Pattern formation at a fluid-ferrofluid interface

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Abstract

We establish the existence of static doubly periodic patterns (in particular rolls, rectangles and hexagons) and symmetric defects ('walls' between two rotated roll patterns) on the free surface of a ferrofluid near onset of the Rosensweig instability, assuming a general (nonlinear) magnetisation law.

To show the existence of static doubly periodic patterns, we formulate the ferrohydrostatic equations in terms of Dirichlet-Neumann operators for nonlinear elliptic boundary-value problems. We demonstrate the analyticity of these operators in suitable function spaces and solve the ferrohydrostatic problem using an analytic version of Crandall-Rabinowitz local bifurcation theory. Criteria are derived for the bifurcations to be sub-, super- or transcritical with respect to a dimensionless physical parameter.

To show the existence of symmetric defects, we use a Hamiltonian version of a spatial dynamics theory for domain walls by Haragus and Iooss [17, 18]. We formulate the ferrohydrodynamic problem as a spatial Hamiltonian system. A centre-manifold reduction technique converts the problem for small solutions near onset to an equivalent Hamiltonian system with six degrees of freedom. We show that the reduced system has a heteroclinic connection between two periodic solutions corresponding to rotated rolls.

Zusammenfassung

Wir zeigen die Existenz doppelperiodischer statischer Muster (insbesondere Rollen, Rechtecke und Hexagone) und symmetrischer Defekte ('Wände' zwischen zwei rotierten Rollenmustern) an der freien Oberfläche eines Ferrofluids kurz vor Eintritt der Rosensweig-Instabilität, wobei wir ein allgemeines (nichtlineares) Magnetisierungsgesetz verwenden.

Um die Existenz doppelperiodischer statischer Muster zu zeigen, formulieren wir die ferrohydrostatischen Gleichungen mit Dirichlet-Neumann-Operatoren für nichtlineare elliptische Randwertprobleme um. Wir weisen die Analytizität dieser Operatoren in geeigneten Funktionenräumen nach und lösen das ferrohydrostatische Problem mithilfe einer analytischen Version der lokalen Bifurkationstheorie von Crandall und Rabinowitz. Wir leiten Bedingungen dafür her, dass die Bifurkation sub-, super- oder transkritisch bezüglich eines dimensionlosen physikalischen Parameters ist.

Um die Existenz symmetrischer Defekte zu zeigen, nutzen wir eine Hamiltonische Version einer Theorie von räumlicher Dynamik für Domänenwände von Haragus und Iooss [17, 18]. Dafür formulieren wir das ferrohydrodynamische Problem als räumliches Hamiltonisches System. Mithilfe einer Zentrumsmannigfaltigkeitreduktion wandeln wir das Problem für kleine Lösungen bei Eintritt der Rosensweig Instabilität zu einem äquivalenten Hamiltonischen System mit sechs Freiheitsgraden um. Wir zeigen, dass das reduzierte System eine heterokline Verbindung zwischen zwei periodischen Lösungen besitzt, die rotierten Rollen entsprechen.

Acknowledgements

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Contents

Chapter 1

Introduction

1.1 The physical problem

We begin by introducing the physical problem studied in this thesis. Consider two fluids with undisturbed depth d which are incompressible, inviscid, isotropic and free from internal currents. The upper fluid has unit relative permeability and lies in the fluid domain

$$\Omega' := \{(x, y, z) : \eta(x, z) < y < d\},$$

where gravity acts in the negative y-direction. The lower fluid is a ferrofluid with a general (nonlinear) magnetisation law and relative permeability μ greater than unity (further assumptions on μ are given below) which lies in the fluid domain

$$\Omega := \{(x, y, z) : -d < y < \eta(x, z)\}.$$

When these fluids are subjected to a vertically directed magnetic field of sufficient strength, patterns occur at the interface, and we now derive the equations for this phenomenon following Rosensweig [38, Chapters 3 and 5].

The relations between the magnetic fields $\mathbf{H}', \mathbf{H}$ and the induction fields $\mathbf{B}', \mathbf{B}$ are given by the identities

$$\begin{aligned} \mu_0 \mathbf{H}' &= \mathbf{B}' && \text{in } \Omega', \\ \mu_0(\mathbf{H} + \mathbf{M}(|\mathbf{H}|)) &= \mathbf{B} && \text{in } \Omega, \end{aligned}$$

where μ_0 is the vacuum permeability and $\mathbf{M}$ is the magnetic intensity of the ferrofluid. We suppose that

$$\mathbf{M}(|\mathbf{H}|) = m(|\mathbf{H}|)\frac{\mathbf{H}}{|\mathbf{H}|},$$

where m is a (prescribed) nonnegative function, so that in particular $\mathbf{M}$ and $\mathbf{H}$ are collinear.

The fields obey Maxwell's equations

$$\operatorname{div}\mathbf{B}' = 0 \qquad \text{in } \Omega', \tag{1.1}$$

$$\operatorname{curl}\mathbf{H}' = \mathbf{0} \qquad \text{in } \Omega', \tag{1.2}$$

$$\operatorname{div}\mathbf{B} = 0 \qquad \text{in } \Omega, \tag{1.3}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{0} \qquad \text{in } \Omega. \tag{1.4}$$

From equations (1.2) and (1.4) it follows that magnetic potential functions ϕ', ϕ exist with

$$-\operatorname{grad}\phi' = \mathbf{H}' \qquad \text{in } \Omega',$$
$$-\operatorname{grad}\phi = \mathbf{H} \qquad \text{in } \Omega.$$

Using equations (1.1) and (1.3), one finds that these potentials satisfy the equations

$$\operatorname{div}(\operatorname{grad}\phi') = 0 \qquad \text{in } \Omega',$$
$$\operatorname{div}(\mu(|\operatorname{grad}\phi|)\operatorname{grad}\phi) = 0 \qquad \text{in } \Omega,$$

in which

$$\mu(s) = 1 + \frac{m(s)}{s};$$

we assume that $\mu : (0,\infty) \to \mathbb{R}$ is analytic and satisfies $\mu(1) + \dot{\mu}(1) > 0$ (so that the linearised version of the equation for ϕ is elliptic).

At the interface we have the magnetic conditions

$$\mathbf{H}'\cdot\mathbf{t}_1 = \mathbf{H}\cdot\mathbf{t}_1, \qquad \mathbf{H}'\cdot\mathbf{t}_2 = \mathbf{H}\cdot\mathbf{t}_2, \tag{1.5}$$

and

$$\mathbf{B}'\cdot\mathbf{n} = \mathbf{B}\cdot\mathbf{n}, \tag{1.6}$$

where

$$\mathbf{t}_1 = \frac{(1,\eta_x,0)^{\mathrm{T}}}{\sqrt{1+\eta_x^2}}, \qquad \mathbf{t}_2 = \frac{(0,\eta_z,1)^{\mathrm{T}}}{\sqrt{1+\eta_z^2}}$$

and

$$\mathbf{n} = \frac{(-\eta_x,1,-\eta_z)^{\mathrm{T}}}{\sqrt{1+\eta_x^2+\eta_z^2}}$$

are the tangent and normal vectors to the interface; it follows that

$$\phi' - \phi = 0 \qquad \text{for } y = \eta(x,z), \tag{1.7}$$

$$\phi_n' - \mu(|\operatorname{grad}\phi|)\phi_n = 0 \qquad \text{for } y = \eta(x,z). \tag{1.8}$$

The static ferrohydrodynamic Euler equation, which balances the pressure and the forces resulting from gravity and magnetism in the fluid, is derived by Rosensweig [38, Section 5.1] and given by

$$-\operatorname{grad}(p' + \rho' g y) = \mathbf{0} \qquad \text{in } \Omega',$$
$$\mu_0(\mathbf{M}(|\mathbf{H}|)\cdot(\partial_x,\partial_z,\partial_y)^{\mathrm{T}})\mathbf{H} - \operatorname{grad}(p^\star + \rho g y) = \mathbf{0} \qquad \text{in } \Omega,$$

in which g is the acceleration due to gravity, p' is the pressure in the upper fluid, $p^\star$ is a composite of the magnetostrictive and fluid-magnetic pressures and ρ', ρ are the densities of the upper and lower fluid. The calculation

$$\begin{aligned}
\left(\mathbf{M}(|\mathbf{H}|) \cdot (\partial_x, \partial_y, \partial_z)^{\mathrm{T}}\right) \mathbf{H} &= (\mu(|\mathbf{H}|) - 1) \left(\mathbf{H} \cdot (\partial_x, \partial_y, \partial_z)^{\mathrm{T}}\right) \mathbf{H} \\
&= (\mu(|\mathbf{H}|) - 1) \left(\operatorname{grad}\left(\frac{1}{2}|\mathbf{H}|^2\right) - \mathbf{H} \wedge \operatorname{curl} \mathbf{H}\right) \\
&= (\mu(|\mathbf{H}|) - 1)|\mathbf{H}| \operatorname{grad} |\mathbf{H}| \\
&= m(|\mathbf{H}|) \operatorname{grad} |\mathbf{H}| \\
&= \operatorname{grad}\left(\int_0^{|\mathbf{H}|} m(t)\,\mathrm{d}t\right)
\end{aligned}$$

shows that these equations are equivalent to

$$\begin{aligned}
-\operatorname{grad}(p' + \rho' g y) &= \mathbf{0} && \text{in } \Omega', \\
\operatorname{grad}\left(\mu_0 \int_0^{|\mathbf{H}|} m(t)\,\mathrm{d}t - (p^\star + \rho g y)\right) &= \mathbf{0} && \text{in } \Omega
\end{aligned}$$

or, after integrating, to

$$-(p' + \rho' g y) = C_0' \qquad \text{in } \Omega', \tag{1.9}$$

$$\mu_0 \int_0^{|\mathbf{H}|} m(t)\,\mathrm{d}t - (p^\star + \rho g y) = C_0 \qquad \text{in } \Omega, \tag{1.10}$$

where C_0', C_0 are constants.

The ferrohydrodynamic boundary condition, which balances the composite pressure and the forces resulting from the surface tension and magnetism at the interface (see Figure 1.1), is derived by Rosensweig [38, Section 5.2] and given by

$$p^\star + \frac{\mu_0}{2}(\mathbf{M}(|\mathbf{H}|) \cdot \mathbf{n})^2 = p' + 2\sigma\kappa,$$

in which $\sigma > 0$ is the coefficient of surface tension and

$$2\kappa = -\frac{\eta_{xx}(1 + \eta_z^2) + \eta_{zz}(1 + \eta_x^2) - 2\eta_x\eta_z\eta_{xz}}{(1 + \eta_x^2 + \eta_z^2)^{3/2}}$$

is the mean curvature of the interface.

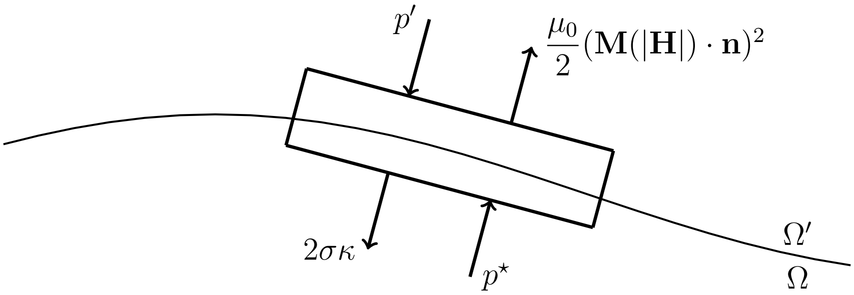


Figure 1.1:
Balance of forces in the ferrohydrodynamic boundary condition.

Using the fact that the ferrofluid is magnetically linear and substituting equations (1.9) and (1.10) into this boundary condition shows that

$$\frac{\mu_0}{2}(\mathbf{M}(|\mathbf{H}|)\cdot\mathbf{n})^2 + \mu_0\int_0^{|\mathbf{H}|} m(t)\,\mathrm{d}t + C + (\rho'-\rho)g\eta - 2\sigma\kappa = 0, \tag{1.11}$$

where $C = C_0' - C_0$.

From equation (1.7) we find that

$$\begin{aligned}
0 &= \partial_x(\phi'(x,\eta(x,z),z) - \phi(x,\eta(x,z),z))\\
&= \phi_x'(x,\eta(x,z),z) + \eta_x(x,z)\phi_y'(x,\eta(x,z),z)\\
&\qquad - (\phi_x(x,\eta(x,z),z) + \eta_x(x,z)\phi_y(x,\eta(x,z),z)),\\
0 &= \partial_z(\phi'(x,\eta(x,z),z) - \phi(x,\eta(x,z),z))\\
&= \phi_z'(x,\eta(x,z),z) + \eta_z(x,z)\phi_y'(x,\eta(x,z),z)\\
&\qquad - (\phi_z(x,\eta(x,z),z) + \eta_z(x,z)\phi_y(x,\eta(x,z),z))
\end{aligned}$$

and hence obtain the identities

$$\begin{aligned}
\phi_x' - \phi_x &= -\eta_x\left(\phi_y' - \phi_y\right) &&\text{for } y = \eta(x,z),\\
\phi_z' - \phi_z &= -\eta_z\left(\phi_y' - \phi_y\right) &&\text{for } y = \eta(x,z).
\end{aligned}$$

Substituting these identities into the equation

$$\phi_n' - \phi_n = \frac{-\eta_x(\phi_x'-\phi_x) - \eta_z(\phi_z'-\phi_z) + \phi_y' - \phi_y}{\sqrt{1+\eta_x^2+\eta_z^2}} \qquad \text{for } y = \eta(x,z)$$

shows that

$$\phi_n' - \phi_n = \sqrt{1+\eta_x^2+\eta_z^2}\left(\phi_y' - \phi_y\right) \qquad \text{for } y = \eta(x,z). \tag{1.12}$$

Using condition (1.5) and the facts that

$$\begin{aligned}
(\mathbf{M}(|\mathbf{H}|)\cdot\mathbf{n})^2 &= (\mu(|\mathbf{H}|)-1)^2(\mathbf{H}\cdot\mathbf{n})^2 &&\text{in } \Omega,\\
\int_0^{|\mathbf{H}|} m(t)\,\mathrm{d}t &= \int_0^{|\mathbf{H}|}(t\mu(t) - t)\,\mathrm{d}t\\
&= \int_0^{|\mathbf{H}|} t\mu(t)\,\mathrm{d}t - \frac{1}{2}|\mathbf{H}|^2 &&\text{in } \Omega
\end{aligned}$$

and

$$\mathbf{H}'\cdot\mathbf{n} = \mu(|\mathbf{H}|)\mathbf{H}\cdot\mathbf{n}$$

at the interface (because of the relations

$$\begin{aligned}
\mu_0\mathbf{H}' &= \mathbf{B}' &&\text{in } \Omega',\\
\mu_0\mu(|\mathbf{H}|)\mathbf{H} &= \mathbf{B} &&\text{in } \Omega
\end{aligned}$$

and condition (1.6)), we derive the identity

$$
\begin{aligned}
-|\mathbf{H}|^2 + (\mu(|\mathbf{H}|)-1)^2(\mathbf{H}\cdot\mathbf{n})^2 &= -|\mathbf{H}|^2 + (\mathbf{H}\cdot\mathbf{n})^2 - 2\mu(|\mathbf{H}|)(\mathbf{H}\cdot\mathbf{n})^2 + \mu(|\mathbf{H}|)^2(\mathbf{H}\cdot\mathbf{n})^2 \\
&= -|\mathbf{H}'|^2 + (\mathbf{H}'\cdot\mathbf{n})^2 - 2(\mathbf{H}'\cdot\mathbf{n})(\mathbf{H}\cdot\mathbf{n}) + (\mathbf{H}'\cdot\mathbf{n})^2 \\
&= -|\mathbf{H}'|^2 + 2\mathbf{H}'\cdot\mathbf{n}\,(\mathbf{H}'\cdot\mathbf{n} - \mathbf{H}\cdot\mathbf{n})
\end{aligned}
$$

at the interface, and from equations (1.8) and (1.12) it follows that

$$
\begin{aligned}
\mathbf{H}'\cdot\mathbf{n}\,(\mathbf{H}'\cdot\mathbf{n} - \mathbf{H}\cdot\mathbf{n}) &= \phi_n'\,(\phi_n' - \phi_n) \\
&= \phi_n'\left(\sqrt{1+\eta_x^2+\eta_z^2}\,(\phi_y' - \phi_y)\right) \\
&= \sqrt{1+\eta_x^2+\eta_z^2}(\phi_y'\phi_n' - \mu(|\mathrm{grad}\,\phi|)\phi_y\phi_n) \qquad \text{for } y=\eta(x,z),
\end{aligned}
$$

so that equation (1.11) can be written as

$$
\begin{aligned}
&C - (\rho-\rho')g\eta - 2\sigma\kappa \\
&\quad - \mu_0\left(\frac{1}{2}|\mathrm{grad}\,\phi'|^2 - M(|\mathrm{grad}\,\phi|)\right) \\
&\quad + \mu_0\sqrt{1+\eta_x^2+\eta_z^2}(\phi_y'\phi_n' - \mu(|\mathrm{grad}\,\phi|)\phi_y\phi_n) = 0 \qquad \text{for } y=\eta(x,z)
\end{aligned}
$$

with

$$
M(s) = \int_0^s t\mu(t)\,\mathrm{d}t.
$$

Altogether the mathematical problem is to solve the equations

$$
\mathrm{div}(\mathrm{grad}\,\phi') = 0 \qquad \text{in } \Omega', \tag{1.13}
$$

$$
\mathrm{div}(\mu(|\mathrm{grad}\,\phi|)\mathrm{grad}\,\phi) = 0 \qquad \text{in } \Omega \tag{1.14}
$$

with boundary conditions

$$
\phi' - \phi = 0 \qquad \text{for } y=\eta(x,z), \tag{1.15}
$$

$$
\phi_n' - \mu(|\mathrm{grad}\,\phi|)\phi_n = 0 \qquad \text{for } y=\eta(x,z), \tag{1.16}
$$

and

$$
\begin{aligned}
&C - (\rho-\rho')g\eta - 2\sigma\kappa \\
&\quad - \mu_0\left(\frac{1}{2}|\mathrm{grad}\,\phi'|^2 - M(|\mathrm{grad}\,\phi|)\right) \\
&\quad + \mu_0\sqrt{1+\eta_x^2+\eta_z^2}(\phi_y'\phi_n' - \mu(|\mathrm{grad}\,\phi|)\phi_y\phi_n) = 0 \qquad \text{for } y=\eta(x,z).
\end{aligned} \tag{1.17}
$$

The requirement that a uniform magnetic field and flat interface solves the physical problem, that is

$$
\eta_0 = 0, \qquad \phi_0' = \mu(h)hy, \qquad \phi_0 = hy
$$

is a solution to (1.13)–(1.17), leads us to choose

$$
C = -\mu_0 M(h) - \mu_0\mu(h)\left(\frac{\mu(h)}{2} - 1\right)h^2.
$$

Introducing the constant $\tilde{\rho} = \rho - \rho'$, and the variables

$$\tilde{\eta} = \eta - \eta_0, \qquad \tilde{\phi}' = \phi' - {\phi'}_0, \qquad \tilde{\phi} = \phi - \phi_0,$$

one obtains the equations

$$\begin{aligned} \operatorname{div}(\operatorname{grad}\phi') &= 0 && \text{in } \Omega', \\ \operatorname{div}(\mu(|\operatorname{grad}(\phi + hy)|)\operatorname{grad}(\phi + hy)) &= 0 && \text{in } \Omega \end{aligned}$$

with boundary conditions

$$\begin{aligned} \phi' - \phi + (\mu(h) - 1)\, h\eta &= 0 && \text{for } y = \eta(x,z), \\ (\phi' + \mu(h)hy)_n - \mu(|\operatorname{grad}(\phi + hy)|)(\phi + hy)_n &= 0 && \text{for } y = \eta(x,z), \end{aligned}$$

and

$$\begin{aligned} &-\mu_0 M(h) - \mu_0\mu(h)\left(\frac{\mu(h)}{2} - 1\right)h^2 - \rho g\eta - 2\sigma\kappa \\ &-\mu_0\left(\frac{1}{2}|\operatorname{grad}(\phi' + \mu(h)hy)|^2 - M(|\operatorname{grad}(\phi + hy)|)\right) \\ &+\mu_0\sqrt{1 + \eta_x^2 + \eta_z^2}(\phi'_y + \mu(h)h)(\phi' + \mu(h)hy)_n \\ &-\mu_0\sqrt{1 + \eta_x^2 + \eta_z^2}\mu(|\operatorname{grad}(\phi + hy)|)(\phi_y + h)(\phi + hy)_n = 0 \qquad \text{for } y = \eta(x,z), \end{aligned}$$

where the tildes have been dropped for notational simplicity. Finally, we specify Neumann boundary conditions

$$\begin{aligned} \phi'_y &= 0 && \text{for } y = d, \\ \mu(|\operatorname{grad}(\phi + hy)|)(\phi_y + h) - \mu(h)h &= 0 && \text{for } y = -d. \end{aligned}$$

The next step is to introduce dimensionless variables

$$(\hat{x}, \hat{y}, \hat{z}) = \frac{\mu_0 h^2}{\sigma}(x, y, z), \qquad \hat{\phi}' = \frac{\mu_0 h}{\sigma}\phi', \qquad \hat{\phi} = \frac{\mu_0 h}{\sigma}\phi, \qquad \hat{\eta} = \frac{\mu_0 h^2}{\sigma}\eta$$

and functions

$$\hat{\mu}(s) = \mu(hs), \qquad \hat{M}(s) = M(hs).$$

Dropping the hats for notational simplicity, we find that

$$\begin{aligned} \operatorname{div}(\operatorname{grad}\phi') &= 0 && \text{in } \Omega', && (1.18) \\ \operatorname{div}(\mu(|\operatorname{grad}(\phi + y)|)\operatorname{grad}(\phi + y)) &= 0 && \text{in } \Omega && (1.19) \end{aligned}$$

with boundary conditions

$$\begin{aligned} \phi'_y &= 0 && \text{for } y = \tfrac{1}{\beta}, && (1.20) \\ \mu(|\operatorname{grad}(\phi + y)|)(\phi_y + 1) - \mu(1) &= 0 && \text{for } y = -\tfrac{1}{\beta}, && (1.21) \\ \phi' - \phi + (\mu(1) - 1)\,\eta &= 0 && \text{for } y = \eta(x,z), && (1.22) \\ (\phi' + \mu(1)y)_n - \mu(|\operatorname{grad}(\phi + y)|)(\phi + y)_n &= 0 && \text{for } y = \eta(x,z), && (1.23) \end{aligned}$$

and

$$\begin{aligned}
&-M(1)-\mu(1)\left(\frac{\mu(1)}{2}-1\right)-\gamma\eta-2\kappa\\
&\quad-\left(\frac{1}{2}|\mathrm{grad}(\phi'+\mu(1)y)|^2-M(|\mathrm{grad}(\phi+y)|)\right)\\
&\quad+\sqrt{1+\eta_x^2+\eta_z^2}(\phi_y'+\mu(1))(\phi'+\mu(1)y)_n\\
&\quad-\sqrt{1+\eta_x^2+\eta_z^2}\mu(|\mathrm{grad}(\phi+y)|)(\phi_y+1)(\phi+y)_n=0 \qquad \text{for } y=\eta(x,z), \qquad (1.24)
\end{aligned}$$

where

$$\alpha=\frac{\rho g d}{\mu_0 h^2}, \qquad \beta=\frac{\sigma}{\mu_0 h^2 d}, \qquad \gamma=\alpha\beta. \qquad (1.25)$$

We use the parameter γ as a bifurcation parameter and fix the value β_0 of β, noting that $\eta=0, \phi'=0, \phi=0$ is a solution to (1.18)–(1.24) for every $\gamma\in\mathbb{R}$ and that the limit $\beta_0\to 0$ corresponds to fluids of infinite depth.

1.2 Small-amplitude doubly periodic patterns

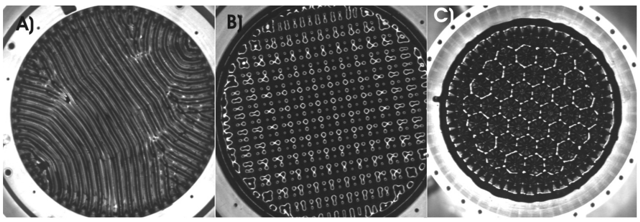


Figure 1.2:
Experimental observation of static doubly periodic patterns at the fluid-ferrofluid interface: A) rolls; B) rectangles; C) hexagons (Fachrichtung Physik, Universität des Saarlandes).

In Chapter 2 we present an existence theory for small amplitude, doubly periodic solutions to (1.18)–(1.24), that is solutions with

$$\eta(\mathbf{x}+\mathbf{l})=\eta(\mathbf{x}), \qquad \phi'(\mathbf{x}+\mathbf{l},y)=\phi'(\mathbf{x},y), \qquad \phi(\mathbf{x}+\mathbf{l},y)=\phi(\mathbf{x},y)$$

for every $\mathbf{l}\in\mathscr{L}$, where $\mathbf{x}=(x,z)$ and $\mathscr{L}$ is the lattice given by

$$\mathscr{L}=\{m\mathbf{l}_1+n\mathbf{l}_2:\ m,n\in\mathbb{Z}\}$$

with $|\mathbf{l}_1|=|\mathbf{l}_2|$. We are especially interested in three patterns which are observed in experiments (Figure 1.2), namely rolls, rectangles and hexagons (see Figure 1.3).

(i) For rolls we seek functions that are independent of the z-direction and choose $\mathbf{l}=(\frac{2\pi}{\omega},0)$, so that the periodic base cell is given by

$$\left\{x:|x|<\frac{\pi}{\omega}\right\}.$$

(ii) For rectangles we choose $\mathbf{l}_1 = (\frac{2\pi}{\omega}, 0)$, $\mathbf{l}_2 = (0, \frac{2\pi}{\omega})$, so that the periodic base cell is given by

$$\left\{(x,z) : |x|\,,\ |z| < \frac{\pi}{\omega}\right\}.$$

(iii) For hexagons we choose $\mathbf{l}_1 = \frac{2\pi}{\omega}(1, -\frac{1}{\sqrt{3}})$, $\mathbf{l}_2 = \frac{2\pi}{\omega}(0, \frac{2}{\sqrt{3}})$, so that we obtain an additional periodic direction $\mathbf{l}_3 = \mathbf{l}_1 + \mathbf{l}_2 = \frac{2\pi}{\omega}(1, \frac{1}{\sqrt{3}})$ and the periodic base cell is given by

$$\left\{(x,z) : |x| < \frac{2\pi}{\omega},\ \left|x - \sqrt{3}z\right| < \frac{4\pi}{\omega}, \left|x + \sqrt{3}z\right| < \frac{4\pi}{\omega}\right\}.$$

Notice that each of these patterns exhibits a rotational symmetry: the shape of the free surface is invariant under a rotation of the (x,z)-plane through respectively (i) $\frac{\pi}{2}$, (ii) $\frac{\pi}{3}$ and (iii) π.

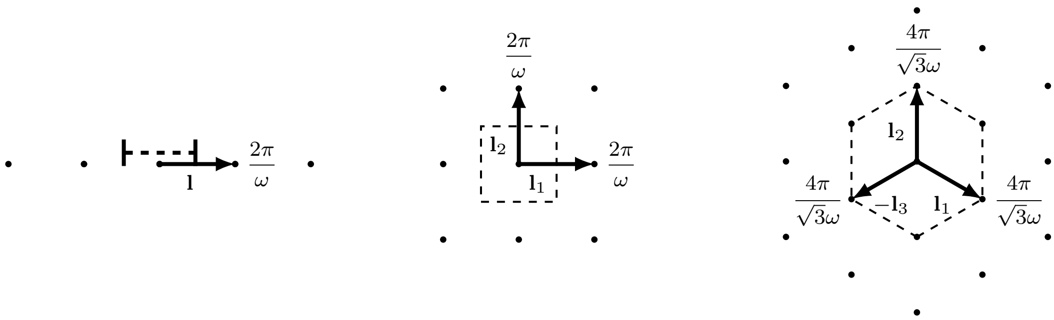


Figure 1.3:
The lattice $\mathscr{L}$ and periodic base cell for rolls (left), squares (centre) and hexagons (right).

This problem was first studied by Cowley and Rosensweig [8]. Using a linear stability analysis, they found that, as the strength h of the magnetic field exceeds a critical value h_c, the flat surface destabilises and a hexagonal pattern of peaks appears. This phenomenon is known as the *Rosensweig instability*. A mathematically rigorous treatment of the problem was given by Twombly and Thomas [44], who used coordinate transformations to 'flatten' the free surface by transforming the *a priori* unknown domains Ω' and Ω into fixed strips. Applying Lyapunov-Schmidt reduction reduces these transformed equations for rotationally symmetric patterns (see below) to a locally equivalent one-dimensional equation which is solved using the implicit-function theorem; the result is the existence, for values of h near h_c, of rolls and rectangles in addition to the hexagonal pattern. Twombly and Thomas's work is however flawed by some miscalculations and mathematical inconsistencies, and is also restricted to linear magnetisation laws. In this article we present a more systematic approach which is motivated by the corresponding study of doubly periodic travelling water waves by Craig and Nicholls [10]; we also consider general nonlinear magnetisation laws.

We work with the dimensionless variables introduced in Section 1.1 above, and 'flatten' the equations using *Dirichlet-Neumann formalism*. The *Dirichlet-Neumann operator* G' for the upper fluid domain (given by $\{\eta(x,z) < y < \frac{1}{\beta_0}\}$ in dimensionless variables) is defined as follows. Fix $\Phi' = \Phi'(x,z)$, solve the *linear* boundary-value problem

$$\phi'_{xx} + \phi'_{yy} + \phi'_{zz} = 0, \qquad \eta < y < \tfrac{1}{\beta_0}, \tag{1.26}$$

$$\phi' = \Phi', \qquad y = \eta, \tag{1.27}$$

$$\phi'_y = 0, \qquad y = \tfrac{1}{\beta_0}, \tag{1.28}$$

and define

$$G'(\eta, \Phi') = -(1+\eta_x^2+\eta_z^2)^{\frac{1}{2}}\phi'_n\big|_{y=\eta} = -(\phi'_y - \eta_x\phi'_x - \eta_z\phi'_z)\big|_{y=\eta}. \tag{1.29}$$

The *Dirichlet-Neumann operator* G for the lower fluid domain $\{-\frac{1}{\beta_0} < y < \eta(x,z)\}$ is similarly defined as

$$\begin{aligned} G(\eta,\Phi) &= (1+\eta_x^2+\eta_z^2)^{\frac{1}{2}}\mu(|\mathrm{grad}\,(\phi+y)|)\phi_n\big|_{y=\eta} \\ &= \mu(|\mathrm{grad}\,(\phi+y)|)(\phi_y - \eta_x\phi_x - \eta_z\phi_z)\big|_{y=\eta}, \end{aligned} \tag{1.30}$$

where ϕ is the solution of the (in general *nonlinear*) boundary-value problem

$$\mathrm{div}(\mu(|\mathrm{grad}\,(\phi+y)|)\mathrm{grad}\,(\phi+y)) = 0, \qquad -\tfrac{1}{\beta_0} < y < \eta, \tag{1.31}$$

$$\phi = \Phi, \qquad y = \eta, \tag{1.32}$$

$$\mu(|\mathrm{grad}\,(\phi+y)|)(\phi_y+1) = \mu(1), \qquad y = -\tfrac{1}{\beta_0}. \tag{1.33}$$

The nonlinearity of (1.31)–(1.33) is inherited from that of the magnetisation law $\mathbf{M} = \mathbf{M}(\mathbf{H})$ (for a linear magnetisation law the value of μ is constant and (1.31), (1.33) are replaced by respectively Laplace's equation and a linear Neumann boundary condition).

In Section 2.2 we show that G' and G are analytic functions of respectively (η, Φ') and (η, Φ) in the following sense.

Theorem 1. *Suppose that $s > 5/2$. The formulae*

$$\begin{aligned} G'(\eta,\Phi') &= -(\phi'_y - \eta_x\phi'_x - \eta_z\phi'_z)\big|_{y=\eta}, \\ G(\eta,\Phi) &= (\mu(|\mathrm{grad}(\phi+y)|)(\phi_y - \eta_x\phi_x - \eta_z\phi_z))\big|_{y=\eta}, \end{aligned}$$

and

$$H'(\eta,\Phi') = \phi'_y|_{y=\eta}, \qquad H(\eta,\Phi) = \phi_y|_{y=\eta},$$

where ϕ' and ϕ are the solutions to the boundary-value problems (1.26)–(1.28) *and* (1.31)–(1.33)*, define mappings G', G : $H^s_{\mathrm{per}}(\Gamma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma) \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ and H', H : $H^s_{\mathrm{per}}(\Gamma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma) \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ that are analytic at the origin.*

Using this Dirichlet-Neumann formalism, we can recast the governing equations as

$$\Phi' - \Phi + (\mu(1)-1)\eta = 0, \tag{1.34}$$

$$G'(\eta,\Phi') + G(\eta,\Phi) + \mu^\star - \mu(1) = 0 \tag{1.35}$$

and

$$\begin{aligned} &-\gamma\eta + \nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) \\ &\quad + \frac{1}{2}\left((1+|\nabla\eta|^2)\,H'(\eta,\Phi')^2 - |\nabla\Phi'|^2\right) \\ &\quad - (\mu(1)G'(\eta,\Phi') + G(\eta,\Phi) + \mu^\star - \mu(1)) \\ &\quad + (M^\star - M(1) - \mu^\star H(\eta,\Phi) - H(\eta,\Phi)G(\eta,\Phi)) = 0, \end{aligned} \tag{1.36}$$

in which $\nabla = (\partial_x, \partial_z)^{\mathrm{T}}$,

$$\mu^\star = \mu\Big(\big(|\nabla\Phi|^2 + 2(1-\nabla\eta\cdot\nabla\Phi)H(\eta,\Phi) + (1+|\nabla\eta|^2)H(\eta,\Phi)^2 + 1\big)^{1/2}\Big),$$
$$M^\star = M\Big(\big(|\nabla\Phi|^2 + 2(1-\nabla\eta\cdot\nabla\Phi)H(\eta,\Phi) + (1+|\nabla\eta|^2)H(\eta,\Phi)^2 + 1\big)^{1/2}\Big).$$

The mathematical problem is thus to solve the equation

$$\mathcal{G}(\eta,\Phi',\Phi;\gamma) = 0, \tag{1.37}$$

where $\mathcal{G}$ is given explicitly by the left-hand sides of (1.34)–(1.36).

Lemma 2. *Suppose that $s > 5/2$, let Γ be the parallelogram defined by $\mathbf{l}_1$ and $\mathbf{l}_2$ (or $\mathbf{l}$ in the case of rolls), and define*

$$X_0 := H^{s+1/2}_{\mathrm{per}}(\Gamma) \times \bar{H}^s_{\mathrm{per}}(\Gamma) \times H^s_{\mathrm{per}}(\Gamma),$$
$$Y_0 := H^s_{\mathrm{per}}(\Gamma) \times \bar{H}^{s-1}_{\mathrm{per}}(\Gamma) \times H^{s-3/2}_{\mathrm{per}}(\Gamma),$$

where

$$\bar{H}^r_{\mathrm{per}}(\Gamma) = \left\{\zeta : \int_\Gamma \zeta = 0\right\}.$$

The left-hand sides of (1.34)–(1.36) *define a mapping $\mathcal{G} : X_0 \times \mathbb{R} \to Y_0$ which is analytic at the origin.*

Observe that (1.37) exhibits rotational symmetry: it is invariant under rotations through respectively π, $\frac{\pi}{2}$ and $\frac{\pi}{3}$ for rolls, rectangles and hexagons, and one may therefore replace X_0 and Y_0 by their subspaces of functions that are invariant under these rotations (denoted by X_{sym} and Y_{sym}).

In Section 2.4 we discuss the existence of small-amplitude solutions to (1.37) within the framework of *analytic Crandall-Rabinowitz local bifurcation theory* (see Buffoni and Toland [6, Chapter 8]), using γ as a bifurcation parameter. According to that theory values γ_0 of γ at which non-trivial solutions bifurcate from zero (clearly $\mathcal{G}(0,0,0;\gamma) = 0$ for all values of γ) necessarily have the property that the kernel of the linear operator $L_0 := \mathrm{d}_1\mathcal{G}[0,0,0;\gamma_0] : X_0 \to Y_0$ is non-trivial. We show that ker L_0 is non-trivial if and only if

$$\gamma_0 = r(|\mathbf{k}|) := \left(\mu_1(\mu_1-1)^2\left(\mu_1|\mathbf{k}|\coth\frac{|\mathbf{k}|}{\beta_0} + S_1|\mathbf{k}|\coth\frac{S_1|\mathbf{k}|}{\beta_0}\right)^{-1} - 1\right)|\mathbf{k}|^2 \tag{1.38}$$

for some $\mathbf{k} \in \mathscr{L}^* \setminus \{\mathbf{0}\}$, where $\mu_1 = \mu(1)$, $\dot{\mu}_1 = \dot{\mu}(1)$, $S_1 = (\mu_1/(\mu_1+\dot{\mu}_1))^{1/2}$ and $\mathscr{L}^*$ is the dual lattice to $\mathscr{L}$. The function $|\mathbf{k}| \mapsto r(|\mathbf{k}|)$, which satisfies $r(0) = 0$ and $r(|\mathbf{k}|) \to -\infty$ as $|\mathbf{k}| \to \infty$, takes only negative values for $\beta_0 > \mu_1(\mu_1-1)^2/(\mu_1+1)$, while for $\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ it has a unique maximum ω with $r(\omega) > 0$ (see Figure 1.4); we choose $\gamma_0 = r(\omega)$ and note the relationships

$$\beta_0 = \frac{\mu_1(\mu_1-1)^2}{2\tilde{\omega}}\left(\frac{h(\tilde{\omega}) - \tilde{\omega}\dot{h}(\tilde{\omega})}{h(\tilde{\omega})^2}\right), \qquad \gamma_0 = \left(\frac{\mu_1(\mu_1-1)^2}{\omega h(\tilde{\omega})} - 1\right)\omega^2,$$

where $\tilde{\omega} = \omega/\beta_0$ and $h(\tilde{\omega}) = \mu_1\coth\tilde{\omega} + S_1\coth S_1\tilde{\omega}$.

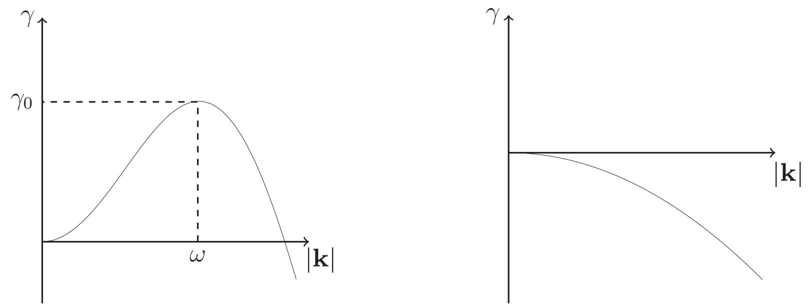


Figure 1.4:
The graph of the function $|\mathbf{k}| \mapsto r(|\mathbf{k}|)$ for $\beta_0 > \mu_1(\mu_1-1)^2/(\mu_1+1)$ (left) and $\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ (right).

This value (β_0, γ_0) of (β, γ) corresponds to the Rosensweig instability. The dimension of ker L_0 is therefore determined by the number of vectors in $\mathscr{L}^*$ with length ω; for rolls, rectangles and hexagons we find that $\dim\ker L_0$ is respectively 2, 4 and 6. Because the kernel of L_0 is multidimensional, one can not use Crandall-Rabinowitz local bifurcation theory directly. To overcome this problem we replace X_0 and Y_0 by X_{sym} and Y_{sym}, thus restricting to solutions that are invariant under rotations through respectively π, $\frac{\pi}{2}$ and $\frac{\pi}{3}$ for rolls, rectangles and hexagons. These restrictions ensure that $\dim\ker L_0 = 1$ with ker $L_0 = \langle v_0 \rangle$, where

$$v_0 = \begin{pmatrix} \dfrac{1}{\mu_1 - 1}\left(\mu_1 S_1^{-1} \tanh S_1\tilde{\omega} \coth\tilde{\omega} + 1\right) \\ -\mu_1 S_1^{-1} \tanh S_1\tilde{\omega} \coth\tilde{\omega} \\ 1 \end{pmatrix} e_1(x,z)$$

and

$$e_1(x,z) = \begin{cases} \cos\omega x & \text{(rolls)} \\ \cos\omega x + \cos\omega z & \text{(rectangles)} \\ \cos\omega x + \cos\frac{\omega}{2}\left(x + \sqrt{3}z\right) + \cos\frac{\omega}{2}\left(x - \sqrt{3}z\right) & \text{(hexagons).} \end{cases}$$

Verifying the remaining conditions in the analytic Crandall-Rabinowitz local bifurcation theorem yields the following result.

Theorem 3. *The point* $(\gamma_0, 0)$ *is a local bifurcation point for* (2.15)*, that is there exist* $\varepsilon > 0$*, open neighbourhoods* W_{sym} *of* $(\gamma_0, 0)$ *in* $\mathbb{R} \times X_{\mathrm{sym}}$ *and* V_{sym} *of* 0 *in* X_{sym} *and analytic functions* $w : (-\varepsilon, \varepsilon) \to V_{\mathrm{sym}}$*,* $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ *with* $\gamma(0) = \gamma_0, w(0) = v_0$ *such that* $\mathcal{G}(sw(s); \gamma(s)) = 0$ *for every* $s \in (-\varepsilon, \varepsilon)$*. Furthermore*

$$W_{\mathrm{sym}} \cap N = \{(\gamma(s), sw(s)) : \ 0 < |s| < \varepsilon\},$$

where

$$N = \{(\gamma, v) \in \mathbb{R} \times (V_{\mathrm{sym}} \setminus \{0\}) : \mathcal{G}(v; \gamma) = 0\}.$$

In Section 2.5 we examine the bifurcating branches identified in Theorem 3.

Theorem 4. *Branches of small-amplitude doubly periodic solutions to the ferrohydrostatic problem bifurcate from the trivial solution at $\gamma = \gamma_0$. The bifurcation is*

(i) transcritical in the case of hexagons,

(ii) super- or subcritical in the case of rolls and rectangles, depending upon the sign of a coefficient γ_2 which is determined by μ and ω/β_0. (Explicit formulae for the coefficient γ_2 are given in some special cases in Section 2.5.)

Finally, we note that supercritical bifurcation of rolls is associated with (supercritical) bifurcation of spatially localised patterns, whose existence has been established by dynamical-systems arguments by Groves, Lloyd and Stylianou [14]. A simplified version of the material in Chapter 2, which deals only with a linear magnetisation law and Dirichlet-Neumann operators for linear boundary-value problems, has appeared in a previous thesis (Horn [25]).

1.3 Small-amplitude symmetric corner defects

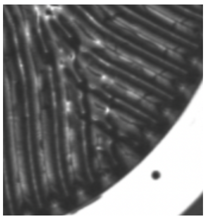


Figure 1.5:
Experimental observation of a static symmetric corner defect at the fluid-ferrofluid interface (Fachrichtung Physik, Universität des Saarlandes).

In Chapter 3 we present an existence theory for symmetric corner defects between rolls (see Figure 1.5). A *corner defect* is a transition between two periodic (moving or static) wave trains with different wavenumbers or directions. They have been studied in several different settings, in particular as solutions of reaction-diffusion equations (see Haragus and Scheel [19, 20, 21], Sandstede and Scheel [39], and Scheel and Wu [41]) and the Swift-Hohenberg equation (see Haragus and Scheel [22, 23, 24] and Lloyd and Scheel [31]). Recently Haragus and Iooss [17, 18] have examined them as patterns in the Bénard-Rayleigh convection problem using spatial dynamics. In this thesis we consider them as patterns at the fluid-ferrofluid interface and study them using a Hamiltonian version of the Haragus-Iooss theory.

We seek solutions of (1.18)–(1.24) which are $\frac{2\pi}{\nu}$-periodic in the z-direction and formulate these equations as a spatial Hamiltonian system with the x-coordinate as the time-like variable (see Groves and Haragus [13] for a comprehensive discussion of this method in the context of water waves). Our starting point is the observation that (1.18)–(1.24) follow from the variational principle

$$\delta \int_{-\infty}^{\infty} L(\eta, \phi', \phi, \eta_x, \phi'_x, \phi_x)\,\mathrm{d}x = 0$$

with Lagrangian

$$
\begin{aligned}
&L(\eta, \phi', \phi, \eta_x, \phi'_x, \phi_x) \\
&= \int_0^{2\pi} \left\{ \int_{-\frac{1}{\beta_0}}^{\eta} M\left(\left(\phi_x^2 + \nu^2\phi_z^2 + (\phi_y+1)^2\right)^{1/2} \right) \mathrm{d}y \right. \\
&\qquad + \int_{\eta}^{\frac{1}{\beta_0}} \frac{1}{2}\left(\phi_x'^2 + \nu^2\phi_z'^2 + (\phi'_y + \mu(1))^2\right) \mathrm{d}y \\
&\qquad + \mu(1)\left(\phi\big|_{y=-\frac{1}{\beta_0}} - \phi'\big|_{y=\frac{1}{\beta_0}}\right) - \left(M(1) + \mu(1)\left(\frac{\mu(1)}{2} - 1\right)\right)\eta \\
&\qquad \left. - \left(\frac{1}{2}(\gamma_0+\varepsilon)\eta^2 + (\sqrt{1+\eta_x^2+\nu^2\eta_z^2} - 1)\right) \right\} \mathrm{d}z,
\end{aligned}
$$

where the variations are taken with respect to η, ϕ' and ϕ satisfying

$$
(\phi' - \phi)\big|_{y=0} + (\mu(1) - 1)\,\eta = 0.
$$

Here we have scaled the z-coordinate so that η, ϕ, ϕ' are 2π-periodic in this variable (and ν appears as a parameter) and introduced a bifurcation parameter ε by writing $\gamma = \gamma_0 + \varepsilon$. The next step is to use the ‘flattening’ transformation

$$
\begin{aligned}
\tilde{y} &= \frac{y-\eta}{1+\beta_0\eta}, &\qquad \chi(x,\tilde{y},z) &= \phi(x,y,z) &\qquad &\text{for } -\tfrac{1}{\beta_0} < y < \eta, \\
\tilde{y} &= \frac{y-\eta}{1-\beta_0\eta}, &\qquad \chi'(x,\tilde{y},z) &= \phi'(x,y,z) &\qquad &\text{for } \eta < y < \tfrac{1}{\beta_0}
\end{aligned}
$$

to map the variable domains

$$
\left\{(x,y,z):\ -\frac{1}{\beta_0} < y < \eta\right\}, \qquad \left\{(x,y,z):\ \eta < y < \frac{1}{\beta_0}\right\}
$$

into the rigid domains

$$
\left\{(x,y,z):\ -\frac{1}{\beta_0} < y < 0\right\}, \qquad \left\{(x,y,z):\ 0 < y < \frac{1}{\beta_0}\right\},
$$

and the free interface $\{(x,y,z):\ y=\eta\}$ to $\{(x,y,z):\ y=0\}$. In the new coordinates the variational principle takes the form

$$
\delta \int_{-\infty}^{\infty} L(\eta, \chi', \chi, \eta_x, \chi'_x, \chi_x)\,\mathrm{d}x = 0,
$$

where the variations are taken with respect to η, χ' and χ satisfying

$$
(\chi' - \chi)\big|_{y=0} + (\mu(1) - 1)\,\eta = 0.
$$

We proceed by carrying out a formal Legendre transformation. Define new variables

$$
\rho = \frac{\delta H}{\delta \eta_x}, \qquad \xi' = \frac{\delta H}{\delta \chi'_x}, \qquad \xi = \frac{\delta H}{\delta \chi_x},
$$

solve these equations for ρ_x, χ'_x, χ_x as functions of $\eta, \chi', \chi, \rho, \xi', \xi$, and set

$$\begin{aligned} H\left(\eta, \chi', \chi, \rho, \xi', \xi\right) = & \int_0^{2\pi} \int_{-\frac{1}{\beta_0}}^{0} \chi_x \xi \, \mathrm{d}y \, \mathrm{d}z + \int_0^{2\pi} \int_0^{\frac{1}{\beta_0}} \chi'_x \xi' \, \mathrm{d}y \, \mathrm{d}z \\ & + \int_0^{2\pi} \eta_x \rho \, \mathrm{d}z - L\left(\eta, \chi', \chi, \rho, \xi', \xi\right). \end{aligned}$$

In Section 3.1 we confirm rigorously that

$$\eta_x = \frac{\delta H}{\delta \rho}, \qquad \chi'_x = \frac{\delta H}{\delta \xi'}, \qquad \chi_x = \frac{\delta H}{\delta \xi}, \tag{1.39}$$

$$\rho_x = -\frac{\delta H}{\delta \eta}, \qquad \xi'_x = -\frac{\delta H}{\delta \chi'}, \qquad \xi_x = -\frac{\delta H}{\delta \chi}, \tag{1.40}$$

represent Hamilton's equations for a spatial Hamiltonian formulation of the ferrohydrostatic problem. Equations (1.39), (1.40) have the disadvantage that they are accompanied by nonlinear boundary conditions, and we construct a near-identity change of variable which transforms those conditions into their linearisations. The result is a quasilinear evolutionary equation of the form

$$u_x = Lu + N^\varepsilon(u) \tag{1.41}$$

with phase space

$$\begin{aligned} X_0 = \{ & u \in H^1_{\mathrm{per}}(0, 2\pi) \times H^1_{\mathrm{per}}(\Sigma') \times H^1_{\mathrm{per}}(\Sigma) \times L^2_{\mathrm{per}}(0, 2\pi) \times L^2_{\mathrm{per}}(\Sigma') \times L^2_{\mathrm{per}}(\Sigma) : \\ & (\chi' - \chi)|_{y=0} + (\mu_1 - 1)\,\eta = 0 \}, \end{aligned}$$

where $u = (\eta, \chi', \chi, \rho, \xi', \xi)$, $\Sigma' = (0, 2\pi) \times (0, \frac{1}{\beta_0})$, $\Sigma = (0, 2\pi) \times (-\frac{1}{\beta_0}, 0)$ and the subscript 'per' on the standard Sobolev spaces indicates that the functions are 2π-periodic in z. The vector field on the right hand side of (1.41) (whose linear and parameter-dependent nonlinear parts are denoted by L and N^ε) maps a neighbourhood of the origin in

$$\begin{aligned} \mathcal{D}(L) = \Big\{ & u \in H^2_{\mathrm{per}}(0, 2\pi) \times H^2_{\mathrm{per}}(\Sigma') \times H^2_{\mathrm{per}}(\Sigma) \times H^1_{\mathrm{per}}(0, 2\pi) \times H^1_{\mathrm{per}}(\Sigma') \times H^1_{\mathrm{per}}(\Sigma) : \\ & (\chi' - \chi)|_{y=0} + (\mu_1 - 1)\,\eta = 0, \mu_1 S_1^{-2} \chi_y|_{y=-\frac{1}{\beta_0}} = 0, \chi'_y|_{y=\frac{1}{\beta_0}} = 0, \\ & (\mu_1 S_1^{-2} \chi_y - \chi'_y)|_{y=0} = 0, \left(\frac{\xi}{\mu_1} - \xi' \right) \Big|_{y=0} + (\mu_1 - 1)\rho = 0 \Big\} \end{aligned}$$

analytically into X_0. (Here $\mu_1 = \mu(1), \dot{\mu}_1 = \dot{\mu}(1), S_1 = (\mu_1/(\mu_1 + \dot{\mu}_1))^{1/2}$.) Equation (1.41) is reversible, that is invariant under the transformation $(\eta, \chi', \chi, \rho, \xi', \xi)(x) \mapsto S(\eta, \chi', \chi, \rho, \xi', \xi)(-x)$, where the *reverser* is defined by

$$S(\eta, \chi', \chi, \rho, \xi', \xi) = (\eta, \chi', \chi, -\rho, -\xi', -\xi).$$

Furthermore it is invariant under the transformation $(\chi', \chi) \mapsto (\chi' + c, \chi + c)$, $c \in \mathbb{R}$, the reflection $T : z \mapsto -z$ and the translation $R_\alpha : z \mapsto z + \alpha$, $\alpha \in \mathbb{R}$.

The linear operator L is regular (that is, its spectrum consists entirely of isolated eigenvalues of finite algebraic multiplicity with no accumulation points). A purely imaginary eigenvalue is with corresponding eigenvector in the k-th Fourier mode satisfies $\gamma_0 = r(\tilde{\sigma})$, where $\tilde{\sigma}^2 = s^2 + k^2\nu^2$

and r is defined in equation (1.38). We choose $\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ and $\gamma_0 = r(\omega)$ as before (see the discussion below equation (1.38)). With these choices of β_0 and γ_0 we find that $\pm\mathrm{i}\omega$ are mode 0 eigenvalues of L, and an additional pair $\pm\mathrm{i}s$ of mode k eigenvalues, which are always geometrically double, arises whenever $\gamma_0 = r(\sqrt{s^2+k^2\nu^2})$ (so that $s = \omega_k := \sqrt{\omega^2 - k^2\nu^2}$). By choosing ν with $\frac{\omega}{m+1} < \nu < \frac{\omega}{m}$, $m \in \mathbb{N}$, we find therefore m additional pairs of eigenvalues (see Figure 1.6).

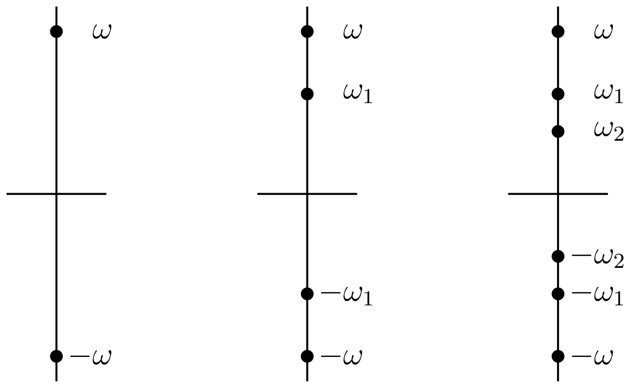


Figure 1.6:
The purely imaginary non-zero eigenvalues of L for ν such that $\nu > \omega$ (left), $\frac{\omega}{2} < \nu < \omega$ (centre) and $\frac{\omega}{3} < \nu < \frac{\omega}{2}$ (right).

We proceed by choosing $\frac{\omega}{2} < \nu < \omega$, so that $\pm\mathrm{i}\omega$ and $\pm\mathrm{i}\omega_1$ are respectively mode 0 and mode 1 eigenvalues of L. Straightforward calculations show that each of these eigenvalues has an associated Jordan chain of length 2; we find that

$$\begin{aligned} (L - \mathrm{i}\omega_1 I)f_j^1 &= 0, & (L - \mathrm{i}\omega_1 I)f_j^2 &= f_j^1, \quad j = 1,2, \\ (L + \mathrm{i}\omega_1 I)\overline{f_j^1} &= 0, & (L + \mathrm{i}\omega_1 I)\overline{f_j^2} &= \overline{f_j^1}, \quad j = 1,2, \\ (L - \mathrm{i}\omega I)f_3^1 &= 0, & (L - \mathrm{i}\omega I)f_3^2 &= f_3^1, \\ (L + \mathrm{i}\omega I)\overline{f_3^1} &= 0, & (L + \mathrm{i}\omega I)\overline{f_3^2} &= \overline{f_3^1}. \end{aligned}$$

Furthermore 0 is a mode 0 geometrically simple eigenvalue whose algebraic multiplicity is 2; we find that $Lf_0^1 = 0, Lf_0^2 = f_0^1$. (Explicit formulae for the (generalised) eigenvectors are given in Section 3.3.) We normalise these vectors so that they form a symplectic basis for the centre subspace of L with respect to the 2-form Ω, that is $\Omega(f_0^1, \overline{f_0^2}) = 1$,

$$\begin{aligned} \Omega(f_j^1, \overline{f_j^2}) &= \Omega(\overline{f_j^1}, f_j^2) = 1, \\ \Omega(f_j^2, \overline{f_j^1}) &= \Omega(\overline{f_j^2}, f_j^1) = -1 \end{aligned}$$

for $j = 1,2,3$ and the symplectic products of all other combinations are zero. We denote the coordinates in the $f_0^1, f_0^2, \ldots, f_3^1, f_3^2, \overline{f_1^1}, \overline{f_1^2}, \ldots, \overline{f_3^1}, \overline{f_3^2}$ directions by $q_0, p_0, a_1, b_1, \ldots, a_3, b_3, \overline{a_1}, \overline{b_1}, \ldots, \overline{a_3}, \overline{b_3}$ and observe that the actions of the actions of the reverser S, the reflection $T: z \mapsto -z$ and the translation (rotation) $R_\alpha : z \mapsto z + \alpha$, $\alpha \in \mathbb{R}$ on this space are given by

$$\begin{aligned} S(q_0, p_0, a_1, b_1, a_2, b_2, a_3, b_3) &= (-q_0, p_0, \overline{a}_1, -\overline{b}_1, \overline{a}_2, -\overline{b}_2, \overline{a}_3, -\overline{b}_3), \\ T(q_0, p_0, a_1, b_1, a_2, b_2, a_3, b_3) &= (q_0, p_0, a_2, b_2, a_1, b_1, a_3, b_3), \\ R_\alpha(q_0, p_0, a_1, b_1, a_2, b_2, a_3, b_3) &= (q_0, p_0, \mathrm{e}^{\mathrm{i}\alpha}a_1, \mathrm{e}^{\mathrm{i}\alpha}b_1, \mathrm{e}^{-\mathrm{i}\alpha}a_2, \mathrm{e}^{-\mathrm{i}\alpha}b_2, a_3, b_3). \end{aligned}$$

Figure 1.7 shows how the eigenvalue configuration changes as γ_0 is varied through $r(\omega)$: the Jordan chains associated with the eigenvalues $\pm\mathrm{i}\omega$ and $\pm\mathrm{i}\omega_1$ resolve into pairs of imaginary eigenvalues for $\gamma_0 < r(\omega)$ and into pairs of complex eigenvalues for $\gamma_0 > r(\omega)$. (Note that $\sigma(L)$ is always symmetric with respect to the real and imaginary axes.)

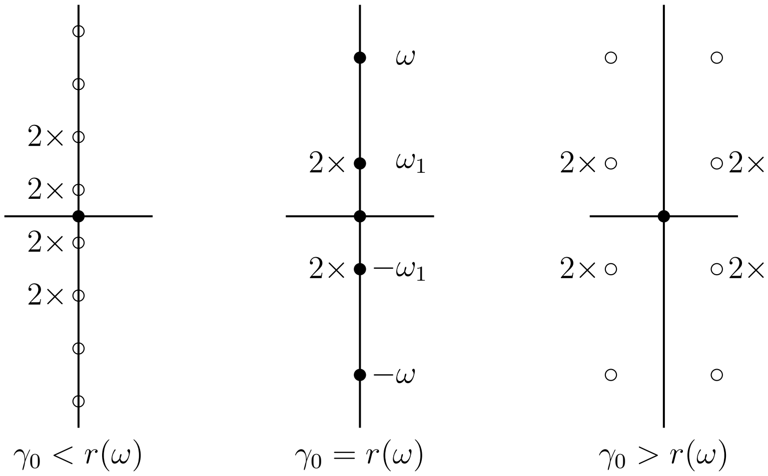


Figure 1.7:
Eigenvalues of L for $\gamma_0 < r(\omega)$ (left), $\gamma_0 = r(\omega)$ (centre) and $\gamma_0 > r(\omega)$ (right); hollow and solid dots denote eigenvalues with an associated Jordan chain of length 1 and 2 respectively.

In Sections 3.2 and 3.3 we apply Mielke's centre-manifold reduction theory (see Theorem 33) and show that (1.41) is locally equivalent to the reduced Hamiltonian system

$$p_{0x} = \frac{\partial \tilde{H}^\varepsilon}{\partial q_0}(\mathbf{a},\mathbf{b},\overline{\mathbf{a}},\overline{\mathbf{b}},p_0), \qquad q_{0x} = -\frac{\partial \tilde{H}^\varepsilon}{\partial p_0}(\mathbf{a},\mathbf{b},\overline{\mathbf{a}},\overline{\mathbf{b}},p_0), \tag{1.42}$$

$$a_{jx} = \frac{\partial \tilde{H}^\varepsilon}{\partial \overline{b_j}}(\mathbf{a},\mathbf{b},\overline{\mathbf{a}},\overline{\mathbf{b}},p_0), \qquad b_{jx} = -\frac{\partial \tilde{H}^\varepsilon}{\partial \overline{a_j}}(\mathbf{a},\mathbf{b},\overline{\mathbf{a}},\overline{\mathbf{b}},p_0), \qquad j=1,2,3, \tag{1.43}$$

where $\mathbf{a} = (a_1,a_2,a_3), \mathbf{b} = (b_1,b_2,b_3)$ and

$$\begin{aligned}\tilde{H}^\varepsilon(\mathbf{a},\mathbf{b},\overline{\mathbf{a}},\overline{\mathbf{b}},p_0) &= \mathrm{i}\omega_1(a_1\overline{b_1} - \overline{a_1}b_1 + a_2\overline{b_2} - \overline{a_2}b_2) + \mathrm{i}\omega(a_3\overline{b_3} - \overline{a_3}b_3)\\ &\quad + |\mathbf{b}|^2 + \frac{1}{2}p_0^2 + O(|(\varepsilon,p_0,\mathbf{a},\mathbf{b})||(p_0,\mathbf{a},\mathbf{b})|^2).\end{aligned}$$

The reduced system inherits the reversibility, reflection and translation symmetries of (1.41); in particular $\tilde{H}^\varepsilon$ is invariant unter S, T and R_α for all $\alpha \in \mathbb{R}$. The invariance of (1.41) under $(\chi',\chi) \mapsto (\chi'+c, \chi+c), c \in \mathbb{R}$ is reflected in the fact that $\tilde{H}^\varepsilon$ does not depend upon q_0 (which is a cyclic variable), so that p_0 is a conserved quantity. Finally, we note that

$$\{(q_0,p_0,\mathbf{a},\mathbf{b},\overline{\mathbf{a}},\overline{\mathbf{b}}) : a_1,a_2,b_1,b_2 = 0\}$$

is an invariant subspace for the equations (1.42) and (1.43), solutions in which correspond to two-dimensional, that is z-independent, solutions of the physical problem.

According to the classical theory, the next step is to lower the dimension of the reduced system by two by setting $p_0 = 0$, solving the resulting decoupled system (3.39) for $a_1, b_1, \ldots, a_3, b_3$,

$\overline{a_1}, \overline{b_1}, \ldots, \overline{a_3}, \overline{b_3}$ and recovering q_0 from (1.42) by quadrature. The lower-order system is typically studied using a canonical, symmetry preserving change of variables which simplifies its Hamiltonian $\tilde{H}^\varepsilon\big|_{p_0=0}$ (a 'normal-form' transformation) by transforming it to

$$\begin{aligned}&\mathrm{i}\omega_1(A_1\overline{B}_1 - \overline{A}_1 B_1 + A_2\overline{B}_2 - \overline{A}_2 B_2) + \mathrm{i}\omega(A_3\overline{B}_3 - \overline{A}_3 B_3)\\ &\qquad + |\mathbf{B}|^2 + \tilde{K}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) + O(|(\mathbf{A}, \mathbf{B})|^2|(\varepsilon, \mathbf{A}, \mathbf{B})|^{n_0})\end{aligned} \tag{1.44}$$

for some $n_0 \geq 2$, where $\tilde{K}^\varepsilon$ is a polynomial of order $n_0 + 1$ of its arguments and ε with

$$\tilde{K}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) = O(|(\mathbf{A}, \mathbf{B})|^2|(\varepsilon, \mathbf{A}, \mathbf{B})|).$$

Theorem 5. *The polynomial $\tilde{K}^\varepsilon$ admits a unique representation of the form*

$$\tilde{K}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) = p_1^\varepsilon(M_1, \ldots, M_8) + p_2^\varepsilon(M_1, \ldots, M_8)M_9,$$

where p_j^ε is a real polynomial function of its arguments and ε with

$$\begin{aligned}&p_j^\varepsilon(M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8)\\ &\qquad = p_j^\varepsilon(M_3, M_4, M_1, M_2, M_5, M_6, M_8, M_7), \qquad j = 1, 2,\end{aligned}$$

where

$$\begin{aligned}&M_1 = A_1\overline{A}_1, \qquad M_2 = \mathrm{i}(A_1\overline{B}_1 - \overline{A}_1 B_1),\\ &M_3 = A_2\overline{A}_2, \qquad M_4 = \mathrm{i}(A_2\overline{B}_2 - \overline{A}_2 B_2),\\ &M_5 = A_3\overline{A}_3, \qquad M_6 = \mathrm{i}(A_3\overline{B}_3 - \overline{A}_3 B_3),\\ &M_7 = (A_1 B_3 - A_3 B_1)(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1),\\ &M_8 = (A_2 B_3 - A_3 B_2)(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2).\end{aligned}$$

In Sections 3.5 and 3.6 we find solutions of the reduced Hamiltonian system

$$A_{jx} = \frac{\partial \tilde{H}^\varepsilon}{\partial \overline{B}_j}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, 0), \qquad B_{jx} = -\frac{\partial \tilde{H}^\varepsilon}{\partial \overline{A}_j}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, 0), \qquad j = 1, 2, 3, \tag{1.45}$$

where $\mathbf{A} = (A_1, A_2, A_3)$, $\mathbf{B} = (B_1, B_2, B_3)$, making assumptions on the coefficients $c_1^1, c_3^1, c_1, c_2, c_3, c_4$ of $\varepsilon M_1, \varepsilon M_5, M_1^2, M_1 M_3, M_1 M_5, M_5^2$ in p_j^ε. We begin by examining the invariant subspace

$$R := \{(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) : A_1, A_2, B_1, B_2 = 0\}$$

and proving the following theorem by reworking the geometric arguments given by Iooss and Pérouème [26, §§III.1 and VI.1] from a functional-analytic perspective.

Theorem 6. *Suppose that $c_3^1 > 0$ and $c_4 < 0$ and set $\delta = \varepsilon^{\frac{1}{2}}$. Equations* (3.59), (3.60) *admit a family $\{\mathbf{Z}_R^{\delta,\theta}\}$ of reversible periodic solutions smoothly parametrised by $\delta \in (-\delta_0, \delta_0)$, $\theta \in (-\theta_0, \theta_0)$ for some $\delta_0, \theta_0 > 0$. The solution $\mathbf{Z}_R^{\delta,\theta}$ has period $2\pi/(\omega + {c_3^1}^{1/2}\delta\theta)$ and satisfies*

$$\mathbf{Z}_R^{\delta,\theta}(x) = \delta\left(\frac{c_3^1}{-2c_4}\right)^{\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(\omega + {c_3^1}^{1/2}\delta\theta)x} + O(\delta^2)$$

uniformly over $x \in \mathbb{R}$ as $\delta \to 0$.

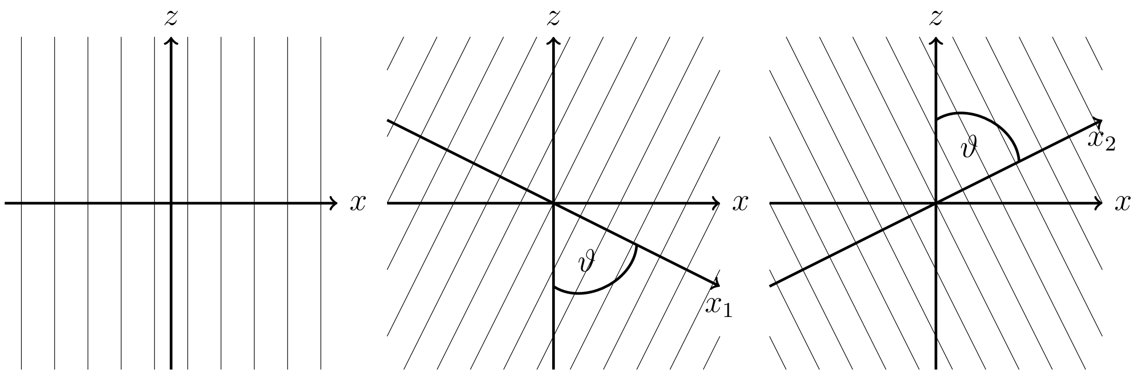


Figure 1.8:
Rolls perpendicular to the directions x (left), $x_1 = \sin(\vartheta)x + \cos(\vartheta)z$ (centre)
and $x_2 = -\sin(\vartheta)x + \cos(\vartheta)z$ (right).

Tracing back the various changes of variables, we find that the above solutions correspond to solutions of the 'flattened' physical problem of the form

$$\begin{pmatrix}\eta\\ \chi'\\ \chi\end{pmatrix} = 2\delta\left(\frac{c_3^1\kappa_\omega^{-1}}{-2c_4}\right)^{\frac{1}{2}}\mathbf{e}_\omega\cos(\omega + c_3^{1^{1/2}}\delta\theta)x + \kappa^{-\frac{1}{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}q_0(x) + O(\delta^2)$$

which depend on the single horizontal variable x. (Explicit formulae for the constant κ_ω and vector $\mathbf{e}_\omega$ are given in Chapter 3.) These solutions are rolls perpendicular to the x-direction (see Figure 1.8). Because the physical problem is rotationally invariant, replacing x by x_1 in the above formula generates rolls perpendicular to an arbitrary direction x_1. In particular, writing $x_1 = \sin(\vartheta)x + \cos(\vartheta)z$ and noting that

$$\kappa_\omega c_1^1 = \kappa_{\omega_1}c_3^1, \qquad \kappa_\omega^2 c_4 = \kappa_{\omega_1}^2 c_1$$

(see Section 3.7), we obtain the rotated rolls

$$\begin{aligned}\begin{pmatrix}\eta_\vartheta\\ \chi'_\vartheta\\ \chi_\vartheta\end{pmatrix} &= 2\delta\left(\frac{c_1^1\kappa_{\omega_1}^{-1}}{-2c_1}\right)^{\frac{1}{2}}\mathbf{e}_\omega\cos\Big((\omega + c_3^{1^{1/2}}\delta\theta)(\sin(\vartheta)x + \cos(\vartheta)z)\Big)\\ &\quad + \kappa^{-\frac{1}{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}q_0(\sin(\vartheta)x + \cos(\vartheta)z) + O(\delta^2),\end{aligned}$$

which are periodic in x with wavenumber $(\omega + c_3^{1^{1/2}}\delta\theta)\sin(\vartheta)$ and in z with wavenumber $(\omega + c_3^{1^{1/2}}\delta\theta)\cos(\vartheta)$ (see Figure 1.8).

Defining ϑ, θ_1 so that

$$\begin{aligned}(\omega + c_3^{1^{1/2}}\delta\theta)\sin(\vartheta) &= \omega_1 + c_1^{1^{1/2}}\delta\theta_1,\\ (\omega + c_3^{1^{1/2}}\delta\theta)\cos(\vartheta) &= \nu,\end{aligned}$$

we can write the rotated rolls as

$$\begin{pmatrix}\eta_\vartheta\\ \chi'_\vartheta\\ \chi_\vartheta\end{pmatrix} = 2\delta\left(\frac{c_1^1\kappa_{\omega_1}^{-1}}{-2c_1}\right)^{\frac{1}{2}} \mathbf{e}_\omega \cos\Big((\omega_1 + c_1^{1^{1/2}}\delta\theta_1)(x+\nu z)\Big)$$
$$+\kappa^{-\frac{1}{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} q_0(\sin(\vartheta)x+\cos(\vartheta)z) + O(\delta^2).$$

We can repeat this procedure with $x_2 = -\sin(\vartheta)x + \cos(\vartheta)z$ to obtain a second family of rolls which are rotated in the opposite direction (see Figure 1.8). Reapplying the changes of variables and the centre-manifold reduction leads to the following existence result.

Lemma 7. *Suppose that $c_1^1 > 0$ and $c_1 < 0$ and set $\delta = \varepsilon^{\frac{1}{2}}$. Equations* (3.53)–(3.58) *admit families $\{\mathbf{Z}_P^{\delta,\theta}\}, \{\mathbf{Z}_Q^{\delta,\theta}\}$ of reversible periodic solutions smoothly parametrised by $\delta \in (-\delta_0, \delta_0)$ and $\theta \in (-\theta_0, \theta_0)$. The solutions $\mathbf{Z}_P^{\delta,\theta}$ and $\mathbf{Z}_Q^{\delta,\theta}$ have period $2\pi/(\omega_1 + c_1^{1^{1/2}}\delta\theta)$ and satisfy*

$$\mathbf{Z}_P^{\delta,\theta}(x) = \delta\left(\frac{c_1^1}{-2c_1}\right)^{\frac{1}{2}}\begin{pmatrix}1\\0\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(\omega_1+c_1^{1^{1/2}}\delta\theta)x} + O(\delta^2),$$
$$\mathbf{Z}_Q^{\delta,\theta}(x) = \delta\left(\frac{c_1^1}{-2c_1}\right)^{\frac{1}{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(\omega_1+c_1^{1^{1/2}}\delta\theta)x} + O(\delta^2)$$

uniformly over $x \in \mathbb{R}$ as $\delta \to 0$. (Obviously $\mathbf{Z}_Q^{\delta,\theta} = T\mathbf{Z}_P^{\delta,\theta}$.)

Finally, we construct a heteroclinic solution to equations (1.45) which connects a periodic solution $\mathbf{Z}_P^{\delta,\theta}$ with a periodic solution $\mathbf{Z}_Q^{\delta,\theta}$. We use the method of Haragus and Iooss [17, 18], giving full details for completeness since their work defers to several other sources.

Introducing scaled variables $\hat{x} = (c_1^1\varepsilon)^{\frac{1}{2}}x$ and

$$A_j(x) = \left(\frac{c_1^1\varepsilon}{-2c_1}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega_1 x}C_j(\hat{x}), \qquad j=1,2,$$
$$A_3(x) = \left(\frac{c_1^1\varepsilon}{-2c_1}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega x}C_3(\hat{x}),$$

we write equations (1.45) as

$$\mathcal{H}(\mathbf{C},\mathbf{C}_x,\mathbf{C}_{xx},x,\delta) := \begin{pmatrix}C_{1xx}\\C_{2xx}\\C_{3xx}\end{pmatrix} - \mathcal{H}_1(\mathbf{C}) - \mathcal{H}_2(\mathbf{C},\mathbf{C}_x,x,\delta) = 0, \tag{1.46}$$

in which $\mathbf{C} = (C_1, C_2, C_3)$, the hats have been dropped for notational simplicity,

$$\mathcal{H}_1(\mathbf{C}) = \begin{pmatrix}(-1+|C_1|^2+d_1|C_2|^2+d_2|C_3|^2)C_1\\ (-1+d_1|C_1|^2+|C_2|^2+d_2|C_3|^2)C_2\\ (d_3+d_2(|C_1|^2+|C_2|^2)+d_4|C_3|^2)C_3\end{pmatrix},$$

and

$$
\begin{gathered}
|\mathcal{H}_2(\mathbf{C}, \mathbf{C}_x, x, \delta)| = O(\delta|(\mathbf{C}, \mathbf{C}_x)|), \\
|\partial_{\mathbf{C}}\mathcal{H}_2(\mathbf{C}, \mathbf{C}_x, x, \delta)| = O(\delta), \qquad |\partial_{\mathbf{C}_x}\mathcal{H}_2(\mathbf{C}, \mathbf{C}_x, x, \delta)| = O(\delta)
\end{gathered}
$$

as $(\mathbf{C}, \mathbf{C}_x, \delta) \to 0$, uniformly over $x \in \mathbb{R}$. Note that the formulae

$$
\begin{aligned}
\mathbf{P}^{\delta,\theta}(x) &= \mathrm{e}^{-\mathrm{i}\omega_1 x/(c_1^{1\,1/2}\delta)}\mathbf{Z}_P^{\delta,\theta}\left(\frac{x}{c_1^{1\,1/2}\delta}\right), \\
\mathbf{Q}^{\delta,\theta}(x) &= \mathrm{e}^{-\mathrm{i}\omega_1 x/(c_1^{1\,1/2}\delta)}\mathbf{Z}_Q^{\delta,\theta}\left(\frac{x}{c_1^{1\,1/2}\delta}\right)
\end{aligned}
$$

define (quasiperiodic) solutions to (1.46). The following result was proved by van der Berg [45, Theorem 5].

Lemma 8. *There exists a smooth real solution $\mathbf{C}^\star = (C_1^\star, C_2^\star, 0)$ to the equation*

$$
\begin{pmatrix} C_{1xx} \\ C_{2xx} \\ C_{3xx} \end{pmatrix} = \mathcal{H}_1(\mathbf{C})
$$

such that

(i) $\lim_{x\to\infty} \mathbf{C}^\star = (1, 0, 0)$ *and* $\lim_{x\to-\infty} \mathbf{C}^\star = (0, 1, 0)$,

(ii) $C_1^\star(x), C_2^\star(x) \geq 0$ *for all* $x \in \mathbb{R}$,

(iii) $C_1^\star(x) = C_2^\star(-x)$ *for all* $x \in \mathbb{R}$.

Defining

$$
\mathbf{H}(\theta, \delta)(x) := \mathrm{e}^{\mathrm{i}\theta x}\mathbf{C}^\star(x) + \chi(x)(\mathbf{P}^{\delta,\theta}(x) - (1, 0, 0)\mathrm{e}^{\mathrm{i}\theta x}) + \chi(-x)(\mathbf{Q}^{\delta,\theta}(x) - (0, 1, 0)\mathrm{e}^{\mathrm{i}\theta x}),
$$

where $\chi : \mathbb{R} \to [0, 1]$ is a smooth function with

$$
\chi(x) = \begin{cases} 1, & x \geq M, \\ 0, & x \leq m \end{cases}
$$

for some positive constants $m < M$, and substituting the *Ansatz*

$$
\mathbf{C} = \mathbf{H}(\theta, \delta) + \mathbf{V}
$$

into (1.46), we obtain the equation

$$
\mathcal{G}(\mathbf{V}, \theta, \delta) = 0.
$$

Lemma 9. *Define*

$$
\begin{aligned}
L_\eta^2(\mathbb{R}) &= \{f : ||f||_\eta < \infty\}, \qquad ||f||_\eta^2 := \int_{-\infty}^{\infty} \cosh^2(\eta x)|f(x)|^2 \mathrm{d}x, \\
H_\eta^2(\mathbb{R}) &= \left\{f : f, f_x, f_{xx} \in L_\eta^2(\mathbb{R})\right\}
\end{aligned}
$$

with norm

$$||f||_{2,\eta}^2 = ||f||_\eta^2 + ||f_x||_\eta^2 + ||f_{xx}||_\eta^2$$

and

$$\mathcal{X}_\eta = \left\{ \mathbf{C} \in (L_\eta^2(\mathbb{R}))^6 : C_1(x) = \overline{C_2(-x)}, C_3(x) = \overline{C_3(-x)} \text{ for } x \in \mathbb{R} \right\},$$
$$\mathcal{Y}_\eta = \mathcal{X}_\eta \cap (H_\eta^2(\mathbb{R}))^6.$$

For each sufficiently small $\eta > 0$ there exists an open neighbourhood $V_\mathcal{G}$ of the origin in $\mathcal{Y}_\eta$ such that the function $\mathcal{G} : V_\mathcal{G} \times (-\theta_0, \theta_0) \times (-\delta_0, \delta_0) \to \mathcal{X}_\eta$ is well defined and continuously differentiable.

Our final theorem is obtained using the implicit-function theorem (which necessitates a detailed study of the linear operator $\mathrm{d}_1\mathcal{G}[\mathbf{0}, 0, 0]$).

