\overline{z}_i^N-z_i^{N,M}\right\|_{i,M}^2\right]\le\Delta\Bigg(&\inf_{\psi\in\mathcal{K}_{z,i}}E\left[\left|\psi\left(X_{t_i}\right)-\overline{z}_i^N\left(X_{t_i}\right)\right|^2\right]\\&+(1+\kappa^{-1})C_{q,i}^2\frac{\mathcal{D}K_{z,i}}{\Delta M_i}+(1+\kappa)E\left[(\xi_i^z(X_{t_i}))^2\right]\Bigg).\end{aligned}\tag{1.8}$$

Now using the tower property and a zero addition, we get

$$\begin{aligned}\Delta E\left[(\xi_i^z(X_{t_i}))^2\right]&=\Delta E\left[E\left[\frac{\Delta W_{i+1}}{\Delta}\left(\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\right)\Big|\mathcal{F}_0^M,X_{t_i}\right]^2\right]\\&=\Delta E\left[E\left[\frac{\Delta W_{i+1}}{\Delta}E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,\mathcal{F}_{t_{i+1}}\right]\Big|\mathcal{F}_0^M,X_{t_i}\right]^2\right]\\&=\Delta E\Bigg[E\Bigg[\frac{\Delta W_{i+1}}{\Delta}\Big(E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,\mathcal{F}_{t_{i+1}}\right]\\&\qquad-E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,X_{t_i}\right]\Big)\Big|\mathcal{F}_0^M,X_{t_i}\Bigg]^2\Bigg]\\&\le\mathcal{D}E\Bigg[E\Bigg[\Big(E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,\mathcal{F}_{t_{i+1}}\right]\\&\qquad-E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,X_{t_i}\right]\Big)^2\Big|\mathcal{F}_0^M,X_{t_i}\Bigg]\Bigg]\\&\le\mathcal{D}E\left[E\left[E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,\mathcal{F}_{t_{i+1}}\right]^2\Big|\mathcal{F}_0^M,X_{t_i}\right]\\&\qquad-\mathcal{D}E\left[\Xi_i^{N,M}(\underline{X}_{t_i})-\Xi_i^N(\underline{X}_{t_i})\Big|\mathcal{F}_0^M,X_{t_i}\right]^2\right].\end{aligned}$$

For the next step we have to distinguish between the time points again. If t_i and t_{i+1} are in the same segment, i.e., at all time points t_i such that $t_{i+1}\notin\overline{\pi}_\alpha$, it holds $\tau_\alpha(i)=\tau_\alpha(i+1)$

and we get for a $\Gamma > 0$ which will be specified later that

$$\begin{aligned}
&\Delta E\left[(\xi_i^z(X_{t_i}))^2\right] \\
&\leq \mathcal{D}E\left[E\left[\overline{q}_{\tau_\alpha(i)}(X_{t_{\tau_\alpha(i)}}) - q^{N,M}_{\tau_\alpha(i)}(X_{t_{\tau_\alpha(i)}}) + \sum_{j=i+1}^{\tau_\alpha(i)} \int_{t_j}^{t_{j+1}} f(s, X_s, Y_s, Z_s)\right.\right. \\
&\quad \left.\left. -f\left(t_j, X_{t_j}, q_j^{N,M}, z_j^{N,M}\right) ds \middle| \mathcal{F}_0^M, \mathcal{F}_{t_{i+1}}\right]^2\right] - \mathcal{D}E\left[E\left[\Xi_i^{N,M}(\underline{X}_i) - \Xi_i^N(\underline{X}_i)\middle|\mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\leq (1+\Gamma\Delta)\mathcal{D}E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right] \\
&\quad + \mathcal{D}(1+\frac{1}{\Delta\Gamma})E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f(s, X_s, Y_s, Z_s) - f(t_{i+1}, X_{t_{i+1}}, q_{i+1}^{N,M}, z_{i+1}^{N,M}) ds \middle| \mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\quad - \mathcal{D}E\left[(\xi_i^q(X_{t_i}))^2\right].
\end{aligned}$$

Like in the derivation of the error of the approximation of $\overline{q}^N$, the last inequality in the calculations above does not hold true when considering $E[\xi_i^z(X_{t_i})^2]$ at time points t_i at the end of a segment, i.e., t_i such that $t_{i+1} \in \overline{\pi}_\alpha$. However, by adding and subtracting a multiple of $E[(\xi_{i+1}^q(X_{t_{i+1}}))^2]$, we again get the similar bound

$$\begin{aligned}
&\Delta E\left[(\xi_i^z(X_{t_i}))^2\right] \\
&\leq \mathcal{D}(1+\Gamma\Delta)(1+\kappa)(1+\epsilon)E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right] \\
&\quad + \mathcal{D}(1+\frac{1}{\Delta\Gamma})E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f(s, X_s, Y_s, Z_s) - f(t_{i+1}, X_{t_{i+1}}, q_{i+1}^{N,M}, z_{i+1}^{N,M}) ds \middle| \mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\quad - \mathcal{D}E\left[(\xi_i^q(X_{t_i}))^2\right] \\
&\quad + \mathcal{D}(1+\Gamma\Delta)\left(E\left[\left\|\overline{q}_{i+1}^N - q_{i+1}^{N,M}\right\|_{i+1,\infty}^2\right] - (1+\kappa)(1+\epsilon)E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right]\right)_+
\end{aligned}$$

for these time points, with an additional error term that depends on the approximation of $\overline{q}^N$ at the next time point t_{i+1}. As argued in the analysis of the approximation of $\overline{q}^N$, this term results from the single use of an ODP step in the discretization scheme that is used to connect two time segments.

Next we want to derive a global error bound for the approximation of $\overline{z}^N$. Since $\overline{z}^N$ appears in the discretization scheme only as argument of the driver, we state this term as an averaged sum over the time steps rather than the maximum.

Lemma 1.5.8. *Let Γ be a positive constants and set* $\lambda_i := (1+\Gamma\Delta)^i((1+N^{\alpha-1})^{2|\{j\leq i: t_j \in \overline{\pi}_\alpha\}|}$

for $i \in \{0, \dots, N-1\}$. *Then*

$$\sum_{i=0}^{N-1} \Delta\lambda_i E\left[(\xi_i^z(X_{t_i}))^2\right]$$
$$\leq \mathcal{D}\sum_{i=0}^{N-2}\lambda_i(1+\frac{1}{\Delta\Gamma})E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f(s,X_s,Y_s,Z_s) - f(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M})ds\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]$$
$$+\lambda_N\mathcal{D}\lceil N^{1-\alpha}\rceil\max_{j\in\mathcal{J}}\left((1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,j}} E\left[\left|\psi(X_{t_j}-\overline{q}_i^N(X_{t_j})\right|^2\right]+(1+N^{1-\alpha})\frac{C_{q,j}^2K_{q,j}}{M_j}\right)\right.$$
$$\left.+\frac{N^{1-\alpha}C_1K_{q,j}\log(C_2M_j)}{M_j}\right).$$

Proof. First, note that $E[(\xi_{N-1}^z(X_{t_{N-1}}))^2] = 0$ by definition since $\Xi_{N-1}^{N,M} = \Xi_{N-1}^N$. Then, by plugging in the estimate for $\Delta E[(\xi_i^z(X_{t_i}))^2]$ from the analysis of the error propagation where we choose $\epsilon = \kappa = N^{\alpha-1}$ for all i and summing up we get

$$\sum_{i=0}^{N-2}\Delta\lambda_i E\left[(\xi_i^z(X_{t_i}))^2\right]$$
$$\leq \mathcal{D}\sum_{i=0}^{N-2}\left(\lambda_{i+1}E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right]\right.$$
$$+\lambda_i(1+\frac{1}{\Delta\Gamma})E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f(s,X_s,Y_s,Z_s)-f(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M})ds\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]$$
$$-\lambda_iE\left[(\xi_i^q(X_{t_i}))^2\right]$$
$$\left.+\lambda_i(1+\Gamma\Delta)\left(E\left[\|\overline{q}_{i+1}^N-q_{i+1}^{N,M}\|_{i+1,\infty}^2\right]-(1+N^{\alpha-1})^2E\left[\left(\xi_{i+1}^q(X_{t_{i+1}})\right)^2\right]\right)_+\mathbb{1}_{\mathcal{J}}(i+1)\right)$$
$$\leq \mathcal{D}\sum_{i=0}^{N-2}\lambda_i(1+\frac{1}{\Delta\Gamma})E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f(s,X_s,Y_s,Z_s)-f(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M})ds\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]$$
$$+\mathcal{D}\lambda_N\lceil N^{1-\alpha}\rceil\sup_{j\in\mathcal{J}}\left(\inf_{\psi\in\mathcal{K}_{q,j}}E\left[\left|\psi(X_{t_j}-\overline{q}_j^N(X_{t_j})\right|^2\right]-(1+N^{\alpha-1})^2E\left[\left(\xi_j^q(X_{t_j})\right)^2\right]\right)_+.$$

Plugging in the estimate for $E[\|\overline{q}_i^N-q_i^{N,M}\|_{i,\infty}^2]$ from (1.6) derived in the proof of Lemma 1.5.7 finishes the proof. □

1.5.4 Bounds for the driver-dependent terms

In this section, we derive a bound for the sum of the terms

$$E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)ds\middle|\mathcal{F}^M,X_{t_i}\right]^2\right]$$

over the time steps that appears in the bounds of Lemma 1.5.7 and 1.5.8 both. For this purpose, note that for any $i\in\{0,\dots,N-1\}$, it holds by Fubini's theorem

$$\begin{aligned}
&E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)ds\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]\\
&\le E\Bigg[\Bigg(\int_{t_{i+1}}^{t_{i+2}} E_i\left[f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^{N},\overline{z}_{i+1}^{N}\right)\right]\\
&\quad+E\left[f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^{N},\overline{z}_{i+1}^{N}\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)\middle|\mathcal{F}_0^M,X_{t_i}\right]ds\Bigg)^2\Bigg]\\
&\le 2E\left[\left(\int_{t_{i+1}}^{t_{i+2}} E_i\left[f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^{N},\overline{z}_{i+1}^{N}\right)\right]ds\right)^2\right]\\
&\quad+2\Delta^2E\left[E\left[f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^{N},\overline{z}_{i+1}^{N}\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right].
\end{aligned}$$

Then, by the Lipschitz assumption on f we get

$$\begin{aligned}
&E\left[E\left[f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^{N},\overline{z}_{i+1}^{N}\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]\\
&\le +2L_f^2E\Bigg[E\left[\left|\overline{q}_{i+1}^{N}(X_{t_{i+1}})-q_{i+1}^{N,M}(X_{t_{i+1}})\right|\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\\
&\quad+E\left[\left|\overline{z}_{i+1}^{N}(X_{t_{i+1}})-z_{i+1}^{N,M}(X_{t_{i+1}})\right|\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\Bigg]\\
&=2L_f^2\left(E\left[\left\|\overline{q}_{i+1}^{N}-q_{i+1}^{N,M}\right\|_{i,\infty}^2\right]+E\left[\left\|\overline{z}_{i+1}^{N}-z_{i+1}^{N,M}\right\|_{i,\infty}^2\right]\right).
\end{aligned}$$

Hence it holds

$$\sum_{i=0}^{N-2}\lambda_i E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)ds\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]$$
$$\leq \sum_{i=0}^{N-2}\lambda_i 2E\left[\left(\int_{t_{i+1}}^{t_{i+2}} E_i\left[f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^N,\overline{z}_{i+1}^N\right)\right]ds\right)^2\right]$$
$$+\sum_{i=0}^{N-2}\lambda_i 4\Delta^2 L_f^2\left(E\left[\left\|\overline{q}_{i+1}^N-q_{i+1}^{N,M}\right\|_{i,\infty}^2\right]+E\left[\left\|\overline{z}_{i+1}^N-z_{i+1}^{N,M}\right\|_{i,\infty}^2\right]\right)$$
$$\leq \sum_{i=0}^{N-2}\lambda_i 2E\left[\left(\int_{t_{i+1}}^{t_{i+2}} E_i\left[f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^N,\overline{z}_{i+1}^N\right)\right]ds\right)^2\right]$$
$$+4\Delta L_f^2 T\max_{0\leq i\leq N-1}\lambda_i E\left[\left\|\overline{q}_{i+1}^N-q_{i+1}^{N,M}\right\|_{i,\infty}^2\right]+4\Delta L_f^2\sum_{i=1}^{N-1}\lambda_i\Delta E\left[\left\|\overline{z}_{i+1}^N-z_{i+1}^{N,M}\right\|_{i,\infty}^2\right]$$
$$\leq 2\lambda_N\mathcal{R}^N+4\Delta L_f^2 T\max_{0\leq i\leq N-1}\lambda_i E\left[\left\|\overline{q}_{i+1}^N-q_{i+1}^{N,M}\right\|_{i,\infty}^2\right]+4\Delta L_f^2\sum_{i=1}^{N-1}\lambda_i\Delta E\left[\left\|\overline{z}_{i+1}^N-z_{i+1}^{N,M}\right\|_{i,\infty}^2\right]$$

with

$$\mathcal{R}^N:=\sum_{i=0}^{N-2}E\left[\left(\int_{t_{i+1}}^{t_{i+2}} E_i\left[f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},\overline{q}_{i+1}^N,\overline{z}_{i+1}^N\right)\right]ds\right)^2\right]$$

as defined in Theorem 1.3.1. The term $\mathcal{R}^N$ does not depend on our approximation of the BSDE but only on the real solution Y,Z, the semi-continuous versions $\overline{q}^N,\overline{z}^N$ and the solution of the forward SDE X. It can be bounded in different ways depending on the regularity of these functions, which leads to the different bounds of the total quadratic error in Theorem 1.3.2 and 1.3.3. The different bounds will be derived at the end of the next section.

1.5.5 Final error bounds

Using the bounds derived throughout this section, we are now ready to proof Theorem 1.3.1.

Proof. Proof of Theorem 1.3.1:
In the following calculations, c denotes a positive constant that does not depend on N and

may change from line to line. First, we can write the quadratic error as

$$\max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(X_{t_i}) - q_i^{N,M}(X_{t_i})\right|^2\right] + \sum_{i=0}^{N-1} \Delta E\left[\left|\overline{z}^N(X_{t_i}) - z_i^{N,M}(X_{t_i})\right|^2\right]$$
$$\le \max_{0\le i\le N-1} \lambda_i E\left[\left|\overline{q}_i^N(X_{t_i}) - q_i^{N,M}(X_{t_i})\right|^2\right] + \sum_{i=0}^{N-2} \Delta\lambda_i E\left[\left|\overline{z}^N(X_{t_i}) - z_i^{N,M}(X_{t_i})\right|^2\right]$$
$$= \max_{0\le i\le N-1} \lambda_i E\left[\left\|\overline{q}_i^N - q_i^{N,M}\right\|_{i,\infty}^2\right] + \sum_{i=0}^{N-1} \Delta\lambda_i E\left[\left\|\overline{z}_i^N - z_i^{N,M}\right\|_{i,\infty}^2\right].$$

We can now estimate this term by Lemma 1.5.5 with $\epsilon = 1$ as

$$\max_{0\le i\le N-1} \lambda_i E\left[\left\|\overline{q}_i^N - q_i^{N,M}\right\|_{i,\infty}^2\right] + \sum_{i=0}^{N-1} \Delta\lambda_i E\left[\left\|\overline{z}_i^N - z_i^{N,M}\right\|_{i,\infty}^2\right]$$
$$\le \max_{0\le i\le N-1} \left(2\lambda_i E\left[\left\|\overline{q}_i^N - q_i^{N,M}\right\|_{i,M}^2\right] + \lambda_i \frac{C_1 K_{q,i}\log(C_2 M_i)}{M_i}\right)$$
$$+ \sum_{i=0}^{N-2} 2\Delta\lambda_i E\left[\left\|\overline{z}_i^N - z_i^{N,M}\right\|_{i,M}^2\right] + \Delta\lambda_i \frac{C_1 K_{z,i}\log(C_2 M_i)}{\Delta M_i}.$$

By the inequalities (1.5) and (1.8) we then get with the choice $\kappa = 1$

$$\max_{0\le i\le N-1} \lambda_i E\left[\left\|\overline{q}_i^N - q_i^{N,M}\right\|_{i,\infty}^2\right] + \sum_{i=0}^{N-1} \Delta\lambda_i E\left[\left\|\overline{z}_i^N - z_i^{N,M}\right\|_{i,\infty}^2\right]$$
$$\le \lambda_N \max_{0\le i\le N-1} \left(4\frac{C_{q,i}K_{q,i}}{M_i} + 2\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + \frac{C_1 K_{q,i}\log(C_2 M_i)}{M_i}\right)$$
$$+ \sum_{i=0}^{N-1} \Delta\lambda_i \left(4\frac{C_{q,i}^2 K_{z,i}}{\Delta M_i} + 2\inf_{\psi\in\mathcal{K}_{z,i}} E\left[\left|\overline{z}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + \frac{C_1 K_{z,i}\log(C_2 M_i)}{\Delta M_i}\right)$$
$$+ 4\left(\max_{0\le i\le N-1} \lambda_i E\left[(\xi_i^q(X_{t_i}))^2\right] + \sum_{i=0}^{N-1} \Delta\lambda_i E\left[(\xi_i^z(X_{t_i}))^2\right]\right).$$

Now the bounds in Lemma 1.5.7 and Lemma 1.5.8 yield

$$
\begin{aligned}
&\max_{0\le i\le N-1} \lambda_i E\left[\left(\xi_i^q(X_{t_i})\right)^2\right] + \sum_{i=0}^{N-1} \Delta\lambda_i E\left[\left(\xi_i^z(X_{t_i})\right)^2\right] \\
&\le \sum_{i=0}^{N-2} (1+\frac{1}{\Delta\Gamma})\lambda_i E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right) - f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\quad + \lceil N^{1-\alpha}\rceil \lambda_N \max_{j\in\mathcal{J}} \left((1+N^{\alpha-1}) \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,i}^2 K_{q,j}}{M_j}\right)\right. \\
&\quad \left. + \frac{N^{1-\alpha} C_1 K_{q,j}\log(C_2 M_j)}{M_j}\right) \\
&\quad + \mathcal{D}\sum_{i=0}^{N-2} \lambda_i (1+\frac{1}{\Delta\Gamma}) E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right) - f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\quad + \lambda_N \mathcal{D}\lceil N^{1-\alpha}\rceil \max_{j\in\mathcal{J}} \left((1+N^{\alpha-1}) \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,i}^2 K_{q,j}}{M_j}\right)\right. \\
&\quad \left. + \frac{N^{1-\alpha} C_1 K_{q,j}\log(C_2 M_j)}{M_j}\right) \\
&= \sum_{i=0}^{N-2} (1+\mathcal{D})(1+\frac{1}{\Delta\Gamma})\lambda_i E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right) - f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right) ds \middle| \mathcal{F}_0^M, X_{t_i}\right]^2\right] \\
&\quad + (1+\mathcal{D})\lceil N^{1-\alpha}\rceil \lambda_N \max_{j\in\mathcal{J}} \left((1+N^{\alpha-1}) \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,i}^2 K_{q,j}}{M_j}\right)\right. \\
&\quad \left. + \frac{N^{1-\alpha} C_1 K_{q,j}\log(C_2 M_j)}{M_j}\right).
\end{aligned}
$$

By plugging in the bounds derived in Section 1.5.4, we can estimate this term further as

$$
\begin{aligned}
&\max_{0\le i\le N-1}\lambda_i E\left[\left(\xi_i^q(X_{t_i})\right)^2\right]+\sum_{i=0}^{N-1}\Delta\lambda_i E\left[\left(\xi_i^z(X_{t_i})\right)^2\right]\\
&\le \sum_{i=0}^{N-2}(1+\mathcal{D})(1+\frac{1}{\Delta\Gamma})\lambda_i E\left[E\left[\int_{t_{i+1}}^{t_{i+2}} f\left(s,X_s,Y_s,Z_s\right)-f\left(t_{i+1},X_{t_{i+1}},q_{i+1}^{N,M},z_{i+1}^{N,M}\right)ds\middle|\mathcal{F}_0^M,X_{t_i}\right]^2\right]\\
&\quad+(1+\mathcal{D})\lceil N^{1-\alpha}\rceil\lambda_N\max_{j\in\mathcal{J}}\left((1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,i}}E\left[\left|\overline{q}_i^N(X_{t_i})-\psi(X_{t_i})\right|^2\right]+(1+N^{1-\alpha})\frac{C_{q,i}^2K_{q,j}}{M_j}\right)\right.\\
&\quad\left.+\frac{N^{1-\alpha}C_1K_{q,j}\log(C_2M_j)}{M_j}\right)\\
&\le\left[(\Delta+\frac{1}{\Gamma})(1+\mathcal{D})\right]4(T\vee 1)L_f^2\left(\max_{0\le i\le N-1}\lambda_i E\left[\left\|\overline{q}_i^N-q_i^{N,M}\right\|_{i,\infty}^2\right]+\sum_{i=0}^{N-2}\Delta\lambda_i E\left[\left\|\overline{z}_i^N-z_i^{N,M}\right\|_{i,\infty}^2\right]\right)\\
&\quad+\left[(\Delta+\frac{1}{\Gamma})(1+\mathcal{D})\right]\Delta^{-1}\lambda_N\mathcal{R}^N\\
&\quad+\lceil N^{1-\alpha}\rceil\lambda_N(1+\mathcal{D})\max_{j\in\mathcal{J}}\left((1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,i}}E\left[\left|\overline{q}_i^N(X_{t_i})-\psi(X_{t_i})\right|^2\right]+(1+N^{1-\alpha})\frac{C_{q,j}^2K_{q,j}}{M_j}\right)\right.\\
&\quad\left.+\frac{N^{1-\alpha}C_1K_{q,j}\log(C_2M_j)}{M_j}\right).
\end{aligned}
$$

Now, assuming that N and Γ are sufficiently large such that $[(\Delta+\frac{1}{\Gamma})(1+\mathcal{D})]16L^2(T\vee 1)\le\frac{1}{2}$, we have

$$
\begin{aligned}
&\max_{0\le i\le N-1} E\left[\left\|\overline{q}_i^N - q_i^{N,M}\right\|_{i,\infty}^2\right]\lambda_i + \sum_{i=0}^{N-1}\lambda_i E\left[\left\|\overline{z}_i^N - z_i^{N,M}\right\|_{i,\infty}^2\right]\Delta \\
&\le \frac{1}{2}\left(\max_{0\le i\le N-1} E\left[\left\|\overline{q}_i^N - q_i^{N,M}\right\|_{i,\infty}^2\right]\lambda_i + \sum_{i=0}^{N-1}\lambda_i E\left[\left\|\overline{z}_i^N - z_i^{N,M}\right\|_{i,\infty}^2\right]\Delta\right) \\
&+ c\lambda_N \lceil N^{1-\alpha}\rceil \max_{j\in\mathcal{J}}\left((1+N^{\alpha-1})\left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + (1+N^{1-\alpha})\frac{C_{q,j}^2 K_{q,j}}{M_j}\right)\right. \\
&\quad\left. + \frac{N^{1-\alpha}C_1 K_{q,j}\log(C_2 M_j)}{M_j}\right) \\
&\quad + c\lambda_N \max_{0\le i\le N-1}\left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\overline{q}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + \frac{K_{q,i}}{M_i} + \frac{K_{q,i}\log(M_i)}{M_i}\right. \\
&\quad\left. + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[\left|\overline{z}_i^N(X_{t_i}) - \psi(X_{t_i})\right|^2\right] + \frac{K_{z,i}}{\Delta M_i} + \frac{K_{z,i}\log(M_i)}{\Delta M_i}\right) \\
&\quad + c\Delta^{-1}\lambda_N \mathcal{R}^N.
\end{aligned}
$$

Considering that λ_N is bounded by a constant independent of N, since

$$
\begin{aligned}
\lambda_N &= \left(1+\frac{T\Gamma}{N}\right)^N \left(1+N^{\alpha-1}\right)^{2\lceil N^{1-\alpha}\rceil} \\
&\le e^{T\Gamma N^{-1}N} e^{2\lceil N^{1-\alpha}\rceil N^{\alpha-1}} = e^{T\Gamma+4},
\end{aligned}
$$

this implies

$$
\begin{aligned}
&\max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(X_{t_i}) - q_i^{N,M}(X_{t_i})\right|^2\right] + \sum_{i=0}^{N-2}\Delta E\left[\left|\overline{z}_i^N(X_{t_i}) - z_i^{N,M}(X_{t_i})\right|^2\right] \\
&\le c\max_{i\in\mathcal{J}}\left(N^{1-\alpha}\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})\right|^2\right] + N^{2-2\alpha}\frac{K_{q,i}}{M_i} + N^{2-2\alpha}\frac{K_{q,i}\log(C_2 M_i)}{M_i}\right) \\
&\quad + c\max_{0\le i\le N-1}\left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[\left|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})\right|^2\right] + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[\left|\psi(X_{t_i}) - \overline{z}_i^N(X_{t_i})\right|^2\right]\right. \\
&\quad\left. + \frac{K_{q,i}}{M_i} + N\frac{K_{z,i}}{M_i} + \frac{K_{q,i}\log(C_2 M_i)}{M_i} + N\frac{K_{z,i}\log(C_2 M_i)}{M_i}\right) \\
&\quad + cN\mathcal{R}^N
\end{aligned}
$$

what finishes the proof. □

It remains to derive the results of Theorem 1.3.2 and Theorem 1.3.3 from this error representation. The remaining estimates needed for this depend only on the true BSDE solution rather than the approximation obtained with the algorithm and we hence have to distinguish depending on the regularity assumptions.

Proof. Proof of Theorem 1.3.2:
We first have to show that

$$\max_{0\le i\le N-1} E\left[|q_i^{N,M}(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} |z_i^{N,M}(X_{t_i}) - \overline{z}(s, X_s)|^2 ds\right]$$
$$\le c \max_{i\in\mathcal{I}} \left(N^{1-\alpha} \inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + N^{2-2\alpha}\frac{K_{q,i}}{M_i} + N^{2-2\alpha}\frac{K_{q,i}\log(M_i)}{M_i}\right)$$
$$+ c \max_{0\le i\le N-1} \left(\inf_{\psi\in\mathcal{K}_{q,i}} E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] + \inf_{\psi\in\mathcal{K}_{z,i}} E\left[|\psi(X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2\right]\right.$$
$$\left. + \frac{K_{q,i}}{M_i} + N\frac{K_{z,i}}{M_i} + \frac{K_{q,i}\log(M_i)}{M_i} + N\frac{K_{z,i}\log(M_i)}{M_i}\right)$$
$$+ cN^{-1}$$

under the standing assumptions. Since

$$\max_{0\le i\le N-1} E\left[\left|q_i^{N,M}(X_{t_i}) - \overline{y}(t_i, X_{t_i})\right|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} \left|z_i^{N,M}(X_{t_i}) - \overline{z}(s, X_s)\right|^2 ds\right]$$
$$\le 2\Bigg(\max_{0\le i\le N-1} E\left[\left|q_i^{N,M}(X_{t_i}) - \overline{q}_i^N(X_{t_i})\right|^2\right] + \max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(X_{t_i}) - \overline{y}(t_i, X_{t_i})\right|^2\right]$$
$$+ \sum_{i=0}^{N-1} \Delta E\left[\left|z_i^{N,M}(X_{t_i}) - \overline{z}_i^N(X_{t_i})\right|^2 ds\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} \left|\overline{z}_i^N(X_{t_i}) - \overline{z}(s, X_s)\right|^2 ds\right]\Bigg),$$

this follows directly from Theorem 1.3.1 if we can prove the bounds

$$\max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(X_{t_i}) - \overline{y}(t_i, X_{t_i})\right|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} \left|\overline{z}_i^N(X_{t_i}) - \overline{z}(s, X_s)\right|^2 ds\right] \le c\Delta$$

and $\mathcal{R}^N \le c\Delta^2$. We start with the bound for $\mathcal{R}^N$ and use Hölder's inequality to get

$$\mathcal{R}^N = \sum_{i=0}^{N-2} E\left[\left(\int_{t_{i+1}}^{t_{i+2}} E_i\left[f(s, X_s, Y_s, Z_s) - f\left(t_{i+1}, X_{t_{i+1}}, \overline{q}_{i+1}^N, \overline{z}_{i+1}^N\right)\right] ds\right)^2\right]$$
$$\le \sum_{i=0}^{N-2} \Delta E\left[\int_{t_{i+1}}^{t_{i+2}} E_i\left[\left(f(s, X_s, Y_s, Z_s) - f\left(t_{i+1}, X_{t_{i+1}}, \overline{q}_{i+1}^N, \overline{z}_{i+1}^N\right)\right)^2\right] ds\right].$$

Then, due to the Lipschitz continuity (respectively Hölder continuity in t) of f, it holds

$$
\begin{aligned}
\mathcal{R}^N &\leq \sum_{i=0}^{N-2} \Delta E\left[\int_{t_{i+1}}^{t_{i+2}} E_i\left[\left(f\left(s, X_s, Y_s, Z_s\right) - f\left(t_{i+1}, X_{t_{i+1}}, \overline{q}_{i+1}^N, \overline{z}_{i+1}^N\right)\right)^2\right] ds\right] \\
&\leq \sum_{i=0}^{N-2} \Delta E\Bigg[\int_{t_{i+1}}^{t_{i+2}} E_i\Bigg[L_f^2\Big(|s-t_{i+1}|^{\frac{1}{2}} + |X_s - X_{t_{i+1}}| + |Y_s - \overline{q}_{i+1}^N(X_{t_{i+1}})| \\
&\quad + |Z_s - \overline{z}_{i+1}^N(X_{t_{i+1}})|\Big)^2\Bigg] ds\Bigg] \\
&\leq \sum_{i=0}^{N-2} 4L_f^2 \Delta E\Bigg[\int_{t_{i+1}}^{t_{i+2}} |s-t_{i+1}| + E_i\left[(X_s - X_{t_{i+1}})^2\right] + E_i\left[(Y_s - \overline{q}_{i+1}^N(X_{t_{i+1}}))^2\right] \\
&\quad + E_i\left[(Z_s - \overline{z}_{i+1}^N(X_{t_{i+1}}))^2\right] ds\Bigg]
\end{aligned}
$$

and we consider the terms in the integrand separately for an arbitrary $s \in [t_{i+1}, t_{i+2}]$:
By choice of the time grid, it obviously holds that $|s-t_{i+1}| \leq \Delta$ and, under the assumptions on b and σ, it follows that $E_i[(X_s - X_{t_{i+1}})^2] \leq c(s-t_{i+1}) \leq c\Delta$ (see e.g. Kloeden and Platen, 1992). Then, by the definition of $\overline{q}_{i+1}^N$ and a zero addition, we get:

$$
\begin{aligned}
&E_i\left[\left(Y_s - \overline{q}_{i+1}^N(X_{i+1})\right)^2\right] \\
&= E_i\left[\left(Y_s - Y_{t_{i+1}} + Y_{t_{i+1}} + E_{i+1}\left[Y_{t_{i+2}}\right]\right)^2\right] \\
&\leq 4 \max_{0\leq i\leq N-1} \sup_{s\in[t_{i+1},t_{i+2}]} E_i\left[\left(Y_s - Y_{t_{i+1}}\right)^2\right].
\end{aligned}
\tag{1.9}
$$

To estimate the difference $Z_s - \overline{z}_{i+1}^N(X_{t_{i+1}})$, we define for each $i \in \{0, \ldots, N-1\}$ the random variable

$$
\tilde{Z}_i := \frac{1}{\Delta} E_i\left[\int_{t_i}^{t_{i+1}} Z_s ds\right],
$$

which can be used to express the quadratic difference as

$$
\begin{aligned}
&E_i\left[\left(Z_s - \overline{z}^N_{i+1}(X_{t_{i+1}})\right)^2\right]\\
&= E_i\left[\left(Z_s - E_{i+1}\left[\frac{\Delta W_{i+2}}{\Delta}Y_{t_{i+2}}\right]\right)^2\right]\\
&= E_i\left[\left(Z_s - E_{i+1}\left[\frac{\Delta W_{i+2}}{\Delta}\left(Y_{t_{i+1}} - \int_{t_{i+1}}^{t_{i+2}} f(l, X_l, Y_l, Z_l)dl + \int_{t_{i+1}}^{t_{i+2}} Z_l dW_l\right)\right]\right)^2\right]\\
&= E_i\left[\left(Z_s - \frac{1}{\Delta}E_{i+1}\left[\int_{t_{i+1}}^{t_{i+2}} Z_l dl\right] + \frac{1}{\Delta}E_i\left[\Delta W_{i+2}\int_{t_{i+1}}^{t_{i+2}} f(l, X_l, Y_l, Z_l)dl\right]\right)^2\right]\\
&\leq 2E_i\left[\left|Z_s - \tilde{Z}_{i+1}\right|^2\right] + 2E_i\left[\left(\frac{1}{\Delta}E_{i+1}\left[\Delta W^2_{i+2}\right]^{\frac{1}{2}}E_{i+1}\left[(C_f\Delta)^2\right]^{\frac{1}{2}}\right)^2\right]\\
&\leq 2E_i\left[\left|Z_s - \tilde{Z}_{i+1}\right|^2\right] + c\Delta.
\end{aligned}
\tag{1.10}
$$

Here, the second equality follows by the Itô-isometry and the measurablility of $Y_{t_{i+1}}$, the following inequality due to the boundedness of f and Hölder's inequality. Plugging in the obtained bounds we have

$$
\begin{aligned}
\mathcal{R}^N \leq& \sum_{i=0}^{N-2} 4L_f^2\Delta E\Bigg[\int_{t_{i+1}}^{t_{i+2}} |s - t_{i+1}| + E_i\left[\left(X_s - X_{t_{i+1}}\right)^2\right] + E_i\left[\left(Y_s - \overline{q}^N_{i+1}(X_{t_{i+1}})\right)^2\right]\\
&+ E_i\left[\left(Z_s - \overline{z}_{i+1}(X_{t_{i+1}})\right)^2\right] ds\Bigg]\\
\leq& \sum_{i=0}^{N-2} 4L_f^2\Delta E\Bigg[\int_{t_{i+1}}^{t_{i+2}} \Delta + c\Delta + 4\max_{0\leq j\leq N-1}\sup_{l\in[t_{j+1},t_{j+2}]} E\left[\left(Y_l - Y_{t_{j+1}}\right)^2\right]\\
&+ 2E_i\left[\left(Z_s - \tilde{Z}_{i+1}\right)^2\right] + c\Delta ds\Bigg]\\
\leq& \sum_{i=0}^{N-2} 4L_f^2\Delta E\Bigg[\Delta\left(c\Delta + 4\max_{0\leq j\leq N-1}\sup_{l\in[t_{j+1},t_{j+2}]} E\left[\left(Y_l - Y_{t_{j+1}}\right)^2\right]\right)\\
&+ 2\int_{t_{i+1}}^{t_{i+2}} E_i\left[\left(Z_s - \tilde{Z}_{i+1}\right)^2\right] ds\Bigg]\\
\leq& T4L_f^2\left(c\Delta^2 + \Delta 4\max_{0\leq j\leq N-1}\sup_{l\in[t_{j+1},t_{j+2}]} E\left[\left(Y_l - Y_{t_{j+1}}\right)^2\right]\right)\\
&+ 8L_f^2\Delta\sum_{i=0}^{N-2}\int_{t_{i+1}}^{t_{i+2}} E\left[\left(Z_s - \tilde{Z}_{i+1}\right)^2\right] ds.
\end{aligned}
$$

Then, using the the L^2-regularity of BSDEs (see Zhang, 2001), which states

$$\max_{0\le i\le N}\sup_{t_i\le s\le t_{i+1}} E\left[\left|Y_s - Y_{t_i}\right|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} \left|Z_s - \tilde{Z}_i\right|^2 ds\right] \le c\Delta,$$

it follows

$$\begin{aligned}\mathcal{R}^N &\le T4L_f^2\left(c\Delta^2 + 4\Delta \max_{0\le j\le N-1}\sup_{l\in[t_{j+1},t_{j+2}]} E\left[\left(Y_l - Y_{t_{j+1}}\right)^2\right]\right)\\ &\quad + 8L_f^2\Delta\sum_{i=0}^{N-2}\int_{t_{i+1}}^{t_{i+2}} E\left[\left(Z_s - \tilde{Z}_{i+1}\right)^2\right] ds\\ &\le c\Delta^2,\end{aligned}$$

what proves the bound for $\mathcal{R}^N$. Note that the inequalities (1.10) and (1.9) together with the L^2 regularity of BSDEs (see Zhang, 2001) in particular also imply that

$$\begin{aligned}&\max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(X_{t_i}) - Y_{t_i}\right|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} \left|\overline{z}_i^N(X_{t_i}) - Z_s\right|^2 ds\right]\\ &\le 4\max_{0\le i\le N-1}\sup_{l\in[t_{j+1},t_{j+2}]} E\left[\left|Y_l - Y_{t_i}\right|^2\right]\\ &\quad + \sum_{i=0}^{N-1} 2E\left[\int_{t_i}^{t_{i+1}} \left|Z_s - \tilde{Z}_i(X_{t_i})\right|^2 + c\Delta ds\right]\\ &\le c\Delta,\end{aligned}$$

which shows the second bound.
For the additional statement of Theorem 1.3.2, note that

$$N^{1-\alpha}E\left[|\psi(X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right] \le 2N^{1-\alpha}\left(E\left[|\psi(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] + E\left[|\overline{y}(t_i, X_{t_i}) - \overline{q}_i^N(X_{t_i})|^2\right]\right)$$

and analogously

$$E\left[|\psi(X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2\right] \le 2\left(E\left[|\psi(X_{t_i}) - \overline{z}(t_i, X_{t_i})|^2\right] + E\left[|\overline{z}(t_i, X_{t_i}) - \overline{z}_i^N(X_{t_i})|^2\right]\right)$$

for all $\psi \in \mathcal{K}_{q,i}$ or $\psi \in \mathcal{K}_{z,i}$ respectively. Hence it suffices to show that the bounds

$$E\left[|q_i^N(X_{t_i}) - \overline{y}(t_i, X_{t_i})|^2\right] \le \Delta^2, \quad E\left[|\overline{z}_i^N(X_{t_i}) - \overline{z}(t_i, X_{t_i})|^2\right] \le \Delta$$

hold true for each $i \in \{0, \dots, N-1\}$, whenever z is Lipschitz continuous in x and $\frac{1}{2}$-Hölder continuous in t. For the bound regarding y, we directly get by the definition of $\overline{q}_i^N$ and the boundedness assumption on f that

$$\begin{aligned}
&E\left[\left|q_i^N(X_{t_i})-\overline{y}(t_i,X_{t_i})\right|^2\right] \\
&= E\left[\left|E_i\left[\overline{y}(t_i,X_{t_i})-\int_{t_i}^{t_{i+1}} f(t,X_t,Y_t,Z_t)dt+\int_{t_i}^{t_{i+1}} Z_t dW_t\right]-\overline{y}(t_i,X_{t_i})\right|^2\right] \\
&\leq E\left[\left|\int_{t_i}^{t_{i+1}} C_f dt\right|^2\right] \\
&\leq c\Delta^2.
\end{aligned}$$

For the bound concerning $\overline{z}$, we get by inequality (1.10)

$$\begin{aligned}
E\left[\left|\overline{z}_i^N(X_{t_i})-\overline{z}(t_i,X_{t_i})\right|^2\right] &\leq 2E\left[\left|\overline{z}(t_i,X_{t_i})-\tilde{Z}_{i+1}\right|^2\right]+c\Delta \\
&= 2E\left[\left|\frac{1}{\Delta}E_i\left[\int_{t_i}^{t_{i+1}} \overline{z}(t_i,X_{t_i})-\overline{z}(l,X_l)dl\right]\right|^2\right]+c\Delta \\
&\leq \frac{2}{\Delta^2}E\left[E_i\left[\int_{t_i}^{t_{i+1}} \left|\overline{z}(t_i,X_{t_i})-\overline{z}(l,X_l)\right|dl\right]^2\right]+c\Delta \\
&\leq \frac{2}{\Delta}\int_{t_i}^{t_{i+1}} E\left[E_i\left[L_z(|t_i-l|^{\frac{1}{2}}+|X_{t_i}-X_l|)dl\right]^2\right]+c\Delta \\
&\leq \frac{2}{\Delta}\int_{t_i}^{t_{i+1}} E\left[c\Delta^{\frac{1}{2}}dl\right]^2+c\Delta \\
&\leq c\Delta
\end{aligned}$$

where we used Hölder's inequality in the first step, the continuity assumptions on $\overline{z}$ along with Fubini's theorem in the second inequality and denote the Lipschitz constant of $\overline{z}$ with L_z. □

The remainder of this section is dedicated to the proof of Theorem 1.3.3, which shows that an asymptotic convergence rate of order N^{-2} is possible under stronger regularity assumptions on the components.

Proof. Proof of Theorem 1.3.3:

Similar to the proof of Theorem 1.3.2, we have

$$
\begin{aligned}
&\max_{0\le i\le N-1} E\left[\left|q_i^{N,M}(W_{t_i})-\overline{y}(t_i,W_{t_i})\right|^2\right]+\sum_{i=0}^{N-1}\Delta E\left[\left|\overline{z}(t_i,W_{t_i})-z_i^{N,M}(W_{t_i})\right|^2 ds\right]\\
&\le 2\left(\max_{0\le i\le N-1} E\left[\left|q_i^{N,M}(W_{t_i})-\overline{q}_i^N(W_{t_i})\right|^2\right]+\max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(W_{t_i})-\overline{y}(t_i,W_{t_i})\right|^2\right]\right.\\
&\quad\left.+\sum_{i=0}^{N-1}\Delta E\left[\left|\overline{z}(t_i,W_{t_i})-\overline{z}_i^N(W_{t_i})\right|^2 ds\right]+\sum_{i=0}^{N-1}\Delta E\left[\left|z_i^{N,M}(W_s)-\overline{z}_i^N(W_{t_i})\right|^2 ds\right]\right)
\end{aligned}
$$

and it suffices to prove the bounds

$$
\max_{0\le i\le N-1} E\left[\left|\overline{q}_i^N(W_{t_i})-\overline{y}(t_i,W_{t_i})\right|^2\right]+\sum_{i=0}^{N-1}\Delta E\left[\left|\overline{z}(t_i,W_{t_i})-\overline{z}_i^N(W_{t_i})\right|^2\right]\le c\Delta^2 \tag{1.11}
$$

and $\mathcal{R}^N\le c\Delta^3$ for the first statement of Theorem 1.3.3. We focus on the bound on $\mathcal{R}^N$ and derive the bounds in (1.11) along the way. It suffices to show that

$$
\left|E_i\left[f\left(s,W_s,Y_s,Z_s\right)-f\left(t_i,W_{t_i},\overline{q}_i^N,\overline{z}_i^N\right)\right]\right|\le c\Delta \tag{1.12}
$$

for any $t_i\in\pi$ and $s\in[t_i,t_{i+1}]$, since then

$$
\begin{aligned}
\mathcal{R}^N&=\sum_{i=0}^{N-1}E\left[\left(\int_{t_i}^{t_{i+1}}E_i\left[f\left(s,W_s,Y_s,Z_s\right)-f\left(t_i,W_{t_i},\overline{q}_i^N,\overline{z}_i^N\right)\right]ds\right)^2\right]\\
&\le\sum_{i=0}^{N-1}E\left[\left(\int_{t_i}^{t_{i+1}}c\Delta ds\right)^2\right]\\
&\le c\Delta^3.
\end{aligned}
$$

In order to prove (1.12), we set for arbitrary but fixed $t_i\in\pi$ and $s\in[t_i,t_{i+1}]$

$$
\begin{aligned}
a&:=\left(t_i,W_{t_i}^{(1)},\dots,W_{t_i}^{(\mathcal{D})},\overline{q}_i^N(W_{t_i}),\overline{z}_i^{N,(1)}(W_{t_i}),\dots,\overline{z}_i^{N,(\mathcal{D})}(W_{t_i})\right)^T\\
\tilde{a}&:=\left(s,W_s^{(1)},\dots,W_s^{(\mathcal{D})},Y_s,Z_s^{(1)},\dots,Z_s^{(\mathcal{D})}\right)^T.
\end{aligned}
$$

Then, a Taylor expansion of f yields

$$
\begin{aligned}
&E_i\left[f\left(s,W_s,Y_s,Z_s\right)-f\left(t_i,W_{t_i},\overline{q}_i^N,\overline{z}_i^N\right)\right]\\
&=E_i\left[\nabla f(a)^T(a-\tilde{a})+\frac{1}{2}\int_0^1(1-\Theta)(a-\tilde{a})^T\operatorname{Hess}_f(a+\Theta(\tilde{a}-a))(a-\tilde{a})d\Theta\right],
\end{aligned}
$$

where ∇f denotes the gradient and Hess_f the Hessian matrix of f. Using that f has bounded derivatives and a is $\mathcal{F}_i$-measurable we obtain

$$\begin{aligned}
&E_i\left[f\left(s, W_s, Y_s, Z_s\right) - f\left(t_i, W_{t_i}, \overline{q}_i^N, \overline{z}_i^N\right)\right] \\
&\leq \nabla f(a)^T E_i\left[(a-\tilde{a})\right] + \frac{1}{2}\sup_{\Theta\in[0,1]}\left|E_i\left[(a-\tilde{a})^T H_f(a+\Theta(\tilde{a}-a))(a-\tilde{a})\right]\right| \\
&\leq C_f \sum_{l=1}^{2\mathcal{D}+2}\left|E_i\left[a^{(l)}-\tilde{a}^{(l)}\right]\right| + \frac{1}{2}C_f E_i\left[\sum_{l,k=1}^{2\mathcal{D}+2}\left|a^{(l)}-\tilde{a}^{(l)}\right|\left|a^{(k)}-\tilde{a}^{(k)}\right|\right] \\
&\leq C_f \sum_{l=1}^{2\mathcal{D}+2}\left|E_i\left[a^{(l)}-\tilde{a}^{(l)}\right]\right| + \frac{1}{2}C_f \sum_{l,k=1}^{2\mathcal{D}+2} E_i\left[\left|a^{(l)}-\tilde{a}^{(l)}\right|^2\right]^{\frac{1}{2}} E_i\left[\left|a^{(k)}-\tilde{a}^{(k)}\right|^2\right]^{\frac{1}{2}}
\end{aligned}$$

and it suffices to show that it holds $|E_i[(a^{(l)} - \tilde{a}^{(l)})^p]| \leq c\Delta$ for $l \in \{1, \ldots, 2\mathcal{D}+2\}$ and $p \in \{1, 2\}$. This is trivial for $l = 1, \ldots, \mathcal{D}+1$, since W is a Brownian motion and the step width of the time grid is Δ. For the remaining values of l, we either have $a^{(l)} - \tilde{a}^{(l)} = \overline{y}(s, W_s) - \overline{q}_i^N(W_{t_i})$ or $a^{(l)} - \tilde{a}^{(l)} = \overline{z}^{(d)}(s, W_s) - \overline{z}_i^{N,(d)}(W_{t_i})$ for a $d \in \{1, \ldots, \mathcal{D}\}$.
We first consider the terms $\overline{y}(s, W_s) - \overline{q}_i^N(W_{t_i})$.
By the definition of $\overline{q}_i^N$, we get for $p \in \{1, 2\}$ with Hölder's inequality that

$$\begin{aligned}
&E_i\left[\left(\overline{y}(s, W_s) - \overline{q}_i^N(W_{t_i})\right)^p\right] \\
&= E_i\left[\left(\overline{y}(s, W_s) - E_i\left[\overline{y}(t_{i+1}, W_{t_{i+1}})\right]\right)^p\right] \\
&= E_i\left[\left(\overline{y}(s, W_s) - E_i\left[\overline{y}(t_i, W_{t_i}) - \int_{t_i}^{t_{i+1}} f(l, W_l, Y_l, Z_l)dl + \int_{t_i}^{t_{i+1}} Z_l dW_l\right]\right)^p\right] \\
&= E_i\left[\left(\overline{y}(s, W_s) - E_i\left[\overline{y}(t_i, W_{t_i}) - \int_{t_i}^{t_{i+1}} f(l, W_l, Y_l, Z_l)dl\right]\right)^p\right] \\
&= pE_i\left[\left(\overline{y}(s, W_s) - \overline{y}(t_i, W_{t_i})\right)^p\right] + pE_i\left[\left(\int_{t_i}^{t_{i+1}} f(l, W_l, Y_l, Z_l)dl\right)^p\right] \\
&\leq pE_i\left[\left(\overline{y}(s, W_s) - \overline{y}(t_i, W_{t_i})\right)^p\right] + p\Delta^p C_f^p
\end{aligned}$$

where we used that f is uniformly bounded by C_f in the last step. Note that for $s = t_i$, this shows in particular that

$$E_i\left[\left(\overline{y}(s, W_s) - \overline{q}_i^N(W_{t_i})\right)^2\right] \leq c\Delta^2$$

which is the first part of the bound in (1.11). We now set $\tilde{a}_y := (s, W_s^{(1)}, \ldots, W_s^{(\mathcal{D})})^T$ and

$a_y := (t_i, W_{t_i}^{(1)}, \dots, W_{t_i}^{(\mathcal{D})})^T$. Then, for $p = 2$, a Taylor expansion on y yields

$$\begin{aligned} E_i\left[(\overline{y}(s,W_s) - \overline{y}(t_i,W_{t_i}))^2\right] &= E_i\left[\left(\int_0^1 (1-\Theta)\nabla\overline{y}(a_y + \Theta(\tilde{a}_y - a_y))(a_y - \tilde{a}_y)d\Theta\right)^2\right] \\ &\le E_i\left[\left(\sup_{\Theta\in[0,1]} \nabla\overline{y}(a_y + \Theta(\tilde{a}_y - a_y))(a_y - \tilde{a}_y)\right)^2\right] \\ &\le C_y^2 E_i\left[|a_y - \tilde{a}_y|^2\right] \le c\Delta. \end{aligned}$$

Here we used that the derivatives of $\overline{y}$ are bounded by a constant C_y and that it holds for the entries of $a_y - \tilde{a}_y$

$$\begin{aligned} E_i\left[|a_y^{(d)} - \tilde{a}_y^{(d)}|^2\right] &= \begin{cases} E_i\left[|s-t_i|^2\right] & d = 1 \\ E_i\left[W_s^{(d-1)} - W_{t_i}^{(d-1)}|^2\right] & d > 1 \end{cases} \\ &\le \begin{cases} \Delta^2 & d = 1 \\ \Delta & d > 1 \end{cases}, \end{aligned}$$

since W is a Brownian motion. In the case $p = 1$, we have to continue the Taylor expansion an additional step and get similarly:

$$\begin{aligned} &E_i\left[(\overline{y}(s,W_s) - \overline{y}(t_i,W_{t_i}))\right] \\ &= E_i\left[\nabla\overline{y}(a_y)^T(a_y - \tilde{a}_y) + \frac{1}{2}\int_0^1 (1-\Theta)(a_y - \tilde{a}_y)^T \operatorname{Hess}_{\overline{y}}(a_y + \Theta(\tilde{a}_y - a_y))(a_y - \tilde{a}_y)d\Theta\right] \\ &\le E_i\left[\nabla\overline{y}(a_y)^T(a_y - \tilde{a}_y)\right] + \frac{1}{2}E_i\left[\sup_{\Theta\in[0,1]} (a_y - \tilde{a}_y)^T \operatorname{Hess}_{\overline{y}}(a_y + \Theta(\tilde{a}_y - a_y))(a_y - \tilde{a}_y)\right]. \end{aligned}$$

Now $\nabla\overline{y}(a_y)$ is $\mathcal{F}_i$-measurable and $E_i[(a_y - \tilde{a}_y)] = (s - t_i, 0, \dots, 0)^T$, since $W_s - W_{t_i}$ is independent of $\mathcal{F}_{t_i}$ and has expectation 0. Additionally, using that $\overline{y}$ has bounded derivatives, we conclude

$$\begin{aligned} E_i\left[(\overline{y}(s,W_s) - \overline{y}(t_i,W_{t_i}))\right] &\le C_y\Delta + \frac{1}{2}\sum_{d,l=1}^{1+\mathcal{D}} C_f E_i\left[|(a_y^{(d)} - \tilde{a}_y^{(d)})|^2\right]^{\frac{1}{2}} E_i\left[|(a_y^{(l)} - \tilde{a}_y^{(l)})|^2\right]^{\frac{1}{2}} \\ &\le c\Delta. \end{aligned}$$

It remains to show that $E_i[\left(z^{(d)}(s,W_s) - \overline{z}_i^N(W_{t_i})^{(d)}\right)^p]$ is bounded by a multiple of Δ for $p \in \{0,1\}$. For this, we first rewrite the $d-th$ component of $\overline{z}_i^N$ by a Taylor expansion on

y as

$$
\begin{aligned}
\overline{z}_i^{N,(d)} &= E_i\left[\frac{\Delta W_{i+1}^{(d)}}{\Delta}\overline{y}(t_{i+1}, W_{t_{i+1}})\right] \\
&= E_i\Bigg[\frac{\Delta W_{i+1}^{(d)}}{\Delta}\Bigg(\overline{y}(t_i, W_{t_i}) + \sum_{e=1}^{\mathcal{D}}\frac{\partial}{\partial x^{(e)}}\overline{y}(t_i, W_{t_i})(\Delta W_{i+1}^{(e)}) + \frac{\partial}{\partial t}\overline{y}(t_i, W_{t_i})\Delta \\
&\quad + \frac{1}{2}\sum_{e,l=1}^{\mathcal{D}}\frac{\partial^2}{\partial x^{(e)}\partial x^{(l)}}\overline{y}(t_i, W_{t_i})(\Delta W_{i+1}^{(e)})(\Delta W_{i+1}^{(l)}) + \frac{1}{2}\sum_{e=1}^{\mathcal{D}}\frac{\partial^2}{\partial t\partial x^{(e)}}\overline{y}(t_i, W_{t_i})(\Delta W_{i+1}^{(e)})\Delta \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial^2 t}\overline{y}(t_i, W_{t_i})\Delta^2 + \frac{1}{6}\int_0^1(1-\Theta)\Bigg(\frac{\partial^3}{\partial^3 t}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})\Delta^3 \\
&\quad + \sum_{e,l,k=1}^{\mathcal{D}}\frac{\partial^3}{\partial x^{(e)}\partial x^{(l)}\partial x^{(k)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})(\Delta W_{i+1}^{(l)})(\Delta W_{i+1}^{(k)}) \\
&\quad + \sum_{e,l=1}^{\mathcal{D}}\frac{\partial^3}{\partial t\partial x^{(e)}\partial x^{(l)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})(\Delta W_{i+1}^{(l)})\Delta \\
&\quad + \sum_{e=1}^{\mathcal{D}}\frac{\partial^3}{\partial^2 t\partial x^{(e)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})\Delta^2\Bigg)d\Theta\Bigg)\Bigg].
\end{aligned}
$$