Theorem 10. *Let*

$$d_1 = \frac{c_2}{2c_1}, \qquad d_2 = \frac{c_3}{2c_1}, \qquad d_3 = \frac{c_3^1}{c_1^1}$$

and suppose that $d_1 \in (1, 4 + \sqrt{13})$ and $d_2 > -d_3$. There exist $\delta_0 > 0$, open neighbourhoods V, W of respectively the origin in $\mathcal{Y}_\eta \times \mathbb{R}$ and $\mathcal{X}_\eta \times \mathbb{R} \times \mathbb{R}$ and a continuously differentiable mapping $(\mathbf{V}, \theta) : (-\delta_0, \delta_0) \to V$ with $(\mathbf{V}(0), \theta(0)) = (\mathbf{0}, 0)$ such that

- $\mathcal{G}(\mathbf{V}(\delta), \theta(\delta), \delta) = 0$ *for all* $\delta \in (-\delta_0, \delta_0)$,
- $(\mathbf{V}, \theta) = (\mathbf{V}(\delta), \theta(\delta))$ *whenever* $(\mathbf{V}, \theta, \delta) \in W$ *satisfies* $\mathcal{G}(\mathbf{V}, \theta, \delta) = 0$.

Furthermore, the solution

$$\mathbf{C}_\delta = \mathbf{H}(\theta(\delta), \delta) + \mathbf{V}(\delta)$$

to the system (1.46) *is a reversible heteroclinic solution connecting $\mathbf{P}^{\delta,\theta(\delta)}$ with $\mathbf{Q}^{\delta,\theta(\delta)}$, that is*

$$\mathbf{C}_\delta - \mathbf{P}^{\delta,\theta(\delta)} = o(\mathrm{e}^{-\eta x})$$

as $x \to \infty$ and

$$\mathbf{C}_\delta - \mathbf{Q}^{\delta,\theta(\delta)} = o(\mathrm{e}^{\eta x})$$

as $x \to -\infty$.

Altogether we have established the following result.

Theorem 11. *Suppose that $\beta_0 < \mu_1(\mu_1 - 1)^2/(\mu_1 + 1)$, $\gamma_0 = r(\omega)$, $\gamma = \gamma_0 + \delta^2$ and the normal-form coefficients $c_1^1, c_3^1, c_1, c_2, c_3, c_4$ satisfy the conditions stated in Theorem 6, Lemma 7 and Theorem 10. (Explicit formulae for these coefficients are given in some special cases in Chapter 3.) For each sufficiently small value of δ equations* (1.18)–(1.24) *admit roll patterns R_+^δ and R_-^δ together with an angle ϑ_δ such that R_+^δ, R_-^δ are perpendicular to the directions making angles $\pm\left(\frac{\pi}{2} - \vartheta_\delta\right)$ with the x-direction. The rolls meet in a corner defect and the entire pattern is symmetric with respect to the x-direction.*

Chapter 2

Doubly periodic patterns

2.1 Dirichlet-Neumann formalism

In this chapter we present an existence theory for small amplitude, doubly periodic solutions to (1.18)–(1.24), that is solutions with

$$\eta(\mathbf{x}+\mathbf{l}) = \eta(\mathbf{x}), \qquad \phi'(\mathbf{x}+\mathbf{l},y) = \phi'(\mathbf{x},y), \qquad \phi(\mathbf{x}+\mathbf{l},y) = \phi(\mathbf{x},y)$$

for every $\mathbf{l} \in \mathscr{L}$ (with a slight abuse of notation), where $\mathbf{x} = (x,z)$ and $\mathscr{L}$ is the lattice given by

$$\mathscr{L} = \{m\mathbf{l}_1 + n\mathbf{l}_2 : \; m,n \in \mathbb{Z}\}$$

with $|\mathbf{l}_1| = |\mathbf{l}_2|$. Choose $\mathbf{k}_1, \mathbf{k}_2$ with $\mathbf{k}_i \cdot \mathbf{l}_j = 2\pi\delta_{ij}$ for $i,j = 1,2$ and define the dual lattice $\mathscr{L}^*$ to $\mathscr{L}$ by

$$\mathscr{L}^* = \{m\mathbf{k}_1 + n\mathbf{k}_2 : \; m,n \in \mathbb{Z}\},$$

so that our periodic functions can be written as

$$\eta(\mathbf{x}) = \sum_{\mathbf{k}\in\mathscr{L}^*} \eta_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \phi'(\mathbf{x},y) = \sum_{\mathbf{k}\in\mathscr{L}^*} \phi'_{\mathbf{k}}(y)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \phi(\mathbf{x},y) = \sum_{\mathbf{k}\in\mathscr{L}^*} \phi_{\mathbf{k}}(y)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},$$

where $\eta_{-\mathbf{k}} = \bar{\eta}_{\mathbf{k}}, \phi'_{-\mathbf{k}} = \bar{\phi}'_{\mathbf{k}}, \phi_{-\mathbf{k}} = \bar{\phi}_{\mathbf{k}}$ and $\mathbf{k} = (k_1,k_2)$. We are especially interested in three periodic patterns, namely rectangles, hexagons and rolls.

(i) For rectangles we choose $\mathbf{l}_1 = (\frac{2\pi}{\omega},0)$ and $\mathbf{l}_2 = (0,\frac{2\pi}{\omega})$, so that the dual lattice $\mathscr{L}^*$ is generated by $\mathbf{k}_1 = (\omega,0)$ and $\mathbf{k}_2 = (0,\omega)$ and the periodic base cell is given by

$$\left\{(x,z) : |x|\,,\; |z| < \frac{\pi}{\omega}\right\}$$

(see Figure 2.1). Furthermore, equations (1.18)–(1.24) are invariant under rotations through $\frac{\pi}{2}$ in the (x,z)-plane.

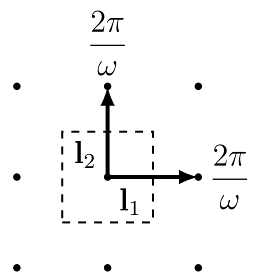


Figure 2.1:
The lattice $\mathcal{L}$ and periodic base cell for rectangles.

(ii) For hexagons we choose $\mathbf{l}_1 = \frac{2\pi}{\omega}(1, \frac{-1}{\sqrt{3}})$ and $\mathbf{l}_2 = \frac{2\pi}{\omega}(0, \frac{2}{\sqrt{3}})$, so that we obtain an additional periodic direction $\mathbf{l}_3 = \mathbf{l}_1 + \mathbf{l}_2 = \frac{2\pi}{\omega}(1, \frac{1}{\sqrt{3}})$. The dual lattice $\mathscr{L}^*$ is generated by $\mathbf{k}_1 = (\omega, 0)$ and $\mathbf{k}_2 = \omega(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and the periodic base cell is given by

$$\left\{(x,z) : |x| < \frac{2\pi}{\omega},\ \left|x - \sqrt{3}z\right| < \frac{4\pi}{\omega} \text{ and } \left|x + \sqrt{3}z\right| < \frac{4\pi}{\omega}\right\}$$

(see Figure 2.2). Furthermore, equations (1.18)–(1.24) are invariant under rotations through $\frac{\pi}{3}$ in the (x,z)-plane.

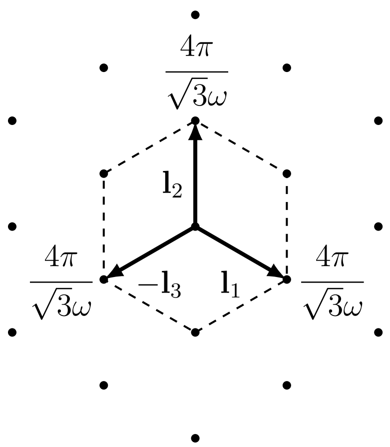


Figure 2.2:
The lattice $\mathcal{L}$ and periodic base cell for hexagons.

(iii) For rolls we seek functions that are independent of the z-direction and we choose $\mathbf{l} = (\frac{2\pi}{\omega}, 0)$, so that the dual lattice $\mathscr{L}^*$ is generated by $\mathbf{k} = (\omega, 0)$ and the periodic base cell is given by

$$\left\{x : |x| < \frac{\pi}{\omega}\right\}$$

(see Figure 2.3). Furthermore, the z-independent versions of equations (1.18)–(1.24) are invariant under the reflection $x \mapsto -x$ (which corresponds to a rotation through π in the (x,z)-plane).

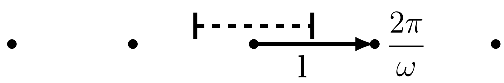


Figure 2.3:
The lattice $\mathcal{L}$ and periodic base cell for rolls.

The mathematical problem is to solve Laplace's equation (1.18) and the equation (1.19) with boundary conditions (1.20)–(1.24) for periodic functions η, ϕ' and ϕ in the domains $\Gamma, \Omega'_{\text{per}}$ and Ω_{per}, where

$$\Omega'_{\text{per}} := \{(x,y,z) : (x,z) \in \Gamma\} \cap \Omega',$$
$$\Omega_{\text{per}} := \{(x,y,z) : (x,z) \in \Gamma\} \cap \Omega$$

and Γ is the parallelogram defined by $\mathbf{l}_1$ and $\mathbf{l}_2$ (or by $\mathbf{l}$ in the case of rolls).

Our formulation of the ferrohydrodynamic problem has the disadvantage that it is posed in the *a priori* unknown domains Ω'_{per} and Ω_{per}. We overcome this difficulty using *Dirichlet-Neumann formalism*. The *Dirichlet-Neumann operator* G' for the upper fluid domain given by $\{\eta(x,z) < y < \frac{1}{\beta_0}\}$ is defined as follows. Fix $\Phi = \Phi(x,z)$, solve the boundary-value problem

$$\operatorname{div}(\mu(|\operatorname{grad}(\phi+y)|)\operatorname{grad}(\phi+y)) = 0 \qquad \text{in } \Omega_{\text{per}}, \tag{2.1}$$
$$\phi - \Phi = 0 \qquad \text{for } y = \eta, \tag{2.2}$$
$$\mu(|\operatorname{grad}(\phi+y)|)(\phi_y + 1) - \mu(1) = 0 \qquad \text{for } y = -\tfrac{1}{\beta_0} \tag{2.3}$$

and define

$$\begin{aligned} G(\eta,\Phi) &= \sqrt{1+\eta_x^2+\eta_z^2}\big(\mu(|\operatorname{grad}(\phi+y)|)\,\phi_n\big)\big|_{y=\eta} \\ &= \big(\mu(|\operatorname{grad}(\phi+y)|)(\phi_y - \eta_x\phi_x - \eta_z\phi_z)\big)\big|_{y=\eta}; \end{aligned} \tag{2.4}$$

note that a solution ϕ to (2.1)–(2.3) and hence the operator G depend *nonlinearly* upon Φ.

The Dirichlet-Neumann operator for the domain Ω'_{per} is defined as

$$\begin{aligned} G'(\eta,\Phi') &= -\sqrt{1+\eta_x^2+\eta_z^2}\,\phi'_n\big|_{y=\eta} \\ &= -(\phi'_y - \eta_x\phi'_x - \eta_z\phi'_z)\big|_{y=\eta}, \end{aligned} \tag{2.5}$$

where ϕ' is the solution to the boundary-value problem

$$\operatorname{div}(\operatorname{grad}\phi') = 0 \qquad \text{in } \Omega'_{\text{per}}, \tag{2.6}$$
$$\phi' - \Phi' = 0 \qquad \text{for } y = \eta, \tag{2.7}$$
$$\phi'_y = 0 \qquad \text{for } y = \tfrac{1}{\beta_0}; \tag{2.8}$$

here a solution ϕ' to (2.6)–(2.8) and hence the operator G' depend *linearly* upon Φ. It is also convenient to introduce auxiliary operators H' and H given by

$$H'(\eta,\Phi') = \phi'_y|_{y=\eta}, \qquad H(\eta,\Phi) = \phi_y|_{y=\eta}, \tag{2.9}$$

where ϕ' and ϕ are the solutions to the boundary-value problems (2.6)–(2.8) and (2.1)–(2.3).

Using this Dirichlet-Neumann formalism, we can write (1.18)–(1.24) as

$$\Phi' - \Phi + (\mu(1) - 1)\eta = 0, \tag{2.10}$$
$$G'(\eta,\Phi') + G(\eta,\Phi) + \mu^\star - \mu(1) = 0 \tag{2.11}$$

and

$$
\begin{aligned}
-\gamma\eta + \nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)& \\
+\frac{1}{2}\left(\left(1+|\nabla\eta|^2\right)H'(\eta,\Phi')^2-|\nabla\Phi'|^2\right)& \\
-\left(\mu(1)G'(\eta,\Phi')+G(\eta,\Phi)+\mu^\star-\mu(1)\right)& \\
+\left(M^\star-M(1)-\mu^\star H(\eta,\Phi)-H(\eta,\Phi)G(\eta,\Phi)\right)&=0, \qquad (2.12)
\end{aligned}
$$

in which $\nabla=(\partial_x,\partial_z)^{\mathrm{T}}$,

$$
\begin{aligned}
\mu^\star &= \mu\Big(\left(|\nabla\Phi|^2+2(1-\nabla\eta\cdot\nabla\Phi)H(\eta,\Phi)+(1+|\nabla\eta|^2)H(\eta,\Phi)^2+1\right)^{1/2}\Big),\\
M^\star &= M\Big(\left(|\nabla\Phi|^2+2(1-\nabla\eta\cdot\nabla\Phi)H(\eta,\Phi)+(1+|\nabla\eta|^2)H(\eta,\Phi)^2+1\right)^{1/2}\Big).
\end{aligned}
$$

We study equations (2.10)–(2.12) in the standard Sobolev spaces

$$
H^r_{\mathrm{per}}(\Gamma)=\left\{\zeta=\sum_{\mathbf{k}\in\mathscr{L}^*}\zeta_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}:\zeta_{-\mathbf{k}}=\overline{\zeta_{\mathbf{k}}},\ \|\zeta\|_r<\infty\right\}
$$

with norm

$$
\|\zeta\|_r^2:=C(\Gamma)\sum_{\mathbf{k}\in\mathscr{L}^*}\left(1+|\mathbf{k}|^2\right)^r|\zeta_{\mathbf{k}}|^2
$$

and their subspaces

$$
\bar{H}^r_{\mathrm{per}}(\Gamma)=\left\{\zeta\in H^r_{\mathrm{per}}(\Gamma):\zeta_{\mathbf{0}}=0\right\}
$$

consisting of functions with zero mean (the value of the normalisation constant $C(\Gamma)$ is $2\pi/\omega$ for rolls, $4(\pi/\omega)^2$ for squares and $8/\sqrt{3}(\pi/\omega)^2$ for hexagons). We also use the Sobolev spaces $H^r_{\mathrm{per}}(\Sigma)$ defined for $r\in\mathbb{N}_0$ by

$$
H^r_{\mathrm{per}}(\Sigma)=\left\{u\in L^2_{\mathrm{per}}(\Sigma):\|u\|_r<\infty\right\},
$$

where

$$
\|u\|_r^2=\sum_{0\le\alpha_1+\alpha_2+\alpha_3\le r}\left\|\partial_x^{\alpha_1}\partial_z^{\alpha_2}\partial_y^{\alpha_3}u\right\|_0^2,
$$

and for $r\notin\mathbb{N}_0$ by interpolation in the sense of Lions and Magenes [29, Volume 1, Chapter 1, Section 9.1]. In Section 2.2 below we establish the following theorem.

Theorem 12. *Suppose that* $s>5/2$. *Formulae* (2.5), (2.4) *and* (2.9) *define mappings* G', $G:H^s_{\mathrm{per}}(\Gamma)\times H^{s-1/2}_{\mathrm{per}}(\Gamma)\to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ *and* H', $H:H^s_{\mathrm{per}}(\Gamma)\times H^{s-1/2}_{\mathrm{per}}(\Gamma)\to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ *which are analytic at the origin.*

Here we use the following definition of analyticity (see Buffoni and Toland [6, Definition 4.3.1]).

Definition 13. *Let X and Y be Banach spaces, V be a non-empty, open subset of X and $\mathcal{L}_{\mathrm{s}}^k(X,Y)$ be the space of bounded, k-linear symmetric operators $X^k \to Y$ with norm*

$$|||m||| := \inf\left\{c : \left\|m\big(\{f\}^{(k)}\big)\right\|_Y \leq c\,\|f\|_X^k \text{ for all } f \in X\right\}.$$

*A function $F : V \to Y$ is **analytic at a point** $x_0 \in V$ if there exist real numbers $\delta, r > 0$ and a sequence $\{m_k\}$, where $m_k \in \mathcal{L}_{\mathrm{s}}^k(X,Y)$ for $k \in \mathbb{N}_0$, with the properties that*

$$F(x) = \sum_{k=0}^{\infty} m_k\big(\{x - x_0\}^{(k)}\big)$$

for $x \in B_\delta(x_0)$ and

$$\sup_{k \geq 0} r^k |||m_k||| < \infty.$$

*The function is **analytic** in V if it is analytic at each point $x_0 \in V$.*

Remark 14. Any function $F : V \to Y$ which is analytic at a point $x_0 \in V$ also defines an analytic function in an open neighbourhood of x_0 in V, so that by choosing V smaller if necessary, we may assume that F is analytic in V.

Using the fact that $H_{\mathrm{per}}^{s-3/2}(\Gamma)$ is a Banach algebra for $s > 5/2$, we find that the left-hand sides of equations (2.10)–(2.12) define an analytic function $\mathcal{G} : V_{\mathcal{G}} \times \mathbb{R} \to Y_0$, where $V_{\mathcal{G}}$ is an open neighbourhood of the origin in X chosen small enough that $(\eta, \Phi) \in V$ and $(\eta, \Phi') \in V'$ for each $(\eta, \Phi', \Phi) \in V_{\mathcal{G}}$. Here

$$X := H_{\mathrm{per}}^{s+1/2}(\Gamma) \times H_{\mathrm{per}}^{s}(\Gamma) \times H_{\mathrm{per}}^{s}(\Gamma),$$

and

$$Y_0 := H_{\mathrm{per}}^{s}(\Gamma) \times \bar{H}_{\mathrm{per}}^{s-1}(\Gamma) \times H_{\mathrm{per}}^{s-3/2}(\Gamma) \tag{2.13}$$

(the calculation

$$\begin{aligned}
&\int_\Gamma (G'(\eta,\Phi') + G(\eta,\Phi) + \mu^\star - \mu(1))\,\mathrm{d}x\,\mathrm{d}z \\
&= -\int_\Gamma \sqrt{1+\eta_x^2+\eta_z^2}((\phi' + \mu(1)y)_n - \mu(|\mathrm{grad}(\phi+y)|)(\phi+y)_n)\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_{\Omega'_{\mathrm{per}}} \mathrm{div}(\mathrm{grad}\,\phi')\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y - \int_\Gamma \sqrt{1+\eta_x^2+\eta_z^2}(\mu(1)y)_n\,\mathrm{d}x\,\mathrm{d}z \\
&\qquad + \int_{\Omega_{\mathrm{per}}} \mathrm{div}(\mu(|\mathrm{grad}(\phi+y)|)\mathrm{grad}(\phi+y))\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\
&\qquad + \int_\Gamma \sqrt{1+\eta_x^2+\eta_z^2}(\mu(|\mathrm{grad}(\phi+y)|)(\phi+y)_n)\big|_{y=-\frac{1}{\beta_0}}\,\mathrm{d}x\,\mathrm{d}z \\
&= 0
\end{aligned}$$

explains the choice of functions with zero mean in the second component of the target space Y_0).

Finally, we replace X by

$$X_0 := H_{\mathrm{per}}^{s+1/2}(\Gamma) \times \bar{H}_{\mathrm{per}}^{s}(\Gamma) \times H_{\mathrm{per}}^{s}(\Gamma), \tag{2.14}$$

so that the kernel of the linear operator $\mathrm{d}_2\mathcal{G}[0,0,0;\gamma]$ does not contain any constant terms for any $\gamma \in \mathbb{R}$ (see Section 2.4). The formulation of the mathematical problem is thus to solve

$$\mathcal{G}(\eta, \Phi', \Phi; \gamma) = 0 \tag{2.15}$$

for $(\eta, \Phi', \Phi) \in V_0$ and $\gamma \in \mathbb{R}$, where $V_0 := V_{\mathcal{G}} \cap X_0$.

2.2 Analyticity

In this section we prove Theorem 12, beginning with the operators G and H. We study the boundary-value problem (2.1)–(2.3) by transforming it to an equivalent problem in a fixed domain (cf. Nicholls and Reitich [37] and Twombly and Thomas [44]). The change of variable

$$\tilde{y} = \frac{y-\eta}{1+\beta_0\eta}, \qquad u(x,\tilde{y},z) = \phi(x,y,z) \qquad \text{for } -\tfrac{1}{\beta_0} < y < \eta$$

transforms the variable domain Ω_{per} into the rigid domain

$$\Sigma = \left\{ (x,y,z) : (x,z) \in \Gamma, y \in \left(-\frac{1}{\beta_0}, 0 \right) \right\}$$

and the boundary-value problem (2.1)–(2.3) into

$$\begin{aligned}
\operatorname{div}(\mu^\dagger(\operatorname{grad}(u+y) - F(\eta,u))) &= 0 && \text{in } \Sigma, && (2.16)\\
u - \Phi &= 0 && \text{for } y = 0, && (2.17)\\
\mu^\dagger(\operatorname{grad}(u+y) - F(\eta,u)) \cdot (0,1,0)^{\mathrm{T}} - \mu(1) &= 0 && \text{for } y = -\tfrac{1}{\beta_0}, && (2.18)
\end{aligned}$$

where

$$\begin{aligned}
F(\eta,u) &= (F_1(\eta,u), F_3(\eta,u), F_2(\eta,u))^{\mathrm{T}}, && (2.19)\\
F_1(\eta,u) &= -\beta_0\eta u_x + (1+\beta_0 y)\eta_x u_y, && (2.20)\\
F_2(\eta,u) &= -\beta_0\eta u_z + (1+\beta_0 y)\eta_z u_y, && (2.21)\\
F_3(\eta,u) &= \frac{\beta_0\eta u_y}{1+\beta_0\eta} + (1+\beta_0 y)(\eta_x u_x + \eta_z u_z) - \frac{(1+\beta_0 y)^2}{1+\beta_0\eta}(\eta_x^2 + \eta_z^2)u_y && (2.22)
\end{aligned}$$

and

$$\mu^\dagger = \mu\left(\left| \frac{1}{1+\beta_0\eta}\left(\operatorname{grad} u - (F_1(\eta,u), 0, F_2(\eta,u))^{\mathrm{T}} \right) + (0,1,0)^{\mathrm{T}} \right| \right), \tag{2.23}$$

and we have again dropped the tildes for notational simplicity.

Recall that $H^{s-1}_{\mathrm{per}}(\Sigma)$ is a Banach algebra for $s > 5/2$, that the trace map $u \mapsto u|_{y=0}$ defines a continuous operator $H^s_{\mathrm{per}}(\Sigma) \to H^{s-1/2}_{\mathrm{per}}(\Gamma)$ for $s > 1/2$, and that compositions of analytic functions and continuous linear functions are analytic. Combining these facts, we find that the left-hand sides of equations (2.16)–(2.18) define an analytic function

$$\mathcal{H} : V_{\mathcal{H}} \to H^{s-2}_{\mathrm{per}}(\Sigma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma) \times H^{s-3/2}_{\mathrm{per}}(\Gamma)$$

for $s > 5/2$, where $V_{\mathcal{H}}$ is an open neighbourhood of the origin in $H^s_{\rm per}(\Sigma) \times H^s_{\rm per}(\Gamma) \times H^{s-1/2}_{\rm per}(\Gamma)$. We now consider the equation

$$\mathcal{H}(u, \eta, \Phi) = 0 \tag{2.24}$$

and prove the following theorem.

Theorem 15. *Suppose that $s > 5/2$. There exist open neighbourhoods V of the origin in $H^s_{\rm per}(\Gamma) \times H^{s-1/2}_{\rm per}(\Gamma)$ and U of the origin in $H^s_{\rm per}(\Sigma)$ such that equation* (2.24)*, and hence the boundary-value problem* (2.16)–(2.18)*, has a unique solution $u = u(\eta, \Phi)$ in U for each $(\eta, \Phi) \in V$, that is*

$$\{(u, \eta, \Phi) \in U \times V : \mathcal{H}(u, \eta, \Phi) = 0\} = \{(u(\eta, \Phi), \eta, \Phi) : (\eta, \Phi) \in V\}.$$

Furthermore, $u(\eta, \Phi)$ depends analytically upon η and Φ.

We begin by considering the boundary-value problem

$$\mu_1 \left(u_{xx} + u_{zz} + S_1^{-2} u_{yy}\right) = \psi \qquad \text{in } \Sigma, \tag{2.25}$$

$$\mu_1 S_1^{-2} u = \zeta \qquad \text{for } y = 0, \tag{2.26}$$

$$\mu_1 S_1^{-2} u_y = \xi \qquad \text{for } y = -\tfrac{1}{\beta_0} \tag{2.27}$$

for $(\psi, \zeta, \xi) \in H^{s-2}_{\rm per}(\Sigma) \times H^{s-1/2}_{\rm per}(\Gamma) \times H^{s-3/2}_{\rm per}(\Gamma)$, where

$$\mu_1 = \mu(1), \qquad \dot{\mu}_1 = \dot{\mu}(1), \qquad S_1 = \left(\frac{\mu_1}{\mu_1 + \dot{\mu}_1}\right)^{1/2}.$$

Writing

$$\psi(\mathbf{x}, y) = \sum_{\mathbf{k} \in \mathscr{L}^\star} \psi_{\mathbf{k}}(y) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \zeta(\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{L}^\star} \zeta_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \xi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{L}^\star} \xi_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},$$

we find that the solution to (2.25)–(2.27) is given by the explicit formula

$$u(\mathbf{x}, y) = \sum_{\mathbf{k} \in \mathscr{L}^\star} \left(\int_{-\frac{1}{\beta_0}}^0 G(y, \tilde{y}) \psi_{\mathbf{k}}(\tilde{y})\, \mathrm{d}\tilde{y} \ + G_{\tilde{y}}(y, 0)\zeta_{\mathbf{k}} + G(y, -\tfrac{1}{\beta_0})\xi_{\mathbf{k}} \right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{2.28}$$

where the Green's function G is given by

$$G(y, \tilde{y}) = \begin{cases} \dfrac{S_1 \cosh \beta_0^{-1} S_1 |\mathbf{k}|(1 + \beta_0 y) \sinh S_1 |\mathbf{k}| \tilde{y}}{\mu_1 |\mathbf{k}| \cosh \beta_0^{-1} S_1 |\mathbf{k}|}, & -\frac{1}{\beta_0} \le y \le \tilde{y} \le 0, \\[2ex] \dfrac{S_1 \cosh \beta_0^{-1} S_1 |\mathbf{k}|(1 + \beta_0 \tilde{y}) \sinh S_1 |\mathbf{k}| y}{\mu_1 |\mathbf{k}| \cosh \beta_0^{-1} S_1 |\mathbf{k}|}, & -\frac{1}{\beta_0} \le \tilde{y} \le y \le 0, \end{cases}$$

for $\mathbf{k} \neq 0$ and

$$G(y, \tilde{y}) = \begin{cases} \dfrac{S_1^2}{\mu_1} \tilde{y}, & -\frac{1}{\beta_0} \le y \le \tilde{y} \le 0, \\[2ex] \dfrac{S_1^2}{\mu_1} y, & -\frac{1}{\beta_0} \le \tilde{y} \le y \le 0, \end{cases}$$

for $\mathbf{k} = 0$.

Proposition 16. *The estimates*

$$\int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}|^2 |G(y,\tilde{y})|\,\mathrm{d}y \lesssim 1, \qquad \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}|^2 |G(y,\tilde{y})|\,\mathrm{d}\tilde{y} \lesssim 1,$$
$$\int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}| |G_y(y,\tilde{y})|\,\mathrm{d}y \lesssim 1, \qquad \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}| |G_y(y,\tilde{y})|\,\mathrm{d}\tilde{y} \lesssim 1$$

hold for all $\mathbf{k} \in \mathscr{L}^\star \setminus \{\mathbf{0}\}$.

Proof. The estimates follow from the calculations

$$\begin{aligned}
\mu_1 \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}|^2 |G(y,\tilde{y})|\,\mathrm{d}y &= -\mu_1 \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}|^2 G(y,\tilde{y})\,\mathrm{d}y \\
&= 1 - \frac{\cosh \beta_0^{-1} S_1 |\mathbf{k}|(1+\beta_0\tilde{y})}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq 1, \\
\mu_1 \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}|^2 |G(y,\tilde{y})|\,\mathrm{d}\tilde{y} &= \mu_1 \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}|^2 |G(y,\tilde{y})|\,\mathrm{d}y \\
&\leq 1, \\
\mu_1 \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}| |G_y(y,\tilde{y})|\,\mathrm{d}y &= \frac{(1 - 2\cosh \beta_0^{-1} S_1 |\mathbf{k}|(1+\beta_0\tilde{y}))\sinh \beta_0^{-1} S_1 |\mathbf{k}|\tilde{y}}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq -\frac{2\cosh \beta_0^{-1} S_1 |\mathbf{k}|(1+\beta_0\tilde{y})\sinh \beta_0^{-1} S_1 |\mathbf{k}|\tilde{y}}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq -\frac{\sinh \beta_0^{-1} S_1 |\mathbf{k}|(1+2\beta_0\tilde{y}) - \sinh \beta_0^{-1} S_1 |\mathbf{k}|}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq \frac{2\sinh \beta_0^{-1} S_1 |\mathbf{k}|}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq 2, \\
\mu_1 \int_{-\frac{1}{\beta_0}}^{0} |\mathbf{k}| |G_y(y,\tilde{y})|\,\mathrm{d}\tilde{y} &= \frac{(-1 + 2\cosh \beta_0^{-1} S_1 |\mathbf{k}|\beta_0 y)\sinh \beta_0^{-1} S_1 |\mathbf{k}|(y+\beta_0)}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq \frac{2\cosh \beta_0^{-1} S_1 |\mathbf{k}|\beta_0 y \sinh \beta_0^{-1} S_1 |\mathbf{k}|(y+\beta_0}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq \frac{\sinh \beta_0^{-1} S_1 |\mathbf{k}|(1+2\beta_0 y) + \sinh \beta_0^{-1} S_1 |\mathbf{k}|}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq \frac{2\sinh \beta_0^{-1} S_1 |\mathbf{k}|}{\cosh \beta_0^{-1} S_1 |\mathbf{k}|} \\
&\leq 2
\end{aligned}$$

for $\mathbf{k} \in \mathscr{L}^\star \setminus \{\mathbf{0}\}$. □

Lemma 17. *Suppose that* $s \geq 2$. *The boundary-value problem* (2.25)–(2.27) *has a unique solution* $u \in H^s_{\mathrm{per}}(\Sigma)$ *which depends continuously on* $(\psi,\zeta,\xi) \in H^{s-2}_{\mathrm{per}}(\Sigma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma) \times H^{s-3/2}_{\mathrm{per}}(\Gamma)$.

Proof. We show that

$$\|u\|_s \lesssim \|\psi\|_{s-2} + \|\zeta\|_{s-1/2} + \|\xi\|_{s-3/2}$$

for $s \in \mathbb{R}$ with $s \geq 2$ by examining the formula (2.28).

For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0$ and $\mathbf{k} \in \mathscr{L}^\star \setminus \{\mathbf{0}\}$ we have that

$$\left|\left(\partial_x^{\alpha_1}\partial_z^{\alpha_2}\partial_y^{\alpha_3}(G_{\tilde{y}}(y,0)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}})\right)\right| = \left|\frac{k_1^{\alpha_1}k_2^{\alpha_2}S_1^{\alpha_3}|\mathbf{k}|^{\alpha_3}}{\cosh\beta_0^{-1}S_1|\mathbf{k}|}\begin{Bmatrix}\cosh\beta_0^{-1}S_1|\mathbf{k}|(1+\beta_0 y)\\ \sinh\beta_0^{-1}S_1|\mathbf{k}|(1+\beta_0 y)\end{Bmatrix}\right|.$$

Using the estimates

$$\frac{1}{\cosh^2\beta_0^{-1}S_1|\mathbf{k}|} \leq \frac{1}{\sinh^2\beta_0^{-1}S_1|\mathbf{k}|},$$

$$\frac{|\mathbf{k}|^2}{\sinh^2\beta_0^{-1}S_1|\mathbf{k}|}\left\{\mp\frac{1}{2\beta_0} + \frac{\sinh 2\beta_0^{-1}S_1|\mathbf{k}|}{4S_1|\mathbf{k}|}\right\} \lesssim \left(1+|\mathbf{k}|^2\right)^{1/2}$$

and $|k_1|, |k_2| \leq |\mathbf{k}|$, we find that

$$\begin{aligned}\left\|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}(G_{\tilde{y}}(y,0)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}})\right\|_0^2 &\lesssim \frac{|\mathbf{k}|^{2(\alpha_1+\alpha_2+\alpha_3)}}{\cosh^2\beta_0^{-1}S_1|\mathbf{k}|}\int_{-\frac{1}{\beta_0}}^0 \left|\begin{Bmatrix}\cosh\beta_0^{-1}S_1|\mathbf{k}|(1+\beta_0 y)\\ \sinh\beta_0^{-1}S_1|\mathbf{k}|(1+\beta_0 y)\end{Bmatrix}\right|^2 \mathrm{d}y \\ &\lesssim \left(1+|\mathbf{k}|^2\right)^{(\alpha_1+\alpha_2+\alpha_3-1/2)}\end{aligned}$$

and in a similar fashion

$$\left\|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}(G(y,-\tfrac{1}{\beta_0})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}})\right\|_0^2 \lesssim \left(1+|\mathbf{k}|^2\right)^{(\alpha_1+\alpha_2+\alpha_3-3/2)}.$$

Note that

$$\begin{aligned}\int_{-\frac{1}{\beta_0}}^0 \partial_y^{2m}G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y})\,\mathrm{d}\tilde{y} &= \int_{-\frac{1}{\beta_0}}^0 (S_1|\mathbf{k}|)^{2m}G(y,\tilde{y})\psi_{\mathbf{k}}\,\mathrm{d}\tilde{y} \\ &\quad + \sum_{j=1}^m S_1^2(S_1|\mathbf{k}|)^{2j-2}\left(\partial_y^{2m-2j}\psi_{\mathbf{k}}\right), \\ \int_{-\frac{1}{\beta_0}}^0 \partial_y^{2m+1}G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y})\,\mathrm{d}\tilde{y} &= \int_{-\frac{1}{\beta_0}}^0 (S_1|\mathbf{k}|)^{2m}\left(G_y(y,\tilde{y})\psi_{\mathbf{k}}\right)\mathrm{d}\tilde{y} \\ &\quad + \sum_{j=1}^m S_1^2(S_1|\mathbf{k}|)^{2j-2}\left(\partial_y^{2m-2j+1}\psi_{\mathbf{k}}\right)\end{aligned}$$

for $m \in \mathbb{N}_0$ and $\mathbf{k} \in \mathscr{L}^\star \setminus \{\mathbf{0}\}$ and that

$$\begin{aligned}&\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^0 \left|\sum_{j=1}^m S_1^2(S_1|\mathbf{k}|)^{2j-2}\left(\partial_y^{2m-2j}\psi_{\mathbf{k}}\right)\right|^2 \mathrm{d}y \\ &\qquad \lesssim \sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^0 \sum_{j=1}^m \left|S_1^{2j}(k_1^{2j-2}+k_2^{2j-2})\left(\partial_y^{2m-2j}\psi_{\mathbf{k}}\right)\right|^2 \mathrm{d}y\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{j=1}^{m}\Big(\big\|\partial_x^{2j-2}\partial_y^{2m-2j}\psi\big\|_0^2+\big\|\partial_z^{2j-2}\partial_y^{2m-2j}\psi\big\|_0^2\Big)\\&\lesssim \|\psi\|_{2m-2}^2\end{aligned}$$

and

$$\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}\left|\sum_{j=1}^{m}S_1^2(S_1|\mathbf{k}|)^{2j-2}\Big(\partial_y^{2m-2j+1}\psi_{\mathbf{k}}\Big)\right|^2\,\mathrm{d}y\lesssim\|\psi\|_{2m-2}^2$$

for $m\in\mathbb{N}_0$ and $\mathbf{k}\in\mathscr{L}^\star\setminus\{\mathbf{0}\}$.

Using the above estimates and Proposition 16, one finds that

$$\begin{aligned}&\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}\left|\int_{-\frac{1}{\beta_0}}^{0}(S_1|\mathbf{k}|)^{2m}G(y,\tilde{y})\psi_{\mathbf{k}}\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y\\
&\quad=\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}S_1^2(S_1|\mathbf{k}|)^{4m-2}\left|\int_{-\frac{1}{\beta_0}}^{0}|\mathbf{k}|^{1/2}|G(y,\tilde{y})|^{1/2}|\mathbf{k}|^{1/2}|G(y,\tilde{y})|^{1/2}\psi_{\mathbf{k}}\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y\\
&\quad\lesssim\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}S_1^2(S_1|\mathbf{k}|)^{4m-4}\int_{-\frac{1}{\beta_0}}^{0}|G(y,\tilde{y})|\,\mathrm{d}\tilde{y}\int_{-\frac{1}{\beta_0}}^{0}|G(y,\tilde{y})|\,|\psi_{\mathbf{k}}|^2\,\mathrm{d}\tilde{y}\,\mathrm{d}y\\
&\quad\lesssim\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}S_1^2(S_1|\mathbf{k}|)^{4m-2}\left|\int_{-\frac{1}{\beta_0}}^{0}\mathrm{e}^{-|\mathbf{k}||y-\tilde{y}|}\psi_{\mathbf{k}}\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y\\
&\quad\lesssim\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}S_1^2(S_1|\mathbf{k}|)^{4m-4}\,|\psi_{\mathbf{k}}|^2\,\mathrm{d}y\\
&\quad\lesssim\|\psi\|_{2m-2}^2\end{aligned}$$

and in a similar fashion that

$$\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}\left|\int_{-\frac{1}{\beta_0}}^{0}(S_1|\mathbf{k}|)^{2m}\Big(G_y(y,\tilde{y})\psi_{\mathbf{k}}\Big)\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y\lesssim\|\psi\|_{2m-1}^2\,.$$

The above estimates yield

$$\left\|\sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}}\int_{-\frac{1}{\beta_0}}^{0}\partial_y^m(G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y}))\,\mathrm{d}\tilde{y}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right\|_0^2\lesssim\|\psi\|_{m-2}^2$$

for $m\in\mathbb{N}$ and using the identity

$$\int_{-\frac{1}{\beta_0}}^{0}\partial_x^{\alpha_1}\partial_z^{\alpha_2}\partial_y^{\alpha_3}(G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}})\,\mathrm{d}\tilde{y}=\mathrm{i}^{\alpha_1+\alpha_2}k_1^{\alpha_1}k_2^{\alpha_2}\int_{-\frac{1}{\beta_0}}^{0}\partial_y^{\alpha_3}(G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y}))\,\mathrm{d}\tilde{y}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},$$

one finds by an analogous calculation that

$$\left\| \sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}} \int_{-\frac{1}{\beta_0}}^0 \partial_x^{\alpha_1}\partial_z^{\alpha_2}\partial_y^{\alpha_3}(G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}})\,\mathrm{d}\tilde{y} \right\|_0^2 \lesssim \|\psi\|_{\alpha_1+\alpha_2+\alpha_3-2}^2$$

for $\alpha_1,\alpha_2,\alpha_3\in\mathbb{N}_0$. Altogether we have established that

$$\left\| \sum_{\substack{\mathbf{k}\in\mathscr{L}^\star\\ \mathbf{k}\neq\mathbf{0}}} \int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\psi_{\mathbf{k}}(\tilde{y})\,\mathrm{d}\tilde{y}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \right\|_s^2 \lesssim \|\psi\|_{s-2}^2$$

for $s\in\mathbb{N}$ with $s\geq 2$ and for $s\in\mathbb{R}$ the result follows by interpolation.

For $\mathbf{k}=\mathbf{0}$ we have that

$$\begin{aligned}(u)_{\mathbf{0}} &= \int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\psi_{\mathbf{0}}(\tilde{y})\,\mathrm{d}\tilde{y}\ + G_{\tilde{y}}(y,0)\zeta_{\mathbf{0}} + G(y,-\tfrac{1}{\beta_0})\xi_{\mathbf{0}}\\ &= \frac{S_1^2}{\mu_1}\left(\int_y^0 \tilde{y}\psi_{\mathbf{0}}(\tilde{y})\,\mathrm{d}\tilde{y} + \int_{-\frac{1}{\beta_0}}^y y\psi_{\mathbf{0}}(\tilde{y})\,\mathrm{d}\tilde{y} + \zeta_{\mathbf{0}} + \xi_{\mathbf{0}}y\right)\end{aligned}$$

and obtain the estimate

$$\begin{aligned}\sum_{m=0}^n \left\| \left(\partial_y^m u\right)_{\mathbf{0}} \right\|_0^2 &\lesssim \|\psi\|_{n-2}^2 + \|\zeta\|_0^2 + \|\xi\|_{n-1}^2\\ &\lesssim \|\psi\|_{n-2}^2 + \|\zeta\|_{n-1/2}^2 + \|\xi\|_{n-3/2}^2\end{aligned}$$

for $n\in\mathbb{N}$ with $n\geq 2$.

Altogether we have established that

$$\|u\|_s \lesssim \|\psi\|_{s-2} + \|\zeta\|_{s-1/2} + \|\xi\|_{s-3/2}$$

for $s\in\mathbb{N}$ with $s\geq 2$ and for $s\in\mathbb{R}$ the result follows by interpolation; that is $u\in H^s_{\mathrm{per}}(\Sigma)$ for $(\psi,\zeta,\xi)\in H^{s-2}_{\mathrm{per}}(\Sigma)\times H^{s-1/2}_{\mathrm{per}}(\Gamma)\times H^{s-3/2}_{\mathrm{per}}(\Gamma)$. □

Theorem 15 follows from the analytic implicit-function theorem (see Buffoni and Toland [6, Theorem 4.5.4]) using the facts that $\mathcal{H}(0,0,0)=0$ and

$$\mathrm{d}_1\mathcal{H}[0,0,0](u) = \begin{pmatrix}\mu_1\left(u_{xx}+u_{zz}+S_1^{-2}u_{yy}\right)\\ u|_{y=0}\\ \mu_1 S_1^{-2}u_y|_{y=-\frac{1}{\beta_0}}\end{pmatrix}$$

is an isomorphism $H^s_{\mathrm{per}}(\Sigma)\to H^{s-2}_{\mathrm{per}}(\Sigma)\times H^{s-1/2}_{\mathrm{per}}(\Gamma)\times H^{s-3/2}_{\mathrm{per}}(\Gamma)$ for $s\geq 2$ (see Lemma 17).

In keeping with the notation in Theorem 15, we now introduce the rigorous definition of the Dirichlet-Neumann operator G; its analyticity is a direct consequence of Theorem 15.

Definition 18. *For each $s > 5/2$ the **Dirichlet-Neumann operator** for the boundary-value problem* (2.1)–(2.3) *is the analytic operator $G : V \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ defined by*

$$(\eta, \Phi) \mapsto \left(\mu^\dagger \left(\frac{u_y}{1+\beta_0\eta} - (\eta_x u_x + \eta_z u_z) + \frac{u_y}{1+\beta_0\eta}(\eta_x^2 + \eta_z^2) \right) \right) \Big|_{y=0},$$

where u is the solution to (2.16)–(2.18) *in U.*

For the discussion of the ferrohydrodynamic problem it is necessary to compute the term $\phi_y|_{y=\eta}$, where ϕ solves the boundary-value problem (2.1)–(2.3), for which we have no explicit formula in terms of G, η and Φ. To overcome this difficulty we introduce the auxiliary operator H given by

$$H(\eta, \Phi) = \phi_y\big|_{y=\eta}$$

which is rigorously defined as follows.

Definition 19. *Suppose that $s > 5/2$. The analytic operator $H : V \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ is defined by*

$$H(\eta, \Phi) = \left(\frac{u_y}{1+\beta_0\eta} \right) \Big|_{y=0},$$

where u is the solution to (2.16)–(2.18) *in U.*

Next we discuss the Dirichlet-Neumann operator G' for the domain Ω'_{per} and note that $\phi'_y\big|_{y=\eta}$ is given by the explicit formula

$$\phi'_y\big|_{y=\eta} = \frac{(-G'(\eta, \Phi') + \eta_x \Phi'_x + \eta_z \Phi'_z)}{1 + \eta_x^2 + \eta_z^2}.$$

We proceed as above by transforming the boundary-value problem (2.6)–(2.8) to an equivalent problem in a fixed domain. The change of variable

$$\tilde{y} = \frac{y - \eta}{1 - \beta_0\eta}, \qquad u'(x, \tilde{y}, z) = \phi'(x, z, y), \qquad \text{for } \eta < y < \tfrac{1}{\beta_0}$$

transforms the variable domain Ω'_{per} into the rigid domain

$$\Sigma' = \left\{ (x, y, z) : (x, z) \in \Gamma, y \in \left(0, \frac{1}{\beta_0}\right) \right\}$$

and the boundary-value problem (2.6)–(2.8) into

$$\operatorname{div}(\operatorname{grad} u' - F'(\eta, u')) = 0 \qquad \text{in } \Sigma', \tag{2.29}$$

$$u' - \Phi' = 0 \qquad \text{for } y = 0, \tag{2.30}$$

$$u'_y = 0 \qquad \text{for } y = \tfrac{1}{\beta_0}, \tag{2.31}$$

where

$$\begin{aligned}
F'(\eta,u') &= (F_1'(\eta,u'), F_3'(\eta,u'), F_2'(\eta,u'))^{\mathrm{T}}, \qquad (2.32)\\
F_1'(\eta,u') &= \beta_0\eta u_x' + (1-\beta_0 y)\eta_x u_y',\\
F_2'(\eta,u') &= \beta_0\eta u_z' + (1-\beta_0 y)\eta_z u_y',\\
F_3'(\eta,u') &= \frac{-\beta_0\eta u_y'}{1-\beta_0\eta} + (1-\beta_0 y)(\eta_x u_x' + \eta_z u_z') - \frac{(1-\beta_0 y)^2}{1-\beta_0\eta}(\eta_x^2+\eta_z^2)u_y',
\end{aligned}$$

and we have again dropped the tildes for notational simplicity. We find that the left-hand sides of equations (2.29)–(2.31) define an analytic function $\mathcal{H}' : V_{\mathcal{H}'} \to H^{s-2}_{\mathrm{per}}(\Sigma') \times H^{s-1/2}_{\mathrm{per}}(\Gamma) \times H^{s-3/2}_{\mathrm{per}}(\Gamma)$ for $s > 5/2$, where $V_{\mathcal{H}'}$ is an open neighbourhood of the origin in $H^s_{\mathrm{per}}(\Sigma') \times H^s_{\mathrm{per}}(\Gamma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma)$. Considering the equation

$$\mathcal{H}'(u',\eta,\Phi') = 0, \qquad (2.33)$$

we obtain the following result in an analogous way to Theorem 15.

Theorem 20. *Suppose that $s > 5/2$. There exist open neighbourhoods V' of the origin in $H^s_{\mathrm{per}}(\Gamma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma)$ and U' of the origin in $H^s_{\mathrm{per}}(\Sigma')$ such that equation* (2.33)*, and hence the boundary-value problem* (2.29)–(2.31)*, has a unique solution $u' = u'(\eta,\Phi')$ in U' for each $(\eta,\Phi') \in V'$, that is*

$$\{(u',\eta,\Phi') \in U' \times V' : \mathcal{H}'(u',\eta,\Phi') = 0\} = \{(u'(\eta,\Phi'),\eta,\Phi') : (\eta,\Phi') \in V'\}.$$

Furthermore, $u'(\eta,\Phi')$ depends analytically upon η and Φ'.

Next we introduce rigorous definitions of the Dirichlet-Neumann operator G' and the operator H'.

Definition 21. *For each $s > 5/2$ the* ***Dirichlet-Neumann operator*** *for the boundary-value problem* (2.6)–(2.8) *is the analytic operator $G' : V' \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ defined by*

$$(\eta,\Phi') \mapsto -\left(\frac{u_y'}{1-\beta_0\eta} - (\eta_x u_x' + \eta_z u_z') + \frac{u_y'}{1-\beta_0\eta}(\eta_x^2+\eta_z^2)\right)\Bigg|_{y=0},$$

where u' is the solution to (2.29)–(2.31) *in U'.*

Remark 22. We note that G' depends linearly upon Φ' and so can be extended to an analytic operator $V_\eta' \times H^{s-1/2}_{\mathrm{per}}(\Gamma) \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ (which is linear in its second argument), where V_η' is an open neighbourhood of the origin in $H^s_{\mathrm{per}}(\Gamma)$. The formula

$$\eta \mapsto (\Phi' \mapsto G'(\eta,\Phi'))$$

therefore defines an analytic operator $V_\eta' \to \mathcal{L}(H^{s-1/2}_{\mathrm{per}}(\Gamma), H^{s-3/2}_{\mathrm{per}}(\Gamma))$.

Definition 23. *Suppose that $s > 5/2$. The analytic operator $H' : V' \to H^{s-3/2}_{\mathrm{per}}(\Gamma)$ is defined by*

$$\begin{aligned}
H'(\eta,\Phi') &= \frac{(-G'(\eta,\Phi') + \eta_x\Phi_x' + \eta_z\Phi_z')}{1+\eta_x^2+\eta_z^2}\\
&= \left(\frac{u_y'}{1-\beta_0\eta}\right)\Bigg|_{y=0},
\end{aligned}$$

where u' is the solution to (2.29)–(2.31) *in U'.*

2.3 Taylor-series representation

To compute the Taylor-series representations of u and G we begin with the function $\nu : (H^{s-1}_{\mathrm{per}}(\Sigma))^3 \to H^{s-1}_{\mathrm{per}}(\Sigma)$ defined by

$$\nu(T) = \mu(|T + (0,1,0)^{\mathrm{T}}|).$$

Observing that ν is analytic at the origin, we write its Taylor series as

$$\nu(T) = \sum_{j=0}^{\infty} \nu^j(\{T\}^{(j)}), \tag{2.34}$$

where $\nu^j \in \mathcal{L}^j_{\mathrm{s}}((H^{s-1}_{\mathrm{per}}(\Sigma))^3, H^{s-1}_{\mathrm{per}}(\Sigma))$ is given by

$$\nu^j(T_1,\dots,T_j) = \frac{1}{j!}\mathrm{d}^j\nu[0](T_1,\dots,T_j)$$

and may be computed explicitly from μ (note in particular that $\nu^0 = \mu_1$). The functions $u^n \in \mathcal{L}^n_{\mathrm{s}}(H^s_{\mathrm{per}}(\Gamma) \times H^{s-1/2}_{\mathrm{per}}(\Gamma), H^s_{\mathrm{per}}(\Sigma))$ (with $u^0 = 0$) in the corresponding series

$$u(\eta,\Phi) = \sum_{n=0}^{\infty} u^n\big(\{(\eta,\Phi)\}^{(n)}\big) \tag{2.35}$$

may be computed recursively by substituting the *Ansatz* (2.34), (2.35) into equations (2.16)–(2.18). Consistently abbreviating $m^n\big(\{(\eta,\Phi)\}^{(n)}\big)$ to m^n for notational simplicity, one finds after a lengthy but straightforward calculation that

$$\begin{aligned}
\operatorname{div}(L \operatorname{grad} u^1) &= 0, & \operatorname{div}(L \operatorname{grad} u^n - F^n) &= 0, & -\tfrac{1}{\beta_0} < y < 0,\\
u^1 - \Phi &= 0, & u^n &= 0, & y = 0,\\
u^1_y &= 0, & (L \operatorname{grad} u^n - F^n)\cdot(0,1,0)^{\mathrm{T}} &= 0, & y = -\tfrac{1}{\beta_0}
\end{aligned}$$

for $n \geq 2$, where

$$\begin{aligned}
L &= \begin{pmatrix} 1 & 0 & 0\\ 0 & S_1^{-2} & 0\\ 0 & 0 & 1\end{pmatrix},\\
\mu_1 F^n &= \mu_1\left((F_1^n, F_3^n, F_2^n)^{\mathrm{T}} - \frac{\dot{\mu}_1}{\mu_1}\sum_{j=1}^{n}(-\beta_0\eta)^j\, u_y^{n-j}(0,1,0)^{\mathrm{T}}\right)\\
&\quad - \sum_{j=0}^{n}\nu^1(T^j)(\operatorname{grad} u^{n-j} - (F_1^{n-j}, F_3^{n-j}, F_2^{n-j})^{\mathrm{T}})\\
&\quad - R^n(0,1,0)^{\mathrm{T}} - \sum_{j=0}^{n} R^j(\operatorname{grad} u^{n-j} - (F_1^{n-j}, F_3^{n-j}, F_2^{n-j})^{\mathrm{T}})
\end{aligned}$$

and

$$F_1^n = -\beta_0\eta u_x^{n-1} + (1+\beta_0 y)\eta_x u_y^{n-1}, \qquad F_2^n = -\beta_0\eta u_z^{n-1} + (1+\beta_0 y)\eta_z u_y^{n-1},$$

$$F_3^n = \beta_0\eta \sum_{j=0}^{n-1} (-\beta_0\eta)^j\, u_y^{n-1-j} + (1+\beta_0 y)(\eta_x u_x^{n-1} + \eta_z u_z^{n-1})$$
$$\qquad - (1+\beta_0 y)^2(\eta_x^2+\eta_z^2) \sum_{j=0}^{n-2} (-\beta_0\eta)^j\, u_y^{n-2-j},$$
$$T^n = \sum_{j=0}^{n} (-\beta_0\eta)^j \left(\text{grad } u^{n-j} - (F_1^{n-j}, 0, F_2^{n-j})^{\mathrm{T}}\right), \qquad R^n = \sum_{\substack{2\le j\le n,\\ h_1+\ldots+h_j=n}} \nu^j(T^{h_1},\ldots,T^{h_j}).$$

The Taylor-series representations of G and H are thus given by

$$G(\eta,\Phi) = \sum_{n=0}^{\infty} G_n\big(\{(\eta,\Phi)\}^{(n)}\big), \qquad H(\eta,\Phi) = \sum_{n=0}^{\infty} H_n\big(\{(\eta,\Phi)\}^{(n)}\big),$$

where

$$G_n = \mu_1 I^n + \sum_{j=0}^{n} \nu^1(T^j) I^{n-j} + \sum_{j=0}^{n} R^j I^{n-j}\Bigg|_{y=0}, \qquad H_n = \sum_{j=0}^{n} (-\beta_0\eta)^j u_y^{n-j}\Bigg|_{y=0},$$

and

$$I^n = \sum_{j=0}^{n} (-\beta_0\eta)^j\, u_y^{n-j} + \sum_{j=0}^{n-2} (-\beta_0\eta)^j\, (\eta_x^2+\eta_z^2) u_y^{n-2-j} - (\eta_x u_x^{n-1} + \eta_z u_z^{n-1}).$$

For later use we record the formulae

$$G_1 = \mu_1 u_y^1\big|_{y=0},$$
$$G_2 = \mu_1 \left(\sum_{j=0}^{2} (-\beta_0\eta)^j\, u_y^{2-j} - (\eta_x u_x^1 + \eta_z u_z^1) \right) + \dot\mu_1 (u_y^1)^2 \Bigg|_{y=0},$$
$$G_3 = \mu_1 \left(\sum_{j=0}^{3} (-\beta_0\eta)^j\, u_y^{3-j} + (\eta_x^2+\eta_z^2) u_y^1 - (\eta_x u_x^2 + \eta_z u_z^2) \right) + \tfrac{1}{2}\dot\mu_1 \big((u_x^1)^2 + (u_z^1)^2\big) u_y^1$$
$$\qquad + \dot\mu_1 \left(2\sum_{j=0}^{2} (-\beta_0\eta)^j\, u_y^{2-j} - (\eta_x u_x^1 + \eta_z u_z^1) \right) u_y^1 + \tfrac{1}{2}\ddot\mu_1 (u_y^1)^3 \Bigg|_{y=0},$$

where $\ddot\mu_1 = \ddot\mu(1)$, and

$$H_1 = u_y^1\big|_{y=0}, \qquad H_2 = \sum_{j=0}^{2} (-\beta_0\eta)^j\, u_y^{2-j}\Bigg|_{y=0}, \qquad H_3 = \sum_{j=0}^{3} (-\beta_0\eta)^j\, u_y^{3-j}\Bigg|_{y=0}$$

for the first few terms in these series.

The functions $u'^n \in \mathcal{L}_{\mathrm{s}}^n(H_{\mathrm{per}}^s(\Gamma) \times H_{\mathrm{per}}^{s-1/2}(\Gamma), H_{\mathrm{per}}^s(\Sigma'))$ (with $u'^0 = 0$) in the Taylor series

$$u'(\eta,\Phi') = \sum_{n=1}^{\infty} u'^n\big(\{(\eta,\Phi')\}^{(n)}\big)$$

are computed recursively by substituting this *Ansatz* into equations (2.29)–(2.31); one finds that

$$\begin{aligned} \operatorname{div}(\operatorname{grad} u'^1) &= 0, & \operatorname{div}(\operatorname{grad} u'^n - (F_1'^n, F_3'^n, F_2'^n)^{\mathrm{T}}) &= 0, & 0 < y < \tfrac{1}{\beta_0}, \\ u'^1 - \Phi' &= 0, & u'^n &= 0, & y = 0, \\ u_y'^1 &= 0, & u_y'^n &= 0, & y = \tfrac{1}{\beta_0} \end{aligned}$$

for $n \geq 2$, where

$$F_1'^n = \eta u_x'^{n-1} + (1-\beta_0 y)\eta_x u_y'^{n-1}, \qquad F_2'^n = \eta u_z'^{n-1} + (1-\beta_0 y)\eta_z u_y'^{n-1},$$

$$\begin{aligned} F_3'^n = &-\beta_0\eta \sum_{j=0}^{n-1} (\beta_0\eta)^j\, u_y'^{n-1-j} + (1-\beta_0 y)(\eta_x u_x'^{n-1} + \eta_z u_z'^{n-1}) \\ &- (1-\beta_0 y)^2(\eta_x^2+\eta_z^2)\sum_{j=0}^{n-2}(\beta_0\eta)^j u_y'^{n-2-j}. \end{aligned}$$

The Taylor-series representations of G' and H' are given by

$$G'(\eta,\Phi') = \sum_{n=1}^{\infty} G_n'\big(\{(\eta,\Phi')\}^{(n)}\big), \qquad H'(\eta,\Phi') = \sum_{n=1}^{\infty} H_n'\big(\{(\eta,\Phi')\}^{(n)}\big)$$

with

$$\begin{aligned} G_n' &= -\sum_{j=0}^{n-1} (\beta_0\eta)^j\, u_y'^{n-j} - \sum_{j=0}^{n-3} (\beta_0\eta)^j\, (\eta_x^2+\eta_z^2) u_y'^{n-2-j} + (\eta_x u_x'^{n-1} + \eta_z u_z'^{n-1})\Big|_{y=0}, \\ H_n' &= \sum_{j=0}^{n-1} (\beta_0\eta)^j\, u_y'^{n-j}\Big|_{y=0}, \end{aligned}$$

and in particular we find that

$$G_1' = -u_y'^1\big|_{y=0}, \qquad G_2' = -\sum_{j=0}^{1} (\beta_0\eta)^j\, u_y'^{2-j} + (\eta_x u_x'^1 + \eta_z u_z'^1)\Big|_{y=0},$$

$$G_3' = -\sum_{j=0}^{2} (\beta_0\eta)^j\, u_y'^{3-j} - (\eta_x^2+\eta_z^2) u_y'^1 + (\eta_x u_x'^2 + \eta_z u_z'^2)\Big|_{y=0}.$$

and

$$H_1' = u_y'^1\big|_{y=0}, \qquad H_2' = \sum_{j=0}^{1} (\beta_0\eta)^j\, u_y'^{2-j}\Big|_{y=0}, \qquad H_3' = \sum_{j=0}^{2} (\beta_0\eta)^j\, u_y'^{3-j}\Big|_{y=0}.$$

2.4 Existence theory

Next we introduce the Crandall-Rabinowitz theorem (cf. Buffoni and Toland [6, Theorem 8.3.1]), an application of which yields a local bifurcation point of the equation

$$\mathcal{G}(\eta, \Phi', \Phi; \gamma) = 0, \tag{2.36}$$

where the components of $\mathcal{G} : V_0 \times \mathbb{R} \to Y_0$ are given by the left-hand sides of (2.10)–(2.12).

Theorem 24 (Crandall-Rabinowitz theorem)**.** *Let X and Y be Banach spaces, V be an open neighbourhood of the origin in X and $\mathcal{F} : V \times \mathbb{R} \to Y$ be an analytic function with $\mathcal{F}(v;\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Suppose also that*

(i) $L := \mathrm{d}_1\mathcal{F}[0;\lambda_0] : X \to Y$ *is a Fredholm operator of index zero,*

(ii) $\ker L = \langle v_0 \rangle$ *for some* $v_0 \in X$,

(iii) the transversality condition $P(\mathrm{d}_1\mathrm{d}_2\mathcal{F}[0;\lambda_0](v_0;1)) \neq 0$ *holds, where* $P : Y \to Y$ *is a projection with* $\mathrm{Im}\, L = \ker P$.

The point $(\lambda_0, 0)$ is a local bifurcation point, that is there exist $\varepsilon > 0$, an open neighbourhood W of $(\lambda_0, 0)$ in $\mathbb{R} \times X$ and analytic functions $w : (-\varepsilon, \varepsilon) \to V$, $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $\lambda(0) = \lambda_0$, $w(0) = v_0$ such that $\mathcal{F}(sw(s);\lambda(s)) = 0$ for every $s \in (-\varepsilon, \varepsilon)$. Furthermore

$$W \cap N = \{(\lambda(s), sw(s)) : 0 < |s| < \varepsilon\},$$

where

$$N = \{(\lambda, v) \in \mathbb{R} \times (V \setminus \{0\}) : \mathcal{F}(v;\lambda) = 0\}.$$

The first step is to determine the maximal positive value γ_0 of the parameter γ for which the kernel of the linear operator $L_0 := \mathrm{d}_2\mathcal{G}[0,0,0;\gamma_0] : X_0 \to Y_0$, which is given by the explicit formula

$$L_0 \begin{pmatrix} \eta \\ \Phi' \\ \Phi \end{pmatrix} = \begin{pmatrix} \Phi' - \Phi + (\mu_1 - 1)\eta \\ G_1'(\eta, \Phi') + \mu G_1(\eta, \Phi) + \dot{\mu}_1 H_1(\eta, \Phi) \\ \eta_{xx} + \eta_{zz} - \gamma_0\eta - (\mu_1 G_1'(\eta, \Phi') + G_1(\eta, \Phi) + \dot{\mu}_1 H_1(\eta, \Phi)) \end{pmatrix} \tag{2.37}$$

with

$$G_1'(\eta, \Phi') = \sum_{\mathbf{k} \in \mathscr{L}^*} |\mathbf{k}| \tanh \frac{|\mathbf{k}|}{\beta_0} \Phi'_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad G_1(\eta, \Phi) = \mu_1 \sum_{\mathbf{k} \in \mathscr{L}^*} S_1|\mathbf{k}| \tanh \frac{S_1|\mathbf{k}|}{\beta_0} \Phi_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

and $G_1(\eta, \Phi) = \mu_1 H_1(\eta, \Phi)$, is non-trivial. Writing $v \in X_0$ as

$$v(\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{L}^*} \mathbf{v}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.38}$$

with $\mathbf{v}_{\mathbf{k}} = (\eta_{\mathbf{k}}, \Phi'_{\mathbf{k}}, \Phi_{\mathbf{k}})^{\mathrm{T}}$ and $\mathbf{v}_{-\mathbf{k}} = \bar{\mathbf{v}}_{\mathbf{k}}$, we find that

$$L_0 v = \sum_{\mathbf{k} \in \mathscr{L}^*} L_0(|\mathbf{k}|) \mathbf{v}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{2.39}$$

where

$$L_0(|\mathbf{k}|) = \begin{pmatrix} \mu_1 - 1 & 1 & -1 \\ 0 & |\mathbf{k}| \tanh \dfrac{|\mathbf{k}|}{\beta_0} & \mu_1 S_1^{-1} |\mathbf{k}| \tanh \dfrac{S_1|\mathbf{k}|}{\beta_0} \\ -|\mathbf{k}|^2 - \gamma_0 & -\mu_1 |\mathbf{k}| \tanh \dfrac{|\mathbf{k}|}{\beta_0} & -\mu_1 S_1^{-1} |\mathbf{k}| \tanh \dfrac{S_1|\mathbf{k}|}{\beta_0} \end{pmatrix}$$

for $\mathbf{k} \neq \mathbf{0}$ and

$$L_0(0) = \begin{pmatrix} \mu_1 - 1 & 1 & -1 \\ 0 & 0 & 0 \\ -\gamma_0 & 0 & 0 \end{pmatrix}$$

(where we have identified the subspace $\{(\eta_{\mathbf{o}}, \Phi'_{\mathbf{o}}, \Phi_{\mathbf{o}})^{\mathrm{T}} : \Phi'_{\mathbf{o}} = 0\}$ of $\mathbb{R}^3$ with $\mathbb{R}^2$).

From this observation it follows that $\ker L_0$ is non-trivial if

$$\begin{aligned} \det L_0(|\mathbf{k}|) = \mu_1(\mu_1 - 1)^2 S_1^{-1} |\mathbf{k}|^2 \tanh \frac{|\mathbf{k}|}{\beta_0} \tanh \frac{S_1|\mathbf{k}|}{\beta_0} \\ - (|\mathbf{k}|^2 + \gamma_0)\left(\mu_1 |\mathbf{k}| \tanh \frac{|\mathbf{k}|}{\beta_0} + S_1 |\mathbf{k}| \tanh \frac{S_1|\mathbf{k}|}{\beta_0}\right) = 0, \end{aligned}$$

that is

$$\gamma_0 = r(|\mathbf{k}|) := \left(\mu_1(\mu_1 - 1)^2 \left(\mu_1 |\mathbf{k}| \coth \frac{|\mathbf{k}|}{\beta_0} + S_1 |\mathbf{k}| \coth \frac{S_1|\mathbf{k}|}{\beta_0}\right)^{-1} - 1\right) |\mathbf{k}|^2$$

for some $\mathbf{k} \neq \mathbf{0}$. The function $|\mathbf{k}| \mapsto r(|\mathbf{k}|)$, which satisfies $r(0) = 0$ and $r(|\mathbf{k}|) \to -\infty$ as $|\mathbf{k}| \to \infty$, takes only negative values for $\beta_0 > \mu_1(\mu_1 - 1)^2/(\mu_1 + 1)$, while for $\beta_0 < \mu_1(\mu_1 - 1)^2/(\mu_1 + 1)$ it has a unique maximum ω with $r(\omega) > 0$ (see Figure 2.4); we choose $\gamma_0 = r(\omega)$ and note the relationships

$$\beta_0 = \frac{\mu_1(\mu_1 - 1)^2}{2\tilde{\omega}} \left(\frac{h(\tilde{\omega}) - \tilde{\omega}\dot{h}(\tilde{\omega})}{h(\tilde{\omega})^2}\right), \qquad \gamma_0 = \left(\frac{\mu_1(\mu_1 - 1)^2}{\omega h(\tilde{\omega})} - 1\right)\omega^2,$$

where $\tilde{\omega} = \omega/\beta_0$ and $h(\tilde{\omega}) = \mu_1 \coth \tilde{\omega} + S_1 \coth S_1\tilde{\omega}$.

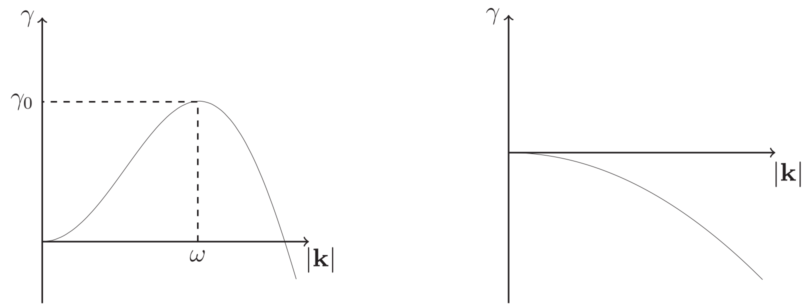


Figure 2.4:
The graph of the function $|\mathbf{k}| \mapsto r(|\mathbf{k}|)$ for $\beta_0 > \mu_1(\mu_1-1)^2/(\mu_1+1)$ (left) and $\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ (right).

Noting that $\ker L_0(\omega) = \langle \mathbf{v} \rangle$, where

$$\mathbf{v} = \begin{pmatrix} \dfrac{1}{\mu_1 - 1}\big(\mu_1 S_1^{-1} \tanh S_1\tilde{\omega} \coth \tilde{\omega} + 1\big) \\ -\mu_1 S_1^{-1} \tanh S_1\tilde{\omega} \coth \tilde{\omega} \\ 1 \end{pmatrix}, \tag{2.40}$$

we find that

$$\ker L_0 = \langle \{ \mathbf{v} \sin(\mathbf{k}\cdot\mathbf{x}), \mathbf{v}\cos(\mathbf{k}\cdot\mathbf{x}) : \mathbf{k} \in \mathscr{L}^\star \text{ with } |\mathbf{k}| = \omega \} \rangle$$

(see Figure 2.5).

(i) For rolls the dual lattice $\mathscr{L}^\star$ is generated by $\mathbf{k} = (\omega, 0)$, so that $|\mathbf{k}|, |-\mathbf{k}| = \omega$ and hence $\dim \ker L_0 = 2$.

(ii) For squares the dual lattice $\mathscr{L}^\star$ is generated by $\mathbf{k}_1 = (\omega, 0)$ and $\mathbf{k}_2 = (0, \omega)$, so that $|\mathbf{k}_1|, |-\mathbf{k}_1|, |\mathbf{k}_2|, |-\mathbf{k}_2| = \omega$ and hence $\dim\ker L_0 = 4$.

(iii) For hexagons the dual lattice $\mathscr{L}^\star$ is generated by $\mathbf{k}_1 = (\omega, 0)$ and $\mathbf{k}_2 = \omega(\frac{1}{2}, \frac{\sqrt{3}}{2})$, so that $|\mathbf{k}_1|, |-\mathbf{k}_1|, |\mathbf{k}_2|, |-\mathbf{k}_2|, |\mathbf{k}_3|, |-\mathbf{k}_3| = \omega$, where $\mathbf{k}_3 = \mathbf{k}_2 - \mathbf{k}_1$, and hence $\dim\ker L_0 = 6$.

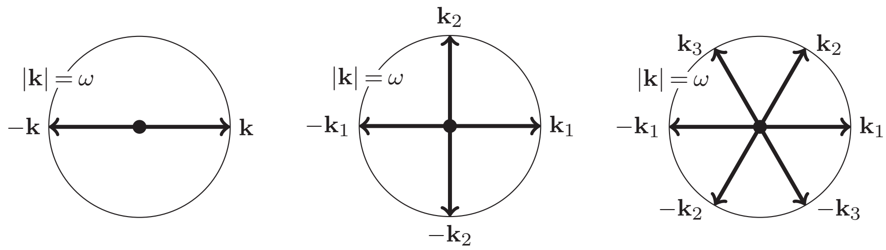


Figure 2.5:
The vectors generated by $\mathscr{L}^\star$ with length ω in the case of rolls (left), squares (centre) and hexagons (right).

Define the projection $P_\omega : X_0 \to X_0, Y_0 \to Y_0$ by

$$P_\omega v = \sum_{\substack{\mathbf{k}\in\mathscr{L}^*,\\ |\mathbf{k}|=\omega}} \mathbf{v}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},$$

where v is given by formula (2.38), so that

$$X_0 = X_\omega \oplus X_\mathrm{r}, \qquad Y_0 = Y_\omega \oplus Y_\mathrm{r}$$

with

$$X_\omega = P_\omega[X_0], \qquad X_\mathrm{r} = (I - P_\omega)[X_0], \qquad Y_\omega = P_\omega[Y_0], \qquad Y_\mathrm{r} = (I-P_\omega)[Y_0].$$

Using (2.39), one finds that $\operatorname{Im} L_0\big|_{X_\omega} \subseteq Y_\omega$ and $\operatorname{Im} L_0\big|_{X_\mathrm{r}} \subseteq Y_\mathrm{r}$ and we prove that $L_0 : X_0 \to Y_0$ is a Fredholm operator of index zero in two steps.

Lemma 25. *The mapping L_0 is an isomorphism $X_\mathrm{r} \to Y_\mathrm{r}$.*

Proof. The mapping $L_0\big|_{X_\mathrm{r}}$ is formally invertible on Y_r with

$$L_0^{-1}(\chi, \Psi', \Psi)^\mathrm{T} = \sum_{\substack{k\in\mathscr{L}^*,\\ |\mathbf{k}|\neq\omega}} L_0(|\mathbf{k}|)^{-1}(\chi_\mathbf{k}, \Psi'_\mathbf{k}, \Psi_\mathbf{k})^\mathrm{T} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{2.41}$$

where

$$L_0(|\mathbf{k}|)^{-1} = \frac{\mu_1 S_1^{-1}|\mathbf{k}|\tanh\dfrac{S_1|\mathbf{k}|}{\beta_0}}{\det L_0(|\mathbf{k}|)}\begin{pmatrix} (\mu_1-1)|\mathbf{k}|\tanh\dfrac{|\mathbf{k}|}{\beta_0} & 1 & 1 \\ -(\gamma_0+|\mathbf{k}|^2) & -(\mu_1-1) & -(\mu_1-1) \\ 0 & 0 & 0 \end{pmatrix}$$

$$+\frac{|\mathbf{k}|\tanh\dfrac{|\mathbf{k}|}{\beta_0}}{\det L_0(|\mathbf{k}|)}\begin{pmatrix} 0 & \mu_1 & 1 \\ 0 & 0 & 0 \\ \gamma_0+|\mathbf{k}|^2 & \mu_1(\mu_1-1) & \mu_1-1 \end{pmatrix} - \frac{\gamma_0+|\mathbf{k}|^2}{\det L_0(|\mathbf{k}|)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

for $\mathbf{k} \in \mathscr{L}^*$ with $|\mathbf{k}| > \omega$ and

$$L_0(0)^{-1}\begin{pmatrix}\chi_{\mathbf{0}} \\ 0 \\ \Psi_{\mathbf{0}}\end{pmatrix} = \begin{pmatrix} -\gamma_0^{-1}\Psi_{\mathbf{0}} \\ 0 \\ -\chi_{\mathbf{0}} - (\mu_1-1)\gamma_0^{-1}\Psi_{\mathbf{0}}\end{pmatrix}$$

Denoting the right-hand side of equation (2.41) by $(\eta, \Phi', \Phi)^{\mathrm{T}}$ and using the estimates

$$S_1^{-1}|\mathbf{k}|\tanh\frac{S_1|\mathbf{k}|}{\beta_0} \lesssim |\mathbf{k}|\tanh\frac{|\mathbf{k}|}{\beta_0}, \qquad \frac{|\mathbf{k}|\tanh\dfrac{|\mathbf{k}|}{\beta_0}}{|\det L_0(|\mathbf{k}|)|} \lesssim |\mathbf{k}|^{-2}$$

for $|\mathbf{k}| > \omega$, one finds that

$$\begin{aligned}\|\eta\|_{s+1/2}^2 &\lesssim (|\chi_{\mathbf{0}}| + |\Psi_{\mathbf{0}}|)^2 + \sum_{\substack{k\in\mathscr{L}^*,\\ |\mathbf{k}|\neq\omega,0}} |\mathbf{k}|^{2s+1}\frac{\left(|\mathbf{k}|\tanh\dfrac{|\mathbf{k}|}{\beta_0}\right)^2}{(\det L_0(|\mathbf{k}|))^2}\left(|\mathbf{k}|\tanh\frac{|\mathbf{k}|}{\beta_0}\,|\chi_{\mathbf{k}}| + |\Psi'_{\mathbf{k}}| + |\Psi_{\mathbf{k}}|\right)^2 \\ &\lesssim |\chi_{\mathbf{0}}|^2 + |\Psi_{\mathbf{0}}|^2 + \sum_{\substack{k\in\mathscr{L}^*,\\ |\mathbf{k}|\neq\omega,0}} |\mathbf{k}|^{2s-3}\left(|\mathbf{k}|^2|\chi_{\mathbf{k}}|^2 + |\Psi'_{\mathbf{k}}|^2 + |\Psi_{\mathbf{k}}|^2\right) \\ &\lesssim \|\chi\|_s^2 + \|\Psi'\|_{s-1}^2 + \|\Psi\|_{s-3/2}^2\,;\end{aligned}$$

similar calculations yield

$$\|\Phi'\|_s^2 \lesssim \|\chi\|_s^2 + \|\Psi'\|_{s-1}^2 + \|\Psi\|_{s-3/2}^2\,, \qquad \|\Phi\|_s^2 \lesssim \|\chi\|_s^2 + \|\Psi'\|_{s-1}^2 + \|\Psi\|_{s-3/2}^2\,.$$

We conclude that $L_0^{-1} : Y_{\mathrm{r}} \to X_{\mathrm{r}}$ exists and is continuous. □

Proposition 26. *The operator $L_0 : X_0 \to Y_0$ is a Fredholm operator of index zero.*

Proof. A straightforward calculation shows that $\mathbf{v} \notin \mathrm{Im}\, L_0(\omega)$ (so that $\ker\,(L_0(\omega))^2 = \ker L_0(\omega)$) and hence

$$Y_\omega = \bigoplus_{\substack{\mathbf{k}\in\mathscr{L}^*,\\ |\mathbf{k}|=\omega}} \Big(\big(\mathrm{Im}\, L_0(\omega) \oplus \ker L_0(\omega)\big) \sin(\mathbf{k}\cdot\mathbf{x}) \oplus \big(\mathrm{Im}\, L_0(\omega) \oplus \ker L_0(\omega)\big) \cos(\mathbf{k}\cdot\mathbf{x}) \Big)$$
$$= \mathrm{Im}\, L_0\big|_{X_\omega} \oplus \ker L_0.$$

Using this decomposition and Lemma 25, we find that

$$Y_0 = Y_\omega \oplus Y_\mathrm{r} = \Big(\mathrm{Im}\, L_0\big|_{X_\omega} \oplus \ker L_0 \Big) \oplus \mathrm{Im}\, L_0\big|_{X_\mathrm{r}} = \mathrm{Im}\, L_0 \oplus \ker L_0.$$

It follows that $\mathrm{Im}\, L_0$ is closed and $\operatorname{codim} \mathrm{Im}\, L_0 = \dim \ker L_0$, so that $L_0 : X_0 \to Y_0$ is a Fredholm operator of index zero. □

Because the kernel of L_0 is multidimensional, we can not use Theorem 24 directly. To overcome this problem, we recall that $\mathcal{G}$ (and hence L_0) is invariant under certain rotations (see below) and seek solutions to (2.36) in X_0 that have this rotational symmetry, denoting the relevant subspaces of $H^r_\mathrm{per}(\Gamma), \bar{H}^r_\mathrm{per}(\Gamma), X_0$ and Y_0 by $H^r_\mathrm{sym}(\Gamma), \bar{H}^r_\mathrm{sym}(\Gamma), X_\mathrm{sym}$ and Y_sym, so that

$$X_\mathrm{sym} = H^{s+1/2}_\mathrm{sym}(\Gamma) \times \bar{H}^s_\mathrm{sym}(\Gamma) \times H^s_\mathrm{sym}(\Gamma), \qquad Y_\mathrm{sym} = H^s_\mathrm{sym}(\Gamma) \times \bar{H}^{s-1}_\mathrm{sym}(\Gamma) \times H^{s-3/2}_\mathrm{sym}(\Gamma)$$

for $s > 5/2$. Note that X_sym and Y_sym are invariant under P_ω, so that according to the above analysis $L_0 : X_\mathrm{sym} \to Y_\mathrm{sym}$ is a Fredholm operator of index zero with one-dimensional kernel and

$$Y_\mathrm{sym} = \mathrm{Im}\, L_0 \oplus \ker L_0.$$

(i) For rectangles we work with the subspace

$$H^r_\mathrm{sym}(\Gamma) = \big\{ \zeta \in H^r_\mathrm{per}(\Gamma) :\ \zeta\,(x,z) = \zeta\,(z,-x) \big\}$$

of functions which are invariant under rotations through $\pi/2$. Functions $\zeta \in H^r_\mathrm{sym}(\Gamma)$ admit the unique trigonometric series representation

$$\begin{aligned}\zeta = C_0^\zeta + \sum_{n=1}^\infty \Bigg(& C_n^\zeta\,(\cos n\omega x + \cos n\omega z) \\ & + A_{n,n}^\zeta\,(\cos n\omega(x+z) + \cos n\omega(x-z)) \\ & + \sum_{m=1}^{n-1} \Big(A_{n,m}^\zeta\,(\cos(n\omega x + m\omega z) + \cos(m\omega x - n\omega z)) \\ & \qquad + A_{m,n}^\zeta\,(\cos(m\omega x + n\omega z) + \cos(n\omega x - m\omega z)) \Big) \Bigg),\end{aligned}$$

where $C_0^\zeta, C_n^\zeta, A_{n,m}^\zeta \in \mathbb{R}$ for $n, m \in \mathbb{N}$; the norm

$$\begin{aligned}\|\zeta\|^2_{\mathrm{sym},r} := C_0^{\zeta 2} + \sum_{n=1}^\infty \Bigg(& (1 + n^2\omega^2)^s C_n^{\zeta 2} + (1 + 2n^2\omega^2)^s A_{n,n}^{\zeta 2} \\ & + \sum_{m=1}^{n-1} \left(1 + \left(n^2 + m^2\right)\omega^2\right)^s \left(A_{n,m}^{\zeta 2} + A_{m,n}^{\zeta 2}\right) \Bigg)\end{aligned}$$

is equivalent to the norm $\|\cdot\|_r$ for this space.