Since the derivatives of $\overline{y}$ are $\mathcal{F}_i$-measurable when evaluated in (t_i, W_{t_i}) and the components of $W_{t_{i+1}} - W_{t_i}$ are independent with mean 0 each, most terms in the right hand side of the equality above vanish and we get

$$\begin{aligned}
\overline{z}_i^{N,(d)} &= E_i\left[\frac{\Delta W_{i+1}^{(d)}}{\Delta}\frac{\partial}{\partial x^{(d)}}\overline{y}(t_i, W_{t_i})\Delta W_{i+1}^{(d)}\right] \\
&\quad + \frac{1}{2}E_i\left[\frac{\Delta W_{i+1}^{(d)}}{\Delta}\frac{\partial^2}{\partial t\partial x^{(d)}}\overline{y}(t_i, W_{t_i})\Delta W_{i+1}^{(d)}\Delta\right] \\
&\quad + \frac{1}{6}E_i\Bigg[\frac{\Delta W_{i+1}^{(d)}}{\Delta}\int_0^1(1-\Theta)\bigg(\frac{\partial^3}{\partial^3 t}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})\Delta^3 \\
&\quad + \sum_{e,l,k=1}^{\mathcal{D}}\frac{\partial^3}{\partial x^{(e)}\partial x^{(l)}\partial x^{(k)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})(\Delta W_{i+1}^{(l)})(\Delta W_{i+1}^{(k)}) \\
&\quad + \sum_{e,l=1}^{\mathcal{D}}\frac{\partial^3}{\partial t\partial x^{(e)}\partial x^{(l)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})(\Delta W_{i+1}^{(l)})\Delta \\
&\quad + \sum_{e=1}^{\mathcal{D}}\frac{\partial^3}{\partial^2 t\partial x^{(e)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})\Delta^2\bigg)d\Theta\Bigg] \\
&= \frac{\partial}{\partial x^{(d)}}\overline{y}(t_i, W_{t_i}) + R_T
\end{aligned}$$

where we set

$$\begin{aligned}
R_T := \frac{1}{2}\frac{\partial^2}{\partial t\partial x^{(d)}}\overline{y}(t_i, W_{t_i})\Delta + \frac{1}{6}E_i\Bigg[\frac{\Delta W_{i+1}^{(d)}}{\Delta}\int_0^1(1-\Theta)\bigg(\frac{\partial^3}{\partial^3 t}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})\Delta^3 \\
+ \sum_{e,l,k=1}^{\mathcal{D}}\frac{\partial^3}{\partial x^{(e)}\partial x^{(l)}\partial x^{(k)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{t_{i+1}}^{(e)})(\Delta W_{t_{i+1}}^{(l)})(\Delta W_{t_{i+1}}^{(k)}) \\
+ \sum_{e,l=1}^{\mathcal{D}}\frac{\partial^3}{\partial t\partial x^{(e)}\partial x^{(l)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})(\Delta W_{i+1}^{(l)})\Delta \\
+ \sum_{e=1}^{\mathcal{D}}\frac{\partial^3}{\partial^2 t\partial x^{(e)}}\overline{y}(t_i+\Theta\Delta, W_{t_i}+\Theta\Delta W_{i+1})(\Delta W_{i+1}^{(e)})\Delta^2\bigg)d\Theta\Bigg].
\end{aligned}$$

Now since the derivatives of y are bounded by assumption, we get

$$\begin{aligned}
|R_T| \le cC_y \Bigg(& \Delta + \Delta^2 E_i\left[|\Delta W_{i+1}^{(d)}|\right] \\
& + \sum_{e,l,k=1}^{\mathcal{D}} \frac{1}{\Delta} E_i \left[|W_{t_{i+1}}^{(d)} - W_{t_i}^{(d)}||W_{t_{i+1}}^{(e)} - W_{t_i}^{(e)}||W_{t_{i+1}}^{(l)} - W_{t_i}^{(l)}||W_{t_{i+1}}^{(k)} - W_{t_i}^{(k)}| \right] \\
& + \sum_{e,l=1}^{\mathcal{D}} E_i \left[|W_{t_{i+1}}^{(d)} - W_{t_i}^{(d)}||W_{t_{i+1}}^{(e)} - W_{t_i}^{(e)}||W_{t_{i+1}}^{(l)} - W_{t_i}^{(l)}| \right] \\
& + \sum_{e=1}^{\mathcal{D}} \Delta E_i \left[|W_{t_{i+1}}^{(d)} - W_{t_i}^{(d)}||W_{t_{i+1}}^{(e)} - W_{t_i}^{(e)}| \right] \Bigg) \\
\le c & \left(\Delta + \Delta^{\frac{5}{2}} + \mathcal{D}^3\Delta + \mathcal{D}^2\Delta^{\frac{3}{2}} + \mathcal{D}\Delta^2 \right) \\
\le c & \Delta.
\end{aligned}$$

Then, since $\overline{z}(s, W_s) = \nabla_x \overline{y}(s, W_s)$, where $\nabla_x \overline{y}$ denotes the vector of first degree partial derivatives of $\overline{y}$ with respect to $x^{(1)}, \dots, x^{(\mathcal{D})}$, it follows

$$\begin{aligned}
& E_i \left[\left(\overline{z}^{(d)}(s, W_s) - \overline{z}_i^{N,(d)} \right)^p \right] \\
& = E_i \left[\left(\frac{\partial}{\partial x^{(d)}} \overline{y}(s, W_s) - \frac{\partial}{\partial x^{(d)}} \overline{y}(t_i, W_{t_i}) - R_T \right)^p \right] \\
& \le pE_i \left[\left(\frac{\partial}{\partial x^{(d)}} \overline{y}(s, W_s) - \frac{\partial}{\partial x^{(d)}} \overline{y}(t_i, W_{t_i}) \right)^p \right] + pE_i\left[|R|^p\right] \\
& \le pE_i \left[\left(\frac{\partial}{\partial x^{(d)}} \overline{y}(s, W_s) - \frac{\partial}{\partial x^{(d)}} \overline{y}(t_i, W_{t_i}) \right)^p \right] + c\Delta^p.
\end{aligned} \tag{1.13}$$

The term $E_i[\frac{\partial}{\partial x^{(d)}} \overline{y}(s, W_s) - \frac{\partial}{\partial x^{(d)}} \overline{y}(t_i, W_{t_i}))^p]$ is for $p \in \{1, 2\}$ bounded by $c\Delta$ for a constant c not depending on Δ, which follows by the same calculations used for the term $E_i[(\overline{y}(s, W_s) - \overline{y}(t_i, W_{t_i}))^p]$ where we have to replace $\overline{y}$ by its first partial derivative $\frac{\partial}{\partial x^{(d)}} \overline{y}$. Note that the Taylor expansion than uses the derivatives of $\overline{y}$ up to degree 3, which still all exist are bounded by assumption. This finishes the proof of (1.12) and hence the bound on $\mathcal{R}_i^N$. Also, note that (1.13) for $s = t_i$ shows in particular that

$$E_i \left[\left(\overline{z}^{(d)}(t_i, W_{t_i}) - \overline{z}_i^{N,(d)} \right)^p \right] \le c\Delta$$

which completes the proof of the bound in (1.11).
It remains to show that, whenever $\overline{y}$ bounded and $s + 1$ times differentiable with bounded derivatives, the functions $\overline{q}_i^N$ and $\overline{z}_i^N$ are bounded as well and are respectively $s + 1$ and s times differentiable with bounded derivatives. For this, we can simply use that the

components of ΔW_{i+1} are independent and Gaussian-distributed with mean 0 and variance Δ each. Hence we have

$$\begin{aligned}\overline{q}_i^N(x) &= E_i\left[\overline{y}(t_{i+1}, W_{t_{i+1}})\middle|W_{t_i}=x\right] = E_i\left[\overline{y}(t_{i+1}, \Delta W_{i+1} + W_{t_i})|W_{t_i}=x\right] \\ &= \int_{\mathbb{R}^{\mathcal{D}}} \overline{y}(t_{i+1}, \tilde{x}+x)\frac{1}{(\sqrt{2\pi\Delta})^{\mathcal{D}}}e^{-\frac{1}{2}\sum_{d=1}^{\mathcal{D}}\frac{(\tilde{x}_i)^2}{\Delta}}l(d\tilde{x}).\end{aligned}$$

Then, since $\overline{y}$ is differentiable in x with bounded derivative, we can partial differentiate under the integral and get

$$\begin{aligned}\frac{\partial}{\partial x^{(d)}}\overline{q}_i^N(x) &= \int_{\mathbb{R}^{\mathcal{D}}} \frac{\partial}{\partial x^{(d)}}\overline{y}(t_{i+1}, \tilde{x}+x)\frac{1}{(\sqrt{2\pi\Delta})^{\mathcal{D}}}e^{-\frac{1}{2}\sum_{d=1}^{\mathcal{D}}\frac{(\tilde{x}_i)^2}{\Delta}}l(d\tilde{x}) \\ &= E\left[\frac{\partial}{\partial x^{(d)}}\overline{y}(t_i, W_{t_{i+1}})\middle|W_{t_i}=x\right]\end{aligned}$$

for each $d \in \{1, \dots, \mathcal{D}\}$, which shows that $\overline{q}_i^N$ is continuous differentiable with bounded derivative. The same argumentation can be applied for the higher order derivatives.
Next, we consider the first coordinate of $\overline{z}_i^N$ as the derivatives of the others follow analogously. By the definition of $\overline{z}_i^N$ and Fubini's law, we have

$$\begin{aligned}\overline{z}_i^{N,(1)}(x) &= E_i\left[\frac{\Delta W_{i+1}^{(1)}}{\Delta}\overline{y}(t_{i+1}, W_{t_{i+1}})\middle|W_{t_i}=x\right] \\ &= \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\prod_{d=2}^{\mathcal{D}}\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{\left(\tilde{x}^{(d)}\right)^2}{2\Delta}}\int_{\mathbb{R}}\frac{\tilde{x}^{(1)}}{\Delta}\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{\left(\tilde{x}^{(1)}\right)^2}{2\Delta}}\overline{y}(t_{i+1}, x+\tilde{x})d\tilde{x}^{(1)}d\tilde{x}^{(2)}\dots d\tilde{x}^{(\mathcal{D})}.\end{aligned}$$

Since $\overline{y}$ is bounded by assumption, integration by parts leads to

$$\begin{aligned}\overline{z}_i^{N,(1)}(x) &= \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\prod_{d=2}^{\mathcal{D}}\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{\left(\tilde{x}^{(d)}\right)^2}{2\Delta}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{\left(\tilde{x}^{(1)}\right)^2}{2\Delta}}\frac{\partial}{\partial x^{(1)}}\overline{y}(t_{i+1}, x+\tilde{x})d\tilde{x}^{(1)}d\tilde{x}^{(2)}\dots d\tilde{x}^{(\mathcal{D})} \\ &= E_i\left[\frac{\partial}{\partial x^{(1)}}\overline{y}(t_{i+1}, x+\tilde{x})\middle|W_{t_i}=x\right] = \frac{\partial}{\partial x^{(1)}}\overline{q}^N(x).\end{aligned}$$

Hence the statement for $\overline{z}_i^N$ follows by the one for $\overline{q}_i^N$. □

Chapter 2

Convergence rates lower bounds for BSDEs with convex driver

In this section, we analyze an algorithm for constructing lower bounds for BSDEs with a convex driver. Here the algorithm itself is not new, as it was already presented in a similar form in Belomestny et al. (2014) or Bender et al. (2017b). Usually, it is paired with a second algorithm that returns an upper bound which allows the construction of confidence intervals for the true solution. While this method is well researched in the special case of Bermudan option pricing (see e.g. Andersen and Broadie, 2004, Belomestny et al., 2014, Belomestny, 2011), where it was initially introduced, results regarding convergence rates in the general case are still lacking. The aim of this chapter is a detailed error analysis in order to provide such results in a general setting.
In Section 2.1, we introduce the setting under which we perform the analysis. Here we start with a dynamic programming equation linked to a time discretization of a BSDE with a convex driver and review how this equation can be presented as a maximization problem. Then in Section 2.2, we present the algorithm for solving this maximization problem approximately, which returns lower bounds for the true solution. Here we also state the rate of convergence of the algorithm, which is the main result of this chapter. Section 2.3 is then dedicated to the proof of this result which we accomplish by a complete error analysis where we derive bounds for the bias and the variance of the approximation. We illustrate our results in a numerical example in Section 2.4, where we test the algorithm with an option pricing problem under credit value adjustment.
This is continued work from the Master thesis "Multi-Level-Monte-Carlo für nicht-lineare Optionsbewertungsprobleme" (Meyer, 2017). There, most components of the equation were assumed to be bounded. The focus was on a multi-level approach in a setting where the maximization problem was controlled by a process with only two different states, much similar to the work of Belomestny et al. (2015) focused on Bermudan option pricing. In this chapter, in comparison, we refrain from a multi-level approximation. We focus only on

a detailed error analysis but consider a much more general class of dynamic programming equations with weaker integrability assumptions instead of bounded coefficients and allow a much more general class of control processes.

Notation:

Since we focus on a discrete time setting in this chapter, the notation may vary slightly from the one used in Chapter 1. We try, however, to keep the notation mostly consistent and denote similar objects analog to their corresponding terms in the continuous time setting in Chapter 1.

2.1 Main setting

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probablilty space in discrete time. Furthermore, let X be a D-dimensional Markovian Process in discrete time defined by

$$\begin{aligned} X_0 &:= x_0 \\ X_j &:= h\left(X_{j-1}, B_j\right) \quad j = 1, \dots, J \end{aligned}$$

for a deterministic function $h : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^D$ and initial value $x_0 \in \mathbb{R}^D$. We suppose that the process B is at any time j given by

$$B_j := \left(\Delta W_j^{(1)}, \dots, \Delta W_j^{(D)}\right)$$

for independent Gaussian random variables $\Delta W_j^{(d)}$, $d = 1, \dots, D$ with mean zero and variance Δ each, such that $W_j^{(d)}$ is $\mathcal{F}_j$-measurable and independent of $\mathcal{F}_{j-1}$. Throughout this chapter, we then consider dynamic programming equations of the form

$$\begin{aligned} Y_J &:= \xi(X_J) \\ Y_j &:= E\left[Y_{j+1}|\mathcal{F}_j\right] + \Delta f_j\left(X_j, E\left[\beta_{j+1} Y_{j+1}|\mathcal{F}_j\right]\right), \quad j = J-1, \dots, 0 \end{aligned} \tag{2.1}$$

for measurable functions $\xi : \mathbb{R}^D \to \mathbb{R}$ and $f_j : \mathbb{R}^D \times \mathbb{R}^{D+1} \to \mathbb{R}$, where the process β is defined as

$$\beta_j := \left(1, \frac{\left[\Delta W_j^{(1)}\right]_{\varsigma\sqrt{\Delta}}}{\Delta}, \dots, \frac{\left[\Delta W_j^{(D)}\right]_{\varsigma\sqrt{\Delta}}}{\Delta}\right)^T \quad j = 1, \dots, J$$

and $[.]_{\varsigma\sqrt{\Delta}} : \mathbb{R} \to [-\varsigma\sqrt{\Delta}, \varsigma\sqrt{\Delta}]$ is the truncation function defined by

$$[x]_{\varsigma\sqrt{\Delta}} = \operatorname{sign}(x) \min\left\{\varsigma\sqrt{\Delta}, |x|\right\}$$

for a constant $\varsigma > 0$.

Remark 2.1.1. Equations of this type not only appear in numerous stochastic control problems but are also directly linked to BSDEs in the following way:
Consider a forward-backward stochastic differential equation with terminal time T, driven by a D-dimensional Brownian motion $\mathcal{W}$ of the form

$$\begin{aligned} \mathcal{X}_0 &= x_0 \\ d\mathcal{X}_t &= b(t, X_t)dt + \sigma(t, X_t)d\mathcal{W}_t \\ -d\mathcal{Y}_t &= f(t, \mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)dt - \mathcal{Z}_t d\mathcal{W}_t \\ \mathcal{Y}_T &= \xi(\mathcal{X}_T) \end{aligned}$$

and an equidistant time grid $\pi = \{t_0 = 0, t_1 = \Delta, \dots, t_J = T\}$ with step width Δ. By the usual steps in discretizing the BSDE, we can write

$$\begin{aligned} \mathcal{Y}_{t_i} &= E\left[\mathcal{Y}_{t_i} | \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right] \\ &= E\left[\mathcal{Y}_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(s, \mathcal{X}_s, \mathcal{Y}_s, \mathcal{Z}_s)ds \middle| \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right] \\ &\approx E\left[\mathcal{Y}_{t_{i+1}} | \sigma((\mathcal{W}_s)_{s \le t_i})\right] + \Delta f\left(t_i, \mathcal{X}_{t_i}, E\left[\mathcal{Y}_{t_{i+1}} | \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right], \mathcal{Z}_{t_i}\right) \end{aligned}$$

or, similarly without the conditional expectation

$$\mathcal{Y}_{t_i} \approx \mathcal{Y}_{t_{i+1}} + \Delta f(t_i, \mathcal{X}_{t_i}, \mathcal{Y}_{t_i}, \mathcal{Z}_{t_i}) - \mathcal{Z}_{t_i}(\mathcal{W}_{t_{i+1}} - \mathcal{W}_{t_i}). \tag{2.2}$$

Multiplying both sides in (2.2) with the Brownian increment and taking conditional expectation afterward leads to

$$\mathcal{Z}_{t_i} \approx E\left[\frac{\mathcal{W}_{t_{i+1}} - \mathcal{W}_{t_i}}{\Delta} \mathcal{Y}_{t_{i+1}} \middle| \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right]$$

and we obtain the discretization scheme

$$\begin{aligned} Y_J &:= \xi(\mathcal{X}_T) \\ Y_j &:= E\left[Y_{j+1} | \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right] + \Delta f\left(t_j, \mathcal{X}_{t_j}, E\left[Y_{j+1} | \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right], Z_j\right) \quad j = 0, \dots, J-1 \\ Z_j &:= E\left[\frac{\mathcal{W}_{t_{i+1}} - \mathcal{W}_{t_i}}{\Delta} Y_{j+1} \middle| \sigma\left((\mathcal{W}_s)_{s \le t_i}\right)\right] \quad j = 0, \dots, J-1. \end{aligned}$$

Here, similarly to Chapter 1, it holds under appropriate assumptions on the components that

$$\lim_{N\to\infty}\left(\max_{i=0,\dots,N-1} E\left[|Y_i - \mathcal{Y}_t|^2\right] + \sum_{i=0}^{N-1} E\left[\int_{t_i}^{t_{i+1}} |Z_i - \mathcal{Z}_s| ds^2\right]\right) = 0,$$

see e.g. Zhang (2001). Then, by truncating the Brownian increments in the definition of Z_j and setting

$$\tilde{f}_j(x,y) := f(t_j, x, y^{(1)}, (y^{(2)}, \dots, y^{(D+1)})^T)$$

for $x \in \mathbb{R}^D$ and $y \in \mathbb{R}^{D+1}$ and

$$\tilde{\beta}_{j+1} := \left(1, \frac{\left[\mathcal{W}^{(1)}_{t_{i+1}} - \mathcal{W}^{(1)}_{t_i}\right]_{\varsigma\sqrt{\Delta}}}{\Delta}, \dots, \frac{\left[\mathcal{W}^{(D)}_{t_{i+1}} - \mathcal{W}^{(D)}_{t_i}\right]_{\varsigma\sqrt{\Delta}}}{\Delta}\right)^T,$$

we can rewrite the discretization scheme in the form of the dynamic programming equation (2.1) as

$$\begin{aligned} Y_J &= \xi(\mathcal{X}_J) \\ Y_j &= E\left[Y_{j+1}\middle|\sigma\left((\mathcal{W}_s)_{s\le t_i}\right)\right] + \Delta\tilde{f}_j\left(\mathcal{X}_{t_j}, E\left[\tilde{\beta}_{j+1}Y_{j+1}\middle|\sigma\left((\mathcal{W}_s)_{s\le t_i}\right)\right]\right). \end{aligned} \tag{2.3}$$

By these considerations, we can interpret Δ as the step length of an equidistant time grid on the interval $[0,T]$ for $T = J\Delta$ and B_j as the increment of a D-dimensional Brownian motion between two time points in the grid.

Throughout this chapter the following assumptions are in charge:

Assumptions 2.1.2. Standing assumptions

(A_c) For each $x \in \mathbb{R}^D$ and $j \in \{0,\dots,J-1\}$, the map $y \mapsto f_j(x,y)$ is convex.

(A_f) The functions f_j are uniformly Lipschitz continuous with constant L_f and it holds

$$\sum_{j=0}^{J-1} E[|f_j(X_j,0)|^{2+\epsilon}] < \infty$$

for some $\epsilon > 0$.

(A_ξ) The function $\xi : \mathbb{R}^D \to \mathbb{R}$ is deterministic and it holds

$$E[|\xi(X_J)|^{2+\epsilon}] < \infty$$

for some $\epsilon > 0$.

(A_M) The parameter Δ is sufficiently small such that $1 \ge L_f\sqrt{\Delta^2 + \varsigma^2 D\Delta}$.

When viewed as discretization of a BSDE, the Lipschitz continuity of the functions f_j results from the usual conditions for BSDEs. Paired with the integrability conditions on ξ and f_j, this especially ensures that the P-almost surely unique solution to (2.1) satisfies

$$\sum_{j=0}^{J-1} E\left[|Y_j|^{2+\epsilon}\right] < \infty$$

where the integrability follows easily by a backward induction using Minkowski's inequality and the Lipschitz continuity of the functions f_j. Assumption (A_M) can be thought of a monotony assumption, as it ensures the following comparison principle.

Corollary 2.1.3. *Let Y and $\tilde{Y}$ be F_{j+1}-measurable random variables such that $Y \geq \tilde{Y}$ P-almost surely. Then*

$$E\big[Y\big|\mathcal{F}_j\big] + \Delta f_j\big(X_j, E\big[\beta_{j+1}, Y\big|\mathcal{F}_j\big]\big) \geq E\big[\tilde{Y}\big|\mathcal{F}_j\big] + \Delta f_j\big(X_j, E\big[\beta_{j+1}\tilde{Y}\big|\mathcal{F}_j\big]\big).$$

Proof. By the Lipschitz continuity of the functions f_j, Hölder's inequality and assumption (A_M), we have

$$\begin{aligned}
&E[Y|\mathcal{F}_j] - E[\tilde{Y}|\mathcal{F}_j] + \Delta\left(f_j(X_j, E[\beta_{j+1}, Y|\mathcal{F}_j]) - f_j(X_j, E[\beta_{j+1}, \tilde{Y}|\mathcal{F}_j]\right) \\
&\geq E_j[Y - \tilde{Y}|\mathcal{F}_j] - \Delta L_f \left|E[\beta_{j+1}, Y|\mathcal{F}_j] - E[\beta_{j+1}, \tilde{Y}|\mathcal{F}_j]\right| \\
&\geq E[(Y - \tilde{Y})|\mathcal{F}_j] - \Delta L_f E[|\beta_{j+1}|(Y - \tilde{Y})|\mathcal{F}_j] \\
&\geq E[(Y - \tilde{Y})|\mathcal{F}_j]\left(1 - \Delta L_f \sqrt{1 + \varsigma^2 \frac{D}{\Delta}}\right) \\
&= E[(Y - \tilde{Y})|\mathcal{F}_j]\left(1 - L_f \sqrt{\Delta^2 + \varsigma^2 D\Delta}\right) \geq 0
\end{aligned}$$

where we used that $Y - \tilde{Y} \geq 0$ P-almost surely. □

The convexity assumption is crucial for the algorithm we will present in the next section. It does not seem to be too restrictive since equations of the form (2.3) were mainly studied due to their application in financial mathematics, where numerous non-linear option pricing problems can be related to (2.1) such that the convexity assumption holds true. In the following, we give some examples of such option pricing problems.

Example 2.1.4.

(i) American/Bermudan option pricing:
The simplest framework that fits in the setting and lead to equations of the form (2.1) in the first place (see e.g. Andersen and Broadie, 2004 and Haugh and Kogan, 2004) is the pricing of American options. Consider a standard Black-Scholes market with

one stock $\mathcal{X}$ given by

$$d\mathcal{X}_t = r\mathcal{X}_t + \sigma\mathcal{X}_t d\mathcal{W}_t$$

under the risk-free measure, where r is the interest rate in the market, σ the volatility of the stock and $\mathcal{W}$ is a Brownian motion. It is well known that the value function of an American option on $\mathcal{X}$ is then given by

$$\mathcal{Y}_t(x) = \max_{t\le\tau\le T} E[g(\tau, \mathcal{X}_\tau)|\mathcal{X}_t = x]$$

where the maximum runs over all $\mathbb{F}$-stopping times τ and g is the discounted payoff function in case of execution of the option (see e.g. Haugh and Kogan, 2004). Then typically, the value function is approximated by the one of a Bermudan option with the possible exercise dates $t_1, \dots t_J$ for large J, which leads to

$$\mathcal{Y}_{t_j}(x) \approx \sup_{\tau\in\mathcal{T}_J} E[g(\mathcal{X}_\tau)|\mathcal{X}_{t_j} = x].$$

Here $\mathcal{T}_J$ is the set of all $\mathbb{F}$-stopping times taking values in $\{t_0, t_1, \dots, t_J\}$. Then making use of the finite possible exercise dates, we can approximate the price process as

$$\begin{aligned} Y_j &= \max\{g_j(X_j, E[Y_{j+1}|X_j]\} \\ &= E[Y_{j+1}|X_j] + \Delta f_j(X_j, E[\beta_{j+1}Y_{j+1}]) \end{aligned}$$

with $X_j := \mathcal{X}_{t_j}$, $f_j(x,y) := \frac{1}{\Delta}\max_{\rho\in\{0,1\}} \rho(g(x) - y^{(1)})$ and β defined as in (2.1).