(ii) For hexagons we work with the subspace

$$H^r_{\text{sym}}(\Gamma) = \left\{ \zeta \in H^r_{\text{per}}(\Gamma) : \ \zeta(x,z) = \zeta\left(\frac{1}{2}\left(x - \sqrt{3}z\right), \frac{1}{2}\left(\sqrt{3}x + z\right)\right)\right\}$$

of functions which are invariant under rotations through $\pi/3$. Functions $\zeta \in H^r_{\text{sym}}(\Gamma)$ admit the unique trigonometric series representation

$$\begin{aligned}
\zeta = C_0^\zeta &+ \sum_{n=1}^{\infty} C_n^\zeta \left(\cos n\omega x + \cos\frac{n}{2}\omega\left(x + \sqrt{3}z\right) + \cos\frac{n}{2}\omega\left(x - \sqrt{3}z\right)\right) \\
&+ \sum_{n=1}^{\infty} A_{n,n}^\zeta \left(\cos n\sqrt{3}\omega z + \cos\frac{n}{2}\omega\left(3x + \sqrt{3}z\right) + \cos\frac{n}{2}\omega\left(3x - \sqrt{3}z\right)\right) \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{n-1} \Bigg(A_{n,m}^\zeta \Big(\cos\left(n\omega x + \frac{m}{2}\omega\left(x + \sqrt{3}z\right)\right) \\
&\qquad\qquad + \cos\left(m\omega x + \frac{n}{2}\omega\left(x - \sqrt{3}z\right)\right) \\
&\qquad\qquad + \cos\left(\frac{n}{2}\omega\left(x + \sqrt{3}z\right) - \frac{m}{2}\omega\left(x - \sqrt{3}z\right)\right)\Big) \\
&\qquad + A_{m,n}^\zeta \Big(\cos\left(m\omega x + \frac{n}{2}\omega\left(x + \sqrt{3}z\right)\right) \\
&\qquad\qquad + \cos\left(n\omega x + \frac{m}{2}\omega\left(x - \sqrt{3}z\right)\right) \\
&\qquad\qquad + \cos\left(\frac{m}{2}\omega\left(x + \sqrt{3}z\right) - \frac{n}{2}\omega\left(x - \sqrt{3}z\right)\right)\Big)\Bigg),
\end{aligned}$$

where $C_0^\zeta, C_n^\zeta, A_{n,m}^\zeta \in \mathbb{R}$ for $n, m \in \mathbb{N}$; the norm

$$\begin{aligned}
\|\zeta\|^2_{\text{sym},r} := C_0^{\zeta^2} + \sum_{n=1}^{\infty} \Bigg(&(1 + n^2\omega^2)^s C_n^{\zeta^2} + (1 + 3n^2\omega^2)^s A_{n,n}^{\zeta 2} \\
&+ \sum_{m=1}^{n-1} (1 + \left(n^2 + nm + m^2\right)\omega^2)^s \left(A_{n,m}^{\zeta 2} + A_{m,n}^{\zeta 2}\right)\Bigg)
\end{aligned}$$

is equivalent to the norm $\|\cdot\|_r$ for this space.

(iii) For rolls we consider functions that are independent of the z-coordinate and lie in the subspace

$$H^r_{\text{sym}}(\Gamma) = \left\{ \zeta \in H^r_{\text{per}}(\Gamma) : \ \zeta(x) = \zeta(-x)\right\}$$

of functions which are invariant under rotations through π. Functions $\zeta \in H^r_{\text{sym}}(\Gamma)$ admit the unique trigonometric series representation

$$\zeta = \sum_{n=0}^{\infty} C_n^\zeta \cos n\omega x,$$

where $C_n^\zeta \in \mathbb{R}$ for $n \in \mathbb{N}_0$; the norm

$$\|\zeta\|^2_{\text{sym},r} := \sum_{n=0}^{\infty} (1 + n^2\omega^2)^s C_n^{\zeta^2}$$

is equivalent to the norm $\|\cdot\|_r$ for this space.

(Note that $C_0^\zeta = 0$ for $\zeta \in \bar{H}^r_{\rm sym}(\Gamma)$ in the above notation.) These restrictions ensure that $\dim \ker L_0 = 1$ with $\ker L_0 = \langle v_0 \rangle$, where $v_0 = \mathbf{v} e_1(x,z)$ with

$$e_1(x,z) = \begin{cases} \cos \omega x & \text{(rolls)} \\ \cos \omega x + \cos \omega z & \text{(rectangles)} \\ \cos \omega x + \cos \frac{\omega}{2}\left(x + \sqrt{3}z\right) + \cos \frac{\omega}{2}\left(x - \sqrt{3}z\right) & \text{(hexagons).} \end{cases}$$

The projection $P : Y_{\rm sym} \to Y_{\rm sym}$ onto $\ker L_0$ along $\operatorname{Im} L_0$ is given by

$$P(\chi, \Psi', \Psi)^{\rm T} = C^\star((\chi_{(\omega,0)}, \Psi'_{(\omega,0)}, \Psi_{(\omega,0)})^{\rm T} \cdot \mathbf{v}^\star)\mathbf{v}\, e_1(x,z), \tag{2.42}$$

where

$$\mathbf{v}^\star = \begin{pmatrix} \dfrac{\gamma_0 + \omega^2}{\mu_1 - 1} \\ \dfrac{\gamma_0 + \omega^2}{(\mu_1 - 1)^2 \omega}\left(S_1^{-1} \coth S_1 \tilde{\omega} + \coth \tilde{\omega}\right) \\ 1 \end{pmatrix},$$

$$C^\star = \mu_1 \omega S_1^{-1} \tanh S_1 \tilde{\omega} - \mu_1^2 S_1^{-1} \frac{\tanh S_1 \tilde{\omega}\big(\coth \tilde{\omega} + S_1 \coth S_1 \tilde{\omega}\big)}{\tanh \tilde{\omega}\big(\mu_1 \coth \tilde{\omega} + S_1 \coth S_1 \tilde{\omega}\big)} + 1$$

($\mathbf{v}^\star \in \mathbb{R}^3$ solves the equation $L_0(\omega)^{\rm T}\mathbf{v}^\star = \mathbf{0}$ and $C^\star = (\mathbf{v} \cdot \mathbf{v}^\star)^{-1}$).

Lemma 27. *The transversality condition* $P(\mathrm{d}_1\mathrm{d}_2\mathcal{G}[0,0,0;\gamma_0](v_0;1)) \neq 0$ *is satisfied.*

Proof. It follows from the calculation $\mathrm{d}_1\mathrm{d}_2\mathcal{G}[0,0,0;\gamma_0]((\eta, \Phi', \Phi)^{\rm T}; 1) = (0,0,-\eta)^{\rm T}$ and the formula (2.42) for P that

$$P(\mathrm{d}_1\mathrm{d}_2\mathcal{G}[0,0,0;\gamma_0](v_0;1)) = -\frac{C^\star}{(\mu_1 - 1)S_1} \tanh S_1 \tilde{\omega}(\mu_1 \coth \tilde{\omega} + S_1 \coth S_1 \tilde{\omega})\mathbf{v}\, e_1(x,z).$$

□

The facts established above confirm that the hypotheses of Theorem 24 are satisfied, an application of which yields the following result.

Theorem 28. *The point* $(\gamma_0, 0)$ *is a local bifurcation point for* (2.15)*, that is there exist* $\varepsilon > 0$*, open neighbourhoods* $W_{\rm sym}$ *of* $(\gamma_0, 0)$ *in* $\mathbb{R} \times X_{\rm sym}$ *and* $V_{\rm sym}$ *of* 0 *in* $X_{\rm sym}$ *and analytic functions* $w : (-\varepsilon, \varepsilon) \to V_{\rm sym}$*,* $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ *with* $\gamma(0) = \gamma_0, w(0) = v_0$ *such that* $\mathcal{G}(sw(s); \gamma(s)) = 0$ *for every* $s \in (-\varepsilon, \varepsilon)$*. Furthermore*

$$W_{\rm sym} \cap N = \{(\gamma(s), sw(s)) :\ 0 < |s| < \varepsilon\},$$

where

$$N = \{(\gamma, v) \in \mathbb{R} \times (V_{\rm sym} \setminus \{0\}) : \mathcal{G}(v;\gamma) = 0\}.$$

2.5 The bifurcating solution branches

In this section we examine the bifurcating solution branches identified in Theorem 28 by applying the following supplement to the Crandall-Rabinowitz theorem.

Theorem 29. *Suppose that the hypotheses of Theorem 24 hold. In the notation of that theorem, let* $Q : X \to X$ *be a projection with* $\operatorname{Im} Q = \ker L$ *and the Taylor series of the functions* $w : (-\varepsilon, \varepsilon) \to V$, $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}$ *be given by*

$$\lambda(s) = \lambda_0 + s\lambda_1 + s^2\lambda_2 + \dots, \qquad w(s) = v_0 + sw_1 + \dots,$$

where $\lambda_1, \lambda_2, \dots \in \mathbb{R}$ *and* $w_1, w_2, \dots \in \ker Q$.

(i) The coefficient λ_1 *satisfies the equation*

$$P\left(\frac{1}{2!}\mathrm{d}_1^2\mathcal{F}[0;\lambda_0](v_0,v_0)\right) + \lambda_1 P(\mathrm{d}_1\mathrm{d}_2\mathcal{F}[0;\lambda_0](v_0;1)) = 0$$

and the bifurcation is transcritical if λ_1 *is non-zero (see Figure 2.6).*

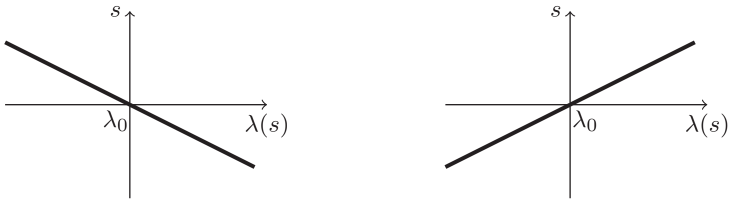


Figure 2.6:
Transcritical Crandall-Rabinowitz bifurcation for $\lambda_1 < 0$ (left) and $\lambda_1 > 0$ (right).

(ii) Suppose that λ_1 *is zero. The coefficient* λ_2 *satisfies the equation*

$$P\left(\mathrm{d}_1^2\mathcal{F}[0;\lambda_0](v_0,w_1) + \frac{1}{3!}\mathrm{d}_1^3\mathcal{F}[0;\lambda_0](v_0,v_0,v_0)\right) + \lambda_2 P\left(\mathrm{d}_1\mathrm{d}_2\mathcal{F}[0;\lambda_0](v_0;1)\right) = 0,$$

where $w_1 \in \ker Q$ *solves the equation*

$$\mathrm{d}_1\mathcal{F}[0;\lambda_0](w_1) = -\frac{1}{2!}\mathrm{d}_1^2\mathcal{F}[0;\lambda_0](v_0,v_0).$$

The bifurcation is supercritical for $\lambda_2 > 0$ *and subcritical for* $\lambda_2 < 0$ *(see Figure 2.7).*

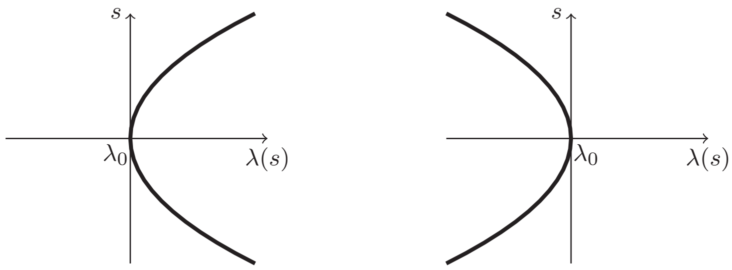


Figure 2.7:
Crandall-Rabinowitz bifurcation for $\lambda_1 = 0$, $\lambda_2 > 0$ (supercritical, left) and $\lambda_1 = 0$, $\lambda_2 < 0$ (subcritical,right).

To apply this theorem we write the Taylor series of the analytic functions $w:(-\varepsilon,\varepsilon)\to V_{\mathrm{sym}}$, $\gamma:(-\varepsilon,\varepsilon)\to\mathbb{R}$ given in Theorem 28 as

$$\gamma(s)=\gamma_0+s\gamma_1+s^2\gamma_2+\dots,\qquad w(s)=v_0+sw_1+\dots$$

with $\gamma_1,\gamma_2,\ldots\in\mathbb{R}$ and $w_1,w_2,\ldots\in\ker(I-P)$ and introduce the operators

$$L_1=\mathrm{d}_1\mathrm{d}_2\mathcal{G}[0,0,0;\gamma_0](\cdot;1),\qquad Q_0=\frac{1}{2!}\mathrm{d}_1^2\mathcal{G}[0,0,0;\gamma_0],\qquad C_0=\frac{1}{3!}\mathrm{d}_1^3\mathcal{G}[0,0,0;\gamma_0].$$

One finds that $L_1(v)=(0,0,-\eta)^{\mathrm{T}}$ and

$$Q_0(v,v)$$
$$=\begin{pmatrix} 0 \\ G_2'+G_2+\dot{\mu}_1H_2+\frac{1}{2}\left(\ddot{\mu}_1H_1^2+\dot{\mu}_1|\nabla\Phi|^2\right) \\ \frac{1}{2}\left(G_1'^2-|\nabla\Phi'|^2+(\mu_1-\dot{\mu}_1)H_1^2+\mu_1|\nabla\Phi|^2-2G_1H_1-\ddot{\mu}_1H_1^2-\dot{\mu}_1|\nabla\Phi|^2\right) \\ -\mu_1G_2'-G_2-\dot{\mu}_1H_2 \end{pmatrix}$$

$$C_0(v,v,v)$$
$$=\begin{pmatrix} 0 \\ G_3'+G_3+\dot{\mu}_1H_3+\ddot{\mu}_1H_1H_2+\frac{1}{2}\left(\ddot{\mu}_1-\dot{\mu}_1\right)|\nabla\Phi|^2H_1-\dot{\mu}_1\nabla\eta\cdot\nabla\Phi\ H_1+\frac{1}{6}\dddot{\mu}_1H_1^3 \\ -\mu_1G_3'-G_3-\dot{\mu}_1H_3-\ddot{\mu}_1H_1H_2-\frac{1}{2}\left(\ddot{\mu}_1-\dot{\mu}_1\right)|\nabla\Phi|^2H_1+\dot{\mu}_1\nabla\eta\cdot\nabla\Phi\ H_1-\frac{1}{6}\dddot{\mu}_1H_1^3 \\ +G_1'(G_2'-\nabla\eta\cdot\nabla\Phi')-H_1(G_2+\mu_1\nabla\eta\cdot\nabla\Phi)+(\mu_1-\dot{\mu}_1)H_1H_2+\frac{1}{3}(\dot{\mu}_1-\ddot{\mu}_1)H_1^3 \\ -G_1H_2+\eta_x^2\eta_{zz}+\eta_z^2\eta_{xx}-2\eta_x\eta_z\eta_{xz}-\frac{3}{2}|\nabla\eta|^2\eta \end{pmatrix},$$

where $v=(\eta,\Phi',\Phi)^{\mathrm{T}}$ and $\dddot{\mu}_1=\dddot{\mu}(1)$. Theorem 29 shows that

$$\gamma_1=-\frac{[Q_0(v_0,v_0)]_1\cdot\mathbf{v}^\star}{[L_1v_0]_1\cdot\mathbf{v}^\star,}$$

where

$$[\zeta]_1=\zeta_{(\omega,0)}=\int_\Gamma\zeta e_1$$

(with componentwise extension), and

$$\gamma_2=-\frac{[2Q_0(v_0,w_1)+C_0(v_0,v_0,v_0)]_1\cdot\mathbf{v}^\star}{[L_1v_0]_1\cdot\mathbf{v}^\star}$$

for $\gamma_1=0$, where $w_1\in\ker(I-P)$ solves the equation

$$L_0w_1=-Q_0(v_0,v_0).$$

A straightforward calculation shows $Q_0(v_0,v_0)$ can be written as a sum in which each summand is a constant vector multiplied by either e_1^2 or $|\nabla e_1|^2$. For hexagons we find that γ_1 generally does not vanish, while for rolls

$$\left[e_1^2\right]_1=\left[\frac{1}{2}+\frac{1}{2}\cos 2\omega x\right]_1=0,\qquad\left[|\nabla e_1|^2\right]_1=\left[\frac{1}{2}-\frac{1}{2}\cos 2\omega x\right]_1=0,$$

and for rectangles

$$
\begin{aligned}
\left[e_1^2\right]_1 &= \left[\frac{1}{2} + \cos\omega(x+z) + \cos\omega(x-z) + \frac{1}{2}(\cos 2\omega x + \cos 2\omega z)\right]_1 = 0, \\
\left[|\nabla e_1|^2\right]_1 &= \left[1 - \frac{1}{2}(\cos 2\omega x \ + \cos 2\omega z)\right]_1 = 0,
\end{aligned}
$$

so that in both cases $\gamma_1 = 0$.

For rectangles the solution to the equation

$$
\begin{aligned}
-L_0 w_1 &= Q(v_0, v_0) \\
&= \begin{pmatrix} 0 \\ B_{\sqrt{2},1} + B_{\sqrt{2},S} + D_{\sqrt{2}} \\ A_{\sqrt{2}} - \left(\mu_1 B_{\sqrt{2},1} + B_{\sqrt{2},S} + D_{\sqrt{2}}\right) \end{pmatrix} e_{\sqrt{2}}(x,z) \\
&\quad + \begin{pmatrix} 0 \\ B_{2,1} + B_{2,S} + D_2 \\ A_2 - (\mu_1 B_{2,1} + B_{2,S} + D_2) \end{pmatrix} e_2(x,z) \\
&\quad + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -\omega^2 \left(c_{\Phi'}^2 \left(1 - \tanh^2 \dfrac{\omega}{\beta_0}\right) - \mu_1 c_\Phi^2 \left(1 - \tanh^2 \dfrac{S_1\omega}{\beta_0}\right)\right) \end{pmatrix},
\end{aligned}
$$

where $(c_\eta, c_{\Phi'}, c_\Phi)^{\mathrm{T}} = \mathbf{v}_0$,

$$
e_{\sqrt{2}}(x,z) = \cos\omega(x+z) + \cos\omega(x-z), \qquad e_2(x,z) = \cos 2\omega x + \cos 2\omega z,
$$

and

$$
\begin{aligned}
A_2 &= \frac{\omega}{4}\left(c_{\Phi'}^2 \left(1 + \tanh^2 \frac{\omega}{\beta_0}\right) - \mu_1 c_\Phi^2 \left(1 + \tanh^2 \frac{S_1\omega}{\beta_0}\right)\right), \\
B_{2,1} &= -c_{\Phi'} c_\eta \omega^2 \left(1 - \tanh \frac{\omega}{\beta_0}\ \tanh \frac{2\omega}{\beta_0}\right), \\
B_{2,S} &= \mu_1 c_\Phi c_\eta \omega^2 \left(1 - \tanh \frac{S_1\omega}{\beta_0}\ \tanh \frac{2S_1\omega}{\beta_0}\right), \\
D_2 &= \frac{c_\Phi^2 S_1^2 \omega^2}{8\mu_1} \left(\dot\mu_1^2 + \mu_1 \ddot\mu_1 + 3\mu_1 \dot\mu_1\right) \tanh^3 \frac{S_1\omega}{\beta_0}\ \tanh \frac{S_1\omega}{\beta_0}, \\
A_{\sqrt{2}} &= \frac{\omega}{2}\left(c_{\Phi'}^2 \left(\tanh^2 \frac{\omega}{\beta_0}\right) - \mu_1 c_\Phi^2 \left(\tanh^2 \frac{S_1\omega}{\beta_0}\right)\right), \\
B_{\sqrt{2},1} &= -c_{\Phi'} c_\eta \omega^2 \left(1 - \sqrt{2}\tanh \frac{\omega}{\beta_0}\ \tanh \frac{\sqrt{2}\omega}{\beta_0}\right), \\
B_{\sqrt{2},S} &= \mu_1 c_\Phi c_\eta \omega^2 \left(1 - \sqrt{2}\tanh \frac{S_1\omega}{\beta_0}\ \tanh \frac{\sqrt{2}S_1\omega}{\beta_0}\right), \\
D_{\sqrt{2}} &= -\frac{c_\Phi^2 S_1^2 \omega^2}{8\mu_1} \Bigg(\Big(2\mu_1\ddot\mu_1 + 4\mu_1\dot\mu_1 + \left(2\mu_1\ddot\mu_1 - 4\dot\mu_1^2\right)\cosh \frac{2S_1\omega}{\beta_0}\Big)\Big(1 - \tanh^2 \frac{S_1\omega}{\beta_0}\Big)
\end{aligned}
$$

$$
+\left(\dot{\mu}_1^2-\mu_1\ddot{\mu}_1-\mu_1\dot{\mu}_1\right) \times 4\left(\frac{1-\tanh^2\dfrac{S_1\omega}{\beta_0}}{\cosh\dfrac{\sqrt{2}S_1\omega}{\beta_0}}+\sqrt{2}\tanh\frac{S_1\omega}{\beta_0}\tanh\frac{\sqrt{2}S_1\omega}{\beta_0}\right)\Bigg),
$$

is given by

$$
\begin{aligned}
w_1 = C_{\sqrt{2}} &\begin{pmatrix}
\begin{aligned}
&(\mu_1-1)\Big(\left(D_{\sqrt{2}}+B_{\sqrt{2},S}\right)\sqrt{2}\omega\tanh\frac{\sqrt{2}\omega}{\beta_0} \\
&\qquad -\mu_1 B_{\sqrt{2},1}\sqrt{2}S_1^{-1}\omega\tanh\frac{\sqrt{2}S_1\omega}{\beta_0}\Big) \\
&\quad +\left(\mu_1\sqrt{2}S_1^{-1}\omega\tanh\frac{\sqrt{2}S_1\omega}{\beta_0}+\sqrt{2}\omega\tanh\frac{\sqrt{2}\omega}{\beta_0}\right)A_{\sqrt{2}}
\end{aligned} \\[2ex]
\begin{aligned}
&\mu_1\left(\mu_1-1\right)^2 B_{\sqrt{2},1}\sqrt{2}S_1^{-1}\omega\tanh\frac{\sqrt{2}S_1\omega}{\beta_0} \\
&\quad -\mu_1\left(\mu_1-1\right)A_{\sqrt{2}}\sqrt{2}S_1^{-1}\omega\tanh\frac{\sqrt{2}S_1\omega}{\beta_0}-b_{\sqrt{2}}
\end{aligned} \\[2ex]
\begin{aligned}
&\left(\mu_1-1\right)^2\left(B_{\sqrt{2},S}+D_{\sqrt{2}}\right)\sqrt{2}\omega\tanh\frac{\sqrt{2}\omega}{\beta_0} \\
&\quad +\left(\mu_1-1\right)A_{\sqrt{2}}\sqrt{2}\omega\tanh\frac{\sqrt{2}\omega}{\beta_0}-b_{\sqrt{2}}
\end{aligned}
\end{pmatrix} e_{\sqrt{2}}(x,z) \\
+ C_2 &\begin{pmatrix}
\begin{aligned}
&(\mu_1-1)\Big(\left(D_2+B_{2,S}\right)2\omega\tanh\frac{2\omega}{\beta_0} \\
&\qquad -\mu_1 B_{2,1}2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}\Big) \\
&\quad +\left(\mu_1 2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}+2\omega\tanh\frac{2\omega}{\beta_0}\right)A_2
\end{aligned} \\[2ex]
\begin{aligned}
&\mu_1\left(\mu_1-1\right)^2 B_{2,1}2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0} \\
&\quad -\mu_1\left(\mu_1-1\right)A_2 2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}-b_2
\end{aligned} \\[2ex]
\begin{aligned}
&\left(\mu_1-1\right)^2\left(B_{2,S}+D_2\right)2\omega\tanh\frac{2\omega}{\beta_0} \\
&\quad +\left(\mu_1-1\right)A_2 2\omega\tanh\frac{2\omega}{\beta_0}-b_2
\end{aligned}
\end{pmatrix} e_2(x,z) \\
-\frac{\omega^2}{2\gamma_0}&\left(c_{\Phi'}^2\left(1-\tanh^2\frac{\omega}{\beta_0}\right)-\mu_1 c_\Phi^2\left(1-\tanh^2\frac{S_1\omega}{\beta_0}\right)\right)\begin{pmatrix}1\\0\\\mu_1-1\end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{\sqrt{2}} = -\Bigg(&\left(\gamma_0+\omega^2\right)\left(\omega\tanh\frac{\omega}{\beta_0}+\mu_1 S_1^{-1}\omega\tanh\frac{S_1\omega}{\beta_0}\right)\frac{2\tanh\dfrac{\sqrt{2}\omega}{\beta_0}\tanh\dfrac{\sqrt{2}S_1\omega}{\beta_0}}{\tanh\dfrac{\omega}{\beta_0}\tanh\dfrac{S_1\omega}{\beta_0}} \\
&-\left(\gamma_0+2\omega^2\right)\left(\sqrt{2}\omega\tanh\frac{\sqrt{2}\omega}{\beta_0}+\mu_1\sqrt{2}S_1^{-1}\omega\tanh\frac{\sqrt{2}S_1\omega}{\beta_0}\right)\Bigg)^{-1}, \\
C_2 = -\Bigg(&\left(\gamma_0+\omega^2\right)\left(\omega\tanh\frac{\omega}{\beta_0}+\mu_1 S_1^{-1}\omega\tanh\frac{S_1\omega}{\beta_0}\right)\frac{4\tanh\dfrac{2\omega}{\beta_0}\tanh\dfrac{2S_1\omega}{\beta_0}}{\tanh\dfrac{\omega}{\beta_0}\tanh\dfrac{S_1\omega}{\beta_0}} \\
&-\left(\gamma_0+4\omega^2\right)\left(2\omega\tanh\frac{2\omega}{\beta_0}+\mu_1 2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}\right)\Bigg)^{-1}, \\
b_{\sqrt{2}} = &\left(2\omega^2+\gamma_0\right)\left(B_{\sqrt{2},1}+B_{\sqrt{2},S}+D_{\sqrt{2}}\right), \\
b_2 = &\left(4\omega^2+\gamma_0\right)\left(B_{2,1}+B_{2,S}+D_2\right).
\end{aligned}
$$

For rolls the solution to the equation

$$
\begin{aligned}
-L_0 w_1 &= Q(v_0,v_0) \\
&= \begin{pmatrix} 0 \\ B_{2,1}+B_{2,S}+D_2 \\ A_2-\left(\mu_1 B_{2,1}+B_{2,S}+D_2\right)\end{pmatrix} e_2(x,z) \\
&\quad+\frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ -\omega^2\left(c_{\Phi'}^2\left(1-\tanh^2\dfrac{\omega}{\beta_0}\right)-\mu_1 c_\Phi^2\left(1-\tanh^2\dfrac{S_1\omega}{\beta_0}\right)\right)\end{pmatrix},
\end{aligned}
$$

where $e_2(x,z)=\cos 2\omega x$, is given by

$$
\begin{aligned}
w_1 = C_2 &\begin{pmatrix}
\begin{aligned}(\mu_1-1)\Big(&\left(D_2+B_{2,S}\right)2\omega\tanh\frac{2\omega}{\beta_0} \\ &-\mu_1 B_{2,1}2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}\Big) \\ &+\left(\mu_1 2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}+2\omega\tanh\frac{2\omega}{\beta_0}\right)A_2\end{aligned} \\[2ex]
\begin{aligned}\mu_1\left(\mu_1-1\right)^2 &B_{2,1}2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0} \\ &-\mu_1\left(\mu_1-1\right)A_2 2S_1^{-1}\omega\tanh\frac{2S_1\omega}{\beta_0}-b_2\end{aligned} \\[2ex]
\begin{aligned}\left(\mu_1-1\right)^2 &\left(B_{2,S}+D_2\right)2\omega\tanh\frac{2\omega}{\beta_0} \\ &+\left(\mu_1-1\right)A_2 2\omega\tanh\frac{2\omega}{\beta_0}-b_2\end{aligned}
\end{pmatrix} e_2(x,z) \\
&-\frac{\omega^2}{4\gamma_0}\left(c_{\Phi'}^2\left(1-\tanh^2\frac{\omega}{\beta_0}\right)-\mu_1 c_\Phi^2\left(1-\tanh^2\frac{S_1\omega}{\beta_0}\right)\right)\begin{pmatrix}1\\0\\\mu_1-1\end{pmatrix}.
\end{aligned}
$$

Using the above calculations, we can determine the nature of the local bifurcation at $\gamma = \gamma_0$. For hexagons we have that $\gamma_1 \neq 0$ in general, so that the bifurcation is transcritical, while for rectangles and rolls it is sub- or supercritical depending upon the sign of γ_2, which we now determine for small values of β_0; noting that the limit $\beta_0 \to 0$ corresponds to fluids of infinite depth. Recall that $\omega, \gamma_0, \beta_0$ satisfy the equation

$$\gamma_0 = f_{\beta_0}(\omega),$$

where

$$f_{\beta_0}(s) = g_{\beta_0}(s)s^2 \qquad g_{\beta_0}(s) = \frac{\mu_1(\mu_1-1)^2}{\mu_1 s \coth \dfrac{s}{\beta_0} + S_1 s \coth \dfrac{S_1 s}{\beta_0}} - 1$$

and ω is the unique maximum of the mapping f_{β_0}, and note that f_{β_0} converges pointwise to $f_0 : [0, \infty)$ given by

$$f_0(s) = \frac{\mu_1(\mu_1-1)^2}{\mu_1 + S_1} s - s^2$$

as $\beta_0 \to 0$. From the fact that γ_0, ω satisfy the equations

$$\gamma_0 = f_0(\omega), \qquad 0 = f_0'(\omega),$$

where the prime denotes differentiation with respect to s, it follows that

$$\omega = \frac{\mu_1(\mu_1-1)^2}{2(\mu_1 + S_1)} + o(1), \qquad \gamma_0 = \left(\frac{\mu_1(\mu_1-1)^2}{2(\mu_1 + S_1)}\right)^2 + o(1)$$

as $\beta_0 \to 0$. Attempting to compute explicit general expressions for γ_2 leads to unwieldy formulae (it appears more appropriate to calculate them numerically for a specific choice of μ, that is a specific magnetisation law). Here we confine ourselves to stating the values of the coefficients for two particular special cases.

(i) *Constant relative permeability μ (corresponding to a linear magnetisation law):* We find that

$$\begin{aligned}\gamma_2 = -\mu\frac{\mu-1}{\mu+1}\omega^2\Bigg(&-\mu C_2\Bigg(\left(8(\mu+1)\omega^2\frac{4\omega^2+\gamma_0}{t_2} - 4(\mu^2-1)^2\omega^3\right)(1-t_1t_2)^2 \\ &\qquad + \omega^3(\mu-1)^4\left(2(1+t_1^2)(1-t_1t_2) - \frac{1}{4}(1+t_1^2)^2\right)\Bigg) \\ &- \frac{\mu\omega^2}{4\gamma_0}\frac{(\mu-1)^3}{\mu+1}(1-t_1^2)^2 - \frac{(\mu+1)^2\omega}{\mu-1}t_1\left(\frac{3\,\omega^2}{8(\gamma_0+\omega^2)} - \frac{3}{2} + t_1t_2\right)\Bigg)\end{aligned}$$

for rolls and

$$\begin{aligned}\gamma_2 = -\mu\frac{\mu-1}{\mu+1}\omega^2\Bigg(&-\mu C_{\sqrt{2}}\Bigg(\left(8(\mu+1)\omega^2\frac{2\omega^2+\gamma_0}{t_{\sqrt{2}}} - 2\sqrt{2}(\mu^2-1)^2\omega^3\right)(1-\sqrt{2}t_1t_{\sqrt{2}})^2 \\ &\qquad - \frac{\omega^3(\mu-1)^4}{2}\sqrt{2}\left(t_1^4 - 4t_1^2(1-\sqrt{2}t_1t_{\sqrt{2}})\right)\Bigg)\end{aligned}$$

$$
\begin{aligned}
&- \mu C_2 \Bigg(\bigg(8(\mu+1)\omega^2 \frac{4\omega^2+\gamma_0}{t_2} - 4(\mu^2-1)^2\omega^3 \bigg) (1-t_1t_2)^2 \\
&\qquad + \omega^3(\mu-1)^4 \bigg(2(1+t_1^2)(1-t_1t_2) - \frac{1}{4}(1+t_1^2)^2 \bigg) \Bigg) \\
&- \frac{\omega(\mu+1)^2}{2(\mu-1)} t_1 \bigg(\frac{5\omega^2}{4(\gamma_0+\omega^2)} - 9 + 2t_1t_2 + 4\sqrt{2} t_1 t_{\sqrt{2}} \bigg) \\
&- \frac{\omega^2\mu(\mu-1)^3}{2\gamma_0(\mu+1)} (1-t_1^2)^2 \Bigg)
\end{aligned}
$$

for rectangles, where $t_1 = \tanh\tilde{\omega}$, $t_{\sqrt{2}} = \tanh\sqrt{2}\tilde{\omega}$, $t_2 = \tanh 2\tilde{\omega}$, $\tilde{\omega} = \omega/\beta_0$ and

$$
\begin{aligned}
C_{\sqrt{2}} &= \frac{1}{\sqrt{2}(\mu^2-1)\omega} \left(\sqrt{2}(\omega^2+\gamma_0)\frac{t_{\sqrt{2}}}{t_1} - \gamma_0 - 2\omega^2 \right)^{-1}, \\
C_2 &= \frac{1}{2(\mu^2-1)\omega} \left(2(\gamma_0+\omega^2)\frac{t_2}{t_1} - \gamma_0 - 4\omega^2 \right)^{-1}
\end{aligned}
$$

The sign of γ_2 clearly depends upon μ and $\tilde{\omega}$ (see Figure 2.8).

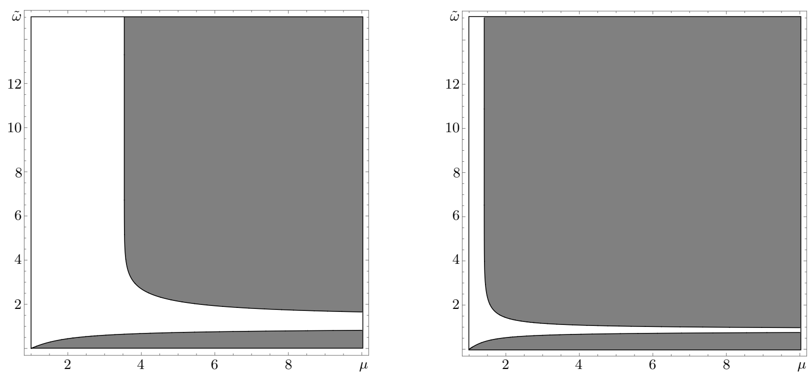


Figure 2.8:
The sign of the coefficient γ_2 as a function of μ and $\tilde{\omega}$ for a linear magnetisation law for rolls (left) and rectangles (right). The shaded and white areas show the regions in which the bifurcation is respectively super- and subcritical.

(ii) *Small values of β_0 (corresponding to deep fluids):* Abbreviating $\mu(1), \dot{\mu}(1), \ddot{\mu}(1), \dddot{\mu}(1)$ to respectively $\mu_1, \dot{\mu}_1, \ddot{\mu}_1, \dddot{\mu}_1$, one finds that

$$
\begin{aligned}
\gamma_2 = \Big(& t\big(2\mu_1^3\dddot{\mu}_1 + 2\mu_1^2\dot{\mu}_1\dddot{\mu}_1 - 2\mu_1^2\dddot{\mu}_1 - 2\mu_1\dot{\mu}_1\dddot{\mu}_1 - \tfrac{11}{2}\mu_1^2\ddot{\mu}_1^2 + \tfrac{11}{2}\mu_1\ddot{\mu}_1^2 + 42\mu_1^3\ddot{\mu}_1 + 49\mu_1^2\dot{\mu}_1\ddot{\mu}_1 - 8\mu_1^2\dot{\mu}_1 \\
&\quad + 29\mu_1\dot{\mu}_1^2\ddot{\mu}_1 - 10\mu_1^2\ddot{\mu}_1 + 15\mu_1\dot{\mu}_1\ddot{\mu}_1 + 3\dot{\mu}_1^2\ddot{\mu}_1 + 8\dot{\mu}_1\mu_1^3 + 16\mu_1^2\dot{\mu}_1^2 + 8\mu_1\dot{\mu}_1^3 - 16\mu_1\dot{\mu}_1^2 - 8\dot{\mu}_1^3\big) \\
& + 16\mu_1^4\ddot{\mu}_1 + 32\mu_1^3\dot{\mu}_1\ddot{\mu}_1 + 16\mu_1^2\dot{\mu}_1^2\ddot{\mu}_1 + 30\mu_1^2\dot{\mu}_1\ddot{\mu}_1 + 5\mu_1^2\ddot{\mu}_1^2 + 10\mu_1\dot{\mu}_1^2\ddot{\mu}_1 - 16\mu_1^2\ddot{\mu}_1 - 46\mu_1\dot{\mu}_1\ddot{\mu}_1 \\
& - 5\mu_1\ddot{\mu}_1^2 - 10\dot{\mu}_1^2\ddot{\mu}_1\Big) \frac{(\mu_1-1)^7\mu_1^2}{512t^6(\mu_1+\dot{\mu}_1)} \\
+ \Big(& t\big(96\mu_1^6 + 636\mu_1^5\dot{\mu}_1 + 1129\mu_1^4\dot{\mu}_1^2 + 850\mu_1^3\dot{\mu}_1^3 + 217\mu_1^2\dot{\mu}_1^4 - 512\mu_1^5 - 1720\mu_1^4\dot{\mu}_1 - 1810\mu_1^3\dot{\mu}_1^2 \\
&\quad - 740\mu_1^2\dot{\mu}_1^3 - 50\mu_1\dot{\mu}_1^4 + 96\mu_1^4 + 220\dot{\mu}_1\mu_1^3 + 73\mu_1^2\dot{\mu}_1^2 - 14\mu_1\dot{\mu}_1^3 - 7\dot{\mu}_1^4\big) \\
& + 88\mu_1^8 + 440\mu_1^7\dot{\mu}_1 + 880\mu_1^6\dot{\mu}_1^2 + 880\mu_1^5\dot{\mu}_1^3 + 440\mu_1^4\dot{\mu}_1^4 + 88\mu_1^3\dot{\mu}_1^5 - 256\mu_1^7 - 928\mu_1^6\dot{\mu}_1 - 1312\mu_1^5\dot{\mu}_1^2 \\
& - 864\mu_1^4\dot{\mu}_1^3 - 224\mu_1^3\dot{\mu}_1^4 - 80\mu_1^6 - 416\mu_1^5\dot{\mu}_1 - 614\mu_1^4\dot{\mu}_1^2 - 420\mu_1^3\dot{\mu}_1^3 - 102\mu_1^2\dot{\mu}_1^4 + 256\mu_1^5 + 672\mu_1^4\dot{\mu}_1 \\
& - 8\mu_1^4 + 460\mu_1^3\dot{\mu}_1^2 + 104\mu_1^2\dot{\mu}_1^3 - 20\mu_1\dot{\mu}_1^4 + 72\dot{\mu}_1\mu_1^3 + 194\mu_1^2\dot{\mu}_1^2 + 84\mu_1\dot{\mu}_1^3 + 10\dot{\mu}_1^4\Big) \frac{(\mu_1-1)^6\mu_1}{1024t^6(\mu_1+\dot{\mu}_1)} \\
+ o(1) &
\end{aligned}
$$

as $\beta_0 \to 0$ for rolls and

$$
\begin{aligned}
\gamma_2 = \Big(t\big(&(42\sqrt{2}-60)\mu_1\ddot{\mu}_1\dot{\mu}_1-(42\sqrt{2}-60)\mu_1^2\ddot{\mu}_1\dot{\mu}_1+(\tfrac{411}{\sqrt{2}}-289)\mu_1\ddot{\mu}_1^2+(4461\sqrt{2}-6190)\ddot{\mu}_1\mu_1^2\dot{\mu}_1\\
&+(3413\sqrt{2}-4766)\ddot{\mu}_1\mu_1\dot{\mu}_1^2-(429\sqrt{2}-558)\ddot{\mu}_1\mu_1\dot{\mu}_1+(168\sqrt{2}-240)\dot{\mu}_1^3+(168\sqrt{2}-240)\mu_1^2\dot{\mu}_1\\
&+(336\sqrt{2}-480)\mu_1\dot{\mu}_1^2+(1870\sqrt{2}-2580)\ddot{\mu}_1\mu_1^3+(82\sqrt{2}-172)\ddot{\mu}_1\mu_1^2-(1333\sqrt{2}-1886)\ddot{\mu}_1\dot{\mu}_1^2\\
&+(42\sqrt{2}-60)\mu_1^2\ddot{\mu}_1-(\tfrac{411}{\sqrt{2}}-289)\mu_1^2\ddot{\mu}_1^2-(168\sqrt{2}-240)\mu_1\dot{\mu}_1^3-(168\sqrt{2}-240)\dot{\mu}_1\mu_1^3\\
&-(336\sqrt{2}-480)\mu_1^2\dot{\mu}_1^2-(42\sqrt{2}-60)\mu_1^3\ddot{\mu}_1\big)\\
&+(976\sqrt{2}-1376)\mu_1^4\ddot{\mu}_1+(1952\sqrt{2}-2752)\dot{\mu}_1\mu_1^3\ddot{\mu}_1+(976\sqrt{2}-1376)\mu_1^2\dot{\mu}_1^2\ddot{\mu}_1+(445\sqrt{2}-622)\mu_1^2\ddot{\mu}_1^2\\
&+(366\sqrt{2}-468)\mu_1^2\dot{\mu}_1\ddot{\mu}_1-(1414\sqrt{2}-2020)\mu_1\dot{\mu}_1^2\ddot{\mu}_1-(445\sqrt{2}-622)\mu_1\ddot{\mu}_1^2-(976\sqrt{2}-1376)\mu_1^2\ddot{\mu}_1\\
&-(1342\sqrt{2}-1844)\mu_1\dot{\mu}_1\ddot{\mu}_1+(1414\sqrt{2}-2020)\dot{\mu}_1^2\ddot{\mu}_1\Big)\frac{(\mu_1-1)^7\mu_1^2}{512(10-7\sqrt{2})t^6(\mu_1+\dot{\mu}_1)}\\
+\Big(t\big(&(3300\sqrt{2}-3288)\dot{\mu}_1\mu_1^3+(5745\sqrt{2}-7110)\mu_1^2\dot{\mu}_1^2+(2410\sqrt{2}-3260)\mu_1\dot{\mu}_1^3\\
&+(6084\sqrt{2}-6168)\mu_1^5\dot{\mu}_1+(7377\sqrt{2}-6406)\mu_1^4\dot{\mu}_1^2-(3190\sqrt{2}-7300)\mu_1^3\dot{\mu}_1^3\\
&-(6799\sqrt{2}-10298)\mu_1^2\dot{\mu}_1^4-(58696\sqrt{2}-79024)\mu_1^4\dot{\mu}_1-(61154\sqrt{2}-81804)\mu_1^3\dot{\mu}_1^2\\
&-(11604\sqrt{2}-14456)\mu_1^2\dot{\mu}_1^3+(11550\sqrt{2}-16500)\mu_1\dot{\mu}_1^4+(672\sqrt{2}-448)\mu_1^6\\
&-(17408\sqrt{2}-23552)\mu_1^5-(2351\sqrt{2}-3322)\dot{\mu}_1^4+(672\sqrt{2}-448)\mu_1^4\big)\\
&+(2344\sqrt{2}-3056)\mu_1^3\dot{\mu}_1^5+(11720\sqrt{2}-15280)\mu_1^7\dot{\mu}_1+(23440\sqrt{2}-30560)\mu_1^6\dot{\mu}_1^2\\
&+(23440\sqrt{2}-30560)\mu_1^5\dot{\mu}_1^3+(11720\sqrt{2}-15280)\mu_1^4\dot{\mu}_1^4-(36384\sqrt{2}-49344)\mu_1^6\dot{\mu}_1\\
&-(60832\sqrt{2}-82880)\mu_1^5\dot{\mu}_1^2-(47328\sqrt{2}-64832)\mu_1^4\dot{\mu}_1^3-(14176\sqrt{2}-19520)\mu_1^3\dot{\mu}_1^4\\
&+(4344\sqrt{2}-6608)\dot{\mu}_1\mu_1^3-(478\sqrt{2}-628)\mu_1^2\dot{\mu}_1^2-(4860\sqrt{2}-7016)\mu_1\dot{\mu}_1^3+(2344\sqrt{2}-3056)\mu_1^8\\
&-(8704\sqrt{2}-11776)\mu_1^7-(15392\sqrt{2}-21440)\mu_1^5\dot{\mu}_1-(18422\sqrt{2}-25348)\mu_1^4\dot{\mu}_1^2\\
&-(8196\sqrt{2}-10904)\mu_1^3\dot{\mu}_1^3+(2410\sqrt{2}-3644)\mu_1^2\dot{\mu}_1^4+(27680\sqrt{2}-37568)\mu_1^4\dot{\mu}_1\\
&+(33644\sqrt{2}-46344)\mu_1^3\dot{\mu}_1^2+(18088\sqrt{2}-25328)\mu_1^2\dot{\mu}_1^3-(3700\sqrt{2}-5176)\mu_1\dot{\mu}_1^4\\
&-(4016\sqrt{2}-5664)\mu_1^6+(8704\sqrt{2}-11776)\mu_1^5+(1850\sqrt{2}-2588)\dot{\mu}_1^4\\
&+(1672\sqrt{2}-2608)\mu_1^4\Big)\frac{(\mu_1-1)^6\mu_1}{1024(10-7\sqrt{2})t^6(\mu_1+\dot{\mu}_1)}+o(1)
\end{aligned}
$$

as $\beta_0 \to 0$ for rectangles, where $t = \sqrt{\mu_1(\mu_1+\dot{\mu}_1)}+1$. (Note that

$$\omega = \frac{\mu_1(\mu_1-1)^2}{2(\mu_1+S_1)} + o(1), \qquad \gamma_0 = \frac{\mu_1^2(\mu_1-1)^4}{4(\mu_1+S_1)^2} + o(1)$$

as $\beta_0 \to 0$.)

We note in particular that for constant μ (corresponding to a linear magnetisation law), rolls bifurcate subcritically for $\mu < \mu_c^1$ and supercritically for $\mu > \mu_c^1$, while rectangles bifurcate subcritically for $\mu < \mu_c^2$ and supercritically for $\mu > \mu_c^2$, where

$$\mu_c^1 = \frac{21}{11} + \frac{8}{11}\sqrt{5}, \qquad \mu_c^2 = \frac{115+160\sqrt{2}+8\sqrt{184+11\sqrt{2}}}{141+128\sqrt{2}}.$$

The same values were obtained by Silber and Knobloch [42] in a discussion of normal forms for this bifurcation problem and confirmed by Lloyd, Gollwitzer, Rehberg and Richter [30] as part of a wider numerical and experimental investigation.

Figure 2.9 shows the sign of γ_2 for the Langevin magnetisation law

$$\mu(s) = 1 + \frac{M}{s}\left(\coth(\gamma s) - \frac{1}{\gamma s}\right) \tag{2.43}$$

in the limit $\beta_0 \to 0$, where M and χ_0 are respectively the magnetic saturation and initial susceptibility of the ferrofluid and $\gamma = 3\chi_0/M$.

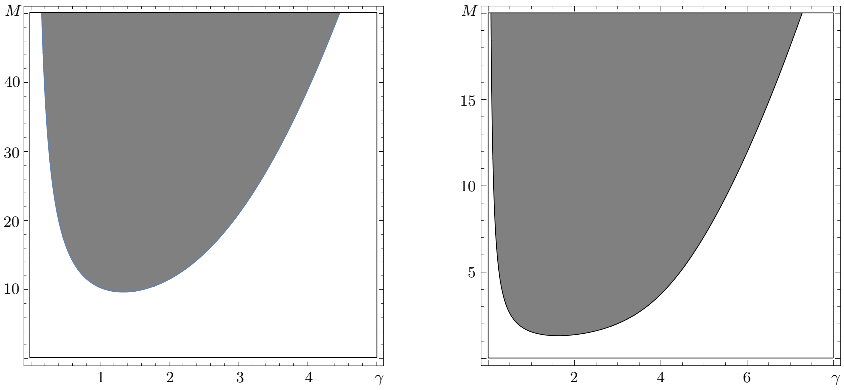


Figure 2.9:
The sign of the coefficient γ_2 as a function of M and γ for the Langevin magnetisation law (2.43) for rolls (left) and rectangles (right) in a ferrofluid of great depth. The shaded and white areas show the regions in which the bifurcation is respectively super- and subcritical.

Chapter 3

Symmetric corner defects

3.1 Spatial Hamiltonian formalism

In this chapter we present an existence theory for symmetric corner defects using a Hamiltonian version of a spatial dynamics theory for domain walls by Haragus and Iooss [17, 18]. To this end we formulate the ferrohydrodynamic problem as a spatial Hamiltonian system in which the horizontal spatial x-coordinate plays the role of time and (y, z) are the space-like coordinates (see Groves, Lloyd and Stylianou [14] for the corresponding formulation of the two-dimensional problem).

We seek solutions of (1.18)–(1.24) which are $\frac{2\pi}{\nu}$-periodic in the z-direction. It is convenient to to use the change of variable

$$\tilde{\eta}(x, \tilde{z}) = \eta(z), \quad \tilde{\phi}'(x, y, \tilde{z}) = \phi'(x, y, z), \quad \tilde{\phi}(x, y, \tilde{z}) = \phi(x, y, z),$$

where $\tilde{z} = \nu z$, so that these functions are 2π-periodic in $\tilde{z}$. Dropping the tildes for notational simplicity, we arrive at the equations

$$\phi'_{xx} + \phi'_{yy} + \nu^2\phi'_{zz} = 0 \qquad \text{in } \Omega', \tag{3.1}$$

$$\mu^\star(\phi_{xx} + \phi_{yy} + \nu^2\phi_{zz}) + \mu^\star_x\phi_x + \mu^\star_y(\phi_y + 1) + \nu^2\mu^\star_z\phi_z = 0 \qquad \text{in } \Omega \tag{3.2}$$

with boundary conditions

$$\phi'_y = 0 \qquad \text{for } y = \tfrac{1}{\beta_0}, \tag{3.3}$$

$$\mu^\star(\phi_y + 1) - \mu(1) = 0 \qquad \text{for } y = -\tfrac{1}{\beta_0}, \tag{3.4}$$

$$\phi' - \phi + (\mu(1) - 1)\,\eta = 0 \qquad \text{for } y = \eta(x, z), \tag{3.5}$$

$$\phi'_y + \mu(1) - \eta_x\phi'_x - \nu^2\eta_z\phi'_z - \mu^\star(\phi_y + 1 - \eta_x\phi_x - \nu^2\eta_z\phi_z) = 0 \qquad \text{for } y = \eta(x, z) \tag{3.6}$$

and

$$
\begin{aligned}
&- M(1) - \mu(1)\left(\frac{\mu(1)}{2} - 1\right) - \gamma\eta \\
&- \left(\frac{1}{2}\left(\phi_x'^2 + (\phi_y' + \mu(1))^2 + \nu^2\phi_z'^2\right) - M^\star\right) \\
&- \mu^\star(\phi_y + 1)(\phi_y + 1 - \eta_x\phi_x - \nu^2\eta_z\phi_z) \\
&+ (\phi_y' + \mu(1))(\phi_y' + \mu(1) - \eta_x\phi_x' - \nu^2\eta_z\phi_z') \\
&+ \left(\frac{\eta_x}{\sqrt{1+\eta_x^2+\nu^2\eta_z^2}}\right)_x + \nu^2\left(\frac{\eta_z}{\sqrt{1+\eta_x^2+\nu^2\eta_z^2}}\right)_z = 0 \qquad \text{for } y = \eta(x,z), \qquad (3.7)
\end{aligned}
$$

where

$$
\mu^\star = \mu\left(\left(\phi_x^2 + (\phi_y+1)^2 + \nu^2\phi_z^2\right)^{1/2}\right), \qquad M^\star = M\left(\left(\phi_x^2 + (\phi_y+1)^2 + \nu^2\phi_z^2\right)^{1/2}\right).
$$

Starting with a variational principle for the hydrodynamic problem, we perform a formal Legendre transform and show that solutions to the resulting Hamilton system lead to solutions of (3.1)–(3.7). We observe that equations (3.1)–(3.7) follow from the formal variational principle

$$
\delta \int_{-\infty}^{\infty} L(\eta, \phi', \phi, \eta_x, \phi_x', \phi_x)\,\mathrm{d}x = 0
$$

with Lagrangian

$$
\begin{aligned}
&L(\eta, \phi', \phi, \eta_x, \phi_x', \phi_x) \\
&= \int_0^{2\pi} \Bigg\{ \int_{-\frac{1}{\beta_0}}^{\eta} M\left(\left(\phi_x^2 + \nu^2\phi_z^2 + (\phi_y+1)^2\right)^{1/2}\right)\mathrm{d}y \\
&\qquad + \int_{\eta}^{\frac{1}{\beta_0}} \frac{1}{2}\left(\phi_x'^2 + \nu^2\phi_z'^2 + (\phi_y' + \mu(1))^2\right)\mathrm{d}y \\
&\qquad + \mu(1)\left(\phi\big|_{y=-\frac{1}{\beta_0}} - \phi'\big|_{y=\frac{1}{\beta_0}}\right) - \left(M(1) + \mu(1)\left(\frac{\mu(1)}{2} - 1\right)\right)\eta \\
&\qquad - \left(\frac{\gamma}{2}\eta^2 + (\sqrt{1+\eta_x^2+\nu^2\eta_z^2} - 1)\right)\Bigg\}\,\mathrm{d}z,
\end{aligned}
$$

where the variations are taken with respect to η, ϕ' and ϕ satisfying

$$
(\phi' - \phi)\big|_{y=0} + (\mu(1) - 1)\,\eta = 0.
$$

The next step is to use the ‘flattening’ transformation

$$
\begin{aligned}
\tilde{y} &= \frac{y-\eta}{1+\beta_0\eta}, \qquad & \chi(x,\tilde{y},z) &= \phi(x,y,z) \qquad & &\text{for } -\tfrac{1}{\beta_0} < y < \eta, \\
\tilde{y} &= \frac{y-\eta}{1-\beta_0\eta}, \qquad & \chi'(x,\tilde{y},z) &= \phi'(x,y,z) \qquad & &\text{for } \eta < y < \tfrac{1}{\beta_0}
\end{aligned}
$$

to map the variable domains

$$
\left\{(x,y,z):\, -\frac{1}{\beta_0} < y < \eta\right\}, \qquad \left\{(x,y,z):\, \eta < y < \frac{1}{\beta_0}\right\}
$$

into the rigid domains

$$\left\{(x,y,z):\, -\frac{1}{\beta_0}<y<0\right\}, \qquad \left\{(x,y,z):\, 0<y<\frac{1}{\beta_0}\right\},$$

the free interface $\{(x,y,z):\ y=\eta\}$ to $\{(x,y,z):\ y=0\}$ and transform L into

$$\begin{aligned}
&L(\eta,\chi',\chi,\eta_x,\chi'_x,\chi_x)\\
&=\int_0^{2\pi}\Bigg\{\int_{-\frac{1}{\beta_0}}^0 M\left(\left((\chi_x-K_1\eta_x\chi_y)^2+\nu^2\left(\chi_z-K_1\eta_z\chi_y\right)^2+(K_2\chi_y+1)^2\right)^{1/2}\right)K_2^{-1}\,\mathrm{d}y\\
&\qquad+\int_0^{\frac{1}{\beta_0}}\frac{1}{2}\left(\left(\chi'_x-K'_1\eta_x\chi'_y\right)^2+\nu^2\left(\chi'_z-K'_1\eta_z\chi'_y\right)^2+\left(K'_2\chi'_y+\mu(1)\right)^2\right)(K'_2)^{-1}\,\mathrm{d}y\\
&\qquad+\mu(1)\left(\chi\big|_{y=-\frac{1}{\beta_0}}-\chi'\big|_{y=\frac{1}{\beta_0}}\right)-\left(M(1)+\mu(1)\left(\frac{\mu(1)}{2}-1\right)\right)\eta\\
&\qquad-\left(\frac{\gamma}{2}\eta^2+(\sqrt{1+\eta_x^2+\nu^2\eta_z^2}-1)\right)\Bigg\}\,\mathrm{d}z
\end{aligned}$$

(with a slight abuse of notation), where

$$K_1=\frac{1+\beta_0 y}{1+\beta_0\eta}, \qquad K'_1=\frac{1-\beta_0 y}{1-\beta_0\eta}, \qquad K_2=\frac{1}{1+\beta_0\eta}, \qquad K'_2=\frac{1}{1-\beta_0\eta}$$

and the variations are taken with respect to η, χ' and χ satisfying

$$(\chi'-\chi)\big|_{y=0}+(\mu(1)-1)\,\eta=0.$$

We exploit this variational principle by deriving a canonical Hamiltonian formulation of (3.1)–(3.7) by means of the Legendre transform. We introduce the variables

$$\begin{aligned}
\rho=\frac{\delta L}{\delta\eta_x}=&-\frac{\eta_x}{\sqrt{1+\eta_x^2+\nu^2\eta_z^2}}-\int_{-\frac{1}{\beta_0}}^0\mu^\dagger\left(\chi_x-K_1\eta_x\chi_y\right)K_1K_2^{-1}\chi_y\,\mathrm{d}y\\
&-\int_0^{\frac{1}{\beta_0}}\left(\chi'_x-K'_1\eta_x\chi'_y\right)K'_1(K'_2)^{-1}\chi'_y\,\mathrm{d}y
\end{aligned}\tag{3.8}$$

and

$$\xi=\frac{\delta L}{\delta\chi_x}=f(\chi_x-K_1\eta_x\chi_y,\nu^2\left(\chi_z-K_1\eta_z\chi_y\right)^2+(K_2\chi_y+1)^2)K_2^{-1},\tag{3.9}$$

$$\xi'=\frac{\delta L}{\delta\chi'_x}=\left(\chi'_x-K'_1\eta_x\chi'_y\right)(K'_2)^{-1}\tag{3.10}$$

with

$$f(s,t)=\mu(\sqrt{s^2+t})s.$$

Noting that $f(0,1)=0$, $\partial_1 f(0,1)=\mu(1)>0$, we obtain the following result from the inverse-function theorem.

Proposition 30. *There exist open neighbourhoods V_1, V_3 of the origin and V_2 of unity in $\mathbb{R}$ such that $f(\cdot, t) : V_1 \to V_3$ is a bijection for each $t \in V_2$. Furthermore $f : V_1 \times V_2 \to V_3$ and $f^{-1} : V_3 \times V_2 \to V_1$ are analytic.*

Using Proposition 30, we find from (3.8)–(3.10) that

$$\eta_x = \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W, \qquad \chi'_x = K'_2\xi' + K'_1\eta_x\chi'_y, \qquad \chi_x = f^{-1}(K_2\xi) + K_1\eta_x\chi_y$$

and we define the Hamiltonian function by

$$\begin{aligned} H\left(\eta, \chi', \chi, \rho, \xi', \xi\right) = & \int_0^{2\pi}\int_{-\frac{1}{\beta_0}}^{0} \chi_x\xi \,\mathrm{d}y\,\mathrm{d}z + \int_0^{2\pi}\int_0^{\frac{1}{\beta_0}} \chi'_x\xi' \,\mathrm{d}y\,\mathrm{d}z \\ & + \int_0^{2\pi} \eta_x\rho \,\mathrm{d}z - L\left(\eta, \chi', \chi, \rho, \xi', \xi\right) \\ = & \int_{-\frac{1}{\beta_0}}^{0} \left(\frac{K_2}{\mu^\dagger}\xi^2 - M^\dagger K_2^{-1}\right) \mathrm{d}y + \int_0^{\frac{1}{\beta_0}} \frac{1}{2}\left(\xi'^2 - \chi_y'^2\right) K'_2 \,\mathrm{d}y \\ & - \int_0^{\frac{1}{\beta_0}} \frac{\nu^2}{2}\left(\chi'_z - K'_1\eta_z\chi'_y\right)^2 (K'_2)^{-1}\,\mathrm{d}y + \mu(1)\left(\chi\big|_{y=0} - \chi\big|_{y=-\frac{1}{\beta_0}}\right) \\ & + M(1)\eta + \frac{\gamma}{2}\eta^2 + \sqrt{1+\nu^2\eta_z^2}\sqrt{1-W^2} - \left(\frac{\mu(1)^2}{2} + 1\right), \end{aligned}$$

where

$$\begin{aligned} \mu^\dagger &= \mu\left(\left(f^{-1}(K_2\xi, \nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + (K_2\chi_y+1)^2\right)^2\right. \\ &\qquad \left. +\nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + (K_2\chi_y+1)^2\right)^{1/2}\Big), \\ M^\dagger &= M\left(\left(f^{-1}(K_2\xi, \nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + (K_2\chi_y+1)^2\right)^2\right. \\ &\qquad \left. +\nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + (K_2\chi_y+1)^2\right)^{1/2}\Big), \\ W &= -\rho - \int_{-\frac{1}{\beta_0}}^{0} K_1\xi\chi_y\,\mathrm{d}y - \int_0^{\frac{1}{\beta_0}} K'_1\xi'\chi'_y\,\mathrm{d}y. \end{aligned}$$

Hamilton's equations are given explicitly by

$$\eta_x = \frac{\delta H}{\delta\rho} = \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W, \tag{3.11}$$

$$\chi'_x = \frac{\delta H}{\delta\chi'} = K'_2\xi' + K'_1\frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W\chi'_y, \tag{3.12}$$

$$\chi_x = \frac{\delta H}{\delta\chi} = \frac{K_2\xi}{\mu^\dagger} + K_1\frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W\chi_y \tag{3.13}$$

and

$$
\begin{aligned}
\rho_x = &-\frac{\delta H}{\delta \eta} \\
= &\int_0^{\frac{1}{\beta_0}} \nu^2 \left(K_1'(K_2')^{-1}\left(\chi_z' - K_1'\eta_z\chi_y'\right)\chi_y'\right)_z \,\mathrm{d}y \\
&- \int_0^{\frac{1}{\beta_0}} \left(\frac{\beta_0}{2}\left(\nu^2\left(\chi_z'^2 + (K_1'\eta_z\chi_y')^2\right) + \left(\xi'^2 - \chi_y'^2\right)K_2'^2\right)\right) \mathrm{d}y \\
&+ \beta_0 \int_{-\frac{1}{\beta_0}}^0 \left(M^\dagger + \mu^\dagger\left(\nu^2\left(\chi_z - K_1\eta_z\chi_y\right)K_1\eta_z\chi_y - (K_2\chi_y + 1)K_2\chi_y\right)\right) \mathrm{d}y \\
&+ \int_{-\frac{1}{\beta_0}}^0 \nu^2\left(\mu^\dagger K_1 K_2^{-1}\left(\chi_z - K_1\eta_z\chi_y\right)\chi_y\right)_z \mathrm{d}y + \nu^2\left(\frac{\sqrt{1-W^2}}{\sqrt{1+\nu^2\eta_z^2}}\eta_z\right)_z \\
&- \beta_0 \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W \left(\int_0^{\frac{1}{\beta_0}} K_1'K_2'\xi'\chi_y' \,\mathrm{d}y - \int_{-\frac{1}{\beta_0}}^0 K_1K_2\xi\chi_y \,\mathrm{d}y\right) - M(1) - \gamma\eta \\
&-(\mu(1)-1)\left(-K_2'\chi_y' + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W K_2'\xi' + \nu^2(\chi_z' - K_2'\eta_z\chi_y')\eta_z\right)\Bigg|_{y=0}, \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
\xi_x' = -\frac{\delta H}{\delta \chi'} &= -K_2'\chi_{yy}' + \left(\nu^2 K_1'(K_2')^{-1}\left(\chi_z' - K_1'\eta_z\chi_y'\right)\eta_z\right)_y \\
&\quad - \nu^2\left((K_2')^{-1}\left(\chi_z' - K_1'\eta_z\chi_y'\right)\right)_z + \frac{\sqrt{1+\eta_z^2}}{\sqrt{1-W^2}} W\left(K_1'\xi'\right)_y, \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
\xi_x = -\frac{\delta H}{\delta \chi} &= \left(\mu^\dagger\left(\nu^2 K_1 K_2^{-1}\left(\chi_z - K_1\eta_z\chi_y\right)\eta_z - K_2\chi_y - 1\right)\right)_y \\
&\quad - \nu^2\left(\mu^\dagger K_2^{-1}\left(\chi_z - K_1\eta_z\chi_y\right)\right)_z + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W\left(K_1\xi\right)_y \qquad (3.16)
\end{aligned}
$$

with boundary conditions

$$
\chi_y' = 0 \qquad \text{for } y = \tfrac{1}{\beta_0}, \qquad (3.17)
$$

$$
\mu^\dagger K_2\chi_y + \mu^\dagger - \mu(1) = 0 \qquad \text{for } y = -\tfrac{1}{\beta_0}, \qquad (3.18)
$$

and

$$
\chi' - \chi + (\mu(1)-1)\,\eta = 0, \qquad (3.19)
$$

$$
\begin{aligned}
&\mu^\dagger\left(\nu^2\left(\chi_z - K_2\eta_z\chi_y\right)\eta_z - K_2\chi_y\right) - \left(\mu^\dagger - \mu(1)\right) \\
&\quad - \left(\nu^2\left(\chi_z' - K_2'\eta_z\chi_y'\right)\eta_z - K_2'\chi_y'\right) + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W\left(K_2\xi - K_2'\xi'\right) = 0, \qquad (3.20)
\end{aligned}
$$

$$
K_2'\xi' - \frac{K_2\xi}{\mu^\dagger} + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W\left(K_2'\chi_y' - K_2\chi_y + (\mu(1)-1)\right) = 0 \qquad (3.21)
$$

for $y = 0$.

We note that the equations (3.11)–(3.21) are reversible, that is invariant under the transformation $(\eta, \chi', \chi, \rho, \xi', \xi)(x) \mapsto S(\eta, \chi', \chi, \rho, \xi', \xi)(-x)$, where the *reverser* is defined by

$$
S(\eta, \chi', \chi, \rho, \xi', \xi) = (\eta, \chi', \chi, -\rho, -\xi', -\xi).
$$

Furthermore we note that these equations are invariant under the transformation $(\chi', \chi) \mapsto (\chi' + c, \chi + c), c \in \mathbb{R}$, the reflection $T : z \mapsto -z$ and the translation $R_\alpha : z \mapsto z + \alpha$, $\alpha \in \mathbb{R}$.

Now we discuss the rigorous mathematics. First we recall the differential-geometric definitions of a Hamiltonian system.

Definition 31. *A* ***Hamiltonian system*** *consists of a triple $(M, \Omega, \mathcal{H})$, where M is a manifold, $\Omega : TM \times TM \to \mathbb{R}$ is a closed, weakly nondegenerate bilinear form (the* ***symplectic 2–form****) and the* ***Hamiltonian*** *$\mathcal{H} : N \to \mathbb{R}$ is a smooth function on a manifold domain N of M (that is, a manifold N which is smoothly embedded in M and has the property that $TN\big|_n$ is densely embedded in $TM\big|_n$ for each $n \in N$). Its* ***Hamiltonian vector field*** *$v_\mathcal{H}$ with domain $\mathcal{D}(v_\mathcal{H}) \subseteq N$ is defined as follows. The point $n \in N$ belongs to $\mathcal{D}(v_\mathcal{H})$ with $v_\mathcal{H}\big|_n := w \in TM\big|_n$ if and only if*

$$\Omega\big|_n(w, v) = \mathbf{d}\mathcal{H}\big|_n(v)$$

for all tangent vectors $v \in TM\big|_n$ (by construction $\mathbf{d}\mathcal{H}\big|_n \in T^*N\big|_n$ admits a unique extension $\mathbf{d}\mathcal{H}\big|_n \in T^*M\big|_n$). ***Hamilton's equations*** *for $(M, \Omega, \mathcal{H})$ are the differential equations*

$$u_x = v_\mathcal{H}(u) := v_\mathcal{H}\big|_u$$

which determine the trajectories $u \in C^1(\mathbb{R}, M) \cap C(\mathbb{R}, N)$ of its Hamiltonian vector field.

To apply this definition to the Hamiltonian system derived above we define

$$\begin{aligned} X_t = \{&u \in H^{t+1}_{\mathrm{per}}(0, 2\pi) \times H^{t+1}_{\mathrm{per}}(\Sigma') \times H^{t+1}_{\mathrm{per}}(\Sigma) \times H^{t}_{\mathrm{per}}(0, 2\pi) \times H^{t}_{\mathrm{per}}(\Sigma') \times H^{t}_{\mathrm{per}}(\Sigma) : \\ &(\chi' - \chi)|_{y=0} + (\mu(1) - 1)\,\eta = 0\} \end{aligned}$$

for $t \in \mathbb{N}_0$, where $u = (\eta, \chi', \chi, \rho, \xi', \xi)$, $\Sigma' = (0, 2\pi) \times (0, \frac{1}{\beta_0})$, $\Sigma = (0, 2\pi) \times (-\frac{1}{\beta_0}, 0)$ and the subscript 'per' on the standard Sobolev spaces indicates that the functions are 2π-periodic in z. Let $M = X_0$ and N be a neighbourhood of the origin in X_1 such that

$$|\eta| < \tfrac{1}{\beta_0}, \qquad |W| < 1, \qquad K_2\chi_y(y) + 1 \in V_2, \qquad K_2\xi \in V_3$$

for all $z \in \mathbb{R}$ and $y \in [-\frac{1}{\beta_0}, 0]$. We observe that $H : N \to \mathbb{R}$ is an analytic function and N is a manifold domain of M. The formula

$$\begin{aligned} \Omega(u_1, u_2) \;=&\; \int_0^{2\pi} (\eta_1\rho_2 - \rho_1\eta_2)\,\mathrm{d}z + \int_0^{2\pi}\int_0^{\frac{1}{\beta_0}} (\chi_1'\xi_2' - \xi_1'\chi_2')\,\mathrm{d}y\,\mathrm{d}z \\ &+ \int_0^{2\pi}\int_{-\frac{1}{\beta_0}}^{0} (\chi_1\xi_2 - \xi_1\chi_2)\,\mathrm{d}y\,\mathrm{d}z \end{aligned}$$

defines a closed, weakly nondegenerate bilinear form $TM \times TM \to \mathbb{R}$, so that the triple (M, Ω, H) is a Hamiltonian system.

Lemma 32. *Consider the Hamiltonian system (M, Ω, H). The corresponding Hamiltonian vector field $v_H : \mathcal{D}(v_H) \to M$ is defined by the right-hand sides of equations* (3.11)–(3.16)*, where $\mathcal{D}(v_H) := \{u \in N : B(u) = 0\}$ and the analytic function $B : N \to (H^{1/2}_{\mathrm{per}}(0, 2\pi))^4$ is defined by the left-hand sides of the boundary conditions* (3.17)*,* (3.18)*,* (3.20)*,* (3.21)*.*

Proof. A direct calculation shows that

$$
\begin{aligned}
&\mathbf{d}H\big|_n(\hat{u}) \\
&= \int_0^{2\pi} \Bigg\{ \int_0^{\frac{1}{\beta_0}} \left(K_2'\xi' + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W K_1' \chi_y' \right) \hat{\xi}' \,\mathrm{d}y \\
&\qquad + \int_{-\frac{1}{\beta_0}}^0 \left(\frac{K_2\xi}{\mu^\dagger} + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W K_1 \chi_y \right) \hat{\xi} \,\mathrm{d}y \\
&\qquad + \int_0^{\frac{1}{\beta_0}} \Bigg(K_2'\chi_{yy}' + \nu^2 \left(\left(\chi_z' - K_1'\eta_z\chi_y'\right)(K_2')^{-1} \right)_z \\
&\qquad\qquad - \nu^2 \left(\left(\chi_z' - K_1'\eta_z\chi_y'\right)(K_2')^{-1} K_1'\eta_z \right)_y - \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W (K_1\xi)_y \Bigg) \hat{\chi}' \,\mathrm{d}y \\
&\qquad + \int_{-\frac{1}{\beta_0}}^0 \Bigg(\left(\mu^\dagger K_2^{-1} \left(\left(K_2\chi_y + 1\right) K_2 - \nu^2 \left(\chi_z - K_1\eta_z\chi_y\right) K_1\eta_z \right) \right)_y \\
&\qquad\qquad + \nu^2 \left(\mu^\dagger K_2^{-1} \left(\chi_z - K_1\eta_z\chi_y\right) \right)_z - \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W (K_1\xi)_y \Bigg) \hat{\chi} \,\mathrm{d}y \\
&\qquad + \Bigg(\int_0^{\frac{1}{\beta_0}} \frac{1}{2} \left(\xi'^2 - \chi_y'^2 \right) \beta_0 K_2'^2 \,\mathrm{d}y - \int_{-\frac{1}{\beta_0}}^0 M^\dagger \beta_0 \,\mathrm{d}y \\
&\qquad\quad - \int_{-\frac{1}{\beta_0}}^0 \nu^2 \left(\mu^\dagger K_2^{-1} \left(\chi_z - K_1\eta_z\chi_y\right) K_1\chi_y \right)_z \,\mathrm{d}y \\
&\qquad\quad - \int_{-\frac{1}{\beta_0}}^0 \mu^\dagger \left(\nu^2 \left(\chi_z - K_1\eta_z\chi_y\right) \beta_0 K_1\eta_z\chi_y - \left(K_2\chi_y + 1\right) \beta_0 K_2\chi_y \right) \,\mathrm{d}y \\
&\qquad\quad + \int_0^{\frac{1}{\beta_0}} \left(\frac{\beta_0}{2} \left(\nu^2\chi_z'^2 - \nu^2 (K_1'\eta_z\chi_y')^2 \right) - \nu^2 \left(\left(\chi_z' - K_1'\eta_z\chi_y'\right)(K_2')^{-1} K_1'\chi_y' \right)_z \right) \mathrm{d}y \\
&\qquad\quad + M(1) + \gamma\eta - \nu^2 \left(\frac{\sqrt{1-W^2}}{\sqrt{1+\nu^2\eta_z^2}} \eta_z \right)_z \\
&\qquad\quad + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W \beta_0 \left(\int_0^{\frac{1}{\beta_0}} K_1' K_2' \xi' \chi_y' \,\mathrm{d}y - \int_{-\frac{1}{\beta_0}}^0 K_1 K_2 \xi \chi_y \,\mathrm{d}y \right) \\
&\qquad\quad + (\mu(1) - 1) \left(\nu^2 \left(\chi_z' - K_1'\eta_z\chi_y'\right) \eta_z - K_2'\chi_y' + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W K_2' \xi' \right) \Bigg) \hat{\eta} \\
&\qquad + \Bigg(\frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W (K_2\xi - K_2'\xi') - \nu^2 \left(\chi_z' - K_1'\eta_z\chi_y'\right) \eta_z + K_2'\chi_y' \\
&\qquad\quad + \mu^\dagger \left(\nu^2 \left(\chi_z - K_2\eta_z\chi_y\right) \eta_z - \left(K_2\chi_y + 1\right) \right) + \mu(1) \Bigg) \hat{\chi}\big|_{y=0} \\
&\qquad + \left(\mu^\dagger \left(K_2\chi_y + 1\right) - \mu(1) \right) \hat{\chi}\big|_{y=-\frac{1}{\beta_0}} - K_2'\chi_y'\hat{\chi}'\big|_{y=\frac{1}{\beta_0}} + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}} W \hat{\rho} \Bigg\} \,\mathrm{d}z
\end{aligned}
$$

for $n = (\eta, \chi', \chi, \rho, \xi', \xi)$ and $\hat{u} = (\hat{\eta}, \hat{\chi}', \hat{\chi}, \hat{\rho}, \hat{\xi}', \hat{\xi})$, where we have integrated by parts and we used the facts that

$$\frac{\delta}{\delta\eta}K_2 = -\beta_0 K_2^2, \qquad \frac{\delta}{\delta\eta}K_1 = -\beta_0 K_1 K_2, \qquad \frac{\delta}{\delta\eta}K_2^{-1} = \beta_0,$$
$$\frac{\delta}{\delta\eta}K_2' = \beta_0 K_2'^2, \qquad \frac{\delta}{\delta\eta}K_1' = \beta_0 K_1' K_2', \qquad \frac{\delta}{\delta\eta}(K_2')^{-1} = -\beta_0$$

and

$$\hat{\chi}'\big|_{y=0} = \hat{\chi}\big|_{y=0} - (\mu(1)-1)\,\hat{\eta}.$$

Note that $\mathbf{d}H\big|_n \in T^*N\big|_n \cong X_1$ admits a unique extension $\mathbf{d}H\big|_n \in T^*M\big|_n \cong X_0$ which depends analytically upon $n \in N$. Recall that the point $n \in N$ belongs to $\mathcal{D}(v_H)$ with $v_H\big|_n = \bar{v}\big|_n$ if and only if

$$\Omega(\bar{v}\big|_n, \hat{u}) = \mathbf{d}H\big|_n(\hat{u})$$

for all $\hat{u} \in TM\big|_n$. Using this criterion and the above calculation we obtain equations (3.11)–(3.20). The condition $(\chi' - \chi)|_{y=0} + (\mu(1)-1)\,\eta = 0$ in the definition of M forces one to impose the compatibility condition (3.21) so that $v_H\big|_n$ maps N into M. □

3.2 Centre-manifold reduction

Next we introduce a parametrised, Hamiltonian version of a reduction principle for quasilinear evolutionary equations presented by Mielke [34, Theorem 4.1] (see Mielke [33, Theorem 4.1] and Buffoni, Groves and Toland [4, Theorem 4.1]). Using this result we reduce the problem for small solutions to equations (3.11)–(3.21) to an equivalent Hamiltonian system with finitely many degrees of freedom.