(ii) Option pricing with credit value adjustment:
A second example is option pricing in a model with a default risk of the trading partner. Consider a BSDE of the form

$$d\mathcal{Y}_t = r\mathcal{Y}_t - (\lambda(R-1)\mathcal{Y}_t)_- dt + \mathcal{Z}^T d\tilde{W}_t, \quad \mathcal{Y}_T = g(\mathcal{X}_T)$$

for constants $\lambda > 0$, $r \ge 0$, $R \in [0,1]$ and a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$. Once again we suppose that $\mathcal{X}$ is a (possibly multidimensional) Black-Scholes model, i.e., the components of $\mathcal{X}$ are given by

$$d\mathcal{X}_t^{(d)} = \mathcal{X}_t^{(d)}\mu dt + \mathcal{X}_t^{(d)}\sigma d\mathcal{W}_t^{(d)}$$

where $\mathcal{W}^{(d)}$ is the d-th component of a D-dimensional Brownian motion. Then $-(\mathcal{Y}_t)_{t\in[0,T]}$ describes the price process of an option with payoff-function $-g$ at the maturity T with credit value adjustment, assuming that default of the trading partner did not occur prior to time t and that default occurs at the first jump of a Poisson process with intensity λ (see e.g. Crépey, 2015, Duffie et al., 1996). Here r is the risk-free interest rate, σ the volatility of the stocks and R is the recovery rate in case of default. By discretizing the BSDE as described in Remark 2.1.1 we obtain

$$\begin{aligned}
\mathcal{Y}_T &\approx Y_J = g(\mathcal{X}_T) \\
\mathcal{Y}_{t_j} &\approx Y_j := E_j[Y_{j+1}] + \Delta(-rE_j[Y_{j+1}] + (\lambda(R-1)E_j[Y_{j+1}])_-)
\end{aligned}$$

which is in the form of (2.1) with $f_j(x,y) = \max_{\rho\in\{-r,-(r+\lambda(1-R))\}} \rho y^{(1)}$. The formulation in terms of the negated price process is necessary to obtain convex functions f_j. This example will be covered in more detail in Section 2.4 as it is used as a numerical example to illustrate the theoretical results.

(iii) Option pricing under funding costs:
As a final example, we consider option pricing under funding costs in a financial market with the different interest rates R^l for lending money and R^b for borrowing money. Suppose there are D risky stocks $(\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(D)})$ which are modeled independently with the same drift μ and volatility σ as a solution of the SDEs

$$d\mathcal{X}_t^{(d)} = \mathcal{X}_t^{(d)}\mu dt + \mathcal{X}_t^{(d)}\sigma d\mathcal{W}_t^{(d)}$$

where $\mathcal{W}^{(d)}$ is the d-th component of a D-dimensional Brownian motion. Consider an option on those D stocks with square-integrable payoff $g(\mathcal{X}_T)$ at maturity T. Then, the value process $\mathcal{Y}$ and a replicant portfolio $\mathcal{Z}$ for the claim $g(\mathcal{X}_T)$ are given by the solution $(\mathcal{Y},\mathcal{Z})$ of the BSDE

$$\begin{aligned}
\mathcal{Y}_T &= g(\mathcal{X}_T) \\
d\mathcal{Y}_t &= -f(t,\mathcal{Y}_t,\mathcal{Z}_t)dt + \mathcal{Z}_t d\mathcal{W}_t
\end{aligned}$$

with

$$f(t,y,z) = -R^l y - \frac{\mu - R^l}{\sigma}\sum_{d=1}^{D} z^d + (R^b - R^l)\left(y - \sum_{d=1}^{D}\frac{1}{\sigma}z^d\right)_-,$$

see e.g. Bergman (1995) and El Karoui et al. (1997). Here $\mathcal{Y}$ models the price of the portfolio and $\mathcal{Z}$ the amount of money held in the D stocks. Discretizing the BSDE on an equidistant time grid leads to

$$\begin{aligned}
\mathcal{Y}_T &= Y_J = g(X_J) \\
\mathcal{Y}_{t_j} &\approx Y_j = E_j[Y_{j+1}] + \Delta\Big(- R^l e_1^T E[\beta_{j+1}Y_{j+1}] \\
&\quad - \sum_{d=2}^{D+1}\frac{\mu - R^l}{\sigma} e_d^T E[\beta_{j+1}Y_{j+1}] + (R^b - R^l)\left(e_1 - \sum_{d=2}^{D+1}\frac{e_d}{\sigma}\right)^T E[\beta_{j+1}Y_{j+1}])_-\Big)
\end{aligned}$$

where we denote the $i-th$ canonical basis vector in $\mathbb{R}^{D+1}$ with e_i and set

$$\beta_{j+1} := \left(1, \frac{\left[\mathcal{W}^{(1)}_{t_{i+1}} - \mathcal{W}^{(1)}_{t_i}\right]_{\varsigma\sqrt{\Delta}}}{\Delta}, \dots, \frac{\left[\mathcal{W}^{(D)}_{t_{i+1}} - \mathcal{W}^{(D)}_{t_i}\right]_{\varsigma\sqrt{\Delta}}}{\Delta}\right)^T$$

for $j = 0, \dots, J$. Through rewriting the negative part, we get the representation

$$\begin{aligned} Y_J &= g(X_J) \\ Y_j &= E_j[Y_{j+1}] + \Delta f_j(x, E_j[\beta_{j+1} Y_{j+1}]) \end{aligned}$$

with $f(x,y) := \max\{\rho^T y, \tilde{\rho}^T y\}$ and

$$\begin{aligned} \rho &:= \left(-R^l, -\frac{\mu - R^l}{\sigma}, \dots, -\frac{\mu - R^l}{\sigma}\right)^T, \\ \tilde{\rho} &:= \left(-R^b, -\frac{\mu - R^b}{\sigma}, \dots, -\frac{\mu - R^b}{\sigma}\right)^T. \end{aligned}$$

In all the examples above, the functions $f_j(x,.)$ are not just convex but can, for fixed x, even be written as the pointwise maximum of a finite set of affine functions. Motivated by this, we will consider (2.1) once under the standing assumptions defined above and once additionally in the more restrictive setting where we replace assumption (A_c) with the following stronger assumption.

Assumptions 2.1.5. Special case

(A_S) For every $j = 0, \dots, J-1$, the function f_j is given by

$$f_j(x,y) = \max_{k \in \mathcal{K}} k^T y + k^T b_j(x) + a_j(x)$$

for a finite set $\mathcal{K} = \{k_1, \dots, k_\kappa\} \subset \mathbb{R}^{D+1}$ and deterministic, bounded functions $b_j : \mathbb{R}^D \to \mathbb{R}^{D+1}$ and $a_j : \mathbb{R}^D \to \mathbb{R}$.

The algorithm we will present in the next chapter is based on an alternative representation of Y in the form of an optimization problem using the convex conjugates of the functions f_j, which are defined pointwise for each $x \in \mathbb{R}^D$ as

$$f_j^\#(x,\rho) = \sup_{y \in \mathbb{R}^{D+1}} \rho^T y - f_j(x,y)$$

where we restrict these functions on their effective domain

$$D_f^{(j,x)} := \left\{\rho \in \mathbb{R}^{D+1} : f_j^\#(x,\rho) < \infty\right\}.$$

Using the convex conjugates the following alternative representation of Y can be derived.

Theorem 2.1.6. *Under the standing assumptions it holds for all $j \in 0, \ldots, J$*

$$Y_j = \sup_{\rho \in D_{f\#}} E\left[\xi \prod_{i=j}^{J-1} \left(\Delta\rho_i^T \beta_{i+1} + 1\right) - \sum_{l=j}^{J-1} \Delta f_l^{\#}(X_l, \rho_l) \prod_{i=j}^{l-1} \left(\Delta\rho_i^T \beta_{i+1} + 1\right) \middle| \mathcal{F}_j \right] \tag{2.4}$$

where $D_{f\#}$ is the set of all $\mathbb{F}$-adapted processes $\rho = (\rho_0, \ldots, \rho_{J-1})$ such that

$$\sum_{i=0}^{J-1} E\left[|f_i^{\#}(X_i, \rho_i)|^{2+\epsilon}\right] < \infty.$$

Furthermore, there exists a process $\rho^* \in D_{f\#}$ that maximizes (2.4).

A proof under weaker assumptions (which translate to a weaker integrability of the convex conjugates) can be found in Bender et al. (2017b). We state the analog proof again as it illustrates important features of the representation. We will split up the proof and first show one key argument in a general formulation which will be particularly useful later on.

Lemma 2.1.7. *Let $g : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{G} \times \mathcal{B}(\mathbb{R}^d)$-measurable for a σ-field $\mathcal{G}$ on Ω such that the map $y \mapsto g(\omega, y)$ is convex and Lipschitz continuous for all $\omega \in \Omega$ and $E[|g(0)|^p] \leq \infty$ for a $p > 1$. Furthermore, let C be $\mathcal{G}$-measurable, $\mathbb{R}^d$-valued random variable in $\mathcal{L}^p(\Omega, \mathcal{G}, P)$. Then there exists a $\mathcal{G}$-measurable, $\mathbb{R}^d$-valued random variable ρ such that*

$$g(\omega, C) = \rho^T C - g^{\#}(\omega, \rho)$$

and it holds $E[|g^{\#}(\rho)|^p] < \infty$.

Proof. Let C be an arbitrary $\mathbb{R}^d$-valued random variable in $L^p(\Omega, \mathcal{G}, P)$. Since $g(\omega, .)$ is convex and Lipschitz continuous for all $\omega \in \Omega$ and hence closed, it follows by the Fenchel-Moreau theorem that

$$(g^{\#})^{\#}(\omega, .) = g(\omega, .)$$

pointwise for all $\omega \in \Omega$. Hence we have for any $\mathcal{G}$-measurable, $\mathbb{R}^d$-valued random variable $\tilde{\rho}$ that

$$g(\omega, C) = \sup_{r \in \mathbb{R}^d} r^T C - g^{\#}(\omega, r) \geq \tilde{\rho}^T C - g^{\#}(\omega, \tilde{\rho}) \tag{2.5}$$

pointwise for all $\omega \in \Omega$. On the other hand, it holds due to the Lipschitz continuity of g that

$$g(\omega, C) - g(\omega, C + \tilde{C}) \leq |g(\omega, C) - g(\omega, C + \tilde{C})| \leq L|\tilde{C}|,$$

which implies

$$g(\omega, C + \tilde{C}) \geq g(\omega, C) - L|\tilde{C}|$$

for all $\mathbb{R}^d$-valued random variables $\tilde{C} \in L^p(\Omega, \mathcal{G}, P)$. It follows from Theorem 7.10 in Cheridito et al. (2015) that there exists a $\mathcal{G}$-measurable subgradient of g at C, i.e., a random variable ρ such that

$$g(\omega, C + \tilde{C}) - g(\omega, C) \geq \rho^T \tilde{C}$$

for all $\mathcal{G}$-measurable, $\mathbb{R}^d$-valued random variables $\tilde{C}$. In particular, it holds for any $r \in \mathbb{R}^d$ and $\tilde{C} := r - C$ that

$$\rho^T C - g(\omega, C) \geq \rho^T r - g(\omega, r).$$

Now taking the supremum over all $r \in \mathbb{R}^d$, we conclude

$$\rho^T C - g^{\#}(\omega, \rho) \geq g(\omega, C)$$

and equality follows by (2.5). For the integrability condition, note that we have by the Lipschitz continuity of g that

$$\begin{aligned} E\left[|g^{\#}(\rho)|^p\right]^{\frac{1}{p}} &= E\left[|\rho^T C - g(C)|^p\right]^{\frac{1}{p}} \\ &\leq E\left[|\rho^T C|^p\right]^{\frac{1}{p}} + E\left[|g(0) - g(C)|^p\right]^{\frac{1}{p}} + E\left[|g(0)|^p\right]^{\frac{1}{p}} \\ &\leq 2LE\left[|C|^p\right]^{\frac{1}{p}} + E\left[|g(0)|^p\right]^{\frac{1}{p}} < \infty \end{aligned}$$

by Minkowski's inequality and the integrability assumptions on g and C. Here we used that $|\rho| \leq L$ for each ρ in the effective domain of $g^{\#}$, where we denote the Lipschitz constant of g with L. This fact is proven later on in Lemma 2.3.1. Note that this shows in particular that $\rho(\omega)$ is for P-almost every ω an element of the effective domain of $g^{\#}$, i.e., it holds $g^{\#}(\omega, \rho) < \infty$ P-almost surely. □

Proof. Proof of Theorem 2.1.6
Define for each $\rho \in D_{f^{\#}}$ the discrete time processes $(\Theta(\rho)_j)_{j=0,\ldots,J}$ and $(\hat{Y}_j(\rho))_{j=0,\ldots,J}$ as

$$\begin{aligned} \Theta_J(\rho) &= \xi(X_J) \\ \Theta_j(\rho) &= \xi(X_J) \prod_{i=j}^{J-1} \left(\Delta \rho_i^T \beta_{i+1} + 1\right) - \sum_{l=j}^{J-1} \Delta f_l^{\#}(X_l, \rho_l) \prod_{i=j}^{l-1} \left(\Delta \rho_i^T \beta_{i+1} + 1\right) \quad j = 0, \ldots, J-1 \\ \hat{Y}_j(\rho) &= E\left[\Theta_j(\rho) | \mathcal{F}_j\right] \quad j = 0, \ldots, J. \end{aligned}$$

Then, by the tower property of the conditional expectation and the Fenchel-Moreau theorem, we have for any $j \in \{0, \dots, J-1\}$

$$\begin{aligned}
\hat{Y}_j(\rho) &= E\left[\Theta_j(\rho)|\mathcal{F}_j\right] \\
&= E\left[\xi(X_J)\prod_{i=j}^{J-1}\left(\Delta\rho_i^T\beta_{i+1}+1\right) - \sum_{l=j}^{J-1}\Delta f_l^{\#}(X_l,\rho_l)\prod_{i=j}^{l-1}\left(\Delta\rho_i^T\beta_{i+1}+1\right)\middle|\mathcal{F}_j\right] \\
&= E\left[\Theta_{j+1}(\rho)|\mathcal{F}_j\right] + \Delta\rho_j^T E\left[\beta_{j+1}\Theta_{j+1}(\rho)|\mathcal{F}_j\right] - \Delta f_j^{\#}(X_j,\rho_j) \\
&= E\left[\hat{Y}_{j+1}(\rho)\middle|\mathcal{F}_j\right] + \Delta\rho_j^T E\left[\beta_{j+1}\hat{Y}_{j+1}(\rho)\middle|\mathcal{F}_j\right] - \Delta f_j^{\#}(X_j,\rho_j) \\
&\leq E\left[\hat{Y}_{j+1}(\rho)\middle|\mathcal{F}_j\right] + \Delta f_j\left(X_j, E\left[\beta_{j+1}\hat{Y}_{j+1}(\rho)\middle|\mathcal{F}_j\right]\right).
\end{aligned} \tag{2.6}$$

On the other hand, it follows for the true solution Y of (2.1) recursively backward in time by Lemma 2.1.7 that there exists a process $\rho^* = (\rho_0^*, \dots, \rho_{J-1}^*)$ such that ρ_j^* is $\mathcal{F}_j$-measurable and solves the equation

$$(\rho_j^*)^T E\left[\beta_{j+1}Y_{j+1}|\mathcal{F}_j\right] - f_j^{\#}(X_j,\rho_j^*) = f_j\left(X_j, E\left[\beta_{j+1}, Y_{j+1}|\mathcal{F}_j\right]\right).$$

The,n since $\hat{Y}_J = \xi(X_J) = Y_J$ for all $\rho \in D_{f^{\#}}$, this yields recursively backward in time that

$$\begin{aligned}
Y_j = \hat{Y}_j(\rho^*) &= E\left[\xi\prod_{i=j}^{J-1}\left(\Delta(\rho_i^*)^T\beta_{i+1}+1\right) - \sum_{l=j}^{J-1}\Delta f_l^{\#}(X_l,\rho_l^*)\prod_{i=j}^{l-1}\left(\Delta(\rho_i^*)^T\beta_{i+1}+1\right)\middle|\mathcal{F}_j\right] \\
&= \sup_{\rho\in D_{f^{\#}}} E\left[\xi\prod_{i=j}^{J-1}\left(\Delta\rho_i^T\beta_{i+1}+1\right) - \sum_{l=j}^{J-1}\Delta f_l^{\#}(X_l,\rho_l)\prod_{i=j}^{l-1}\left(\Delta\rho_i^T\beta_{i+1}+1\right)\middle|\mathcal{F}_j\right]
\end{aligned}$$

for all $j \in \{0, \dots, J-1\}$ where $\rho^* \in D_{f^{\#}}$ follows from Lemma 2.1.7. □

Note that for any $\rho \in D_{f^{\#}}$, we can define the process $\hat{Y}(\rho)$ as a low biased approximation of Y due to (2.6) and the comparison principle in Corollary 2.1.3. Furthermore, the proof shows that at any time j, a control ρ_j is optimal if and only if it is a solution to the equation

$$\rho_j^T E\left[\beta_{j+1}Y_{j+1}|\mathcal{F}_j\right] - f_j^{\#}(X_j,\rho_j))E\left[Y_{j+1}|\mathcal{F}_j\right] = f_j\left(X_j, E\left[\beta_{j+1}Y_{j+1}|\mathcal{F}_j\right]\right)$$

where Lemma 2.1.7 guarantees the existence and that we can choose the solution $\mathcal{F}_j$-measurable. Throughout the rest of this chapter, we denote with ρ^* such an adapted optimal control process, i.e., a process that maximizes (2.4).

2.2 Algorithm and complexity

In this section, we state an algorithm for constructing a lower biased approximation for Y using the representation in Theorem 2.1.6 and state the complexity of this algorithm once under the standing assumptions and once under the additional assumption (A_S). Those rates are the main result of this chapter and will be derived in detail in the next section. We note again that the algorithm was already presented by Bender et al. (2017b) and the contribution of this chapter is the detailed error analysis and derivation of convergence rates which, to our best knowledge, has only been done for the special case of American option pricing so far.

The idea of the algorithm is to first construct approximations $\hat{\rho}$ of the optimal control process ρ^* and use these to derive approximations $\hat{Y}(\hat{\rho})$ for Y. As shown in the previous section, in dependence on ω, the value ρ_j^* is given by a solution to the equation

$$\rho_j^T E\left[\beta_{j+1} Y_{j+1} | \mathcal{F}_j\right] - f_j^{\#}(X_j, \rho_j) = f_j\left(X_j, E\left[\beta_{j+1}, Y_{j+1} | \mathcal{F}_j\right]\right).$$

Assuming that the functions f_j and $f_j^{\#}$ can be evaluated, this allows us to calculate ρ^* for given values of the conditional expectation $E[\beta_{j+1} Y_{j+1} | \mathcal{F}_j]$. However, since the conditional expectations can, in general, not be calculated in a closed form, they have to be approximated. For this purpose, note that there exist deterministic functions C_j and y_j such that

$$\begin{aligned} Y_j &= y_j(X_j), \quad j = 0, \ldots, J \\ E\left[\beta_{j+1} Y_{j+1} | \mathcal{F}_j\right] &= C_j(X_j), \quad j = 0, \ldots, J-1, \end{aligned}$$

which can be shown by first setting $y_J := \xi$. Then, assuming that $Y_j = y_j(X_j)$ for a fixed $j \in \{1, \ldots, J\}$, we get by the factorization lemma that

$$\begin{aligned} E\left[\beta_j Y_j | \mathcal{F}_{j-1}\right] &= E\left[\beta_j y_j(X_j) | \mathcal{F}_{j-1}\right] \\ &= E\left[\beta_j y_j(h(X_{j-1}, B_j)) | \mathcal{F}_{j-1}\right] \\ &= E\left[\beta_j y_j(h(x, B_j) | \mathcal{F}_{j-1}\right] \Big|_{x = X_{j-1}} =: C_{j-1}(X_{j-1}) \end{aligned}$$

where we used the definition of X and that β_j and B_j are independent of $\mathcal{F}_{j-1}$ per definition. Similarly, we get

$$\begin{aligned} Y_{j-1} &= E\left[Y_j | \mathcal{F}_{j-1}\right] + \Delta f_j\left(X_{j-1}, E\left[\beta_j Y_j | \mathcal{F}_{j-1}\right]\right) \\ &= E\left[y_j(h(x, B_j)) | \mathcal{F}_{j-1}\right] \Big|_{x = X_{j-1}} + \Delta f_j\left(X_{j-1}, C_{j-1}(X_{j-1})\right) =: y_{j-1}(X_{j-1}). \end{aligned}$$

Combined with Lemma 2.1.7 this especially implies that the optimal control ρ^* is already adapted to the filtration generated by the process X, i.e., ρ_j^* is $\sigma(X_j)$-measurable. The conditional expectations $E[\beta_j Y_j|\mathcal{F}_{j-1}]$ can then be approximated by replacing the functions C_j by some approximation operator $C_{M,j}$ and evaluate them using simulations of the process X. In the algorithm, we assume that we can construct "suitable" input approximations $C_{M,j}$ using simulations of the process X as well, where we specify later on what properties of these approximations we classify as "suitable". Under this assumption, we analyze the following algorithm, like it was introduced by Bender et al. (2017b) in a similar setting or by Belomestny (2011) in the special case of pricing Bermudan style options:

Algorithm 2.2.1.

- For an $M \in \mathbb{N}$, construct approximation operators $C_{M,j}(.)$ for $j = 0, \dots, J-1$ with a suitable algorithm using M independent "training paths"

$$\left(\hat{X}_0^{[m]}, \dots, \hat{X}_J^{[m]}\right)_{m=1,\dots,M}.$$

  Here the distribution of these random variables may vary depending on the chosen algorithm.

- For an $N \in \mathbb{N}$, simulate N new independent copies

$$B_1^{[n]}, \dots, B_J^{[n]}$$

  of the process $(B_j)_{j=1,\dots,J}$ which impose samples

$$\left(X_0^{[n]}, \dots X_J^{[n]}\right)_{n=1,\dots,N}$$

  and

$$\left(\beta_1^{[n]}, \dots, \beta_J^{[n]}\right)_{n=1,\dots,N}$$

  of the processes $(X_j)_{j\in\{0,\dots,J\}}$ and $(\beta_j)_{j\in\{1,\dots,J\}}$ to which we further refer to as "evaluation paths". Then, given those simulations, set $\rho_j^{[n]} = \rho_j^{[n]}(X_j^{[n]})$ as a solution to the equation

$$\rho_j^T C_{M,j}\left(X_j^{[n]}\right) - f_j^{\#}\left(X_j^{[n]}, \rho_j\right) = f_j\left(X_j^{[n]}, C_{M,j}\left(X_j^{[n]}\right)\right)$$

  for $n = 1, \dots, N$ and $j = 0, \dots, J-1$.

- Calculate

$$Y_0^{N,M} := \frac{1}{N}\sum_{n=1}^{N} \Theta_0\left(\rho^{[n]}\right)$$

with

$$\Theta_0\left(\rho^{[n]}\right) := \xi\left(X_J^{[n]}\right)\prod_{j=0}^{J-1}\left(\Delta\left(\rho_j^{[n]}\right)^T\beta_{j+1}^{[n]}+1\right)$$
$$-\sum_{j=0}^{J-1}\Delta f_j^{\#}\left(X_j^{[n]},\rho_j^{[n]}\right)\prod_{l=0}^{j-1}\left(\Delta\left(\rho_l^{[n]}\right)^T\beta_{l+1}^{[n]}+1\right)$$

as approximation for Y_0.

There are several methods for obtaining the input approximations in the first step of the algorithm available in the literature, for example the mesh method (see Broadie et al., 2004) regression based methods like least squares Monte-Carlo (see e.g. Lemor et al., 2006 or Bender and Denk, 2007), quantization methods (see e.g. Bally et al., 2003) or Malliavin Monte-Carlo (see e.g. Bouchard and Touzi, 2004). Since the input approximation is not the focus of this chapter, we will not go into detail here, but we will briefly describe how input approximations can be obtained with the mesh method in the appendix to this chapter for the sake of completeness.
For now, we suppose that the input approximations in the first step of the algorithm are given and discuss the remaining steps of the algorithm. For any fixed outcome of the training paths such that $C_{M,j}(X_j)$ is integrable for each j, there exists a solution ρ_j^M to the equation

$$(\rho_j^M)^T C_{M,j}(X_j) - f_j^{\#}(X_j,\rho_j^M) = f_j\left(X_j, C_{M,j}(X_j)\right),$$

which is $\sigma((\hat{X}_j^{[m]})_{m=1,\dots,M;j=0,\dots,J})\vee\sigma(X_j)$-measurable by Lemma 2.1.7. Hence the approximation operators $C_{M,j}$ impose an approximate distribution $\rho_j^M := \rho_j^M(X_j)$ of ρ_j^* for all $j\in\{0,\dots,J\}$. Assuming that the evaluation paths are independent of the training path, the values $\rho_j^{[n]}$ for $n=1,\dots,N$ in the second step of the algorithm are then copies of $\rho_j^M(X_j)$ which are sampled via the evaluation paths and are conditionally independent given the outcome of the training paths. The same logic applies to the random variables $\Theta_0(\rho^{[n]})$ in the last step of the algorithm and hence, conditioned on the training paths, the approximation $Y_0^{N,M}$ is the empiric mean of conditionally independent and identically distributed random variables and will therefore converge to their expectation, i.e.,

$$Y_0^{N,M}\to E\left[\Theta_0\left(\rho^M\right)\middle|\sigma((\hat{X}_j^{[m]})_{m=1,\dots,M;j=0,\dots,J})\right]$$

in the limit as N goes to infinity. Since $Y_0 = E\left[\Theta_0\left(\rho^*\right)|\mathcal{F}_0\right]$, the quality of the approximation then depends on the sample size N and the difference between ρ^M and ρ^*, which is determined by the input approximations. Convergence properties of the algorithm will therefore depend on the input approximations. For our analysis, we hence extend the standing assumptions and assume for the rest of this chapter that the following conditions hold true.

Assumptions 2.2.2. Extension to the standing assumptions:

$(B1)$ For each $j \in \{0, \ldots, J\}$, the σ-fields

$$\mathcal{H} := \sigma\left((\hat{X}_j^{[m]})_{m=1,\ldots,M;j=0,\ldots,J}\right)$$
$$\mathcal{G}_j := \sigma\left(\left(B_i^{[n]}\right)_{i=1,\ldots,j;n=1,\ldots,N}\right)$$
$$\tilde{\mathcal{F}}_j = \sigma\left((B_i)_{i=1,\ldots,j}\right)$$

are independent and the filtration is given by

$$\mathcal{F}_j = \sigma\left(\mathcal{H} \cup \mathcal{G}_j \cup \tilde{\mathcal{F}}_j\right) \quad j = 0, \ldots, J.$$

$(B2)$ It holds for a pair $(p, q) \in (1, \infty) \times [1, \infty)$ that

$$\left(\sum_{j=0}^{J-1} \Delta E\left[E\left[|C_{M,j}(X_j) - C_j(X_j)|^p|\mathcal{F}_0\right]^{\frac{q}{p}}\right]\right)^{\frac{1}{q}} \leq K_{B_2}(p,q)M^{-\mu}$$

for positive constants μ and $K_{B_2}(p, q)$. Furthermore, it holds

$$\max_{j\in\{0,\ldots,J-1\}} E[|C_{M,j}(X_j)|^{2(1+\epsilon)}] \leq K'_{B_2}$$

for a constant K'_{B_2} independent of M.

$(B3)$ The costs of constructing the approximations $C_{M,j}$ on the M training paths are of order $M^{1+\chi_1}$ for a constant $\chi_1 > 0$. The costs of evaluating the approximations $C_{M,j}$ in a point $x \in \mathbb{R}^{D+1}$ not on the training paths is of order M^{χ_2} for a constant $\chi_2 > 0$.

To further utilize the additional information about the functions f_j in the special case where assumption (A_s) holds true, we assume that the following condition holds true in that case.

Extension to the special case:

$(B4)$ There exist constants $K_{B_4}, \alpha > 0$ such that

$$P\left(\left|\frac{(k_i - k_l)}{|k_i - k_l|}^T (C_j(X_j) + b_j(X_j))\right| \leq \delta\right) \leq K_{B_4}\delta^\alpha$$

for any $\delta > 0$, $j = 0, \ldots, J - 1$ and $k_i \neq k_l \in \mathcal{K}$.