Theorem 33. *Consider the differential equation*

$$\dot{u} = \mathcal{L}u + \mathcal{N}(u;\lambda), \tag{3.22}$$

which represents Hamilton's equations for the reversible Hamiltonian system $(M, \Omega^\lambda, \mathcal{H}^\lambda)$*. Here* u *belongs to a Hilbert space* $\mathcal{X}$*,* $\lambda \in \mathbb{R}^l$ *is a parameter and* $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ *is a densely defined, closed linear operator. Regarding* $\mathcal{D}(\mathcal{L})$ *as a Hilbert space equipped with the graph norm, suppose that* 0 *is an equilibrium point of* (3.22) *when* $\lambda = 0$ *and that*

(H1) The part of the spectrum $\sigma(\mathcal{L})$ *of* $\mathcal{L}$ *which lies on the imaginary axis of a finite number of eigenvalues of finite multiplicity and is separated from the rest of* $\sigma(\mathcal{L})$ *in the sense of Kato, so that* $\mathcal{X}$ *admits the decomposition* $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$*, where* $\mathcal{X}_1 = \mathcal{P}(\mathcal{X})$*,* $\mathcal{X}_2 = (I - \mathcal{P})(\mathcal{X})$ *and* $\mathcal{P}$ *is the spectral projection corresponding the purely imaginary part of* $\sigma(\mathcal{L})$*.*

(H2) The operator $\mathcal{L}_2 = \mathcal{L}|_{\mathcal{X}_2}$ *satisfies the estimate*

$$\left\|(\mathcal{L}_2 - \mathrm{i}sI)^{-1}\right\|_{\mathcal{X}_2\to\mathcal{X}_2} \lesssim \frac{1}{1+|s|}, \qquad s \in \mathbb{R}.$$

(H3) There exist a natural number k and neighbourhoods $\Lambda \subset \mathbb{R}^l$ of 0 and $U \subset \mathcal{D}(\mathcal{L})$ of 0 such that $\mathcal{N}$ is $(k+1)$ times continuously differentiable on $U \times \Lambda$, its derivatives are bounded and uniformly continuous on $U \times \Lambda$ and $\mathcal{N}(0;0) = 0, \mathrm{d}_1\mathcal{N}[0;0] = 0$.

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ of 0 and $\tilde{U}_1 \subset U \cap \mathcal{X}_1$, $\tilde{U}_2 \subset U \cap \mathcal{X}_2$ of 0 and a reduction function $r : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2$ with the following properties. The reduction function r is k times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$ and $r(0;0) = 0$, $\mathrm{d}_1 r[0;0] = 0$. The graph $\tilde{M}^\lambda = \{u_1 + r(u_1;\lambda) \in \mathcal{X}_1 \oplus \mathcal{X}_2 : u_1 \in \tilde{U}_1\}$ is a Hamiltonian centre manifold for (3.22)*, so that*

(i) $\tilde{M}^\lambda$ is a locally invariant manifold of (3.22)*: trough every point in $\tilde{M}^\lambda$ there passes a unique solution of* (3.22) *that remains on $\tilde{M}^\lambda$ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$.*

(ii) Every small bounded solution $u(x)$, $x \in \mathbb{R}$ of (3.22) *that satisfies $(u_1(x), u_2(x)) \in \tilde{U}_1 \times \tilde{U}_2$ lies completely in $\tilde{M}^\lambda$.*

(iii) Every solution $u_1 : (x_1, x_2) \to \tilde{U}_1$ of the reduced equation

$$\dot{u}_1 = \mathcal{L}u_1 + \mathcal{P}\mathcal{N}(u_1 + r(u_1;\lambda);\lambda) \tag{3.23}$$

generates a solution

$$u(x) = u_1(x) + r(u_1(x);\lambda)$$

of the full equation (3.22)*.*

(iv) $\tilde{M}^\lambda$ is a symplectic submanifold of M and the flow determined by the Hamiltonian system $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{\mathcal{H}}^\lambda)$, where the tilde denotes restriction to $\tilde{M}^\lambda$, coincides with the flow on $\tilde{M}^\lambda$ determined by $(M, \Omega^\lambda, \mathcal{H}^\lambda)$. The reduced equation (3.23) *is reversible and represents Hamilton's equations for $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{\mathcal{H}}^\lambda)$.*

We introduce the parameter ε by setting $\gamma = \gamma_0 + \varepsilon$ and denote the Hamiltonian H with this parameter choice by H^ε, so that β_0, γ_0 are fixed values of the dimensionless variables β and γ (to be selected later) and ε plays the role of a bifurcation parameter. We consider the equation

$$u_x = v_{H^\varepsilon}(u), \tag{3.24}$$

but cannot use Theorem 33 directly due to the nonlinearity of the boundary conditions in the definition of the domain $\mathcal{D}(v_{H^\varepsilon})$ of the Hamiltonian vector field. To overcome this difficulty we use a change of variable $\tilde{u} = G(u)$ which transforms the nonlinear boundary conditions in $\mathcal{D}(v_{H^\varepsilon})$ into their linearisations $B_l\tilde{u} = 0$, where $B_l = \mathrm{d}B[0]$, $u = (\eta, \chi', \chi, \rho, \xi', \xi)^{\mathrm{T}}$ and $\tilde{u} = (\eta, \tau', \tau, \rho, \zeta', \zeta)^{\mathrm{T}}$.

Define analytic functions $F_1, F_1', F_2, F_2' : N \to \mathbb{R}$ by

$$\begin{aligned}(\mu(1) + \dot{\mu}(1))F_1(u) = (1 + \beta_0 y)\left(\mu^\dagger \nu^2 (\chi_z - K_2\eta_z\chi_y)\,\eta_z + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}WK_2\xi\right)\\ - (K_2\chi_y - \mu(1)\chi_y) - (\mu^\dagger - (\mu(1) + \dot{\mu}(1)\chi_y)),\end{aligned}$$

$$\begin{aligned}F_1'(u) &= (1-\beta_0 y)\left(\nu^2\left(\chi_z' - K_2'\eta_z\chi_y'\right)\eta_z + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}WK_2'\xi'\right) \\ &\quad - (K_2'\chi_y' - \chi_y'), \\ \frac{F_2(u)}{\mu(1)} &= \frac{K_2\xi}{\mu^\dagger} - \frac{\xi}{\mu(1)} + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}WK_2\chi_y, \\ F_2'(u) &= K_2'\xi' - \xi' + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W\left(K_2'\chi_y' + (\mu(1)-1)\right) + (\mu(1)-1)\rho,\end{aligned}$$

so that the boundary conditions (3.17), (3.18), (3.20), (3.21) can be written as

$$\begin{aligned}\chi_y' &= F_1'(u) \qquad \text{for } y = \tfrac{1}{\beta_0}, \\ (\mu(1)+\dot\mu(1))\chi_y &= (\mu(1)+\dot\mu(1))F_1(u) \qquad \text{for } y = -\tfrac{1}{\beta_0},\end{aligned}$$

and

$$\begin{aligned}\chi_y' - (\mu(1)+\dot\mu(1))\chi_y - (F_1'(u) - (\mu(1)+\dot\mu(1))F_1(u)) &= 0, \\ \frac{\xi}{\mu(1)} + \frac{F_2(u)}{\mu(1)} - (\xi' + F_2'(u) - (\mu(1)-1)\rho) &= 0\end{aligned}$$

for $y=0$. We use the change of variable

$$\tau = \chi + \psi_y, \qquad \tau' = \chi' + \psi_y', \qquad \zeta = \xi + F_2(u), \zeta' = \xi' + F_2'(u),$$

where $\psi \in H^3_{\mathrm{per}}(\Sigma), \psi' \in H^3_{\mathrm{per}}(\Sigma')$ solve the boundary-value problems

$$\begin{aligned}-(\psi_{yy}+\nu^2\psi_{zz}) &= F_1(u) && \text{in } \Sigma, & -(\psi_{yy}'+\nu^2\psi_{zz}') &= F_1'(u) && \text{in } \Sigma', \\ \psi &= 0 && \text{for } y=0, & \psi' &= 0 && \text{for } y=0, \\ \psi &= 0 && \text{for } y=-\tfrac{1}{\beta_0}, & \psi' &= 0 && \text{for } y=\tfrac{1}{\beta_0}.\end{aligned}$$

The formula G with $G(u) = \tilde u$ defines an analytic, near identity mapping $N \to X_1$. Its derivative is given by

$$\mathrm{d}G[u](\hat u) = (\hat\eta, \hat\chi' + \hat\psi_y', \hat\chi + \hat\psi_y, \hat\rho, \hat\xi' + \mathrm{d}F_2'[u](\hat u), \hat\xi + \mathrm{d}F_2[u](\hat u)),$$

where $\hat u = (\hat\eta, \hat\chi', \hat\chi, \hat\rho, \hat\xi', \hat\xi)$ and $\hat\psi \in H^3(\Sigma), \hat\psi' \in H^3(\Sigma')$ solve the boundary-value problems

$$\begin{aligned}-(\hat\psi_{yy}+\nu^2\hat\psi_{zz}) &= \mathrm{d}F_1[u](\hat u) && \text{in } \Sigma, \\ \hat\psi &= 0 && \text{for } y=0, \\ \hat\psi &= 0 && \text{for } y=-\tfrac{1}{\beta_0}, \\ -(\hat\psi_{yy}'+\nu^2\hat\psi_{zz}') &= \mathrm{d}F_1'[u](\hat u) && \text{in } \Sigma', \\ \hat\psi' &= 0 && \text{for } y=0, \\ \hat\psi' &= 0 && \text{for } y=\tfrac{1}{\beta_0}\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{d}F_1[u](\hat{u}) = {}& \frac{(1+\beta_0 y)}{\mu(1)+\dot{\mu}(1)}\nu^2\left(\chi_z - K_2\eta_z\chi_y\right)\eta_z \mathrm{d}\mu^\dagger[u]\hat{u} \\
&+ \frac{(1+\beta_0 y)}{\mu(1)+\dot{\mu}(1)}\left(\mu^\dagger\nu^2\left(\hat{\chi_z} - K_2\eta_z\chi_y + \beta_0 K_2^2\eta_z\chi_y\hat{\eta} - K_2\hat{\eta_z}\chi_y - K_2\eta_z\hat{\chi_y}\right)\eta_z\right) \\
&+ \frac{(1+\beta_0 y)}{\mu(1)+\dot{\mu}(1)}\frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W\left(K_2\hat{\xi} - \beta_0 K_2^2\xi\hat{\eta}\right) \\
&+ \frac{(1+\beta_0 y)}{\mu(1)+\dot{\mu}(1)}\left(\frac{\nu^2 W K_2\xi\eta_z\hat{\eta_z}}{\sqrt{1+\nu^2\eta_z^2}\sqrt{1-W^2}} + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}^3}K_2\xi \mathrm{d}W[u](\hat{u})\right) \\
&- \frac{K_2\hat{\chi_y} - \beta_0 K_2^2\chi_y\hat{\eta} - \mu(1)\hat{\chi_y}}{\mu(1)+\dot{\mu}(1)} - \frac{\mathrm{d}\mu^\dagger[u](\hat{u}) - \dot{\mu}(1)\hat{\chi_y}}{\mu(1)+\dot{\mu}(1)}, \\
\mathrm{d}F_1'[u](\hat{u}) = {}& (1-\beta_0 y)\nu^2\left(\hat{\chi_z'}\eta_z + \left(\chi_z' - K_2\eta_z\chi_y'\right)\hat{\eta_z}\right) \\
&- (1-\beta_0 y)\nu^2\left(\left(\beta_0 K_2'^2\eta_z\chi_y'\hat{\eta} + K_2'\hat{\eta_z}\chi_y' + K_2'\eta_z\hat{\chi_y'}\right)\right)\eta_z \\
&+ (1-\beta_0 y)\frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}}W\left(K_2'\hat{\xi'} + \beta_0 K_2'^2\xi'\hat{\eta}\right) \\
&+ (1-\beta_0 y)\left(\frac{\nu^2 W K_2'\xi'\eta_z\hat{\eta_z}}{\sqrt{1+\nu^2\eta_z^2}\sqrt{1-W^2}} + \frac{\sqrt{1+\nu^2\eta_z^2}}{\sqrt{1-W^2}^3}K_2'\xi' \mathrm{d}W[u](\hat{u})\right) \\
&- (\beta_0 K_2'^2\chi_y'\hat{\eta} + K_2'\hat{\chi_y'} - \hat{\chi_y'}), \\
\mathrm{d}\mu^\dagger[u](\hat{u}) = {}& (\mu^\dagger)'\left(\frac{K_2^2\xi^2}{\mu^{\dagger 2}} + \nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + \left(K_2\chi_y+1\right)^2\right)^{-1/2} \\
&\times 2\Bigg(\nu^2\left(\chi_z - K_1\eta_z\chi_y\right)\left(\hat{\chi_z} + K_1K_2\eta_z\chi_y\hat{\eta} - K_1\hat{\eta_z}\chi_y - K_1\eta_z\hat{\chi_y}\right) \\
&\quad + \frac{K_2\xi\hat{\xi} - \beta_0 K_2^3\xi^2\hat{\eta}}{\mu^\dagger + \dfrac{(\mu^\dagger)'K_2^2\xi^2}{\left((\mu^\dagger)^{-2}K_2^2\xi^2 + \nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + \left(K_2\chi_y+1\right)^2\right)^{1/2}}} \\
&\quad + \left(K_2\chi_y+1\right)\left(K_2\hat{\chi_y} - \beta_0 K_2^2\chi_y\hat{\eta}\right)\Bigg), \\
(\mu^\dagger)' = {}& \mu'\Big(\left(f^{-1}(K_2\xi, \nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + (K_2\chi_y+1)^2\right)^2 \\
&\quad + \nu^2\left(\chi_z - K_1\eta_z\chi_y\right)^2 + (K_2\chi_y+1)^2\big)^{1/2}\Big), \\
\mathrm{d}W[u](\hat{u}) = {}& -\hat{\rho} - \int_{-\frac{1}{\beta_0}}^{0}\left(-\beta_0 K_1K_2\xi\chi_y\hat{\eta} + K_1\hat{\xi}\chi_y + K_1\xi\hat{\chi_y}\right)\mathrm{d}y \\
&- \int_0^{\frac{1}{\beta_0}}\left(K_1'K_2'\xi'\chi_y'\hat{\eta} + K_1'\hat{\xi'}\chi_y' + K_1'\xi'\hat{\chi_y'}\right)\mathrm{d}y.
\end{aligned}
$$

Noting that $\mathrm{d}F_1[u], \mathrm{d}F_1'[u] \in \mathcal{L}(X_1)$ extend to mappings $\widehat{\mathrm{d}F_1[u]}, \widehat{\mathrm{d}F_1'[u]} \in \mathcal{L}(X_0)$ which depend analytically upon $u \in N$, we obtain the following result by the inverse-function theorem.

Lemma 34.

(i) There exists an open neighbourhood $\tilde{N}$ of the origin in X_1 such that $G : N \to \tilde{N}$ is a diffeomorphism.

(ii) For each $u \in N$ the operator $\mathrm{d}G[u] \in \mathcal{L}(X_1)$ extends to an isomorphism $\widehat{\mathrm{d}G}[u] : X_0 \to X_0$. The operators $\widehat{\mathrm{d}G}[u], \widehat{\mathrm{d}G}[u]^{-1} \in \mathcal{L}(X_0)$ depend analytically upon $u \in N$.

Using the calculations

$$\begin{aligned}(\mu(1)+\dot{\mu}(1))\tau_y &= (\mu(1)+\dot{\mu}(1))(\chi_y+\psi_{yy})\\ &= (\mu(1)+\dot{\mu}(1))(\chi_y-(F_1(u)+\nu^2\psi_{zz})),\\ \tau_y' &= \chi_y'-(F_1'(u)+\nu^2\psi_{zz}'),\end{aligned}$$

we find that

$$\begin{aligned}(\mu(1)+\dot{\mu}(1))\tau_y &= (\mu(1)+\dot{\mu}(1))(\chi_y-F_1(u)), &&\text{for } y=-\tfrac{1}{\beta_0},\\ \tau_y' &= \chi_y'-F_1'(u), &&\text{for } y=\tfrac{1}{\beta_0}\end{aligned}$$

and

$$\begin{aligned}\tau_y'-(\mu(1)+\dot{\mu}(1))\tau_y &= \chi_y'-F_1'(u)-(\mu(1)+\dot{\mu}(1))(\chi_y-F_1(u))\\ \frac{\zeta}{\mu(1)}-(\zeta'-(\mu(1)-1)\rho) &= \frac{\xi}{\mu(1)}+\frac{F_2(u)}{\mu(1)}-(\xi'+F_2'(u)-(\mu(1)-1)\rho)\end{aligned}$$

for $y=0$ because $\psi_{zz}=0$ on $y=-\frac{1}{\beta_0},0$ and $\psi_{zz}'=0$ on $y=0,\frac{1}{\beta_0}$. It follows that $B(u)=0$ if and only if $B_1\tilde{u}=0$.

The diffeomorphism G transforms (3.24) into the equation

$$\tilde{u}_x = v_{\tilde{H}^\varepsilon}(\tilde{u}) = L\tilde{u}+\tilde{N}^\varepsilon(\tilde{u}) \tag{3.25}$$

with $\mathcal{D}(v_{\tilde{H}^\varepsilon}) := \{\tilde{u}\in\tilde{N} : \tilde{B}_1\tilde{u}=0\}$, and $v_{\tilde{H}^\varepsilon} : \tilde{N}\to M$ is the analytic vector field defined by

$$v_{\tilde{H}^\varepsilon}(\tilde{u}) = \widehat{\mathrm{d}G}[G^{-1}(\tilde{u})](v_{H^\varepsilon}(G^{-1}(\tilde{u}))),$$

and L is the linearisation of $v_{\tilde{H}^\varepsilon}$ at $\varepsilon=0$, while $\tilde{N}^\varepsilon(\tilde{u}) = v_{\tilde{H}^\varepsilon}(\tilde{u})-L\tilde{u}$. Equation (3.25) represents Hamilton's equations for the Hamiltonian system $(M,\tilde{\Omega},\tilde{H}^\varepsilon)$, where

$$\tilde{\Omega}\big|_{\tilde{u}}(\hat{v},\hat{w}) = \Omega(\widehat{\mathrm{d}G}[G^{-1}(\tilde{u})](\hat{v}),\widehat{\mathrm{d}G}[G^{-1}(\tilde{u})](\hat{w}))$$

for $\hat{v},\hat{w}\in TM\big|_{\tilde{u}}\cong X_0$ and $\tilde{H}^\varepsilon(\tilde{u}) = H^\varepsilon(G^{-1}(\tilde{u}))$ for $\tilde{u}\in\tilde{N}$. We henceforth drop the tildes for notational simplicity and apply Theorem 33 to equation (3.25); note that (H3) is satisfied for any $k\in\mathbb{N}$ since $v_{\tilde{H}^\varepsilon}$ is analytic.

Now we turn to the spectral hypotheses on $L : \mathcal{D}(L)\subseteq X_0\to X_0$, which is given by the explicit formula

$$L\begin{pmatrix}\eta\\ \tau'\\ \tau\\ \rho\\ \zeta'\\ \zeta\end{pmatrix} = \begin{pmatrix}-\rho\\ \zeta'\\ \frac{\zeta}{\mu_1}\\ -\gamma_0\eta+\nu^2\eta_{zz}+(\mu(1)-1)\tau_y'\big|_{y=0}\\ -(\nu^2\tau_{zz}'+\tau_{yy}')\\ -\mu_1(\nu^2\tau_{zz}+S_1^{-2}\tau_{yy})\end{pmatrix}$$

and

$$\mathcal{D}(L) := \{u \in X_1 : B_\mathrm{l} u = 0\},$$

where

$$\mu_1 = \mu(1), \qquad \dot{\mu}_1 = \dot{\mu}(1), \qquad S_1 = \left(\frac{\mu_1}{\mu_1 + \dot{\mu}_1}\right)^{1/2}.$$

Writing $u \in X_1$ as

$$u(y,z) = \sum_{k\in\mathbb{Z}} u_k \mathrm{e}^{\mathrm{i}kz}$$

with $u_k = (\eta_k, \tau_k'(y), \tau_k(y), \rho_k, \zeta_k'(y), \zeta_k(y))^\mathrm{T}$, we find that

$$Lu = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} L_k u_k \mathrm{e}^{\mathrm{i}kz}, \qquad B_\mathrm{l} u = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} B_{\mathrm{l}k} u_k \mathrm{e}^{\mathrm{i}kz},$$

where

$$L_k u_k = \begin{pmatrix} -\rho_k \\ \zeta_k' \\ \frac{\zeta_k}{\mu_1} \\ -(\gamma_0 + k^2\nu^2)\eta_k + (\mu_1 - 1)\tau'_{k_y}\big|_{y=0} \\ k^2\nu^2\tau_k' - \tau'_{k_{yy}} \\ \mu_1(k^2\nu^2\tau_k - S_1^{-2}\tau_{k_{yy}}) \end{pmatrix}, \qquad B_{\mathrm{l}k} u_k = \begin{pmatrix} \mu_1 S_1^{-2}\tau_{k_y} \\ \tau'_{k_y} \\ \mu_1 S_1^{-2}\tau_{k_y} - \tau'_{k_y} \\ \frac{\zeta_k}{\mu_1} - \zeta_k' + (\mu_1 - 1)\rho_k \end{pmatrix}.$$

We note that $L_k = L_{-k}$ for $k \in \mathbb{N}$ and that the norms of the Sobolev spaces $H^t_\mathrm{per}(\Sigma)$, $H^t_\mathrm{per}(\Sigma')$, $H^r_\mathrm{per}(0, 2\pi)$ can be written as

$$\begin{aligned} \|w\|_t^2 &= \sum_{0\le\alpha_1+\alpha_2\le t} \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{\alpha_2} \|\partial_y^{\alpha_1} w_k\|^2_{L^2(-\frac{1}{\beta_0},0)}, \\ \|w'\|_t^2 &= \sum_{0\le\alpha_1+\alpha_2\le t} \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^{\alpha_2} \|\partial_y^{\alpha_1} w_k'\|^2_{L^2(0,\frac{1}{\beta_0})}, \\ \|\zeta\|_r^2 &= \sum_{k\in\mathbb{Z}} \left(1+k^2\right)^r |\zeta_k|^2. \end{aligned}$$

First we discuss the eigenvalues of L. We consider the equations $(L_k - \lambda I)u_k = 0$ for $\lambda \in \mathbb{C}$, that is

$$\begin{aligned} -\rho_k - \lambda\eta_k &= 0, \\ \zeta_k' - \lambda\tau_k' &= 0, \\ \zeta_k - \mu_1\lambda\tau_k &= 0, \\ -(\gamma_0 + k^2\nu^2)\eta_k + (\mu_1 - 1)\tau'_{k_y}\big|_{y=0} - \lambda\rho_k &= 0, \\ k^2\nu^2\tau_k' - \tau'_{k_{yy}} - \lambda\zeta_k' &= 0, \\ \mu_1(k^2\nu^2\tau_k - S_1^{-2}\tau_{k_{yy}}) - \lambda\zeta_k &= 0, \end{aligned}$$

so that

$$
\begin{aligned}
(\sigma^2-\gamma_0)\eta_k+(\mu_1-1)\tau'_{k y}\big|_{y=0}&=0,\\
\sigma^2\tau'_k+\tau'_{k yy}&=0,\\
\mu_1(\sigma^2\tau_k+S_1^{-2}\tau_{k yy})&=0
\end{aligned}
$$

with boundary conditions $B_{1k}u_k=0$, where $\sigma^2=\lambda^2-k^2\nu^2$.

- For values of λ and k with $\sigma\neq 0$ satisfying

$$
\gamma_0=-\left(\mu_1(\mu_1-1)^2\left(\mu_1\sigma\cot\frac{\sigma}{\beta_0}+S_1\sigma\cot\frac{S_1\sigma}{\beta_0}\right)^{-1}-1\right)\sigma^2 \tag{3.26}
$$

  the kernel of $(L_k-\lambda I)$ is spanned by $(\eta_k,\tau'_k,\tau_k,-\lambda\eta_k,\lambda\tau'_k,\mu_1\lambda\tau_k)^{\mathrm{T}}$ with

$$
\begin{aligned}
\eta_k&=\frac{\sin\dfrac{\sigma}{\beta_0}\cos\dfrac{S_1\sigma}{\beta_0}+\dfrac{\mu_1}{S_1}\sin\dfrac{S_1\sigma}{\beta_0}\cos\dfrac{\sigma}{\beta_0}}{\mu_1-1},\\
\tau'_k&=-\frac{\mu_1}{S_1}\sin\frac{S_1\sigma}{\beta_0}\cos\sigma\left(y-\frac{1}{\beta_0}\right),\\
\tau_k&=\sin\frac{\sigma}{\beta_0}\cos S_1\sigma\left(y+\frac{1}{\beta_0}\right).
\end{aligned}
$$

- For values of λ and k with $\sigma=0$ the kernel of $(L_k-\lambda I)$ is spanned by $(0,1,1,0,\lambda,\mu_1\lambda)^{\mathrm{T}}$.

- The kernel of $(L_k-\lambda I)$ is trivial for all other values of λ and k.

Next we consider the operator $L-\mathrm{i}sI$ with $s\in\mathbb{R}$ in more detail, noting that $\sigma=\pm\mathrm{i}\tilde\sigma$, where $\tilde\sigma^2=s^2+k^2\nu^2$, and give the solution u to the resolvent equation $(L-\mathrm{i}sI)u=u^\star$ with $u^\star\in M$, where $\tilde\sigma\in\mathbb{R}$ is non-zero and does not satisfy (3.26) for any k. Writing $u^\star\in M$ as

$$
u^\star(y,z)=\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}u^\star_k\mathrm{e}^{\mathrm{i}kz}
$$

with $u^\star_k=(\eta^\star_k,\tau^{\star\prime}_k(y),\tau^\star_k(y),\rho^\star_k,\zeta^{\star\prime}_k(y),\zeta^\star_k(y))^{\mathrm{T}}$ we write the resolvent equation

$$
(L_k-\mathrm{i}sI)u_k=u^\star_k
$$

as

$$
\begin{aligned}
-\rho_k-\mathrm{i}s\eta_k&=\eta^\star_k, &(3.27)\\
\zeta'_k-\mathrm{i}s\tau'_k&=\tau'^\star_k, &(3.28)\\
\zeta_k-\mu_1\mathrm{i}s\tau_k&=\mu_1\tau^\star_k, &(3.29)\\
-(\gamma_0+k^2\nu^2)\eta_k+(\mu_1-1)\tau'_{k y}\big|_{y=0}-\mathrm{i}s\rho_k&=\rho^\star_k, &(3.30)\\
k^2\nu^2\tau'_k-\tau'_{k yy}-\mathrm{i}s\zeta'_k&=\zeta'^\star_k, &(3.31)\\
\mu_1(k^2\nu^2\tau_k-S_1^{-2}\tau_{k yy})-\mathrm{i}s\zeta_k&=\zeta^\star_k, &(3.32)
\end{aligned}
$$

with boundary conditions $B_{lk}u = 0$. Under the assumption that (3.26) is not satisfied, the solutions to these equations are given by the explicit formulae

$$\eta_k = \frac{1}{(\tilde{\sigma}^2 + \gamma_0)} \left(\mu_1(\mu_1 - 1)A - \rho_k^\star + \mathrm{i}s\eta_k^\star\right), \tag{3.33}$$

$$\begin{aligned} \tau_k'(y) &= \int_0^{\frac{1}{\beta_0}} G'(y,\tilde{y}) \left(\zeta_k'^\star(\tilde{y}) + \mathrm{i}s\tau_k'^\star(\tilde{y})\right) \,\mathrm{d}\tilde{y} \; - \mu_1 G'(y,0)A, \\ \tau_k(y) &= \int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y}) \left(\frac{\zeta_k^\star(\tilde{y})}{\mu_1} + \mathrm{i}s\tau_k^\star(\tilde{y})\right) \,\mathrm{d}\tilde{y} \; + G(y,0)A, \end{aligned} \tag{3.34}$$

$$\rho_k = \frac{-\mathrm{i}s}{(\tilde{\sigma}^2 + \gamma_0)} \left(\mu_1(\mu_1 - 1)A - \rho_k^\star\right) - \frac{\gamma_0 + k^2}{(\tilde{\sigma}^2 + \gamma_0)} \eta_k^\star, \tag{3.35}$$

$$\begin{aligned} \zeta_k' &= \tau_k'^\star + \mathrm{i}s \int_0^{\frac{1}{\beta_0}} G'(y,\tilde{y}) \left(\zeta_k'^\star(\tilde{y}) + \mathrm{i}s\tau_k'^\star(\tilde{y})\right) \,\mathrm{d}\tilde{y} \; - \mathrm{i}s\mu_1 G'(y,0)A, \\ \zeta_k &= \mu_1\tau_k^\star + \mu_1 \mathrm{i}s \int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y}) \left(\frac{\zeta_k^\star(\tilde{y})}{\mu_1} + \mathrm{i}s\tau_k^\star(\tilde{y})\right) \,\mathrm{d}\tilde{y} \; + \mathrm{i}s\mu_1 G(y,0)A, \end{aligned} \tag{3.36}$$

where the Green's functions G, G' are given by

$$G(y,\tilde{y}) = \begin{cases} \dfrac{S_1 \cosh S_1\tilde{\sigma}\left(y + \dfrac{1}{\beta_0}\right) \cosh S_1\tilde{\sigma}\tilde{y}}{\tilde{\sigma}\sinh \dfrac{S_1\tilde{\sigma}}{\beta_0}}, & \dfrac{1}{\beta_0} \le y \le \tilde{y} \le 0, \\[3ex] \dfrac{S_1 \cosh S_1\tilde{\sigma}\left(\tilde{y} + \dfrac{1}{\beta_0}\right) \cosh S_1\tilde{\sigma}y}{\tilde{\sigma}\sinh \dfrac{S_1\tilde{\sigma}}{\beta_0}}, & \dfrac{1}{\beta_0} \le \tilde{y} \le y \le 0, \end{cases}$$

$$G'(y,\tilde{y}) = \begin{cases} \dfrac{\cosh \tilde{\sigma}\left(y - \dfrac{1}{\beta_0}\right) \cosh \tilde{\sigma}\tilde{y}}{\tilde{\sigma}\sinh \dfrac{\tilde{\sigma}}{\beta_0}}, & 0 \le \tilde{y} \le y \le \dfrac{1}{\beta_0}, \\[3ex] \dfrac{\cosh \tilde{\sigma}\left(\tilde{y} - \dfrac{1}{\beta_0}\right) \cosh \tilde{\sigma}y}{\tilde{\sigma}\sinh \dfrac{\tilde{\sigma}}{\beta_0}}, & 0 \le y \le \tilde{y} \le \dfrac{1}{\beta_0}, \end{cases}$$

and

$$\begin{aligned} A =& C_A \left(\rho_k^\star - \mathrm{i}s\eta_k^\star + (\gamma_0 - \tilde{\sigma}^2) \int_{-\frac{1}{\beta_0}}^0 \frac{G(0,\tilde{y})}{\mu_1 - 1} \left(\frac{\zeta_k^\star(\tilde{y})}{\mu_1} + \mathrm{i}s\tau_k^\star(\tilde{y})\right) \,\mathrm{d}\tilde{y} \right. \\ & \left. - (\gamma_0 - \tilde{\sigma}^2) \int_0^{\frac{1}{\beta_0}} \frac{G'(0,\tilde{y})}{\mu_1 - 1} \left(\zeta_k'^\star(\tilde{y}) + \mathrm{i}s\tau_k'^\star(\tilde{y})\right) \,\mathrm{d}\tilde{y}\right) \end{aligned} \tag{3.37}$$

with

$$C_{A} = \left(\mu_1(\mu_1 - 1) - (\tilde{\sigma}^2 + \gamma_0) \frac{\sinh \dfrac{\tilde{\sigma}}{\beta_0} \cosh \dfrac{S_1\tilde{\sigma}}{\beta_0} + \dfrac{\mu_1}{S_1} \sinh \dfrac{S_1\tilde{\sigma}}{\beta_0} \cosh \dfrac{\tilde{\sigma}}{\beta_0}}{(\mu_1 - 1)\tilde{\sigma} \sinh \dfrac{S_1\tilde{\sigma}}{\beta_0} \sinh \dfrac{\tilde{\sigma}}{\beta_0}} \right)^{-1}.$$

The convergence of the Fourier series $\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} u_{k} \mathrm{e}^{\mathrm{i}kz}$ is discussed below.

We now estimate the Green's functions G and G'.

Lemma 35. *The estimates*

$$\int_{-\frac{1}{\beta_0}}^{0} |G(y,\tilde{y})| \, \mathrm{d}y + \int_{-\frac{1}{\beta_0}}^{0} |G(y,\tilde{y})| \, \mathrm{d}\tilde{y} + \int_{0}^{\frac{1}{\beta_0}} |G'(y,\tilde{y})| \, \mathrm{d}y + \int_{0}^{\frac{1}{\beta_0}} |G'(y,\tilde{y})| \, \mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-2},$$

$$\int_{-\frac{1}{\beta_0}}^{0} |G_y(y,\tilde{y})| \, \mathrm{d}y + \int_{-\frac{1}{\beta_0}}^{0} |G_y(y,\tilde{y})| \, \mathrm{d}\tilde{y} + \int_{0}^{\frac{1}{\beta_0}} |G'_y(y,\tilde{y})| \, \mathrm{d}y + \int_{0}^{\frac{1}{\beta_0}} |G'_y(y,\tilde{y})| \, \mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-1}$$

$$\int_{-\frac{1}{\beta_0}}^{0} |G(0,\tilde{y})|^2 \, \mathrm{d}\tilde{y} + \int_{0}^{\frac{1}{\beta_0}} |G'(0,\tilde{y})|^2 \, \mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-3},$$

$$\int_{-\frac{1}{\beta_0}}^{0} |G_y(y,0)|^2 \, \mathrm{d}y + \int_{0}^{\frac{1}{\beta_0}} |G'_y(y,0)|^2 \, \mathrm{d}y \lesssim \tilde{\sigma}^{-1}$$

hold for all $\tilde{\sigma} \gtrsim 1$.

Proof. The estimates for G' follow from the calculations

$$\int_{0}^{\frac{1}{\beta_0}} |G'(y,\tilde{y})| \, \mathrm{d}\tilde{y} = \frac{1}{\tilde{\sigma}^2},$$

$$\begin{aligned}
\int_{0}^{\frac{1}{\beta_0}} |G'_y(y,\tilde{y})| \, \mathrm{d}\tilde{y} &= \frac{\cosh \dfrac{\tilde{\sigma}}{\beta_0} - \cosh \tilde{\sigma} \left(\dfrac{1}{\beta_0} - 2y \right)}{\tilde{\sigma} \sinh \dfrac{\tilde{\sigma}}{\beta_0}} \\
&\le \frac{\cosh \dfrac{\tilde{\sigma}}{\beta_0}}{\tilde{\sigma} \sinh \dfrac{\tilde{\sigma}}{\beta_0}} \\
&\lesssim \frac{1}{\tilde{\sigma}},
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\frac{1}{\beta_0}} |G'(0,\tilde{y})|^2 \, \mathrm{d}\tilde{y} &= \frac{\dfrac{2\tilde{\sigma}}{\beta_0} + \sinh \dfrac{2\tilde{\sigma}}{\beta_0}}{4\tilde{\sigma}^3 \sinh^2 \dfrac{\tilde{\sigma}}{\beta_0}} \\
&\le \frac{\sinh \dfrac{2\tilde{\sigma}}{\beta_0}}{2\tilde{\sigma}^3 \sinh^2 \dfrac{\tilde{\sigma}}{\beta_0}} \\
&\lesssim \frac{1}{\tilde{\sigma}^3},
\end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{1}{\beta_0}} |G'_y(y,0)|^2 \, \mathrm{d}y &= \frac{\sinh \dfrac{2\tilde{\sigma}}{\beta_0} - \dfrac{2\tilde{\sigma}}{\beta_0}}{\tilde{\sigma} \sinh^2 \dfrac{\tilde{\sigma}}{\beta_0}} \\
&\leq \frac{\sinh \dfrac{2\tilde{\sigma}}{\beta_0}}{\tilde{\sigma} \sinh^2 \dfrac{\tilde{\sigma}}{\beta_0}} \\
&\lesssim \frac{1}{\tilde{\sigma}}
\end{aligned}$$

for $\tilde{\sigma} \gtrsim 1$. The estimates for G are obtained in a similar fashion. □

In the estimates below it is helpful to move the derivatives of G, G' to another function by integrating by parts. For this purpose we use the formulae

$$\begin{aligned}
\int_{-\frac{1}{\beta_0}}^0 G_y(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y} &= \int_{-\frac{1}{\beta_0}}^0 H_{\tilde{y}}(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y}, \\
\int_0^{\frac{1}{\beta_0}} G'_y(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y} &= \int_0^{\frac{1}{\beta_0}} H'_{\tilde{y}}(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y}
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y} &= \int_{-\frac{1}{\beta_0}}^0 F_{\tilde{y}}(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y}, \\
\int_0^{\frac{1}{\beta_0}} G'(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y} &= \int_0^{\frac{1}{\beta_0}} F'_{\tilde{y}}(y,\tilde{y}) f(\tilde{y}) \, \mathrm{d}\tilde{y},
\end{aligned}$$

where

$$H(y,\tilde{y}) = \begin{cases} \dfrac{S_1 \sinh S_1 \tilde{\sigma} \left(y + \dfrac{1}{\beta_0} \right) \sinh S_1 \tilde{\sigma} \tilde{y}}{\tilde{\sigma} \sinh \dfrac{S_1 \tilde{\sigma}}{\beta_0}}, & -\dfrac{1}{\beta_0} \leq y \leq \tilde{y} \leq 0, \\[6ex] \dfrac{S_1 \sinh S_1 \tilde{\sigma} \left(\tilde{y} + \dfrac{1}{\beta_0} \right) \sinh S_1 \tilde{\sigma} y}{\tilde{\sigma} \sinh \dfrac{S_1 \tilde{\sigma}}{\beta_0}}, & -\dfrac{1}{\beta_0} \leq \tilde{y} \leq y \leq 0, \end{cases}$$

$$H'(y,\tilde{y}) = \begin{cases} \dfrac{\sinh \tilde{\sigma} \left(y - \dfrac{1}{\beta_0} \right) \sinh \tilde{\sigma} \tilde{y}}{\tilde{\sigma} \sinh \dfrac{\tilde{\sigma}}{\beta_0}}, & 0 \leq \tilde{y} \leq y \leq \dfrac{1}{\beta_0}, \\[6ex] \dfrac{\sinh \tilde{\sigma} \left(\tilde{y} - \dfrac{1}{\beta_0} \right) \sinh \tilde{\sigma} y}{\tilde{\sigma} \sinh \dfrac{\tilde{\sigma}}{\beta_0}}, & 0 \leq y \leq \tilde{y} \leq \dfrac{1}{\beta_0}, \end{cases}$$

$$F(y,\tilde{y}) = \begin{cases} \dfrac{\cosh S_1\tilde{\sigma}\left(y+\dfrac{1}{\beta_0}\right)\sinh S_1\tilde{\sigma}\tilde{y}}{\tilde{\sigma}^2\sinh\dfrac{S_1\tilde{\sigma}}{\beta_0}}, & -\dfrac{1}{\beta_0}\leq y\leq\tilde{y}\leq 0,\\[2ex] \dfrac{\sinh S_1\tilde{\sigma}\left(\tilde{y}+\dfrac{1}{\beta_0}\right)\cosh S_1\tilde{\sigma}y}{\tilde{\sigma}^2\sinh\dfrac{S_1\tilde{\sigma}}{\beta_0}}, & -\dfrac{1}{\beta_0}\leq\tilde{y}\leq y\leq 0,\end{cases}$$

$$F'(y,\tilde{y}) = \begin{cases} \dfrac{\cosh\tilde{\sigma}\left(y-\dfrac{1}{\beta_0}\right)\sinh\tilde{\sigma}\tilde{y}}{\tilde{\sigma}^2\sinh\dfrac{\tilde{\sigma}}{\beta_0}}, & 0\leq\tilde{y}\leq y\leq\dfrac{1}{\beta_0},\\[2ex] \dfrac{\sinh\tilde{\sigma}\left(\tilde{y}-\dfrac{1}{\beta_0}\right)\cosh\tilde{\sigma}y}{\tilde{\sigma}^2\sinh\dfrac{\tilde{\sigma}}{\beta_0}}, & 0\leq y\leq\tilde{y}\leq\dfrac{1}{\beta_0}.\end{cases}$$

The following result is proved in the same way as Lemma 35.

Lemma 36. *The estimates*

$$\int_{-\frac{1}{\beta_0}}^0 |H(y,\tilde{y})|\,\mathrm{d}y, + \int_{-\frac{1}{\beta_0}}^0 |H(y,\tilde{y})|\,\mathrm{d}\tilde{y} + \int_0^{\frac{1}{\beta_0}} |H'(y,\tilde{y})|\,\mathrm{d}y + \int_0^{\frac{1}{\beta_0}} |H'(y,\tilde{y})|\,\mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-2},$$
$$\int_{-\frac{1}{\beta_0}}^0 |F(y,\tilde{y})|\,\mathrm{d}y + \int_{-\frac{1}{\beta_0}}^0 |F(y,\tilde{y})|\,\mathrm{d}\tilde{y} + \int_0^{\frac{1}{\beta_0}} |F'(y,\tilde{y})|\,\mathrm{d}y + \int_0^{\frac{1}{\beta_0}} |F'(y,\tilde{y})|\,\mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-3},$$
$$\int_{-\frac{1}{\beta_0}}^0 |F(0,\tilde{y})|^2\,\mathrm{d}\tilde{y} + \int_0^{\frac{1}{\beta_0}} |F'(0,\tilde{y})|^2\,\mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-5}$$

hold for all $\tilde{\sigma} \gtrsim 1$.

Lemma 37. *The estimates*

$$||(L-\mathrm{i}sI)^{-1}||_{X_0\to X_0} \lesssim \frac{1}{|s|}, \qquad ||(L-\mathrm{i}sI)^{-1}||_{X_0\to X_1} \lesssim 1$$

hold for all sufficiently large values of $|s|$.

Proof. We use the formulae for solutions to $(L_k - \mathrm{i}sI)u_k = u_k^\star$ derived above. We begin by estimating the constant A (given in equation (3.37)). Integrating by parts, we find that

$$\tau_k^\star(0) - \int_{-\frac{1}{\beta_0}}^0 \tilde{\sigma}^2 G(0,\tilde{y})\tau_k^\star(\tilde{y})\,\mathrm{d}\tilde{y} = \tilde{\sigma}^2\int_{-\frac{1}{\beta_0}}^0 F(0,\tilde{y})\tau_{k\tilde{y}}^\star(\tilde{y})\,\mathrm{d}\tilde{y}$$

and by Lemma 36 that

$$\left|\tau_{k}^{\star}(0) - \int_{-\frac{1}{\beta_0}}^{0} \tilde{\sigma}^2 G(0,\tilde{y})\tau_{k}^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right| \lesssim \tilde{\sigma}^2 \left(\int_0^{\frac{1}{\beta_0}} |F(0,\tilde{y})|^2\,\mathrm{d}\tilde{y}\right)^{\frac{1}{2}} \|\tau_{k_{\tilde{y}}}^{\star}\|_{L^2(-\frac{1}{\beta_0},0)}$$
$$\lesssim \tilde{\sigma}^{-\frac{1}{2}}\|\tau_{k_{\tilde{y}}}^{\star}\|_{L^2(-\frac{1}{\beta_0},0)}.$$

We estimate

$$\int_0^{\frac{1}{\beta_0}} |G'(0,\tilde{y})\left(\zeta_{k}'^{\star}(\tilde{y}) + \mathrm{i}s\tau_{k}'^{\star}(\tilde{y})\right)|\,\mathrm{d}\tilde{y}$$
$$\lesssim \left(\int_0^{\frac{1}{\beta_0}} |G'(0,\tilde{y})|^2\,\mathrm{d}\tilde{y}\right)^{\frac{1}{2}} \left(\|\zeta_{k}'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + |s|\|\tau_{k}'^{\star}\|_{L^2(0,\frac{1}{\beta_0})}\right)$$
$$\lesssim \tilde{\sigma}^{-\frac{3}{2}}\left(\|\zeta_{k}'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + |s|\|\tau_{k}'^{\star}\|_{L^2(0,\frac{1}{\beta_0})}\right),$$

and find in a similar fashion that

$$\int_{-\frac{1}{\beta_0}}^{0} |G(0,\tilde{y})\left(\zeta_{k}^{\star}(\tilde{y}) + \mathrm{i}s\tau_{k}^{\star}(\tilde{y})\right)|\,\mathrm{d}\tilde{y} \lesssim \tilde{\sigma}^{-\frac{3}{2}}\left(\|\zeta_{k}^{\star}\|_{L^2(-\frac{1}{\beta_0},0)} + |s|\|\tau_{k}^{\star}\|_{L^2(-\frac{1}{\beta_0},0)}\right).$$

Using the fact that

$$\eta_k^{\star} = \frac{\tau_{k}^{\star}(0) - \tau_{k}'^{\star}(0)}{\mu_1 - 1},$$

we write

$$\begin{aligned}
AC_{A}^{-1} = {}& \rho_{k}^{\star} + \gamma_0 \int_{-\frac{1}{\beta_0}}^{0} \frac{G(0,\tilde{y})}{\mu_1 - 1}\left(\frac{\zeta_{k}^{\star}(\tilde{y})}{\mu_1} + \mathrm{i}s\tau_{k}^{\star}(\tilde{y})\right)\mathrm{d}\tilde{y} + \tilde{\sigma}^2 \int_{-\frac{1}{\beta_0}}^{0} \frac{G(0,\tilde{y})}{\mu_1 - 1}\frac{\zeta_{k}^{\star}(\tilde{y})}{\mu_1}\,\mathrm{d}\tilde{y} \\
& - \gamma_0 \int_0^{\frac{1}{\beta_0}} \frac{G'(0,\tilde{y})}{\mu_1 - 1}\left(\zeta_{k}'^{\star}(\tilde{y}) + \mathrm{i}s\tau_{k}'^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right) - \tilde{\sigma}^2 \int_0^{\frac{1}{\beta_0}} \frac{G'(0,\tilde{y})}{\mu_1 - 1}\zeta_{k}'^{\star}(\tilde{y})\,\mathrm{d}\tilde{y} \\
& - \frac{\mathrm{i}s}{\mu_1 - 1}\left(\tau_{k}^{\star}(0) - \tilde{\sigma}^2 \int_{-\frac{1}{\beta_0}}^{0} G(0,\tilde{y})\tau_{k}^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right) \\
& + \frac{\mathrm{i}s}{\mu_1 - 1}\left(\tau_{k}'^{\star}(0) - \tilde{\sigma}^2 \int_0^{\frac{1}{\beta_0}} G'(0,\tilde{y})\tau_{k}'^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right).
\end{aligned}$$

Using the calculation

$$\begin{aligned}
&|C_{A}| \\
&= \left|\frac{(\mu_1 - 1)\tilde{\sigma}\sinh\frac{S_1\tilde{\sigma}}{\beta_0}\sinh\frac{\tilde{\sigma}}{\beta_0}}{\mu_1(\mu_1 - 1)^2\tilde{\sigma}\sinh\frac{S_1\tilde{\sigma}}{\beta_0}\sinh\frac{\tilde{\sigma}}{\beta_0} - (\tilde{\sigma}^2 + \gamma_0)\left(\sinh\frac{\tilde{\sigma}}{\beta_0}\cosh\frac{S_1\tilde{\sigma}}{\beta_0} + \frac{\mu_1}{S_1}\sinh\frac{S_1\tilde{\sigma}}{\beta_0}\cosh\frac{\tilde{\sigma}}{\beta_0}\right)}\right| \\
&= \left|\frac{(\mu_1 - 1)\tilde{\sigma}}{(\tilde{\sigma}^2 + \gamma_0)\left(\coth\frac{S_1\tilde{\sigma}}{\beta_0} + \frac{\mu_1}{S_1}\coth\frac{\tilde{\sigma}}{\beta_0}\right) - \mu_1(\mu_1 - 1)^2\tilde{\sigma}}\right| \\
&\lesssim \frac{1}{\tilde{\sigma}}
\end{aligned}$$

for sufficiently large $\tilde{\sigma}$ and noting that $\frac{|s|}{\tilde{\sigma}} \lesssim 1$, we conclude that

$$
\begin{aligned}
|A| \lesssim &\tilde{\sigma}^{-1}|\rho_k^\star| + \tilde{\sigma}^{-\frac{1}{2}}\|\zeta_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-\frac{1}{2}}\|\tau_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-\frac{1}{2}}\|\zeta_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} \\
&+ \tilde{\sigma}^{-\frac{1}{2}}\|\tau_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} + \tilde{\sigma}^{-\frac{1}{2}}\|\tau_{ky}'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-\frac{1}{2}}\|\tau_{ky}^\star\|_{L^2(-\frac{1}{\beta_0},0)}.
\end{aligned}
$$

Next we estimate the components of u, beginning with η. Equation (3.33) together with the estimate for A yields

$$
\begin{aligned}
|\eta_k| \lesssim &\tilde{\sigma}^{-\frac{5}{2}}\|\zeta_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-\frac{5}{2}}\|\tau_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-\frac{5}{2}}\|\zeta_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} \\
&+ \tilde{\sigma}^{-\frac{5}{2}}\|\tau_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} + \tilde{\sigma}^{-2}|\rho_k^\star| + \tilde{\sigma}^{-1}|\eta_k^\star|.
\end{aligned}
$$

We consider $|s||\eta_k|$, $|s||k||\eta_k|$ and $|k|^2|\eta_k|$ to obtain estimates for $|s|\|\eta\|_0$, $|s|\|\eta_z\|_0$ and $\|\eta_{zz}\|_0$. From the above estimate for $|\eta_k|$ we find that

$$
\begin{aligned}
|s||\eta_k| &\lesssim \|\zeta_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \|\tau_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \|\zeta_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} \\
&\quad + \|\tau_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} + |\rho_k^\star| + |\eta_k^\star|, \\
|k||s||\eta_k| &\lesssim \|\zeta_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + |k|\|\tau_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \|\zeta_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} \\
&\quad + |k|\|\tau_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} + |\rho_k^\star| + |k||\eta_k^\star|, \\
|k|^2|\eta_k| &\lesssim \|\zeta_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + |k|\|\tau_k'^\star\|_{L^2(0,\frac{1}{\beta_0})} + \|\zeta_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} \\
&\quad + |k|\|\tau_k^\star\|_{L^2(-\frac{1}{\beta_0},0)} + |\rho_k^\star| + |k||\eta_k^\star|
\end{aligned}
$$

and conclude that

$$
\|\eta\|_0 + \|\eta_z\|_0 \lesssim \frac{1}{|s|}\|u^\star\|_{X_0}, \qquad \|\eta\|_0 + \|\eta_z\|_0 + \|\eta_{zz}\|_0 \lesssim \|u^\star\|_{X_0}.
$$

Using equation (3.35), we find in an analogous way to the above estimates for η that

$$
\|\rho\|_0 \lesssim \frac{1}{|s|}\|u^\star\|_{X_0}, \qquad \|\rho\|_0 + \|\rho_x\|_0 \lesssim \|u^\star\|_{X_0}.
$$

From Lemma 35 it follows that

$$
\begin{aligned}
\int_{-\frac{1}{\beta_0}}^0 \left| \int_{-\frac{1}{\beta_0}}^0 G\left(\zeta_k^\star + \mu_1 \mathrm{i} s \tau_k^\star\right)\, \mathrm{d}\tilde{y} \right|^2 \mathrm{d}y &\lesssim \int_{-\frac{1}{\beta_0}}^0 \int_{-\frac{1}{\beta_0}}^0 |G|\, \mathrm{d}\tilde{y} \int_{-\frac{1}{\beta_0}}^0 |G||\zeta_k^\star + \mu_1 \mathrm{i} s \tau_k^\star|^2\, \mathrm{d}\tilde{y}\, \mathrm{d}y \\
&\lesssim \tilde{\sigma}^{-2} \int_{-\frac{1}{\beta_0}}^0 \int_{-\frac{1}{\beta_0}}^0 |G|\, \mathrm{d}y\, |\zeta_k^\star + \mu_1 \mathrm{i} s \tau_k^\star|^2\, \mathrm{d}\tilde{y} \\
&\lesssim \tilde{\sigma}^{-4} \int_{-\frac{1}{\beta_0}}^0 \left(\left|\zeta_k^\star\right|^2 + |s|^2 \left|\tau_k^\star\right|^2 \right) \mathrm{d}\tilde{y},
\end{aligned}
$$

and using the formula for τ_k given in equation (3.34), the estimate for A and Lemma 35, we find that

$$\|\tau_{k}\|^2_{L^2(-\frac{1}{\beta_0},0)} \lesssim \tilde{\sigma}^{-4}\|\zeta'^{\star}_{k}\|^2_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-2}\|\tau'^{\star}_{k}\|^2_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-4}\|\zeta^{\star}_{k}\|^2_{L^2(-\frac{1}{\beta_0},0)}$$
$$+ \tilde{\sigma}^{-2}\|\tau^{\star}_{k}\|^2_{L^2(-\frac{1}{\beta_0},0)} + \tilde{\sigma}^{-4}|\rho^{\star}_{k}|^2 + \tilde{\sigma}^{-4}\|\tau'^{\star}_{k_y}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-4}\|\tau^{\star}_{k_y}\|_{L^2(-\frac{1}{\beta_0},0)},$$

which shows that

$$\|\tau\|_0 + \|\tau_z\|_0 \lesssim \frac{1}{|s|}\|u^\star\|_{X_0}, \qquad \|\tau\|_0 + \|\tau_z\|_0 + \|\tau_{zz}\|_0 \lesssim \|u^\star\|_{X_0}.$$

Integrating by parts, we find that

$$\tau^{\star}_{k}(y) - \int_{-\frac{1}{\beta_0}}^0 s^2 G(y,\tilde{y})\tau^{\star}_{k}(\tilde{y})\,\mathrm{d}\tilde{y} = \frac{k^2}{\tilde{\sigma}^2}\tau^{\star}_{k}(y) + s^2\left(\int_{-\frac{1}{\beta_0}}^0 F(y,\tilde{y})\tau^{\star}_{k_{\tilde{y}}}(\tilde{y})\,\mathrm{d}\tilde{y}\right),$$

so that

$$\int_{-\frac{1}{\beta_0}}^0 \left|\tau^{\star}_{k}(y) - \int_{-\frac{1}{\beta_0}}^0 s^2 G(y,\tilde{y})\tau^{\star}_{k}(\tilde{y})\,\mathrm{d}\tilde{y}\right|^2 \mathrm{d}y \lesssim \frac{k^2}{\tilde{\sigma}^2}\int_{-\frac{1}{\beta_0}}^0 |\tau^{\star}_{k}(y)|^2\,\mathrm{d}y + \frac{s^2}{\tilde{\sigma}^4}\int_{-\frac{1}{\beta_0}}^0 |\tau^{\star}_{k_{\tilde{y}}}(\tilde{y})|^2\,\mathrm{d}\tilde{y}.$$

Using the formula for ζ_{k} given in equation (3.36) and noting that

$$\begin{aligned}
|\zeta_{k}|^2 &= \left|\mu_1\tau^{\star}_{k} + \mu_1\mathrm{i}s\int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\left(\frac{\zeta^{\star}_{k}(\tilde{y})}{\mu_1} + \mathrm{i}s\tau^{\star}_{k}(\tilde{y})\right)\,\mathrm{d}\tilde{y}\, + \mathrm{i}s\mu_1 G(y,0)A\right|^2 \\
&= \left|\mu_1\left(\tau^{\star}_{k} - s^2\int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\tau^{\star}_{k}(\tilde{y})\,\mathrm{d}\tilde{y}\right) + \mu_1\mathrm{i}s\int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\frac{\zeta^{\star}_{k}(\tilde{y})}{\mu_1}\,\mathrm{d}\tilde{y} + \mathrm{i}s\mu_1 G(y,0)A\right|^2 \\
&\lesssim \left|\tau^{\star}_{k} - s^2\int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\tau^{\star}_{k}(\tilde{y})\,\mathrm{d}\tilde{y}\right|^2 + |s|\left|\int_{-\frac{1}{\beta_0}}^0 G(y,\tilde{y})\zeta^{\star}_{k}(\tilde{y})\,\mathrm{d}\tilde{y}\right|^2 + |s|\,|G(y,0)A|^2,
\end{aligned}$$

we find that

$$\|\zeta_{k}\|_{L^2(-\frac{1}{\beta_0},0)} \lesssim \tilde{\sigma}^{-\frac{3}{2}}|\rho^{\star}_{k}| + \tilde{\sigma}^{-1}\|\zeta'^{\star}_{k}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-1}|k|\|\tau'^{\star}_{k}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-1}\|\zeta^{\star}_{k}\|_{L^2(-\frac{1}{\beta_0},0)}$$
$$+ \tilde{\sigma}^{-1}|k|\|\tau^{\star}_{k}\|_{L^2(-\frac{1}{\beta_0},0)} + \tilde{\sigma}^{-1}\|\tau'^{\star}_{k_y}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-1}\|\tau^{\star}_{k_y}\|_{L^2(-\frac{1}{\beta_0},0)},$$

so that

$$\|\zeta\|_0 \lesssim \frac{1}{|s|}\|u^\star\|_{X_0}, \qquad \|\zeta\|_0 + \|\zeta_z\|_0 \lesssim \|u^\star\|_{X_0}.$$

Next we estimate the derivatives of τ', τ, ζ' and ζ with respect to y. We note that

$$\begin{aligned}
\tau_{k_y}(y) &= \int_{-\frac{1}{\beta_0}}^0 G_y(y,\tilde{y})\left(\frac{\zeta^{\star}_{k}(\tilde{y})}{\mu_1} + \tilde{\sigma}\tau^{\star}_{k}(\tilde{y})\right)\,\mathrm{d}\tilde{y}\, + G_y(y,0)A, \\
\zeta_{k_y}(y) &= \mu_1\tau^{\star}_{k_y}(y) + \mu_1\mathrm{i}s\int_{-\frac{1}{\beta_0}}^0 G_y(y,\tilde{y})\left(\frac{\zeta^{\star}_{k}(\tilde{y})}{\mu_1} + \mathrm{i}s\tau^{\star}_{k}(\tilde{y})\right)\,\mathrm{d}\tilde{y}\, + \mu_1\mathrm{i}sG_y(y,0)A.
\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}\int_{-\frac{1}{\beta_0}}^{0}\left|\int_{-\frac{1}{\beta_0}}^{0} \mathrm{i}sG_y(y,\tilde{y})\tau_{k}^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y &= \int_{-\frac{1}{\beta_0}}^{0}\left|\int_{-\frac{1}{\beta_0}}^{0} \mathrm{i}sH(y,\tilde{y})\tau_{k_{\tilde{y}}}^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y \\ &\lesssim \tilde{\sigma}^{-4}\int_{-\frac{1}{\beta_0}}^{0}\left|s\tau_{k_{\tilde{y}}}^{\star}(\tilde{y})\right|^2\,\mathrm{d}\tilde{y}, \\ &\lesssim \tilde{\sigma}^{-2}\int_{-\frac{1}{\beta_0}}^{0}\left|\tau_{k_{\tilde{y}}}^{\star}(\tilde{y})\right|^2\,\mathrm{d}\tilde{y}.\end{aligned}$$

The above calculation yields

$$\begin{aligned}\|\tau_{k_y}\|_{L^2(-\frac{1}{\beta_0},0)} \lesssim\ & \tilde{\sigma}^{-\frac{3}{2}}|\rho_k^{\star}| + \tilde{\sigma}^{-1}\|\zeta_k'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-1}|k|\|\tau_k'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-1}\|\zeta_k^{\star}\|_{L^2(-\frac{1}{\beta_0},0)} \\ & + \tilde{\sigma}^{-1}|k|\|\tau_k^{\star}\|_{L^2(-\frac{1}{\beta_0},0)} + \tilde{\sigma}^{-1}\|\tau_{k_y}'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + \tilde{\sigma}^{-1}\|\tau_{k_y}^{\star}\|_{L^2(-\frac{1}{\beta_0},0)}, \\ \|\zeta_{k_y}\|_{L^2(-\frac{1}{\beta_0},0)} \lesssim\ & \tilde{\sigma}^{-\frac{1}{2}}|\rho_k^{\star}| + \|\zeta_k'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + |k|\|\tau_k'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + \|\zeta_k^{\star}\|_{L^2(-\frac{1}{\beta_0},0)} \\ & + |k|\|\tau_k^{\star}\|_{L^2(-\frac{1}{\beta_0},0)} + \|\tau_{k_y}'^{\star}\|_{L^2(0,\frac{1}{\beta_0})} + \|\tau_{k_y}^{\star}\|_{L^2(-\frac{1}{\beta_0},0)}\end{aligned}$$

and hence

$$\|\tau_y\|_0 \lesssim \frac{1}{|s|}\|u^{\star}\|_{X_0}, \qquad \|\tau_y\|_0 + \|\tau_{yz}\|_0 \lesssim \|u^{\star}\|_{X_0}, \qquad \|\zeta_y\|_0 \lesssim \|u^{\star}\|_{X_0}.$$

Finally, we note that

$$\tau_k^{\star}(y) - \int_{-\frac{1}{\beta_0}}^{0}\tilde{\sigma}^2 G(y,\tilde{y})\tau_k^{\star}(\tilde{y})\,\mathrm{d}\tilde{y} = \tilde{\sigma}^2\left(\int_{-\frac{1}{\beta_0}}^{y} F(y,\tilde{y})\tau_{k_{\tilde{y}}}^{\star}(\tilde{y})\,\mathrm{d}\tilde{y} + \int_{y}^{0} F(y,\tilde{y})\tau_{k_{\tilde{y}}}^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right)$$

and hence

$$\int_{-\frac{1}{\beta_0}}^{0}\left|\tau_k^{\star}(y) - \int_{-\frac{1}{\beta_0}}^{0}\tilde{\sigma}^2 G(y,\tilde{y})\tau_k^{\star}(\tilde{y})\,\mathrm{d}\tilde{y}\right|^2\,\mathrm{d}y \lesssim \frac{1}{\tilde{\sigma}^2}\int_{-\frac{1}{\beta_0}}^{0}|\tau_{k_{\tilde{y}}}^{\star}(\tilde{y})|^2\,\mathrm{d}\tilde{y}.$$

From equations (3.29) and (3.32) we obtain

$$-(\tilde{\sigma}^2\tau_k + S_1^{-2}\tau_{k_{yy}}) = \frac{\zeta_k^{\star}}{\mu_1} + \mathrm{i}s\tau_k^{\star}$$

and find that

$$S_1^{-2}\tau_{k_{yy}} = \tilde{\sigma}^2\tau_k - \frac{\zeta_k^{\star}}{\mu_1} - \mathrm{i}s\tau_k^{\star}$$

which shows that

$$\|\tau_{yy}\|_0 \lesssim \|u^{\star}\|_{X_0}.$$

The remaining estimates

$$\begin{aligned}&\|\tau'\|_0 + \|\tau_z'\|_0 + \|\zeta'\|_0 + \|\tau_y'\|_0 \lesssim \frac{1}{|s|}\|u^{\star}\|_{X_0}, \\ &\|\tau'\|_0 + \|\tau_z'\|_0 + \|\tau_{zz}'\|_0 + \|\zeta'\|_0 + \|\zeta_z\|_0 + \|\tau_y'\|_0 + \|\tau_{yz}'\|_0 + \|\zeta_y'\|_0 + \|\tau_{yy}'\|_0 \lesssim \|u^{\star}\|_{X_0}.\end{aligned}$$

are derived in an analogous way. □

Lemma 37 shows in particular that $\pm \mathrm{i}s \in \rho(L)$ for sufficiently large values of $|s|$ and that the resolvent estimate (H2) is satisfied. Since X_1 is compactly embedded in X_0 we conclude that $(L - \mathrm{i}sI)^{-1} : X_0 \to X_1$ is continuous and $(L - \mathrm{i}sI)^{-1} : X_0 \to X_0$ is compact for sufficiently large values of $|s|$. It follows that L is a regular operator, that is its spectrum consists entirely of isolated eigenvalues of finite multiplicity with no accumulation points (see Kato [28, Theorem III.6.29]). Moreover $\sigma(L) \cap \mathrm{i}\mathbb{R}$ is a finite set, so that (H1) is satisfied.

3.3 The reduced Hamiltonian system

In this section we choose appropriate coordinates for the reduced Hamiltonian system. We keep the notation of Theorem 33 and note that the space X_0 admits the decomposition $X_0 = X_{0,1} \oplus X_{0,2}$, where $X_{0,1} = P(X_0)$, $X_{0,2} = (I - P)(X_0)$ and P is the spectral projection corresponding the purely imaginary part of $\sigma(L)$. The centre manifold $\tilde{M}^\varepsilon$ is equipped with the single coordinate chart $\tilde{U}_1 \subset X_{0,1}$ and coordinate map $\pi : \tilde{M}^\varepsilon \to \tilde{U}_1$ defined by $\pi^{-1}(u_1) = u_1 + r(u_1; \varepsilon)$. (We choose k large enough so that the reduction function r and hence the reduced Hamiltonian is sufficiently smooth for all subsequent calculations.) It is however more convenient to use an alternative map for calculations. According to a parameter-dependent version of Darboux's theorem (e.g. see Buffoni and Groves [3, Theorem 4]) there exists a near-identity change of variable

$$u_1 = \hat{u}_1 + D(\hat{u}_1; \varepsilon)$$

which transforms Ω^ε into $\hat{\Omega}$, where

$$\hat{\Omega}(v, w) = \tilde{\Omega}^0|_0(v, w) = \Omega(v, w).$$

We define the function $\tilde{r} : \hat{U}_1 \times \tilde{\Lambda} \to \tilde{U}_1 \times \tilde{U}_2$ with $\hat{U}_1 = PG^{-1}(\tilde{U}_1 \times \tilde{U}_2)$ (which in general has components in $X_{0,1}$ and $X_{0,2}$) by the formula

$$\hat{u}_1 + \tilde{r}(\hat{u}_1; \varepsilon) = G^{-1} \left(\hat{u}_1 + D(\hat{u}_1; \varepsilon) + r(\hat{u}_1 + D(\hat{u}_1; \varepsilon); \varepsilon) \right),$$

where $\tilde{r}(0; 0) = 0$, $\mathrm{d}_1\tilde{r}[0; 0] = 0$, and equip $\tilde{M}^\varepsilon$ with the coordinate map $\tilde{\pi} : \tilde{M}^\varepsilon \to \hat{U}_1$ given by $\tilde{\pi}^{-1}(\hat{u}_1) = \hat{u}_1 + \tilde{r}(\hat{u}_1; \varepsilon)$, so that

$$\begin{aligned} \tilde{H}^\varepsilon(\hat{u}_1) &= H^\varepsilon(\hat{u}_1 + \tilde{r}(\hat{u}_1; \varepsilon)), \\ \tilde{\Omega}^\varepsilon|_{\hat{u}_1}(v, w) &= \Omega(v, w). \end{aligned}$$

To find a suitable basis for $X_{0,1}$ we determine the purely imaginary eigenvalues and corresponding (generalised) eigenvectors of L. Substituting $\lambda = \mathrm{i}s$ with $s \in \mathbb{R}$ into equation (3.26) yields the equation

$$\gamma_0 = r(\tilde{\sigma}) := \left(\mu_1(\mu_1 - 1)^2 \left(\mu_1 \tilde{\sigma} \coth \frac{\tilde{\sigma}}{\beta_0} + S_1 \tilde{\sigma} \coth \frac{S_1 \tilde{\sigma}}{\beta_0} \right)^{-1} - 1 \right) \tilde{\sigma}^2$$

for $\tilde{\sigma}^2 = s^2 + k^2\nu^2$. The function $\tilde{\sigma} \mapsto r(\tilde{\sigma})$, which satisfies $r(0) = 0$ and $r(\tilde{\sigma}) \to -\infty$ as $\tilde{\sigma} \to \infty$, takes only negative values for $\beta_0 > \mu_1(\mu_1 - 1)^2/(\mu_1 + 1)$, while for

$\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ it has a unique maximum ω with $r(\omega) > 0$ (see Figure 3.1); we choose $\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ and $\gamma_0 = r(\omega)$, and note the relationships

$$\beta_0 = \frac{\mu_1(\mu_1-1)^2}{2\tilde{\omega}}\left(\frac{h(\tilde{\omega}) - \tilde{\omega}\dot{h}(\tilde{\omega})}{h(\tilde{\omega})^2}\right), \qquad \gamma_0 = \left(\frac{\mu_1(\mu_1-1)^2}{\omega h(\tilde{\omega})} - 1\right)\omega^2,$$

where $\tilde{\omega} = \omega/\beta_0$ and $h(\tilde{\omega}) = \mu_1 \coth\tilde{\omega} + S_1 \coth S_1\tilde{\omega}$.

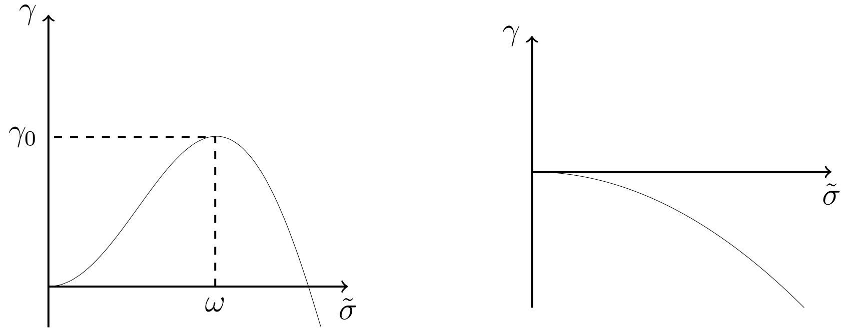


Figure 3.1:
The graph of the function $\tilde{\sigma} \mapsto r(\tilde{\sigma})$ for $\beta_0 > \mu_1(\mu_1-1)^2/(\mu_1+1)$ (left) and $\beta_0 < \mu_1(\mu_1-1)^2/(\mu_1+1)$ (right).

With the above choices of β_0 and γ_0 we find that $\pm\mathrm{i}\omega$ are mode 0 eigenvalues of L, and an additional pair $\pm\mathrm{i}s$ of mode k eigenvalues, which are always geometrically double, arises whenever $\gamma_0 = r(\sqrt{s^2+k^2\nu^2})$ (so that $s = \omega_k := \sqrt{\omega^2 - k^2\nu^2}$). By choosing ν with $\frac{\omega}{m+1} < \nu < \frac{\omega}{m}$, $m \in \mathbb{N}$, we find therefore m additional pairs of eigenvalues (see Figure 3.2).

We proceed by choosing $\frac{\omega}{2} < \nu < \omega$, so that $\pm\mathrm{i}\omega$ and $\pm\mathrm{i}\omega_1$ are respectively mode 0 and mode 1 eigenvalues of L. Straightforward calculations show that each of these eigenvalues has an associated Jordan chain of length 2; we find that $(L_0 - \mathrm{i}\omega)e_{\omega,1} = 0$, $(L_0 - \mathrm{i}\omega)e_{\omega,2} = e_{\omega,1}$, $(L_1 - \mathrm{i}\omega_1)e_{\omega_1,1} = 0$, $(L_1 - \mathrm{i}\omega_1)e_{\omega_1,2} = e_{\omega_1,1}$ and $(L_0 + \mathrm{i}\omega)\overline{e_{\omega,1}} = 0$, $(L_0 + \mathrm{i}\omega)\overline{e_{\omega,2}} = \overline{e_{\omega,1}}$, $(L_1 + \mathrm{i}\omega_1)\overline{e_{\omega_1,1}} = 0$, $(L_1 + \mathrm{i}\omega_1)\overline{e_{\omega_1,2}} = \overline{e_{\omega_1,1}}$, where

$$e_{s,1} = \begin{pmatrix} \dfrac{\sinh\frac{\omega}{\beta_0}\cosh\frac{S_1\omega}{\beta_0} + \frac{\mu_1}{S_1}\sinh\frac{S_1\omega}{\beta_0}\cosh\frac{\omega}{\beta_0}}{\mu_1 - 1} \\[2ex] -\dfrac{\mu_1}{S_1}\sinh\dfrac{S_1\omega}{\beta_0}\cosh\omega\left(y - \dfrac{1}{\beta_0}\right) \\[2ex] \sinh\dfrac{\omega}{\beta_0}\cosh S_1\omega\left(y + \dfrac{1}{\beta_0}\right) \\[2ex] -\mathrm{i}s\dfrac{\sinh\frac{\omega}{\beta_0}\cosh\frac{S_1\omega}{\beta_0} + \frac{\mu_1}{S_1}\sinh\frac{S_1\omega}{\beta_0}\cosh\frac{\omega}{\beta_0}}{\mu_1 - 1} \\[2ex] -\mathrm{i}s\dfrac{\mu_1}{S_1}\sinh\dfrac{S_1\omega}{\beta_0}\cosh\omega\left(y - \dfrac{1}{\beta_0}\right) \\[2ex] \mathrm{i}\mu_1 s\sinh\dfrac{\omega}{\beta_0}\cosh S_1\omega\left(y + \dfrac{1}{\beta_0}\right) \end{pmatrix}$$

and

$$
e_{s,2} = \begin{pmatrix}
\begin{array}{l}
-\mathrm{i}\dfrac{s}{\omega\beta_0(\mu_1-1)}\left((\mu_1+1)\cosh\dfrac{\omega}{\beta_0}\cosh\dfrac{S_1\omega}{\beta_0}\right.\\
\qquad\left. +\, S_1\left(1+\dfrac{\mu_1}{S_1^2}\right)\sinh\dfrac{\omega}{\beta_0}\sinh\dfrac{S_1\omega}{\beta_0}\right)
\end{array}\\[2ex]
\begin{array}{l}
\mathrm{i}\dfrac{s\mu_1}{S_1\omega}\left(y-\dfrac{1}{\beta_0}\right)\sinh\dfrac{S_1\omega}{\beta_0}\sinh\omega\left(y-\dfrac{1}{\beta_0}\right)\\
\qquad +\mathrm{i}\dfrac{s\mu_1}{\omega\beta_0}\cosh\dfrac{S_1\omega}{\beta_0}\cosh\omega\left(y-\dfrac{1}{\beta_0}\right)
\end{array}\\[2ex]
\begin{array}{l}
-\mathrm{i}\dfrac{sS_1}{\omega}\left(y+\dfrac{1}{\beta_0}\right)\sinh\dfrac{\omega}{\beta_0}\sinh S_1\omega\left(y+\dfrac{1}{\beta_0}\right)\\
\qquad -\mathrm{i}\dfrac{s}{\omega\beta_0}\cosh\dfrac{\omega}{\beta_0}\cosh S_1\omega\left(y+\dfrac{1}{\beta_0}\right)
\end{array}\\[2ex]
\begin{array}{l}
-\dfrac{s^2}{\omega\beta_0(\mu_1-1)}\left((\mu_1+1)\cosh\dfrac{\omega}{\beta_0}\cosh\dfrac{S_1\omega}{\beta_0}\right.\\
\qquad\left. +\, S_1\left(1+\dfrac{\mu_1}{S_1^2}\right)\sinh\dfrac{\omega}{\beta_0}\sinh\dfrac{S_1\omega}{\beta_0}\right)\\
-\dfrac{\sinh\dfrac{\omega}{\beta_0}\cosh\dfrac{S_1\omega}{\beta_0}+\dfrac{\mu_1}{S_1}\sinh\dfrac{S_1\omega}{\beta_0}\cosh\dfrac{\omega}{\beta_0}}{\mu_1-1}
\end{array}\\[2ex]
\begin{array}{l}
-\dfrac{s^2\mu_1}{S_1\omega}\left(y-\dfrac{1}{\beta_0}\right)\sinh\dfrac{S_1\omega}{\beta_0}\sinh\omega\left(y-\dfrac{1}{\beta_0}\right)\\
\qquad -\dfrac{s^2\mu_1}{\omega\beta_0}\cosh\dfrac{S_1\omega}{\beta_0}\cosh\omega\left(y-\dfrac{1}{\beta_0}\right)\\
\qquad -\dfrac{\mu_1}{S_1}\sinh\dfrac{S_1\omega}{\beta_0}\cosh\omega\left(y-\dfrac{1}{\beta_0}\right)
\end{array}\\[2ex]
\begin{array}{l}
\dfrac{s^2S_1}{S_1\omega}\left(y+\dfrac{1}{\beta_0}\right)\sinh\dfrac{\omega}{\beta_0}\sinh S_1\omega\left(y+\dfrac{1}{\beta_0}\right)\\
\qquad +\dfrac{s^2}{\omega\beta_0}\cosh\dfrac{\omega}{\beta_0}\cosh S_1\omega\left(y+\dfrac{1}{\beta_0}\right)\\
\qquad +\sinh\dfrac{\omega}{\beta_0}\cosh S_1\omega\left(y+\dfrac{1}{\beta_0}\right)
\end{array}
\end{pmatrix}.
$$

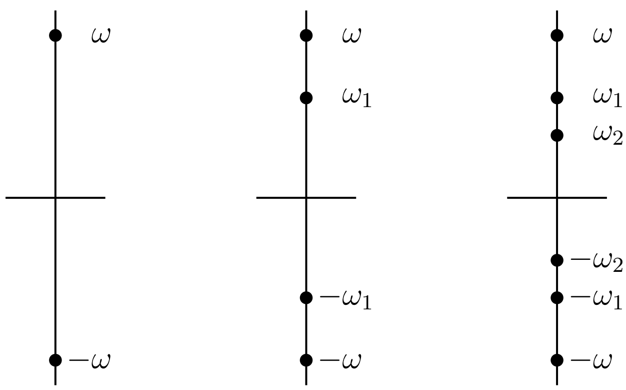


Figure 3.2:
The purely imaginary non-zero eigenvalues of L for ν such that $\nu > \omega$ (left), $\frac{\omega}{2} < \nu < \omega$ (centre) and $\frac{\omega}{3} < \nu < \frac{\omega}{2}$ (right).

Furthermore 0 is a mode 0 geometrically simple eigenvalue whose algebraic multiplicity is 2; we find that

$$e_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \mu_1 \end{pmatrix},$$

where $Le_1 = 0, Le_2 = e_1$. Figure 3.3 shows how the eigenvalue configuration changes as γ_0 is varied through $r(\omega)$: the Jordan chains associated with the eigenvalues $\pm\mathrm{i}\omega$ and $\pm\mathrm{i}\omega_1$ resolve into pairs of imaginary eigenvalues for $\gamma_0 < r(\omega)$ and into pairs of complex eigenvalues for $\gamma_0 > r(\omega)$. (Note that $\sigma(L)$ is always symmetric with respect to the real and imaginary axes.)