Condition ($B1$) formally describes the measurability of all the used simulations. In particular, we assume that the training paths are independent of the evaluation paths and X. Condition ($B2$) describes a convergence rate of the approximation operators $C_{M,j}$ towards the functions C_j in relation to the used evaluation paths, while condition (B3) bounds the required computation costs for constructing and evaluating those approximation operators. We will show in the error analysis that the optimal control ρ^* can only take values in $\mathcal{K}$ under assumption (A_S) and that $\rho_j = k_i$ is optimal if

$$k_i C_j(X_j) + b_j(X_j) > k_l C_j(X_j) + b_j(X_j)$$

for all $k_l \neq k_i \in \mathcal{K}$. Hence the set

$$\mathcal{E} := \{(j,x) : \exists k_i \neq k_l \in \mathcal{K} : (k_i - k_l)^T (C_j(x) + b_j(x)) = 0\}$$

describes the boundary region between two different values of the optimal control ρ^*. Condition ($B4$) hence characterizes the probability of X taking values close to this critical decision region. Similar conditions were considered in statistical classification problems (see e.g. Mammen and Tsybakov, 1999) and later adapted to the pricing of Bermudan options by Belomestny (2011). For smooth functions C_j and b_j with non-vanishing derivatives in the vicinity of $\mathcal{E}$, the condition holds true for $\alpha = 1$, see e.g. Mammen and Tsybakov (1999) and Belomestny (2011). For other examples of problems with different values of $\alpha \in [1, \infty)$, we refer to Belomestny (2011).
When choosing the number of training and evaluation paths, one has to consider a trade-off between the computation time and the accuracy of the approximation. For a fast convergence of the approximation, we therefore analyze how to optimally choose the number of paths N and M in relation to each other. More precisely, we analyze how to choose those paramters to bound the mean squared error

$$E\left[\left(Y_0 - Y_0^{N,M}\right)^2\right]$$

(asymptotically) by a constant $\epsilon > 0$ with smallest computation time possible. By condition ($B3$), the computation costs for a fixed number of paths N and M are given by

$$M^{\chi_1+1} + M^{\chi_2} N,$$

so we are interested in

$$\mathcal{C}(\epsilon) := \min_{M,N\in\mathbb{N}} \left\{ M^{\chi_1+1} + M^{\chi_2} N : E\left[\left(Y_0 - Y_0^{N,M}\right)^2\right] \leq c\epsilon \right\}$$

for some constant $c > 0$ not depending on ϵ. Through the error analysis in the next section, we derive the following rates for $\mathcal{C}(\epsilon)$, which are the main result of this chapter.

Theorem 2.2.3. *Under the standing assumptions, suppose we can choose* $(p,q) = (p,2)$ *in assumption* $(B2)$ *for a* $p > 1$. *Then*

$$\mathcal{C}(\epsilon) \in \mathcal{O}\left(\epsilon^{-\max\left\{\frac{\chi_1+1}{2\mu},1+\frac{\chi_2}{2\mu}\right\}}\right).$$

If condition (A_S) *and* (B_4) *hold true and we can choose* $(p,q) = (p, 2(1+\frac{\alpha}{\vartheta}))$ *for some* $p > 1$ *and* $\vartheta \in (\frac{p+\alpha}{p-1}, \infty)$ *in assumption* $(B2)$, *the complexity improves to*

$$\mathcal{C}(\epsilon) \in \mathcal{O}\left(\epsilon^{-1\max\left\{\frac{\chi_1+1}{2\mu\left(1+\frac{\alpha}{\vartheta}\right)};1+\frac{\chi_2}{2\mu\left(1+\frac{\alpha}{\vartheta}\right)}\right\}}\right).$$

Remark 2.2.4.

(i) In the case of a discretized BSDE, the constants only depend on the step width Δ in the form ΔJ, i.e., the time horizon of the BSDE. Hence, the rate is independent of the time grid and holds for arbitrary fine partitions of the time horizon.

(ii) Our results are consistent with previous ones, since in the special case of Bermudan option pricing (compare Example 2.1.4. (i)), the setting can be simplified by choosing $\beta \equiv 1$ and $\rho_j \in \{0,1\}$ one-dimensional. Then, assuming that Assumption $(B2)$ holds for any $p > 1$, which is implied by assumption (AQ) in Belomestny et al. (2015), we can choose $\vartheta = 1$ in $(B2)$ in the limit for $p \to \infty$ and reproduce the order of the complexity proven in Belomestny et al. (2015) for this example.

2.3 Error analysis

In this section, we provide a detailed error analysis of the algorithm presented in the previous section to prove Theorem 2.2.3. For this purpose, we first derive bounds on the bias and variance of the approximation, once under the standing assumptions only and once in the more restrictive setting where assumptions (A_S) and $(B4)$ hold true. We then use these bounds for the proof of Theorem 2.2.3 by utilizing the usual decomposition of the mean squared error into the squared bias and variance. For a slighter notation, we will omit the dependency of X_j in the functions f_j and their convex conjugates from now on when no simulations are plugged in. Furthermore we from now on notate the conditional expectation given $\mathcal{F}_j$ with $E_j[.]$.
We start by proving the following two standard lemmas which will be used in the analysis of the bias and the variance of the approximation. The first of the two shows that the effective domain of the convex conjugates is a subset of $\{x \in \mathbb{R}^D : |x| \leq L\}$ which gives us a bound for the possible values of the control process ρ^*.

Lemma 2.3.1. *Let $f : \mathbb{R}^{D+1} \to \mathbb{R}$ be a convex and Lipschitz continuous function with Lipschitz constant L. It then holds that*

$$f^{\#}(\rho) = \infty$$

for all $\rho \in \mathbb{R}^{D+1}$ with $|\rho| > L$.

Proof. Let $\rho \in \mathbb{R}^{D+1}$ be arbitrary with $|\rho| > L$ and set $c := |\rho| - L > 0$. By the Lipschitz continuity of f, it holds for all $y \in \mathbb{R}^{D+1}$

$$\begin{aligned} \rho^T y - f(y) + f(0) &\geq \rho^T y - |f(y) - f(0)| \\ &\geq \rho^T y - L|y|. \end{aligned}$$

Hence by the definition of the convex conjugates and the choice $y_n = n\rho$ for $n \in \mathbb{N}$, we get

$$\begin{aligned} f^{\#}(\rho) + f(0) &= \sup_{y \in \mathbb{R}^{D+1}} \rho^T y - f(y) + f(0) \\ &\geq \sup_{y \in \mathbb{R}^{D+1}} \rho^T y - L|y| \\ &\geq \lim_{n \to \infty} \rho^T y_n - L|y_n| \\ &= \lim_{n \to \infty} n|\rho|(|\rho| - L) = \infty. \end{aligned}$$

□

Next we consider the term

$$\Gamma_j(\rho) := \prod_{i=0}^{j-1} \left(1 + \Delta\rho_i^T \beta_{i+1}\right)$$

for $j \in \{0, \ldots, J-1\}$ and any process $\rho \in D_{f^{\#}}$, which is part of the definition of the function $\hat{Y}(\rho)$. Using the results of Lemma 2.3.1, we can derive a bound on the absolute moments of this term.

Lemma 2.3.2. *Let ρ be an arbitrary process in $D_{f^{\#}}$. It then holds for any $j \in \{0, \ldots, J\}$ and $r > 1$ under the standing assumptions that*

$$E_0\left[|\Gamma_j(\rho)|^r\right]^{\frac{1}{r}} \leq K_\Gamma(r) := e^{J\Delta L_f(1 + \frac{1}{2} r D L_f)}.$$

Proof. First note that, since the functions f_j are Lipschitz continuous with Lipschitz constant L_f, it holds that $|\rho_i| \leq L_f$ for each $i \in \{0, \ldots, J\}$ by Lemma 2.3.1. Furthermore, due to assumption (A_M), it holds that $|\Delta\rho_i^T \beta_{i+1}| \leq 1$ and hence each factor in $\Gamma_j(\rho)$ is non-negative and we can omit the absolute value. By the definition of Γ_j and β and the tower property of the conditional expectation, we hence get

$$
\begin{aligned}
E_0\left[|\Gamma_j(\rho)|^r\right] &= E_0\left[\prod_{i=0}^{j-1}\left(1+\Delta\rho_i^{(1)}+\sum_{d=1}^{D}\rho_i^{(d+1)}\left[\Delta W_{i+1}^{(d)}\right]_{\varsigma\sqrt{\Delta}}\right)^r\right] \\
&\leq E_0\left[\prod_{i=0}^{j-1} e^{r\Delta\rho_i^{(1)}} e^{r\sum_{d=1}^{D}\rho_i^{(d+1)}\left[\Delta W_{i+1}^{(d)}\right]_{\varsigma\sqrt{\Delta}}}\right] \\
&\leq e^{J\Delta L_f r} E_0\left[\prod_{i=0}^{j-1} e^{r\sum_{d=1}^{D}\rho_i^{(d+1)}\left[\Delta W_{i+1}^{(d)}\right]_{\varsigma\sqrt{\Delta}}}\right] \\
&= e^{J\Delta L_f r} E_0\left[\prod_{i=0}^{j-2} e^{r\sum_{d=1}^{D}\rho_i^{(d+1)}\left[\Delta W_{i+1}^{(d)}\right]_{\varsigma\sqrt{\Delta}}} E_{j-1}\left[e^{r\sum_{d=1}^{D}\rho_{j-1}^{(d+1)}\left[\Delta W_{j}^{(d)}\right]_{\varsigma\sqrt{\Delta}}}\right]\right]. \quad (2.7)
\end{aligned}
$$

Using the characterization of the exponential function as power series in the inner conditional expectation leads to

$$
\begin{aligned}
E_{j-1}\left[e^{r\sum_{d=1}^{D}\rho_{j-1}^{(d+1)}\left[\Delta W_{j}^{(d)}\right]_{\varsigma\sqrt{\Delta}}}\right] &= \prod_{d=1}^{D} E_{j-1}\left[e^{r\rho_{j-1}^{(d+1)}\left[\Delta W_{j}^{(d)}\right]_{\varsigma\sqrt{\Delta}}}\right] \\
&= \prod_{d=1}^{D}\sum_{l=0}^{\infty}\frac{(r\rho_{j-1}^{(d+1)})^l}{l!} E_{j-1}\left[\left[\Delta W_j^{(d)}\right]_{\varsigma\sqrt{\Delta}}^l\right],
\end{aligned}
$$

where we used that the components of $(\Delta W_j^{(d)})_{d=1,\dots,D}$ are independent and that ρ_{j-1} is $\mathcal{F}_{j-1}$-measurable. It then holds for all even $l \in \mathbb{N}$ that

$$
E_{j-1}\left[\left[\Delta W_j^{(d)}\right]_{\varsigma\sqrt{\Delta}}^l\right] \leq E_{j-1}\left[\left(\Delta W_j^{(d)}\right)^l\right]
$$

since $[\Delta W_j^{(d)}]_{\varsigma\sqrt{\Delta}}^l$ is P-almost surely positive and hence can be increased by dropping the truncation. On the other hand, for any odd $l \in \mathbb{N}$, we have

$$
E_{j-1}\left[\left[\Delta W_j^{(d)}\right]_{\varsigma\sqrt{\Delta}}^l\right] = 0 = E_{j-1}\left[\left(\Delta W_j^{(d)}\right)^l\right]
$$

since $\Delta W_j^{(d)}$ is Gaussian with mean zero. We conclude

$$\begin{aligned}
E_{j-1}\left[e^{r\sum_{d=1}^D \rho_{j-1}^{(d+1)}\left[\Delta W_j^{(d)}\right]_{\varsigma\sqrt{\Delta}}}\right] &\le \prod_{d=1}^{D}\sum_{l=0}^{\infty}\frac{r(\rho_{j-1}^{(d+1)})^l}{l!}E_{j-1}\left[\left(\Delta W_j^{(d)}\right)^l\right]. \\
&= \prod_{d=1}^{D} E_{j-1}\left[e^{r\rho_{j-1}^{(d+1)}\Delta W_j^{(d)}}\right] \\
&\le e^{\frac{1}{2}r^2|\rho_{j-1}|^2\Delta} \\
&\le e^{\frac{1}{2}r^2 L_f^2\Delta}
\end{aligned}$$

since $e^{\Delta W_j^{(d)}}$ is log-normal distributed and ρ_{j-1} is $\mathcal{F}_{j-1}$-measurable and bounded by L_f. The remaining Brownian increments in (2.7) can be bound recursively following the same steps, which leads to

$$\begin{aligned}
E_0\left[\left|\Gamma_j(\rho^M)\right|^r\right]^{\frac{1}{r}} &\le \left(e^{J\Delta L_f r}\prod_{i=0}^{j-2} e^{\frac{1}{2}Dr^2L_f^2\Delta}\right)^{\frac{1}{r}} \\
&\le e^{J\Delta L_f(1+\frac{1}{2}rDL_f)}
\end{aligned}$$

and finishes the proof. □

2.3.1 Bounds for the bias of the approximation

With the preliminaries in the beginning of this section, we are now ready to derive a bound for the bias under the standing assumptions.

Theorem 2.3.3. *Under the standing assumptions, it holds*

$$E\left[Y_0 - Y_0^{N,M}\right] \le K_\Gamma\left(\frac{p}{p-1}\right)2L_f\sum_{j=0}^{J}\Delta E\left[E_0\left[|C_{j,M}-C_j|^p\right]^{\frac{1}{p}}\right] \le K_{4.1}(p)K_{B_2}(p,1)M^{-\mu}$$

with $K_{4.1}(p) = 2L_f K_\Gamma(\frac{p}{p-1})$.

Proof. To prove the first inequality, first recall the processes $\hat{Y}(\rho)$ defined as

$$\begin{aligned}
\hat{Y}_j(\rho) &:= E_j\left[\xi\prod_{i=j}^{J-1}\left(\Delta\rho_i^T\beta_{i+1}+1\right) - \sum_{l=j}^{J-1}\Delta f_l^{\#}(\rho_l)\prod_{i=j}^{l-1}\left(\Delta\rho_i^T\beta_{i+1}+1\right)\right] \\
&= E_j\left[\hat{Y}_{j+1}(\rho)\right] + \Delta\rho_j^T E_j\left[\beta_{j+1}\hat{Y}_{j+1}(\rho)\right] - \Delta f_j^{\#}(\rho_j). \qquad (2.8)
\end{aligned}$$

Note that the samples $((X_j^{[n]},\beta_j^{[n]})_{j=0,\dots,J})_{n=1,\dots,N}$ are independent and identically distributed random variables distributed like $(X_j,\beta_j)_{j=0,\dots,J}$. Hence, conditionally on the training paths, the values $\rho_j^{[n]} = \hat{\rho}_j(X_j^{[n]})$ are independent copies of the random variable $\rho_j^M(X_j)$ because ρ_j^M is random depending on the training paths only. We write short $\rho_j^M := \rho_j^M(X_j)$ and $\rho^M := (\rho_0^M, \ldots, \rho_{J-1}^M)$. By construction, ρ^M is element of $D_{f\#}$. Conditioning on $\mathcal{F}_0$, which holds all the information about the training paths, yields

$$E_0\left[Y_0^{N,M}\right] = E_0\left[\frac{1}{N}\sum_{n=1}^{N}\Theta_0\left(\rho^{[n]}\right)\right] = E_0\left[\Theta_0\left(\rho^M\right)\right] = E_0\left[\hat{Y}_0\left(\rho^M\right)\right].$$

On the other hand, following the arguments in Section 2.1, it holds $Y_0 = \hat{Y}(\rho^*)$. Hence we get with equation (2.8) that

$$\begin{aligned}
&E_0\left[Y_0 - Y_0^{N,M}\right]\\
&= E_0\left[\hat{Y}_0\left(\rho^*\right) - \hat{Y}_0\left(\rho^M\right)\right]\\
&= E_0\left[\left(1+\Delta\left(\rho_0^*\right)^T\beta_1\right)\hat{Y}_1\left(\rho^*\right) - \left(1+\Delta\left(\rho_0^M\right)^T\beta_1\right)\hat{Y}_1\left(\rho^M\right) - \Delta f_0^{\#}\left(\rho_0^*\right) + \Delta f_0^{\#}\left(\rho_0^M\right)\right]\\
&= E_0\Bigg[\left(1+\Delta\left(\rho_0^M\right)^T\beta_1\right)\left(\hat{Y}_1\left(\rho^*\right) - \hat{Y}_1\left(\rho^M\right)\right) - \Delta\left(f_0^{\#}\left(\rho_0^*\right) - f_0^{\#}\left(\rho_0^M\right)\right) - \Delta\left(\rho_0^M - \rho_0^*\right)^T\beta_1\hat{Y}_1\left(\rho^*\right)\Bigg]\\
&= E_0\left[\left(1+\Delta\left(\rho_0^M\right)^T\beta_1\right)\left(\hat{Y}_1\left(\rho^*\right) - \hat{Y}_1\left(\rho^M\right)\right)\right]\\
&\quad - E_0\left[\Delta\left(f_0^{\#}\left(\rho_0^*\right) - f_0^{\#}\left(\rho_0^M\right)\right)\right] - \Delta\left(\rho_0^M - \rho_0^*\right)^T E_0\left[\beta_1\hat{Y}_1\left(\rho^*\right)\right]
\end{aligned}$$

Iterating this step yields

$$\begin{aligned}
E_0\left[Y_0 - Y_0^{N,M}\right] &= E_0\left[\sum_{j=0}^{J-1}\Gamma_j\left(\rho^M\right)\Delta\left(f_j^{\#}\left(\rho_j^M\right) - f_j^{\#}\left(\rho_j^*\right) - \left(\rho_j^M - \rho_j^*\right)^T E_j\left[\beta_{j+1}\hat{Y}_{j+1}\left(\rho^*\right)\right]\right)\right]\\
&= E_0\left[\sum_{j=0}^{J-1}\Gamma_j\left(\rho^M\right)\Delta\left(f_j^{\#}\left(\rho_j^M\right) - f_j^{\#}\left(\rho_j^*\right) - \left(\rho_j^M - \rho_j^*\right)^T C_j(X_j)\right)\right],
\end{aligned}$$

where the recursion ends, since $\hat{Y}_J(\rho^*) = \hat{Y}_J(\rho^M)$ by definition. Now, since f is closed, it holds

$$f_j\left(C_j(X_j)\right) = \left(\rho_j^*\right)^T C_j(X_j) - f_j^{\#}\left(\rho_j^*\right)$$

and respectively

$$f_j\left(C_{j,M}(X_j)\right) = \left(\rho_j^M\right)^T C_{j,M}(X_j) - f_j^{\#}\left(\rho_j^M\right)$$

by construction of ρ_j^M and the Fenchel-Moreau theorem. Paired with a zero addition, this yields

$$
\begin{aligned}
&E_0\left[Y_0 - Y_0^{N,M}\right] \\
&= E_0\Bigg[\sum_{j=0}^{J-1}\Gamma_j\left(\rho^M\right)\Delta\Big(\left(\rho_j^M\right)^T\left(C_{j,M}(X_j) - C_j(X_j) - C_{j,M}(X_j)\right) + f_j\left(C_j(X_j)\right) + f_j^{\#}\left(\rho^M\right)\Big)\Bigg] \\
&= E_0\Bigg[\sum_{j=0}^{J-1}\Gamma_j\left(\rho^M\right)\Delta\Big(\left(\rho_j^M\right)^T\left(C_{j,M}(X_j) - C_j(X_j)\right) + f_j\left(C_j(X_j)\right) - f_j\left(C_{j,M}(X_j)\right)\Big)\Bigg].
\end{aligned}
$$

We can estimate this term further due to the Lipschitz continuity of the functions f_j and Hölder's inequality as

$$
\begin{aligned}
&E_0\left[Y_0 - Y_0(\rho^M)\right] \\
&\leq \sum_{j=1}^{J-1}\Delta E_0\Big[\left|\Gamma_j(\rho^M)\right|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} E_0\Big[\left|\left(\rho_j^M\right)^T\left(C_{j,M} - C_j\right) - f_j\left(C_{j,M}\right) + f_j\left(C_j\right)\right|^p\Big]^{\frac{1}{p}} \\
&\leq \sum_{j=1}^{J-1}\Delta E_0\Big[\left|\Gamma_j\left(\rho^M\right)\right|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} E_0\Big[\Big(\left|\left(\rho_j^M\right)^T\left(C_{j,M} - C_j\right)\right| + L_f\left|C_{j,M} - C_j\right|\Big)^p\Big]^{\frac{1}{p}}.
\end{aligned}
$$

Then, since $\rho_j^M \in D_f^j$ and f is Lipschitz continuous with constant L_f, Lemma 2.3.1 implies that $|\rho_j^M| \leq L_f$ for each j and hence

$$
E_0\Big[\Big(\left|\left(\rho_j^M\right)^T\left(C_{j,M} - C_j\right)\right| + L_f\left|C_{j,M} - C_j\right|\Big)^p\Big]^{\frac{1}{p}} \leq 2L_f E_0\left[|C_{j,M} - C_j|^p\right]^{\frac{1}{p}}.
$$

Together with the bound for $E_0\Big[|\Gamma_j(\rho^M)|^{\frac{p}{p-1}}\Big]$ in Lemma 2.3.2, we get

$$
E_0\left[Y_0 - Y_0^{N,M}\right] \leq K_\Gamma\left(\frac{p}{p-1}\right)2L_f\sum_{j=0}^{J-1}\Delta E_0\left[|C_{j,M} - C_j|^p\right]^{\frac{1}{p}}.
$$

The statement of the theorem then follows by taking expectation and using assumption $(B2)$ where we assume w.l.o.g. that condition $(B2)$ holds true for the pair $(p, 1)$. □

Theorem 2.3.3 shows that the bias is of the same order as the one of the input approximation. However, we get a low biased estimator. If we calculate the input approximations with some method that gives an approximation with a positive bias, like the mesh method, this allows us to construct confidence intervals for Y.
We can even improve the bounds for the bias in the special case when assumption (A_S) is satisfied. Then, the special structure of the function f_j allows us to restrict the choice of the approximate control process ρ^M to adapted processes taking values in $\mathcal{K}$ only and we

can state the convex conjugates for these values explicitly as shown in the following lemma.

Lemma 2.3.4. *Suppose that assumption (A_S) holds true, i.e., the functions f_j are given by*

$$f_j(x,y) := \max_{k\in\mathcal{K}} y^T k + b_j(x)^T k + a_j(x)$$

for deterministic bounded functions b_j and a_j and a finite set $\mathcal{K} = \{k_1, \dots, k_\kappa\} \subset \mathbb{R}^{D+1}$. Then:

(i) It holds pointwise for all $x \in \mathbb{R}^D$, that $f_j^{\#}(x,k) = -b_j^T(x)k - a_j(x)$ for all $k \in \mathcal{K}$.

(ii) It holds

$$\sup_{\rho\in\mathbb{R}^{D+1}} \rho^T C - f_j^{\#}(\tilde{X},\rho) = \max_{k\in\mathcal{K}} k^T C - f_j^{\#}(\tilde{X},k)$$

for all $\mathcal{F}_j$-measurable random variables C and $\tilde{X}$ taking values in $\mathbb{R}^{D+1}$ and $\mathbb{R}^D$ respectively, where the equality holds pointwise for all $\omega \in \Omega$.

Proof. For the first statement, fix a $k_i \in \mathcal{K}$ and a $x \in \mathbb{R}^D$. Then, for any $z \in \mathbb{R}^{D+1}$, set

$$\tilde{k}(z) := \operatorname*{argmax}_{k\in\mathcal{K}} k^T z + k^T b_j(x) - a_j(x).$$

We then have

$$\tilde{k}^T(z)z + \tilde{k}^T(z)b_j(x) + a_j(x) \geq k_i^T z + k_i^T b_j(x) + a_j(x)$$

which is equivalent to

$$(\tilde{k}(z) - k_i)^T z \geq (k_i - \tilde{k}(z))^T b_j(x).$$

We conclude

$$\begin{aligned} k_i z - f_j(x,z) &= k_i^T z - \tilde{k}^T(z)z - \tilde{k}^T(z)b_j(x) - a_j(x) \\ &= (k_i - \tilde{k}(z))^T z - \tilde{k}^T(z)b_j(x) - a_j(x) \\ &\leq (\tilde{k}(z) - k_i - \tilde{k}(z))^T b_j(x) - a_j(x) \\ &= -k_i^T b_j(x) - a_j(x) \end{aligned}$$

where equality holds for all $z \in \mathbb{R}^{D+1}$ for which

$$\tilde{k}(z) = k_i.$$

Hence, taking the supremum over all $z \in \mathbb{R}^{D+1}$ in the inequality above shows that

$$f_j^{\#}(x,k_i) = -k_i^T b_j(x) - a_j(x)$$

for all $k_i \in \mathcal{K}$ and $x \in \mathbb{R}^D$.
For the second statement, we have by the Fenchel-Moreau theorem that

$$\sup_{\rho \in \mathbb{R}^{D+1}} \rho^T C - f_j^{\#}(\tilde{X}, \rho) = f_j(\tilde{X}, C).$$

On the other hand, it holds pointwise for each $\omega \in \Omega$ that

$$f_j(\tilde{X}, C) = k^T C + k^T b_j(\tilde{X}) + a_j(\tilde{X})$$

for a $k = k(\omega) \in \mathcal{K}$. Hence, by the first statement, setting $\rho(\omega) := k(\omega)$ pointwise for all $\omega \in \Omega$, we have

$$\sup_{\rho \in \mathbb{R}^{D+1}} \rho^T C - f_j^{\#}(\tilde{X}, \rho) = f_j(\tilde{X}, C) = \max_{k \in \mathcal{K}} k^T C - f_j^{\#}(\tilde{X}, k)$$

what finishes the proof. This especially shows that it suffices to replace the supremum in (2.4) by the maximum over all adapted processes ρ which take values only in the set $\mathcal{K}$. □

The restriction of the possible values of ρ together with assumption $(B4)$ allows us to improve the bound on the Bias of our approximation as follows.

Theorem 2.3.5. *Suppose that condition (A_S) and (B4) hold true in additional to the standing assumptions where we assume that condition $(B2)$ holds for a pair (p,q) with $p > 1$ and $q = 1 + \frac{\alpha}{\vartheta}$ for some $\vartheta \in (\frac{p+\alpha}{p-1}, \infty)$. Then*

$$E\left[Y_0 - Y_0^{N,M}\right] \leq K_{4.2}(p,q) \sum_{j=0}^{J} E[E_0[|C_{j,M} - C_j|^p]^{\frac{q}{p}}] \leq K_{4.2}(p,q) K_{B_2}(p,q) M^{-\mu q}$$

for a positive constant $K_{4.2}(p,q)$ not depending on Δ, M and N.