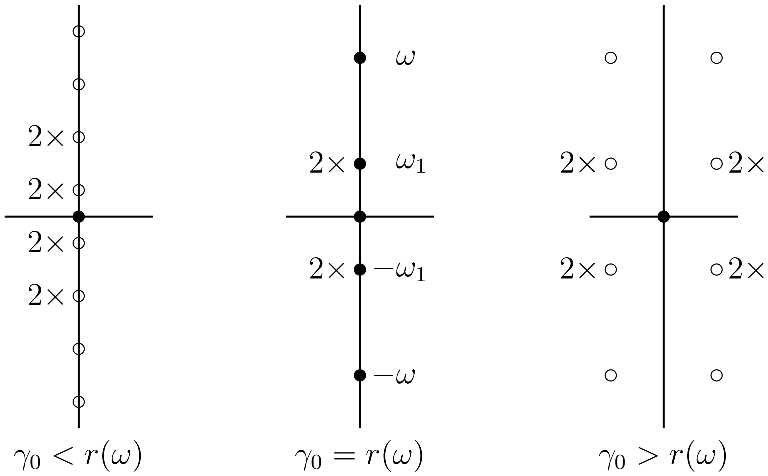


Figure 3.3:
Eigenvalues of L for $\gamma_0 < r(\omega)$ (left), $\gamma_0 = r(\omega)$ (centre) and $\gamma_0 > r(\omega)$ (right); hollow and solid dots denote eigenvalues with an associated Jordan chain of length 1 and 2 respectively.

We use a basis for $X_{0,1}$ with respect to which Ω is the canonical symplectic 2-form (a *symplectic basis*). We choose $f_0^1, f_0^2, \ldots, f_3^1, f_3^2, \overline{f_1^1}, \overline{f_1^2}, \ldots, \overline{f_3^1}, \overline{f_3^2}$ with

$$\begin{aligned}
f_0^1 &= \tilde{\kappa}^{-\frac{1}{2}} e_1, & f_0^2 &= \tilde{\kappa}^{-\frac{1}{2}} e_2, \\
f_1^1 &= \tilde{\kappa}_{\omega_1}^{-\frac{1}{2}} e_{\omega_1,1} \mathrm{e}^{\mathrm{i}z}, & f_1^2 &= \tilde{\kappa}_{\omega_1}^{-\frac{1}{2}} \left(e_{\omega_1,2} - \frac{\mathrm{i}\kappa_{\omega_1}}{2\tilde{\kappa}_{\omega_1}} e_{\omega_1,1} \right) \mathrm{e}^{\mathrm{i}z}, \\
f_2^1 &= \tilde{\kappa}_{\omega_1}^{-\frac{1}{2}} e_{\omega_1,1} \mathrm{e}^{-\mathrm{i}z}, & f_2^2 &= \tilde{\kappa}_{\omega_1}^{-\frac{1}{2}} \left(e_{\omega_1,2} - \frac{\mathrm{i}\kappa_{\omega_1}}{2\tilde{\kappa}_{\omega_1}} e_{\omega_1,1} \right) \mathrm{e}^{-\mathrm{i}z}, \\
f_3^1 &= \tilde{\kappa}_{\omega}^{-\frac{1}{2}} e_{\omega,1}, & f_3^2 &= \tilde{\kappa}_{\omega}^{-\frac{1}{2}} \left(e_{\omega,2} - \frac{\mathrm{i}\kappa_{\omega}}{2\tilde{\kappa}_{\omega}} e_{\omega,1} \right)
\end{aligned}$$

such that $\Omega(f_0^1, \overline{f_0^2}) = 1$,

$$\begin{aligned}
\Omega(f_j^1, \overline{f_j^2}) &= \Omega(\overline{f_j^1}, f_j^2) = 1, \\
\Omega(f_j^2, \overline{f_j^1}) &= \Omega(\overline{f_j^2}, f_j^1) = -1
\end{aligned}$$

for $j = 1, 2, 3$ and the symplectic products of all other combinations are zero (note that Ω acts bilinearly on pairs of complex vectors). The real constants $\tilde{\kappa}$ and $\tilde{\kappa}_s$ are given by

$$
\begin{aligned}
\tilde\kappa &= \frac{\mu_1+1}{\beta_0},\\
\tilde\kappa_s &= \frac{1}{2S_1^2(\mu_1-1)^2\omega\beta_0}\Bigg(\frac{\beta_0^2\omega(S_1^2+\mu_1^2)}{2} - s^2(S_1^2-\mu_1^2)\left(\sinh\frac{2\omega}{\beta_0} - S_1\sinh\frac{2S_1\omega}{\beta_0}\right) - \frac{\beta_0\omega}{2}(S_1^2-\mu_1^2)\left(\cosh\frac{2\omega}{\beta_0}-\cosh\frac{2S_1\omega}{\beta_0}\right)\\
&\qquad - \frac{\beta_0\omega}{2}(S_1^2+\mu_1^2)\cosh\frac{2\omega}{\beta_0}\cosh\frac{2S_1\omega}{\beta_0} - S_1\mu_1\beta_0\omega\sinh\frac{2\omega}{\beta_0}\sinh\frac{2S_1\omega}{\beta_0}\\
&\qquad - s^2(S_1^2+\mu_1^2+2\mu_1S_1^2)\sinh\frac{2\omega}{\beta_0}\cosh\frac{2S_1\omega}{\beta_0} - s^2(S_1^2+\mu_1^2+2\mu_1)S_1\cosh\frac{2\omega}{\beta_0}\sinh\frac{2S_1\omega}{\beta_0}\Bigg)\\
&\quad + \frac{\mu_1\sinh\frac{\omega}{\beta_0}}{2\beta_0^2\omega^3S_1}\Bigg(S_1\beta_0\omega^3\sinh\frac{\omega}{\beta_0} + 2S_1\omega^2s^2\cosh\frac{\omega}{\beta_0} + \left(\omega s^2\beta_0\cosh\frac{\omega}{\beta_0} - \frac{\beta_0(s^2-\omega^2)}{2}\sinh\frac{\omega}{\beta_0}\right)\sinh\frac{2S_1\omega}{\beta_0}\\
&\qquad + S_1\omega s^2\beta_0\sinh\frac{\omega}{\beta_0}\cosh\frac{2S_1\omega}{\beta_0}\Bigg)\\
&\quad + \frac{\mu_1\sinh\frac{S_1\omega}{\beta_0}}{2\beta_0^2\omega^3S_1^2}\Bigg(\mu_1\beta_0\omega^3\sinh\frac{S_1\omega}{\beta_0} + 2\mu_1S_1\omega^2s^2\cosh\frac{S_1\omega}{\beta_0} + \mu_1\left(S_1\omega s^2\beta_0\cosh\frac{S_1\omega}{\beta_0} - \frac{\beta_0(s^2-\omega^2)}{2}\sinh\frac{S_1\omega}{\beta_0}\right)\sinh\frac{2\omega}{\beta_0}\\
&\qquad + \mu_1\omega s^2\beta_0\sinh\frac{S_1\omega}{\beta_0}\cosh\frac{2\omega}{\beta_0}\Bigg),\\
\kappa_s &= \frac{\mu_1 s}{2S_1^2\omega^5\beta_0^2}\Bigg(\left(S_1\omega\beta_0(s^2-\omega^2)\cosh^2\frac{\omega}{\beta_0} - 2s^2S_1\omega^2\sinh\frac{\omega}{\beta_0}\cosh\frac{\omega}{\beta_0}\right)\cosh\frac{2S_1\omega}{\beta_0}\\
&\qquad + \frac{\mu_1}{S_1}\left(\omega\beta_0(s^2-\omega^2)\cosh^2\frac{S_1\omega}{\beta_0} - 2s^2S_1\omega^2\sinh\frac{S_1\omega}{\beta_0}\cosh\frac{S_1\omega}{\beta_0}\right)\cosh\frac{2\omega}{\beta_0}\\
&\qquad + \Bigg(\omega\beta_0(s^2-\omega^2)\cosh\frac{\omega}{\beta_0}\sinh\frac{\omega}{\beta_0} - \left(\frac{\beta_0^2}{2}(s^2-\omega^2) + s^2\omega^2(S_1^2+1)\right)\cosh^2\frac{\omega}{\beta_0}\\
&\qquad\quad + \frac{\beta_0^2}{2}(s^2-\omega^2) + s^2\omega S_1\beta_0 + S_1\omega^2(S_1s^2-\omega\beta_0)\Bigg)\sinh\frac{2S_1\omega}{\beta_0}\\
&\qquad + \frac{\mu_1}{S_1}\Bigg(\omega S_1\beta_0(s^2-\omega^2)\cosh\frac{S_1\omega}{\beta_0}\sinh\frac{S_1\omega}{\beta_0} - \left(\frac{\beta_0^2}{2}(s^2-\omega^2)+s^2\omega^2(S_1^2+1)\right)\cosh^2\frac{S_1\omega}{\beta_0}\\
&\qquad\quad + \frac{\beta_0^2}{2}(s^2-\omega^2) + s^2\omega\beta_0 + \omega^2(s^2-\omega\beta_0)\Bigg)\sinh\frac{2\omega}{\beta_0} - S_1\omega\beta_0(s^2-\omega^2)\left(\cosh\frac{2S_1\omega}{\beta_0}+\sinh\frac{2S_1\omega}{\beta_0}\right)\\
&\qquad - \frac{\mu_1}{S_1}\omega\beta_0(s^2-\omega^2)\left(\cosh\frac{2\omega}{\beta_0}+\sinh\frac{2\omega}{\beta_0}\right)\Bigg)\\
&\quad + \frac{s}{3S_1^2\omega^2\beta_0^3(\mu_1-1)^2}\Bigg(6\beta_0s^2\left((S_1^2+1)(\mu_1^2+S_1^2)+4S_1^2\mu_1\right)\cosh^2\frac{\omega}{\beta_0}\cosh^2\frac{S_1\omega}{\beta_0}\\
&\qquad + 6\beta_0\omega(2S_1^2\mu_1+S_1^2+\mu_1^2)\cosh\frac{\omega}{\beta_0}\sinh\frac{\omega}{\beta_0}\cosh^2\frac{S_1\omega}{\beta_0}\\
&\qquad + 6\beta_0\omega S_1(2\mu_1+S_1^2+\mu_1^2)\cosh\frac{S_1\omega}{\beta_0}\sinh\frac{S_1\omega}{\beta_0}\cosh^2\frac{\omega}{\beta_0}\\
&\qquad + \left(s^2\mu_1^2(\mu_1-1)^2(1-3S_1^2) - 6s^2\beta_0(S^2+\mu_1)^2\right)\cosh^2\frac{S_1\omega}{\beta_0}\\
&\qquad + \left(s^2\mu_1^2S_1^2(\mu_1-1)^2(S_1^2-3) - 6s^2\beta_0(S^2+\mu_1)^2\right)\cosh^2\frac{\omega}{\beta_0}\\
&\qquad + 12s^2S_1\beta_0(\mu_1+1)(S_1+\mu_1)\cosh\frac{\omega}{\beta_0}\sinh\frac{\omega}{\beta_0}\cosh\frac{S_1\omega}{\beta_0}\sinh\frac{S_1\omega}{\beta_0}\\
&\qquad - 3\omega\left(2\beta_0^2\mu_1(S_1^2+\mu_1) + \beta_0S_1^2\mu_1(\mu_1-1)^2\right)\cosh\frac{\omega}{\beta_0}\sinh\frac{\omega}{\beta_0}\\
&\qquad - 3\omega\left(2S_1\beta_0^2(S_1^2+\mu_1) + \beta_0S_1\mu_1^2(\mu_1-1)^2\right)\cosh\frac{S_1\omega}{\beta_0}\sinh\frac{S_1\omega}{\beta_0}\\
&\qquad - s^2(S_1^2(S_1^2(\mu_1^2(\mu_1-1)^2-6\beta_0)-12\beta_0\mu_1)-6\mu_1^2\beta_0)\\
&\qquad - s^2\mu_1^2(\mu_1-1)^2\Bigg).
\end{aligned}
$$

The coordinates $q_0, p_0, a_1, b_1, \ldots, a_3, b_3, \overline{a_1}, \overline{b_1}, \ldots, \overline{a_3}, \overline{b_3}$ in the $f_0^1, f_0^2, \ldots, f_3^1, f_3^2, \overline{f_1^1}, \overline{f_1^2}, \ldots, \overline{f_3^1}, \overline{f_3^2}$ directions are canonical coordinates for the reduced Hamiltonian system and the actions of the reverser S, the reflection $T : z \mapsto -z$ and the translation (rotation) $R_\alpha : z \mapsto z + \alpha$,

$\alpha \in \mathbb{R}$ on this space are given by

$$
\begin{aligned}
S(q_0, p_0, a_1, b_1, a_2, b_2, a_3, b_3) &= (-q_0, p_0, \overline{a}_1, -\overline{b}_1, \overline{a}_2, -\overline{b}_2, \overline{a}_3, -\overline{b}_3),\\
T(q_0, p_0, a_1, b_1, a_2, b_2, a_3, b_3) &= (q_0, p_0, a_2, b_2, a_1, b_1, a_3, b_3),\\
R_\alpha(q_0, p_0, a_1, b_1, a_2, b_2, a_3, b_3) &= (q_0, p_0, \mathrm{e}^{\mathrm{i}\alpha}a_1, \mathrm{e}^{\mathrm{i}\alpha}b_1, \mathrm{e}^{-\mathrm{i}\alpha}a_2, \mathrm{e}^{-\mathrm{i}\alpha}b_2, a_3, b_3).
\end{aligned}
$$

The reduced system inherits the reversibility, reflection and translation symmetries of (3.11)–(3.21); in particular $\tilde{H}^\varepsilon$ is invariant under S, T and R_α for all $\alpha \in \mathbb{R}$. Furthermore, equations (3.11)–(3.21) are invariant under the transformation $(\chi', \chi) \mapsto (\chi' + c, \chi + c), c \in \mathbb{R}$, that is these equations and $\tilde{H}^\varepsilon$ not depend upon q_0 (which is a cyclic variable), so that p_0 is a conserved quantity. We write the reduced Hamiltonian system as

$$
p_{0x} = \frac{\partial \tilde{H}^\varepsilon}{\partial q_0}(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, p_0), \qquad q_{0x} = -\frac{\partial \tilde{H}^\varepsilon}{\partial p_0}(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, p_0), \tag{3.38}
$$

$$
a_{jx} = \frac{\partial \tilde{H}^\varepsilon}{\partial \overline{b_j}}(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, p_0), \qquad b_{jx} = -\frac{\partial \tilde{H}^\varepsilon}{\partial \overline{a_j}}(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, p_0), \qquad j = 1, 2, 3, \tag{3.39}
$$

where $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$ and

$$
\begin{aligned}
\tilde{H}^\varepsilon(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, p_0) &= \mathrm{i}\omega_1(a_1\overline{b_1} - \overline{a_1}b_1 + a_2\overline{b_2} - \overline{a_2}b_2) + \mathrm{i}\omega(a_3\overline{b_3} - \overline{a_3}b_3)\\
&\qquad + |\mathbf{b}|^2 + \frac{1}{2}p_0^2 + O(|(\varepsilon, p_0, \mathbf{a}, \mathbf{b})||(p_0, \mathbf{a}, \mathbf{b})|^2).
\end{aligned}
$$

Note that

$$
\{(q_0, p_0, \mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}) : a_1, a_2, b_1, b_2 = 0\}
$$

is an invariant subspace for the equations (3.38) and (3.39), solutions in which correspond to two-dimensional, that is z-independent, solutions of the physical problem.

According to the classical theory, the next step is to lower the dimension of the reduced system by two by setting $p_0 = 0$, solving the resulting decoupled system (3.39) for $a_1, b_1, \ldots, a_3, b_3$, $\overline{a_1}, \overline{b_1}, \ldots, \overline{a_3}, \overline{b_3}$ and recovering q_0 from (3.38) by quadrature. The lower-order system is typically studied using a canonical change of variables which simplifies its Hamiltonian $\tilde{H}^\varepsilon\big|_{p_0=0}$ (a 'normal-form' transformation) by transforming it to

$$
\begin{aligned}
&\mathrm{i}\omega_1(A_1\overline{B}_1 - \overline{A}_1B_1 + A_2\overline{B}_2 - \overline{A}_2B_2) + \mathrm{i}\omega(A_3\overline{B}_3 - \overline{A}_3B_3)\\
&\qquad + |\mathbf{B}|^2 + \tilde{K}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) + O(|(\mathbf{A}, \mathbf{B})|^2|(\varepsilon, \mathbf{A}, \mathbf{B})|^{n_0})
\end{aligned} \tag{3.40}
$$

for some $n_0 \geq 2$, where $\tilde{K}^\varepsilon$ is a polynomial of order $n_0 + 1$ of its arguments and ε with

$$
\tilde{K}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) = O(|(\mathbf{A}, \mathbf{B})|^2|(\varepsilon, \mathbf{A}, \mathbf{B})|).
$$

In the present context it is convenient to use a normal-form transformation before lowering the order of the system since it can be 'absorbed' into the changes of variable. The following result (see Groves and Nilsson [16, Theorem 4.4]) shows that this procedure is possible.

Theorem 38. *There exists a near-identity, parameter-dependent canonical change of variables* $(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, q_0, p_0) \mapsto (\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, Q_0, P_0)$ *for* (3.38), (3.39) *with the properties that Q_0 is cyclic, $P_0 = p_0$ and the lower-order Hamiltonian system*

$$A_{jx} = \frac{\partial \tilde{H}^\varepsilon}{\partial \overline{B_j}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, 0), \qquad B_{jx} = -\frac{\partial \tilde{H}^\varepsilon}{\partial \overline{A_j}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, 0), \qquad j = 1, 2, 3,$$

adopts its usual normal form, that is $\tilde{H}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, 0)$ *is given by* (3.40). *(Here, with a slight abuse of notation, we denote the transformed Hamiltonian by* $\tilde{H}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, P_0)$.*)*

Proof. We write $\mathbf{q} = (\mathbf{a}, \overline{\mathbf{a}}), \mathbf{p} = (\overline{\mathbf{b}}, \mathbf{b})$ and consider the six-degree of freedom Hamiltonian system

$$a_{jx} = \frac{\partial \tilde{H}^\varepsilon}{\partial \overline{b_j}}, \qquad b_{jx} = -\frac{\partial \tilde{H}^\varepsilon}{\partial \overline{a_j}}, \qquad j = 1, 2, 3, \tag{3.41}$$

in which p_0 and ε are parameters. The standard theory asserts the existence of a canonical change of variables

$$\mathbf{Q} = \mathbf{q} + \mathbf{h}_1^\varepsilon(\mathbf{q}, \mathbf{p}, p_0), \qquad \mathbf{P} = \mathbf{p} + \mathbf{h}_2^\varepsilon(\mathbf{q}, \mathbf{p}, p_0)$$

with

$$\mathbf{h}_j^\varepsilon(\mathbf{q}, \mathbf{p}, p_0) = (|(\varepsilon, p_0, \mathbf{q}, \mathbf{p})||(\mathbf{q}, \mathbf{p})|), \qquad j = 1, 2,$$

which converts (3.41) into its parameter-dependent normal form. Note that

$$M_1^{\mathrm{T}} J_1 M_1 = J_1,$$

where

$$M_1 = \begin{pmatrix} \mathbf{Q_q} & \mathbf{Q_p} \\ \mathbf{P_q} & \mathbf{P_p} \end{pmatrix} = \begin{pmatrix} I + \partial_\mathbf{q} \mathbf{h}_1^\varepsilon & \partial_\mathbf{p} \mathbf{h}_1^\varepsilon \\ \partial_\mathbf{q} \mathbf{h}_2^\varepsilon & I + \partial_\mathbf{p} \mathbf{h}_2^\varepsilon \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and the operators $\partial_\mathbf{q} = (\partial_{a_1}, \partial_{a_2}, \partial_{a_3}, \partial_{\overline{a_1}}, \partial_{\overline{a_2}}, \partial_{\overline{a_3}})$, $\partial_\mathbf{p} = (\partial_{\overline{b_1}}, \partial_{\overline{b_2}}, \partial_{\overline{b_3}}, \partial_{b_1}, \partial_{b_2}, \partial_{b_3})$ are applied elementwise; this condition may be written as

$$(I + (\partial_\mathbf{q} \mathbf{h}_1^\varepsilon)^{\mathrm{T}})(I + \partial_\mathbf{p} \mathbf{h}_2^\varepsilon) - (\partial_\mathbf{q} \mathbf{h}_2^\varepsilon)^{\mathrm{T}} \partial_\mathbf{p} \mathbf{h}_1^\varepsilon = I, \tag{3.42}$$

$$(I + (\partial_\mathbf{q} \mathbf{h}_1^\varepsilon)^{\mathrm{T}}) \partial_\mathbf{q} \mathbf{h}_2^\varepsilon - (\partial_\mathbf{q} \mathbf{h}_2^\varepsilon)^{\mathrm{T}} (I + \partial_\mathbf{q} \mathbf{h}_1^\varepsilon) = 0, \tag{3.43}$$

$$(\partial_\mathbf{p} \mathbf{h}_1^\varepsilon)^{\mathrm{T}} (I + \partial_\mathbf{p} \mathbf{h}_2^\varepsilon) - (I + (\partial_\mathbf{p} \mathbf{h}_2^\varepsilon)^{\mathrm{T}}) \partial_\mathbf{p} \mathbf{h}_1^\varepsilon = 0. \tag{3.44}$$

We seek a change of variable for (3.38), (3.39) of the form

$$\mathbf{Q} = \mathbf{q} + \mathbf{h}_1^\varepsilon(\mathbf{q}, \mathbf{p}, p_0), \qquad \mathbf{P} = \mathbf{p} + \mathbf{h}_2^\varepsilon(\mathbf{q}, \mathbf{p}, p_0), \qquad Q_0 \;\; = q_0 + h_3^\varepsilon(\mathbf{q}, \mathbf{p}, p_0), \qquad P_0 = p_0,$$

where

$$\mathbf{Q} = (A_1, A_2, A_3, \overline{A}_1, \overline{A}_2, \overline{A_3})^{\mathrm{T}}, \qquad \mathbf{P} = (\overline{B}_1, \overline{B}_2, \overline{B_3}, B_1, B_2, B_3)^{\mathrm{T}};$$

the function

$$h_3^\varepsilon(\mathbf{q}, \mathbf{p}, p_0) = (|(\varepsilon, p_0, \mathbf{q}, \mathbf{p})||(\mathbf{q}, \mathbf{p})|)$$

is subject to the requirement that

$$M_2^{\mathrm{T}} J_2 M_2 = J_2, \tag{3.45}$$

where

$$M_2 = \begin{pmatrix} \mathbf{Q}_\mathbf{q} & \mathbf{Q}_\mathbf{p} & \mathbf{Q}_{q_0} & \mathbf{Q}_{p_0} \\ \mathbf{P}_\mathbf{q} & \mathbf{P}_\mathbf{p} & \mathbf{P}_{q_0} & \mathbf{P}_{p_0} \\ Q_{0\mathbf{q}} & Q_{0\mathbf{p}} & Q_{0q_0} & Q_{0p_0} \\ P_{0\mathbf{q}} & P_{0\mathbf{p}} & P_{0q_0} & P_{0p_0} \end{pmatrix} = \begin{pmatrix} I + \partial_\mathbf{q}\mathbf{h}_1^\varepsilon & \partial_\mathbf{p}\mathbf{h}_1^\varepsilon & 0 & \partial_{p_0}\mathbf{h}_1^\varepsilon \\ \partial_\mathbf{q}\mathbf{h}_2^\varepsilon & I + \partial_\mathbf{p}\mathbf{h}_2^\varepsilon & 0 & \partial_{p_0}\mathbf{h}_2^\varepsilon \\ \partial_\mathbf{q}h_3^\varepsilon & \partial_\mathbf{p}h_3^\varepsilon & 1 & \partial_{p_0}h_3^\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and equation (3.45) may be written as (3.42)–(3.44) which are automatically satisfied, and

$$(\partial_{p_0}\mathbf{h}_2^\varepsilon)^\mathrm{T}(I + \partial_\mathbf{q}\mathbf{h}_1^\varepsilon) - (\partial_{p_0}\mathbf{h}_1^\varepsilon)^\mathrm{T}\partial_\mathbf{q}\mathbf{h}_2^\varepsilon = \partial_\mathbf{q}h_3^\varepsilon, \tag{3.46}$$

$$(\partial_{p_0}\mathbf{h}_2^\varepsilon)^\mathrm{T}\partial_\mathbf{p}\mathbf{h}_1^\varepsilon - (\partial_{p_0}\mathbf{h}_1^\varepsilon)^\mathrm{T}(I + \partial_\mathbf{p}\mathbf{h}_2^\varepsilon) = \partial_\mathbf{p}h_3^\varepsilon. \tag{3.47}$$

We can solve equation (3.46) for h_3^ε if and only if h_3^ε is conservative, that is

$$\partial_\mathbf{q}^\mathrm{T}\partial_\mathbf{q}h_3^\varepsilon = (\partial_\mathbf{q}^\mathrm{T}\partial_\mathbf{q}h_3^\varepsilon)^\mathrm{T} \tag{3.48}$$

and equation (3.47) is solvable for h_3^ε if and only if

$$\partial_\mathbf{p}^\mathrm{T}\partial_\mathbf{p}h_3^\varepsilon = (\partial_\mathbf{p}^\mathrm{T}\partial_\mathbf{p}h_3^\varepsilon)^\mathrm{T}. \tag{3.49}$$

The compatibility condition that (3.46), (3.47) can be simultaneously solved for h_3^ε is

$$\partial_\mathbf{p}^\mathrm{T}\partial_\mathbf{q}h_3^\varepsilon = (\partial_\mathbf{q}^\mathrm{T}\partial_\mathbf{p}h_3^\varepsilon)^\mathrm{T}. \tag{3.50}$$

Using that

$$\begin{aligned}
\partial_\mathbf{p}^\mathrm{T}(\mathbf{g}^\mathrm{T}M) &= \partial_\mathbf{p}^\mathrm{T}(\mathbf{g}.\mathbf{c}_1(M)|\dots|\mathbf{g}.\mathbf{c}_3(M)) \\
&= \Big((\partial_\mathbf{p}^\mathrm{T}\mathbf{g}^\mathrm{T})\mathbf{c}_1(M) + (\partial_\mathbf{p}^\mathrm{T}\mathbf{c}_1(M)^\mathrm{T})\mathbf{g}\Big|\dots\Big|(\partial_\mathbf{p}^\mathrm{T}\mathbf{g}^\mathrm{T})\mathbf{c}_3(M) + (\partial_\mathbf{p}^\mathrm{T}\mathbf{c}_3(M)^\mathrm{T})\mathbf{g}\Big) \\
&= \Big((\partial_\mathbf{p}^\mathrm{T}\mathbf{g}^\mathrm{T})\mathbf{c}_1(M)\Big|\dots\Big|(\partial_\mathbf{p}^\mathrm{T}\mathbf{g}^\mathrm{T})\mathbf{c}_3(M)\Big) + \Big((\partial_\mathbf{p}^\mathrm{T}\mathbf{c}_1(M)^\mathrm{T})\mathbf{g}\Big|\dots\Big|(\partial_\mathbf{p}^\mathrm{T}\mathbf{c}_3(M)^\mathrm{T})\mathbf{g}\Big) \\
&= (\partial_\mathbf{p}^\mathrm{T}\mathbf{g}^\mathrm{T})M + \Big(\partial_\mathbf{p}^\mathrm{T}\mathbf{c}_1(M)^\mathrm{T}|\dots|\partial_\mathbf{p}^\mathrm{T}\mathbf{c}_3(M)^\mathrm{T}\Big)(I \otimes \mathbf{g}) \\
&= (\partial_\mathbf{p}\mathbf{g})^\mathrm{T}M + (\partial_\mathbf{p}\operatorname{vec}M)^\mathrm{T}(I \otimes \mathbf{g}), \\
\partial_\mathbf{q}^\mathrm{T}(\mathbf{g}^\mathrm{T}M) &= (\partial_\mathbf{q}\mathbf{g})^\mathrm{T}M + (\partial_\mathbf{q}\operatorname{vec}M)^\mathrm{T}(I \otimes \mathbf{g}),
\end{aligned}$$

where $M \in \mathbb{C}^{3\times3}$, $\mathbf{c}_j(M)$ denotes the j-th column vector of $M = M(\mathbf{q}, \mathbf{p}, p_0)$ and $\mathbf{g} = \mathbf{g}(\mathbf{q}, \mathbf{p}, p_0)$ is a three-dimensional vector, we find that

$$\begin{aligned}
\partial_\mathbf{p}^\mathrm{T}\partial_\mathbf{q}h_3^\varepsilon &= (\partial_\mathbf{p}\partial_{p_0}\mathbf{h}_2^\varepsilon)^\mathrm{T}(I + \partial_\mathbf{q}\mathbf{h}_1^\varepsilon) + (\partial_\mathbf{p}\operatorname{vec}\partial_\mathbf{q}\mathbf{h}_1^\varepsilon)^\mathrm{T}(I \otimes \partial_{p_0}\mathbf{h}_2^\varepsilon) \\
&\quad - (\partial_\mathbf{p}\partial_{p_0}\mathbf{h}_1^\varepsilon)^\mathrm{T}\partial_\mathbf{q}\mathbf{h}_2^\varepsilon - (\partial_\mathbf{p}\operatorname{vec}\partial_\mathbf{q}\mathbf{h}_2^\varepsilon)^\mathrm{T}(I \otimes \partial_{p_0}\mathbf{h}_1^\varepsilon), \\
\partial_\mathbf{q}^\mathrm{T}\partial_\mathbf{p}h_3^\varepsilon &= (\partial_\mathbf{q}\partial_{p_0}\mathbf{h}_2^\varepsilon)^\mathrm{T}\partial_\mathbf{p}\mathbf{h}_1^\varepsilon + (\partial_\mathbf{q}\operatorname{vec}\partial_\mathbf{p}\mathbf{h}_1^\varepsilon)^\mathrm{T}(I \otimes \partial_{p_0}\mathbf{h}_2^\varepsilon) \\
&\quad - (\partial_\mathbf{q}\partial_{p_0}\mathbf{h}_1^\varepsilon)^\mathrm{T}(I + \partial_\mathbf{p}\mathbf{h}_2^\varepsilon) - (\partial_\mathbf{q}\operatorname{vec}\partial_\mathbf{p}\mathbf{h}_2^\varepsilon)^\mathrm{T}(I \otimes \partial_{p_0}\mathbf{h}_1^\varepsilon)
\end{aligned}$$

Observing that

$$
\begin{aligned}
(\partial_{\mathbf{q}} \operatorname{vec} \partial_{\mathbf{p}} \mathbf{f})^{\mathrm{T}}(I \otimes \mathbf{g}) &= \Big((\partial_{\mathbf{q}}^{\mathrm{T}} \mathbf{c}_1 (\partial_{\mathbf{p}} \mathbf{f})^{\mathrm{T}}) \mathbf{g} \Big| \dots \Big| (\partial_{\mathbf{q}}^{\mathrm{T}} \mathbf{c}_3 (\partial_{\mathbf{p}} \mathbf{f})^{\mathrm{T}}) \mathbf{g} \Big) \\
&= \Big((\partial_{\mathbf{q}}^{\mathrm{T}} (\partial_{p_1} \mathbf{f})^{\mathrm{T}}) \mathbf{g} \Big| \dots \Big| (\partial_{\mathbf{q}}^{\mathrm{T}} (\partial_{p_3} \mathbf{f})^{\mathrm{T}}) \mathbf{g} \Big) \\
&= \left(\sum_{j=1}^{3} ((\partial_{q_k} \partial_{p_l} f_j) g_j) \right)_{k,l=1,2,3} \\
&= [(\partial_{\mathbf{p}} \operatorname{vec} \partial_{\mathbf{q}} \mathbf{f})^{\mathrm{T}}(I \otimes \mathbf{g})]^{\mathrm{T}}
\end{aligned}
$$

for all three-dimensional vectors $\mathbf{f} = \mathbf{f}(\mathbf{q}, \mathbf{p}, p_0), \mathbf{g} = \mathbf{g}(\mathbf{q}, \mathbf{p}, p_0)$, we find that equation (3.50) is satisfied if and only if

$$
(\partial_{\mathbf{p}} \partial_{p_0} \mathbf{h}_2^{\varepsilon})^{\mathrm{T}} (I + \partial_{\mathbf{q}} \mathbf{h}_1^{\varepsilon}) + (I + (\partial_{\mathbf{p}} \mathbf{h}_2^{\varepsilon})^{\mathrm{T}}) \partial_{\mathbf{q}} \partial_{p_0} \mathbf{h}_1^{\varepsilon} - (\partial_{\mathbf{p}} \partial_{p_0} \mathbf{h}_1^{\varepsilon})^{\mathrm{T}} \partial_{\mathbf{q}} \mathbf{h}_2^{\varepsilon} - (\partial_{\mathbf{p}} \mathbf{h}_1^{\varepsilon})^{\mathrm{T}} \partial_{\mathbf{q}} \partial_{p_0} \mathbf{h}_2^{\varepsilon} = 0,
$$

which holds since it is the transpose of the derivative of equation (3.42) with respect to p_0.

In a similar fashion to the above we have that

$$
\begin{aligned}
(\partial_{\mathbf{q}} \operatorname{vec} \partial_{\mathbf{q}} \mathbf{f})^{\mathrm{T}}(I \otimes \mathbf{g}) &= [(\partial_{\mathbf{q}} \operatorname{vec} \partial_{\mathbf{q}} \mathbf{f})^{\mathrm{T}}(I \otimes \mathbf{g})]^{\mathrm{T}}, \\
(\partial_{\mathbf{p}} \operatorname{vec} \partial_{\mathbf{p}} \mathbf{f})^{\mathrm{T}}(I \otimes \mathbf{g}) &= [(\partial_{\mathbf{p}} \operatorname{vec} \partial_{\mathbf{p}} \mathbf{f})^{\mathrm{T}}(I \otimes \mathbf{g})]^{\mathrm{T}}
\end{aligned}
$$

for all three-dimensional vectors $\mathbf{f} = \mathbf{f}(\mathbf{q}, \mathbf{p}, p_0), \mathbf{g} = \mathbf{g}(\mathbf{q}, \mathbf{p}, p_0)$ and equations (3.48), (3.49) are satisfied if and only if

$$
\begin{aligned}
(\partial_{\mathbf{q}} \partial_{p_0} \mathbf{h}_2^{\varepsilon})^{\mathrm{T}} (I + \partial_{\mathbf{q}} \mathbf{h}_1^{\varepsilon}) + (\partial_{\mathbf{q}} \mathbf{h}_2^{\varepsilon})^{\mathrm{T}} \partial_{\mathbf{q}} \partial_{p_0} \mathbf{h}_1^{\varepsilon} - (I + (\partial_{\mathbf{q}} \mathbf{h}_1^{\varepsilon})^{\mathrm{T}}) \partial_{\mathbf{q}} \partial_{p_0} \mathbf{h}_2^{\varepsilon} - (\partial_{\mathbf{q}} \partial_{p_0} \mathbf{h}_1^{\varepsilon})^{\mathrm{T}} \partial_{\mathbf{q}} \mathbf{h}_2^{\varepsilon} = 0, \\
(\partial_{\mathbf{p}} \partial_{p_0} \mathbf{h}_1^{\varepsilon})^{\mathrm{T}} (I + \partial_{\mathbf{p}} \mathbf{h}_2^{\varepsilon}) + (\partial_{\mathbf{p}} \mathbf{h}_1^{\varepsilon})^{\mathrm{T}} \partial_{\mathbf{p}} \partial_{p_0} \mathbf{h}_2^{\varepsilon} - (I + (\partial_{\mathbf{p}} \mathbf{h}_2^{\varepsilon})^{\mathrm{T}}) \partial_{\mathbf{p}} \partial_{p_0} \mathbf{h}_1^{\varepsilon} - (\partial_{\mathbf{p}} \partial_{p_0} \mathbf{h}_2^{\varepsilon})^{\mathrm{T}} \partial_{\mathbf{p}} \mathbf{h}_1^{\varepsilon} = 0.
\end{aligned}
$$

These equations hold since they are the derivative of equations (3.43) and (3.44) with respect to p_0 respectively. □

Applying Theorem 38, we find that, after a canonical change of variables,

$$
\begin{aligned}
\tilde{H}^{\varepsilon}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0) = {}& \mathrm{i}\omega_1 (A_1 \overline{B}_1 - \overline{A}_1 B_1 + A_2 \overline{B}_2 - \overline{A}_2 B_2) + \mathrm{i}\omega (A_3 \overline{B}_3 - \overline{A}_3 B_3) \\
& + |\mathbf{B}|^2 + \frac{1}{2} p_0^2 + \tilde{H}_{\mathrm{nl}}^{\varepsilon}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{H}_{\mathrm{nl}}^{\varepsilon}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0) = {}& \tilde{H}_{\mathrm{NF}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0, \varepsilon) + \tilde{H}_{\mathrm{r}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0, \varepsilon) \\
& + O(|(\mathbf{A}, \mathbf{B}, p_0)|^2 |(\varepsilon, \mathbf{A}, \mathbf{B}, p_0)|^{n_0});
\end{aligned}
$$

here $\tilde{H}_{\mathrm{NF}}$ is a polynomial function of its arguments which satisfies

$$
\tilde{H}_{\mathrm{NF}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0, \varepsilon) = O(|(\mathbf{A}, \mathbf{B})|^2 |(\varepsilon, \mathbf{A}, \mathbf{B}, p_0)|)
$$

and

$$
\tilde{H}_{\mathrm{NF}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, 0, \varepsilon) = \tilde{K}^{\varepsilon}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}),
$$

while $\tilde{H}_{\mathrm{r}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0, \varepsilon)$ is an affine function of its first four arguments which satisfies

$$
\tilde{H}_{\mathrm{r}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, p_0, \varepsilon) = O(|(\mathbf{A}, \mathbf{B}, p_0)||p_0||(\varepsilon, p_0)|).
$$

3.4 Normal-form theory

In this section we discuss the normal form for the lower-order Hamiltonian system

$$a_{jx} = \frac{\partial \tilde{H}^\varepsilon}{\partial \overline{b_j}}(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, 0), \qquad b_{jx} = -\frac{\partial \tilde{H}^\varepsilon}{\partial \overline{a_j}}(\mathbf{a}, \mathbf{b}, \overline{\mathbf{a}}, \overline{\mathbf{b}}, 0), \qquad j = 1, 2, 3. \tag{3.51}$$

We write the linear part of equations (3.51) as

$$\mathbf{z}_x = L\mathbf{z},$$

where

$$\mathbf{z} = (a_1, b_1, \ldots, a_3, a_3, \overline{a_1}, \overline{b_1}, \ldots, \overline{a_3}, \overline{b_3})^{\mathrm{T}}$$

and

$$L = \begin{pmatrix}
\mathrm{i}\omega_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathrm{i}\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathrm{i}\omega_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{i}\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{i}\omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{i}\omega & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega_1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega
\end{pmatrix}.$$

The following result was proved by Elphick [12].

Proposition 39. *Let $n_0 \geq 2$. There exists a near-identity, canonical change of variables $(\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{A}, \mathbf{B})$ which transforms the Hamiltonian to*

$$\begin{aligned}
&\mathrm{i}\omega_1(A_1\overline{B}_1 - \overline{A}_1 B_1 + A_2\overline{B}_2 - \overline{A}_2 B_2) + \mathrm{i}\omega(A_3\overline{B}_3 - \overline{A}_3 B_3) \\
&\qquad + |\mathbf{B}|^2 + \tilde{K}^\varepsilon(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) + O(|(\mathbf{A}, \mathbf{B})|^2 |(\varepsilon, \mathbf{A}, \mathbf{B})|^{n_0}),
\end{aligned}$$

where the complexification of $\tilde{K}^\varepsilon$ lies in $\ker \mathcal{L}_{L^*}$*, and $\mathcal{L}_{L^*} : \mathbb{C}[\mathbf{Z}] \to \mathbb{C}[\mathbf{Z}]$ is defined by*

$$(\mathcal{L}_{L^*} p)(\mathbf{Z}) = L^* \mathbf{Z}.\nabla p(\mathbf{Z}),$$

where the coefficients of the polynomials in the complex polynomial rings depend upon ε and the gradient is taken with respect to $\mathbf{Z} = (A_1, B_1, \ldots, A_3, B_3, \overline{A}_1, \overline{B}_1, \ldots, \overline{A}_3, \overline{B}_3)^{\mathrm{T}}$.

We proceed by characterising $\ker \mathcal{L}_{L^*}$ using the following three results given by Murdock [35, Lemma 3.4.8], Malonza [32, Lemma 4, Theorem 9] and Billera, Cushman and Sanders [1, Section 4] respectively.

Proposition 40. *Let S be defined by*

$$S = \mathrm{i}\,\mathrm{diag}(\omega_1, \omega_1, \omega_1, \omega_1, \omega, \omega, -\omega_1, -\omega_1, -\omega_1, -\omega_1, -\omega, -\omega)$$

and $N = L - S$.

(i) The kernel of $\mathcal{L}_{L^*} : \mathbb{C}[\mathbf{Z}] \to \mathbb{C}[\mathbf{Z}]$ is given by

$$\ker \mathcal{L}_{L^*} = \ker \mathcal{L}_{N^*} \cap \ker \mathcal{L}_{S^*},$$

where $\mathcal{L}_{M^*} : \mathbb{C}[\mathbf{Z}] \to \mathbb{C}[\mathbf{Z}]$ is defined by

$$(\mathcal{L}_{M^*}p)(\mathbf{Z}) = M^*\mathbf{Z}.\nabla p(\mathbf{Z})$$

for $M \in \mathbb{C}^{12\times 12}$.

(ii) The kernel of $\mathcal{L}_{N^*} : \mathbb{C}[\mathbf{Z}] \to \mathbb{C}[\mathbf{Z}]$ is given by

$$\ker \mathcal{L}_{N^*} = \mathbb{C}[I_1, \ldots, I_6, J_{12}, \ldots, J_{16}, J_{23}, \ldots, J_{26}, J_{34}, \ldots, J_{36}, J_{45}, J_{46}, J_{56}],$$

where

$$\begin{array}{llllll}
I_1 = \overline{A}_1, & I_2 = A_1, & I_3 = A_2, & I_4 = \overline{A}_2, & I_5 = A_3, & I_6 = \overline{A}_3,
\end{array}$$
$$\begin{array}{lll}
J_{12} = \overline{A}_1 B_1 - A_1 \overline{B}_1, & J_{13} = \overline{A}_1 B_2 - A_2 \overline{B}_1, & J_{14} = \overline{A}_1 \overline{B}_2 - \overline{A}_2 \overline{B}_1, \\
J_{15} = \overline{A}_1 B_3 - A_3 \overline{B}_1, & J_{16} = \overline{A}_1 \overline{B}_3 - \overline{A}_3 \overline{B}_1, & J_{23} = A_1 B_2 - A_2 B_1, \\
J_{24} = A_1 \overline{B}_2 - \overline{A}_2 B_1, & J_{25} = A_1 B_3 - A_3 B_1, & J_{26} = A_1 \overline{B}_3 - \overline{A}_3 B_1, \\
J_{34} = A_2 \overline{B}_2 - \overline{A}_2 B_2, & J_{35} = A_2 B_3 - A_3 B_2, & J_{36} = A_2 \overline{B}_3 - \overline{A}_3 B_2, \\
J_{45} = \overline{A}_2 B_3 - A_3 \overline{B}_2, & J_{46} = \overline{A}_2 \overline{B}_3 - \overline{A}_3 \overline{B}_2, & J_{56} = A_3 \overline{B}_3 - \overline{A}_3 B_3.
\end{array}$$

(iii) Suppose that $\frac{\omega_1}{\omega} \notin \mathbb{Q}$. The kernel of $\mathcal{L}_{S^*} : \mathbb{C}[\mathbf{Z}] \to \mathbb{C}[\mathbf{Z}]$ is given by

$$\ker \mathcal{L}_{S^*} = \mathbb{C}[P_{11}, \ldots, P_{14}, P_{21}, \ldots, P_{24}, P_{31}, \ldots, P_{34}, P_{41}, \ldots, P_{44}, P_{55}, P_{56}, P_{65}, P_{66}],$$

where

$$\begin{array}{llll}
P_{11} = A_1 \overline{A_1}, & P_{12} = A_1 \overline{A_2}, & P_{13} = A_1 \overline{B_1}, & P_{14} = A_1 \overline{B_2}, \\
P_{21} = A_2 \overline{A_1}, & P_{22} = A_2 \overline{A_2}, & P_{23} = A_2 \overline{B_1}, & P_{24} = A_2 \overline{B_2}, \\
P_{31} = B_1 \overline{A_1}, & P_{32} = B_1 \overline{A_2}, & P_{33} = B_1 \overline{B_1}, & P_{34} = B_1 \overline{B_2}, \\
P_{41} = B_2 \overline{A_1}, & P_{42} = B_2 \overline{A_2}, & P_{43} = B_2 \overline{B_1}, & P_{44} = B_2 \overline{B_2}, \\
P_{55} = A_3 \overline{A_3}, & P_{56} = A_3 \overline{B_3}, & P_{65} = B_3 \overline{A_3}, & P_{66} = B_3 \overline{B_3},
\end{array}$$

Corollary 41. *The kernel of $\mathcal{L}_{L^*} : \mathbb{C}[\mathbf{Z}] \to \mathbb{C}[\mathbf{Z}]$ is given by*

$$\ker \mathcal{L}_{L^*} = \mathbb{C}[L_1, \ldots, L_{18}],$$

where

$$L_1 = A_1 \overline{A}_1, \qquad L_2 = \mathrm{i}(A_1 \overline{B}_1 - \overline{A}_1 B_1),$$
$$L_3 = A_1 \overline{A}_2 + \overline{A}_1 A_2, \qquad L_4 = \mathrm{i}(A_1 \overline{A}_2 - \overline{A}_1 A_2),$$

$$
\begin{aligned}
L_5 &= \overline{A}_1 B_2 - \overline{B}_1 A_2 + A_1 \overline{B}_2 - \overline{A}_2 B_1,\\
L_6 &= \mathrm{i}(\overline{A}_1 B_2 - \overline{B}_1 A_2 - A_1 \overline{B}_2 + \overline{A}_2 B_1),\\
L_7 &= A_2 \overline{A}_2, \qquad L_8 = \mathrm{i}(A_2 \overline{B}_2 - \overline{A}_2 B_2),\\
L_9 &= A_3 \overline{A}_3, \qquad L_{10} = \mathrm{i}(A_3 \overline{B}_3 - \overline{A}_3 B_3),\\
L_{11} &= A_1 A_3(\overline{A}_1 \overline{B}_3 - \overline{A}_3 \overline{B}_1) + \overline{A}_1 \overline{A}_3 (A_1 B_3 - A_3 B_1),\\
L_{12} &= (A_1 B_3 - A_3 B_1)(\overline{A}_1 \overline{B}_3 - \overline{A}_3 \overline{B}_1),\\
L_{13} &= A_2 A_3(\overline{A}_1 \overline{B}_3 - \overline{A}_3 \overline{B}_1) + \overline{A}_2 \overline{A}_3 (A_1 B_3 - A_3 B_1),\\
L_{14} &= \mathrm{i}(A_2 A_3(\overline{A}_1 \overline{B}_3 - \overline{A}_3 \overline{B}_1) - \overline{A}_2 \overline{A}_3 (A_1 B_3 - A_3 B_1)),\\
L_{15} &= (\overline{A}_1 B_3 - A_3 \overline{B}_1)(A_2 \overline{B}_3 - \overline{A}_3 B_2) + (A_1 \overline{B}_3 - \overline{A}_3 B_1)(\overline{A}_2 B_3 - A_3 \overline{B}_2),\\
L_{16} &= \mathrm{i}((\overline{A}_1 B_3 - A_3 \overline{B}_1)(A_2 \overline{B}_3 - \overline{A}_3 B_2) - (A_1 \overline{B}_3 - \overline{A}_3 B_1)(\overline{A}_2 B_3 - A_3 \overline{B}_2)),\\
L_{17} &= A_2 A_3(\overline{A}_2 \overline{B}_3 - \overline{A}_3 \overline{B}_2) + \overline{A}_2 \overline{A}_3 (A_2 B_3 - A_3 B_2),\\
L_{18} &= (A_2 B_3 - A_3 B_2)(\overline{A}_2 \overline{B}_3 - \overline{A}_3 \overline{B}_2).
\end{aligned}
$$

Proof. From Proposition 40 (ii) and (iii) and a straightforward calculation it follows that

$$\ker \mathcal{L}_{N^*} \cap \ker \mathcal{L}_{S^*} = \mathbb{C}[K_1, \ldots, K_{15}, Q_1, \ldots, Q_{26}],$$

where

$$
\begin{array}{llll}
K_1 = I_1 I_2, & K_2 = I_1 I_3, & K_3 = I_4 I_2, & K_4 = I_3 I_4,\\
K_5 = J_{12}, & K_6 = J_{13}, & K_7 = J_{24}, & K_8 = J_{34},\\
K_9 = I_2 I_2 J_{14}, & K_{10} = I_2 I_3 J_{14}, & K_{11} = I_3 I_3 J_{14}, & K_{12} = I_1 I_1 J_{23},\\
K_{13} = I_1 I_4 J_{23}, & K_{14} = I_4 I_4 J_{23}, & K_{15} = J_{14} J_{23}, & \\
Q_1 = I_5 I_6, & Q_2 = J_{56}, & & \\
Q_3 = I_2 I_5 J_{16}, & Q_4 = I_3 I_5 J_{16}, & Q_5 = I_1 I_5 J_{26}, & Q_6 = I_4 I_5 J_{26},\\
Q_7 = I_1 I_5 J_{36}, & Q_8 = I_4 I_5 J_{36}, & Q_9 = I_2 I_5 J_{46}, & Q_{10} = I_3 I_5 J_{46},\\
Q_{11} = I_2 I_6 J_{15}, & Q_{12} = I_3 I_6 J_{15}, & Q_{13} = I_1 I_6 J_{25}, & Q_{14} = I_4 I_6 J_{25},\\
Q_{15} = I_1 I_6 J_{35}, & Q_{16} = I_4 I_6 J_{35}, & Q_{17} = I_2 I_6 J_{45}, & Q_{18} = I_3 I_6 J_{45},\\
Q_{19} = J_{25} J_{16}, & Q_{20} = J_{35} J_{16}, & Q_{21} = J_{15} J_{26}, & Q_{22} = J_{45} J_{26},\\
Q_{23} = J_{15} J_{36}, & Q_{24} = J_{45} J_{36}, & Q_{25} = J_{25} J_{46}, & Q_{26} = J_{35} J_{46}.
\end{array}
$$

Furthermore, the relations

$$
\begin{array}{lll}
K_9 = K_1 K_7 + K_3 K_5, & K_{10} = K_2 K_7 + K_4 K_5, & K_{11} = K_2 K_8 + K_4 K_6,\\
K_{12} = K_1 K_6 - K_2 K_5, & K_{13} = K_3 K_6 - K_4 K_5, & K_{14} = K_4 K_7 - K_3 K_8.\\
K_{15} = K_6 K_7 - K_5 K_8 & &
\end{array}
$$

show that $K_9, \ldots, K_{15}$ are polynomial functions of $K_1, \ldots, K_8$. Using the relations

$$
\begin{array}{lll}
Q_5 = Q_3 - K_5 Q_1, & Q_6 = Q_{14} + K_3 Q_2, & Q_7 = Q_4 - K_6 Q_1,\\
Q_8 = Q_{10} + K_8 Q_1, & Q_9 = Q_{14} + K_3 Q_2 - K_7 Q_1, & Q_{11} = Q_3 - K_1 Q_2,\\
Q_{12} = Q_4 - K_2 Q_2, & Q_{13} = Q_3 - K_5 Q_1 - K_1 Q_2, & Q_{15} = Q_7 - K_2 Q_2,\\
Q_{16} = Q_{10} + K_8 Q_1 - K_4 Q_2, & Q_{17} = Q_{14} - K_7 Q_1, & Q_{18} = Q_{10} - K_4 Q_2,
\end{array}
$$

$$Q_{20} = Q_{23} - K_6 Q_2, \qquad Q_{21} = Q_{19} + K_5 Q_2, \qquad Q_{24} = Q_{26} - K_8 Q_2,$$
$$Q_{25} = Q_{22} + K_7 Q_2,$$

we find that $Q_5, \ldots, Q_9, Q_{11}, Q_{12}, Q_{13}, Q_{15}, \ldots, Q_{18}, Q_{20}, Q_{21}, Q_{24}, Q_{25}$ are polynomial functions of $K_1, \ldots, K_8, Q_1, \ldots, Q_4, Q_{10}, Q_{14}, Q_{19}, Q_{22}, Q_{23}, Q_{26}$.

Finally, we introduce the quantities

$$\begin{array}{llll} L_1 = K_1, & L_2 = -\mathrm{i}K_5, & L_3 = K_3 + K_2, & L_4 = \mathrm{i}(K_3 - K_2), \\ L_5 = K_6 + K_7, & L_6 = \mathrm{i}(K_6 - K_7), & L_7 = K_4, & L_8 = \mathrm{i}K_8. \\ L_9 = Q_1, & L_{10} = \mathrm{i}Q_2, & L_{11} = Q_3 + Q_{13}, & L_{12} = Q_{19}, \\ L_{13} = Q_4 + Q_{14}, & L_{14} = \mathrm{i}(Q_4 - Q_{14}), & L_{15} = Q_{23} + Q_{22}, & \\ L_{16} = \mathrm{i}(Q_{23} - Q_{22}), & L_{17} = Q_{10} + Q_{16}, & L_{18} = Q_{26}, & \end{array}$$

and observe that

$$\begin{aligned} \ker \mathcal{L}_{N^*} \cap \ker \mathcal{L}_{S^*} &= \mathbb{C}[K_1, \ldots, K_{15}, Q_1, \ldots, Q_{26}] \\ &= \mathbb{C}[K_1, \ldots, K_8, Q_1, \ldots, Q_4, Q_{10}, Q_{14}, Q_{19}, Q_{22}, Q_{23}, Q_{26}] \\ &= \mathbb{C}[L_1, \ldots, L_{18}]. \end{aligned}$$

□

We now examine the subset S of $\mathbb{C}[L_1, \ldots, L_{18}]$ consisting of polynomials which are invariant under the reverser

$$\begin{aligned} S(A_1, &B_1, A_2, B_2, A_3, B_3, \overline{A}_1, \overline{B}_1, \overline{A}_2, \overline{B}_2, \overline{A}_3, \overline{B}_3) \\ &= (\overline{A}_1, -\overline{B}_1, \overline{A}_2, -\overline{B}_2, \overline{A}_3, -\overline{B}_3, A_1, -B_1, A_2, -B_2, A_3, -B_3) \end{aligned}$$

and rotation

$$\begin{aligned} R_\alpha(A_1, &B_1, A_2, B_2, A_3, B_3, \overline{A}_1, \overline{B}_1, \overline{A}_2, \overline{B}_2, \overline{A}_3, \overline{B}_3) \\ &= (\mathrm{e}^{\mathrm{i}\alpha} A_1, \mathrm{e}^{\mathrm{i}\alpha} B_1, \mathrm{e}^{-\mathrm{i}\alpha} A_2, \mathrm{e}^{-\mathrm{i}\alpha} B_2, A_3, B_3, \mathrm{e}^{-\mathrm{i}\alpha}\overline{A}_1, \mathrm{e}^{-\mathrm{i}\alpha}\overline{B}_1, \mathrm{e}^{\mathrm{i}\alpha}\overline{A}_2, \mathrm{e}^{\mathrm{i}\alpha}\overline{B}_2, \overline{A}_3, \overline{B}_3) \end{aligned}$$

for all $\alpha \in \mathbb{R}$.

Lemma 42. *One has that*

$$S = \mathbb{C}[M_1, \ldots, M_9]$$

where

$$\begin{aligned} M_1 &= A_1\overline{A}_1, \qquad M_2 = \mathrm{i}(A_1\overline{B}_1 - \overline{A}_1 B_1), \\ M_3 &= A_2\overline{A}_2, \qquad M_4 = \mathrm{i}(A_2\overline{B}_2 - \overline{A}_2 B_2), \\ M_5 &= A_3\overline{A}_3, \qquad M_6 = \mathrm{i}(A_3\overline{B}_3 - \overline{A}_3 B_3), \\ M_7 &= (A_1B_3 - A_3B_1)(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1), \\ M_8 &= (A_2B_3 - A_3B_2)(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2), \\ M_9 &= A_1A_3(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1)A_2A_3(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2) \\ &\quad + \overline{A}_1\overline{A}_3(A_1B_3 - A_3B_1)A_2A_3(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2) \\ &\quad + A_1A_3(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1)\overline{A}_2\overline{A}_3(A_2B_3 - A_3B_2) \\ &\quad + \overline{A}_1\overline{A}_3(A_1B_3 - A_3B_1)\overline{A}_2\overline{A}_3(A_2B_3 - A_3B_2). \end{aligned}$$

Proof. Observe that

$$
\begin{aligned}
R_\alpha L_1 &= A_1\overline{A}_1, \qquad R_\alpha L_2 = \mathrm{i}(A_1\overline{B}_1 - \overline{A}_1 B_1),\\
R_\alpha L_3 &= A_1\overline{A}_2 \mathrm{e}^{2\mathrm{i}\alpha} + \overline{A}_1 A_2 \mathrm{e}^{-2\mathrm{i}\alpha}, \qquad R_\alpha L_4 = \mathrm{i}(A_1\overline{A}_2 \mathrm{e}^{2\mathrm{i}\alpha} - \overline{A}_1 A_2 \mathrm{e}^{-2\mathrm{i}\alpha}),\\
R_\alpha L_5 &= (\overline{A}_1 B_2 - \overline{B}_1 A_2)\mathrm{e}^{-2\mathrm{i}\alpha} + (A_1\overline{B}_2 - \overline{A}_2 B_1)\mathrm{e}^{2\mathrm{i}\alpha},\\
R_\alpha L_6 &= \mathrm{i}((\overline{A}_1 B_2 - \overline{B}_1 A_2)\mathrm{e}^{-2\mathrm{i}\alpha} - (A_1\overline{B}_2 + \overline{A}_2 B_1)\mathrm{e}^{2\mathrm{i}\alpha}),\\
R_\alpha L_7 &= A_2\overline{A}_2, \qquad R_\alpha L_8 = \mathrm{i}(A_2\overline{B}_2 - \overline{A}_2 B_2),\\
R_\alpha L_9 &= A_3\overline{A}_3, \qquad R_\alpha L_{10} = \mathrm{i}(A_3\overline{B}_3 - \overline{A}_3 B_3),\\
R_\alpha L_{11} &= A_1 A_3(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1) + \overline{A}_1\overline{A}_3(A_1 B_3 - A_3 B_1),\\
R_\alpha L_{12} &= (A_1 B_3 - A_3 B_1)(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1),\\
R_\alpha L_{13} &= A_2 A_3(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1)\mathrm{e}^{-2\mathrm{i}\alpha} + \overline{A}_2\overline{A}_3(A_1 B_3 - A_3 B_1)\mathrm{e}^{2\mathrm{i}\alpha},\\
R_\alpha L_{14} &= \mathrm{i}(A_2 A_3(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1)\mathrm{e}^{-2\mathrm{i}\alpha} - \overline{A}_2\overline{A}_3(A_1 B_3 - A_3 B_1)\mathrm{e}^{2\mathrm{i}\alpha}),\\
R_\alpha L_{15} &= (\overline{A}_1 B_3 - A_3\overline{B}_1)(A_2\overline{B}_3 - \overline{A}_3 B_2)\mathrm{e}^{-2\mathrm{i}\alpha} + (A_1\overline{B}_3 - \overline{A}_3 B_1)(\overline{A}_2 B_3 - A_3\overline{B}_2)\mathrm{e}^{2\mathrm{i}\alpha},\\
R_\alpha L_{16} &= \mathrm{i}((\overline{A}_1 B_3 - A_3\overline{B}_1)(A_2\overline{B}_3 - \overline{A}_3 B_2)\mathrm{e}^{-2\mathrm{i}\alpha} - (A_1\overline{B}_3 - \overline{A}_3 B_1)(\overline{A}_2 B_3 - A_3\overline{B}_2)\mathrm{e}^{2\mathrm{i}\alpha}),\\
R_\alpha L_{17} &= A_2 A_3(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2) + \overline{A}_2\overline{A}_3(A_2 B_3 - A_3 B_2),\\
R_\alpha L_{18} &= (A_2 B_3 - A_3 B_2)(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2).
\end{aligned}
$$

This calculation shows that any element of $\mathbb{C}[L_1, \ldots, L_{18}]$ which depends upon $L_3, L_4, L_5, L_6, L_{13}, L_{14}, L_{15}$ or L_{16} is not invariant under R_α for all $\alpha \in \mathbb{R}$.

Computing the action of S on $L_1, L_2, L_7, \ldots, L_{12}, L_{17}, L_{18}$ shows that

$$
\begin{aligned}
SL_1 &= A_1\overline{A}_1, \qquad SL_2 = \mathrm{i}(A_1\overline{B}_1 - \overline{A}_1 B_1),\\
SL_7 &= A_2\overline{A}_2, \qquad SL_8 = \mathrm{i}(A_2\overline{B}_2 - \overline{A}_2 B_2),\\
SL_9 &= A_3\overline{A}_3, \qquad SL_{10} = \mathrm{i}(A_3\overline{B}_3 - \overline{A}_3 B_3),\\
SL_{11} &= -(A_1 A_3(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1) + \overline{A}_1\overline{A}_3(A_1 B_3 - A_3 B_1)),\\
SL_{12} &= (A_1 B_3 - A_3 B_1)(\overline{A}_1\overline{B}_3 - \overline{A}_3\overline{B}_1),\\
SL_{17} &= -(A_2 A_3(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2) + \overline{A}_2\overline{A}_3(A_2 B_3 - A_3 B_2)),\\
SL_{18} &= (A_2 B_3 - A_3 B_2)(\overline{A}_2\overline{B}_3 - \overline{A}_3\overline{B}_2),
\end{aligned}
$$

so that $L_1, L_2, L_7, \ldots, L_{10}, L_{12}, L_{18}$ are reversible while L_{11} and L_{17} are antireversible. Any reversible polynomial function of $L_1, L_2, L_7, \ldots, L_{12}, L_{17}, L_{18}$ therefore depends upon L_{11} and L_{17} only through the combinations L_{11}^2, L_{17}^2 and $L_{11}L_{17}$. Furthermore, the combinations L_{11}^2 and L_{17}^2 can be eliminated using the relations

$$
\begin{aligned}
L_{11}^2 &= -L_1^2 L_{10}^2 + 2L_1 L_2 L_9 L_{10} - L_2^2 L_9^2 + 4L_1 L_9 L_{12},\\
L_{17}^2 &= -L_7^2 L_{10}^2 + 2L_7 L_8 L_9 L_{10} - L_8^2 L_9^2 + 4L_7 L_9 L_{18}.
\end{aligned}
$$

The result now follows by defining

$$
\begin{aligned}
&M_1 = L_1, \qquad M_2 = L_2, \qquad M_3 = L_7, \qquad M_4 = L_8, \qquad M_5 = L_9, \qquad M_6 = L_{10},\\
&M_7 = L_{12}, \qquad M_8 = L_{18}, \qquad M_9 = L_{11}L_{17}.
\end{aligned}
$$

□

Theorem 43. *The polynomial $\tilde{K}^\varepsilon$ admits a unique representation of the form*

$$\tilde{K}^\varepsilon(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) = p_1^\varepsilon(M_1,\dots,M_8) + p_2^\varepsilon(M_1,\dots,M_8)M_9,$$

where p_j^ε is a real polynomial function of its arguments and ε with

$$\begin{aligned} p_j^\varepsilon(M_1,&M_2,M_3,M_4,M_5,M_6,M_7,M_8) \\ &= p_j^\varepsilon(M_3,M_4,M_1,M_2,M_5,M_6,M_8,M_7), \qquad j=1,2. \end{aligned} \tag{3.52}$$

Proof. We use the result that the ideal I of relations between the polynomials $M_1,\dots,M_9$ in $\mathbb{C}[\mathbf{Z}]$ is given by

$$J \cap \mathbb{C}[m_1,\dots,m_9],$$

where $J = < M_1 - m_1,\dots,M_9 - m_9 > \subseteq \mathbb{C}[\mathbf{Z}, m_1,\dots,m_9]$; furthermore, if G is a Gröbner basis for J, then $G \cap \mathbb{C}[m_1,\dots,m_9]$ is a Gröbner basis for I (see Sturmfels and White [43]). Using this method we obtain the Gröbner basis

$$\Big\{ m_9^2 - (-m_1^2m_6^2 + 2m_1m_2m_5m_6 - m_2^2m_5^2 + 4m_1m_5m_7)(-m_3^2m_6^2 + 2m_3m_4m_5m_6) \\ - (-m_1^2m_6^2 + 2m_1m_2m_5m_6 - m_2^2m_5^2 + 4m_1m_5m_7)(-m_4^2m_5^2 + 4m_3m_5m_8) \Big\}$$

for I, and therefore

$$\mathbb{C}[M_1,\dots,M_9]/I \cong \mathbb{C}[M_1,\dots,M_8] \oplus \mathbb{C}[M_1,\dots,M_8]M_9,$$

(see Sturmfels and White [43]).

The final result is obtained by observing that $\tilde{K}^\varepsilon$ and $M_1,\dots,M_9$ are real, so that

$$\tilde{K}^\varepsilon \in \mathbb{R}[M_1,\dots,M_9]/I \cong \mathbb{R}[M_1,\dots,M_8] \oplus \mathbb{R}[M_1,\dots,M_8]M_9,$$

where the coefficients of the polynomials in the real polynomial rings depend upon ε. Equation (3.52) follows from the fact that $\tilde{K}^\varepsilon$ is invariant under the reflection

$$\begin{aligned} T(A_1,&B_1,A_2,B_2,A_3,B_3\overline{A}_1,\overline{B}_1,\overline{A}_2,\overline{B}_2,\overline{A}_3,\overline{B}_3) \\ &= (A_2,B_2,A_1,B_1,A_3,B_3\overline{A}_2,\overline{B}_2,\overline{A}_1,\overline{B}_1,\overline{A}_3,\overline{B}_3), \end{aligned}$$

so that

$$TM_1 = M_3, \quad TM_2 = M_4, \quad TM_5 = M_5, \quad TM_6 = M_6, \quad TM_7 = M_8, \quad TM_9 = M_9.$$

□

Remark 44. Note that $\tilde{K}^\varepsilon$ is invariant under the second rotation

$$\begin{aligned} \hat{R}_\alpha(A_1,&B_1,A_2,B_2,A_3,B_3,\overline{A}_1,\overline{B}_1,\overline{A}_2,\overline{B}_2,\overline{A}_3,\overline{B}_3) \\ &= (\mathrm{e}^{\mathrm{i}\omega_1\alpha}A_1,\mathrm{e}^{\mathrm{i}\omega_1\alpha}B_1,\mathrm{e}^{\mathrm{i}\omega_1\alpha}A_2,\mathrm{e}^{\mathrm{i}\omega_1\alpha}B_2,\mathrm{e}^{\mathrm{i}\omega\alpha}A_3,\mathrm{e}^{\mathrm{i}\omega\alpha}B_3, \\ &\qquad \mathrm{e}^{-\mathrm{i}\omega_1\alpha}\overline{A}_1,\mathrm{e}^{-\mathrm{i}\omega_1\alpha}\overline{B}_1,\mathrm{e}^{-\mathrm{i}\omega_1\alpha}\overline{A}_2,\mathrm{e}^{-\mathrm{i}\omega_1\alpha}\overline{B}_2,\mathrm{e}^{-\mathrm{i}\omega\alpha}\overline{A}_3,\mathrm{e}^{-\mathrm{i}\omega\alpha}\overline{B}_3) \end{aligned}$$

for all $\alpha \in \mathbb{R}$.

Finally, we introduce the concrete notation

$$
\begin{aligned}
\tilde{K}_2^1(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) &= c_1^1(M_1+M_3)+c_2^1(M_2+M_4)+c_3^1M_5+c_4^1M_6,\\
\tilde{K}_4^0(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) &= c_1(M_1^2+M_3^2)+c_2M_1M_3+c_3M_5(M_1+M_3)+c_4M_5^2\\
&\quad + c_5(M_1M_2+M_3M_4)+c_6(M_2^2+M_4^2)\\
&\quad + c_7(M_1M_4+M_2M_3)+c_8M_2M_4\\
&\quad + c_9M_5(M_2+M_4)+c_{10}M_6(M_2+M_4)\\
&\quad + c_{11}M_6(M_1+M_3)+c_{12}M_5M_6+c_{13}M_6^2\\
&\quad + c_{14}(M_7+M_8-M_6(M_2+M_4)),
\end{aligned}
$$

where all constants are real and we denote the part of $\tilde{K}^\varepsilon$ which is homogeneous of order j in ε and n in $(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}})$ by $\varepsilon^j\tilde{K}_n^j(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}})$. The cubic approximation to the lower-order reduced system is

$$
\begin{aligned}
A_{1x} &= \mathrm{i}\omega_1A_1+B_1+(\varepsilon\partial_{\overline{B}_1}\tilde{K}_2^1+\partial_{\overline{B}_1}\tilde{K}_4^0)+R_{A_1}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)\\
&= \mathrm{i}\omega_1A_1+B_1+\mathrm{i}A_1(c_2^1\varepsilon+c_5M_1+2c_6M_2+c_7M_3+c_8M_4+c_9M_5+c_{10}M_6)\\
&\quad + c_{14}A_3(\overline{A_3}B_1-A_1\overline{B_3})+R_{A_1}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), && (3.53)\\
B_{1x} &= \mathrm{i}\omega_1B_1-(\varepsilon\partial_{\overline{A}_1}\tilde{K}_2^1+\partial_{\overline{A}_1}\tilde{K}_4^0)+R_{B_1}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)\\
&= \mathrm{i}\omega_1B_1+\mathrm{i}B_1(c_2^1\varepsilon+c_5M_1+2c_6M_2+c_7M_3+c_8M_4+c_9M_5+c_{10}M_6)\\
&\quad - A_1(2c_1M_1+c_5M_2+c_2M_3+c_6M_4+c_3M_5+c_{10}M_6)\\
&\quad + c_{14}B_3(\overline{A_3}B_1-A_1\overline{B_3})+R_{B_1}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), && (3.54)\\
A_{2x} &= \mathrm{i}\omega_1A_2+B_2+(\varepsilon\partial_{\overline{B}_2}\tilde{K}_2^1+\partial_{\overline{B}_2}\tilde{K}_4^0)+R_{A_2}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)\\
&= \mathrm{i}\omega_1A_2+B_2+\mathrm{i}A_2(c_2^1\varepsilon+c_7M_1+c_8M_2+c_5M_3+2c_6M_4+c_9M_5+c_{10}M_6)\\
&\quad + c_{14}A_3(\overline{A_3}B_2-A_2\overline{B_3})+R_{A_2}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), && (3.55)\\
B_{2x} &= \mathrm{i}\omega_1B_2-(\varepsilon\partial_{\overline{A}_2}\tilde{K}_2^1+\partial_{\overline{A}_2}\tilde{K}_4^0)+R_{B_2}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)\\
&= \mathrm{i}\omega_1B_2+\mathrm{i}B_2(c_2^1\varepsilon+c_7M_1+c_8M_2+c_5M_3+2c_6M_4+c_9M_5+c_{10}M_6)\\
&\quad - A_2(c_2M_1+c_6M_2+2c_1M_3+c_5M_4+c_3M_5+c_{10}M_6)\\
&\quad + c_{14}B_3(\overline{A_3}B_2-A_2\overline{B_3})+R_{B_2}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), && (3.56)\\
A_{3x} &= \mathrm{i}\omega A_3+B_3+(\varepsilon\partial_{\overline{B}_3}\tilde{K}_2^1+\partial_{\overline{B}_3}\tilde{K}_4^0)+R_{A_3}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)\\
&= \mathrm{i}\omega A_3+B_3+\mathrm{i}A_3\left(c_4^1\varepsilon+c_{11}(M_1+M_3)+c_{10}(M_2+M_4)+c_{12}M_5+2c_{13}M_6\right)\\
&\quad - c_{14}(A_1(A_3\overline{B}_1-\overline{A}_1B_3)+A_2(A_3\overline{B}_2-\overline{A}_2B_3))+R_{A_3}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), && (3.57)\\
B_{3x} &= \mathrm{i}\omega B_3-(\varepsilon\partial_{\overline{A}_3}\tilde{K}_2^1+\partial_{\overline{A}_3}\tilde{K}_4^0)+R_{B_3}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)\\
&= \mathrm{i}\omega B_3+\mathrm{i}\omega B_3\left(c_4^1\varepsilon+c_{11}(M_1+M_3)+c_{10}(M_2+M_4)+c_{12}M_5+2c_{13}M_6\right)\\
&\quad - A_3\left(c_3^1\varepsilon+c_3(M_1+M_3)+c_9(M_2+M_4)+2c_4M_5+c_{12}M_6\right)\\
&\quad - c_{14}(B_1(A_3\overline{B}_1-\overline{A}_1B_3)+B_2(A_3\overline{B}_2-\overline{A}_2B_3))+R_{B_3}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), && (3.58)
\end{aligned}
$$

where

$$
\begin{aligned}
R_{A_j}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) &= O(|(\varepsilon,\mathbf{A},\mathbf{B})||(\mathbf{A},\mathbf{B})|^3),\\
R_{B_j}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) &= O(|(\varepsilon,\mathbf{A},\mathbf{B})||(\mathbf{A},\mathbf{B})|^3)
\end{aligned}
$$

and

$$\mathrm{d}_1 R_{A_j}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon] = O(|(\varepsilon,\mathbf{A},\mathbf{B})||(\mathbf{A},\mathbf{B})|^2),$$
$$\mathrm{d}_1 R_{B_j}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon] = O(|(\varepsilon,\mathbf{A},\mathbf{B})||(\mathbf{A},\mathbf{B})|^2)$$

as $(\mathbf{A},\mathbf{B},\varepsilon)\to\mathbf{0}$.

Note that we can write

$$R(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) := \begin{pmatrix} R_{A_1}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) \\ R_{B_1}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) \\ R_{A_2}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) \\ R_{B_2}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) \\ R_{A_3}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) \\ R_{B_3}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) \end{pmatrix}$$

as

$$R(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) = R^{(1)}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) + R^{(2)}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon),$$

where $R^{(1)}$ is a polynomial function of its arguments arising from derivatives of $\tilde{K}^\varepsilon$, so that

$$R^{(1)}(\hat{R}_\alpha(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}});\varepsilon) = \hat{R}_\alpha R^{(1)}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon)$$

for all $\alpha\in\mathbb{R}$, while

$$R^{(2)}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) = O(|(\mathbf{A},\mathbf{B})||(\varepsilon,\mathbf{A},\mathbf{B})|^{n_0}),$$
$$\mathrm{d}_1 R^{(2)}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon] = O(|(\varepsilon,\mathbf{A},\mathbf{B})|^{n_0})$$

as $(\mathbf{A},\mathbf{B},\varepsilon)\to\mathbf{0}$. We may assume that $R^{(2)}$ and all its derivatives up to any sufficiently high order exist and are bounded, uniformly continuous functions of $\mathbf{Z}\in B_M(\mathbf{0})$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ for sufficiently small $M>0$ and $\varepsilon_0>0$ (see the comments at the beginning of Section 3.3). Functions of this kind are referred to as *smooth functions* in Sections 3.5 and 3.6 below.

3.5 Existence of periodic solutions

We begin by proving the existence of a family of periodic solutions to equations (3.53)–(3.58) in the invariant subspace

$$R := \{(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) : A_1, A_2, B_1, B_2 = 0\}.$$

These solutions satisfy the equations

$$A_{3x} = \mathrm{i}A_3\left(\omega + c_4^1\varepsilon + c_{12}M_5 + 2c_{13}M_6\right) + B_3 + R_{A_3}(A_3,B_3,\overline{A_3},\overline{B_3};\varepsilon), \tag{3.59}$$
$$\begin{aligned} B_{3x} = {} & \mathrm{i}B_3\left(\omega + c_4^1\varepsilon + c_{12}M_5 + 2c_{13}M_6\right) - A_3\left(c_3^1\varepsilon + 2c_4M_5 + c_{12}M_6\right) \\ & + R_{B_3}(A_3,B_3,\overline{A_3},\overline{B_3};\varepsilon). \end{aligned} \tag{3.60}$$

Note that (3.59), (3.60) is reversible with reverser

$$S(A_3,B_3,\overline{A_3},\overline{B_3}) = (\overline{A}_3,-\overline{B}_3,A_3,-B_3).$$

Our main result is the following theorem.

Theorem 45. *Suppose that* $c_3^1 > 0$ *and* $c_4 < 0$ *and set* $\delta = \varepsilon^{\frac{1}{2}}$. *Equations* (3.59), (3.60) *admit a family* $\{\mathbf{Z}_R^{\delta,\theta}\}$ *of reversible periodic solutions smoothly parametrised by* $\delta \in (-\delta_0, \delta_0)$, $\theta \in (-\theta_0, \theta_0)$. *The solution* $\mathbf{Z}_R^{\delta,\theta}$ *has period* $2\pi/(\omega + c_3^{1^{1/2}}\delta\theta)$ *and satisfies*

$$\mathbf{Z}_R^{\delta,\theta}(x) = \delta \left(\frac{c_3^1}{-2c_4}\right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}(\omega + c_3^{1^{1/2}}\delta\theta)x} + O(\delta^2)$$

uniformly over $x \in \mathbb{R}$ *as* $\delta \to 0$.

We prove this theorem using a functional-analytic version of the geometric arguments given by Iooss and Pérouème [26, §§III.1 and VI.1]. The first step is to introduce scaled variables

$$A_3(x) = \left(\frac{c_3^1\varepsilon}{-2c_4}\right)^{\frac{1}{2}} \tilde{A}_3(x), \qquad B_3(x) = \frac{c_3^1\varepsilon}{(-2c_4)^{\frac{1}{2}}} \tilde{B}_3(x),$$

which convert equations (3.59), (3.60) into

$$\tilde{A}_{3x} = \mathrm{i}\omega\tilde{A}_3 + \delta\tilde{B}_3 + \delta^2\tilde{R}_{\tilde{A}_3}(\tilde{A}_3, \tilde{B}_3, \overline{\tilde{A}_3}, \overline{\tilde{B}_3}; \delta), \tag{3.61}$$

$$\tilde{B}_{3x} = \mathrm{i}\omega\tilde{B}_3 + \delta(-1 + |\tilde{A}_3|^2)\tilde{A}_3 + \delta^2\tilde{R}_{\tilde{B}_3}(\tilde{A}_3, \tilde{B}_3, \overline{\tilde{A}_3}, \overline{\tilde{B}_3}; \delta), \tag{3.62}$$

where $\delta = (c_3^1\varepsilon)^{\frac{1}{2}}$ and

$$\tilde{R}_{\tilde{A}_3}(\tilde{A}_3, \tilde{B}_3, \overline{\tilde{A}_3}, \overline{\tilde{B}_3}; \delta) = \delta^{-2}R_{A_3}(A_3, B_3, \overline{A_3}, \overline{B_3}; \varepsilon),$$
$$\tilde{R}_{\tilde{B}_3}(\tilde{A}_3, \tilde{B}_3, \overline{\tilde{A}_3}, \overline{\tilde{B}_3}; \delta) = \delta^{-2}R_{B_3}(A_3, B_3, \overline{A_3}, \overline{B_3}; \varepsilon).$$

Dropping the tildes, we define $F, R_F : B_M(\mathbf{0}) \times (-\delta_0, \delta_0) \to \mathbb{R}^4$, where $\delta_0 = (c_3^1\varepsilon_0)^{\frac{1}{2}}$, in complex coordinates by

$$F(\mathbf{Z};\delta) = \begin{pmatrix} \mathrm{i}\omega A_3 + \delta B_3 \\ \mathrm{i}\omega B_3 + \delta(-1 + |A_3|^2)A_3 \\ -\mathrm{i}\omega\overline{A_3} + \delta\overline{B_3} \\ -\mathrm{i}\omega\overline{B_3} + \delta(-1 + |A_3|^2)\overline{A_3} \end{pmatrix}, \qquad R_F(\mathbf{Z};\delta) = \begin{pmatrix} R_{A_3}(A_3, B_3, \overline{A_3}, \overline{B_3}; \delta) \\ R_{B_3}(A_3, B_3, \overline{A_3}, \overline{B_3}; \delta) \\ \overline{R_{A_3}(A_3, B_3, \overline{A_3}, \overline{B_3}; \delta)} \\ \overline{R_{B_3}(A_3, B_3, \overline{A_3}, \overline{B_3}; \delta)} \end{pmatrix}$$

and write the system (3.61), (3.62) as

$$\mathbf{Z}_x = F(\mathbf{Z};\delta) + \rho R_F(\mathbf{Z};\delta), \tag{3.63}$$

in which $\mathbf{Z} = (A_3, B_3, \overline{A}_3, \overline{B}_3)$ and $\rho = \delta^2$ is considered as an independent parameter.