Proof. By Lemma 2.3.4, we can restrict the approximation of ρ^* to all processes $\rho \in D_{f\#}$ taking values in $\mathcal{K}$ only. Additionally, for each $\rho_j \in \mathcal{K}$, the convex conjugate $f_j^{\#}$ is given by

$$f_j^{\#}(\rho_j) = -\rho_j^T b_j - a_j.$$

Following the first steps of the proof of Theorem 2.3.3, we get

$$\begin{aligned} E_0\left[Y_0 - Y_0^{N,M}\right] &\leq E_0\left[\sum_{j=0}^{J-1} \Gamma_0\left(\rho^M\right) \Delta \left(f_j^{\#}\left(\rho_j^M\right) - f_j^{\#}\left(\rho_j^*\right) - \left(\rho_j^M - \rho_j^*\right)^T C_j(X_j)\right)\right] \\ &= E_0\left[\sum_{j=0}^{J-1} \Gamma_j\left(\rho^M\right) \Delta \left(\left(\rho_j^* - \rho_j^M\right)^T \left(C_j(X_j) + b_j\right)\right)\right]. \end{aligned}$$

Now choose an arbitrary $\epsilon \in (0, p-1-\frac{\alpha+p}{\vartheta})$ and set $r = \frac{(p-\epsilon)\vartheta}{p+\alpha+\vartheta}$. Note that by the assumptions on p and ϑ, the set $(0, p-1-\frac{\alpha+p}{\vartheta})$ is non-empty and it holds $r > 1$. Hence we can use Hölder's inequality and get

$$\begin{aligned}
&E_0\left[\sum_{j=0}^{J-1}\Gamma_j\left(\rho^M\right)\Delta\left(\left(\rho_j^*-\rho_j^M\right)^T\left(C_j(X_j)+b_j\right)\right)\right]\\
&\leq \sum_{j=0}^{J-1}\Delta E_0\left[\left|\Gamma_j\left(\rho^M\right)\right|^{\frac{r}{r-1}}\right]^{\frac{r-1}{r}} E_0\left[\left|\left(\rho_j^*-\rho_j^M\right)^T\left(C_j(X_j)+b_j\right)\right|^r\right]^{\frac{1}{r}}\\
&\leq K_\Gamma\left(\frac{r}{r-1}\right)\sum_{j=0}^{J-1}\Delta E_0\left[\left|\left(\rho_j^*-\rho_j^M\right)^T\left(C_j(X_j)+b_j\right)\right|^r\right]^{\frac{1}{r}}
\end{aligned}$$

where we used the bound from Lemma 2.3.2 in the last step. Now let γ_j be a positive, $\mathcal{F}_0$-measurable random variable which we will specify later on and set

$$Z_j := \begin{cases} \frac{(\rho_j^*-\rho_j^M)^T}{|\rho_j^*-\rho_j^M|}(C_j+b_j) & \rho_j^*\neq\rho_j^M \\ 0 & \rho_j^*=\rho^M \end{cases}.$$

We can then split up the appearing conditional expectation in

$$\begin{aligned}
&E_0\left[\left|\left(\rho_j^*-\rho_j^M\right)^T\left(C_j+b_j\right)\right|^r\right]^{\frac{1}{r}}\\
&= E_0\left[\left|\left(\rho_j^*-\rho_j^M\right)^T\left(C_j+b_j\right)\right|^r \mathbb{1}_{\left\{\rho_j^*\neq\rho_j^M\right\}}\right]^{\frac{1}{r}}\\
&\leq E_0\left[\left|\rho_j^*-\rho_j^M\right|^r\left(\mathbb{1}_{\{|Z_j|\leq\gamma_j\}}\mathbb{1}_{\left\{\rho_j^*\neq\rho_j^M\right\}}+\sum_{l=1}^{\infty}\mathbb{1}_{\left\{2^{l-1}\gamma_j<|Z_j|\leq 2^l\gamma_j\right\}}\mathbb{1}_{\left\{\rho_j^*\neq\rho_j^M\right\}}\right)|Z_j|^r\right]^{\frac{1}{r}}.
\end{aligned}$$

It now holds by construction that $(\rho_j^M)^T(C_{j,M}+b_j) \geq \rho^T(C_{j,M}+b_j)$ for all $\rho \in \mathcal{K}$, in particular for $\rho = \rho^*$. We conclude for all ω in the set $\{\rho_j^* \neq \rho_j^M\}$, that

$$Z_j \leq Z_j - \frac{(\rho_j^*-\rho_j^M)^T(C_{j,M}+b_j)}{|\rho_j^M-\rho_j^*|} \leq |C_j - C_{j,M}|$$

and consequently

$$\left\{Z_j > 2^l\gamma_j\right\} \cap \left\{\rho_j^* \neq \rho_j^M\right\} \subset \left\{|C_j - C_{j,M}| > 2^l\gamma_j\right\}.$$

Note that, since $p-\epsilon-r > (p-\epsilon) - \frac{(p-\epsilon)\vartheta}{\vartheta} = 0$, it holds $1 < \frac{p+\alpha}{p-r-\epsilon} =: s$, such that Hölder's inequality yields

$$
\begin{aligned}
&E_0\left[\left|\left(\rho_j^*-\rho_j^M\right)^T(C_j+b_j)\right|^r\right]^{\frac{1}{r}}\\
&\leq E_0\left[|\rho_j^*-\rho_j^M|^r\left(\mathbb{1}_{\{|Z_j|\leq\gamma_j\}}\mathbb{1}_{\{\rho_j^*\neq\rho_j^M\}}+\sum_{l=1}^{\infty}\mathbb{1}_{\{2^{l-1}\gamma_j<|C_j-C_{j,M}|\}}\mathbb{1}_{\{|Z_j|\leq 2^l\gamma_j\}}\mathbb{1}_{\{\rho_j^*\neq\rho_j^M\}}\right)|Z_j|^r\right]^{\frac{1}{r}}\\
&\leq 2L_f\gamma_j\Bigg(P_0\left(\{|Z_j|\leq\gamma_j\}\cap\{\rho_j^*\neq\rho_j^M\}\right)\\
&\quad+\sum_{l=1}^{\infty}2^{lr}P_0\left(\{|Z_j|\leq 2^l\gamma_j\}\cap\{\rho_j^*\neq\rho_j^M\}\right)^{\frac{1}{s}}P_0\left(\{|C_j-C_{j,M}|>2^{l-1}\gamma_j\}\right)^{\frac{s-1}{s}}\Bigg)^{\frac{1}{r}}.
\end{aligned}
$$

Here we used again that $|\rho_j^*|$ and $|\rho_j^M|$ are bounded by L_f and denote with P_0 the conditional probability on the σ-field $\mathcal{F}_0$. We can now estimate the probabilites of the sets $\{|Z_j|\leq 2^l\gamma_j\}\cap\{\rho_j^*\neq\rho_j^M\}$ using assumptions (B4) and the independence of C_j+b_j of $\mathcal{F}_0$ by

$$
\begin{aligned}
P_0\left(\{|Z_j|\leq 2^l\gamma_j\}\cap\{\rho_j^*\neq\rho_j^M\}\right)&\leq\sum_{k_1\neq k_2\in\mathcal{K}}P_0\left(\left\{\left|\frac{(k_1-k_2)^T}{|k_1-k_2|}(C_j+b_j)\right|\leq 2^l\gamma_j\right\}\right)\\
&=\sum_{k_1\neq k_2\in\mathcal{K}}P\left(\left\{\left|\frac{(k_1-k_2)^T}{|k_1-k_2|}(C_j+b_j)\right|\leq x\right\}\right)\Bigg|_{x=2^l\gamma_j}\\
&\leq|\mathcal{K}|^2K_{B_4}\left(2^l\gamma_j\right)^{\alpha}
\end{aligned}
$$

for all $l\in\mathbb{N}_0$. Then, since

$$
P_0\left(\{|Z_j|\leq\gamma_j\}\cap\{\rho_j^*\neq\rho_j^M\}\right)\leq\left(P_0\left(\{|Z_j|\leq\gamma_j\}\cap\{\rho_j^*\neq\rho_j^M\}\right)\right)^{\frac{1}{s}},
$$

we get

$$
\begin{aligned}
&E_0\left[\left|\left(\rho_j^*-\rho_j^M\right)^T(C_j+b_j)\right|^r\right]^{\frac{1}{r}}\\
&\leq 2L_f\gamma_j\left(|\mathcal{K}|^{\frac{2}{s}}K_{B_4}^{\frac{1}{s}}\gamma_j^{\frac{\alpha}{s}}+\sum_{l=1}^{\infty}|\mathcal{K}|^{\frac{2}{s}}K_{B_4}^{\frac{1}{s}}(2^l\gamma_j)^{\frac{\alpha}{s}}2^{lr}P_0\left(\{|C_j-C_{j,M}|>2^{l-1}\gamma_j\}\right)^{\frac{s-1}{s}}\right)^{\frac{1}{r}}\\
&\leq 2L_f|\mathcal{K}|^{\frac{2}{sr}}K_{B_4}^{\frac{1}{sr}}\gamma_j^{1+\frac{\alpha}{sr}}\left(1+\sum_{l=1}^{\infty}2^{l\left(r+\frac{\alpha}{s}\right)}P_0\left(\{|C_j-C_{j,M}|>2^{l-1}\gamma_j\}\right)^{\frac{s-1}{s}}\right)^{\frac{1}{r}}.
\end{aligned}
$$

By using Hölder's inequality once more, we can bound the series in the term above as

$$\begin{aligned}
&\sum_{l=1}^{\infty} 2^{l\left(r+\frac{\alpha}{s}\right)} P_0\left(\{|C_j - C_{j,M}| > 2^{l-1}\gamma_j\}\right)^{\frac{s-1}{s}} \\
&= \sum_{l=1}^{\infty} 2^{-l\varepsilon} 2^{l\left(r+\frac{\alpha}{s}+\varepsilon\right)} P_0\left(\{|C_j - C_{j,M}| > 2^{l-1}\gamma_j\}\right)^{\frac{s-1}{s}} \\
&\le \left(\sum_{l=1}^{\infty} 2^{-l\varepsilon s}\right)^{\frac{1}{s}} \left(\sum_{l=1}^{\infty} 2^l 2^{l\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}-l} P_0\left(\left\{\frac{4|C_j - C_{j,M}|}{\gamma_j} > 2^{l+1}\right\}\right)\right)^{\frac{s-1}{s}} \\
&\le \left(\frac{1}{1-2^{-s\varepsilon}}\right)^{\frac{1}{s}} \left(\sum_{l=1}^{\infty} \int_{2^l}^{2^{l+1}} x^{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}-1} P_0\left(\left\{\frac{4|C_j - C_{j,M}|}{\gamma_j} > x\right) dx\right\}\right)^{\frac{s-1}{s}} \\
&\le \left(\frac{1}{1-2^{-s\varepsilon}}\right)^{\frac{1}{s}} \left(\int_0^{\infty} x^{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}-1} \left(1 - F_\chi(x)\right) dx\right)^{\frac{s-1}{s}},
\end{aligned}$$

where F_χ denotes the distribution function of $\frac{4|C_j - C_{j,M}|}{\gamma_j}$. Now by construction it holds

$$\begin{aligned}
(r + \frac{\alpha}{s} + \epsilon)\frac{s}{s-1} &= (\frac{r+\alpha+\epsilon}{s} + \frac{s-1}{s}(r+\epsilon))\frac{s}{s-1} \\
&= (r+\alpha+\epsilon)\frac{p-r-\epsilon}{\alpha+r+\epsilon} + \epsilon + r = p
\end{aligned}$$

and, since $E_0[|C_{j,M} - C_j|^p]$ exists by assumption $(B2)$, integration by parts yields

$$\begin{aligned}
&\sum_{l=1}^{\infty} 2^{l\left(r+\frac{\alpha}{s}\right)} P_0\left(\{|C_j - C_{j,M}| > 2^{l-1}\gamma_j\}\right)^{\frac{s-1}{s}} \\
&\le \left(\frac{1}{1-2^{-s\varepsilon}}\right)^{\frac{1}{s}} \left(\frac{s-1}{\left(r+\frac{\alpha}{s}+\varepsilon\right)s} E_0\left[\left(\frac{4|C_j - C_{j,M}|}{\gamma_j}\right)^{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}}\right]\right)^{\frac{s-1}{s}} \\
&= \left(\frac{1}{1-2^{-s\varepsilon}}\right)^{\frac{1}{s}} \left(\frac{4^{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}}}{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}} E_0\left[|C_j - C_{j,M}|^{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}}\right] \frac{1}{\gamma_j^{\left(r+\frac{\alpha}{s}+\varepsilon\right)\frac{s}{s-1}}}\right)^{\frac{s-1}{s}}.
\end{aligned}$$

By choosing $\gamma_j = E_0[|C_{j,M} - C_j|^p]^{\frac{1}{p}}$, we therefore get

$$
\begin{aligned}
&E_0\left[Y_0 - Y_0^{N,M}\right] \\
&\leq \sum_{j=0}^{J-1} \Delta K_\Gamma \left(\frac{r}{r-1}\right) 2L_f \gamma_j^{1+\frac{\alpha}{sr}} |\mathcal{K}|^{\frac{2}{sr}} K_{B_4}^{\frac{1}{sr}} \left(1 + \left(\frac{1}{1-2^{-s\epsilon}}\right)^{\frac{1}{s}} \left(\frac{4^p}{p}\right)^{\frac{s-1}{s}}\right)^{\frac{1}{r}} \\
&= K_\Gamma \left(\frac{r}{r-1}\right) 2L_f |\mathcal{K}|^{\frac{2}{\vartheta}} K_{B_4}^{\frac{1}{\vartheta}} \left(1 + \left(\frac{1}{1-2^{-s\epsilon}}\right)^{\frac{1}{s}} \left(\frac{4^p}{p}\right)^{\frac{s-1}{s}}\right)^{\frac{1}{r}} \sum_{j=0}^{J-1} \Delta E_0\left[|C_j - C_{j,M}|^p\right]^{\frac{1+\frac{\alpha}{\vartheta}}{p}},
\end{aligned}
$$

where we used that

$$
\frac{1}{sr} = \frac{p-\epsilon-r}{p+\alpha}\frac{1}{r} = \frac{p-\epsilon}{p+\alpha}\frac{p+\alpha+\vartheta}{(p-\epsilon)\vartheta} - \frac{1}{p+\alpha} = \frac{1}{\vartheta}.
$$

Then taking expectation and using assumption $(B2)$ yields

$$
E\left[Y_0 - Y_0^{N,M}\right] \leq K_{4.2}(p,q) K_{B_2}(p,q) M^{-\mu(1+\frac{\alpha}{\vartheta})}
$$

where we set

$$
K_{4.2}(p,q) := K_\Gamma \left(\frac{r}{r-1}\right) 2L_f |\mathcal{K}|^{\frac{2}{sr}} K_{B_4}^{\frac{1}{sr}} \left(1 + \left(\frac{1}{1-2^{-s\epsilon}}\right)^{\frac{1}{s}} \left(\frac{4^p}{p}\right)^{\frac{s-1}{s}}\right)^{\frac{1}{r}}
$$

with constants s, ϵ, r depending on p, α and ϑ. □

Remark 2.3.6. The results show that in the general case, the bias of our approximation is at least of the same order as the one of the input approximation. Under assumptions (A_S) and $(B4)$ we can improve the convergence rate from $M^{-\mu}$ to $M^{-\mu\frac{\alpha}{\vartheta}}$, where α depends on the concrete BSDE and ϑ depends on the norm in which we can control the input approximation. In the limit, for $\alpha \to 0$ in Theorem 2.3.5, i.e., if we can not utilize condition $(B4)$ we end up with the rate of Theorem 2.3.3 and the improvement due to the finite possible values of ρ vanishes. In this sense, the results are stable.

2.3.2 Bounds for the variance of the approximation

In this section, we derive the following bounds for the variance of the approximation.

Theorem 2.3.7. *(i) Under the standing assumptions, suppose we can choose $(p,q) = (p,2)$ for a $p > 1$ in assumption $(B2)$. It then holds*

$$
Var\left(Y_0^{N,M}\right) \leq \frac{K_{4.4}}{N} + K_{4.1}^2(p) J \Delta K_{B_2}(p,q) M^{-2\mu}
$$

with a positive constant $K_{4.4}$ not depending on N and M.

(ii) If additionally assumptions (A_S) and $(B4)$ hold true and we can choose $(p,q) = (p, 2(1+\frac{\alpha}{\vartheta}))$ for a $p > 1$ and a $\vartheta \in (\frac{p+\alpha}{p-1}, \infty)$ in assumption $(B2)$, the bound in (i) can be improved to

$$Var\left(Y_0^{N,M}\right) \leq \frac{K_{4.4}}{N} + K_{4.2}^2(p, 1+\frac{\alpha}{\vartheta})J\Delta K_{B_2}(p,q)M^{-2(1+\frac{\alpha}{\vartheta})\mu}.$$

Proof. We prove both bounds simultaneously. Even though the processes $\rho^{[n]}$, $n = 1, \ldots, N$ are calculated using independent simulations of evaluation paths, they still depend on the same set of training paths. Hence they are just conditionally independent given the outcome of the training paths. Therefore, we split up the variance in

$$Var\left(Y_0^{N,M}\right) = Var\left(E_0\left[Y_0^{N,M}\right]\right) + E\left[Var\left(Y_0^{N,M}\middle|\mathcal{F}_0\right)\right] \tag{2.9}$$

using the law of total variance. Then, since Y_0 is deterministic, we have

$$\begin{aligned} Var\left(E_0\left[Y_0^{N,M}\right]\right) &= Var\left(E_0\left[Y_0^{N,M}\right] - Y_0\right) \\ &\leq E\left[\left|E_0\left[Y_0^{N,M} - Y_0\right]\right|^2\right]. \end{aligned}$$

Depending on the setting, we now get direclty from the proof of Theorem 4.1 or Theorem 4.2 respectively that

$$E\left[\left|E_0\left[Y_0^{N,M} - Y_0\right]\right|^2\right] \leq E\left[\left(K_{Bias}\sum_{j=0}^{J-1}\Delta E_0\left[|C_{j,M} - C_j|^p\right]^{\frac{q}{2p}}\right)^2\right]$$

with $K_{Bias} = K_{4.1}(p)$ and $q = 2$ under the standing assumption and $K_{Bias} = K_{4.2}(p, 1+\frac{\alpha}{\vartheta})$ and $q = 2 + 2\frac{\alpha}{\vartheta}$ in the setting of (ii). Hölder's inequality and assumption $(B2)$ then yield in both cases

$$\begin{aligned} E\left[\left|E_0\left[Y_0^{N,M} - Y_0\right]\right|^2\right] &\leq K_{Bias}^2 J\Delta \sum_{j=0}^{J-1}\Delta E\left[E_0\left[|C_{j,M} - C_j|^p\right]^{\frac{q}{p}}\right] \\ &\leq K_{Bias}^2 K_{B_2}(p,q) M^{-q\mu} J\Delta, \end{aligned}$$

which is part of the final bound. We can estimate the second term in (2.9) by

$$\begin{aligned}
E\left[Var\left(Y_0^{N,M}\middle|\mathcal{F}_0\right)\right] &= E\left[Var\left(\frac{1}{N}\sum_{i=1}^{N}\Theta_0\left(\rho^{n,M}\right)\middle|\mathcal{F}_0\right)\right] \\
&= \frac{1}{N}E\left[Var\left(\Theta_0\left(\rho^M\right)|\mathcal{F}_0\right)\right] \\
&\leq \frac{1}{N}E\left[E_0\left[\Theta_0\left(\rho^M\right)^2\right]\right] \\
&= \frac{1}{N}E\left[E_0\left[\left(\Gamma_J\left(\rho^M\right)\xi - \sum_{j=0}^{J-1}\Delta f_j^{\#}\left(\rho_j^M\right)\Gamma_j\left(\rho^M\right)\right)^2\right]\right] \\
&\leq \frac{2}{N}E\left[E_0\left[\left|\Gamma_J\left(\rho^M\right)\xi\right|^2\right]\right] + \frac{2}{N}E\left[E_0\left[\left(\sum_{j=0}^{J-1}\Delta f_j^{\#}\left(\rho_j^M\right)\Gamma_j\left(\rho^M\right)\right)^2\right]\right] \\
&\leq \frac{2}{N}E\left[E_0\left[\left|\Gamma_J\left(\rho^M\right)\xi\right|^2\right]\right] + \frac{2}{N}\Delta J\sum_{i=0}^{J-1}\Delta E\left[E_0\left[\left|f_j^{\#}\left(\rho_j^M\right)\Gamma_j\left(\rho^M\right)\right|^2\right]\right].
\end{aligned}$$

Then by Hölder's inequality, it holds

$$\begin{aligned}
\frac{2}{N}E\left[E_0\left[\left|\Gamma_J\left(\rho^M\right)\xi\right|^2\right]\right] &\leq \frac{2}{N}E\left[E_0\left[\left|\Gamma_J\left(\rho^M\right)\right|^{\frac{2(1+\epsilon)}{\epsilon}}\right]^{\frac{\epsilon}{1+\epsilon}}\right]E\left[E_0\left[|\xi|^{2+2\epsilon}\right]^{\frac{1}{1+\epsilon}}\right] \\
&\leq \frac{2}{N}K_\Gamma^2\left(\frac{2+2\epsilon}{\epsilon}\right)K_\xi
\end{aligned}$$

where we set $K_\xi := E[|\xi|^{2+2\epsilon}]^{\frac{1}{1+\epsilon}}$, which is finite by assumption (A_ξ). Again, by Hölder's inequality and since

$$f_j^{\#}\left(\rho^M\right) = C_{M,j}\rho_j^M - f_j\left(C_{M,j}\right),$$

we get

$$\begin{aligned}
&\frac{2}{N}\Delta J\sum_{i=0}^{J-1}\Delta E\left[E_0\left[\left|f_j^{\#}\left(\rho_j^M\right)\Gamma_j\left(\rho^M\right)\right|^2\right]\right] \\
&\leq \frac{2}{N}\Delta J\sum_{i=0}^{J-1}\Delta E\left[E_0\left[\left|f_j^{\#}\left(\rho_j^M\right)\right|^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}E_0\left[\left|\Gamma_j\left(\rho^M\right)\right|^{\frac{2(1+\epsilon)}{\epsilon}}\right]^{\frac{\epsilon}{1+\epsilon}}\right] \\
&\leq \frac{2}{N}\Delta JK_\Gamma^2\left(\frac{2+2\epsilon}{\epsilon}\right)\sum_{i=0}^{J-1}\Delta E\left[E_0\left[\left|C_{M,j}^T\rho_j^M - f_j(C_{M,j}) + f_j(0) - f_j(0)\right|^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}\right]
\end{aligned}$$

where we used a zero addition in the last step. We can estimate the appearing expectation further as

$$
\begin{aligned}
&E\left[E_0\left[\left|C_{M,j}^T\rho_j^M - f_j(C_{M,j}) + f_j(0) - f_j(0)\right|^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}\right] \\
&\leq E\left[E_0\left[\left(\left|C_{M,j}^T\rho_j^M\right| + |f_j(C_{M,j}) - f_j(0)| + |f_j(0)|\right)^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}\right] \\
&\leq 3^{\frac{1+2\epsilon}{1+\epsilon}}\left(E\left[E_0\left[\left|\rho_j^M C_{M,j}\right|^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}\right] + E\left[E_0\left[|f_j(C_{M,j}) - f_j(0)|^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}\right]\right. \\
&\quad \left. + E\left[E_0\left[|f_j(0)|^{2(1+\epsilon)}\right]^{\frac{1}{1+\epsilon}}\right]\right) \\
&\leq 3^{\frac{1+2\epsilon}{1+\epsilon}}\left(2L_f^2 K_{B2}'^{\frac{1}{1+\epsilon}} + C_f\right),
\end{aligned}
$$

where we used the Lipschitz continuity of f_j, assumptions (A_f) and $(B2)$, and that $|\rho^M| \leq L$ in the last inequality. Here we denote with $C_f := \max_j E[E_0[|f_j(0)|^{2+2\epsilon}]^{\frac{1}{1+\epsilon}}]$. This yields

$$
\begin{aligned}
E\left[Var\left(Y_0^{N,M}\middle|\mathcal{F}_0\right)\right] &\leq \frac{2}{N}K_\Gamma^2\left(\frac{2+2\epsilon}{\epsilon}\right)K_\xi \\
&\quad + \frac{2}{N}\Delta J K_\Gamma^2\left(\frac{2+2\epsilon}{\epsilon}\right)\sum_{i=0}^{J-1}\Delta 3^{\frac{1+2\epsilon}{1+\epsilon}}\left(2L_f^2 K_{B2}'^{\frac{1}{1+\epsilon}} + C_f\right) \\
&= \frac{2}{N}K_\Gamma^2\left(\frac{2+2\epsilon}{\epsilon}\right)\left(K_\xi + J^2\Delta^2 3^{\frac{1+2\epsilon}{1+\epsilon}}\left(2L_f^2 K_{B2}'^{\frac{1}{1+\epsilon}} + C_f\right)\right) \\
&=: \frac{1}{N}K_{4.4}
\end{aligned}
$$

with $K_{4.4} := 2K_\Gamma^2\left(\frac{2+2\epsilon}{\epsilon}\right)\left(K_\xi + J^2\Delta^2 3^{\frac{1+2\epsilon}{1+\epsilon}}\left(2L_f^2 K_{B2}'^{\frac{1}{1+\epsilon}} + C_f\right)\right)$. □

2.3.3 Complexity

In this section, we use the obtained bounds on the bias and the variance of the approximation to prove Theorem 2.2.3. With the usual decomposition of the mean squared error

$$
E\left[\left(Y_0 - Y_0^{N,M}\right)^2\right] = E\left[Y_0 - Y_0^{N,M}\right]^2 + Var\left(Y_0^{N,M}\right),
$$

the following corollary, which bounds the mean squared error for any fixed number of evaluation and training paths, follows directly from previous results:

Corollary 2.3.8. *Under the standing assumptions, suppose we can choose $(p, q) = (p, 2)$ for* *a $p > 1$ in $(B2)$. Then*

$$E\left[\left(Y_0 - Y_0^{N,M}\right)^2\right] \le c\left(N^{-1} + M^{-2\mu}\right).$$

If additionally conditions (A_S) and $(B4)$ hold true and we can choose $(p,q) = (p, 2(1+\frac{\alpha}{\vartheta})$ in $(B2)$ for a $p > 1$ and a $\vartheta \in (\frac{p+\alpha}{p-1}, \infty)$ this bound can be improved to

$$E\left[\left(Y_0 - Y_0^{N,M}\right)^2\right] \le c\left(N^{-1} + M^{-2\mu\left(1+\frac{\alpha}{\vartheta}\right)}\right).$$

With this bound we are ready to prove Theorem 2.2.3:

Proof. Proof of Theorem 2.2.3:
We prove the rates under the standing assumptions and under the additional assumptions (A_S) and $(B4)$ simultaneously. By the results of Corollary 2.3.8 and condition $(B3)$, it suffices to solve the optimization problem

$$\begin{aligned} M^{1+\chi_1} + NM^{\chi_2} &\to \min \\ M^{-2\mu q} &\le \frac{\epsilon}{c} \\ N^{-1} &\le \frac{\epsilon}{c} \end{aligned}$$

for any $\epsilon > 0$ and a constant $c > 0$ not depending on ϵ, where $q = 1$ under the standing assumptions and $q = 1 + \frac{\alpha}{\vartheta}$ for the improved rate under condition (A_S) and $(B4)$. Hence, the optimal choice for the number of training and evaluation paths in dependence of ϵ is given by

$$\begin{aligned} M(\epsilon) &= c\epsilon^{-\frac{1}{2q\mu}} \\ N(\epsilon) &= c\epsilon^{-1}. \end{aligned}$$

for a constant $c > 0$ independent of ϵ. The complexity stated in the theorem then follows in both cases directly by assumption $(B3)$. □