When $\rho = 0$ this equation has the family

$$\mathbf{Z}^{\delta,\theta,0}(x) = S_{(\omega+\delta\theta)x}\mathbf{X}_\theta$$

of reversible periodic solutions with period $2\pi/(\omega + \delta\theta)$ for $\theta \in (-\theta_0, \theta_0)$, $\theta_0 < 1$, where

$$S_\alpha(A_3, B_3, \overline{A}_3, \overline{B}_3) = (\mathrm{e}^{\mathrm{i}\alpha}A_3, \mathrm{e}^{\mathrm{i}\alpha}B_3, \mathrm{e}^{-\mathrm{i}\alpha}\overline{A}_3, \mathrm{e}^{-\mathrm{i}\alpha}\overline{B}_3),$$

$$\mathbf{X}_\theta = \begin{pmatrix} (1-\theta^2)^{1/2} \\ \mathrm{i}\theta(1-\theta^2)^{1/2} \\ (1-\theta^2)^{1/2} \\ -\mathrm{i}\theta(1-\theta^2)^{1/2} \end{pmatrix}.$$

We show that the periodic solutions persist for small positive values of ρ using the following characterisation of reversible periodic solutions (see Iooss and Pérouème [26, p. 76]).

Proposition 46. *A solution of an autonomous reversible system is periodic and reversible if and only if there exist two different fixed points on this solution with respect to the reversibility.*

Linearising (3.63) with respect to $\mathbf{Z}$ at $\mathbf{Z} = \mathbf{Z}^{\delta,\theta,0}$ and $\rho = 0$ yields the equation

$$\mathcal{L}_\theta^\delta \mathbf{Y} = \mathbf{0},$$

where

$$\mathcal{L}_\theta^\delta = \frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{d}_1 F[\mathbf{X}_\theta; \delta].$$

Noting that

$$\begin{aligned} \mathcal{L}_\theta^\delta S_{(\omega+\delta\theta)x} S_{\pi/2} \mathbf{X}_\theta &= 0, \\ \mathcal{L}_\theta^\delta S_{(\omega+\delta\theta)x} \mathbf{X}_\theta' &= -\delta S_{(\omega+\delta\theta)x} S_{\pi/2} \mathbf{X}_\theta, \\ \mathcal{L}_\theta^\delta S_{(\omega+\delta\theta)x} \xi_\theta^\pm &= \pm\delta\lambda_\theta S_{(\omega+\delta\theta)x} \xi_\theta^\pm, \end{aligned}$$

where

$$\mathbf{X}_\theta' = \frac{\mathrm{d}\mathbf{X}_\theta}{\mathrm{d}\theta}, \qquad \lambda_\theta = \sqrt{2-6\theta}$$

and

$$\xi_\theta^- = \begin{pmatrix} (1-\theta^2)(\lambda_\theta - 2\mathrm{i}\theta) \\ (1-\theta^2)(\lambda_\theta - 2\mathrm{i}\theta)(\lambda_\theta + \mathrm{i}\theta) \\ (1-\theta^2)(\lambda_\theta + 2\mathrm{i}\theta) \\ (1-\theta^2)(\lambda_\theta + 2\mathrm{i}\theta)(\lambda_\theta - \mathrm{i}\theta) \end{pmatrix}, \qquad \xi_\theta^+ = \begin{pmatrix} (1-\theta^2)(\lambda_\theta + 2\mathrm{i}\theta) \\ -(1-\theta^2)(\lambda_\theta + 2\mathrm{i}\theta)(\lambda_\theta - \mathrm{i}\theta) \\ (1-\theta^2)(\lambda_\theta - 2\mathrm{i}\theta) \\ -(1-\theta^2)(\lambda_\theta - 2\mathrm{i}\theta)(\lambda_\theta + \mathrm{i}\theta) \end{pmatrix},$$

we find that

$$\left\{ S_{(\omega+\delta\theta)x} S_{\pi/2} \mathbf{X}_\theta, S_{(\omega+\delta\theta)x}(\mathbf{X}_\theta' + \delta x S_{\pi/2}\mathbf{X}_\theta), \mathrm{e}^{\delta\lambda_\theta x} S_{(\omega+\delta\theta)x}\xi_\theta^+, \mathrm{e}^{-\delta\lambda_\theta x} S_{(\omega+\delta\theta)x}\xi_\theta^- \right\}$$

is a fundamental solution set for this equation. This calculation shows that $\mathbf{Z}^{\delta,\theta,0}$ intersects the hyperplanes

$$\begin{aligned} \Sigma_1 &= \left\{ \mathbf{X}_\theta + u\mathbf{X}_\theta' + v\left(\xi_\theta^+ + \xi_\theta^-\right) + w\left(\xi_\theta^+ - \xi_\theta^-\right) : u, v, w \in \mathbb{R} \right\}, \\ \Sigma_2 &= \left\{ S_\pi \mathbf{X}_\theta + u' S_\pi \mathbf{X}_\theta' + v' S_\pi\left(\xi_\theta^+ + \xi_\theta^-\right) + w' S_\pi\left(\xi_\theta^+ - \xi_\theta^-\right) : u', v', w' \in \mathbb{R} \right\} \end{aligned}$$

transversally at the respective points $\mathbf{Z}^{\delta,\theta,0}(0) = \mathbf{X}_\theta$ (here $(u,v,w) = (0,0,0)$) and $\mathbf{Z}^{\delta,\theta,0}(\pi/(\omega+\delta\theta)) = S_\pi \mathbf{X}_\theta$ (here $(u',v',w') = (0,0,0)$) since

$$\begin{aligned}&\{\underbrace{S_{\pi/2}\mathbf{X}_\theta}, \mathbf{X}_\theta', \xi_\theta^+ + \xi_\theta^-, \xi_\theta^+ - \xi_\theta^-\},\\ &= (\omega+\delta\theta)^{-1}\mathbf{Z}_x^{\delta,\theta,0}(0)\\ &\{\underbrace{S_\pi S_{\pi/2}\mathbf{X}_\theta}, S_\pi\mathbf{X}_\theta', S_\pi\left(\xi_\theta^+ + \xi_\theta^-\right), S_\pi\left(\xi_\theta^+ - \xi_\theta^-\right)\}\\ &= (\omega+\delta\theta)^{-1}\mathbf{Z}_x^{\delta,\theta,0}(\pi/(\omega+\delta\theta))\end{aligned}$$

are bases for $\mathbb{R}^4$ (see Figure 3.4).

Note that $\mathbf{Z}^{\delta,\theta,0}$ intersects the *symmetric section* $\operatorname{Fix} S$ at the points $\mathbf{Z}^{\delta,\theta,0}(0)$ and $\mathbf{Z}^{\delta,\theta,0}(\pi/(\omega+\delta\theta))$. It is therefore a reversible $2\pi/(\omega+\delta\theta)$-periodic solution to (3.63) for $\rho=0$. Our strategy is to construct a solution $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}$ to (3.63) with initial data

$$\mathbf{Z}(0) = \mathbf{X}_\theta + u\mathbf{X}_\theta' + v\left(\xi_\theta^+ + \xi_\theta^-\right) + w\left(\xi_\theta^+ - \xi_\theta^-\right) \in \Sigma_1$$

for small $|(u,v,w,\delta,\theta,\rho)|$, so that $\mathbf{Z}(0)$ is close to $\mathbf{Z}^{0,0,0}(0)=\mathbf{X}_0$, and show that this solution intersects Σ_2 at a point $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}(x^\star)$ close to $\mathbf{Z}^{0,0,0}(\pi/\omega) = S_\pi\mathbf{X}_0$ (where $x^\star$ is close to π/ω); see Figure 3.4. We then select u,v,w so that $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}(0)$ and $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}(x^\star)$ lie in $\operatorname{Fix} S$, which implies that $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}$ is a reversible periodic solution of (3.63).

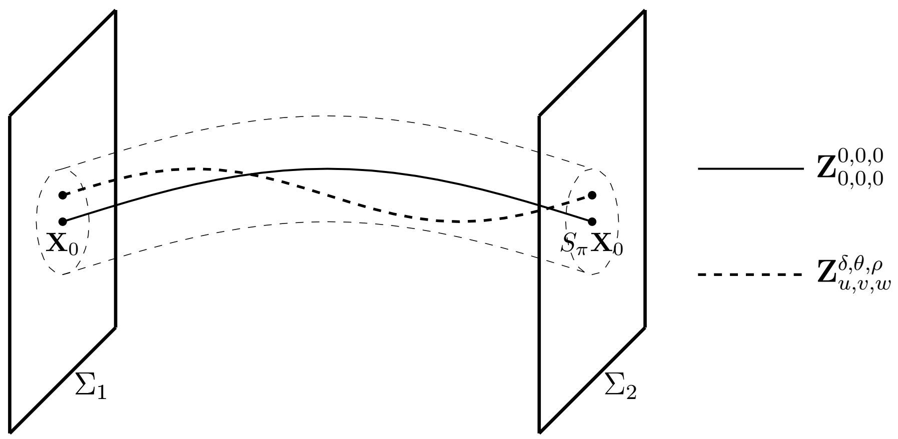


Figure 3.4:
The solution $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}$ lies in a tubular neighbourhood of $\mathbf{Z}_{0,0,0}^{0,0,0}$ with radius $O(|\rho|)+O(|\delta||(u,v,w)|)$ and intersects $\Sigma_1\cap\operatorname{Fix} S$ and $\Sigma_2\cap\operatorname{Fix} S$.

The first step is to show that the initial-value problem for $\mathbf{Z}$ can be solved for x in a sufficiently long interval.

Lemma 47. *Let $\hat{x}>0$. The initial-value problem*

$$\begin{aligned}\mathbf{Z}_x &= F(\mathbf{Z};\delta) + \rho R_F(\mathbf{Z};\delta),\\ \mathbf{Z}(0) &= \mathbf{Z}_0\end{aligned}$$

has a unique solution $\mathbf{Z}\in C[0,\hat{x}]$ for all (δ,ρ) in a neighbourhood of the origin in $\mathbb{R}^2$ and $\mathbf{Z}_0$ in a neighbourhood of $\mathbf{X}_0$ in $\mathbb{R}^4$. The solution and all its derivatives depend continuously upon δ,ρ and $\mathbf{Z}_0$ uniformly over $[0,\hat{x}]$.

Proof. Let

$$\mathrm{C} = \sup_{\mathbf{Z} \in B_M(\mathbf{0})} |F(\mathbf{Z}; 0)|$$

and introduce the norm

$$\|\mathbf{Z}\| = \sup_{x \in [0,\hat{x}]} \mathrm{e}^{-2\mathrm{C}x} |\mathbf{Z}(x)|,$$

which is equivalent to the usual norm on $C([0, \hat{x}]; \mathbb{R}^4)$. Note further that $W = C([0, \hat{x}]; B_M(\mathbf{0}))$ is an open neighbourhood of the origin in $C([0, \hat{x}]; \mathbb{R}^4)$.

Define $\mathcal{F} : W \times \mathbb{R} \times (-\delta_0, \delta_0) \times (-\rho_0, \rho_0) \to C([0, \hat{x}]; \mathbb{R}^4)$ by

$$\mathcal{F}(\mathbf{Z}, \mathbf{Z}_0, \delta, \rho) = \mathbf{Z} - \left(\mathbf{Z}_0 + \int_0^x (F(\mathbf{Z}(t); \delta) + \rho R_F(\mathbf{Z}(t); \delta))\, \mathrm{d}t \right).$$

The zeros of $\mathcal{F}$ are the solutions of

$$\begin{aligned} \mathbf{Z}_x &= F(\mathbf{Z}; \delta) + \rho R_F(\mathbf{Z}; \delta), \\ \mathbf{Z}(0) &= \mathbf{Z}_0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathcal{F}(S_{\omega x}\mathbf{X}_0, \mathbf{X}_0, 0, 0) &= 0, \\ \mathrm{d}_1\mathcal{F}[S_{\omega x}\mathbf{X}_0, \mathbf{X}_0, 0, 0](\mathbf{Y}) &= \mathbf{Y} - \mathcal{G}(\mathbf{Y}), \end{aligned}$$

where

$$\mathcal{G}(\mathbf{Y}) = \int_0^x \mathrm{d}_1 F[S_{\omega t}\mathbf{X}_0; 0](\mathbf{Y}(t))\mathrm{d}t.$$

The estimate

$$\begin{aligned} |(\mathcal{G}\mathbf{Y})(x)|\mathrm{e}^{-2\mathrm{C}x} &\leq \mathrm{C}\mathrm{e}^{-2\mathrm{C}x} \int_0^x \mathrm{e}^{2\mathrm{C}t}\mathrm{d}t \|\mathbf{Y}\| \\ &= \frac{1}{2}\left(1 - \mathrm{e}^{-2\mathrm{C}x}\right) \|\mathbf{Y}\| \end{aligned}$$

shows that $\|\mathcal{G}\| \leq \frac{1}{2}$ and hence that

$$\mathrm{d}_1\mathcal{F}[S_{\omega x}\mathbf{X}_0, \mathbf{X}_0, 0, 0] = I - \mathcal{G}$$

is an isomorphism. Using the implicit-function theorem, we obtain a solution $\mathbf{Z} \in C[0, \hat{x}]$ which is a smooth function of $\mathbf{Z}_0$, δ and ρ for $|\mathbf{Z}_0 - \mathbf{X}_0|$, δ and ρ sufficiently small. □

Corollary 48. *The initial-value problem*

$$\begin{aligned} \mathbf{Z}_x &= F(\mathbf{Z}; \delta) + \rho R_F(\mathbf{Z}; \delta), \\ \mathbf{Z}(0) &= \mathbf{X}_\theta + u\mathbf{X}'_\theta + v\left(\xi_\theta^+ + \xi_\theta^-\right) + w\left(\xi_\theta^+ - \xi_\theta^-\right) \end{aligned}$$

has a unique solution $\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho} \in C[0, 2\pi/\omega]$ *for all* $(u, v, w, \delta, \theta, \rho)$ *in a neighbourhood of the origin in* $\mathbb{R}^6$*. This solution and all its derivatives depend upon* $(u, v, w, \delta, \theta, \rho)$ *uniformly over* $[0, 2\pi/\omega]$*; in particular*

$$|\mathbf{Z}_{u,v,w}^{\delta,\theta,\rho}(x) - \mathbf{Z}_{u,v,w}^{\delta,\theta,0}(x)| = O(|\rho|), \qquad |\mathbf{Z}_{u,v,w}^{\delta,\theta,0}(x) - \mathbf{Z}_{0,0,0}^{\delta,\theta,0}(x)| = O(|(u, v, w)|)$$

uniformly over $x \in [0, 2\pi/\omega]$.

Remark 49. Note the explicit formulae

$$\begin{aligned}\mathbf{Z}^{\delta,\theta,0}_{0,0,0} &= S_{(\omega+\delta\theta)x}\mathbf{X}_\theta,\\ \mathrm{d}\mathbf{Z}^{\delta,\theta,0}_{[0,0,0]}(u,v,w) &= uS_{(\omega+\delta\theta)x}(\mathbf{X}'_\theta + \delta x S_{\pi/2}\mathbf{X}_\theta) + vS_{(\omega+\delta\theta)x}\left(\mathrm{e}^{\delta\lambda_\theta x}\xi^+_\theta + \mathrm{e}^{-\delta\lambda_\theta x}\xi^-_\theta\right)\\ &\quad + wS_{(\omega+\delta\theta)x}\left(\mathrm{e}^{\delta\lambda_\theta x}\xi^+_\theta - \mathrm{e}^{-\delta\lambda_\theta x}\xi^-_\theta\right) S_{(\omega+\delta\theta)x}\mathbf{X}_\theta\end{aligned}$$

and

$$\mathbf{Z}^{0,\theta,0}_{u,v,w} = S_{\omega x}\mathbf{X}_\theta + uS_{\omega x}\mathbf{X}'_\theta + vS_{\omega x}\left(\xi^+_\theta + \xi^-_\theta\right) + wS_{\omega x}\left(\xi^+_\theta - \xi^-_\theta\right).$$

We henceforth consider parameter values $u \in (-u_0, u_0)$, $v \in (-v_0, v_0)$, $w \in (-w_0, w_0)$, $\delta \in (-\delta_0, \delta_0)$, $\theta \in (-\theta_0, \theta_0)$, $\rho \in (-\rho_0, \rho_0)$, replacing u_0, v_0, w_0, δ_0 θ_0, ρ_0 by smaller numbers whenever necessary in the following arguments. (In particular we take $\delta_0^2 < \rho_0$, so that our results are valid for $\rho = \delta^2$.)

Proposition 50. *There exists a smooth function* $x^\star = x^\star(u,v,w,\delta,\theta,\rho)$ *such that* $\mathbf{Z}^{\delta,\theta,\rho}_{u,v,w}(x^\star) \in \Sigma_2$ *and*

$$x^\star(u,v,w,\delta,\theta,0) = \frac{\pi}{\omega+\delta\theta} + \underline{O}(|\delta||(u,v,w)|),$$

where $\underline{O}(|\delta|^m|(u,v,w)|^n)$ *denotes a smooth function of* u,v,w,δ,θ *which is* $O(|\delta|^m|(u,v,w)|^n)$ *and whose first derivatives with respect to* u, v *and* w *are* $O(|\delta|^m|(u,v,w)|^{n-1})$.

Proof. Recall that

$$\left\{S_\pi S_{\pi/2}\mathbf{X}_\theta, S_\pi\mathbf{X}'_\theta, S_\pi(\xi^+_\theta + \xi^-_\theta), S_\pi(\xi^+_\theta - \xi^-_\theta)\right\}$$

is a basis for $\mathbb{R}^4$. Let $\{\mathbf{Z}^\theta_1, \mathbf{Z}^\theta_2, \mathbf{Z}^\theta_3, \mathbf{Z}^\theta_4\}$ be the dual basis (whose elements are by the inverse-function theorem smooth functions of θ in a neighbourhood of the origin) and define

$$\mathcal{F}(x,u,v,w,\delta,\theta,\rho) = \langle \mathbf{Z}^{\delta,\theta,\rho}_{u,v,w}(x) - S_\pi\mathbf{X}_\theta, \mathbf{Z}^\theta_1\rangle,$$

so that $\mathcal{F}$ is a smooth function of x (in a neighbourhood of π/ω) and u,v,w,δ,θ,ρ. (The dependence on x is readily analysed by noting that $\mathbf{Z}^{\delta,\theta,\rho}_{u,v,w}$ solves (3.63).) Since

$$\begin{aligned}\mathcal{F}\left(\frac{\pi}{\omega},0,0,0,0,0,0\right) &= 0,\\ \partial_1\mathcal{F}\left(\frac{\pi}{\omega},0,0,0,0,0,0\right) &= \left\langle \partial_x\mathbf{Z}^{0,0,0}_{0,0,0}\left(\frac{\pi}{\omega}\right), \mathbf{Z}^0_1\right\rangle\\ &= \omega\langle S_{\pi/2}S_\pi\mathbf{X}_0, \mathbf{Z}^0_1\rangle\\ &= \omega,\end{aligned}$$

the existence of $x^\star$ follows by the implicit-function theorem. The estimate for $x^\star(u,v,w,\delta,\theta,0)$ follows from the facts that

$$x^\star(0,0,0,\delta,\theta,0) = \frac{\pi}{\omega+\delta\theta}, \qquad x^\star(u,v,w,0,\theta,0) = \frac{\pi}{\omega},$$

so that in particular $x^\star(u,v,w,0,\theta,0) = x^\star(0,0,0,0,\theta,0)$ (and $\partial_j x^\star(u,v,w,0,\theta,0) = 0$, $j = 1,2,3$). □

Remark 51. Note that

$$|x^\star(u,v,w,\delta,\theta,\rho) - x^\star(u,v,w,\delta,\theta,0)| = O(|\rho|).$$

Observe that

$$\begin{gathered} S\mathbf{X}_\theta = \mathbf{X}_\theta, \qquad S\mathbf{X}'_\theta = \mathbf{X}'_\theta, \\ S\left(\xi_\theta^+ + \xi_\theta^-\right) = \left(\xi_\theta^+ + \xi_\theta^-\right), \qquad S\left(\xi_\theta^+ - \xi_\theta^-\right) = -\left(\xi_\theta^+ - \xi_\theta^-\right). \end{gathered}$$

It follows that

$$\mathbf{Z}_{u,v,0}^{\delta,\theta,\rho}(0) = \mathbf{X}_\theta + u\mathbf{X}'_\theta + v\left(\xi_\theta^+ + \xi_\theta^-\right) \in \Sigma_1,$$

and we now choose u, v such that

$$\begin{aligned} &\mathbf{Z}_{u,v,0}^{\delta,\theta,\rho}(x^\star(u,v,0,\delta,\theta,\rho)) \\ &\quad = S_\pi \mathbf{X}_\theta + u'(u,v,\delta,\theta,\rho) S_\pi \mathbf{X}'_\theta \\ &\qquad + v'(u,v,\delta,\theta,\rho) S_\pi\left(\xi_\theta^+ + \xi_\theta^-\right) + w'(u,v,\delta,\theta,\rho) S_\pi\left(\xi_\theta^+ - \xi_\theta^-\right) \in \Sigma_2. \end{aligned}$$

Lemma 52. *The equation*

$$w'(u,v,\delta,\theta,\delta^2) = 0$$

admits a smooth solution $v = v(u,\delta,\theta)$ *which satisfies* $v(0,0,\theta) = 0$.

Proof. First we derive a (semi-)explicit formula for $w'(u,v,\delta,\theta,\rho)$. It follows from Remark 49 in particular that

$$\mathbf{Z}_{u,v,w}^{0,\theta,0} = \mathbf{Z}_{0,0,0}^{0,\theta,0} + \mathrm{d}\mathbf{Z}_{[0,0,0]}^{0,\theta,0}(u,v,w)$$

(and $\mathrm{d}\mathbf{Z}_{[u,v,w]}^{0,\theta,0} = \mathrm{d}\mathbf{Z}_{[0,0,0]}^{0,\theta,0}$), so that

$$\begin{aligned} \mathbf{Z}_{u,v,w}^{\delta,\theta,0}(x) = {} & S_{(\omega+\delta\theta)x}\mathbf{X}_\theta + uS_{(\omega+\delta\theta)x}(\mathbf{X}'_\theta + \delta x S_{\pi/2}\mathbf{X}_\theta) + vS_{(\omega+\delta\theta)x}\left(\mathrm{e}^{\delta\lambda_\theta x}\xi_\theta^+ + \mathrm{e}^{-\delta\lambda_\theta x}\xi_\theta^-\right) \\ & + wS_{(\omega+\delta\theta)x}\left(\mathrm{e}^{\delta\lambda_\theta x}\xi_\theta^+ - \mathrm{e}^{-\delta\lambda_\theta x}\xi_\theta^-\right) + \underline{O}(|\delta||(u,v,w)|^2), \end{aligned}$$

and hence from

$$x^\star(u,v,w,\delta,\theta,0) = \frac{\pi}{\omega+\delta\theta} + \underline{O}(|\delta||(u,v,w)|)$$

that

$$\begin{aligned} &\mathbf{Z}_{u,v,w}^{\delta,\theta,0}(x^\star(u,v,w,\delta,\theta,0)) - S_\pi\mathbf{X}_\theta \\ &\quad = uS_\pi\left(\mathbf{X}'_\theta + \frac{\delta\pi}{\omega+\delta\theta}S_{\pi/2}\mathbf{X}_\theta\right) \\ &\qquad + \left(v\cosh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right) + w\sinh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right)\right) S_\pi\left(\xi_\theta^+ + \xi_\theta^-\right) \\ &\qquad + \left(v\sinh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right) + w\cosh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right)\right) S_\pi\left(\xi_\theta^+ - \xi_\theta^-\right) \\ &\qquad + \underline{O}(|\delta||(u,v,w)|^2). \end{aligned} \tag{3.64}$$

Finally, using Corollary 48 and Remark 51 and setting $w = 0$, we find that

$$
\begin{aligned}
\mathbf{Z}^{\delta,\theta,\rho}_{u,v,0}&(x^\star(u,v,0,\delta,\theta,\rho)) - S_\pi \mathbf{X}_\theta \\
&= uS_\pi \left(\mathbf{X}'_\theta + \frac{\delta\pi}{\omega+\delta\theta} S_{\pi/2}\mathbf{X}_\theta\right) \\
&\qquad + v\cosh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right) S_\pi \left(\xi^+_\theta + \xi^-_\theta\right) \\
&\qquad + v\sinh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right) S_\pi \left(\xi^+_\theta - \xi^-_\theta\right) \\
&\qquad + O(|\rho|) + \underline{O}(|\delta||(u,v)|^2),
\end{aligned}
$$

where the right-hand side has no component in the $S_\pi \mathbf{X}_\theta$-direction.

We thus find that

$$
\begin{aligned}
w'(u,v,\delta,\theta,\rho) &= v\sinh\left(\frac{\pi\delta\lambda_\theta}{\omega+\delta\theta}\right) + O(|\rho|) + \underline{O}(|\delta||(u,v,w)|^2) \\
&= \frac{\pi\delta\lambda_\theta}{\omega} v + O(|\delta|^2|v|) + O(|\rho|) + \underline{O}(|\delta||(u,v,w)|^2).
\end{aligned}
$$

Define

$$
\begin{aligned}
w''(u,v,\delta,\theta) &= \delta^{-1} w'(u,v,\delta,\theta,\delta^2) \\
&= \frac{\pi\lambda_\theta}{\omega} v + O(|\delta||v|) + O(|\delta|) + \underline{O}(|(u,v,w)|^2).
\end{aligned}
$$

Since

$$
w''(0,0,0,0) = 0, \qquad \partial_2 w''(0,0,0,0) = \frac{\pi\lambda_0}{\omega} \neq 0
$$

the result follows from the implicit-function theorem (evidently $v(0,0,\theta) = 0$ by inspection). $\square$

We have thus established the existence of a family $\{\mathbf{Z}^{\delta,\theta,\delta^2}_{u,v(u,\delta,\theta),0}\}$ of periodic solutions described by δ, θ and their initial values. Note however that different values of (u,θ) can lead to the same initial value. The next result shows that ambiguity can be removed by setting $u = 0$.

Proposition 53. *The mapping*

$$
(\theta, v) \mapsto \mathbf{X}_\theta + u\mathbf{X}'_\theta + v\left(\xi^+_\theta + \xi^-_\theta\right) + w\left(\xi^+_\theta - \xi^-_\theta\right)\big|_{u=0,w=0}
$$

is a smooth injection $\mathbb{R}^2 \to \mathbb{R}^4$ with smooth inverse.

Proof. Changing to real coordinates x_1, x_2, y_1, y_2 given by $A_3 = x_1 + \mathrm{i}x_2$ and $B_3 = y_1 + \mathrm{i}y_2$, one finds that

$$
\mathbf{X}_\theta + v\left(\xi^+_\theta + \xi^-_\theta\right) = \begin{pmatrix} (1-\theta^2)^{1/2} + 2\lambda_\theta(1-\theta^2)v \\ 0 \\ 0 \\ \theta(1-\theta^2)^{1/2} - 2\lambda_\theta\theta(1-\theta^2)v \end{pmatrix},
$$

and the determinant of the Jacobian of the mapping

$$\begin{pmatrix}\theta\\ v\end{pmatrix} \mapsto \begin{pmatrix}(1-\theta^2)^{1/2}+2\lambda_\theta(1-\theta^2)v\\ \theta(1-\theta^2)^{1/2}-2\lambda_\theta\theta(1-\theta^2)v\end{pmatrix}$$

at the origin is $2\sqrt{2}$. This mapping is therefore a local diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. □

Finally, we reparametrise the family $\{\mathbf{Z}^{\delta,\theta,\delta^2}_{0,v(0,\delta,\theta),0}\}$ of periodic solutions.

Proposition 54. *The formula* $(\delta,\theta) \mapsto (\delta,\tilde{\theta})$*, where*

$$\frac{2\pi}{\omega+\delta\tilde{\theta}} = x^\star(0,v(0,\delta,\theta),0,\delta,\theta,\delta^2),$$

defines a local diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$.

Proof. Define $f:(-v_0,v_0)\times(-\delta_0,\delta_0)\times(-\theta_0,\theta_0)\to\mathbb{R}$ by

$$f(v,\delta,\theta) = \frac{1}{\delta}\left(x^\star(0,v(0,\delta,\theta),0,\delta,\theta,\delta^2) - \frac{\pi}{\omega+\delta\theta}\right)$$

so that

$$\tilde{\theta} = \frac{2\pi\theta-\omega^2 f(v(0,\delta,\theta),\delta,\theta)-\omega\delta\theta f(v(0,\delta,\theta),\delta,\theta)}{2\pi+\omega\delta f(v(0,\delta,\theta),\delta,\theta)+\delta^2\theta f(v(0,\delta,\theta),\delta,\theta)}.$$

It follows that

$$\begin{aligned}\frac{\partial\tilde{\theta}}{\partial\theta} &= 1-\frac{\omega^2}{2\pi}\left(\partial_3 f(\underbrace{v(0,0,0)}_{=0},0,0)+\partial_1 f(\underbrace{v(0,0,0)}_{=0},0,0)\partial_3 v(0,0,0)\right)\\ &= 1\end{aligned}$$

because $f(v,\delta,\theta)=O(|v|)$ and $\partial_3 v(0,0,0)=0$.

The result now follows by the inverse-function theorem. □

Define

$$\hat{\mathbf{Z}}^{\delta,\tilde{\theta}} = \mathbf{Z}^{\delta,\theta,\delta^2}_{0,v(0,\delta,\theta),0},$$

so that $\hat{\mathbf{Z}}^{\delta,\tilde{\theta}}$ is periodic with period $2\pi\ (\omega+\delta\tilde{\theta})$.

Proposition 55. *The estimate*

$$\hat{\mathbf{Z}}^{\delta,\tilde{\theta}} = \begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(\omega+\delta\tilde{\theta})x}+O(\delta)$$

holds uniformly over $x\in\mathbb{R}$.

Proof. Let

$$\mathrm{C} = \sup_{\mathbf{Z}\in B_M(\mathbf{0})} |F(\mathbf{Z};0)|$$

and introduce the norm

$$\|\mathbf{Z}\| = \sup_{\tilde{x}\in[0,2\pi]} \mathrm{e}^{-2\mathrm{C}\tilde{x}/\omega}|\mathbf{Z}(\tilde{x})|,$$

which is equivalent to the usual norm on $C([0,2\pi];\mathbb{R}^4)$. Note further that $W = C([0,2\pi];B_M(\mathbf{0}))$ is an open neighbourhood of the origin in $C([0,2\pi];\mathbb{R}^4)$.

Introducing the scaled variable $\tilde{x} = (\omega + \delta\tilde{\theta})x$ converts (3.63) into

$$(\omega + \delta\tilde{\theta})\mathbf{Z}_{\tilde{x}} = F(\mathbf{Z};\delta) + \rho R_F(\mathbf{Z};\delta). \tag{3.65}$$

Define $\mathcal{F} : W \times \mathbb{R} \times (-\delta_0,\delta_0) \times (-\rho_0,\rho_0) \to C([0,2\pi];\mathbb{R}^4)$ by

$$\mathcal{F}(\mathbf{Z},\mathbf{Z}_0,\delta,\rho) = (\omega + \delta\tilde{\theta})(\mathbf{Z} - \mathbf{Z}_0) - \int_0^{\tilde{x}} \left(F(\mathbf{Z}(t);\delta) + \rho R_F(\mathbf{Z}(t);\delta)\right) \mathrm{d}t.$$

The zeros of $\mathcal{F}$ are the solutions of (3.65) with initial data

$$\mathbf{Z}(0) = \mathbf{Z}_0. \tag{3.66}$$

Furthermore,

$$\begin{aligned} \mathcal{F}(S_{\tilde{x}}\mathbf{X}_0,\mathbf{X}_0,0,0) &= 0,\\ \mathrm{d}_1\mathcal{F}[S_{\tilde{x}}\mathbf{X}_0,\mathbf{X}_0,0,0](\mathbf{Y}) &= \omega\mathbf{Y} - \mathcal{G}(\mathbf{Y}), \end{aligned}$$

where

$$\mathcal{G}(\mathbf{Y}) = \int_0^{\tilde{x}} \mathrm{d}_1F[S_t\mathbf{X}_0;0](\mathbf{Y}(t))\mathrm{d}t.$$

The estimate

$$\begin{aligned} |(\mathcal{G}\mathbf{Y})(x)|\mathrm{e}^{-2\mathrm{C}\tilde{x}/\omega} &\le \mathrm{C}\mathrm{e}^{-2\mathrm{C}\tilde{x}/\omega}\int_0^{\tilde{x}} \mathrm{e}^{2\mathrm{C}t/\omega}\mathrm{d}t\|\mathbf{Y}\|\\ &= \frac{\omega}{2}\left(1-\mathrm{e}^{-2\mathrm{C}x}\right)\|\mathbf{Y}\| \end{aligned}$$

shows that $\|\mathcal{G}\| \le \frac{\omega}{2}$ and hence that

$$\mathrm{d}_1\mathcal{F}[S_{\tilde{x}}\mathbf{X}_0,\mathbf{X}_0,0,0] = \omega I - \mathcal{G}$$

is an isomorphism. Using the implicit-function theorem, we obtain a solution $\mathbf{Z} \in C[0,2\pi]$ of (3.65), (3.66) which is a smooth function of $\mathbf{Z}_0$, δ and θ for $|\mathbf{Z}_0 - \mathbf{X}_0|$, δ and θ sufficiently small. In particular

$$|\hat{\mathbf{Z}}^{\delta,\tilde{\theta}}(\tilde{x}) - \underbrace{\hat{\mathbf{Z}}^{0,\tilde{\theta}}(\tilde{x})}_{=S_{\tilde{x}}\mathbf{X}_0}| = O(|\delta|)$$

uniformly over $\tilde{x} \in [0,2\pi]$ and hence all $\tilde{x} \in \mathbb{R}$ (since both functions are 2π-periodic) and $\tilde{\theta} \in (-\tilde{\theta}_0,\tilde{\theta}_0)$ because

$$\lim_{(\delta,\tilde{\theta})\to(0,0)} \hat{\mathbf{Z}}^{\delta,\tilde{\theta}}(0) = \mathbf{X}_0$$

(the solutions to the initial-value problem are unique).

The result now follows by returning to the unscaled variable x. $\square$

Finally, the periodic solutions $\mathbf{Z}_R^{\tilde{\delta},\tilde{\theta}}$ in Theorem 45 are obtained by applying the scaling

$$A_3 = \frac{\delta}{(-2c_4)^{\frac{1}{2}}}\hat{A}_3, \qquad B_3 = \frac{\delta^2}{(-2c_4)^{\frac{1}{2}}}\hat{B}_3,$$

and setting $\delta_0 = (c_3^1 \varepsilon_0)^{\frac{1}{2}}$.

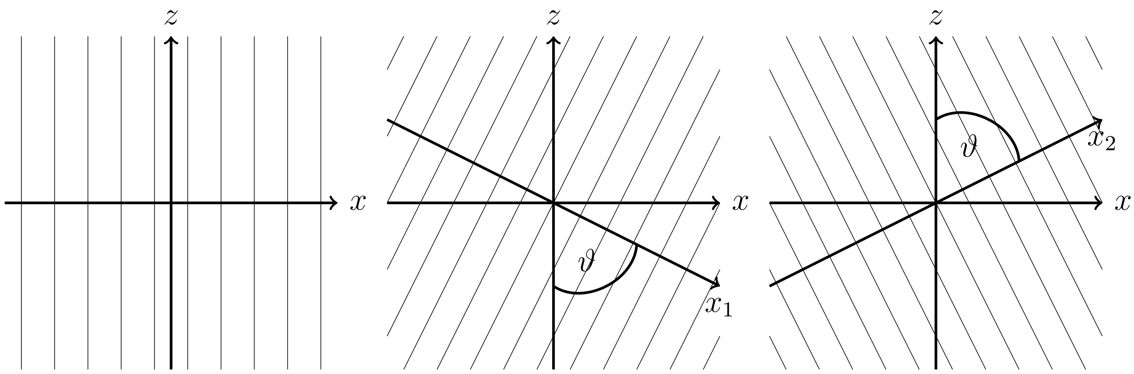


Figure 3.5:
Rolls perpendicular to the directions x (left), $x_1 = \sin(\vartheta)x + \cos(\vartheta)z$ (centre)
and $x_2 = -\sin(\vartheta)x + \cos(\vartheta)z$ (right).

Tracing back the various changes of variables, we find that the above solutions correspond to solutions of the 'flattened' physical problem of the form

$$\begin{pmatrix}\eta\\ \chi'\\ \chi\end{pmatrix} = 2\delta\left(\frac{c_3^1\kappa_\omega^{-1}}{-2c_4}\right)^{\frac{1}{2}}\mathbf{e}_\omega\cos(\omega + c_3^{1^{1/2}}\delta\theta)x + \kappa^{-\frac{1}{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}q_0(x) + O(\delta^2),$$

where

$$\mathbf{e}_\omega = \begin{pmatrix} \dfrac{\sinh\dfrac{\omega}{\beta_0}\cosh\dfrac{S_1\omega}{\beta_0} + \dfrac{\mu_1}{S_1}\sinh\dfrac{S_1\omega}{\beta_0}\cosh\dfrac{\omega}{\beta_0}}{\mu_1 - 1} \\[2ex] -\dfrac{\mu_1}{S_1}\sinh\dfrac{S_1\omega}{\beta_0}\cosh\omega\left(y - \dfrac{1}{\beta_0}\right) \\[2ex] \sinh\dfrac{\omega}{\beta_0}\cosh S_1\omega\left(y + \dfrac{1}{\beta_0}\right)\end{pmatrix},$$

which depend on the single horizontal variable x. These solutions are rolls perpendicular to the x-direction (see Figure 3.5). Because the physical problem is rotationally invariant, replacing x by x_1 in the above formula generates rolls perpendicular to an arbitrary direction x_1. In particular, writing $x_1 = \sin(\vartheta)x + \cos(\vartheta)z$ and noting that

$$\kappa_\omega c_1^1 = \kappa_{\omega_1}c_3^1, \qquad \kappa_\omega^2 c_4 = \kappa_{\omega_1}^2 c_1$$

(see Section 3.7), we obtain the rotated rolls

$$\begin{pmatrix}\eta_\vartheta\\ \chi'_\vartheta\\ \chi_\vartheta\end{pmatrix} = 2\delta\left(\frac{c_1^1\kappa_{\omega_1}^{-1}}{-2c_1}\right)^{\frac{1}{2}}\mathbf{e}_\omega\cos\Big((\omega + c_3^{1^{1/2}}\delta\theta)(\sin(\vartheta)x+\cos(\vartheta)z)\Big)$$
$$+\kappa^{-\frac{1}{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}q_0(\sin(\vartheta)x+\cos(\vartheta)z)+O(\delta^2),$$

which are periodic in x with wavenumber $(\omega + c_3^{1^{1/2}}\delta\theta)\sin(\vartheta)$ and in z with wavenumber $(\omega + c_3^{1^{1/2}}\delta\theta)\cos(\vartheta)$ (see Figure 3.5).

Defining ϑ, θ_1 so that

$$(\omega + c_3^{1^{1/2}}\delta\theta)\sin(\vartheta) = \omega_1 + c_1^{1^{1/2}}\delta\theta_1,$$
$$(\omega + c_3^{1^{1/2}}\delta\theta)\cos(\vartheta) = \nu,$$

we can write the rotated rolls as

$$\begin{pmatrix}\eta_\vartheta\\ \chi'_\vartheta\\ \chi_\vartheta\end{pmatrix} = 2\delta\left(\frac{c_1^1\kappa_{\omega_1}^{-1}}{-2c_1}\right)^{\frac{1}{2}}\mathbf{e}_\omega\cos\Big((\omega_1 + c_1^{1^{1/2}}\delta\theta_1)(x+\nu z)\Big)$$
$$+\kappa^{-\frac{1}{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}q_0(\sin(\vartheta)x+\cos(\vartheta)z)+O(\delta^2).$$

We can repeat this procedure with $x_2 = -\sin(\vartheta)x + \cos(\vartheta)z$ to obtain a second family of rolls which are rotated in the opposite direction (see Figure 3.5). Reapplying the changes of variables and the centre-manifold reduction leads to the following existence result.

Lemma 56. *Suppose that* $c_1^1 > 0$ *and* $c_1 < 0$ *and set* $\delta = \varepsilon^{\frac{1}{2}}$. *Equations* (3.53)–(3.58) *admit families* $\{\mathbf{Z}_P^{\delta,\theta}\}, \{\mathbf{Z}_Q^{\delta,\theta}\}$ *of reversible periodic solutions smoothly parametrised by* $\delta \in (-\delta_0, \delta_0)$ *and* $\theta \in (-\theta_0, \theta_0)$. *The solutions* $\mathbf{Z}_P^{\delta,\theta}$ *and* $\mathbf{Z}_Q^{\delta,\theta}$ *have period* $2\pi/(\omega_1 + c_1^{1^{1/2}}\delta\theta)$ *and satisfy*

$$\mathbf{Z}_P^{\delta,\theta}(x) = \delta\left(\frac{c_1^1}{-2c_1}\right)^{\frac{1}{2}}\begin{pmatrix}1\\0\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(\omega_1+c_1^{1^{1/2}}\delta\theta)x}+O(\delta^2),$$
$$\mathbf{Z}_Q^{\delta,\theta}(x) = \delta\left(\frac{c_1^1}{-2c_1}\right)^{\frac{1}{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(\omega_1+c_1^{1^{1/2}}\delta\theta)x}+O(\delta^2)$$

uniformly over $x \in \mathbb{R}$ *as* $\delta \to 0$.

Remark 57. Obviously $\mathbf{Z}_Q^{\delta,\theta} = T\mathbf{Z}_P^{\delta,\theta}$.

3.6 Existence of heteroclinic solutions

3.6.1 Formulation

In this section we construct a heteroclinic solution to equations (3.53)–(3.58) which connects a periodic solution $\mathbf{Z}_P^{\delta,\theta}$ with a periodic solution $\mathbf{Z}_Q^{\delta,\theta}$. We use the method given by Haragus and

Iooss [17, Section 7], Haragus and Scheel [23, Section 4] and Haragus and Iooss [18, Section 4]. The first step is to formulate (3.53)–(3.58) as an second-order system.

We write (3.53)–(3.58) as

$$\mathbf{A}_{\boldsymbol{\omega}} = \mathbf{B} + \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) + \mathbf{R}^{(2)}_{\mathbf{A}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), \tag{3.67}$$

$$\mathbf{B}_{\boldsymbol{\omega}} = \mathbf{P}^{\varepsilon}_{\mathbf{B}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) + \mathbf{R}^{(2)}_{\mathbf{B}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), \tag{3.68}$$

where

$$\mathbf{A}_{\boldsymbol{\omega}} = (\partial_x - \mathrm{i}\boldsymbol{\omega})\mathbf{A}, \qquad \mathbf{B}_{\boldsymbol{\omega}} = (\partial_x - \mathrm{i}\boldsymbol{\omega})\mathbf{B}, \qquad \boldsymbol{\omega} = \mathrm{diag}(\omega_1,\omega_1,\omega),$$
$$\mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) = \partial_{\overline{\mathbf{B}}}\tilde{K}^{\varepsilon}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}), \qquad \mathbf{P}^{\varepsilon}_{\mathbf{B}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) = -\partial_{\overline{\mathbf{A}}}\tilde{K}^{\varepsilon}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}})$$

and

$$\mathbf{R}^{(2)}_{\mathbf{A}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon), \mathbf{R}^{(2)}_{\mathbf{B}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon) = O(|(\mathbf{A},\mathbf{B})||(\varepsilon,\mathbf{A},\mathbf{B})|^{n_0}),$$
$$\mathrm{d}_1\mathbf{R}^{(2)}_{\mathbf{A}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon], \mathrm{d}_1\mathbf{R}^{(2)}_{\mathbf{B}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}};\varepsilon] = O(|(\varepsilon,\mathbf{A},\mathbf{B})|^{n_0})$$

as $(\mathbf{A},\mathbf{B},\varepsilon) \to \mathbf{0}$. Note that

$$\hat{R}_\alpha \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) = \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\hat{R}_\alpha(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}})), \quad \hat{R}_\alpha \mathbf{P}^{\varepsilon}_{\mathbf{B}}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}) = \mathbf{P}^{\varepsilon}_{\mathbf{B}}(\hat{R}_\alpha(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}}))$$

for all $\alpha \in \mathbb{R}$ (see Remark 44 for the definition of $\hat{R}_\alpha$). All equations are partnered with their complex conjugates and subject to the corresponding reality conditions.

The next step is to recast equations (3.67) and (3.68) as a single second-order equation. Writing

$$\mathbf{B} = \mathbf{A}_{\boldsymbol{\omega}} + \mathbf{D},$$

we find from equation (3.67) that

$$\mathbf{D} + \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}}+\mathbf{D},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}+\overline{\mathbf{D}}) + \mathbf{R}^{(2)}_{\mathbf{A}}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}}+\mathbf{D},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}+\overline{\mathbf{D}};\varepsilon) = \mathbf{0}, \tag{3.69}$$

and we solve this equation for $\mathbf{D} = \mathbf{D}^{\varepsilon}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}})$ in two stages. First we consider the equation

$$\mathbf{D}_1 + \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}}+\mathbf{D}_1,\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}+\overline{\mathbf{D}_1}) = 0.$$

Using the implicit-function theorem we solve this equation for $\mathbf{D}_1 = \mathbf{D}^{\varepsilon}_1(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}})$, where

$$\mathbf{D}^{\varepsilon}_1(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}) = O(|(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}})|^2|(\varepsilon,\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}})|),$$
$$\mathrm{d}\mathbf{D}^{\varepsilon}_1\left[\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}\right] = O(|(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}})||(\varepsilon,\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}})|),$$
$$\mathbf{D}^{\varepsilon}_1(\hat{R}_\alpha(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}})) = \hat{R}_\alpha\mathbf{D}^{\varepsilon}_1(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}).$$

Next we seek a solution to (3.69) of the form $\mathbf{D} = \mathbf{D}^{\varepsilon}_1(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}},\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}) + \mathbf{D}_2$, so that

$$\begin{aligned}\mathbf{D}_2 &+ \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}}+\mathbf{D}^{\varepsilon}_1+\mathbf{D}_2,\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}+\overline{\mathbf{D}^{\varepsilon}_1}+\overline{\mathbf{D}_2})\\ &- \mathbf{P}^{\varepsilon}_{\mathbf{A}}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}}+\mathbf{D}^{\varepsilon}_1,\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}+\overline{\mathbf{D}^{\varepsilon}_1}) + \mathbf{R}^{(2)}_{\mathbf{A}}(\mathbf{A},\mathbf{A}_{\boldsymbol{\omega}}+\mathbf{D}^{\varepsilon}_1+\mathbf{D}_2,\overline{\mathbf{A}},\overline{\mathbf{A}_{\boldsymbol{\omega}}}+\overline{\mathbf{D}^{\varepsilon}_1}+\overline{\mathbf{D}_2};\varepsilon) = 0,\end{aligned}$$

where the arguments of $\mathbf{D}_1^\varepsilon$ have been omitted for notational simplicity. A second application of the implicit-function theorem yields $\mathbf{D}_2 = \mathbf{D}_2^\varepsilon(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega})$ which satisfies

$$\begin{aligned}
\mathbf{D}_2^\varepsilon(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}) &= O(|(\mathbf{A}, \mathbf{A}_\omega)||(\varepsilon, \mathbf{A}, \mathbf{A}_\omega)|^{n_0}),\\
\mathrm{d}\mathbf{D}_2^\varepsilon\left[\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}\right] &= O(|(\varepsilon, \mathbf{A}, \mathbf{A}_\omega)|^{n_0}).
\end{aligned}$$

Substituting

$$\mathbf{B} = \mathbf{A}_\omega + \mathbf{D}_1^\varepsilon + \mathbf{D}_2^\varepsilon$$

into equation (3.68), where the arguments of $\mathbf{D}_1^\varepsilon$, $\mathbf{D}_2^\varepsilon$ have again been omitted for notational simplicity, shows that

$$(\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A} = -(\partial_x - \mathrm{i}\boldsymbol{\omega})(\mathbf{D}_1^\varepsilon + \mathbf{D}_2^\varepsilon) + \tilde{\mathbf{P}}^\varepsilon(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}) + \tilde{\mathbf{R}}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}; \varepsilon), \quad (3.70)$$

in which

$$\begin{aligned}
\tilde{\mathbf{P}}^\varepsilon(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}) &= \mathbf{P}_{\mathbf{B}}^\varepsilon(\mathbf{A}, \mathbf{A}_\omega + \mathbf{D}_1^\varepsilon, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega} + \overline{\mathbf{D}_1^\varepsilon}),\\
\tilde{\mathbf{R}}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}; \varepsilon) &= \mathbf{P}_{\mathbf{B}}^\varepsilon(\mathbf{A}, \mathbf{A}_\omega + \mathbf{D}_1^\varepsilon + \mathbf{D}_2^\varepsilon, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega} + \overline{\mathbf{D}_1^\varepsilon} + \overline{\mathbf{D}_2^\varepsilon})\\
&\qquad - \mathbf{P}_{\mathbf{B}}^\varepsilon(\mathbf{A}, \mathbf{A}_\omega + \mathbf{D}_1^\varepsilon, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega} + \overline{\mathbf{D}_1^\varepsilon})\\
&\qquad + \mathbf{R}_{\mathbf{B}}^{(2)}(\mathbf{A}, \mathbf{A}_\omega + \mathbf{D}_1^\varepsilon + \mathbf{D}_2^\varepsilon, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega} + \overline{\mathbf{D}_1^\varepsilon} + \overline{\mathbf{D}_2^\varepsilon}; \varepsilon).
\end{aligned}$$

It follows that

$$\begin{aligned}
(\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A} = &-\partial_1\mathbf{D}_1^\varepsilon\mathbf{A}_\omega - \partial_2\mathbf{D}_1^\varepsilon(\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A} - \partial_3\mathbf{D}_1^\varepsilon\overline{\mathbf{A}_\omega} - \partial_4\mathbf{D}_1^\varepsilon(\partial_x + \mathrm{i}\boldsymbol{\omega})^2\overline{\mathbf{A}}\\
&- \partial_1\mathbf{D}_2^\varepsilon\mathbf{A}_\omega - \partial_2\mathbf{D}_2^\varepsilon(\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A} - \partial_3\mathbf{D}_2^\varepsilon\overline{\mathbf{A}_\omega} - \partial_4\mathbf{D}_2^\varepsilon(\partial_x + \mathrm{i}\boldsymbol{\omega})^2\overline{\mathbf{A}}\\
&- \mathrm{i}\boldsymbol{\omega}\partial_1\mathbf{D}_2^\varepsilon\mathbf{A} - \mathrm{i}\boldsymbol{\omega}\partial_2\mathbf{D}_2^\varepsilon\mathbf{A}_\omega + \mathrm{i}\boldsymbol{\omega}\partial_3\mathbf{D}_2^\varepsilon\overline{\mathbf{A}} + \mathrm{i}\boldsymbol{\omega}\partial_4\mathbf{D}_2^\varepsilon\overline{\mathbf{A}_\omega} + \mathrm{i}\boldsymbol{\omega}\mathbf{D}_2^\varepsilon\\
&+ \tilde{\mathbf{P}}^\varepsilon(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}) + \tilde{\mathbf{R}}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}; \varepsilon),
\end{aligned}$$

where $\partial_j\mathbf{D}_k^\varepsilon$ is an abbreviation for the matrix $\mathrm{d}_j\mathbf{D}_k^\varepsilon[\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}]$ and we have used the calculation

$$\begin{aligned}
&(\partial_x - \mathrm{i}\boldsymbol{\omega})\mathbf{D}_1^\varepsilon\\
&= (\partial_x - \mathrm{i}\boldsymbol{\omega})\Big(\hat{R}_x\mathbf{D}_1^\varepsilon(\hat{R}_{-x}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}))\Big)\\
&= \hat{R}_x\partial_x\mathbf{D}_1^\varepsilon(\hat{R}_{-x}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}))\\
&= \hat{R}_x\mathrm{d}\mathbf{D}_1^\varepsilon[\hat{R}_{-x}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega})]\Big(\partial_x\hat{R}_{-x}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega})\Big)\\
&= \hat{R}_x\mathrm{d}\mathbf{D}_1^\varepsilon[\hat{R}_{-x}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega})]\Big(\hat{R}_{-x}((\partial_x - \mathrm{i}\boldsymbol{\omega})\mathbf{A}, (\partial_x - \mathrm{i}\boldsymbol{\omega})\mathbf{A}_\omega, (\partial_x + \mathrm{i}\boldsymbol{\omega})\overline{\mathbf{A}}, (\partial_x + \mathrm{i}\boldsymbol{\omega})\overline{\mathbf{A}_\omega})\Big)\\
&= \hat{R}_x\mathrm{d}\mathbf{D}_1^\varepsilon[\hat{R}_{-x}(\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega})]\Big(\hat{R}_{-x}(\mathbf{A}_\omega, (\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A}, \overline{\mathbf{A}_\omega}, (\partial_x + \mathrm{i}\boldsymbol{\omega})^2\overline{\mathbf{A}})\Big)\\
&= \mathrm{d}\mathbf{D}_1^\varepsilon[\mathbf{A}, \mathbf{A}_\omega, \overline{\mathbf{A}}, \overline{\mathbf{A}_\omega}]\Big(\mathbf{A}_\omega, (\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A}, \overline{\mathbf{A}_\omega}, (\partial_x + \mathrm{i}\boldsymbol{\omega})^2\overline{\mathbf{A}}\Big)\\
&= \partial_1\mathbf{D}_1^\varepsilon\mathbf{A}_\omega + \partial_2\mathbf{D}_1^\varepsilon(\partial_x - \mathrm{i}\boldsymbol{\omega})^2\mathbf{A} + \partial_3\mathbf{D}_1^\varepsilon\overline{\mathbf{A}_\omega} + \partial_4\mathbf{D}_1^\varepsilon(\partial_x + \mathrm{i}\boldsymbol{\omega})^2\overline{\mathbf{A}}.
\end{aligned}$$

Finally, write (3.70) and its complex conjugate as

$$\begin{pmatrix} I+\partial_2\mathbf{D}_1^\varepsilon+\partial_2\mathbf{D}_2^\varepsilon & \partial_4\mathbf{D}_1^\varepsilon+\partial_4\mathbf{D}_2^\varepsilon \\ \overline{\partial_4\mathbf{D}_1^\varepsilon+\partial_4\mathbf{D}_2^\varepsilon} & I+\overline{\partial_2\mathbf{D}_1^\varepsilon+\partial_2\mathbf{D}_2^\varepsilon}\end{pmatrix}\begin{pmatrix}(\partial_x-\mathrm{i}\boldsymbol{\omega})^2\mathbf{A}\\ (\partial_x+\mathrm{i}\boldsymbol{\omega})^2\overline{\mathbf{A}}\end{pmatrix}$$
$$=\begin{pmatrix} -\partial_1\mathbf{D}_1^\varepsilon\mathbf{A}_\omega-\partial_3\mathbf{D}_1^\varepsilon\overline{\mathbf{A}_\omega}-\partial_1\mathbf{D}_2^\varepsilon\mathbf{A}_\omega-\partial_3\mathbf{D}_2^\varepsilon\overline{\mathbf{A}_\omega}-\mathrm{i}\boldsymbol{\omega}\partial_1\mathbf{D}_2^\varepsilon\mathbf{A}-\mathrm{i}\boldsymbol{\omega}\partial_2\mathbf{D}_2^\varepsilon\mathbf{A}_\omega \\ \quad +\mathrm{i}\boldsymbol{\omega}\partial_3\mathbf{D}_2^\varepsilon\overline{\mathbf{A}}+\mathrm{i}\boldsymbol{\omega}\partial_4\mathbf{D}_2^\varepsilon\overline{\mathbf{A}_\omega}+\mathrm{i}\boldsymbol{\omega}\mathbf{D}_2^\varepsilon+\tilde{\mathbf{P}}^\varepsilon(\mathbf{A},\mathbf{A}_\omega,\overline{\mathbf{A}},\overline{\mathbf{A}_\omega})+\tilde{\mathbf{R}}(\mathbf{A},\mathbf{A}_\omega,\overline{\mathbf{A}},\overline{\mathbf{A}_\omega};\varepsilon) \\ -\overline{\partial_1\mathbf{D}_1^\varepsilon\mathbf{A}_\omega}-\overline{\partial_3\mathbf{D}_1^\varepsilon}\mathbf{A}_\omega-\overline{\partial_1\mathbf{D}_2^\varepsilon\mathbf{A}_\omega}-\overline{\partial_3\mathbf{D}_2^\varepsilon}\mathbf{A}_\omega+\mathrm{i}\boldsymbol{\omega}\overline{\partial_1\mathbf{D}_2^\varepsilon\mathbf{A}}+\mathrm{i}\boldsymbol{\omega}\overline{\partial_2\mathbf{D}_2^\varepsilon\mathbf{A}_\omega} \\ \quad -\mathrm{i}\boldsymbol{\omega}\overline{\partial_3\mathbf{D}_2^\varepsilon}\mathbf{A}-\mathrm{i}\boldsymbol{\omega}\overline{\partial_4\mathbf{D}_2^\varepsilon}\mathbf{A}_\omega-\mathrm{i}\boldsymbol{\omega}\overline{\mathbf{D}_2^\varepsilon}+\overline{\tilde{\mathbf{P}}^\varepsilon(\mathbf{A},\mathbf{A}_\omega,\overline{\mathbf{A}},\overline{\mathbf{A}_\omega})+\tilde{\mathbf{R}}(\mathbf{A},\mathbf{A}_\omega,\overline{\mathbf{A}},\overline{\mathbf{A}_\omega};\varepsilon)}\end{pmatrix}$$

and note that the matrix on the left-hand side of this equation is invertible by the inverse-function theorem. Its inverse is a perturbation of the identity which depends smoothly upon $\mathbf{A},\mathbf{A}_\omega,\overline{\mathbf{A}},\overline{\mathbf{A}_\omega}$; the perturbation is

$$O(|\mathbf{A}_\omega||(\varepsilon,\mathbf{A},\mathbf{A}_\omega)|)+O(|(\varepsilon,\mathbf{A},\mathbf{A}_\omega)|^{n_0})$$

and is invariant under $\hat{R}_\alpha$ for all $\alpha\in\mathbb{R}$. The final step is to premultiply (3.70) and its complex conjugate by this inverse matrix and introduce the scaled variables $\hat{x}=(c_1^1\varepsilon)^{\frac{1}{2}}x$ and

$$A_j(x)=\left(\frac{c_1^1\varepsilon}{-2c_1}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega_1 x}C_j(\hat{x}),\qquad j=1,2,$$
$$A_3(x)=\left(\frac{c_1^1\varepsilon}{-2c_1}\right)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\omega x}C_3(\hat{x}).$$

We find, after replacing $\hat{x}$ by x for notational simplicity, that

$$\begin{aligned} C_{1xx}&=(-1+|C_1|^2+d_1|C_2|^2+d_2|C_3|^2)C_1+h_1^{(1)}(\mathbf{C},\overline{\mathbf{C}},\mathbf{C}_x,\overline{\mathbf{C}_x},;\delta)\\ &\quad+\mathrm{e}^{-\mathrm{i}\omega_1 x/(c_1^{1\,1/2}\delta)}h_1^{(2)}\big(\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C},\overline{\mathbf{C}}),\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C}_x,\overline{\mathbf{C}_x});\delta\big), \qquad (3.71)\\ C_{2xx}&=(-1+d_1|C_1|^2+|C_2|^2+d_2|C_3|^2)C_2+h_2^{(1)}(\mathbf{C},\overline{\mathbf{C}},\mathbf{C}_x,\overline{\mathbf{C}_x};\delta),\\ &\quad+\mathrm{e}^{-\mathrm{i}\omega_1 x/(c_1^{1\,1/2}\delta)}h_2^{(2)}\big(\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C},\overline{\mathbf{C}}),\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C}_x,\overline{\mathbf{C}_x});\delta\big), \qquad (3.72)\\ C_{3xx}&=(d_3+d_2(|C_1|^2+|C_2|^2)+d_4|C_3|^2)C_3+h_3^{(1)}(\mathbf{C},\overline{\mathbf{C}},\mathbf{C}_x,\overline{\mathbf{C}_x};\delta)\\ &\quad+\mathrm{e}^{-\mathrm{i}\omega x/(c_1^{1\,1/2}\delta)}h_3^{(2)}\big(\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C},\overline{\mathbf{C}}),\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C}_x,\overline{\mathbf{C}_x});\delta\big), \qquad (3.73)\end{aligned}$$

where

$$d_1=\frac{c_2}{2c_1},\qquad d_2=\frac{c_3}{2c_1},\qquad d_3=\frac{c_3^1}{c_1^1},\qquad d_4=\frac{c_4}{c_1},\qquad \delta=\varepsilon^{\frac{1}{2}}$$

and $h_j^{(1)}$ is analytic with

$$h_j^{(1)}(\mathbf{C},\overline{\mathbf{C}},\mathbf{C}_x,\overline{\mathbf{C}_x};\delta)=O(|\delta||(\mathbf{C},\mathbf{C}_x)|),$$

while $h_j^{(2)}$ is a smooth function with

$$h_j^{(2)}\big(\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C},\overline{\mathbf{C}}),\hat{R}_{x/(c_1^{1\,1/2}\delta)}(\mathbf{C}_x,\overline{\mathbf{C}_x});\delta\big)=O(|\delta|^{n_0-2}|(\mathbf{C},\mathbf{C}_x)|)$$

for $n_0\geq 4$. Note that the actions of the reverser S and the reflection $T:z\mapsto -z$ on the space (C_1,C_2,C_3) are given by

$$S(C_1,C_2,C_3)=(\overline{C}_1,\overline{C}_2,\overline{C}_3),\qquad T(C_1,C_2,C_3)=(C_2,C_1,C_3).$$

For notational simplicity we write equations (3.71)–(3.73) as

$$\mathcal{H}(\mathbf{C},\mathbf{C}_x,\mathbf{C}_{xx},x,\delta) := \begin{pmatrix} C_{1xx} \\ C_{2xx} \\ C_{3xx} \end{pmatrix} - \mathcal{H}_1(\mathbf{C}) - \mathcal{H}_2(\mathbf{C},\mathbf{C}_x,x,\delta) = 0, \tag{3.74}$$

in which

$$\mathcal{H}_1(\mathbf{C}) = \begin{pmatrix} (-1+|C_1|^2+d_1|C_2|^2+d_2|C_3|^2)C_1 \\ (-1+d_1|C_1|^2+|C_2|^2+d_2|C_3|^2)C_2 \\ (d_3+d_2(|C_1|^2+|C_2|^2)+d_4|C_3|^2)C_3 \end{pmatrix},$$

and the components of $\mathcal{H}_2(\mathbf{C},\mathbf{C}_x,x,\delta)$ are defined by the final terms in equations (3.71)–(3.73). Note that

$$|\mathcal{H}_2(\mathbf{C},\mathbf{C}_x,x,\delta)| = O(\delta|(\mathbf{C},\mathbf{C}_x)|)$$

as $(\mathbf{C},\mathbf{C}_x,\delta)\to 0$, uniformly over $x\in\mathbb{R}$. We can write equation (3.74) as

$$\mathcal{F}(\mathbf{C};\delta) := \begin{pmatrix} C_{1xx} \\ C_{2xx} \\ C_{3xx} \end{pmatrix} - \mathcal{F}_{\mathrm{r}}(\mathbf{C};\delta) = 0, \tag{3.75}$$

where

$$\mathcal{F}_{\mathrm{r}}(\mathbf{C};\delta) = \mathcal{H}_1(\mathbf{C}) + \mathcal{H}_2(\mathbf{C},\mathbf{C}_x,x,\delta)$$

defines a continuously differentiable function $V_{\mathcal{F}}\times(-\delta_0,\delta_0)\to(C_{\mathrm{b,u}}(\mathbb{R}))^6)$ with

$$\|\mathcal{F}_{\mathrm{r}}(\mathbf{C};\delta)\| = O(\delta\|\mathbf{C}\|)$$

as $(\mathbf{C},\delta)\to 0$ for some open neighbourhood $V_{\mathcal{F}}$ of the origin in $(C^1_{\mathrm{b,u}}(\mathbb{R}))^6$. Note that $\mathcal{F}$ maps functions with the symmetry

$$C_1(x) = \overline{C_2(-x)}, \qquad C_3(x) = \overline{C_3(-x)}$$

for all $x\in\mathbb{R}$ to functions with same symmetry. Our next result follows by defining $\mathbf{P}^{\delta,\theta}$, $\mathbf{Q}^{\delta,\theta}$ by

$$\mathbf{P}^{\delta,\theta}(x) = \mathrm{e}^{-\mathrm{i}\omega_1 x/(c_1^{1\,1/2}\delta)}\mathbf{Z}_P^{\delta,\theta}\left(\frac{x}{c_1^{1\,1/2}\delta}\right),$$
$$\mathbf{Q}^{\delta,\theta}(x) = \mathrm{e}^{-\mathrm{i}\omega_1 x/(c_1^{1\,1/2}\delta)}\mathbf{Z}_Q^{\delta,\theta}\left(\frac{x}{c_1^{1\,1/2}\delta}\right)$$

and using Lemma 56.

Lemma 58. *Suppose that $c_1^1>0$ and $c_1<0$ and set $\delta=\varepsilon^{\frac{1}{2}}$. Equation* (3.75) *admits families $\{\mathbf{P}^{\delta,\theta}\}$, $\{\mathbf{Q}^{\delta,\theta}\}$ of solutions smoothly parametrised by $\delta\in(-\delta_0,\delta_0)$ and $\theta\in(-\theta_0,\theta_0)$ with the following properties.*

(i) The functions

$$\mathrm{e}^{-\mathrm{i}\theta x}\mathbf{P}^{\delta,\theta}(x), \qquad \mathrm{e}^{-\mathrm{i}\theta x}\mathbf{Q}^{\delta,\theta}(x)$$

are periodic with wavenumber

$$\frac{\omega_1}{c_1^{1\,1/2}\delta}+\theta.$$

(ii) The functions $\mathbf{P}^{\delta,\theta}, \mathbf{Q}^{\delta,\theta}$ *satisfy*

$$S\mathbf{P}^{\delta,\theta}(-x) = \mathbf{P}^{\delta,\theta}(x), \qquad S\mathbf{Q}^{\delta,\theta}(-x) = \mathbf{Q}^{\delta,\theta}(x)$$

for all $x \in \mathbb{R}$*.*

(iii) The functions $\mathbf{P}^{\delta,\theta}, \mathbf{Q}^{\delta,\theta}$ *satisfy*

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\theta x}\mathbf{P}^{\delta,\theta}(x) &= (1,0,0) + O(\delta), \\ \mathrm{e}^{-\mathrm{i}\theta x}\mathbf{Q}^{\delta,\theta}(x) &= (0,1,0) + O(\delta) \end{aligned}$$

uniformly over $x \in \mathbb{R}$ *as* $(\delta,\theta) \to (0,0)$*.*

3.6.2 Preparatory results

In this section we record some basic results for Fredholm operators and for the asymptotic behaviour of solutions to linear ordinary differential equations which are used in the theory below.

We begin with the function spaces

$$\begin{aligned} L^2_\eta(\mathbb{R}) &= \{f : ||f||_\eta < \infty\}, \qquad ||f||^2_\eta := \int_{-\infty}^{\infty} \cosh^2(\eta x)|f(x)|^2 \mathrm{d}x, \\ L^2_{-\eta}(\mathbb{R}) &= \{f : ||f||_{-\eta} < \infty\}, \qquad ||f||^2_{-\eta} := \int_{-\infty}^{\infty} \cosh^{-2}(\eta x)|f(x)|^2 \mathrm{d}x \end{aligned}$$

and

$$H^2_\eta(\mathbb{R}) = \left\{f : f, f_x, f_{xx} \in L^2_\eta(\mathbb{R})\right\}, \qquad H^2_{-\eta}(\mathbb{R}) = \left\{f : f, f_x, f_{xx} \in L^2_{-\eta}(\mathbb{R})\right\}$$

with norms

$$||f||^2_{2,\eta} = ||f||^2_\eta + ||f_x||^2_\eta + ||f_{xx}||^2_\eta, \qquad ||f||^2_{2,-\eta} = ||f||^2_{-\eta} + ||f_x||^2_{-\eta} + ||f_{xx}||^2_{-\eta}.$$

Proposition 59. *The mapping*

$$(\Psi g)(f) := \langle f, g\rangle_0$$

defines an isometric isomorphism $\Psi : L^2_{-\eta}(\mathbb{R}) \to (L^2_\eta(\mathbb{R}))^*$ *for* $\eta \geq 0$*, so that*

$$(L^2_\eta(\mathbb{R}))^* \cong L^2_{-\eta}(\mathbb{R}).$$

The following results yield a method for computing the index of a Fredholm operator $L^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ by transforming it to an equivalent operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ (see Neubauer [36, Corollary 5.3]).

Proposition 60. *Suppose that* X, Y, Z *are Banach spaces,* $\mathcal{F}^Y_X : \mathcal{D}(\mathcal{F}^Y_X) \subseteq X \to Y$*,* $\mathcal{F}^Z_Y : \mathcal{D}(\mathcal{F}^Z_Y) \subseteq Y \to Z$ *are Fredholm operators with indices* $\mathrm{ind}(\mathcal{F}^Y_X), \mathrm{ind}(\mathcal{F}^Z_Y)$*. The operator*

$$\mathcal{F}^Z_X = \mathcal{F}^Z_Y\mathcal{F}^Y_X : \mathcal{D}(\mathcal{F}^Z_X) \subseteq X \to Z$$

is Fredholm with index

$$\mathrm{ind}(\mathcal{F}^Z_X) = \mathrm{ind}(\mathcal{F}^Y_X) + \mathrm{ind}(\mathcal{F}^Z_Y).$$

Proposition 61. *Suppose that X, Y are Banach spaces, $\mathcal{B}_X^Y : X \to Y, \mathcal{B}_Y^X : Y \to X$ are isomorphisms. An operator $\mathcal{F}_X^X : \mathcal{D}(\mathcal{F}_X^X) \subseteq X \to X$ is Fredholm with index i if and only if the operator*

$$\mathcal{F}_Y^Y = \mathcal{B}_X^Y \mathcal{F}_X^X \mathcal{B}_Y^X : \mathcal{D}(\mathcal{F}_Y^Y) \subseteq Y \to Y$$

is Fredholm with index i, where $\mathcal{D}(\mathcal{F}_Y^Y) = \mathcal{B}_Y^X[\mathcal{D}(\mathcal{F}_X^X)]$

Proof. Noting that $\mathcal{B}_X^Y : X \to Y, \mathcal{B}_Y^X : Y \to X$ are Fredholm with index 0 it follows from Proposition 60 that the composition $\mathcal{F}_Y^Y : \mathcal{D}(\mathcal{F}_Y^Y) \subseteq Y \to Y$ is Fredholm with index

$$\begin{aligned}
\operatorname{ind}(\mathcal{F}_Y^Y) &= \operatorname{ind}(\mathcal{B}_X^Y \mathcal{F}_X^X) + \operatorname{ind}(\mathcal{B}_Y^X) \\
&= \operatorname{ind}(\mathcal{B}_X^Y) + \operatorname{ind}(\mathcal{F}_X^X) \\
&= \operatorname{ind}(\mathcal{F}_X^X).
\end{aligned}$$

□

Corollary 62. *Define isomorphisms $\mathcal{I}_\eta^0 : L^2(\mathbb{R}) \to L_\eta^2(\mathbb{R})$, $\mathcal{I}_0^\eta : L_\eta^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$\mathcal{I}_\eta^0(U) = \frac{U}{\cosh(\eta x)}, \qquad \mathcal{I}_0^\eta(U) = \cosh(\eta x) U.$$

An operator $\mathcal{M}^\eta : \mathcal{D}(\mathcal{M}^\eta) \subseteq L_\eta^2(\mathbb{R}) \to L_\eta^2(\mathbb{R})$ is Fredholm with index i if and only if

$$\mathcal{M}^0 := \mathcal{I}_0^\eta \mathcal{M}^\eta \mathcal{I}_\eta^0 : \mathcal{D}(\mathcal{M}^0) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is Fredholm with index i, where $\mathcal{D}(\mathcal{M}^0) = \mathcal{I}_0^\eta[\mathcal{D}(\mathcal{M}^\eta)]$.

Using Proposition 59 we identify the dual space $(L_\eta^2(\mathbb{R}))^*$ with $L_{-\eta}^2(\mathbb{R})$. The adjoint operator $\mathcal{K}^* : \mathcal{D}(\mathcal{K}^*) \subseteq L_{-\eta}^2(\mathbb{R}) \to L_{-\eta}^2(\mathbb{R})$ of $\mathcal{K} : \mathcal{D}(\mathcal{K}) \subseteq L_\eta^2(\mathbb{R}) \to L_\eta^2(\mathbb{R})$ satisfies

$$(\mathcal{K}C, D)_0 = (C, \mathcal{K}^* D)_0$$

for all $C \in \mathcal{D}(\mathcal{K})$ and $D \in \mathcal{D}(\mathcal{K}^*)$ by identification. We recall the following results by Kato [28, Theorem IV.5.13, Corollary 5.14] in the present context.

Lemma 63. *Suppose that $\mathcal{K}^* : \mathcal{D}(\mathcal{K}^*) \subseteq L_{-\eta}^2(\mathbb{R}) \to L_{-\eta}^2(\mathbb{R})$ is the adjoint to the closed operator $\mathcal{K} : \mathcal{D}(\mathcal{K}) \subseteq L_\eta^2(\mathbb{R}) \to L_\eta^2(\mathbb{R})$ and set*

$$(\ker \mathcal{K}^*)_\circ = \left\{ C \in L_\eta^2(\mathbb{R}) : (C, D)_0 = 0 \textit{ for all } D \in \ker \mathcal{K}^* \right\}.$$

We have that

$$\operatorname{Im} \mathcal{K} = (\ker \mathcal{K}^*)_\circ.$$

Lemma 64. *Suppose that X, Y are Banach spaces and $\mathcal{F}^* : \mathcal{D}(\mathcal{F}^*) \subseteq X^* \to X^*$ is the adjoint to the closed operator $\mathcal{F} : \mathcal{D}(\mathcal{F}) \subseteq X \to X$ exists. The operator $\mathcal{F}$ is Fredholm if and only if $\mathcal{F}^*$ is Fredholm, and in this case we have that*

$$\dim \ker \mathcal{F} = \operatorname{codim} \operatorname{Im} \mathcal{F}^*, \qquad \dim \ker \mathcal{F}^* = \operatorname{codim} \operatorname{Im} \mathcal{F},$$

so that

$$\operatorname{ind}(\mathcal{F}) = -\operatorname{ind}(\mathcal{F}^*).$$

We study the Fredholm properties of an operator $L^2_\eta(\mathbb{R}) \to L^2_\eta(\mathbb{R})$ by transforming it to an equivalent operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ (see Corollary 62) and applying the following results of Kapitula and Promislow [27, Definition 2.2.5, Theorem 2.2.6 (Weyl essential spectrum theorem), Theorem 3.1.11, Lemma 3.1.7]. These results establish the Fredholm properties of an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by studying its associated *exponentially asymptotic operator*.

Definition 65. *Suppose that X is a Banach space. The operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subseteq X \to X$ is a* ***relatively compact perturbation*** *of $\mathcal{L}_0$ if*

$$(\mathcal{L}_0 - \mathcal{L})(\mathcal{L}_0 - \lambda I)^{-1} : X \to X$$

is compact for some $\lambda \in \rho(\mathcal{L}_0)$.

Theorem 66 (Weyl essential spectrum theorem)**.** *Let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subseteq X \to X$ and $\mathcal{L}_0 : \mathcal{D}(\mathcal{L}_0) \subseteq X \to X$ be closed linear operators in a Banach space X, $\lambda \in \mathbb{C}$, and suppose that $\mathcal{L}$ is a relatively compact perturbation of $\mathcal{L}_0$. The operator $\mathcal{L} - \lambda I$ is Fredholm if and only if $\mathcal{L}_0 - \lambda I$ is Fredholm and in this case*

$$\operatorname{ind}(\mathcal{L} - \lambda I) = \operatorname{ind}(\mathcal{L}_0 - \lambda I).$$

Theorem 67. *Let $\mathcal{L} : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be given by*

$$\mathcal{L}p = \partial_{xx}p + a^{(1)}(x)\partial_x p + a^{(0)}(x)p,$$

where $a^{(0)}, a^{(1)}$ are smooth, real-valued functions. Suppose that $\mathcal{L}$ is ***exponentially asymptotic****, that is, there exists an $r > 0$ and constants $a_+^{(0)}, a_+^{(1)}$, $a_-^{(0)}, a_-^{(1)}$ such that*

$$\lim_{x\to\pm\infty} \mathrm{e}^{r|x|}|a^{(0)}(x) - a_\pm^{(0)}| = 0, \qquad \lim_{x\to\pm\infty} \mathrm{e}^{r|x|}|a^{(1)}(x) - a_\pm^{(1)}| = 0.$$

The closed operator $\mathcal{L}$ is a relatively compact perturbation of its ***exponentially asymptotic operator*** *$\mathcal{L}_\infty$, which is piecewise defined by*

$$\mathcal{L}_\infty p = \begin{cases} \partial_{xx}p + a_+^{(1)}\partial_x p + a_+^{(0)}p, & x \geq 0, \\ \partial_{xx}p + a_-^{(1)}\partial_x p + a_-^{(0)}p, & x < 0. \end{cases}$$

Lemma 68. *Fix $\lambda \in \mathbb{C}$ and for $\mathcal{L}_\infty : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ write the equation $(\mathcal{L}_\infty - \lambda I)p = f$ with*

$$\mathcal{L}_\infty p = \begin{cases} \partial_{xx}p + a_+^{(1)}\partial_x p + a_+^{(0)}p, & x \geq 0, \\ \partial_{xx}p + a_-^{(1)}\partial_x p + a_-^{(0)}p, & x < 0 \end{cases}$$

as

$$\mathbf{P}_x = A_\infty(\lambda)\mathbf{P} + \mathbf{F},$$

where $\mathbf{P} = (p, \partial_x p)$, $\mathbf{F} = (0, f)$ and

$$A_\infty(\lambda) = \begin{cases} A_+(\lambda), & x \geq 0, \\ A_-(\lambda), & x < 0, \end{cases} \qquad A_\pm(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - a_\pm^{(0)} & -a_\pm^{(1)} \end{pmatrix}.$$

Suppose that the matrices $A_\pm(\lambda)$ have no zero eigenvalue.

The operator $\mathcal{L}_\infty - \lambda I$ is Fredholm with index

$$\operatorname{ind}(\mathcal{L}_\infty - \lambda I) = n_+^s(\lambda) + n_-^s(\lambda) - 2,$$

where $n_\pm^s(\lambda)$ is the dimension of the stable subspace of $A_\pm(\lambda)$.

Finally, we give some results for the asymptotic behaviour of solutions to equations of the form

$$\mathbf{U}_x = A_+\mathbf{U} + R_+(x)\mathbf{U}, \tag{3.76}$$
$$\mathbf{U}_x = A_-\mathbf{U} + R_-(x)\mathbf{U}, \tag{3.77}$$

where A_+, A_- are constant, real $m \times m$ matrices and $R_+ : [0,\infty) \to \mathbb{R}^{m\times m}$, $R_- : (-\infty, 0] \to \mathbb{R}^{m\times m}$ are continuous matrix-valued functions (see Coddington and Levinson [7, § 3.8]).

Lemma 69.

(i) Suppose that 0 *is not an eigenvalue of* A_+ *and* A_+ *is diagonalisable, the negative eigenvalues of* A_+ *are given by* $-\lambda_1 \geq -\lambda_2 \geq \ldots \geq -\lambda_p$ *with corresponding eigenvectors* $\mathbf{s}_1, \ldots, \mathbf{s}_p$ *and*

$$\int_0^\infty ||R_+(x)||\, \mathrm{d}x < \infty.$$

Any nontrivial solution of (3.76) *on* $[a,\infty)$ *for* $a > 0$ *sufficiently large with* $\mathbf{U}(x) \to \mathbf{0}$ *as* $x \to \infty$ *satisfies*

$$\mathbf{U}(x) = \alpha_j \mathbf{s}_j(x)\mathrm{e}^{-\lambda_j x} + o(\mathrm{e}^{-\lambda_j x})$$

for some $j \in \{1, 2, \ldots, p\}$ *and constant* $\alpha_j \neq 0$ *as* $x \to \infty$.

(ii) Suppose that 0 *is not an eigenvalue of* A_- *and* A_- *is diagonalisable, the positive eigenvalues of* A_- *are given by* $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$ *with corresponding eigenvectors* $\mathbf{u}_1, \ldots, \mathbf{u}_p$ *and*

$$\int_{-\infty}^0 ||R_-(x)||\, \mathrm{d}x < \infty.$$

Any nontrivial solution of (3.77) *on* $(-\infty, -a]$ *for* $a > 0$ *sufficiently large with* $\mathbf{U}(x) \to \mathbf{0}$ *as* $x \to -\infty$ *satisfies*

$$\mathbf{U}(x) = \beta_j \mathbf{u}_j(x)\mathrm{e}^{\lambda_j x} + o(\mathrm{e}^{\lambda_j x})$$

for some $j \in \{1, 2, \ldots, p\}$ *and constant* $\beta_j \neq 0$ *as* $x \to -\infty$.

Lemma 70.

(i) Suppose that A_+ *is an* $m \times m$ *Jordan block with zero eigenvalue and*

$$\int_1^\infty x^{m-1}||R_+(x)||\, \mathrm{d}x < \infty.$$

No non-trivial solution to equation (3.76) *satisfies* $\mathbf{U}(x) \to \mathbf{0}$ *as* $x \to \infty$.

(ii) Suppose that A_- *is an* $m \times m$ *Jordan block with zero eigenvalue and*

$$\int_{-\infty}^1 x^{m-1}||R_-(x)||\, \mathrm{d}x < \infty.$$

No non-trivial solution to equation (3.77) *satisfies* $\mathbf{U}(x) \to \mathbf{0}$ *as* $x \to -\infty$.

3.6.3 An approximate heteroclinic solution

In this section we study the equations

$$C_{1xx} = (-1 + |C_1|^2 + d_1|C_2|^2 + d_2|C_3|^2)C_1, \tag{3.78}$$
$$C_{2xx} = (-1 + d_1|C_1|^2 + |C_2|^2 + d_2|C_3|^2)C_2, \tag{3.79}$$
$$C_{3xx} = (d_3 + d_2(|C_1|^2 + |C_2|^2) + d_4|C_3|^2)C_3, \tag{3.80}$$

noting that they admit the invariant subspace $\{(C_1, C_2, 0)\}$. The following result by van der Berg [45, Theorem 5] identifies a heteroclinic solution in this subspace.

Lemma 71. *There exists a smooth real solution* $\mathbf{C}^\star = (C_1^\star, C_2^\star, 0)$ *to equations* (3.78)–(3.80) *such that*

(i) $\lim_{x\to\infty} \mathbf{C}^\star = (1, 0, 0)$ *and* $\lim_{x\to-\infty} \mathbf{C}^\star = (0, 1, 0)$,

(ii) $C_1^\star(x), C_2^\star(x) \geq 0$ *for all* $x \in \mathbb{R}$,

(iii) $C_1^\star(x) = C_2^\star(-x)$ *for all* $x \in \mathbb{R}$,

(iv) we have the estimates

$$C_1^\star(x)^2 + C_2^\star(x)^2 \leq 1, \qquad C_1^\star(x) + C_2^\star(x) \geq \min\left\{1, \frac{2}{\sqrt{1+d_1}}\right\}$$

for all $x \in \mathbb{R}$,

(v) we have the identity

$$C_{1x}^\star(x)^2 + C_{2x}^\star(x)^2 = \frac{1}{2}(C_1^\star(x)^2 + C_2^\star(x)^2 - 1)^2 + (d_1 - 1)C_1^\star(x)^2 C_2^\star(x)^2$$

for all $x \in \mathbb{R}$.

The next step is to study the asymptotic behaviour of the functions $C_1^\star, C_2^\star$ and derive a useful estimate (see Haragus and Iooss [17, Lemma 7.1]).

Lemma 72. *For every* $d_1 > 1$ *we have that*

$$C_1^\star(x) = \alpha_{\star 1}\mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad C_2^\star(x) = 1 - \alpha_{\star 2}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}),$$
$$C_{1x}^\star(x) = \sqrt{d_1-1}\alpha_{\star 1}\mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad C_{2x}^\star(x) = -\sqrt{2}\alpha_{\star 2}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})$$

as $x \to -\infty$ *and*

$$C_1^\star(x) = 1 - \beta_{\star 1}\mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), \qquad C_2^\star(x) = \beta_{\star 2}\mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x}),$$
$$C_{1x}^\star(x) = \sqrt{2}\beta_{\star 1}\mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), \qquad C_{2x}^\star(x) = -\sqrt{d_1-1}\beta_{\star 2}\mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x})$$

as $x \to \infty$ *in which* $\alpha_{\star 1}, \beta_{\star 2}$ *are positive and* $\alpha_{\star 2}, \beta_{\star 1}$ *are nonnegative constants.*

Proof. Observe that the equation

$$\mathbf{U}_x = A_\pm \mathbf{U} + R_\pm \mathbf{U},$$

in which

$$A_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d_1-1 & 0 \end{pmatrix}, \qquad A_- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ d_1-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$R_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3(C_1^\star-1)+(C_1^\star-1)^2+d_1 C_2^{\star 2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 C_1^{\star 2}+C_2^{\star 2}-d_1 & 0 \end{pmatrix},$$

$$R_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d_1 C_2^{\star 2}+C_1^{\star 2}-d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3(C_2^\star-1)+(C_2^\star-1)^2+d_1 C_1^{\star 2} & 0 \end{pmatrix},$$

is solved by $\mathbf{U}_\pm = (C_{1\pm}, \partial_x C_{1\pm}, C_{2\pm}, \partial_x C_{2\pm})$ with

$$\begin{aligned} C_{1+}(x) &= C_1^\star(x) - 1, & C_{2+}(x) &= C_2^\star(x), \\ C_{1-}(x) &= C_1^\star(x), & C_{2-}(x) &= C_2^\star(x) - 1. \end{aligned}$$

Noting that the eigenvalues of $A_\pm$ are $\pm\sqrt{2}, \pm\sqrt{d_1-1}$ and that

$$\int_0^\infty ||R_+(x)||\,\mathrm{d}x < \infty, \qquad \int_{-\infty}^0 ||R_-(x)||\,\mathrm{d}x < \infty,$$

we obtain the advertised result from Lemma 69 together with Lemma 71 (iv). $\square$

Lemma 73. *Suppose that* $1 < d_1 < 4 + \sqrt{13}$*. The inequality*

$$3C_1^\star(x)^2 + d_1 C_2^\star(x)^2 - 1 > 0$$

holds for all $x \in \mathbb{R}$*.*

Proof. Suppose that $\frac{3}{2} < d_1 < 4 + \sqrt{13}$. It follows from the estimates in Lemma 71 (iv) that

$$\begin{aligned} 3C_1^\star(x)^2 + d_1 C_2^\star(x)^2 - 1 &= 3\left(C_1^\star(x)^2 + \frac{d_1}{3} C_2^\star(x)^2\right) - 1 \\ &\geq \frac{d_1}{\dfrac{d_1}{3}+1}(C_1^\star(x) + C_2^\star(x))^2 - 1 \\ &\geq \frac{3d_1}{d_1+3}(C_1^\star(x) + C_2^\star(x))^2 - 1 \\ &\geq \frac{3d_1}{d_1+3}\min\left\{1, \frac{2}{\sqrt{1+d_1}}\right\}^2 - 1 \\ &\geq \frac{3d_1}{d_1+3}\min\left\{1, \frac{4}{1+d_1}\right\} - 1 \end{aligned}$$

$$
\begin{aligned}
&\geq \min\left\{\frac{3d_1}{d_1+3}, \frac{12d_1}{(d_1+3)(1+d_1)}\right\} - 1 \\
&\geq \min\left\{\frac{3d_1}{d_1+3}, \frac{12d_1}{d_1^2+4d_1+3}\right\} - 1 \\
&\geq \min\left\{\frac{2d_1-3}{d_1+3}, \frac{8d_1-d_1^2-3}{d_1^2+4d_1+3}\right\} \\
&> 0.
\end{aligned}
$$

Now suppose that $1 < d_1 \leq \frac{3}{2}$. Define the function $f_{d_1} : \mathbb{R} \to \mathbb{R}$ by

$$f_{d_1}(x) = 3C_1^\star(x)^2 + d_1 C_2^\star(x)^2 - 1$$

and note that $(d_1, x) \mapsto f_{d_1}(x)$ depends smoothly upon d_1 and x. Suppose that the closed set

$$N := \left\{(d_1, x) \in \left(1, \tfrac{3}{2}\right] \times \mathbb{R} : f_{d_1}(x) \leq 0\right\}$$

is non-empty and define

$$d_1^\star := \max\{d_1 : (d_1, x) \in N \text{ for some } x \in \mathbb{R}\} \leq \tfrac{3}{2}$$

(note that the set in this equation is compact).

Suppose there exists $x^\dagger \in \mathbb{R}$ such that $f_{d_1^\star}(x^\dagger) < 0$. Noting that $f_{d_1}(x^\dagger)$ depends continuously upon d_1 and $f_{d_1}(x^\dagger) > 0$ for all d_1 with $\frac{3}{2} < d_1 < 4 + \sqrt{13}$, we conclude that there exists $d_1^\dagger \in (d_1^\star, \frac{3}{2}]$ such that $f_{d_1^\dagger}(x^\dagger) = 0$. Hence we have $(d_1^\dagger, x^\dagger) \in N$ with $d_1^\dagger \geq d_1^\star$, which contradicts the definition of $d_1^\star$. It follows that $f_{d_1^\star}(x) \geq 0$ for all $x \in \mathbb{R}$ and there exists $x^\star \in \mathbb{R}$ with $f_{d_1^\star}(x^\star) = 0$. We note that $x^\star \in \mathbb{R}$ is a global minimum of $f_{d_1^\star}$, so that

$$f_{d_1^\star}(x^\star) = 0, \qquad f'_{d_1^\star}(x^\star) = 0, \qquad f''_{d_1^\star}(x^\star) \geq 0.$$

We proceed by computing $f''_{d_1^\star}(x^\star)$ and deriving the contradiction $f''_{d_1^\star}(x^\star) < 0$. Writing

$$U = C_1^\star(x^\star)^2, \qquad V = C_2^\star(x^\star)^2, \qquad X = C_{1x}^\star(x^\star)^2, \qquad Y = C_{2x}^\star(x^\star)^2$$

for notational simplicity, we find from

$$f_{d_1^\star}(x^\star) = f'_{d_1^\star}(x^\star) = 0$$

that

$$3U + d_1^\star V = 1, \qquad 9UX = d_1^{\star 2} VY$$

and from Lemma 71 (v) that

$$X + Y = \frac{1}{2}(U + V - 1)^2 + (d_1^\star - 1)UV.$$

Lemma 71 (iv) shows that $0 \leq U \leq 1$, and $U = 1$ leads to the contradiction $f_{d_1^\star}(x^\star) = 2 \neq 0$, while $U = 0$ leads to the contradiction $f_{d_1^\star}(x^\star) = d_1^\star - 1 \neq 0$; hence $0 < U < 1$.

We can write V, X, Y as functions of U by

$$\begin{aligned}
V &= \frac{1}{d_1^\star}(1-3U), \\
X &= \frac{1}{2}(U+V-1)^2 + (d_1^\star - 1)UV - Y \\
&= \frac{1}{2}\frac{(1-3U)((5d_1^{\star 2}-9)U^2 + 6(1-d_1^\star)U - (d_1^\star - 1)^2)}{d_1^\star(3(d_1^\star - 3)U - d_1^\star)} \\
Y &= \frac{\frac{1}{2}(U+V-1)^2 + (d_1^\star - 1)UV}{d_1^{\star 2}V + 9U} \\
&= \frac{9}{2}\frac{U((5d_1^{\star 2}-9)U^2 + 6(1-d_1^\star)U - (d_1^\star - 1)^2)}{d_1^{\star 2}(3(d_1^\star - 3)U - d_1^\star)},
\end{aligned}$$

and compute

$$\begin{aligned}
f''_{d_1^\star}(x^\star) &= 6(C_1^{\star\prime}(x^\star)^2 + C_1^\star(x^\star)C_1^{\star\prime\prime}(x^\star)) + 2d_1(C_2^{\star\prime}(x^\star)^2 + C_2^\star(x^\star)C_2^{\star\prime\prime}(x^\star)) \\
&= 6(X + U(-1+U+d_1^\star V)) + 2d_1(Y + V(-1+d_1^\star U + V)) \\
&= \frac{18(d_1^\star - 1)(d_1^{\star 2} - 9)U^3 + (12d_1^\star(9-d_1^{\star 2}) - 27(d_1^{\star 2}+3))U^2}{d_1^\star(d_1^\star - 3(d_1^\star - 3)U)} \\
&\qquad + \frac{2d_1^\star(d_1^{\star 2} + 6d_1^\star - 9)U + (d_1^\star - 1)(d_1^\star - 3)}{d_1^\star(d_1^\star - 3(d_1^\star - 3)U)} \\
&< 0
\end{aligned}$$

since $0 < U < 1$. □

Linearising equations (3.78)–(3.80) at $\mathbf{C} = \mathbf{C}^\star$ yields the equation

$$\mathcal{K}\mathbf{C} = 0,$$

where

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{12} & 0 \\ 0 & \mathcal{K}_3 \end{pmatrix} \tag{3.81}$$

and

$$\begin{aligned}
\mathcal{K}_{12}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} C_{1xx} - (-1 + 2C_1^{\star 2} + d_1 C_2^{\star 2})C_1 - C_1^{\star 2}\overline{C_1} - d_1 C_1^\star C_2^\star(C_2 + \overline{C_2}) \\ C_{2xx} - (-1 + d_1 C_1^{\star 2} + 2C_2^{\star 2})C_2 - C_2^{\star 2}\overline{C_2} - d_1 C_1^\star C_2^\star(C_1 + \overline{C_1}) \end{pmatrix}, \\
\mathcal{K}_3\left(C_3\right) &= C_{3xx} - (d_3 + d_2(C_1^{\star 2} + C_2^{\star 2}))C_3.
\end{aligned}$$

We study $\mathcal{K}$ as an operator $\mathcal{D}(\mathcal{K}) \subseteq \mathcal{X}_\eta \to \mathcal{X}_\eta$, $\eta \geq 0$, where $\mathcal{D}(\mathcal{K}) = \mathcal{Y}_\eta$ and the (real) function spaces are

$$\mathcal{X}_\eta = \left\{\mathbf{C} \in (L^2_\eta(\mathbb{R}))^6 : C_1(x) = \overline{C_2(-x)}, C_3(x) = \overline{C_3(-x)} \text{ for } x \in \mathbb{R}\right\}$$

and

$$\mathcal{Y}_\eta = \mathcal{X}_\eta \cap (H^2_\eta(\mathbb{R}))^6.$$

In particular we examine its Fredholm properties and following the steps in Haragus and Iooss [17, Section 7], Haragus and Scheel [23, Section 4] and Haragus and Iooss [18, Section 4].

Introducing the real and imaginary parts of C_1, C_2, C_3 as new variables

$$U_j = \frac{1}{2}(C_j + \overline{C_j}), \qquad V_j = \frac{\mathrm{i}}{2}(C_j - \overline{C_j}) \qquad j = 1, 2, 3$$

transforms $\mathcal{K}$ into

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\mathrm{r}} & 0 & 0 \\ 0 & \mathcal{M}_{\mathrm{i}} & 0 \\ 0 & 0 & \mathcal{M}_3 \end{pmatrix},$$

in which

$$\mathcal{M}_{\mathrm{r}} : \mathcal{Y}^{\mathrm{r}}_\eta \subseteq \mathcal{X}^{\mathrm{r}}_\eta \to \mathcal{X}^{\mathrm{r}}_\eta, \qquad \mathcal{M}_{\mathrm{i}} : \mathcal{Y}^{\mathrm{i}}_\eta \subseteq \mathcal{X}^{\mathrm{i}}_\eta \to \mathcal{X}^{\mathrm{i}}_\eta, \qquad \mathcal{M}_3 : \mathcal{Y}^3_\eta \subseteq \mathcal{X}^3_\eta \to \mathcal{X}^3_\eta$$

are given by

$$\begin{aligned}
\mathcal{M}_{\mathrm{r}} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} &= \begin{pmatrix} U_{1xx} - (-1 + 3C_1^{\star 2} + d_1 C_2^{\star 2})U_1 - 2d_1 C_1^\star C_2^\star U_2 \\ U_{2xx} - (-1 + d_1 C_1^{\star 2} + 3C_2^{\star 2})U_2 - 2d_1 C_1^\star C_2^\star U_1 \end{pmatrix}, \\
\mathcal{M}_{\mathrm{i}} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} &= \begin{pmatrix} V_{1xx} - (-1 + C_1^{\star 2} + d_1 C_2^{\star 2})V_1 \\ V_{2xx} - (-1 + d_1 C_1^{\star 2} + C_2^{\star 2})V_2 \end{pmatrix}, \\
\mathcal{M}_3 \begin{pmatrix} U_3 \\ V_3 \end{pmatrix} &= \begin{pmatrix} U_{3xx} - (d_3 + d_2(C_1^{\star 2} + C_2^{\star 2}))U_3 \\ V_{3xx} - (d_3 + d_2(C_1^{\star 2} + C_2^{\star 2}))V_3 \end{pmatrix}.
\end{aligned}$$

The function spaces are

$$\begin{aligned}
\mathcal{X}^{\mathrm{r}}_\eta &= \left\{(U_1, U_2) \in (L^2_\eta(\mathbb{R}))^2 : U_1(x) = U_2(-x) \text{ for } x \in \mathbb{R}\right\}, \\
\mathcal{X}^{\mathrm{i}}_\eta &= \left\{(V_1, V_2) \in (L^2_\eta(\mathbb{R}))^2 : V_1(x) = -V_2(-x) \text{ for } x \in \mathbb{R}\right\}, \\
\mathcal{X}^3_\eta &= \left\{(U_3, V_3) \in (L^2_\eta(\mathbb{R}))^2 : U_3(x) = U_3(-x), V_3(x) = -V_3(-x) \text{ for } x \in \mathbb{R}\right\},
\end{aligned}$$

with

$$\mathcal{Y}^{\mathrm{r}}_\eta = \mathcal{X}^{\mathrm{r}}_\eta \cap (H^2_\eta(\mathbb{R}))^2, \qquad \mathcal{Y}^{\mathrm{i}}_\eta = \mathcal{X}^{\mathrm{i}}_\eta \cap (H^2_\eta(\mathbb{R}))^2, \qquad \mathcal{Y}^3_\eta = \mathcal{X}^3_\eta \cap (H^2_\eta(\mathbb{R}))^2.$$

We examine the Fredholm properties of the operators $\mathcal{M}_{\mathrm{r}}, \mathcal{M}_{\mathrm{i}}, \mathcal{M}_3$ separately (see Haragus and Iooss [17, Lemma 7.3], Haragus and Scheel [23, Lemma 4.1] and Haragus and Iooss [18, Lemma 4.1]).

Lemma 74. *The operator $\mathcal{M}_{\mathrm{r}} : \mathcal{Y}^{\mathrm{r}}_\eta \subseteq \mathcal{X}^{\mathrm{r}}_\eta \to \mathcal{X}^{\mathrm{r}}_\eta$ is Fredholm with index 0.*

Proof. We write

$$\mathcal{M}_{\mathrm{r}} = \mathcal{A}^{\mathrm{r}}_\pm + \mathcal{R}^{\mathrm{r}}_\pm,$$

where

$$\mathcal{A}^{\mathrm{r}}_+ = \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} - (d_1 - 1) \end{pmatrix}, \qquad \mathcal{R}^{\mathrm{r}}_+ = \begin{pmatrix} 3C_1^{\star 2} + d_1 C_2^{\star 2} - 3 & 2d_1 C_1^\star C_2^\star \\ 2d_1 C_1^\star C_2^\star & d_1 C_1^{\star 2} + 3C_2^{\star 2} - d_1 \end{pmatrix},$$

$$\mathcal{A}^{\mathrm{r}}_{-} = \begin{pmatrix} \partial_{xx} - (d_1 - 1) & 0 \\ 0 & \partial_{xx} \end{pmatrix}, \qquad \mathcal{R}^{\mathrm{r}}_{-} = \begin{pmatrix} 3C_1^{\star 2} + d_1 C_2^{\star 2} - d_1 & 2d_1 C_1^\star C_2^\star \\ 2d_1 C_1^\star C_2^\star & d_1 C_1^{\star 2} + 3C_2^{\star 2} - 3 \end{pmatrix}$$

and the entries of $\mathcal{R}^{\mathrm{r}}_{\pm}$ are $o(\mathrm{e}^{\mp \min\{\sqrt{2},\sqrt{d_1-1}\}x})$ as $x \to \pm\infty$. To discuss its Fredholm properties we define

$$\mathcal{M}_{\mathrm{r}0} := \mathcal{I}_0^\eta \mathcal{M}_{\mathrm{r}} \mathcal{I}_\eta^0 : \mathcal{Y}_0^{\mathrm{r}} \subseteq \mathcal{X}_0^{\mathrm{r}} \to \mathcal{X}_0^{\mathrm{r}}$$

and write

$$\mathcal{M}_{\mathrm{r}0} = \mathcal{A}^{\mathrm{r}0}_{\pm} + \mathcal{R}^{\mathrm{r}0}_{\pm} + \mathcal{R}^{\mathrm{r}}_{\pm},$$

where

$$\mathcal{A}^{\mathrm{r}0}_{\pm} + \mathcal{R}^{\mathrm{r}0}_{\pm} = \mathcal{I}_0^\eta \mathcal{A}^{\mathrm{r}}_{\pm} \mathcal{I}_\eta^0.$$

The new operators are given by the explicit formulae

$$\mathcal{A}^{\mathrm{r}0}_{+} = \begin{pmatrix} \mathcal{D}_+ - 2 & 0 \\ 0 & \mathcal{D}_+ - (d_1 - 1) \end{pmatrix}, \qquad \mathcal{R}^{\mathrm{r}0}_{+} = \begin{pmatrix} \mathcal{R}_+ & 0 \\ 0 & \mathcal{R}_+ \end{pmatrix},$$

$$\mathcal{A}^{\mathrm{r}0}_{-} = \begin{pmatrix} \mathcal{D}_- - (d_1 - 1) & 0 \\ 0 & \mathcal{D}_- - 2 \end{pmatrix}, \qquad \mathcal{R}^{\mathrm{r}0}_{-} = \begin{pmatrix} \mathcal{R}_- & 0 \\ 0 & \mathcal{R}_- \end{pmatrix}$$

and

$$\mathcal{D}_{\pm} = (\partial_x \mp \eta)^2, \qquad \mathcal{R}_{\pm} = \pm 2\eta(1 \mp \tanh \eta x)\partial_x - \frac{2\eta^2}{\cosh^2 \eta x}.$$

We choose $\eta < \min\{\sqrt{2}, \sqrt{d_1 - 1}\}$ and note that

$$\mathcal{A}^{\mathrm{r}0}_{\infty} = \begin{cases} \mathcal{A}^{\mathrm{r}0}_{+}, & x \geq 0, \\ \mathcal{A}^{\mathrm{r}0}_{-}, & x < 0 \end{cases}$$

is the asymptotic operator associated to $\mathcal{M}_{\mathrm{r}0}$ by Theorem 67 because the coefficients of the differential operators $\mathcal{R}_{\pm}$ are $O(\mathrm{e}^{\mp \eta x})$ as $x \to \pm\infty$.

Because $U_1(x) = U_2(-x)$ for all $x \in \mathbb{R}$ it is sufficient to consider the scalar operator

$$\mathcal{A}^{\mathrm{rs}}_{\infty} = \begin{cases} \mathcal{A}^{\mathrm{rs}}_{+}, & x \geq 0, \\ \mathcal{A}^{\mathrm{rs}}_{-}, & x < 0 \end{cases}$$

with

$$\mathcal{A}^{\mathrm{rs}}_{+} = \mathcal{D}_+ - 2, \qquad \mathcal{A}^{\mathrm{rs}}_{-} = \mathcal{D}_- - (d_1 - 1).$$

We rewrite the equation $\mathcal{A}^{\mathrm{rs}}_{\infty} U_1 = f$ as

$$\mathbf{U}_x = A_\infty \mathbf{U} + \mathbf{F},$$

where $\mathbf{U} = (U_1, \partial_x U_1)$, $\mathbf{F} = (0, f)$ and

$$A_\infty = \begin{cases} A_+, & x \geq 0, \\ A_-, & x < 0 \end{cases}$$

with

$$A_+ = \begin{pmatrix} 0 & 1 \\ 2 - \eta^2 & 2\eta \end{pmatrix}, \qquad A_- = \begin{pmatrix} 0 & 1 \\ d_1 - 1 - \eta^2 & -2\eta \end{pmatrix}.$$

Because A_+ has simple eigenvalues $-\eta \pm 2$ and A_- has simple eigenvalues $-\eta \pm \sqrt{d_1 - 1}$, we find from Lemma 68 that $\mathcal{A}^{\mathrm{rs}}_{\infty}$ is Fredholm with index 0. It follows that $\mathcal{M}_{\mathrm{r}0}$ is Fredholm with index 0 (Theorem 66) and that $\mathcal{M}_{\mathrm{r}}$ is Fredholm with index 0 (Corollary 62). □

Next we show that the kernel of $\mathcal{M}_{\mathrm{r}}$ is trivial in three steps.

Lemma 75. *Each solution* $(U_1, U_2) \in (H^2_\eta(\mathbb{R}))^2$ *to*

$$U_{1xx} = (-1 + 3C_1^{\star 2} + d_1 C_2^{\star 2})U_1 + 2d_1 C_1^\star C_2^\star U_2, \tag{3.82}$$
$$U_{2xx} = (-1 + d_1 C_1^{\star 2} + 3C_2^{\star 2})U_2 + 2d_1 C_1^\star C_2^\star U_1. \tag{3.83}$$

satisfies

$$\begin{aligned} U_1(x) &= \alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), & U_2(x) &= \alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}),\\ U_{1x}(x) &= \sqrt{d_1-1}\alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), & U_{2x}(x) &= \sqrt{2}\alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}),\\ U_{1xx}(x) &= (d_1-1)\alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), & U_{2xx}(x) &= 2\alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}) \end{aligned}$$

as $x \to -\infty$ *and*

$$\begin{aligned} U_1(x) &= \beta_1 \mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), & U_2(x) &= \beta_2 \mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x}),\\ U_{1x}(x) &= \sqrt{2}\beta_1 \mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), & U_{2x}(x) &= \sqrt{d_1-1}\beta_2 \mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x}),\\ U_{1xx}(x) &= 2\beta_1 \mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), & U_{2xx}(x) &= (d_1-1)\beta_2 \mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x}) \end{aligned}$$

as $x \to \infty$ *for some vectors*

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}.$$

Proof. We rewrite equations (3.82), (3.83) as the system

$$U_{1x} = W_1, \tag{3.84}$$
$$W_{1x} = (-1 + 3C_1^{\star 2} + d_1 C_2^{\star 2})U_1 + 2d_1 C_1^\star C_2^\star U_2, \tag{3.85}$$
$$U_{2x} = W_2, \tag{3.86}$$
$$W_{2x} = (-1 + d_1 C_1^{\star 2} + 3C_2^{\star 2})U_2 + 2d_1 C_1^\star C_2^\star U_1, \tag{3.87}$$

which can be written as the equation

$$\mathbf{U}_x = A_\pm \mathbf{U} + R_\pm \mathbf{U},$$

where $\mathbf{U} = (U_1, W_1, U_2, W_2)$,

$$A_+ = \begin{pmatrix} 0 & 1 & 0 & 0\\ 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & d_1-1 & 0 \end{pmatrix}, \qquad R_+ = \begin{pmatrix} 0 & 0 & 0 & 0\\ 3C_1^{\star 2} + d_1 C_2^{\star 2} - 3 & 0 & 2d_1 C_1^\star C_2^\star & 0\\ 0 & 0 & 0 & 0\\ 2d_1 C_1^\star C_2^\star & 0 & d_1 C_1^{\star 2} + 3C_2^{\star 2} - d_1 & 0 \end{pmatrix},$$

$$A_- = \begin{pmatrix} 0 & 1 & 0 & 0\\ d_1-1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 2 & 0 \end{pmatrix}, \qquad R_- = \begin{pmatrix} 0 & 0 & 0 & 0\\ 3C_1^{\star 2} + d_1 C_2^{\star 2} - d_1 & 0 & 2d_1 C_1^\star C_2^\star & 0\\ 0 & 0 & 0 & 0\\ 2d_1 C_1^\star C_2^\star & 0 & d_1 C_1^{\star 2} + 3C_2^{\star 2} - 3 & 0 \end{pmatrix}$$

and the coefficients of $R_\pm$ are $o(\mathrm{e}^{\mp \min\{\sqrt{2},\sqrt{d_1-1}\}x})$ as $x \to \pm\infty$.

The eigenvalues of the matrices $A_\pm$ are $\pm\sqrt{d_1-1}, \pm\sqrt{2}$. From Lemma 69 it follows that the asymptotic behaviour of solutions U_1, W_1, U_2, W_2 to (3.84)–(3.87) is given by

$$U_1(x) = \alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad U_2(x) = \alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}),$$
$$W_1(x) = \sqrt{d_1-1}\alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad W_2(x) = \sqrt{2}\alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})$$

as $x \to -\infty$ and

$$U_1(x) = \beta_1 \mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), \qquad U_2(x) = \beta_2 \mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x}),$$
$$W_1(x) = \sqrt{2}\beta_1 \mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), \qquad W_2(x) = \sqrt{d_1-1}\beta_2 \mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x}),$$

as $x \to \infty$ for some real constants $\alpha_1, \alpha_2, \beta_1, \beta_2$. Using this result and Lemma 72, we find from equations (3.85), (3.87) that

$$\begin{aligned}
W_{1x} &= (-1 + 3C_1^{\star 2} + d_1 C_2^{\star 2})U_1 + 2d_1 C_1^\star C_2^\star U_2 \\
&= (d_1-1)U_1 + (3C_1^{\star 2} + d_1 C_2^{\star 2} - d_1)U_1 + 2d_1 C_1^\star C_2^\star U_2 \\
&= (d_1-1)(\alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x})) \\
&\qquad + \Big(3\alpha_{\star 1}^2 \mathrm{e}^{2\sqrt{d_1-1}x} + o(\mathrm{e}^{2\sqrt{d_1-1}x}) \\
&\qquad\qquad + d_1(1 - 2\alpha_{\star 2}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})) - d_1\Big)(\alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x})) \\
&\qquad + 2d_1(\alpha_{\star 1}\mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}))(1 - \alpha_{\star 2}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}))(\alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})) \\
&= (d_1-1)\alpha_1 \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \\
W_{2x} &= 2\alpha_2 \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})
\end{aligned}$$

as $x \to -\infty$ and by similar calculations that

$$W_{1x}(x) = 2\beta_1 \mathrm{e}^{-\sqrt{2}x} + o(\mathrm{e}^{-\sqrt{2}x}), \qquad W_{2x}(x) = (d_1-1)\beta_2 \mathrm{e}^{-\sqrt{d_1-1}x} + o(\mathrm{e}^{-\sqrt{d_1-1}x})$$

as $x \to \infty$. The advertised result follows from the fact that $W_1 = U_{1x}$, $W_2 = U_{2x}$. □

Lemma 76. *The kernel of $\mathcal{M}_\mathrm{r} : (H_\eta^2(\mathbb{R}))^2 \subseteq (L_\eta^2(\mathbb{R}))^2 \to (L_\eta^2(\mathbb{R}))^2$ is at most one-dimensional.*

Proof. Suppose that $\dim\ker\mathcal{M}_\mathrm{r} \geq 2$, so that equations (3.82), (3.83) have two linear independent solutions $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}$. According to Lemma 75 there exist constants $\alpha_1^{(j)}, \alpha_2^{(j)}$ such that

$$U_1^{(j)}(x) = \alpha_1^{(j)} \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad U_2^{(j)}(x) = \alpha_2^{(j)} \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}),$$
$$U_{1x}^{(j)}(x) = \sqrt{d_1-1}\alpha_1^{(j)} \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad U_{2x}^{(j)}(x) = \sqrt{2}\alpha_2^{(j)} \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}),$$
$$U_{1xx}^{(j)}(x) = (d_1-1)\alpha_1^{(j)} \mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}), \qquad U_{2xx}^{(j)}(x) = 2\alpha_2^{(j)} \mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})$$

as $x \to -\infty$, and $U_1^{(j)}(x), U_2^{(j)}(x) \to 0$ as $x \to \infty$. By replacing $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ with linear combinations if necessary, we may assume that $\alpha_1^{(1)} = 0, \alpha_2^{(1)} \neq 0$ if $d_1 - 1 > 2$ or $\alpha_1^{(2)} \neq 0, \alpha_2^{(2)} = 0$ if $d_1 - 1 < 2$. We treat the former case; the other case is handled similarly.

Using Lemma 72 we find that $U_1^{(1)}$ satisfies

$$\begin{aligned}
&U_{1xx}^{(1)} - (d_1 - 1)U_1^{(1)} \\
&\quad = (3C_1^{\star 2} + d_1 C_2^{\star 2} - d_1)U_1^{(1)} + 2d_1 C_1^\star C_2^\star U_2^{(1)} \\
&\quad = (3\alpha_{\star 1}^2 \mathrm{e}^{2\sqrt{d_1-1}x} + o(\mathrm{e}^{2\sqrt{d_1-1}x}) + d_1(1 - 2\alpha_{\star 2}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})) - d_1)o(\mathrm{e}^{\sqrt{d_1-1}x}) \\
&\qquad + 2d_1(\alpha_{\star 1}\mathrm{e}^{\sqrt{d_1-1}x} + o(\mathrm{e}^{\sqrt{d_1-1}x}))(1 - \alpha_{\star 2}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x}))(\alpha_2^{(1)}\mathrm{e}^{\sqrt{2}x} + o(\mathrm{e}^{\sqrt{2}x})) \\
&\quad = 2d_1\alpha_{\star 1}\alpha_2^{(1)}\mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x} + o(\mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x})
\end{aligned}$$

as $x \to -\infty$ and conclude that

$$U_1^{(1)}(x) = \alpha \mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x} + o(\mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x}), \qquad \alpha = \frac{2d_1\alpha_{\star 1}\alpha_2^{(1)}}{(\sqrt{d_1-1}+\sqrt{2})^2 - (d_1-1)}$$

as $x \to -\infty$.

Since $\alpha_{\star 1} > 0$ and hence $\alpha\alpha_2^{(1)} > 0$ the functions $U_1^{(1)}, U_2^{(1)}$ have the same sign for $x < -m_1$ for some $m_1 > 0$. We consider the case that both are positive; the other case is treated in a similar fashion. The calculations

$$\begin{aligned}
U_1^{(1)}(x-h) - U_1^{(1)}(x) &= \alpha \mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})(x-h)} + o(\mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})(x-h)} \\
&\qquad - \left(\alpha \mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x} + o(\mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x})\right), \\
&= \mathrm{e}^{(\sqrt{d_1-1}+\sqrt{2})x}\left(o(\mathrm{e}^{-(\sqrt{d_1-1}+\sqrt{2})h}) - \alpha(1 - \mathrm{e}^{-(\sqrt{d_1-1}+\sqrt{2})h})\right), \\
&\le 0, \\
U_2^{(1)}(x-h) - U_2^{(1)}(x) &= \mathrm{e}^{\sqrt{2}x}\left(o(\mathrm{e}^{-\sqrt{2}h}) - \alpha_2^{(1)}(1 - \mathrm{e}^{-\sqrt{2}h})\right) \\
&\le 0
\end{aligned}$$

as $x \to -\infty$ show that both functions $U_1^{(1)}, U_2^{(1)}$ are monotone increasing for $x < -m_2$ for some $m_2 > 0$. Let $m = \max\{m_1, m_2\}$.

Because $U_2^{(1)} \to 0$ for $x \to \pm\infty$ and $U_2^{(1)}$ is positive and monotone increasing for $x < -m$, it has at least one local maximum $x^\star \in \mathbb{R}$ such that $U_2^{(1)}(x) > 0$ for all $x \le x^\star$ and of course

$$U_{2x}^{(1)}(x^\star) = 0, \qquad U_{2xx}^{(1)}(x^\star) \le 0.$$

Using the inequality $2d_1 C_1^\star(x^\star)C_2^\star(x^\star) \ge 0$, Lemma 71 (ii) and Lemma 73, we find from equation (3.83) that

$$\begin{aligned}
2d_1 C_1^\star(x^\star)C_2^\star(x^\star)U_1^{(1)}(x^\star) &= U_{2xx}^{(1)}(x^\star) - (-1 + d_1 C_1^\star(x^\star)^2 + 3C_2^\star(x^\star))U_2^{(1)}(x^\star) \\
&\le -(-1 + d_1 C_1^\star(x^\star)^2 + 3C_2^\star(x^\star))U_2^{(1)}(x^\star) \\
&= -(-1 + d_1 C_2^\star(-x^\star)^2 + 3C_1^\star(-x^\star))U_2^{(1)}(x^\star) \\
&< 0
\end{aligned}$$

and hence that $U_1^{(1)}(x^\star) < 0$.

Since $U_1^{(1)}(x^\star) < 0$ and $U_1^{(1)}(x) > 0$ for all $x < -m$ it follows that $U_1^{(1)}$ has at least one local maximum $x^{\star\star} < x^\star$ with

$$U_1^{(1)}(x^{\star\star}) > 0, \qquad U_{1x}^{(1)}(x^{\star\star}) = 0, \qquad U_{1xx}^{(1)}(x^{\star\star}) \leq 0.$$

Equation (3.82) shows that

$$2d_1C_1^\star(x^{\star\star})C_2^\star(x^{\star\star})U_2^{(1)}(x^{\star\star}) = U_{1xx}^{(1)}(x^{\star\star}) - (-1 + 3C_1^{\star 2}(x^{\star\star}) + d_1C_2^{\star 2}(x^{\star\star}))U_1^{(1)}(x^{\star\star}) < 0$$

and hence $U_2^{(1)}(x^{\star\star}) < 0$, which contradicts $U_2^{(1)}(x) > 0$ for $x < x^\star$. □

Lemma 77. *The kernel of the Fredholm operator $\mathcal{M}_\mathrm{r} : \mathcal{Y}_\eta^\mathrm{r} \subseteq \mathcal{X}_\eta^\mathrm{r} \to \mathcal{X}_\eta^\mathrm{r}$ is trivial.*

Proof. Differentiating

$$\begin{aligned} C_{1xx}^\star &= (-1 + C_1^{\star 2} + d_1C_2^{\star 2})C_1^\star, \\ C_{2xx}^\star &= (-1 + d_1C_1^{\star 2} + C_2^{\star 2})C_2^\star \end{aligned}$$

with respect to x we find that

$$\begin{aligned} (C_{1x}^\star)_{xx} &= (-1 + 3C_1^{\star 2} + d_1C_2^{\star 2})C_{1x}^\star + 2d_1C_1^\star C_2^\star C_{2x}^\star, \\ (C_{2x}^\star)_{xx} &= (-1 + d_1C_1^{\star 2} + 3C_2^{\star 2})C_{2x}^\star + 2d_1C_1^\star C_2^\star C_{1x}^\star. \end{aligned}$$

Lemma 76 shows that $(C_{1x}^\star, C_{2x}^\star)$ is the only non-trivial solution to equations (3.82), (3.83). However $(C_{1x}^\star, C_{2x}^\star)$ does not lie in $\mathcal{Y}_\eta^\mathrm{r}$ because $C_{1x}^\star(x) = -C_{2x}^\star(-x)$. The kernel of $\mathcal{M}_\mathrm{r} : \mathcal{Y}_\eta^\mathrm{r} \subseteq \mathcal{X}_\eta^\mathrm{r} \to \mathcal{X}_\eta^\mathrm{r}$ is therefore trivial. □

Now we turn to the operator $\mathcal{M}_\mathrm{i}$.

Lemma 78. *The operator $\mathcal{M}_\mathrm{i} : \mathcal{Y}_\eta^\mathrm{i} \subseteq \mathcal{X}_\eta^\mathrm{i} \to \mathcal{X}_\eta^\mathrm{i}$ is Fredholm with index -1 and the kernel of its adjoint $\mathcal{M}_\mathrm{i}^* : \mathcal{Y}_{-\eta}^\mathrm{i} \subseteq \mathcal{X}_{-\eta}^\mathrm{i} \to \mathcal{X}_{-\eta}^\mathrm{i}$ is spanned by $(C_1^\star, -C_2^\star)$.*

Proof. We write

$$\mathcal{M}_\mathrm{i} = \mathcal{A}_\pm^\mathrm{i} + \mathcal{R}_\pm^\mathrm{i},$$

where

$$\begin{aligned} \mathcal{A}_+^\mathrm{i} &= \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} - (d_1 - 1) \end{pmatrix}, & \mathcal{R}_+^\mathrm{i} &= \begin{pmatrix} C_1^{\star 2} + d_1C_2^{\star 2} - 1 & 0 \\ 0 & d_1C_1^{\star 2} + C_2^{\star 2} - d_1 \end{pmatrix}, \\ \mathcal{A}_-^\mathrm{i} &= \begin{pmatrix} \partial_{xx} - (d_1 - 1) & 0 \\ 0 & \partial_{xx} \end{pmatrix}, & \mathcal{R}_-^\mathrm{i} &= \begin{pmatrix} C_1^{\star 2} + d_1C_2^{\star 2} - d_1 & 0 \\ 0 & d_1C_1^{\star 2} + C_2^{\star 2} - 1 \end{pmatrix} \end{aligned}$$

and the entries of $\mathcal{R}_\pm^\mathrm{i}$ are $o(\mathrm{e}^{\mp \min\{\sqrt{2}, \sqrt{d_1 - 1}\}x})$ as $x \to \pm\infty$. We choose $\eta < \min\{\sqrt{2}, \sqrt{d_1 - 1}\}$ and argue as in the proof of Lemma 74 (note the symmetry $V_1(x) = -V_2(-x)$); it is sufficient to consider the operator $\mathcal{A}_\infty^\mathrm{is} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with

$$\mathcal{A}_\infty^\mathrm{is} = \begin{cases} \mathcal{A}_+^\mathrm{is}, & x \geq 0, \\ \mathcal{A}_-^\mathrm{is}, & x < 0, \end{cases}$$

where

$$\mathcal{A}_{+}^{\mathrm{is}} = \mathcal{D}_{+}, \qquad \mathcal{A}_{-}^{\mathrm{is}} = \mathcal{D}_{-} - (d_1 - 1), \qquad \mathcal{D}_{\pm} = (\partial_x \mp \eta)^2.$$

We rewrite the equation $\mathcal{A}_{\infty}^{\mathrm{is}} U_1 = f$ as

$$\mathbf{U}_x = A_\infty \mathbf{U} + \mathbf{F},$$

where $\mathbf{U} = (U_1, \partial_x U_1)$, $\mathbf{F} = (0, f)$ and

$$A_\infty = \begin{cases} A_+, & x \geq 0, \\ A_-, & x < 0 \end{cases}$$

with

$$A_+ = \begin{pmatrix} 0 & 1 \\ -\eta^2 & 2\eta \end{pmatrix}, \qquad A_- = \begin{pmatrix} 0 & 1 \\ d_1 - 1 - \eta^2 & -2\eta \end{pmatrix}.$$

Because A_+ has the geometrically simple, algebraically double eigenvalue η and A_- has simple eigenvalues $-\eta \pm \sqrt{d_1 - 1}$, we find from Lemma 68 that $\mathcal{A}_{\infty}^{\mathrm{is}}$ is Fredholm with index -1. It follows that $\mathcal{M}_{\mathrm{i}}$ is Fredholm with index -1 (Corollary 62, Theorem 66).

We show that the kernel of $\mathcal{M}_{\mathrm{i}}$ is trivial by examining the kernel of the scalar operator

$$\partial_{xx} + C_1^{\star 2} + d_1 C_2^{\star 2} - 1 : H_\eta^2(\mathbb{R}) \subseteq L_\eta^2(\mathbb{R}) \to L_\eta^2(\mathbb{R}).$$

Writing the equation $(\partial_{xx} + C_1^{\star 2} + d_1 C_2^{\star 2} - 1)U_1 = 0$ as

$$\mathbf{U}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{U} + \begin{pmatrix} 0 & C_1^{\star 2} + d_1 C_2^{\star 2} - 1 \\ 0 & 0 \end{pmatrix} \mathbf{U}, \qquad \mathbf{U} = (U_1, \partial_x U_1)$$

and noting that it has no nontrivial solution satisfying $\mathbf{U} = \mathbf{0}$ as $x \to \infty$ by Lemma 70, we find that the kernel of $\mathcal{M}_{\mathrm{i}}$ is trivial.

Next we note that $\mathcal{M}_{\mathrm{i}}$ is an $\mathcal{X}_0^{\mathrm{i}}$-self-adjoint operator. From the facts that $\mathcal{M}_{\mathrm{i}}$ is Fredholm with index -1 and its kernel is trivial we find that the kernel of $\mathcal{M}_{\mathrm{i}}^*$ is one dimensional by Lemma 64. According to Lemma 71 the functions $V_1 = C_1^\star$, $V_2 = C_2^\star$ solve the equations

$$\begin{aligned} V_{1xx} &= (-1 + C_1^{\star 2} + d_1 C_2^{\star 2}) V_1, \\ V_{2xx} &= (-1 + d_1 C_1^{\star 2} + C_2^{\star 2}) V_2 \end{aligned}$$

and $C_1^\star(x) = C_2^\star(-x)$. The above equations are decoupled, so that the kernel of $\mathcal{M}_{\mathrm{i}}^*$ is spanned by $(C_1^\star, -C_2^\star) \in \mathcal{Y}_{-\eta}^{\mathrm{i}}$. □

Finally, we examine the operator $\mathcal{M}_3$.

Lemma 79. *The operator $\mathcal{M}_3 : \mathcal{Y}_\eta^3 \subseteq \mathcal{X}_\eta^3 \to \mathcal{X}_\eta^3$ is Fredholm with index 0 for $d_2 > -d_3$, and the set of values of $d_2 > -d_3$ for which it is not invertible is countable.*

Proof. Arguing as in Lemmata 74 and 78, we find that $\mathcal{M}_3$ is Fredholm with index 0 for $d_2 > -d_3$ and is invertible if and only if $\mathcal{M}_\infty^{3s}[d_2]$ is invertible, where the scalar operator $\mathcal{M}_\infty^{3s}[t] : H_\eta^2(\mathbb{R}) \subseteq L_\eta^2(\mathbb{R}) \to L_\eta^2(\mathbb{R})$ is given by

$$\mathcal{M}_\infty^{3s}[t] = \partial_{xx} - (d_3 + t(C_1^{\star 2} + C_2^{\star 2})), \qquad t > -d_3.$$

Suppose that the kernel of the self-adjoint operator $\mathcal{M}_\infty^{3s}[t^\star] : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is non-trivial for a value $t^\star > -d_3$ and let $\{\xi_1^\star, \ldots, \xi_n^\star\}$ be an orthonormal basis for the (finite-dimensional) kernel of $\mathcal{M}_\infty^{3s}[t^\star]$. The analysis of Kato [28, Chapter VII.3.2] yields the following result. For sufficiently small δ there exist at most n eigenvalues $\lambda_1(\delta), \ldots, \lambda_n(\delta)$ of $\mathcal{M}_\infty^{3s}[t^\star + \delta]$ with corresponding eigenvectors $\xi_1(\delta), \ldots, \xi_n(\delta)$; the functions λ_j, ξ_j depend analytically upon δ and satisfy

$$\lambda_j(\delta) = O(\delta), \qquad \xi_j(\delta) = \xi_j^\star + O(\delta),$$

as $\delta \to 0$ and $j = 1, \ldots, n$. The eigenvalues of $\mathcal{M}_\infty^{3s}[t^\star + \delta]$ are the eigenvalues of the $n \times n$-matrix $M(\delta)$ representing the action of $\mathcal{M}_\infty^{3s}[t^\star + \delta]$ on the basis $\{\xi_1(\delta), \ldots, \xi_n(\delta)\}$. Using the fact that $\mathcal{M}_\infty^{3s}[t^\star + \delta]$ is self-adjoint and the series representations

$$\mathcal{M}_\infty^{3s}[t^\star + \delta] = \mathcal{M}_\infty^{3s}[t^\star] + \mathcal{M}_1 \delta + O(\delta^2), \qquad \mathcal{M}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\infty^{3s}[t] = -(C_1^{\star 2} + C_2^{\star 2})$$

we find that

$$M(\delta)_{ij} = (\mathcal{M}_\infty^{3s}[t^\star + \delta]\xi_i(\delta), \xi_j(\delta))_0,$$

so that

$$M(\delta) = \mathcal{M}_1 \delta I + O(\delta^2)$$

as $\delta \to 0$, where I is the $n \times n$-identity matrix.

The estimates in Lemma 71 yield

$$-(C_1^{\star 2} + C_2^{\star 2}) \leq -\frac{1}{2}\left(\min\left\{1, \frac{2}{\sqrt{2 + d_1}}\right\}\right)^2 < 0,$$

so that 0 is not an eigenvalue of $M(\delta)$ for sufficiently small values of δ, that is $\mathcal{M}_\infty^{3s}[t^\star + \delta]$ is invertible for sufficiently small values of δ. It follows that the set of values $t > -d_3$ for which the operator $\mathcal{M}_\infty^{3s}[t]$ is not invertible has no accumulation point. □

Altogether we have established the following theorem.

Theorem 80. *For $d_1 \in (1, 4 + \sqrt{13})$, $d_2 > -d_3$ and each sufficiently small $\eta > 0$ the operator $\mathcal{K} : \mathcal{Y}_\eta \subseteq \mathcal{X}_\eta \to \mathcal{X}_\eta$ is Fredholm with index -1. The kernel of $\mathcal{K}$ is trivial and the kernel of its adjoint is spanned by $(\mathrm{i}C_1^\star, -\mathrm{i}C_2^\star, 0)$.*

3.6.4 Construction of the heteroclinic solution

Finally, we construct a solution to equation (3.75) which approaches $\mathbf{P}^{\delta,\theta}(x)$ as $x \to \infty$ and $\mathbf{Q}^{\delta,\theta}(x)$ as $x \to -\infty$ following the steps of Scheel and Wu [41, Theorem 2], Haragus and Scheel [23, Theorem 2] and Haragus and Iooss [17, Theorem 2]. Defining

$$\mathbf{H}(\theta,\delta)(x) := \mathrm{e}^{\mathrm{i}\theta x}\mathbf{C}^\star(x) + \chi(x)(\mathbf{P}^{\delta,\theta}(x) - (1,0,0)\mathrm{e}^{\mathrm{i}\theta x}) + \chi(-x)(\mathbf{Q}^{\delta,\theta}(x) - (0,1,0)\mathrm{e}^{\mathrm{i}\theta x}),$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth function with

$$\chi(x) = \begin{cases} 1, & x \geq M, \\ 0, & x \leq m \end{cases}$$

for some positive constants $m < M$, and substituting the *Ansatz*

$$\mathbf{C} = \mathbf{H}(\theta,\delta) + \mathbf{V}$$

with $\mathbf{V} \in \mathcal{Y}_\eta$ into (3.75), we obtain the equation

$$\mathcal{G}(\mathbf{V},\theta,\delta) = 0,$$

in which

$$\mathcal{G}(\mathbf{V},\theta,\delta) = \mathcal{F}(\mathbf{H}(\theta,\delta) + \mathbf{V};\delta).$$

Lemma 81. *For each sufficiently small $\eta > 0$ there exists an open neighbourhood $V_{\mathcal{G}}$ of the origin in $\mathcal{Y}_\eta$ such that the function $\mathcal{G} : V_{\mathcal{G}} \times (-\theta_0,\theta_0) \times (-\delta_0,\delta_0) \to \mathcal{X}_\eta$ is well defined and continuously differentiable.*

Proof. Since $\mathcal{Y}_\eta$ is continuously embedded in $(C^1_{\mathrm{b,u}}(\mathbb{R}))^6$ we can choose $V_{\mathcal{G}}$ such that $\mathbf{V} + \mathbf{H}(\theta,\delta) \in V_{\mathcal{F}}$ for all $\mathbf{V} \in V_{\mathcal{G}}$, $\theta \in (-\theta_0,\theta_0)$, $\delta \in (-\delta_0,\delta_0)$. Denote the norm of $\mathcal{L}((C^1_{\mathrm{b,u}}(\mathbb{R}))^6, (C_{\mathrm{b,u}}(\mathbb{R}))^6)$ by $\|\cdot\|$. Using the estimates

$$\begin{aligned}
&|\mathcal{G}(\mathbf{V},\theta,\delta)(x)| \\
&\quad = |\mathcal{F}(\mathbf{H}(\theta,\delta)(x) + \mathbf{V}(x);\delta)| \\
&\quad = |\mathcal{F}(\mathbf{P}^{\delta,\theta}(x) + \mathrm{e}^{\mathrm{i}\theta x}(\mathbf{C}^\star(x) - (1,0,0)) + \mathbf{V}(x);\delta) - \mathcal{F}(\mathbf{P}^{\delta,\theta}(x);\delta)| \\
&\quad \leq |\partial_{xx}(\mathrm{e}^{\mathrm{i}\theta x}(\mathbf{C}^\star(x) - (1,0,0)) + \mathbf{V}(x))| \\
&\qquad\quad + \sup_{\mathbf{C}\in V_{\mathcal{F}}} \|\mathrm{d}_1\mathcal{F}[\mathbf{C};\delta]\| \Big(|\mathrm{e}^{\mathrm{i}\theta x}(\mathbf{C}^\star(x) - (1,0,0)) + \mathbf{V}(x)| \\
&\qquad\qquad\qquad\qquad + |\partial_x(\mathrm{e}^{\mathrm{i}\theta x}(\mathbf{C}^\star(x) - (1,0,0)) + \mathbf{V}(x))| \Big) \\
&\quad \lesssim |(\mathbf{C}^\star(x) - (1,0,0))| + |\mathbf{C}^\star_x(x)| + |\mathbf{C}^\star_{xx}(x)| \\
&\qquad\quad + |\mathbf{V}(x)| + |\mathbf{V}_x(x)| + |\mathbf{V}_{xx}(x)|
\end{aligned}$$

for $x > M$ and

$$\begin{aligned}
|\mathcal{G}(\mathbf{V},\theta,\delta)(x)| \lesssim{}& |(\mathbf{C}^\star(x) - (0,1,0))| + |\mathbf{C}^\star_x(x)| + |\mathbf{C}^\star_{xx}(x)| \\
&+ |\mathbf{V}(x)| + |\mathbf{V}_x(x)| + |\mathbf{V}_{xx}(x)|
\end{aligned}$$

for $x < -M$ together with Lemma 72, we find that

$$\mathcal{G}(\mathbf{V},\theta,\delta)(x) = o(\mathrm{e}^{\mp\eta x})$$

as $x \to \pm\infty$; that is $\mathcal{G}(\mathbf{V},\theta_0,\delta_0) \in (L^2_\eta(\mathbb{R}))^6$ for $(\mathbf{V},\theta_0,\delta_0) \in V_{\mathcal{G}} \times (-\theta_0,\theta_0) \times (-\delta_0,\delta_0)$.

Note that

$$\begin{aligned}
&|\mathcal{G}(\mathbf{V}_1+\mathbf{V},\theta_1,\delta_1)(x)-\mathcal{G}(\mathbf{V}_1,\theta_1,\delta_1)(x)-\mathrm{d}_1\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathbf{V}(x))| \\
&\quad=|\mathcal{F}(\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1+\mathbf{V};\delta_1)(x)-\mathcal{F}(\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1)(x) \\
&\qquad -\mathrm{d}_1\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathbf{V}(x))| \\
&\quad=\left|\int_0^1\Big(\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1+t\mathbf{V};\delta_1](\mathbf{V}(x))-\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathbf{V}(x))\Big)\mathrm{d}t\right| \\
&\quad\leq\int_0^1\|\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1+t\mathbf{V};\delta_1] \\
&\qquad\quad -\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1]\|(|\mathbf{V}(x)|+|\mathbf{V}_x(x)|)\mathrm{d}t
\end{aligned}$$

for all $x\in\mathbb{R}$ and hence

$$\begin{aligned}
&\|\mathcal{G}(\mathbf{V}_1+\mathbf{V},\theta_1,\delta_1)-\mathcal{G}(\mathbf{V}_1,\theta_1,\delta_1)-\mathrm{d}_1\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathbf{V})\|_{\mathcal{X}_\eta} \\
&\qquad\leq\int_0^1\|\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1+t\mathbf{V};\delta_1]-\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1]\|\,\mathrm{d}t\,\|\mathbf{V}\|_{\mathcal{Y}_\eta}.
\end{aligned}$$

The above estimate shows that $\mathrm{d}_1\mathcal{G}[\cdot]:V_{\mathcal{G}}\times(-\theta_0,\theta_0)\times(-\delta_0,\delta_0)\to\mathcal{L}(\mathcal{Y}_\eta,(L^2_\eta(\mathbb{R}))^6)$ exists and is given by

$$\mathrm{d}_1\mathcal{G}[\mathbf{V}_1,\theta_1,\delta_1](\mathbf{V})=\mathrm{d}_1\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathbf{V}). \tag{3.88}$$

Furthermore

$$\begin{aligned}
&|\mathrm{d}_1\mathcal{G}[\mathbf{V}_1,\theta_1,\delta_1](\mathbf{V}(x))-\mathrm{d}_1\mathcal{G}[\mathbf{V}_2,\theta_2,\delta_2](\mathbf{V}(x))| \\
&\qquad=|\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathbf{V}(x))-\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_2,\delta_2)+\mathbf{V}_2;\delta_2](\mathbf{V}(x))| \\
&\qquad\leq\|\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1]-\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_2,\delta_2)+\mathbf{V}_2;\delta_2]\|\,|(|\mathbf{V}(x)|+|\mathbf{V}_x(x)|)|
\end{aligned}$$

for all $x\in\mathbb{R}$ and hence

$$\begin{aligned}
&\|\mathrm{d}_1\mathcal{G}[\mathbf{V}_1,\theta_1,\delta_1](\mathbf{V})-\mathrm{d}_1\mathcal{G}[\mathbf{V}_2,\theta_2,\delta_2](\mathbf{V})\|_{\mathcal{X}_\eta} \\
&\qquad\leq\|\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1]-\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_2,\delta_2)+\mathbf{V}_2;\delta_2]\|\,\|\mathbf{V}\|_{\mathcal{Y}_\eta}
\end{aligned}$$

for all $V\in\mathcal{Y}_\eta\setminus\{\mathbf{0}\}$. This calculation shows that

$$\begin{aligned}
&\|\mathrm{d}_1\mathcal{G}[\mathbf{V}_1,\theta_1,\delta_1]-\mathrm{d}_1\mathcal{G}[\mathbf{V}_2,\theta_2,\delta_2]\|_{\mathcal{L}(\mathcal{Y}_\eta,\mathcal{X}_\eta)} \\
&\qquad\leq\|\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1]-\mathrm{d}_1\mathcal{F}_{\mathrm{r}}[\mathbf{H}(\theta_2,\delta_2)+\mathbf{V}_2;\delta_2]\|.
\end{aligned}$$

Since $(H^2_\eta(\mathbb{R}))^6$ is continuously embedded in $(C^1_{\mathrm{b,u}}(\mathbb{R}))^6$, it follows from the above inequality that $\mathrm{d}_1\mathcal{G}[\cdot]$ is uniformly continuous.

Similar arguments show that

$$\mathrm{d}_2\mathcal{G}[\mathbf{V}_1,\theta_1,\delta_1](\theta)=\mathrm{d}_1\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathrm{d}_1\mathbf{H}[\theta_1,\delta_1](\theta)), \tag{3.89}$$

$$\begin{aligned}
\mathrm{d}_3\mathcal{G}[\mathbf{V}_1,\theta_1,\delta_1](\delta)&=\mathrm{d}_1\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\mathrm{d}_2\mathbf{H}[\theta_1,\delta_1](\delta)) \\
&\qquad+\mathrm{d}_2\mathcal{F}[\mathbf{H}(\theta_1,\delta_1)+\mathbf{V}_1;\delta_1](\delta)
\end{aligned} \tag{3.90}$$

and that these derivatives are uniformly continuous. Note that

$$\begin{aligned}
\mathrm{d}_1\mathcal{F}[\mathbf{C}_1;\delta_1](\delta) &= \mathbf{C}_{xx} - \mathrm{d}_1\mathcal{H}_1[\mathbf{C}_1](\mathbf{C}) \\
&\qquad - \mathrm{d}_1\mathcal{H}_2[\mathbf{C}_1,\mathbf{C}_{1x},x,\delta_1](\mathbf{C}) - \mathrm{d}_2\mathcal{H}_2[\mathbf{C}_1,\mathbf{C}_{1x},x,\delta_1](\mathbf{C}_x), \\
\mathrm{d}_2\mathcal{F}[\mathbf{C}_1;\delta_1](\delta) &= -\mathrm{d}_4\mathcal{H}_2[\mathbf{C}_1,\mathbf{C}_{1x},x,\delta_1](\delta),
\end{aligned}$$

and these formulae are understood pointwise in formulae (3.88)–(3.90) and the calculations used to derive them. By construction $\mathcal{G}$ maps functions with the symmetry

$$C_1(x) = \overline{C_2(-x)}, \qquad C_3(x) = \overline{C_3(-x)}$$

for all $x \in \mathbb{R}$ to functions with the same symmetry. $\square$

Theorem 82. *Suppose that $d_1 \in (1, 4+\sqrt{13})$ and $d_2 > -d_3$. There exist $\delta_0 > 0$, open neighbourhoods V, W of respectively the origin in $\mathcal{Y}_\eta \times \mathbb{R}$ and $\mathcal{X}_\eta \times \mathbb{R} \times \mathbb{R}$ and a continuously differentiable mapping $(\mathbf{V},\theta) : (-\delta_0,\delta_0) \to V$ with $(\mathbf{V}(0),\theta(0)) = (\mathbf{0},0)$ such that*

- $\mathcal{G}(\mathbf{V}(\delta),\theta(\delta),\delta) = 0$ *for all $\delta \in (-\delta_0,\delta_0)$,*
- $(\mathbf{V},\theta) = (\mathbf{V}(\delta),\theta(\delta))$ *whenever $(\mathbf{V},\theta,\delta) \in W$ satisfies $\mathcal{G}(\mathbf{V},\theta,\delta) = 0$.*

Furthermore, the solution

$$\mathbf{C}_\delta = \mathbf{H}(\theta(\delta),\delta) + \mathbf{V}(\delta)$$

to the system (3.71)–(3.73) is a reversible heteroclinic solution connecting $\mathbf{P}^{\delta,\theta(\delta)}$ with $\mathbf{Q}^{\delta,\theta(\delta)}$, that is

$$\mathbf{C}_\delta - \mathbf{P}^{\delta,\theta(\delta)} = o(\mathrm{e}^{-\eta x})$$

as $x \to \infty$ and

$$\mathbf{C}_\delta - \mathbf{Q}^{\delta,\theta(\delta)} = o(\mathrm{e}^{\eta x})$$

as $x \to -\infty$.

Proof. We note that $\mathcal{G}(\mathbf{0},0,0) = \mathcal{F}(\mathbf{C}^\star;0) = 0$, $\mathrm{d}_1\mathcal{G}[\mathbf{0},0,0] = \mathcal{K}$, where $\mathcal{K} : \mathcal{Y}_\eta \to \mathcal{X}_\eta$ is defined by equation (3.81) and $\mathrm{d}_2\mathcal{G}[\mathbf{0},0,0] : \mathbb{R} \to \mathcal{X}_\eta$ is given by

$$\mathrm{d}_2\mathcal{G}[\mathbf{0},0,0](\theta) = \theta\mathcal{K}\left(\mathrm{i}xC_1^\star, -\mathrm{i}xC_2^\star, 0\right) = \theta(2\mathrm{i}C_1^{\star\prime}, -2\mathrm{i}C_2^{\star\prime}, 0), \quad \theta \in \mathbb{R}.$$

Since $\ker\mathcal{K}^\star = \langle(\mathrm{i}C_1^\star, -\mathrm{i}C_2^\star, 0)\rangle$ (see Lemma 80) and

$$(\mathrm{d}_2\mathcal{G}[\mathbf{0},0,0](\theta), (\mathrm{i}C_1^\star, -\mathrm{i}C_2^\star, 0))_{\mathcal{X}_0} = 2\theta\int_{-\infty}^{\infty}(C_1^{\star 2} - C_2^{\star 2})'\mathrm{d}x = 4\theta, \quad \theta \in \mathbb{R},$$

we find from Lemma 63 that

$$\mathrm{d}_2\mathcal{G}[\mathbf{0},0,0](\theta) \notin (\ker\mathcal{K}^*)_\circ = \mathrm{Im}\,\mathcal{K}$$

for $\theta \neq 0$. It follows from the fact that $\mathcal{K}$ is Fredholm with index -1 (see Theorem 80) and

$$\mathcal{X}_\eta = \mathrm{Im}\,\mathcal{K} \,\oplus \langle\mathrm{d}_2\mathcal{G}[\mathbf{0},0,0](1)\rangle\,.$$

Since the kernel of $\mathcal{K}$ is trivial (see Theorem 80), the mapping

$$\mathrm{d}_1\mathcal{G}[\mathbf{0},0,0] = \mathcal{K} : \mathcal{Y}_\eta \to \mathrm{Im}\,\mathcal{K}$$

is an isomorphism, and writing

$$\mathrm{d}_{(1,2)}\mathcal{G}[\mathbf{0},0,0] : \mathcal{Y}_\eta \times \mathbb{R} \to \mathcal{X}_\eta$$

as

$$\mathrm{d}_{(1,2)}\mathcal{G}[\mathbf{0},0,0](\mathbf{V},\theta) = \mathcal{K}\mathbf{V} + \mathrm{d}_2\mathcal{G}[\mathbf{0},0,0](\theta),$$

we find that $\mathrm{d}_{(1,2)}\mathcal{G}[\mathbf{0},0,0]$ is an isomorphism. The advertised result follows from the implicit-function theorem. □

3.7 Calculation of the normal-form coefficients

We have found periodic solutions under the assumption that $c_1^1 > 0$, $c_1 < 0$ and $c_3^1 > 0$, $c_4 < 0$ (see Section 3.5) and constructed a heteroclinic solution to to equations (3.53)–(3.58) which connects the solutions $\mathbf{P}^{\delta,\theta(\delta)}$ as $x \to \infty$ and $\mathbf{Q}^{\delta,\theta(\delta)}$ as $x \to -\infty$ under the assumption that the coefficients

$$d_1 = \frac{c_2}{2c_1}, \qquad d_2 = \frac{c_3}{2c_1}, \qquad d_3 = \frac{c_3^1}{c_1^1}$$

satisfy $d_1 \in (1, 4+\sqrt{13})$, $d_2 > -d_3$ (see Section 3.6 and Theorem 82). In this section we present an algorithm due to Groves and Nilsson [16, Appendix] for computing $c_1^1, c_3^1, c_1, c_2, c_3, c_4$. First we introduce notation for the terms in the reduced Hamiltonian up to fourth order. We write

$$\begin{aligned}
\tilde{H}_2^1(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0) &= c_1^1(M_1+M_3) + c_2^1(M_2+M_4) + c_3^1 M_5 + c_4^1 M_6 \\
&\quad + c_5^1 p_0^2 + p_0 \sum_{j=1}^{3} (c_{5+j}^1 A_j + \overline{c}_{5+j}^1 \overline{A}_j + c_{8+j}^1 B_j + \overline{c}_{8+j}^1 \overline{B}_j), \\
\tilde{H}_3^0(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0) &= p_0\left(k_1(M_1+M_3) + k_3(M_2+M_4) + k_2 M_5 + k_4 M_6\right) \\
&\quad + p_0 \left(k_5 p_0^2 + p_0 \sum_{j=1}^{3} (k_{5+j} A_j + \overline{k}_{5+j} \overline{A}_j + k_{8+j} B_j + \overline{k}_{8+j} \overline{B}_j) \right), \\
\tilde{H}_4^0(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0) &= c_1(M_1^2+M_3^2) + c_2 M_1 M_3 + c_3 M_5(M_1+M_3) + c_4 M_5^2 \\
&\quad + c_5(M_1M_2 + M_3M_4) + c_6(M_2^2+M_4^2) \\
&\quad + c_7(M_1M_4 + M_2M_3) + c_8 M_2 M_4 \\
&\quad + c_9 M_5(M_2+M_4) + c_{10} M_6(M_2+M_4) \\
&\quad + c_{11} M_6(M_1+M_3) + c_{12} M_5 M_6 + c_{13} M_6^2 \\
&\quad + c_{14}(M_7 + M_8 - M_6(M_2+M_4)) \\
&\quad + p_0^2\left(k_{12}(M_1+M_3) + k_{13}(M_3+M_4) + k_{14} M_5 + k_{15} M_6\right) \\
&\quad + p_0^2 \left(k_{16} p_0^2 + p_0 \sum_{j=1}^{3} (k_{16+j} A_j + \overline{k}_{16+j} \overline{A}_j + k_{19+j} B_j + \overline{k}_{19+j} \overline{B}_j) \right),
\end{aligned}$$

where $\varepsilon^j \tilde{H}_n^j(u)$ denotes the part of $\tilde{H}^\varepsilon$ which is homogeneous of order j in ε and n in u and

$$\begin{aligned}
M_1 &= A_1\overline{A}_1, & M_2 &= \mathrm{i}(A_1\overline{B}_1 - \overline{A}_1 B_1), \\
M_3 &= A_2\overline{A}_2, & M_4 &= \mathrm{i}(A_2\overline{B}_2 - \overline{A}_2 B_2), \\
M_5 &= A_3\overline{A}_3, & M_6 &= \mathrm{i}(A_3\overline{B}_3 - \overline{A}_3 B_3).
\end{aligned}$$

Next we find expressions for the coefficients in the above formulae that depend on solutions to certain boundary-value problems which we derive. Every solution $u_1 : (x_1, x_2) \to \tilde{U}_1$ of the reduced equation

$$u_{1x} = Lu_1 + PN^\varepsilon(u_1 + r(u_1; \varepsilon))$$

with boundary conditions $B(u_1 + r(u_1, \varepsilon)) = 0$ generates a solution

$$u(x) = u_1(x) + r(u_1(x); \varepsilon) \tag{3.91}$$

of the full equation (3.22) with boundary conditions $B(u_1 + r(u_1; \varepsilon)) = 0$ by Theorem 33 (iii). Substituting the *Ansatz* (3.91) into equation (3.22), we find that

$$\begin{aligned} Lr(u_1(x); \varepsilon) - \mathrm{d}_1 r[u_1(x); \varepsilon](Lu_1(x)) = P^\varepsilon(u_1(x)) + \mathrm{d}_1 r[u_1(x); \varepsilon](P^\varepsilon(u_1(x))) \\ - N^\varepsilon(u_1(x) + r(u_1(x); \varepsilon)), \qquad (3.92) \\ B_\mathrm{l} r(u_1(x); \varepsilon) = -B^\varepsilon_\mathrm{nl}(u_1(x) + r(u_1(x); \varepsilon)), \qquad (3.93) \end{aligned}$$

where B_l, $B^\varepsilon_\mathrm{nl}$ are the linear and nonlinear parts of the boundary-value operator B and we have abbreviated $PN^\varepsilon(u_1(x) + r(u_1(x); \varepsilon))$ to $P^\varepsilon(u_1(x))$.

We denote the parts of $B^\varepsilon_\mathrm{nl}(u)$, $N^\varepsilon(u)$ which are homogeneous of order j in ε and n in u by $\varepsilon^j B^j_{\mathrm{nl},n}(u)$, $\varepsilon^j N^j_n(u)$, the part of $r(u_1; \varepsilon)$ which is homogeneous of order j in ε and n in u_1 by $r^j_n(u_1; \varepsilon)$ and note the identity

$$\Omega(Lu + N^\varepsilon(u), v) + (B_\mathrm{l}(u) + B^\varepsilon_\mathrm{nl}(u))v = \mathrm{d}\tilde{H}^\varepsilon[u](v),$$

which implies that

$$\begin{aligned} \Omega(Lw, v) + B_\mathrm{l}(w)v &= 2H^0_2(w, v), \\ \Omega(N^1_1(w), v) + B^1_{\mathrm{nl},1}(w)v &= 2H^1_2(w, v), \\ \Omega(N^0_2(w_1, w_2), v) + B^0_{\mathrm{nl},2}(w_1, w_2)v &= 3H^0_2(w_1, w_2, v), \\ \Omega(N^0_3(w_1, w_2, w_3), v) + B^0_{\mathrm{nl},3}(w_1, w_2, w_3)v &= 4H^0_4(w_1, w_2, w_3, v). \end{aligned}$$

We write

$$\begin{aligned} u_1 &= q_0 f^1_0 + p_0 f^2_0 + A_1 f^1_1 + B_1 f^2_1 + A_2 f^1_2 + B_2 f^2_2 + A_3 f^1_3 + B_3 f^2_3 \\ &\quad + \overline{A_1}\,\overline{f^1_1} + \overline{B_1}\,\overline{f^2_1} + \overline{A_2}\,\overline{f^1_2} + \overline{B_2}\,\overline{f^2_2} + \overline{A_3}\,\overline{f^1_3} + \overline{B_3}\,\overline{f^2_3}, \\ r^j_n(u_1; \varepsilon) &= \sum_{j+|\mathbf{m}|=n} \varepsilon^j r^j_\mathbf{m} A_1^{k_1} B_1^{l_1} \overline{A}_1^{m_1} \overline{B}_1^{n_1} A_2^{k_2} B_2^{l_2} \overline{A}_2^{m_2} \overline{B}_2^{n_2} A_3^{k_3} B_3^{l_3} \overline{A}_3^{m_3} \overline{B}_3^{n_3} p_0^{n_4}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{m} &= (k_1, l_1, m_1, n_1, k_2, l_2, m_2, n_2, k_3, l_3, m_3, n_3, n_4), \\ |\mathbf{m}| &= k_1 + l_1 + m_1 + n_1 + k_2 + l_2 + m_2 + n_2 + k_3 + l_3 + m_3 + n_3 + n_4, \end{aligned}$$

and note that

$$P^{\varepsilon}(\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0) = \sum_{j=1}^{3}\Big(\partial_{\overline{B}_j}\tilde{H}^{\varepsilon}_{\mathrm{nl}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0]f_j^1 + \partial_{B_j}\tilde{H}^{\varepsilon}_{\mathrm{nl}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0]\overline{f_j^1}$$
$$- \Big(\partial_{\overline{A}_j}\tilde{H}^{\varepsilon}_{\mathrm{nl}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0]f_j^2 + \partial_{A_j}\tilde{H}^{\varepsilon}_{\mathrm{nl}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0]\overline{f_j^2}\Big)\Big)$$
$$+ \partial_{p_0}\tilde{H}^{\varepsilon}_{\mathrm{nl}}[\mathbf{A},\mathbf{B},\overline{\mathbf{A}},\overline{\mathbf{B}},p_0]f_0^1.$$

The coefficients

$$c_1^1, \quad c_3^1, \quad c_1, \quad c_2, \quad c_3, \quad c_4$$

are computed by examining the

$$\varepsilon A_1, \quad \varepsilon A_3, \quad A_1|A_1|^2, \quad A_1|A_2|^2, \quad A_1|A_3|^2, \quad A_3|A_3|^2$$

components of equations (3.92), (3.93) respectively.