2.4 Numerical example

In this section, we illustrate our results in a concrete numerical example. We consider a BSDE of the form

$$d\mathcal{Y}_t = r\mathcal{Y}_t - (\lambda(R-1)\mathcal{Y}_t)_- dt + \mathcal{Z}^T d\tilde{W}_t, \quad \mathcal{Y}_T = g(\max_{d=1,\dots,5} \mathcal{X}_T^{(d)})$$

for constants $\lambda > 0$, $r \geq 0$, $R \in [0,1]$ and a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$. Here $\tilde{W}$ is a five-dimensional Brownian Motion and $\mathcal{X}_t^{(d)}$, $d = 1, \ldots 5$ are independent, identically distributed geometric Brownian motions with drift r and volatility $\sigma > 0$ with the same initial value $x_0 > 0$, i.e.,

$$\mathcal{X}_t^{(d)} = x_0^{(d)} \exp\{\sigma \tilde{W}_t^{(d)} + (r - \sigma^2/2)t\}.$$

This example covers an option pricing problem with credit valuation adjustment: Consider two trading partners A and B which trade options with maturity T on the five stocks modeled by $\tilde{X}$, where r is the risk-free interest rate in the market, σ is the volatility of the stocks, and we assume that party B has a risk of default, which occurs when a Poisson-process with intensity λ jumps for the first time. In case of default of party B before the maturity of the options, the momentary value of the options is calculated and A receives a fixed percentage R, the recovery rate, of this value from B (if the value is positive) or has to pay out B (if the current value is negative). If no default of B happens, A receives the payoff $-g$ at maturity. Then, the fair option price at time 0 is given by $-\mathcal{Y}_0$. Note that the algorithm will give us a low biased approximation for $\mathcal{Y}_0$. Hence $-Y_0^{N,M}$ will be an upper bound for the option price. The expression in terms of the negated price is necessary to obtain convex functions f_j when discretizing the BSDE.
Given a time grid $0 = t_0 < t_1 \ldots < t_J = T$ with equidistant steps, a natural time discretization for $\mathcal{Y}_t$ in the form of (2.1) is given by

$$\begin{aligned}
\mathcal{Y}_T \approx Y_J &= g(\max_{d=1,\ldots,5} \mathcal{X}_J^{(d)}) \\
\mathcal{Y}_{t_j} \approx Y_j &= E_j[Y_{j+1}] + \Delta(-rE_j[Y_{j+1}] + (\lambda(R-1)E_j[Y_{j+1}])_-) \\
&=: E_j[Y_{j+1}] + \Delta f_j(E_j[Y_{j+1}]),
\end{aligned}$$

where $\Delta = t_j - t_{j-1}$. Here we use the slight simplification of the setting by setting $\beta \equiv 1$, which is possible since the driver of the BSDE does not depend on $\mathcal{Z}$. This simplification affects only the constants in the error analysis and does not change the asymptotic convergence rates. In addition to all standing assumptions, this problem also satisfies condition (A_S) with $\mathcal{K} = \{-r; -(r + \lambda(1-R))\}$ and $(b_j) = (a_j) \equiv 0$.
To test and illustrate the theoretical results, we will calculate approximations for Y_0 multiple times for an increasing number of training and evaluations paths M and N and calculate the rate at which the mean squared error of the approximation decreases in relation to the required computation time. We will calculate the input approximations with the mesh method as described in the appendix to this chapter and suppose that condition $(B1)$-$(B4)$ holds true with the parameters $\chi_1 = \chi_2 = \alpha = 1$ in Assumption $(B3)$ and $(B4)$ (see remark in the appendix and the discussion of Assumption $(B4)$ in Section 2.2). Since results for the convergence rate of the mesh method are only available in the special

case of Bermudan option pricing (see e.g. Agarwal and Juneja, 2013), we suppose that the same rates apply in our setting and choose $\mu = \frac{1}{2}$ in assumption $(B2)$. Moreover we use the model parameters $r = 0.02$, $\sigma = 0.2$, $T = 2$, $X_0 = (X_0^1, \dots, X_0^5) = (100, \dots, 100)$, $\lambda = 0.02$, $R = 0.5$ and a time grid with 20 equidistant steps, i.e., $\Delta = 0.05$. As a payoff function, we consider

$$-h(x) := 2(x - 115)_+ - (x - 95)_+,$$

which indicates a call spread option. This option was already investigated by Belomestny et al. (2014), who calculated a confidence interval for the option price. We assume that the center of this confidence interval (8.7275) is a good proxy to the true value and use it for the calculation of the mean squared error of our approximations.

We calculate the input approximation with the six different numbers of training paths M

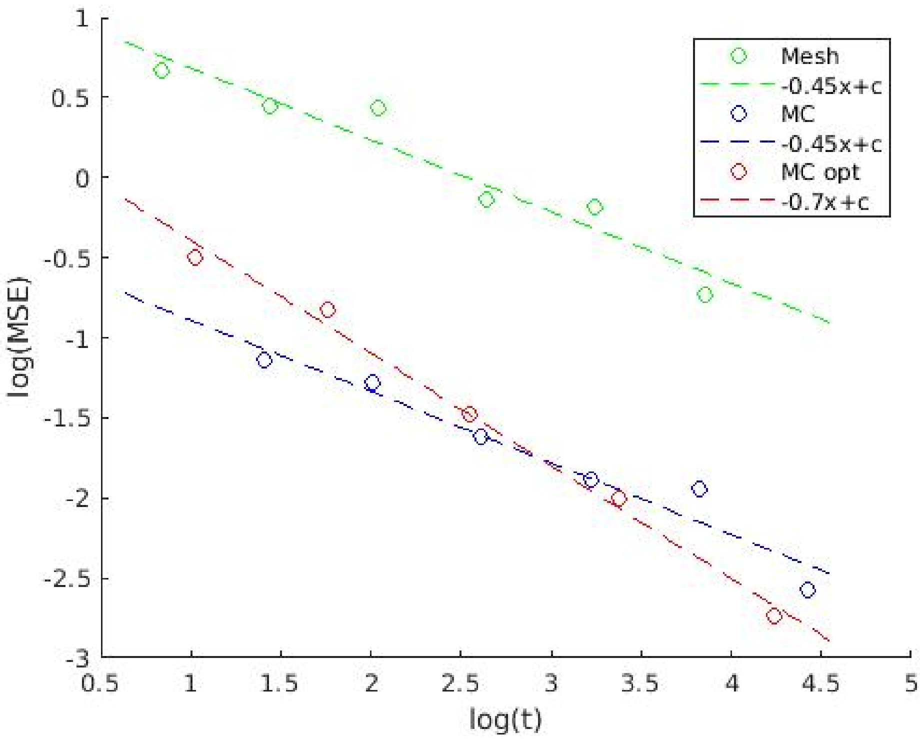

Figure 2.1: Results of Monte Carlo approximation (optimized and plain) and Mesh estimator with 20 time steps.

in $\{512 \times 2^i; i = 0, \dots, 5\}$. Then these input approximations are used to run the Monte-Carlo algorithm from Section 2.2, where we choose the number of evaluation paths N in the optimal relation to M. Since we want to confirm both rates in Theorem 2.2.3, we first neglect that the control ρ^* takes only values from a finite set and choose $N = 10 \times M$. Additional, we run the algorithm with $N = \frac{1}{256} \times M^2$ evaluation paths, which is the optimal relation between M and N in the setting for the second rate in Theorem 2.2.3 under a limit value consideration where we assume that we can choose p arbitrarily large in Assumption $(B2)$, which allows us to choose ϑ approximately as one. We do this using the same input approximation but new simulations of the evaluation paths. We do these calculations (including the calculation of the input approximation) 20 times for each value of M and calculate the mean squared error over the different iterations for each value.

In Figure 2.1, the relation of the mean squared error to the required computation time is plotted in log-log rates. Note that the runtime for the Monte Carlo algorithms includes the required time for constructing the input approximation. In the first calibration of the Monte Carlo algorithm (notated as "MC"), we measured, as suggested by the theoretical results, the same rate of -0.45 as for the mesh method, which matches approximately the expected rate of -0.5. Although the estimates of both converge with the same rate, the mean squared error from the Monte Carlo approximation is much smaller, which is a result of the smaller variance of the Monte Carlo estimates. For the second calibration of the Monte Carlo algorithm (denoted as "MC-opt"), which takes the finite possible values of ρ^* into account, our simulations could confirm the improvement of the convergence rate. We measured a rate of -0.7, which approximately matches the expected rate of $-\frac{2}{3}$ in the optimal case when condition $(B2)$ holds for every pair (p, q). Hence, both theoretical convergence rates could be confirmed by the numerical example.

Appendix A

Appendix to Chapter 1

Proof of Lemma 1.5.3

We essentially follow the proof in Gobet and Turkedjiev (2016) with slight changes in representation. Since $\mathcal{K}$ is finite-dimensional, there exists for each $\omega \in \Omega$ a set of orthonormal (with respect to the L^2-norm induces by $\nu(\omega,.)$) basis function $p_1^\omega, \ldots, p_{\tilde{K}}^\omega \in \mathcal{K}$ with $\tilde{K} \leq K$. We then set

$$\tilde{p}_i(\omega,.) := p_i^\omega(.)$$

for all $i \in \{1, \ldots, \tilde{K}\}$. Then each $\Xi(\omega,.) \in L^2(A, \mathcal{A}, \nu(\omega,.))$ has a ($\nu(\omega,.)$-almost surely unique) best approximation in $\mathcal{K}$ with respect to $\nu(\omega,.)$ which is given by

$$\varphi(\omega,.) := \sum_{k=1}^{\tilde{K}} \tilde{p}_k(\omega,.) \left(\frac{1}{M} \sum_{m=1}^{M} \tilde{p}_k\left(\omega, \chi^{[m]}(\omega)\right)^T \Xi\left(\omega, \chi^{[m]}(\omega)\right) \right).$$

The linearity of the mapping $\Xi \mapsto \varphi$ then follows directly from the linearity of the vector multiplication. Furthermore, by the properties of the best approximation, $\varphi(\omega,.) - \Xi(\omega,.)$ is orthogonal on each element $\tilde{\varphi} \in \mathcal{K}$ with respect to $\nu(\omega,.)$. Hence denoting the L^2- norm with respect to $\nu(\omega,.)$ with $\|.\|$, it holds

$$\|\Xi(\omega,.)\|^2 = \|\varphi(\omega,.) - \Xi(\omega,.) + \varphi(\omega,.)\|^2 \leq \|\varphi(\omega,.) - \Xi(\omega,.)\|^2 + \|\varphi(\omega,.)\|^2.$$

The contraction property in (ii) then follows directly from the inequality above since the norm is non-negative.
For the proof of (iii), note that $\Xi_{\mathcal{G}}$ is an element of $L^2(A, \mathcal{A}, \nu(\omega,.))$ and hence has a best

solution in $\mathcal{K}$ which is given by

$$\sum_{k=1}^{\tilde{K}} \tilde{p}_k(\omega,.) \left(\frac{1}{M} \sum_{m=1}^{M} \tilde{p}_k \left(\omega, \chi^{[m]}(\omega)\right)^T \Xi_{\mathcal{G}} \left(\chi^{[m]}(\omega)\right) \right).$$

On the other hand, $\tilde{p}_k(\chi^{[m]})$ is $\mathcal{G}$-measurable for all k by assumption and hence

$$\begin{aligned}
E[\varphi|\mathcal{G}](\omega,.) &= E\left[\sum_{k=1}^{\tilde{K}} \tilde{p}_k \left(\frac{1}{M} \sum_{m=1}^{M} \tilde{p}(\chi^{[m]})^T, \Xi(\chi^{[m]}) \right) \middle| \mathcal{G} \right] (\omega,.) \\
&= \sum_{k=1}^{\tilde{K}} \tilde{p}_k(\omega,.) \left(\frac{1}{M} \sum_{m=1}^{M} \tilde{p}_k^T \left(\omega, \chi^{[m]})(\omega)\right)^T E\left[\Xi(\chi^{[m]})|\mathcal{G}\right](\omega) \right) \\
&= \sum_{k=1}^{\tilde{K}} \tilde{p}_k(.) \left(\frac{1}{M} \sum_{m=1}^{M} \tilde{p}_k \left(\omega, \chi^{[m]}(\omega)\right)^T \Xi_{\mathcal{G}}(\chi^{[m]}(\omega)) \right)
\end{aligned}$$

and the required equality holds true.
For the proof of (iv), note that

$$\varphi - E[\varphi|\mathcal{G} \vee \mathcal{H}] = \sum_{k=}^{\tilde{K}} \tilde{p}_k \frac{1}{M} \sum_{m=1}^{M} \tilde{p}_k(\chi^{[m]})^T \left(\Xi(\chi^{[m]}) - \Xi_{\mathcal{G}\vee\mathcal{H}}(\chi^{[m]})\right).$$

Hence, we have by the orthogonality of the basis functions that

$$\begin{aligned}
&\|\varphi(\omega,.) - E[\varphi|\mathcal{G} \vee \mathcal{H}](\omega,.)\|^2 \\
&= \sum_{k=1}^{\tilde{K}} \frac{1}{M^2} \sum_{m=1}^{M} \sum_{n=1}^{M} \tilde{p}_k^T(\chi^{[m]})(\Xi(\chi^{[m]}) - E[\Xi(\chi^{[m]})|\mathcal{G} \vee \mathcal{H}])(\Xi(\chi^{[n]}) - E[\Xi(\chi^{[n]})|\mathcal{G} \vee \mathcal{H}])^T \tilde{p}_k(\chi^{[n]}) \\
&= \frac{1}{M^2} \sum_{m,n=1}^{M} \mathrm{Tr}\left(\tilde{p}(\chi^{[m]})\tilde{p}^T(\chi^{[n]})(\Xi(\chi^{[n]}) - E[\Xi(\chi^{[n]})|\mathcal{G} \vee \mathcal{H}])(\Xi(\chi^{[m]}) - E[\Xi(\chi^{[m]})|\mathcal{G} \vee \mathcal{H}])^T\right) \\
&\leq \frac{1}{M^2} \sum_{m,n=1}^{M} \mathrm{Tr}\left(\tilde{p}(\chi^{[m]})\tilde{p}^T(\chi^{[n]})\right) \mathrm{Tr}\left((\Xi(\chi^{[n]}) - E[\Xi(\chi^{[n]})|\mathcal{G} \vee \mathcal{H}])(\Xi(\chi^{[m]}) - E[\Xi(\chi^{[m]})|\mathcal{G} \vee \mathcal{H}])^T\right),
\end{aligned}$$

where we denote with $\tilde{p}$ the $l \times \tilde{K}$-matrix of functions $(\tilde{p}_1, \ldots, \tilde{p}_{\tilde{K}})$. Then, since $\Xi(.)$ is $\mathcal{H}$-measurable and the random variables $\chi^{[m]}$, $m = 1, \ldots M$ are independent, it holds for $m \neq n$ that $(\Xi(\chi^{[m]}) - E[\Xi(\chi^{[m]})|\mathcal{G} \vee \mathcal{H}])$ and $(\Xi(\chi^{[n]}) - E[\Xi(\chi^{[n]})|\mathcal{G} \vee \mathcal{H}])$ are conditionally independent given $\mathcal{G} \vee \mathcal{H}$. Hence, taking conditional expectation yields

$$\begin{aligned}
&E[|\varphi - E[\varphi|\mathcal{G} \vee \mathcal{H}]|^2|\mathcal{G} \vee \mathcal{H}] \\
&\leq \frac{1}{M^2} \sum_{m=1}^{M} \operatorname{Tr}\left(\tilde{p}(\chi^{[m]})\tilde{p}^T(\chi^{[m]})\right) E\left[|\Xi(\chi^{[m]}) - E[\Xi(\chi^{[m]})|\mathcal{G} \vee \mathcal{H}]|^2 \middle| \mathcal{G} \vee \mathcal{H}\right] \\
&\leq \frac{\sigma^2}{M^2} \sum_{m=1}^{M} \operatorname{Tr}\left(\tilde{p}(\chi^{[m]})\tilde{p}^T(\chi^{[m]})\right) = \frac{\tilde{K}\sigma^2}{M} \leq \frac{K\sigma^2}{M}
\end{aligned}$$

where we used that the functions $\tilde{p}_k$, $k = 1, \ldots, \tilde{K}$ are orthonormal.

Proof of Lemma 1.5.5

In this section we prove Lemma 1.5.5. For that, we first prove for a slightly changed version of Theorem 11.6 in Györfi et al. (2006), which was also used in Gobet and Turkedjiev (2016) in the same form. The proof contains so-called covering numbers which are defined as follows.

Definition A.1. *Let $\epsilon > 0$ and $\mathcal{G}$ be a set of functions mapping $\mathbb{R}^D$ into $\mathbb{R}$ and $x = (x_1, \ldots, x_M)$ be a set of (possibly random) points in $\mathbb{R}^D$. Then every collection of functions $g_1, \ldots, g_n : \mathbb{R}^D \to \mathbb{R}$ such that for each $g \in \mathcal{G}$ there exists a $j = j(g) \in \{1, \ldots, n\}$ with*

$$\frac{1}{M} \sum_{i=1}^{M} |g(x_i) - g_j(x_i)| < \epsilon$$

is called an ϵ-cover of $\mathcal{G}$ with respect to the empirical L_1-norm $\|g\|_M := \frac{1}{M}\sum_{i=1}^{M} |g(x_i)|$ of size n.
The ϵ-covering number $\mathcal{N}(\epsilon, \mathcal{G}, x)$ of $\mathcal{G}$ is then defined as the size of the smallest ϵ-cover of $\mathcal{G}$.

With the formal definition of the covering numbers, we can now state the variation of Theorem 11.6 in Györfi et al. (2006) that we will use for the proof of Lemma 1.5.5, just like in Gobet and Turkedjiev (2016).

Theorem A.2. *Let $B \geq 1$, $\alpha > 0$, $\epsilon \in (0,1)$ and let $\mathcal{G}$ be a set of functions mapping $\mathbb{R}^D$ into the set $[0, B]$. Furthermore, let $\chi, \chi_1, \ldots, \chi_n$ be independent and identically distributed $\mathbb{R}^D$-valued random variables. Then, indicating the set of random variables $(\chi_1, \ldots, \chi_n)$ with χ^n, it holds for any $\epsilon \in (0,1)$ and $\alpha > 0$*

$$P\left(\sup_{g \in \mathcal{G}} \frac{Eg - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i)}{\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + Eg} > \epsilon\right) \leq 4E\left[\mathcal{N}_1\left(\frac{\epsilon\alpha}{8}, \mathcal{G}, \chi^n\right)\right] \exp\left(-\frac{\epsilon^2 \alpha n}{B}\frac{6}{169}\right)$$

where we write short $Eg(\chi) := \int g(x) P \circ \chi^{-1}(dx)$ *for any measurable function* $g : \mathbb{R}^D \to [0, B]$.

Proof. Let $\tilde{\chi}^n := \{\tilde{\chi}_1, \ldots, \tilde{\chi}_n\}$ be a second set of independent identically distributed random variables, independent of χ^n, such that $\tilde{\chi}_i$ is distributed like χ. Then, for any fixed $g \in \mathcal{G}$, if the inequalities

$$Eg - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > \epsilon \left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + Eg\right)$$

and

$$Eg - \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) \leq \frac{\epsilon}{4} \left(\alpha + Eg + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i)\right)$$

hold true, it follows that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > -\frac{\epsilon}{4}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) + Eg\right) + \epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + Eg\right) \\
\Leftrightarrow &\frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > \alpha\left(\epsilon - \frac{\epsilon}{4}\right) + \epsilon\frac{1}{n}\sum_{i=1}^{n} g(\chi_i) - \frac{\epsilon}{4}\frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) + \left(\epsilon - \frac{\epsilon}{4}\right) Eg \\
\Leftrightarrow &\left(1 + \frac{5}{8}\epsilon\right)\left(\frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i)\right) > \frac{3}{8}\epsilon\left(\frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) + 2\alpha\right) + \frac{3}{4}\epsilon Eg \\
\Rightarrow &\frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > \frac{3}{13}\epsilon\left(\frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) + 2\alpha\right)
\end{aligned}$$

since $\frac{13}{8} > 1 + \frac{5}{8}\epsilon$ and g is non-negative. Hence the intersection of the sets

$$\left\{Eg - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > \epsilon \left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + Eg\right)\right\}$$

and

$$\left\{Eg - \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) \leq \frac{\epsilon}{4} \left(\alpha + Eg + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i)\right)\right\}$$

is a subset of

$$\left\{\frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > \frac{3}{13}\epsilon\left(\frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) + 2\alpha\right)\right\}.$$

Now define the function g^* pointwise for each ω as a function in $\mathcal{G}$ satisfying $Eg -$

$\frac{1}{n}\sum_{i=1}^n g(\chi_i) > \epsilon(\alpha + \frac{1}{n}\sum_{i=1}^n g(\chi_i) + Eg)$, if such a function exists, and a random function in $\mathcal{G}$ otherwise. Note that g^* depends on the set χ_i^n only. Using the conclusions on the sets above, we get

$$\begin{aligned}
&P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^n g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^n g(\chi_i) > \frac{3}{13}\epsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^n g(\chi_i) + \frac{1}{n}\sum_{i=1}^n g(\tilde{\chi}_i)\right)\right)\\
\geq &P\left(\frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) > \frac{3}{13}\epsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) + \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i)\right)\right)\\
\geq &P\left(\left\{E[g^*|\chi^n] - \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) > \epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) + E[g^*|\chi^n]\right)\right\}\right.\\
&\qquad\left.\cap\left\{E[g^*|\chi^n] - \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i) \leq \frac{\epsilon}{4}\left(\alpha + E[g^*|\chi^n] + \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i)\right)\right\}\right)\\
= &E\left[\mathbb{1}_{\left\{E[g^*|\chi^n] - \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) > \epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) + E[g^*|\chi^n]\right)\right\}}\right.\\
&\qquad\left.\times P\left(E[g^*|\chi^n] - \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i) \leq \frac{\epsilon}{4}\left(\alpha + E[g^*|\chi^n] + \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i)\right)\middle|\chi_1^n\right)\right]
\end{aligned} \tag{A.1}$$

where we used the tower property in the last step to introduce the conditional expectation given the set χ^n and that the indicator function on the right hand side is $\sigma(\chi^n)$-measurable. It now follows directly from Lemma 11.2 in Györfi et al. (2006) that

$$P\left(E[g^*|\chi^n] - \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i) > \frac{\epsilon}{4}\left(\alpha + E[g^*|\chi^n] + \frac{1}{n}\sum_{i=1}^n g^*(\tilde{\chi}_i)\right)\middle|\chi_1^n\right) \leq \frac{4B}{\epsilon^2\alpha n}, \tag{A.2}$$

which essentially follows from Tchebycheff's inequality using that the functions g are non-negative. Hence, for $n > \frac{8B}{\epsilon^2\alpha}$, we have by (A.2) and (A.1) that

$$\begin{aligned}
&P\left(\exists g \in \mathcal{G} : \frac{Eg - \frac{1}{n}\sum_{i=1}^n g(\chi_i)}{\alpha + \frac{1}{n}\sum_{i=1}^n g(\chi_i) + Eg} > \epsilon\right)\\
&= P\left(E[g^*|\chi^n] - \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) > \epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^n g^*(\chi_i) + E[g^*|\chi^n]\right)\right)\\
&\leq 2P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^n g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^n g(\chi_i) > \frac{3}{13}\epsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^n g(\chi_i) + \frac{1}{n}\sum_{i=1}^n g(\tilde{\chi}_i)\right)\right)
\end{aligned}$$

and it suffices to show that

$$
\begin{aligned}
&P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i) - \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) > \frac{3}{13}\epsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i)\right)\right)\\
&\le 2E\left[\mathcal{N}_1\left(\frac{\epsilon\alpha}{8}, \mathcal{G}, \chi^n\right)\right]\exp\left(-\frac{\epsilon^2\alpha n}{B}\frac{6}{169}\right) \qquad \text{(A.3)}
\end{aligned}
$$

in this case. For $n \le \frac{8B}{\alpha\epsilon^2}$, the statement of the theorem is trivial since then the right-hand side is bigger than one.
For the proof of (A.3), let $U_1, \ldots, U_n$ be independent and uniformly $\{-1, 1\}$-distributed random variables that are independent of $\chi_1, \ldots, \chi_n, \tilde{\chi}_1, \ldots, \tilde{\chi}_n$. Since χ_i and $\tilde{\chi}_i$ have the same distribution, it holds that $g(\chi_i) - g(\tilde{\chi}_i)$ is distributed like $U_i(g(\chi_i) - g(\tilde{\chi}_i))$ for any $i \in \{1, \ldots, n\}$. We conclude

$$
\begin{aligned}
&P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) - g(\tilde{\chi}_i) > \frac{3}{13}\epsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i)\right)\right)\\
=&P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i\left(g(\chi_i) - g(\tilde{\chi}_i)\right) > \frac{3}{13}\epsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i) + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i)\right)\right)\\
\le&P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(\chi_i) > \frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i)\right)\right)\\
&\quad + P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(\tilde{\chi}_i) < -\frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\tilde{\chi}_i)\right)\right)\\
=&2P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(\chi_i) > \frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i)\right)\right).
\end{aligned}
$$

Here the inequality holds since the set on the left-hand side is a subset of the conclusion of the sets on the right-hand side of the inequality. The last equality then follows since $-U_i$ has the same distribution as U_i by construction.
We now consider this probability for an arbitrary fixed outcome $z^n = (z_1, \ldots, z_n) \in \mathbb{R}^{D\times n}$ of the random variables χ^n, i.e., we consider

$$
\begin{aligned}
P&\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(\chi_i) > \frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(\chi_i)\right)\middle|\chi^n = z^n\right)\\
&= P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(z_i) > \frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(z_i)\right)\right).
\end{aligned}
$$

For a $\delta > 0$, which we specify later, let $\mathcal{G}_\delta$ be an L_1-δ-cover of $\mathcal{G}$ on the set z^n, i.e., for all

$g \in \mathcal{G}$, we find an $g' \in \mathcal{G}_\delta$ such that

$$\frac{1}{n}\sum_{i=1}^{n} |g(z_i) - g'(z_i)| < \delta.$$

Then, by a zero addition, we get

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} U_i g(z_i) &= \frac{1}{n}\sum_{i=1}^{n} U_i g'(z_i) + \frac{1}{n}\sum_{i=1}^{n} U_i \left(g(z_i) - g'(z_i)\right)\\ &\leq \frac{1}{n}\sum_{i=1}^{n} U_i g'(z_i) + \frac{1}{n}\sum_{i=1}^{n} |g(z_i) - g'(z_i)|\\ &< \frac{1}{n}\sum_{i=1}^{n} U_i g'(z_i) + \delta\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} g(z_i) &= \frac{1}{n}\sum_{i=1}^{n} g'(z_i) - \frac{1}{n}\sum_{i=1}^{n} g'(z_i) + g(z_i)\\ &\geq \frac{1}{n}\sum_{i=1}^{n} g'(z_i) - \frac{1}{n}\sum_{i=1}^{n} |g'(z_i) + g(z_i)|\\ &> \frac{1}{n}\sum_{i=1}^{n} g'(z_i) - \delta.\end{aligned}$$

Hence it holds

$$\begin{aligned}&P\left(\exists g \in \mathcal{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(z_i) > \frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(z_i)\right)\right)\\ \leq &P\left(\exists g \in \mathcal{G}_\delta : \frac{1}{n}\sum_{i=1}^{n} U_i g(z_i) + \delta > \frac{3}{13}\epsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(z_i) - \delta\right)\right)\\ \leq &|\mathcal{G}_\delta| \max_{g \in \mathcal{G}_\delta} P\left(\frac{1}{n}\sum_{i=1}^{n} U_i g(z_i) > \frac{3}{13}\epsilon\alpha - \delta - \frac{3}{13}\epsilon\delta + \frac{3\epsilon}{13n}\sum_{i=1}^{n} g(z_i)\right).\end{aligned}$$

Now choose $\delta = \frac{\epsilon\alpha}{8}$ and suppose that $\mathcal{G}_\delta$ is of minimal size. Then we have

$$\frac{3}{13}\epsilon\alpha - \delta - \epsilon\frac{3}{13}\delta \geq \epsilon\alpha\left(\frac{24}{104} - \frac{13}{104} - \frac{3}{104}\right) = \frac{1}{13}\epsilon\alpha$$

and hence get

$$P\left(\exists g\in\mathcal{G}:\frac{1}{n}\sum_{i=1}^{n}U_ig(z_i)>\frac{3}{13}\epsilon\left(\alpha+\frac{1}{n}\sum_{i=1}^{n}g(z_i)\right)\right)$$
$$\leq\mathcal{N}_1\left(\frac{\epsilon\alpha}{8},\mathcal{G},z_1^n\right)\max_{g\in\mathcal{G}_\delta}P\left(\frac{1}{n}\sum_{i=1}^{n}U_ig(z_i)>\frac{\epsilon\alpha}{13}+\frac{3\epsilon}{13n}\sum_{i=1}^{n}g(z_i)\right).$$

It follows by Hoeffding's inequality that

$$P\left(\frac{1}{n}\sum_{i=1}^{n}U_ig(z_i)>\frac{\epsilon\alpha}{13}+\frac{3\epsilon}{13n}\sum_{i=1}^{n}g(z_i)\right)\leq\exp\left(-\frac{2n^2(\frac{\epsilon\alpha}{13}+\frac{3}{13}\epsilon\frac{1}{n}\sum_{i=1}^n g(z_i))^2}{4\sum_{i=1}^n g(z_i)^2}\right)$$
$$=\exp\left(-\frac{\epsilon^2 9(\frac{\alpha n}{3}+\sum_{i=1}^n g(z_i))^2}{169\times 2B\sum_{i=1}^n g(z_i)}\right).$$

Using that $\frac{(a+y)^2}{a}\geq 4y$ for all $a,y>0$, we obtain

$$\exp\left(-\frac{\epsilon^2 9(\frac{\alpha n}{3}+\sum_{i=1}^n g(z_i))^2}{169\times 2B\sum_{i=1}^n g(z_i)}\right)\leq\exp\left(-\frac{6\alpha\epsilon^2 n}{169B}\right).$$

Then, denoting the distribution of the random set χ^n with P_{χ^n}, it holds

$$P\left(\sup_{g\in\mathcal{G}}\frac{Eg-\frac{1}{n}\sum_{i=1}^n g(\chi_i)}{\alpha+\frac{1}{n}\sum_{i=1}^n g(\chi_i)+Eg}>\epsilon\right)$$
$$\leq 4P\left(\exists g\in\mathcal{G}:\frac{1}{n}\sum_{i=1}^{n}U_ig(\chi_i)>\frac{3}{13}\epsilon\left(\alpha+\frac{1}{n}\sum_{i=1}^{n}g(\chi_i)\right)\middle|Z^n=z^n\right)$$
$$=4\int_{\mathbb{R}^{D\times N}}P\left(\exists g\in\mathcal{G}:\frac{1}{n}\sum_{i=1}^{n}U_ig(z_i)>\frac{3}{13}\epsilon\left(\alpha+\frac{1}{n}\sum_{i=1}^{n}g(z_i)\right)\right)dP_{\chi^n}(dz^n)$$
$$\leq 4\int_{\mathbb{R}^{D\times N}}\mathcal{N}_1\left(\frac{\epsilon\alpha}{8},\mathcal{G},z_1^n\right)\exp\left(-\frac{6\alpha\epsilon^2 n}{169B}\right)dF_{\chi^n}(dz^n)$$
$$=4E\left[\mathcal{N}_1\left(\frac{\epsilon\alpha}{8},\mathcal{G},\chi^n\right)\right]\exp\left(-\frac{6\alpha\epsilon^2 n}{169B}\right)$$

what finishes the proof of Theorem A.2. □

We will now use Theorem A.2 to prove Lemma 1.5.5. For this purpose, let $\mathcal{K}$ be a linear vector space of functions mapping $\mathbb{R}^D$ into $\mathbb{R}$ and let $\upsilon:\mathbb{R}^D\to\mathbb{R}$ be a function satisfying $|\upsilon(x)|\leq C_\upsilon$ for all $x\in\mathbb{R}^d$. We then consider the set of functions $\mathcal{G}:=\{|\mathcal{T}_{C_\upsilon}(\psi(x))-\upsilon(x)|^2:\psi\in\mathcal{K}\}$. By construction, the functions in $\mathcal{G}$ take values in the set $[0,4C_\upsilon^2]$. Using again the notation $Eg:=\int g(x)P_{X_{t_i}}(dx)$ and $\overline{g}:=\frac{1}{M_i}\sum_{i=1}^{M_i}g(X_{t_i}^{[i,m]})$ for any function $g\in\mathcal{G}$, we conclude for any $\beta > 0$ that

$$
\begin{aligned}
&P\left(\sup_{g\in\mathcal{G}} Eg - (1+\epsilon)\overline{g} > \beta\right)\\
=&P\left(\exists g \in \mathcal{G} : \frac{2}{2+\epsilon}Eg - \frac{2+2\epsilon}{2+\epsilon}\overline{g} > \frac{2\beta}{2+\epsilon}\right)\\
=&P\left(\exists g \in \mathcal{G} : Eg - \frac{\epsilon}{2+\epsilon}Eg - \overline{g} - \frac{\epsilon}{2+\epsilon}\overline{g} > \frac{2\beta}{2+\epsilon}\right)\\
=&P\left(\exists g \in \mathcal{G} : Eg - \overline{g} > \frac{\epsilon}{2+\epsilon}\left(Eg + \overline{g} + \frac{2\beta}{\epsilon}\right)\right)\\
=&P\left(\exists g \in \mathcal{G} : \frac{Eg - \overline{g}}{Eg + \overline{g} + \frac{2\beta}{\epsilon}} > \frac{\epsilon}{2+\epsilon}\right).
\end{aligned}
$$

Hence Theorem A.2 yields the bound

$$
\begin{aligned}
&P\left(\sup_{g\in\mathcal{G}} Eg - (1+\epsilon)\overline{g} > \beta\right)\\
\leq&4E\left[\mathcal{N}_1\left(\frac{\beta}{4(2+\epsilon)}, \mathcal{G}, X_i^{M_i}\right)\right]\exp\left(-\frac{3\beta\epsilon M_i}{(2+\epsilon)^2 169 C_v^2}\right).
\end{aligned}
$$

Note that $\mathcal{T}_{C_v}(\psi(x)) - v(x) \in [-2C_v, 2C_v]$ for every $\psi \in \mathcal{K}$ and the function $|.|^2 : [-2C_v, 2C_v] \to \mathbb{R}$, $x \mapsto x^2$ is Lipschitz continuous with Lipschitz constant $4C_v$. Hence it follows for the set of functions $\mathcal{T}_{C_v}\mathcal{K} := \{\mathcal{T}_{C_v}(\psi) : \psi \in \mathcal{K}\}$ that

$$
|(\psi_1(x) - v(x))^2 - (\psi_2(x) - v(x))^2| \leq 4C_v|\psi_1(x) - \psi_2(x)|
$$

for all $\psi_1, \psi_2 \in \mathcal{T}_{C_v}\mathcal{K}$ and $x \in \mathbb{R}^D$. Consequently, it holds

$$
\mathcal{N}_1\left(\frac{\beta}{4(2+\epsilon)}, \mathcal{G}, X_i^{M_i}\right) \leq \mathcal{N}_1\left(\frac{\beta}{16C_v(2+\epsilon)}, \mathcal{T}_{C_v}\mathcal{K}, X_i^{M_i}\right).
$$

By Lemma 9.2, Theroem 9.4 and Theorem 9.5 in Györfi, we get

$$
\mathcal{N}_1\left(\frac{\beta}{16C_v(2+\epsilon)}, \mathcal{T}_{C_v}\mathcal{K}, X_i^{M_i}\right) \leq 3\left(\frac{64eC_v^2(2+\epsilon)}{\beta}\log\left(\frac{96eC_v^2(2+\epsilon)}{\beta}\right)\right)^{K+1}
$$

whenever $\beta < 8C_v^2(2+\epsilon)$ where K is the vector space dimension of $\mathcal{K}$. Then, since for all

$x \geq 4$

$$2ex\log(3ex) = 2ex\log(3ex - 12e + 12e) = 2ex(\log(12e) + log\left(1 + \frac{3ex - 12e}{12e}\right)$$
$$\leq 2ex\left(\log(12e) + \frac{3ex}{12e}\right) \leq 2ex\left(\frac{x\log(12)}{4} + \frac{x}{4}\right) \leq ex^2\left(\frac{\log(12)+1}{2}\right) \leq (3x)^2,$$

it holds

$$\begin{aligned}&P\left(\sup_{g\in\mathcal{G}} Eg - (1+\epsilon)\overline{g} > \beta\right)\\ \leq&12\left(\frac{96eC_v^2(2+\epsilon)}{\beta}\right)^{2(K+1)}\exp\left(-\frac{3\beta\epsilon M_i}{(2+\epsilon)^2 169 C_v^2}\right)\end{aligned}$$

whenever $\beta \leq 8C_v^2(2+\epsilon)$. On the other hand, $P(\sup_{g\in\mathcal{G}} Eg - (1+\epsilon)\overline{g} > \beta) = 0$ for $\beta > 4C_v^2(2+\epsilon)$, since g is bounded by $4C_\nu^2$. Now set $a = 96eC_v^2(2+\epsilon)$, $b = \frac{3\epsilon}{169(2+\epsilon)^2C_v^2}$ and suppose $\beta_0 \geq \frac{a}{M_i(1+ab)}$. Then

$$\begin{aligned}&E\left[\sup_{g\in\mathcal{G}} Eg - (1+\epsilon)\overline{g}\right]\\ \leq&\beta_0 + \int_{\beta_0}^{\infty} 12\left(\frac{a}{\beta}\right)^{2(K+1)}\exp(-bM_i\beta)d\beta\\ \leq&\beta_0 + \frac{12}{bM_i}(M_i(1+ab))^{2(K+1)}\exp(-bM_i\beta_0).\end{aligned}$$

With the choice $\beta_0 = \frac{1}{bM_i}\log(12((1+ab)M_i)^{2(K+1)})$, which satisfies the restiction above (see Gobet and Turkedjiev, 2016), this implies

$$\begin{aligned}&E\left[\sup_{g\in\mathcal{G}} Eg - (1+\epsilon)\overline{g}\right]\\ \leq&\frac{1}{bM_i}\left(1 + \log(12) + 2(K+1)\log((1+ab)M_i)\right)\\ =&\frac{2(K+1)}{bM_i}\log\left((1+ab)\exp\left(\frac{1}{2(K+1)}(1+\log(12))\right)M_i\right)\\ \leq&\frac{(2+\epsilon)^2 169C_\nu^2 2(K+1)}{3\epsilon M_i}\log\left(\left(1 + \frac{288e\epsilon}{169(2+\epsilon)}\right)\exp\left(\frac{1}{4}(1+\log(12))\right)M_i\right).\end{aligned}$$

Now for statement of Lemma 1.5.5 concerning $\overline{q}_i^N$, we get from the calculations above by choosing $\mathcal{K} = \mathcal{K}_{q,i}$ and $\upsilon = \overline{q}_i^N$ that

$$
\begin{aligned}
&E\left[\|\overline{q}_i^N - q_i^{N,M}\|_{i,\infty}^2\right] \\
&\leq (1+\epsilon)E\left[\|\overline{q}_i^N - q_i^{N,M}\|_{i,M_i}^2\right] + E\left[\left(\|\overline{q}_i^N - q_i^{N,M}\|_{i,\infty}^2 - (1+\epsilon)\|\overline{q}_i^N - q_i^{N,M}\|_{i,M_i}^2\right)_+\right] \\
&\leq E\left[\sup_{\psi\in\mathcal{K}_{q,i}} E\psi - (1+\epsilon)\overline{\psi}\right] + (1+\epsilon)E\left[\|\overline{q}_i^N - q_i^{N,M}\|_{i,M_i}^2\right] \\
&\leq \frac{(2+\epsilon)^2 169 C_{q,i}^2 2(K_{q,i}+1)}{3\epsilon M_i}\log\left(\left(1+\frac{288e\epsilon}{169(2+\epsilon)}\right)\exp\left(\frac{1}{4}(1+\log(12))\right)M_i\right) \\
&\quad + (1+\epsilon)E\left[\|\overline{q}_i^N - q_i^{N,M}\|_{i,M_i}^2\right] \\
&\leq \frac{1014C_{q,i}^2(K_{q,i}+1)}{\epsilon M_i}\log\left(\frac{288e}{507}\exp\left(\frac{1}{4}(1+\log(12)\right)M_i\right) + (1+\epsilon)E\left[\|\overline{q}_i^N - q_i^{N,M}\|_{i,M_i}^2\right] \\
&\leq \frac{C_1(K_{q,i}+1)}{\epsilon M_i}\log(C_2 M_i) + (1+\epsilon)E\left[\|\overline{q}_i^N - q_i^{N,M}\|_{i,M_i}^2\right]
\end{aligned}
$$

with

$$
\begin{aligned}
C_1 &:= 1014C_q^2, \\
C_2 &:= \frac{288e}{507}\exp\left(\frac{1}{4}(1+\log(12)\right).
\end{aligned}
$$

For the bound concerning $\overline{z}_i^N$, note that

$$
E\left[\|\overline{z}_i^N - z_i^{N,M}\|_{i,\infty}^2\right] = \sum_{i=1}^{\mathcal{D}} E\left[\|z_i^{N,(d)} - z_i^{N,M,(d)}\|_{i,\infty}^2\right]
$$

and the same arguments can be used for each component by replacing $C_{q,i}$ by $C_z = \frac{C_{q,i}}{\sqrt{\Delta}}$. Thisleads to the additional factor Δ^{-1} in the bound concerning $\overline{z}_i^N$.

Appendix B

Appendix to Chapter 2

Mesh method for constructing input approximations

In the following, we describe how the input approximations used in Algorithm 2.2.1 can be obtained with the mesh method, which was proposed by Broadie et al. (2004) for the pricing of American style derivatives. The idea, however, can be extended in order to cover more general stochastic control problems and the setting in Chapter 2. We first briefly describe the idea of the mesh method in general and then state an algorithm for the implementation used for the numerical example in Chapter 2.
Recall the functions C_j defined as

$$C_j(x) = E[\beta_{j+1} Y_{j+1} | X_j = x]$$

where

$$\beta_j = (1, B_j) \quad X_j = h_j(B_j, X_{j-1})$$

for independent D-dimensional Gaussian random variables B_j and the representation of Y as

$$\begin{aligned} Y_J &= \xi(X_J) \\ Y_j &= \sup_{\rho \in D_f^j} \Big(E_j[Y_{j+1}] + E_j[\Delta \rho^T Y_{j+1} - f_j^{\#}(X_j, \rho)] \Big) \quad j = 0, \ldots, J-1. \end{aligned}$$

We suppose that the function $h_j(x, .)$ is for each $x \in \mathbb{R}^D$ invertible and notate the inverse function with $h_{j,x}^{-1}$. Under this constraint, we can write the process β at any time point t_i

as a function of X_i and X_{i+1}, i.e., by defining $\tilde{\beta}_j : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^{D+1}$ as

$$\begin{aligned}
\tilde{\beta}_j^{(1)}(x, y) &= 1 \\
\tilde{\beta}_j^{(d)}(x, y) &= [h_x^{-1}(y)]^{(d-1)}_{\varsigma\sqrt{\Delta}} \quad d = 2, \dots, D+1,
\end{aligned}$$

we have $\beta_j = \tilde{\beta}(X_j, X_{j+1})$. The central idea of the mesh method is then to simulate nodes $(\hat{X}_j^{[m]})_{j=0,\dots J; m=1,\dots,M}$ in $\mathbb{R}^D$, which are fixed throughout the algorithm and do not necessarily have to be distributed like the SDE solution X. Then, for each pair of nodes $(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[n]})$ with $m, n \in \{1, \dots, M\}$ and $j \in \{0, \dots, J-1\}$, a specific weight $w(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[n]})$ is calculated, which we will specify later on. Given the mesh nodes and the weights, the values of the approximation operators $C_{M,j}$ on the nodes are then constructed recursively backward in time as the weighted average

$$\begin{aligned}
C_{M,J-1}(\hat{X}_{J-1}^{[m]}) &= \frac{1}{M} \sum_{l=1}^{M} \tilde{\beta}_J(\hat{X}_{J-1}^{[m]}, \hat{X}_J^{[l]}) \xi_J(\hat{X}_J^{[l]}) w(\hat{X}_{J-1}^{[m]}, \hat{X}_J^{[l]}) \\
C_{M,j}(\hat{X}_j^{[m]}) &= \frac{1}{M} \sum_{l=1}^{M} \beta_{j+1}(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[l]}) \sup_{\rho \in D_f^j} \Big((e_1 + \rho)^T C_{M,j+1}(\hat{X}_{j+1}^{[l]}) - \Delta f_{j+1}^{\#}(\hat{X}_{j+1}^{[l]}, \rho) \Big) \\
&\quad \times w(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[l]}), \qquad j = 1, \dots, J-2, \qquad \text{(B.1)}
\end{aligned}$$

where e_1 denotes the first canonical basis vector in $\mathbb{R}^{D+1}$. The weight function w can be thought of as a normalization factor which is needed to model the right transition probabilities between the mesh nodes in consecutive time steps. This function has to be chosen in dependence of the distribution of the simulated nodes $\hat{X}$ such that at each time step, the approximations $C_{M,j}$ of C_j on the mesh nodes are unbiased under the constraint that the $C_{M,j+1} = C_{j+1}$, i.e., such that

$$\begin{aligned}
&E\left[\frac{1}{M} \sum_{l=1}^{M} \sup_{\rho \in D_f^{j+1}} \Big(\tilde{\beta}_{j+1}(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[l]})(\Delta\rho + e_1) C_{j+1} - \Delta f_{j+1}^{\#}(\hat{X}_{j+1}^{[l]}, \rho) \Big) w(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[l]}) \middle| (\hat{X}_j^{[k]})_{k=1,\dots,M} \right] \\
&= C_j(\hat{X}_j^{[m]}).
\end{aligned}$$

Note again that due to Lemma 2.1.7, at each time step j and for any mesh node $X_j^{[m]}$, there exists a $\hat{\rho}_{j+1}^{[m]}$ that maximizes the right hand side of (B.1), which is given by a solution to the equation

$$\rho_{j+1}^T C_{M,j+1}(\hat{X}_{j+1}^{[m]}) - f_{j+1}^{\#}(\hat{X}_{j+1}^{[m]}, \rho_{j+1}) = f_{j+1}(\hat{X}_{j+1}^{[m]}, C_{M,j+1}(\hat{X}_{j+1}^{[m]})).$$

Hence the values $C_{M,j}(\hat{X}^{[m]})$ on the Mesh nodes induce samples $\hat{\rho}_j^{[m]}$ of the control process ρ and (B.1) is equivalent to

$$C_{M,j}(\hat{X}_j^{[m]})$$
$$= \frac{1}{M}\sum_{l=1}^{M} \beta_{j+1}(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[l]}) \left((e_1 + \hat{\rho}_{j+1}^{[l]})^T C_{M,j+1}(\hat{X}_{j+1}^{[l]}) - \Delta f_{j+1}^{\#}(\hat{X}_{j+1}^{[l]}, \hat{\rho}_{j+1}^{[l]})\right) w(\hat{X}_j^{[m]}, \hat{X}_{j+1}^{[l]}).$$

Here the values of $\hat{\rho}_j^{[m]}$ only depend on the mesh nodes at the time j and the the approximation operator $C_{M,j}$ evaluated in those mehs nodes. Thus, given the values $C_{M,j}(\hat{X}^{[m]})$ on the mesh nodes, we can define the approximation operators $C_{M,j}$ on any point $x \in \mathbb{R}^D$ not on the mesh nodes as

$$C_{M,J-1}(x) = \frac{1}{M}\sum_{l=1}^{M} \tilde{\beta}_J(x, \hat{X}_J^{[l]})\xi_J(\hat{X}_J^{[l]})w(x, \hat{X}_J^{[l]})$$
$$C_{M,j}(x) = \frac{1}{M}\sum_{l=1}^{M} \left(\tilde{\beta}_{j+1}(x, \hat{X}_{j+1}^{[l]})(\Delta\hat{\rho}_{j+1}^{[l]} + e_1)C_{M,j+1}(\hat{X}_j^{[l]} - \Delta f_{j+1}^{\#}(\hat{X}_{j+1}^{[l]}, \hat{\rho}_{j+1}^{[l]})\right) w(x, \hat{X}_{j+1}^{[l]}).$$

The canonical approach for choosing the distribution of the mesh nodes $\hat{X}$ would be to simulate the nodes $\hat{X}$ according to the distribution of the SDE solution X. With this choice, one should define the mesh weights $w(x, y)$ as the conditional density of X for the transition of x to y in consecutive time steps. Theoretical results, however, show that this choice leads to an exploding variance of the terms $C_{M,j}(\hat{X}_j^{[m]})$ for an increasing number of nodes M, see e.g. Broadie et al. (2004). Therefore, we use a different approach for the calculation of the input approximations in Chapter 2. The idea is to simulate $\hat{X}_{j+1}^{[m]}$ by choosing a random predecessor of the nodes $\hat{X}_j^{[m]}$, $m = 1, \ldots, M$, i.e., a $\hat{X}_j^{[r]}$ for a random $r \in \{1, \ldots, M\}$ and then simulates $\hat{X}_{j+1}^{[l]}$ according to the conditional density of the real SDE solution given $X_j = \hat{X}_j^{[r]}$. Choosing the weights accordingly then results in the following algorithm used for the numerical experiment in Chapter 2.

Algorithm B.1 (Input approximation via mesh method)**.**

(1) Simulate training paths $(\hat{X}_j^{[m]})_{j=1,\ldots,J}$ for $m = 1, \ldots, M$ by setting $\hat{X}_0^{[m]} = X_0$. Then, given the values of $\hat{X}_j^{[m]}$ for $m = 1, \ldots, M$, set

$$\hat{X}_{j+1}^{[m]} = h_j(\hat{X}_j^{[r]}, \Delta\hat{W}_{j+1}^{[m]})$$

where $\Delta\hat{W}_{j+1}^{[m]}$ is a D-dimensional Gaussian distributed random variable with independent components with mean 0 and variance Δ each and r is discrete uniformly distributed on the set $\{1, \ldots, M\}$, which is re-sampled for each node.

(2) *Calculate*

$$C_{M,J-1}(\hat{X}_{J-1}^{[m]}) := \frac{1}{M}\sum_{l=1}^{M}\left(\tilde{\beta}_J(\hat{X}_{J-1}^{[m]},\hat{X}_J^{[l]})\xi(\hat{X}_J^{[l]})\right)\frac{Md_X(J,\hat{X}_{J-1}^{[m]},\hat{X}_J^{[l]})}{\sum_{k=1}^{M} d_X(J,\hat{X}_{J-1}^{[k]},\hat{X}_J^{[l]})}$$

where $d_X(j,x,.)$ *is the conditional density of* X_j *given* $X_{j-1}=x$.
Then, recursive backward in time given the values of $C_{M,j}(\hat{X}_j^{[m]})$, *calculate* $\hat{\rho}_j^{[m]}$ *as a solution of the equation*

$$\rho_j^T C_{M,j}(\hat{X}_j^{[m]}) - f_j^{\#}(\hat{X}_j^{[m]},\rho_j) = f_j(\hat{X}_j^{[m]}, C_{M,j}(\hat{X}_j^{[m]}))$$

and set

$$C_{M,j-1}(\hat{X}_{j-1}^{[m]}) = \frac{1}{M}\sum_{l=1}^{M}\Bigg(\tilde{\beta}_j(\hat{X}_{j-1}^{[m]},\hat{X}_j^{[l]})(\Delta\hat{\rho}_j^{[l]}+e_1)^T C_{M,j}(\hat{X}_j^{[l]}) - \Delta f_j^{\#}(\hat{X}_j^{[l]},\hat{\rho}_j^{[l]}) \times \frac{Md_X(j,\hat{X}_{j-1}^{[m]},\hat{X}_j^{[l]})}{\sum_{k=1}^{M} d_X(j,\hat{X}_{j-1}^{[k]},\hat{X}_j^{[l]})}\Bigg),$$

where e_1 *denotes the first basis vector in* $\mathbb{R}^{D+1}$.

(3) *To evaluate the functions* $C_{M,j}$ *in any point* x *not on the mesh grid, set*

$$C_{M,J-1}(x) := \frac{1}{M}\sum_{m=1}^{M}\left(\tilde{\beta}_J(x,\hat{X}_J^{[m]})\xi(\hat{X}_J^{[m]})\right)\frac{Md_X(J,x,\hat{X}_J^{[m]})}{\sum_{l=1}^{M} d_X(J-1,\hat{X}_J^{[l]},\hat{X}_J^{[m]})}$$

at time $J-1$ *and*

$$C_{M,j-1}(x) = \frac{1}{M}\sum_{m=1}^{M}\Bigg(\tilde{\beta}_j(x,\hat{X}_j^{[m]})(\Delta\hat{\rho}_j^{[m]}+e_1)^T C_{M,j}(\hat{X}_j^{[m]}) - \Delta f_j^{\#}(\hat{X}_j^{[m]},\hat{\rho}_j^{[m]}) \times \frac{Md_X(j,x,\hat{X}_j^{[l]})}{\sum_{l=1}^{M} d_X(j,\hat{X}_{j-1}^{[l]},\hat{X}_j^{[m]})}\Bigg).$$

at any time point $j\in\{0,\dots,J-2\}$ *and all* $x\in\mathbb{R}^D$.

Remark B.0.1. For the construction of the approximation operators on a mesh node $X_j^{[m]}$, we have to calculate the averaged sum over all M nodes in the following time and calculate the mesh weight for each of these nodes, which leads to costs of order M. Hence, the costs for constructing the approximation operators on all mesh nodes are of order M^2. Analogue, for evaluating the operators in any point $x\in\mathbb{R}^D$ not on the mesh nodes, one has again to calculated the averaged sum over the M mesh points in the corresponding time point which again leads to computation costs of order M.

For additional results regarding the mesh method, we refer to Broadie et al. (2004), where the method was introduced for American option pricing, or Agarwal and Juneja (2013), where results regarding the convergence of the estimators can be found in the setting of Bermudan option pricing.

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