- We start with the εA_1 component of equations (3.92), (3.93). The coefficient

$$r^1_{\mathbf{m}_1}, \qquad \mathbf{m}_1 = (1,0,0,0,0,0,0,0,0,0,0,0,0)$$

is found from

$$\begin{aligned}(L - \mathrm{i}\omega_1 I)r^1_{\mathbf{m}_1} &= \mathrm{i}c_2^1 f_1^1 - c_1^1 f_1^2 + c_5^1 f_0^1 - N_1^1(f_1^1),\\ B_{\mathrm{l}} r^1_{\mathbf{m}_1} &= 0.\end{aligned}$$

Noting that

$$\begin{aligned}\Omega((L-\mathrm{i}\omega_1 I)r^1_{\mathbf{m}_1}, \overline{f_1^1}) &= \Omega(Lr^1_{\mathbf{m}_1}, \overline{f_1^1}) - \mathrm{i}\omega_1\Omega(r^1_{\mathbf{m}_1}, \overline{f_1^1})\\ &= 2H_2^0(r^1_{\mathbf{m}_1}, \overline{f_1^1}) - B_1(r^1_{\mathbf{m}_1})\overline{f_1^1} - \mathrm{i}\omega_1\Omega(r^1_{\mathbf{m}_1}, \overline{f_1^1})\\ &= \Omega(L\overline{f_1^1}, r^1_{\mathbf{m}_1}) + B_1(\overline{f_1^1})r^1_{\mathbf{m}_1} - \mathrm{i}\omega_1\Omega(r^1_{\mathbf{m}_1}, \overline{f_1^1})\\ &= -\mathrm{i}\omega_1\Omega(\overline{f_1^1}, r^1_{\mathbf{m}_1}) - \mathrm{i}\omega_1\Omega(r^1_{\mathbf{m}_1}, \overline{f_1^1})\\ &= 0,\\ \Omega(\mathrm{i}c_2^1 f_1^1 - c_1^1 f_1^2 + c_5^1 f_0^1 - N_1^1(f_1^1), \overline{f_1^1}) &= c_1^1 - \Omega(N_1^1(f_1^1), \overline{f_1^1})\\ &= c_1^1 - 2H_2^1(f_1^1, \overline{f_1^1}) + B_1^1(f_1^1)\overline{f_1^1}\\ &= c_1^1 - 2H_2^1(f_1^1, \overline{f_1^1}),\end{aligned}$$

we find that

$$c_1^1 = 2H_2^1(f_1^1, \overline{f_1^1}).$$

- Examining the εA_3 component of equations (3.92), (3.93), we find

$$r^1_{\mathbf{m}_2}, \qquad \mathbf{m}_2 = (0,0,0,0,0,0,0,0,1,0,0,0,0)$$

from

$$\begin{aligned}(L - \mathrm{i}\omega I)r^1_{\mathbf{m}_2} &= \mathrm{i}c_4^1 f_3^1 - c_3^1 f_3^2 + c_7^1 f_0^1 - N_1^1(f_3^1),\\ B_{\mathrm{l}} r^1_{\mathbf{m}_2} &= -B^1_{\mathrm{nl},1}(f_3^1).\end{aligned}$$

Arguing as above, we find that

$$c_3^1 = 2H_2^1(f_3^1, \overline{f_3^1}).$$

- The coefficient

$$r^0_{\mathbf{m}_3}, \qquad \mathbf{m}_3 = (2,0,1,0,0,0,0,0,0,0,0,0,0,0)$$

of $A_1|A_1|^2$ in equations (3.92), (3.93) is found from

$$\begin{aligned}(L-\mathrm{i}\omega_1 I)r^0_{\mathbf{m}_3} &= \mathrm{i}c_5 f^1_1 - 2c_1 f^2_1 - 3N^0_3(f^1_1, f^1_1, \overline{f^1_1}) - 2N^0_2(\overline{f^1_1}, r^0_{\mathbf{m}_4}) - 2N^0_2(f^1_1, r^0_{\mathbf{m}_5}),\\ B_{\mathrm{l}} r^0_{\mathbf{m}_3} &= -3B^0_{\mathrm{nl},3}(f^1_1, f^1_1, \overline{f^1_1}) - 2B^0_{\mathrm{nl},2}(\overline{f^1_1}, r^0_{\mathbf{m}_4}) - 2B^0_{\mathrm{nl},2}(f^1_1, r^0_{\mathbf{m}_5}),\end{aligned}$$

where we additionally examine the A_1^2, $|A_1|^2$ components to find

$$\begin{aligned} &r^0_{\mathbf{m}_4}, \qquad \mathbf{m}_4 = (2,0,0,0,0,0,0,0,0,0,0,0,0,0),\\ &r^0_{\mathbf{m}_5}, \qquad \mathbf{m}_5 = (1,0,1,0,0,0,0,0,0,0,0,0,0,0),\end{aligned}$$

from

$$\begin{aligned}(L-2\mathrm{i}\omega_1 I)r^0_{\mathbf{m}_4} &= -N^0_2(f^1_1, f^1_1),\\ B_{\mathrm{l}} r^0_{\mathbf{m}_4} &= -B^0_{\mathrm{nl},2}(f^1_1, f^1_1),\end{aligned}$$

$$\begin{aligned}L(r^0_{\mathbf{m}_5} - k_1 f^2_0) &= -2N^0_2(f^1_1, \overline{f^1_1}),\\ B_1(r^0_{\mathbf{m}_5} - k_1 f^2_0) &= 2B^0_2(f^1_1, \overline{f^1_1}).\end{aligned}$$

Finally, to determine the value of k_1 we examine the $p_0 A_1$ component of equations (3.92), (3.93). The coefficient

$$r^0_{\mathbf{m}_6}, \qquad \mathbf{m}_6 = (1,0,0,0,0,0,0,0,0,0,0,0,0,1)$$

is found from

$$\begin{aligned}(L-\mathrm{i}\omega_1 I)r^0_{\mathbf{m}_6} &= \mathrm{i}k_3 f^1_1 - k_1 f^2_1 + 2k_6 f^1_0 - 2N^0_2(f^1_1, f^2_0),\\ B_{\mathrm{l}} r^0_{\mathbf{m}_6} &= -2B^0_{\mathrm{nl},2}(f^1_1, f^2_0),\end{aligned}$$

which implies that

$$\begin{aligned}k_1 &= 2\Omega(N^0_2(f^1_1, f^2_0), \overline{f^1_1})\\ &= 6H^0_3(f^1_1, \overline{f^1_1}, f^2_0).\end{aligned}$$

Altogether we have that

$$\begin{aligned}c_1 = 6H^0_4(f^1_1, f^1_1, \overline{f^1_1}, \overline{f^1_1}) &+ 3H^0_3(r^0_{\mathbf{m}_4}, \overline{f^1_1}, \overline{f^1_1})\\ &+ 3H^0_3(r^0_{\mathbf{m}_5} - k_1 f^2_0, f^1_1, \overline{f^1_1}) + 18H^0_3(f^1_1, \overline{f^1_1}, f^2_0)^2.\end{aligned}$$

- The coefficient

$$r^0_{\mathbf{m}_7}, \qquad \mathbf{m}_7 = (1,0,0,0,1,0,1,0,0,0,0,0,0,0)$$

of $A_1|A_2|^2$ in equations (3.92), (3.93) is found from

$$
\begin{aligned}
(L - \mathrm{i}\omega_1 I) r^0_{\mathbf{m}_7} &= \mathrm{i}c_7 f^1_1 - c_2 f^2_1 - 3N^0_3(f^1_1, f^1_2, \overline{f^1_2}) \\
&\quad - 2N^0_2(f^1_1, r^0_{\mathbf{m}_8}) - 2N^0_2(f^1_2, r^0_{\mathbf{m}_9}) \\
&\quad - 2N^0_2(\overline{f^1_2}, r^0_{\mathbf{m}_{10}}), \\
B_\mathrm{l} r^0_{\mathbf{m}_7} &= -3B^0_{\mathrm{nl},3}(f^1_1, f^1_2, \overline{f^1_2}) \\
&\quad - 2B^0_{\mathrm{nl},2}(f^1_1, r^0_{\mathbf{m}_8}) - 2B^0_{\mathrm{nl},2}(f^1_2, r^0_{\mathbf{m}_9}) \\
&\quad - 2B^0_{\mathrm{nl},2}(\overline{f^1_2}, r^0_{\mathbf{m}_{10}}),
\end{aligned}
$$

where we additionally examine the $|A_2|^2$, $A_1\overline{A_2}$, A_1A_2 components to find

$$
\begin{aligned}
r^0_{\mathbf{m}_8}, &\quad \mathbf{m}_8 = (0,0,0,0,1,0,1,0,0,0,0,0,0), \\
r^0_{\mathbf{m}_9}, &\quad \mathbf{m}_9 = (1,0,0,0,0,0,1,0,0,0,0,0,0), \\
r^0_{\mathbf{m}_{10}}, &\quad \mathbf{m}_{10} = (1,0,0,0,1,0,0,0,0,0,0,0,0)
\end{aligned}
$$

from

$$
\begin{aligned}
L(r^0_{\mathbf{m}_8} - k_1 f^2_0) &= -2N^0_2(f^1_2, \overline{f^1_2}), \\
B_1(r^0_{\mathbf{m}_8} - k_1 f^2_0) &= -2B^0_{\mathrm{nl},2}(f^1_2, \overline{f^1_2}),
\end{aligned}
$$

$$
\begin{aligned}
L r^0_{\mathbf{m}_9} &= -2N^0_2(f^1_1, \overline{f^1_2}), \\
B_\mathrm{l} r^0_{\mathbf{m}_9} &= -2B^0_{\mathrm{nl},2}(f^1_1, \overline{f^1_2}),
\end{aligned}
$$

$$
\begin{aligned}
(L - 2\mathrm{i}\omega_1 I) r^0_{\mathbf{m}_{10}} &= -2N^0_2(f^1_1, f^1_2), \\
B_\mathrm{l} r^0_{\mathbf{m}_{10}} &= -2B^0_{\mathrm{nl},2}(f^1_1, f^1_2).
\end{aligned}
$$

It is helpful to determine the value of k_1 in another way by examining the p_0A_2 component of equations (3.92), (3.93). The coefficient

$$
r^0_{\mathbf{m}_{11}}, \qquad \mathbf{m}_{11} = (0,0,0,0,1,0,0,0,0,0,0,0,1)
$$

is found from

$$
\begin{aligned}
(L - \mathrm{i}\omega_1 I) r^0_{\mathbf{m}_{11}} &= \mathrm{i}k_3 f^1_2 - k_1 f^2_2 + 2k_7 f^1_0 - 2N^0_2(f^1_2, f^2_0), \\
B_\mathrm{l} r^0_{\mathbf{m}_{11}} &= -2B^0_{\mathrm{nl},2}(f^1_2, f^2_0),
\end{aligned}
$$

so that

$$
k_1 = 6H^0_3(f^1_2, \overline{f^1_2}, f^2_0) = 6H^0_3(f^1_1, \overline{f^1_1}, f^1_0).
$$

Altogether we have that

$$
\begin{aligned}
c_2 = 12H^0_4(f^1_1, f^1_2, \overline{f^1_1}, \overline{f^1_2}) + 6H^0_3(r^0_{\mathbf{m}_9}, f^1_2, \overline{f^1_1}) + 6H^0_3(r^0_{\mathbf{m}_{10}}, \overline{f^1_1}, \overline{f^1_2}) \\
+ 6H^0_3(r^0_{\mathbf{m}_8} - k_1 f^2_0, f^1_1, \overline{f^1_1}) + 36H^0_3(f^1_1, \overline{f^1_1}, f^2_0)^2.
\end{aligned}
$$

- The coefficient

$$r^0_{\mathbf{m}_{12}}, \qquad \mathbf{m}_{12} = (1,0,0,0,0,0,0,0,1,0,1,0,0)$$

of $A_1|A_3|^2$ in equations (3.92), (3.93) is found from

$$\begin{aligned}
(L - \mathrm{i}\omega_1 I) r^0_{\mathbf{m}_{12}} &= \mathrm{i}c_9 f_1^1 - c_3 f_1^2 - 3N_3^0(f_1^1, f_3^1, \overline{f_3^1}) \\
&\quad - 2N_2^0(f_1^1, r^0_{\mathbf{m}_{13}}) - 2N_2^0(f_3^1, r^0_{\mathbf{m}_{14}}) \\
&\quad - 2N_2^0(\overline{f_3^1}, r^0_{\mathbf{m}_{15}}), \\
B_{\mathrm{l}} r^0_{\mathbf{m}_{12}} &= -3B^0_{\mathrm{nl},3}(f_1^1, f_3^1, \overline{f_3^1}) \\
&\quad - 2B^0_{\mathrm{nl},2}(f_1^1, r^0_{\mathbf{m}_{13}}) - 2B^0_{\mathrm{nl},2}(f_3^1, r^0_{\mathbf{m}_{14}}) \\
&\quad - 2B^0_{\mathrm{nl},2}(\overline{f_3^1}, r^0_{\mathbf{m}_{15}}),
\end{aligned}$$

where we additionally examine the $|A_3|^2$, $A_1\overline{A_3}$, A_1A_3 components to find

$$\begin{aligned}
&r^0_{\mathbf{m}_{13}}, \quad \mathbf{m}_{13} = (0,0,0,0,0,0,0,0,1,0,1,0,0), \\
&r^0_{\mathbf{m}_{14}}, \quad \mathbf{m}_{14} = (1,0,0,0,0,0,0,0,0,0,1,0,0), \\
&r^0_{\mathbf{m}_{15}}, \quad \mathbf{m}_{15} = (1,0,0,0,0,0,0,0,1,0,0,0,0)
\end{aligned}$$

from

$$\begin{aligned}
L(r^0_{\mathbf{m}_{13}} - k_2 f_0^2) &= -2N_2^0(f_3^1, \overline{f_3^1}), \\
B_1(r^0_{\mathbf{m}_{13}} - k_2 f_0^2) &= -2B^0_{\mathrm{nl},2}(f_3^1, \overline{f_3^1}),
\end{aligned}$$

$$\begin{aligned}
(L - \mathrm{i}(\omega_1 - \omega) I) r^0_{\mathbf{m}_{14}} &= -2N_2^0(f_1^1, \overline{f_3^1}), \\
B_{\mathrm{l}} r^0_{\mathbf{m}_{14}} &= -2B^0_{\mathrm{nl},2}(f_1^1, \overline{f_3^1}),
\end{aligned}$$

$$\begin{aligned}
(L - \mathrm{i}(\omega_1 + \omega) I) r^0_{\mathbf{m}_{15}} &= -2N_2^0(f_1^1, f_3^1), \\
B_{\mathrm{l}} r^0_{\mathbf{m}_{15}} &= -2B^0_{\mathrm{nl},2}(f_1^1, f_3^1).
\end{aligned}$$

Finally, to determine the value of k_2 we examine the p_0A_3 component of equations (3.92), (3.93). The coefficient

$$r^0_{\mathbf{m}_{16}}, \qquad \mathbf{m}_{16} = (0,0,0,0,0,0,0,0,1,0,0,0,1)$$

is found from

$$\begin{aligned}
(L - \mathrm{i}\omega_1 I) r^0_{\mathbf{m}_{16}} &= \mathrm{i}k_4 f_3^1 - k_2 f_3^2 + 2k_8 f_0^1 - 2N_2^0(f_3^1, f_0^2), \\
B_{\mathrm{l}} r^0_{\mathbf{m}_{16}} &= -2B^0_{\mathrm{nl},2}(f_3^1, f_0^2),
\end{aligned}$$

which yields

$$k_2 = 6H_3^0(f_3^1, \overline{f_3^1}, f_0^2).$$

Altogether we have that

$$\begin{aligned}
c_3 = 12H_4^0(f_1^1, f_3^1, \overline{f_1^1}, \overline{f_3^1}) + 6H_3^0(r^0_{\mathbf{m}_{14}}, f_3^1, \overline{f_1^1}) + 6H_3^0(r^0_{\mathbf{m}_{15}}, \overline{f_1^1}, \overline{f_3^1}) \\
+ 6H_3^0(r^0_{\mathbf{m}_{13}} - k_2 f_0^2, f_1^1, \overline{f_1^1}) + 36H_3^0(f_3^1, \overline{f_3^1}, f_0^2) H_3^0(f_1^1, \overline{f_1^1}, f_0^2).
\end{aligned}$$

- The coefficient

$$r^0_{\mathbf{m}_{17}}, \qquad \mathbf{m}_{17} = (0,0,0,0,0,0,0,0,0,2,0,1,0,0)$$

of $A_3|A_3|^2$ in equations (3.92), (3.93) is found from

$$\begin{aligned}(L-\mathrm{i}\omega I)r^0_{\mathbf{m}_{17}} &= \mathrm{i}c_{12}f^1_3 - 2c_4 f^2_3 - 3N^0_3(f^1_3, f^1_3, \overline{f^1_3}) - 2N^0_2(\overline{f^1_3}, r^0_{\mathbf{m}_{18}}) - 2N^0_2(f^1_3, r^0_{\mathbf{m}_{19}}),\\ B_\mathrm{l} r^0_{\mathbf{m}_{17}} &= -3B^0_{\mathrm{nl},3}(f^1_3, f^1_3, \overline{f^1_3}) - 2B^0_{\mathrm{nl},2}(\overline{f^1_3}, r^0_{\mathbf{m}_{18}}) - 2B^0_{\mathrm{nl},2}(f^1_3, r^0_{\mathbf{m}_{19}}),\end{aligned}$$

where we additionally examine the A_3^2, $|A_3|^2$ components to find

$$\begin{aligned} &r^0_{\mathbf{m}_{18}}, \qquad \mathbf{m}_{18} = (0,0,0,0,0,0,0,0,2,0,0,0,0),\\ &r^0_{\mathbf{m}_{19}}, \qquad \mathbf{m}_{19} = (0,0,0,0,0,0,0,0,1,0,1,0,0)\end{aligned}$$

from

$$\begin{aligned}(L-2\mathrm{i}\omega I)r^0_{\mathbf{m}_{18}} &= -N^0_2(f^1_3, f^1_3),\\ B_\mathrm{l} r^0_{\mathbf{m}_{18}} &= -B^0_{\mathrm{nl},2}(f^1_3, f^1_3),\end{aligned}$$

$$\begin{aligned}L(r^0_{\mathbf{m}_{19}} - k_2 f^2_0) &= -2N^0_2(f^1_3, \overline{f^1_3}),\\ B_1(r^0_{\mathbf{m}_{19}} - k_2 f^2_0) &= -2B^0_{\mathrm{nl},2}(f^1_3, \overline{f^1_3}).\end{aligned}$$

Altogether we have that

$$\begin{aligned}c_4 = {} & 6H^0_4(f^1_3, f^1_3, \overline{f^1_3}, \overline{f^1_3}) + 3H^0_3(r^0_{\mathbf{m}_{18}}, \overline{f^1_3}, \overline{f^1_3})\\ & + 3H^0_3(r^0_{\mathbf{m}_{19}} - k_2 f^2_0, f^1_3, \overline{f^1_3}) + 18H^0_3(f^1_3, \overline{f^1_3}, f^2_0)^2.\end{aligned}$$

Finally, we record the formulae

$$N^0_2(u) = \begin{pmatrix} -\displaystyle\int_{-\frac{1}{\beta_0}}^0 (1+\beta_0 y)\xi\chi_y \,\mathrm{d}y - \int_0^{\frac{1}{\beta_0}} (1-\beta_0 y)\xi'\chi'_y \,\mathrm{d}y \\[2ex] \beta_0\eta\xi' - (1-\beta_0 y)\rho\chi'_y \\[2ex] -\dfrac{\beta_0\eta\xi}{\mu_1} - \dfrac{\dot\mu_1}{\mu_1^2}\xi\chi_y - (1+\beta_0 y)\rho\chi_y \\[2ex] \begin{aligned}&\int_0^{\frac{1}{\beta_0}} \left(\nu^2(1-\beta_0 y)\left(\chi'_{zz}\chi'_y + \chi'_z\chi'_{yz}\right) - \frac{\beta_0}{2}\left(\nu^2(\chi'_z)^2 + (\xi')^2 - (\chi'_y)^2\right)\right)\mathrm{d}y\\ &+ \int_{-\frac{1}{\beta_0}}^0 \left(\nu^2(1+\beta_0 y)\mu_1\left(\chi_{zz}\chi_y + \chi_z\chi_{yz}\right) + \frac{\beta_0}{2}\mu_1\left(\nu^2\chi_z^2 + \frac{\xi^2}{\mu_1^2} - \frac{\chi_y^2}{S_1^2}\right)\right)\mathrm{d}y\\ &-(\mu_1-1)\left(-\beta_0\eta\chi'_y - \rho\xi' + \nu^2\chi'_z\eta_z\right)\end{aligned} \\[2ex] -\beta_0\eta(\chi'_{yy} - \nu^2\chi'_{zz}) + (1-\beta_0 y)(2\nu^2\eta_z\chi'_{yz} + \nu^2\eta_{zz}\chi'_y - \rho\xi'_y) + \beta_0\rho\xi' \\[2ex] \begin{aligned}&-2\dot\mu_1\chi_{yy}\chi_y - \frac{\dot\mu_1\xi\xi_y}{\mu_1^2} - 2\nu^2\dot\mu_1\chi_{yz}\chi_z + \dot\mu_1\chi_{yy}\beta_0\eta - \ddot\mu_1\chi_{yy}\chi_y\\ &+ 2\nu^2\mu_1\eta_z\chi_{yz}(1+\beta_0 y) + \mu_1\chi_{yy}\beta_0\eta + \nu^2\mu_1\eta_{zz}\chi_y(1+\beta_0 y)\\ &- \nu^2\mu_1\chi_{zz}\eta\beta_0 - \nu^2\dot\mu_1\chi_{zz}\chi_y - \rho\beta_0\xi - \rho\xi_y(1+\beta_0 y)\end{aligned}\end{pmatrix},$$

$$B^0_{\mathrm{nl},2}(u) = \begin{pmatrix} \dot{\mu}_1\chi_y^2 - \mu_1\chi_y\beta_0\eta - \dot{\mu}_1\chi_y\beta_0\eta + \frac{\ddot{\mu}_1\chi_y^2}{2} + \frac{\nu^2\dot{\mu}_1\chi_z^2}{2} + \frac{\dot{\mu}_1\xi^2}{2\mu_1^2} \\ 0 \\ \dot{\mu}_1\chi_y^2 - \mu_1\chi_y\beta_0\eta - \beta_0\eta\chi_y' - \dot{\mu}_1\chi_y\beta_0\eta + \frac{\ddot{\mu}_1\chi_y^2}{2} + \frac{\nu^2\dot{\mu}_1\chi_z^2}{2} + \frac{\dot{\mu}_1\xi^2}{2\mu_1^2} \\ \quad - \nu^2\mu_1\chi_z\eta_z + \nu^2\chi_z'\eta_z - \rho\xi' - \rho\xi \\ \rho\chi_y' - \rho\chi_y - \eta\beta_0\xi' - \frac{\beta_0\eta\xi}{\mu_1} - \frac{\xi\dot{\mu}_1\chi_y}{\mu_1^2} + (\mu_1 - 1)\int_0^{\frac{1}{\beta_0}} \left(\xi'\chi_y'(1-\beta_0 y)\right)\mathrm{d}y \\ \quad + (\mu_1 - 1)\int_{-\frac{1}{\beta_0}}^0 \left(\xi\chi_y(1+\beta_0 y)\right)\mathrm{d}y \end{pmatrix}$$

and

$$H^1_2(u) = \frac{\eta^2}{2},$$

$$\begin{aligned} H^0_3(u) = &\int_{-\frac{1}{\beta_0}}^0 \Bigg(-\frac{\eta\xi^2\beta_0}{2\mu_1} - \frac{\xi^2\dot{\mu}_1\chi_y}{2\mu_1^2} + \frac{\beta_0\eta\chi_y^2\dot{\mu}_1}{2} - \frac{\nu^2\beta_0\eta\mu_1\chi_z^2}{2} \\ &\qquad + \frac{\mu_1\beta_0\eta\chi_y^2}{2} - \frac{\ddot{\mu}_1\chi_y^3}{6} - \frac{\nu^2\dot{\mu}_1\chi_y\chi_z^2}{2} - \frac{\dot{\mu}_1\chi_y^3}{3} + \nu^2(1+\beta_0 y)\mu_1\chi_z\eta_z\chi_y\Bigg)\mathrm{d}y \\ &+ \int_0^{\frac{1}{\beta_0}} \left(-\frac{\beta_0\eta(\chi_y')^2}{2} - \nu^2\eta_z\chi_z'\left(\beta_0 y - 1\right)\chi_y' + \frac{\beta_0\eta\left((\xi')^2 + \nu^2\left(\chi_z'\right)^2\right)}{2}\right)\mathrm{d}y \\ &+ \rho\left(\int_0^{\frac{1}{\beta_0}} \left(\beta_0 y - 1\right)\xi'\chi_y'\mathrm{d}y - \left(\int_{-\frac{1}{\beta_0}}^0 \left(\beta_0 y + 1\right)\xi\chi_y\mathrm{d}y\right)\right), \end{aligned}$$

$$\begin{aligned} H^0_4(u) = &\int_{-\frac{1}{\beta_0}}^0 \Bigg(\frac{\beta_0^2\eta^2\xi^2}{2\mu_1} - \frac{\xi^2\ddot{\mu}_1\chi_y^2}{4\mu_1^2} - \frac{\nu^2\xi^2\dot{\mu}_1\chi_z^2}{4\mu_1^2} + \frac{\xi^2\dot{\mu}_1^2\chi_y^2}{2\mu_1^3} - \frac{\beta_0^2\eta^2\chi_y^2\dot{\mu}_1}{2} + \frac{\beta_0\eta\chi_y^3\ddot{\mu}_1}{3} \\ &\qquad + \frac{2\beta_0\eta\dot{\mu}_1\chi_y^3}{3} - \frac{\beta_0^2\eta^2\mu_1\chi_y^2}{2} - \frac{\nu^2\chi_y^2\chi_z^2\ddot{\mu}_1}{4} - \frac{\xi^4\dot{\mu}_1}{8\mu_1^4} - \frac{\chi_y^4\dddot{\mu}_1}{24} - \frac{\nu^4\chi_z^4\dot{\mu}_1}{8} \\ &\qquad - \frac{\chi_y^4\ddot{\mu}_1}{8} - \frac{\nu^2\mu_1\eta_z^2\chi_y^2}{2} + \nu^2\chi_y^2\eta_z\chi_z\dot{\mu}_1\beta_0 y - \frac{\nu^2\mu_1\eta_z^2\chi_y^2\beta_0^2 y^2}{2} \\ &\qquad - \nu^2\mu_1\eta_z^2\chi_y^2\beta_0 y + \nu^2\chi_y^2\eta_z\chi_z\dot{\mu}_1 + \frac{\xi^2\chi_y\eta\dot{\mu}_1\beta_0}{\mu_1^2}\Bigg)\mathrm{d}y \\ &+ \int_0^{\frac{1}{\beta_0}} \left(\frac{\left(-\left(\beta_0 y - 1\right)^2\nu^2\eta_z^2 - \beta_0^2\eta^2\right)(\chi_y')^2}{2} + \frac{\beta_0^2\eta^2(\xi')^2}{2}\right)\mathrm{d}y \\ &- \frac{1}{2}\left(\int_{-\frac{1}{\beta_0}}^0 \left(\beta_0 y + 1\right)\xi\chi_y\mathrm{d}y\right)^2 - \frac{1}{2}\left(\int_0^{\frac{1}{\beta_0}} \left(\beta_0 y - 1\right)\xi'\chi_y'\mathrm{d}y\right)^2 \\ &+ \left(\rho\beta_0\eta + \left(\int_0^{\frac{1}{\beta_0}} \left(\beta_0 y - 1\right)\xi'\chi_y'\mathrm{d}y\right)\right)\left(\int_{-\frac{1}{\beta_0}}^0 \left(\beta_0 y + 1\right)\xi\chi_y\mathrm{d}y\right) \\ &+ \rho\beta_0\eta\left(\int_0^{\frac{1}{\beta_0}} \left(\beta_0 y - 1\right)\xi'\chi_y'\mathrm{d}y\right) - \frac{1}{8}\left(\nu^2\eta_z^2 + \rho^2\right)^2 \end{aligned}$$

which are used in the above calculations, where we use the associated symmetric multilinear operators. Here $u = (\eta, \chi', \chi, \rho, \xi', \xi)$ and we have abbreviated $\mu(1), \dot{\mu}(1), \ddot{\mu}(1)$ to respectively $\mu_1, \dot{\mu}_1, \ddot{\mu}_1$.

3.8 Results

Attempting to compute explicit general expressions for the coefficients $c_1^1, c_3^1, c_1, c_2, c_3, c_4$ leads to unwieldy formulae, and it appears more appropriate to calculate them for a specific choice of μ, that is a specific magnetisation law. For a constant relative permeability μ (corresponding to a linear magnetisation law) we find that

$$
\begin{aligned}
c_3^1 &= \left(1 + \frac{\mu(\mu-1)^2}{\beta_0(\mu+1)}(q\tanh q - 1)\cosh^{-2} q\right)^{-1}, \\
c_1^1 &= \left(\sin(\alpha)^2\left(1 + \frac{\mu(\mu-1)^2}{\beta_0(\mu+1)}(q\tanh q - 1)\cosh^{-2} q\right)\right)^{-1},
\end{aligned}
$$

where $\alpha = \cos^{-1}\left(\frac{\nu}{\omega}\right) \in (0, \frac{\pi}{3})$, and

$$
\begin{aligned}
\frac{2c_4}{(c_3^1)^2} &= -\frac{q^4\beta_0^4}{4}\frac{\mu^2(\mu-1)^6}{(\mu+1)^4}\frac{(-3-4\cosh(2q)+\cosh(4q))(-2+\mathrm{sech}(q)^2+4\mathrm{sech}(2q))}{\left(2q\beta_0\frac{(\mu-1)^2}{\mu+1}\mu\tanh(2q)-\gamma_0-4q^2\beta_0^2\right)\cosh(q)^2\cosh(2q)} \\
&\quad -\frac{q^3\beta_0^3}{4}\frac{\sinh(q)^{-1}}{(\mu+1)^3}\big(-16\mu(\mu^2-1)^2\cosh(q)^2+16\mu(\mu-1)^4\mathrm{sech}(2q) \\
&\qquad +6q\beta_0(\mu+1)^3\sinh(2q)\big) \\
&\quad -\frac{q^4\beta_0^4}{\gamma_0}\frac{\mu^2(\mu-1)^6}{(\mu+1)^4}\mathrm{sech}(q)^4, \\
\frac{2c_1}{(c_1^1)^2} &= -\frac{q^4\beta_0^4}{4}\frac{\mu^2(\mu-1)^6}{(\mu+1)^4}\frac{(-3-4\cosh(2q)+\cosh(4q))(-2+\mathrm{sech}(q)^2+4\mathrm{sech}(2q))}{\left(2q\beta_0\frac{(\mu-1)^2}{\mu+1}\mu\tanh(2q)-\gamma_0-4q^2\beta_0^2\right)\cosh(q)^2\cosh(2q)} \\
&\quad -\frac{q^3\beta_0^3}{4}\frac{\sinh(q)^{-1}}{(\mu+1)^3}\big(-16\mu(\mu^2-1)^2\cosh(q)^2+16\mu(\mu-1)^4\mathrm{sech}(2q) \\
&\qquad +6q\beta_0(\mu+1)^3\sinh(2q)\big) \\
&\quad -\frac{q^4\beta_0^4}{\gamma_0}\frac{\mu^2(\mu-1)^6}{(\mu+1)^4}\mathrm{sech}(q)^4,
\end{aligned}
$$

where $q = \frac{\omega}{\beta_0}$. The coefficients c_3^1, c_1^1 are evidently both positive, while it is necessary to determine the signs of the (equal) coefficients c_4, c_1 numerically. Figure 3.6 shows the region of (μ, q)-parameter space in which $c_4, c_1 < 0$.

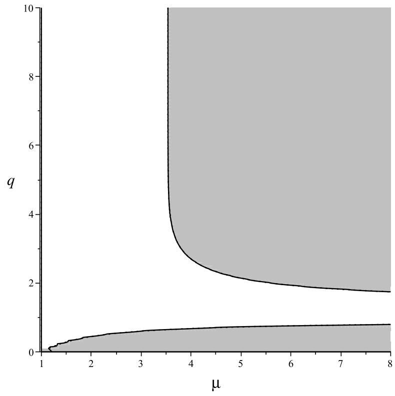


Figure 3.6:
The shaded areas show the regions of (μ, q)-parameter space in which rotated rolls exist.

The remaining coefficients are given by

$$
\begin{aligned}
\frac{c_2}{(c_1^1)^2} = &-\frac{q^3\beta_0^3}{4}\frac{\sinh(q)^{-1}}{(\mu+1)^3}\big(-16\mu(\mu^2-1)^2\cosh(q)^2+16\mu(\mu-1)^4\mathrm{sech}(2q)+6q\beta_0(\mu+1)^3\sinh(2q)\big)\\
&-\frac{q^4\beta_0^4}{\gamma_0}\frac{\mu^2(\mu-1)^6}{(\mu+1)^4}\mathrm{sech}(q)^4\\
&+\Bigg(-6\cosh(q)^4\cosh(2q)\,q^4\beta_0^2\sinh^2(q)\,(\mu+1)^4\sin^2(\alpha)\cos^2(\alpha)\\
&\quad+\Big(\Big(18\cosh^2(q)\,q+\sinh(q)\left(-6q^2-15\right)\cosh(q)+2q^3-15q\Big)\\
&\qquad\times\frac{\beta_0}{4}\sinh^2(q)\,q\cosh(2q)\cosh^2(q)\,\mu\,(\mu-1)^2\,(\mu+1)^3\\
&\qquad+\Big(4q^2\cosh^4(q)-4\sinh(q)\cosh^3(q)\,q-4\cosh^2(q)\,q^2+\cosh^4(q)+2q\sinh(q)\cosh(q)+q^2-\cosh^2(q)\Big)\\
&\qquad\times\frac{3}{16}\sinh^2(q)\cosh(2q)\,\mu^2\,(\mu-1)^6\Big)\cos^2(\alpha)\\
&\quad+\Big(\Big(\left(3\mu^3+\mu^2+\mu+3\right)\left(5\cosh^5(q)-2\cosh^7(q)\right)\\
&\qquad-\left(11\mu^3-3\mu^2-3\mu+11\right)\cosh^3(q)+2\,(\mu+1)\,(\mu-1)^2\cosh(q)\Big)3\sinh(q)\,q^3\mu\,(\mu-1)^2\\
&\qquad-\Big(4\,(\mu+1)^3\sinh(q)\cosh^6(q)-12\,(\mu+1)^3\sinh(q)\cosh^4(q)+5\,(\mu+1)^3\sinh(q)\cosh^2(q)\Big)\\
&\qquad\times\frac{3}{2}\sinh(q)\,q^2\mu\,(\mu-1)^2\\
&\qquad-\frac{3}{2}\sinh^3(q)\cosh^3(q)\left(2\cosh^2(q)-1\right)q\mu\,(\mu-1)^2\,(\mu+1)^3\Big)\beta_0\\
&\quad-\Big(4q^2\cosh^4(q)-4\sinh(q)\cosh^3(q)\,q-4\cosh^2(q)\,q^2+\cosh^4(q)+2q\sinh(q)\cosh(q)+q^2-\cosh^2(q)\Big)\\
&\qquad\times\sinh^2(q)\,(\mu-1)^6\,\mu^2\left(2\cosh^2(q)-1\right)\frac{3}{16}\Bigg)\\
&\quad\times\frac{2\beta_0^2\sinh^2(q)}{3\left(2\cosh^2(q)-1\right)(\mu-1)^4}\\
&+2\left(2k_1^{(1)}h_1^{(1)}+k_1^{(1)}h_r^{(1)}+k_r^{(1)}h_1^{(1)}+k_r^{(1)}h_r^{(1)}+2k_1^{(2)}h_1^{(2)}+k_1^{(2)}h_r^{(2)}+k_r^{(2)}h_1^{(2)}+k_r^{(2)}h_r^{(2)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
h_1^{(1)} = &\Big(\Big(q\beta_0\sin^2(\alpha)\cosh(2q)\,\mu\,(\mu-1)^2+4q\beta_0\sin^2(\alpha)\cosh^2(q)\,\mu\,(\mu-1)\,(\mu+1)\\
&\qquad+2\sinh(q)\cosh(q)\left(4q^2\beta_0^2\sin^2(\alpha)+\gamma_0\right)(\mu+1)-q\beta_0\cos^2(\alpha)\,\mu\,(\mu-1)^2\Big)\\
&\qquad\times\tanh(2\sin(\alpha)\,q)\,(\mu-1)\\
&\quad-4\left(4q^2\beta_0^2\sin^2(\alpha)+\gamma_0\right)\sin(\alpha)\cosh^2(q)\,\mu\,(\mu+1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\beta_0 q\, t(2\sin\alpha)\,(\mu-1)}{\cosh(q)^2\,(\mu+1)^2},\\
h_r^{(1)} &= \left(\frac{\beta_0}{4} q\cosh^2(q)\sin^2(\alpha)\,\mu\,(\mu-1)^2 - \sinh(q)\cosh(q)\left(q^2\beta_0^2\sin^2(\alpha)+\frac{\gamma_0}{4}\right)(\mu+1)+\frac{\beta_0}{8}q\mu\,(\mu-1)^2\right)\\
&\times \frac{8\beta_0 q\tanh(2\sin(\alpha)\,q)\,t(2\sin\alpha)\,(\mu-1)^2}{\cosh(q)^2\,(\mu+1)},\\
k_1^{(1)} &= \Bigg(\Big(\sin^2(\alpha)\,(\mu-1)\cosh(2q) - \big(4\,(\mu+1)\cosh^2(q)+\mu-1\big)\cos^2(\alpha)+4\,(\mu+1)\cosh^2(q)\Big)\\
&\qquad\times\cosh(2\sin(\alpha)\,q)\\
&-4\sin(\alpha)\sinh(q)\cosh(q)\sinh(2\sin(\alpha)\,q)\,(\mu+1)\Bigg)\\
&\times\frac{q^2\beta_0^2\mu\,(\mu-1)}{2\cosh(q)^2\,(\mu+1)^2}\\
k_r^{(1)} &= \Big(\cosh(2q)\cos^2(\alpha)+\cos^2(\alpha)-\cosh(2q)\Big)\frac{q^2\beta_0^2\cosh(2\sin(\alpha)\,q)\,\mu\,(\mu-1)^2}{\cosh(q)^2\,(\mu+1)^2},\\
h_1^{(2)} &= \Bigg(2\cos(\alpha)\cosh^2(q)\left(4q^2\beta_0^2\cos^2(\alpha)+\gamma_0\right)\mu\,(\mu+1)\cosh^2(q\cos(\alpha))\\
&-\left((3\mu+1)\cos^2(\alpha)\,(\mu-1)\,q\beta_0\mu\cosh^2(q)+\left(4q^2\beta_0^2\cos^2(\alpha)+\gamma_0\right)(\mu+1)\sinh(q)\cosh(q)-\frac{\beta_0}{2}q\mu\,(\mu-1)^2\right)\\
&\qquad\times\sinh(q\cos(\alpha))\cosh(q\cos(\alpha))\,(\mu-1)\\
&-\cos(\alpha)\cosh^2(q)\left(4q^2\beta_0^2\cos^2(\alpha)+\gamma_0\right)\mu\,(\mu+1)\Bigg)\\
&\times\frac{4q\beta_0 t(2\cos\alpha)\,(\mu-1)}{\cosh(2q\cos(\alpha))\cosh(q)^2\,(\mu+1)^2},\\
h_r^{(2)} &= \Big(2\left(4q^2\beta_0^2\cos^2(\alpha)+\gamma_0\right)(\mu+1)\sinh(q)\cosh(q)-2\cos^2(\alpha)\,\beta_0\mu q\,(\mu-1)^2\cosh^2(q)-\beta_0\mu q\,(\mu-1)^2\Big)\\
&\times\frac{q\beta_0\tanh(2q\cos(\alpha))\,t(2\cos\alpha)\,(\mu-1)^2}{\cosh(q)^2\,(\mu+1)},\\
k_1^{(2)} &= \Bigg(\left(-4\,(3\mu+1)\cos^2(\alpha)\cosh^2(q)+2\mu-2\right)\cosh^2(q\cos(\alpha))+2\,(3\mu+1)\cos^2(\alpha)\cosh^2(q)\\
&+8\cos(\alpha)\sinh(q)\cosh(q)\sinh(q\cos(\alpha))\,(\mu+1)\cosh(q\cos(\alpha))-(\mu-1)\Bigg)\\
&\times\frac{\mu\, q^2\beta_0^2\,(\mu-1)}{2\cosh(q)^2\,(\mu+1)^2},\\
k_r^{(2)} &= \Big(2\,(\mu-1)\left(2\cosh^2(q\cos(\alpha))-1\right)\left(2\cos^2(\alpha)\cosh^2(q)-1\right)\Big)\frac{q^2\beta_0^2\mu\,(\mu-1)}{2\cosh(q)^2\,(\mu+1)^2},
\end{aligned}
$$

and

$$
t(s)^{-1} = \left(\mu\frac{(\mu-1)^2}{\mu+1}sq\beta_0\tanh sq-(s^2q^2\beta_0^2+\gamma_0)\right)(\mu+1)\sinh sq,
$$

and

$$
\begin{aligned}
\frac{c_3}{c_3^1c_1^1} &= -\frac{q^3\beta_0^3}{4}\frac{\sinh(q)^{-1}}{(\mu+1)^3}\big(-16\mu(\mu^2-1)^2\cosh(q)^2+16\mu(\mu-1)^4\operatorname{sech}(2q)+6q\beta_0(\mu+1)^3\sinh(2q)\big)\\
&-\frac{q^4\beta_0^4}{\gamma_0}\frac{\mu^2(\mu-1)^6}{(\mu+1)^4}\operatorname{sech}(q)^4\\
&-\Bigg(\left(\frac{1}{4}(\mu-1)^4\left(\cos^2(\alpha)-2\right)\left(q^2+\frac{1}{4}\right)\mu\sinh(2q)-\left(\left(q^2+\frac{5}{2}\right)\cos^2(\alpha)+4q^2\right)\beta_0 q\cosh^2(q)\,(\mu+1)^3\right)\\
&\qquad\times 12\cosh^3(2q)\,\mu\,(\mu-1)^2\\
&\quad-3\mu^2q\left(\cos^2(\alpha)-2\right)(\mu-1)^6\cosh^4(2q)\\
&\quad-\Bigg(\left(-\frac{3}{8}\cos^2(\alpha)\,(\mu+1)^2\cosh(q)+\sinh(q)\,q\,(\mu-1)^2\right)q\beta_0\cosh(q)\,(\mu+1)\sinh(2q)\\
&\qquad-\frac{\mu}{32}\left(\cos^2(\alpha)-2\right)(\mu-1)^4\Bigg)\\
&\qquad\times 96\,(\mu-1)^2\,q\beta_0\mu\cosh^2(2q)\\
&\quad+\Bigg(\Big(-192\beta_0\,q^2\left(\beta_0\,q^2\,(\mu+1)\cos^2(\alpha)+2\,(\mu-1)^2\,\mu\right)(\mu+1)^3\cosh^4(q)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad -384\sinh(q)\,(\mu-1)^2\left(q^2+\frac{1}{2}\right)\beta_0 q\mu\,(\mu+1)^3\cosh^3(q)\\
&\qquad\qquad +32\,(\mu-1)^2\left(30+\left(q^2-3\right)\cos^2(\alpha)\right)\beta_0\, q^2\mu\,(\mu+1)^3\cosh^2(q)\\
&\qquad\qquad +384\sinh(q)\,\beta_0\mu\, q^3\,(\mu+1)\,(\mu-1)^4\cosh(q)-3\mu^2\left(\cos^2(\alpha)-2\right)(\mu-1)^6\Big)\\
&\qquad\qquad \times\sinh(2q)\\
&\qquad +\left(\cosh^4(q)\,q^2+\left(\frac{q^2}{16}+\frac{5}{32}\right)\cos^2(\alpha)+\frac{q^2}{4}\right)(\mu-1)^2\beta_0 q\cosh^2(q)\,\mu\,(\mu+1)^3\Big)\\
&\qquad\qquad \times\frac{768}{4}\cosh(2q)\\
&\qquad -192\cosh^4(q)\,\beta_0\mu\, q^3\,(\mu+1)\,(\mu-1)^4\Big)\\
&\qquad \times\frac{\beta_0^2}{96\sinh(q)\cosh(2q)\cosh(q)^5\,(\mu+1)^4}\\
&+2\left(2k_1^{(3)}h_1^{(3)}+k_1^{(3)}h_r^{(3)}+k_r^{(3)}h_1^{(3)}+k_r^{(3)}h_r^{(3)}+2k_1^{(4)}h_1^{(4)}+k_1^{(4)}h_r^{(4)}+k_r^{(4)}h_1^{(4)}+k_r^{(4)}h_r^{(4)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
h_1^{(3)} &= \Bigg(\Bigg(\frac{3}{4}\left(\mu+\frac{1}{3}\right)(\mu-1)\,\mu\left((\sin(\alpha)+2)\cos^2(\alpha)-\frac{9}{4}-\frac{9\sin(\alpha)}{4}\right)q\beta_0\cosh^2(q)\\
&\quad +\sinh(q)\left(\left(\sin(\alpha)\,q^2\beta_0^2+2q^2\beta_0^2+\frac{\gamma_0}{2}\right)\cos^2(\alpha)+\left(-\frac{9q^2\beta_0^2}{4}-\frac{\gamma_0}{2}\right)\sin(\alpha)-\frac{9q^2\beta_0^2}{4}-\frac{5\gamma_0}{8}\right)(\mu+1)\cosh(q)\\
&\quad -\left(\cos^2(\alpha)-\sin(\alpha)-\frac{5}{4}\right)\frac{\beta_0}{4}q\mu\,(\mu-1)^2\Bigg)\\
&\quad \times\sqrt{2+2\sin(\alpha)}\,\tanh\!\left(\sqrt{2+2\sin(\alpha)}\,q\right)(\mu-1)\\
&\quad +\left(\cos^4(\alpha)\,\beta_0^2q^2+\left(\left(-3q^2\beta_0^2-\frac{\gamma_0}{2}\right)\sin(\alpha)-\frac{21q^2\beta_0^2}{4}-\gamma_0\right)\cos^2(\alpha)+\frac{9}{2}\,(1+\sin(\alpha))\left(q^2\beta_0^2+\frac{\gamma_0}{4}\right)\right)\\
&\quad \times 2\cosh^2(q)\,\mu\,(\mu+1)\Bigg)\\
&\quad \times\frac{8q\beta_0\sinh^4(q)\cosh^2(q)\,(\mu+1)^2\,t(\sqrt{2(1+\sin(\alpha))})}{(\mu-1)^3\,(2\sin(\alpha)+1)^2}\\
h_r^{(3)} &= \Bigg(\left(\cos^4(\alpha)+\left(-3\sin(\alpha)-\frac{21}{4}\right)\cos^2(\alpha)+\frac{9}{2}\sin(\alpha)+\frac{9}{2}\right)\frac{\beta_0}{4}q\cosh^2(q)\,\mu\,(\mu-1)^2\\
&\quad -\left(\cos^4(\alpha)\,\beta_0^2q^2+\left(\left(-3q^2\beta_0^2-\frac{\gamma_0}{2}\right)\sin(\alpha)-\frac{21}{4}q^2\beta_0^2-\gamma_0\right)\cos^2(\alpha)+\frac{9}{2}\,(1+\sin(\alpha))\left(q^2\beta_0^2+\frac{\gamma_0}{4}\right)\right)\\
&\quad \times\sinh(q)\,(\mu+1)\cosh(q)\\
&\quad -\left((\sin(\alpha)+2)\cos^2(\alpha)-\frac{9}{4}-\frac{9\sin(\alpha)}{4}\right)\frac{\beta_0}{4}\mu q\,(\mu-1)^2\Bigg)\\
&\quad \times\frac{16q\beta_0\sinh^4(q)\cosh^2(q)\,(\mu+1)^3\tanh\!\left(\sqrt{2+2\sin(\alpha)}\,q\right)t(\sqrt{2(1+\sin(\alpha))})}{(\mu-1)^2\,(2\sin(\alpha)+1)^2\sqrt{2+2\sin(\alpha)}},\\
k_1^{(3)} &= \Bigg(\Big((1+\sin(\alpha))\,(\mu-1)\cosh(2q)+\left((4\mu+4)\cosh^2(q)+\mu-1\right)\sin(\alpha)+(4\mu+4)\cosh^2(q)-\mu+1\Big)\\
&\quad \times\cosh\!\left(\sqrt{2+2\sin(\alpha)}\,q\right)\\
&\quad -4\sinh\!\left(\sqrt{2+2\sin(\alpha)}\,q\right)\sqrt{2+2\sin(\alpha)}\,\sinh(q)\cosh(q)\,(\mu+1)\Bigg)\\
&\quad \times\frac{\beta_0^2q^2\cosh^2(q)\sinh^4(q)\,\mu\,(\mu+1)^2}{4\,(\mu-1)^3},\\
k_r^{(3)} &= -2\cosh\!\left(\sqrt{2+2\sin(\alpha)}\,q\right)\left(\sin(\alpha)\cosh(2q)+\sin(\alpha)+\cosh(2q)-1\right)\frac{\beta_0^2q^2\cosh^2(q)\sinh^4(q)\,\mu\,(\mu+1)^2}{4\,(\mu-1)^2},\\
h_1^{(4)} &= \Bigg(\Bigg((\mu-1)^2\mu\left(\left(\cos^2(\alpha)-\frac{9}{4}\right)\sin(\alpha)-2\cos^2(\alpha)+\frac{9}{4}\right)q\beta_0\cosh(2q)\,\frac{1}{8}\\
&\quad +(\mu-1)\,\mu\,(\mu+1)\left(\left(\cos^2(\alpha)-\frac{9}{4}\right)\sin(\alpha)-2\cos^2(\alpha)+\frac{9}{4}\right)q\beta_0\cosh^2(q)\,\frac{1}{2}\\
&\quad +\sinh(q)\left(\left(\cos^2(\alpha)\,q^2\beta_0^2-\frac{9q^2\beta_0^2}{4}-\frac{\gamma_0}{2}\right)\sin(\alpha)+\left(-2q^2\beta_0^2-\frac{\gamma_0}{2}\right)\cos^2(\alpha)+\frac{9q^2\beta_0^2}{4}+\frac{5\gamma_0}{8}\right)(\mu+1)\cosh(q)
\end{aligned}
$$

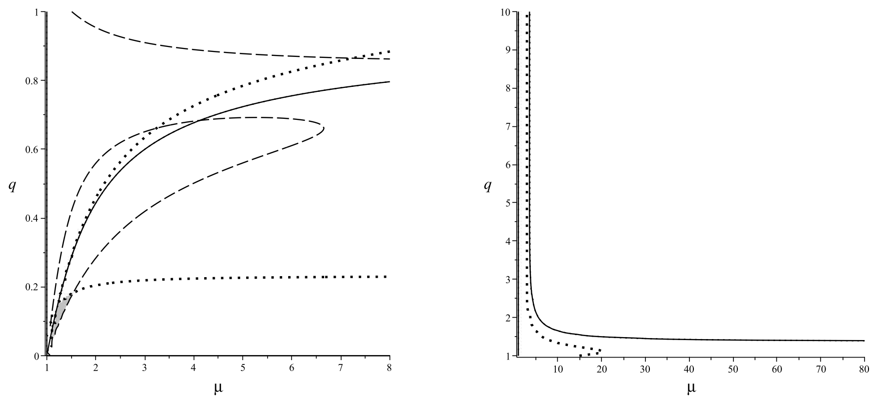
$$
\begin{aligned}
&\qquad + \left(-\frac{1}{4} + \left(\cos^2(\alpha) - \frac{1}{4}\right)\sin(\alpha)\right)(\mu-1)^2\,\mu q\beta_0\,\frac{1}{8}\right) \\
&\qquad \times \sqrt{2-2\sin(\alpha)}\,(\mu-1)\tanh\!\left(\sqrt{2-2\sin(\alpha)}\,q\right) \\
&\quad - \left(\left((3q^2\beta_0^2 + \frac{\gamma_0}{2})\cos^2(\alpha) - \frac{9}{2}q^2\beta_0^2 - \frac{9}{8}\gamma_0\right)\sin(\alpha) + \cos^4(\alpha)\,\beta_0^2 q^2 + \left(-\frac{21}{4}q^2\beta_0^2 - \gamma_0\right)\cos^2(\alpha) + \frac{9}{2}q^2\beta_0^2 + \frac{9}{8}\gamma_0\right) \\
&\qquad \times 2\cosh^2(q)\,\mu\,(\mu+1)\Bigg) \\
&\quad \times \frac{16q\beta_0\sinh^4(q)\cosh^2(q)\,(\mu+1)^2\,t(\sqrt{2(1-\sin(\alpha))})}{\sqrt{2-2\sin(\alpha)}\,(\mu-1)^3\,(4\cos^2(\alpha)+4\sin(\alpha)-5)},
\end{aligned}
$$

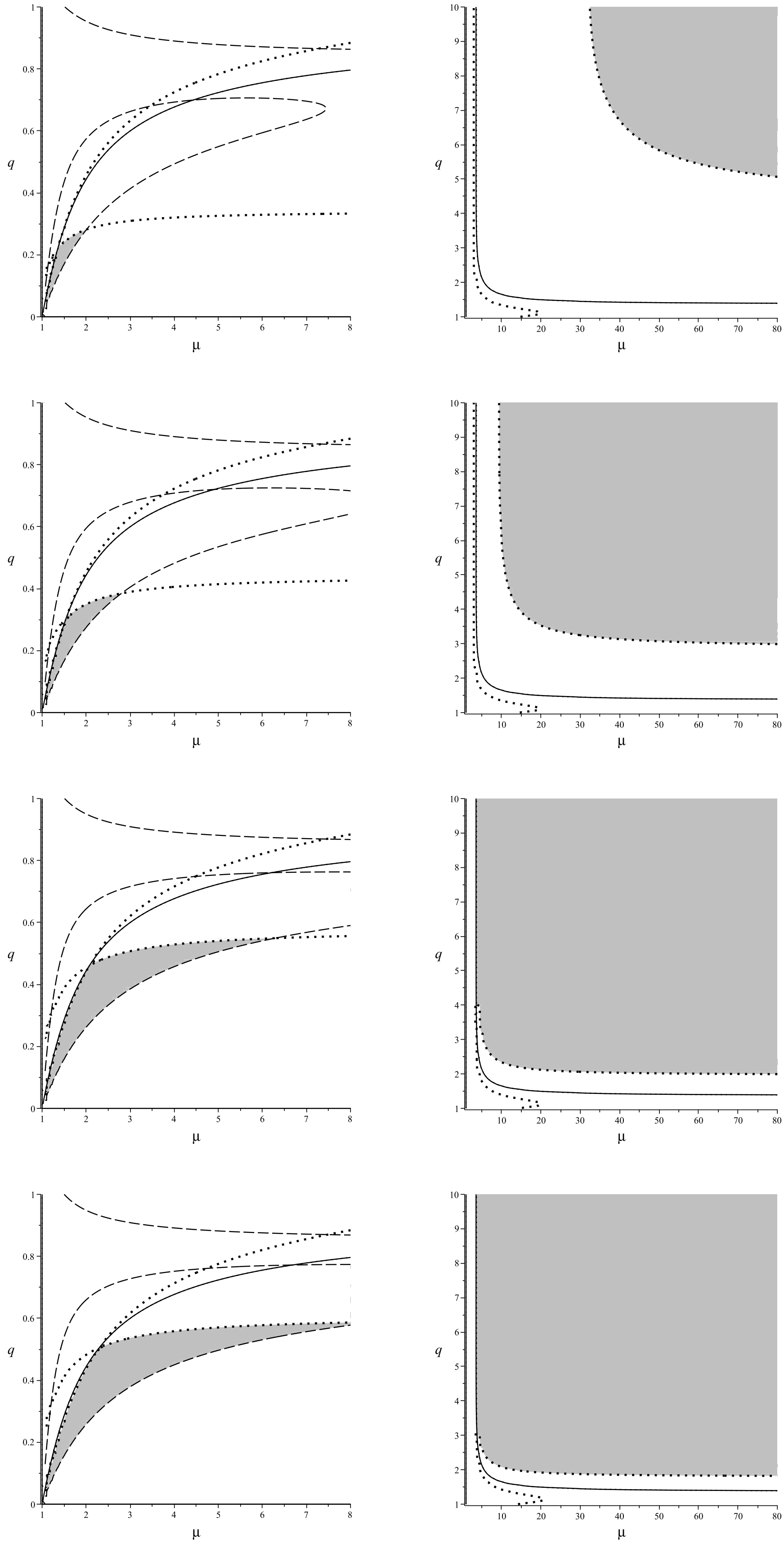
$$
\begin{aligned}
h_r^{(4)} = &\Bigg(-16\,(\mu-1)^2\,\mu\left(\left(\cos^2(\alpha) - \frac{9}{4}\right)\sin(\alpha) - 2\cos^2(\alpha) + \frac{9}{4}\right) q\beta_0\cosh(2q) \\
&+ 64\,(\mu-1)^2\,\mu q\beta_0\left(\left(\cos^2(\alpha) - \frac{9}{4}\right)\sin(\alpha) - 2\cos^2(\alpha) + \frac{9}{4}\right)\cosh^2(q) \\
&- 128\sinh(q)\left(\left(\cos^2(\alpha)\,q^2\beta_0^2 - \frac{9q^2\beta_0^2}{4} - \frac{\gamma_0}{2}\right)\sin(\alpha) + \left(-2q^2\beta_0^2 - \frac{\gamma_0}{2}\right)\cos^2(\alpha) + \frac{9q^2\beta_0^2}{4} + \frac{5\gamma_0}{8}\right)(\mu+1)\cosh(q) \\
&- 16\left(-\frac{1}{4} + \left(\cos^2(\alpha) - \frac{1}{4}\right)\sin(\alpha)\right)(\mu-1)^2\,\mu q\beta_0\Bigg) \\
&\times \frac{q\beta_0\sinh^4(q)\cosh^2(q)\tanh\!\left(\sqrt{2-2\sin(\alpha)}\,q\right) t(\sqrt{2(1-\sin(\alpha))})\,(\mu+1)^3}{8\,(\mu-1)^2\,(4\cos^2(\alpha)+4\sin(\alpha)-5)},
\end{aligned}
$$

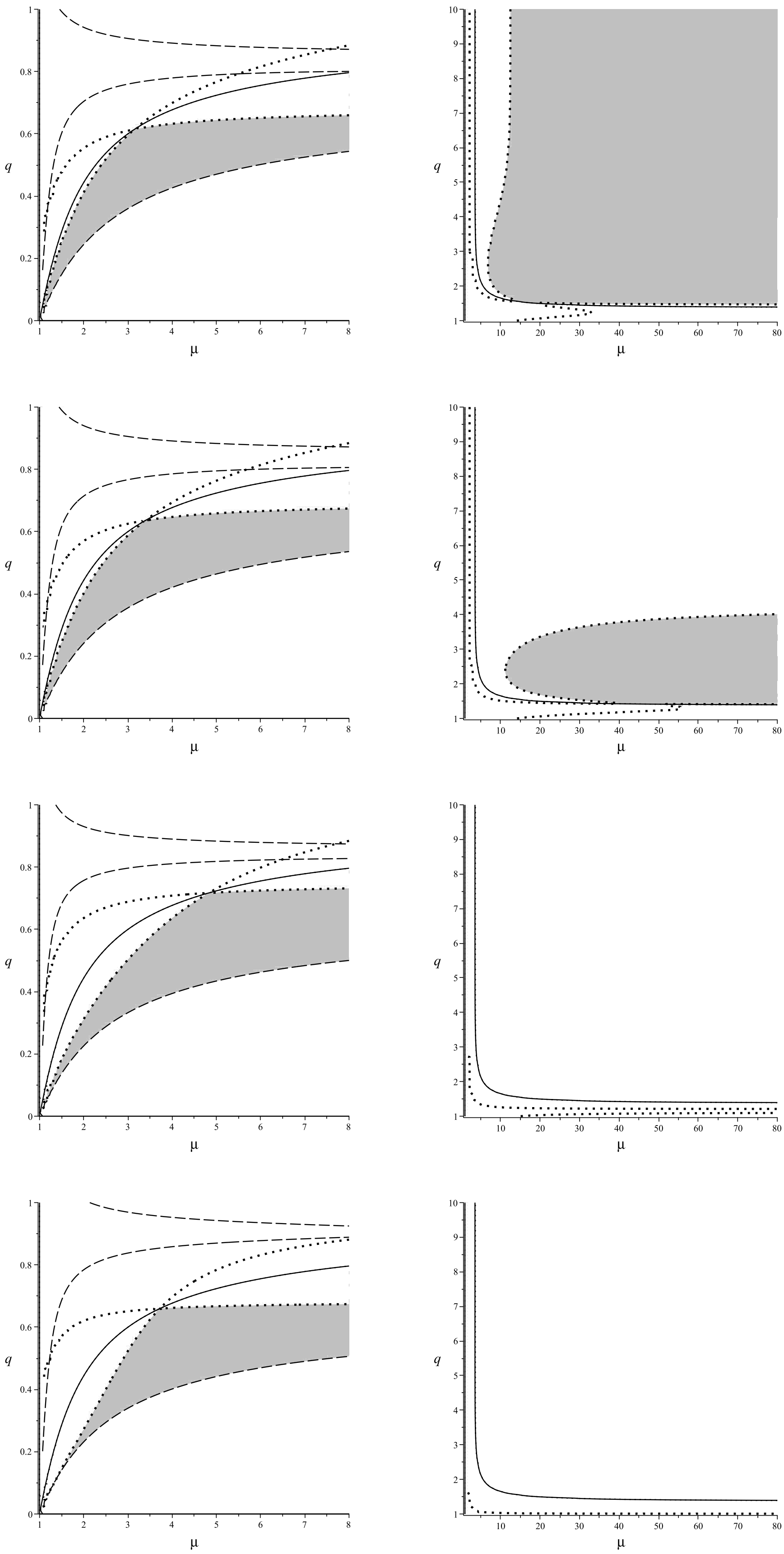
$$
\begin{aligned}
k_1^{(4)} = &\Bigg(\Big((-1+\sin(\alpha))\,(\mu-1)\cosh(2q) + \left((4\mu+4)\cosh^2(q) + \mu - 1\right)\sin(\alpha) + (-4\mu-4)\cosh^2(q) + \mu - 1\Big) \\
&\quad \times \cosh\!\left(\sqrt{2-2\sin(\alpha)}\,q\right) \\
&+ 4\sinh\!\left(\sqrt{2-2\sin(\alpha)}\,q\right)\sinh(q)\cosh(q)\,(\mu+1)\sqrt{2-2\sin(\alpha)}\Bigg) \\
&\times \frac{\beta_0^2 q^2\cosh^2(q)\sinh^4(q)\,\mu\,(\mu+1)^2}{4\,(\mu-1)^3},
\end{aligned}
$$

$$
k_r^{(4)} = -2\cosh\!\left(\sqrt{2-2\sin(\alpha)}\,q\right)\left((-1+\sin(\alpha))\cosh(2q) + \sin(\alpha) + 1\right)\frac{\beta_0^2 q^2\cosh^2(q)\sinh^4(q)\,\mu\,(\mu+1)^2}{4\,(\mu-1)^2}.
$$

The regions of (μ, q)-parameter space in which additionally $d_1 \in (1, 4+\sqrt{13})$ and $d_2 > -d_3$ are shown for various values of α in Figure 3.7.




q
μ


q
μ

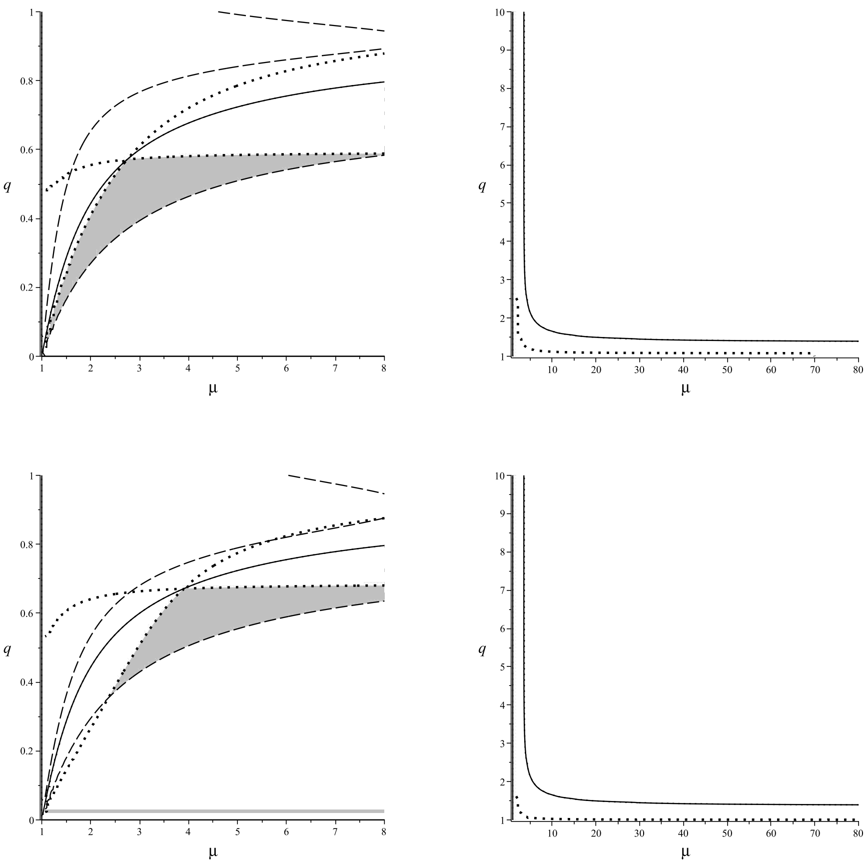


Figure 3.7:
The shaded areas of (μ, q)-parameter space in which symmetric corner defects exist.

The solid, dotted and dashed lines delineate the regions in which $c_1 < 0, d_1 \in (1, 4+\sqrt{13})$ and $d_2 > -d_3$; the 'lower' and 'upper' shaded area in Figure 3.6 are shown on the left and right.

The values of α (from top to bottom) are $\sin^{-1}(0.1), \sin^{-1}(0.15), \sin^{-1}(0.2), \sin^{-1}(0.28), \sin^{-1}(0.3), \sin^{-1}(0.35), \sin^{-1}(0.36), \sin^{-1}(0.4), \sin^{-1}(0.6), \sin^{-1}(0.7), \sin^{-1}(0.8)$.

It is also possible to compute the coefficients for small values of β_0 (the limit $\beta_0 \to 0$ corresponds to fluids of infinite depth). Recall that $\omega, \gamma_0, \beta_0$ satisfy the equation

$$\gamma_0 = f_{\beta_0}(\omega),$$

where

$$f_{\beta_0}(s) = g_{\beta_0}(s)s^2 \qquad g_{\beta_0}(s) = \frac{\mu_1(\mu_1-1)^2}{\mu_1 s \coth\dfrac{s}{\beta_0} + S_1 s \coth\dfrac{S_1 s}{\beta_0}} - 1$$

and ω is the unique maximum of the mapping f_{β_0}, and note that f_{β_0} converges pointwise to

$$f_0(s) = \frac{\mu_1(\mu_1-1)^2}{\mu_1 + S_1} s - s^2$$

as $\beta_0 \to 0$. Because

$$\gamma_0 = f_0(\omega), \qquad 0 = f_0'(\omega),$$

where the prime denotes differentiation with respect to s, it follows that

$$\omega = \sigma + o(1), \qquad \gamma_0 = \sigma^2 + o(1)$$

as $\beta_0 \to 0$, where

$$\sigma = \frac{\mu_1(\mu_1 - 1)^2}{2(\mu_1 + S_1)}, \qquad \dot{\mu}_1 = \dot{\mu}(1), \qquad S_1 = \left(\frac{\mu_1}{\mu_1 + \dot{\mu}_1}\right)^{1/2}.$$

Abbreviating $\mu(1), \dot{\mu}(1), \ddot{\mu}(1), \dddot{\mu}(1)$ to respectively $\mu_1, \dot{\mu}_1, \ddot{\mu}_1, \dddot{\mu}_1$, one finds for small values of β_0 (corresponding to deep fluids) that

$$c_3^1 = 1 + o(1), \qquad c_1^1 = \sin^{-2}(\alpha) + o(1)$$

and

$$\begin{aligned}
\frac{c_4}{(c_3^1)^2} = \frac{c_1}{(c_1^1)^2}& \\
= &-\frac{S_1\sigma^7\ddot{\mu}_1}{\mu_1(\mu_1-1)^4(\mu_1+\dot{\mu}_1)^3} + \frac{\sigma^8\left(1+11S_1(\mu_1+\dot{\mu}_1)\right)\ddot{\mu}_1^2}{2\mu_1(\mu_1-1)^6(\mu_1+\dot{\mu}_1)^5} \\
&+\left(6\dot{\mu}_1^2\mu_1 + 15\dot{\mu}_1\mu_1^2 - 15\mu_1^3 - 6\dot{\mu}_1^2 - 47\mu_1\dot{\mu}_1 - 17\mu_1^2\right)\frac{S_1\sigma^7\ddot{\mu}_1}{3\mu_1^2(\mu_1-1)^5(\mu_1+\dot{\mu}_1)^4} \\
&-\frac{32\sigma^7\ddot{\mu}_1}{3(\mu_1+\dot{\mu}_1)^3(\mu_1-1)^5} \\
&+\Big(3\dot{\mu}_1^4\mu_1^3 + \left(12\mu_1^4+8\mu_1^3+13\mu_1^2\right)\dot{\mu}_1^3 + \left(18\mu_1^5+8\mu_1^4+91\mu_1^3\right)\dot{\mu}_1^2 + \left(12\mu_1^6+24\mu_1^5+111\mu_1^4\right)\dot{\mu}_1 + 3\mu_1^7 + 24\mu_1^6 + 33\mu_1^5\Big) \\
&\quad\times\frac{2\sigma^7}{3\mu_1^3(\mu_1-1)^6(\mu_1+\dot{\mu}_1)^4} \\
&+\Big(\left(27\mu_1^2-18\mu_1+9\right)\dot{\mu}_1^4 + \left(153\mu_1^3-50\mu_1^2+47\mu_1\right)\dot{\mu}_1^3 + \left(291\mu_1^4-62\mu_1^3+137\mu_1^2\right)\dot{\mu}_1^2 + \left(207\mu_1^5+66\mu_1^4+81\mu_1^3\right)\dot{\mu}_1 \\
&\quad + 66\mu_1^6 + 48\mu_1^5 + 6\mu_1^4\Big)\frac{S_1\sigma^7}{3\mu_1^3(\mu_1-1)^6(\mu_1+\dot{\mu}_1)^4} \\
&-\Big(783\dot{\mu}_1^4\mu_1^2 + 3306\dot{\mu}_1^3\mu_1^3 + 4791\dot{\mu}_1^2\mu_1^4 + 2880\dot{\mu}_1\mu_1^5 + 576\mu_1^6 - 222\dot{\mu}_1^4\mu_1 - 2452\dot{\mu}_1^3\mu_1^2 - 5806\dot{\mu}_1^2\mu_1^3 \\
&\quad - 4800\dot{\mu}_1\mu_1^4 - 1152\mu_1^5 + 15\dot{\mu}_1^4 + 298\dot{\mu}_1^3\mu_1 + 1591\dot{\mu}_1^2\mu_1^2 + 1920\dot{\mu}_1\mu_1^3 + 576\mu_1^4\Big) \\
&\quad\times\frac{S_1\sigma^8}{6(\mu_1+\dot{\mu}_1)^4(\mu_1-1)^8\mu_1^3} \\
&-\Big(288\dot{\mu}_1^5\mu_1^3 + 1440\dot{\mu}_1^4\mu_1^4 + 2880\dot{\mu}_1^3\mu_1^5 + 2880\dot{\mu}_1^2\mu_1^6 + 1440\dot{\mu}_1\mu_1^7 + 288\mu_1^8 - 608\dot{\mu}_1^4\mu_1^3 - 2464\dot{\mu}_1^3\mu_1^4 - 3680\dot{\mu}_1^2\mu_1^5 \\
&\quad - 2400\dot{\mu}_1\mu_1^6 - 576\mu_1^7 + 557\dot{\mu}_1^4\mu_1^2 + 2782\dot{\mu}_1^3\mu_1^3 + 4421\dot{\mu}_1^2\mu_1^4 + 2688\dot{\mu}_1\mu_1^5 + 576\mu_1^6 - 282\dot{\mu}_1^4\mu_1 - 2204\dot{\mu}_1^3\mu_1^2 \\
&\quad - 4490\dot{\mu}_1^2\mu_1^3 - 2976\dot{\mu}_1\mu_1^4 - 576\mu_1^5 + 45\dot{\mu}_1^4 + 446\dot{\mu}_1^3\mu_1 + 1445\dot{\mu}_1^2\mu_1^2 + 1248\dot{\mu}_1\mu_1^3 + 288\mu_1^4\Big) \\
&\quad\times\frac{\sigma^8}{6\mu_1^3(\mu_1-1)^8(\mu_1+\dot{\mu}_1)^5} + o(1)
\end{aligned}$$

as $\beta_0 \to 0$, while

$$\begin{aligned}
\frac{c_2}{2(c_1^1)^2} = &-\frac{S_1\sigma^7\ddot{\mu}_1}{\mu_1(\mu_1-1)^4(\dot{\mu}_1+\mu_1)^3} \\
&+\Big(-\Big(4(\sin(\alpha)-1)(\mu_1+1)(\dot{\mu}_1+\mu_1)\cos^2(\alpha) - 5(\mu_1+1)(\dot{\mu}_1+\mu_1)\sin(\alpha) + (5\mu_1+4)(\dot{\mu}_1+\mu_1)\Big)S_1 + 1\Big) \\
&\quad\times\frac{8\sigma^8\sin(\alpha)\ddot{\mu}_1^2}{(\mu_1-1)^6\mu_1(\dot{\mu}_1+\mu_1)^5\left(4\cos^2(\alpha)\sin(\alpha)-\sin(\alpha)-1\right)\cos(\alpha)^2} \\
&-\Big(\Big(4(\dot{\mu}_1+\mu_1)\cos^3(\alpha) - 4(\dot{\mu}_1+\mu_1)\cos^2(\alpha) + (\dot{\mu}_1+\mu_1)\cos(\alpha)\Big)S_1 + 1\Big) \\
&\quad\times\frac{8\sigma^8\cos(\alpha)\ddot{\mu}_1^2}{(\mu_1-1)^6\mu_1(\dot{\mu}_1+\mu_1)^5\sin(\alpha)^2\left(4\cos^3(\alpha)-3\cos(\alpha)+1\right)}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2\left(\cos^4(\alpha)-\cos^2(\alpha)-1\right)S_1\sigma^7\ddot{\mu}_1^2}{(\mu_1-1)^4\,\mu_1\,(\dot{\mu}_1+\mu_1)^4\sin(\alpha)^2\cos(\alpha)^2}\\
&+\Bigg(\bigg(\Big(2\mu_1(\dot{\mu}_1+\mu_1)(4\mu_1^2+6\mu_1+4)\sin(\alpha)+\mu_1^5+(2\dot{\mu}_1-5)\mu_1^4+(\dot{\mu}_1^2-5\dot{\mu}_1-6)\mu_1^3-(6\dot{\mu}_1+5)\mu_1^2-(6\dot{\mu}_1-1)\,\mu_1+\dot{\mu}_1\Big)\\
&\qquad\times 24\,(\dot{\mu}_1+\mu_1)\cos^4(\alpha)\\
&\quad-\Big(\tfrac{10}{3}\mu_1\,(\dot{\mu}_1+\mu_1)\left(\mu_1^2+\tfrac{3}{2}\mu_1+1\right)\sin(\alpha)+\mu_1^5+(2\dot{\mu}_1-4)\,\mu_1^4+\left(\dot{\mu}_1^2-4\dot{\mu}_1-4\right)\mu_1^3\\
&\qquad-\left(4\dot{\mu}_1+\tfrac{23}{6}\right)\mu_1^2+\left(-5\dot{\mu}_1+\tfrac{7}{6}\right)\mu_1+\tfrac{7\dot{\mu}_1}{6}\Big)\\
&\qquad\times 36\,(\dot{\mu}_1+\mu_1)\cos^2(\alpha)\\
&\quad+6\left(\mu_1^2+(2\dot{\mu}_1-1)\,\mu_1-\dot{\mu}_1\right)(\dot{\mu}_1+\mu_1)\sin(\alpha)+6\mu_1^3+(12\dot{\mu}_1-6)\,\mu_1^2+\left(-6\dot{\mu}_1^2-6\dot{\mu}_1\right)\mu_1+12\dot{\mu}_1^2\bigg)\\
&\quad\times\frac{8}{3}\\
&\quad+\bigg(\Big(\Big(\mu_1^3+\left(\tfrac{\dot{\mu}_1}{3}+\tfrac{7}{3}\right)\mu_1^2+\left(\tfrac{4\dot{\mu}_1}{3}+\tfrac{4}{3}\right)\mu_1+\tfrac{2\dot{\mu}_1}{3}\Big)\sin(\alpha)-\tfrac{\mu_1^3}{3}+\left(\dot{\mu}_1-\tfrac{10}{3}\right)\mu_1^2+(-2\dot{\mu}_1-1)\,\mu_1-\dot{\mu}_1\Big)\\
&\qquad\times 12\,(\dot{\mu}_1+\mu_1)\cos^4(\alpha)\\
&\quad+\Big(\left(-15\mu_1^4+(-16\dot{\mu}_1-35)\,\mu_1^3+\left(7\dot{\mu}_1^2-55\dot{\mu}_1-20\right)\mu_1^2+\left(-20\dot{\mu}_1^2-34\dot{\mu}_1\right)\mu_1-22\dot{\mu}_1^2\right)\sin(\alpha)\\
&\qquad+2\mu_1^4+(47-20\dot{\mu}_1)\,\mu_1^3+\left(-30\dot{\mu}_1^2+77\dot{\mu}_1+9\right)\mu_1^2+\left(30\dot{\mu}_1^2+26\dot{\mu}_1\right)\mu_1+25\dot{\mu}_1^2\Big)\\
&\qquad\times\cos^2(\alpha)\\
&\quad+\left((-5\dot{\mu}_1+1)\,\mu_1^3+\left(-15\dot{\mu}_1^2+3\dot{\mu}_1-1\right)\mu_1^2+\left(2\dot{\mu}_1^2+3\dot{\mu}_1\right)\mu_1+14\dot{\mu}_1^2\right)\sin(\alpha)+(5\dot{\mu}_1+1)\,\mu_1^3\\
&\quad+\left(15\dot{\mu}_1^2+2\dot{\mu}_1-1\right)\mu_1^2+\left(-\dot{\mu}_1^2-6\dot{\mu}_1\right)\mu_1-13\dot{\mu}_1^2\bigg)\\
&\quad\times 16\,(\dot{\mu}_1+\mu_1)\,S_1\Bigg)\\
&\quad\times\frac{\sigma^8\sin(\alpha)\,\ddot{\mu}_1}{\mu_1^3\,(\dot{\mu}_1+\mu_1)^5\cos^2(\alpha)\left(4\cos^2(\alpha)\sin(\alpha)-\sin(\alpha)-1\right)(\mu_1-1)^8}\\
&+\Bigg(16\mu_1\,(\mu_1+1)\,(\dot{\mu}_1+\mu_1)^2\cos^5(\alpha)-16\left(\mu_1^3+\left(\dot{\mu}_1+\tfrac{5}{4}\right)\mu_1^2+\left(\tfrac{3\dot{\mu}_1}{2}-\tfrac{1}{4}\right)\mu_1-\tfrac{\dot{\mu}_1}{4}\right)(\dot{\mu}_1+\mu_1)\cos^4(\alpha)\\
&\quad-12\mu_1\,(\mu_1+1)\,(\dot{\mu}_1+\mu_1)^2\cos^3(\alpha)+16\left(\mu_1^3+\left(\dot{\mu}_1+\tfrac{17}{16}\right)\mu_1^2+\left(\tfrac{9\dot{\mu}_1}{8}-\tfrac{1}{16}\right)\mu_1-\tfrac{\dot{\mu}_1}{16}\right)(\dot{\mu}_1+\mu_1)\cos^2(\alpha)\\
&\quad-4\left(\mu_1^3+\left(\dot{\mu}_1+\tfrac{3}{4}\right)\mu_1^2+\left(\tfrac{\dot{\mu}_1}{2}+\tfrac{1}{4}\right)\mu_1+\tfrac{\dot{\mu}_1}{4}\right)(\dot{\mu}_1+\mu_1)\cos(\alpha)+4\mu_1^3+(11\dot{\mu}_1-4)\,\mu_1^2+\left(5\dot{\mu}_1^2-7\dot{\mu}_1\right)\mu_1-\dot{\mu}_1^2\\
&\quad+\bigg(16\,(\dot{\mu}_1+\mu_1)\,(\mu_1+1)\left(\mu_1+\tfrac{\dot{\mu}_1}{2}\right)\cos^5(\alpha)-20\left(\mu_1^2+\left(\tfrac{2\dot{\mu}_1}{5}+\tfrac{3}{5}\right)\mu_1+\tfrac{3\dot{\mu}_1}{5}\right)(\dot{\mu}_1+\mu_1)\cos^4(\alpha)\\
&\qquad+\left(-12\mu_1^3+(-22\dot{\mu}_1-12)\,\mu_1^2+\left(-18\dot{\mu}_1^2-14\dot{\mu}_1\right)\mu_1+6\dot{\mu}_1^2\right)\cos^3(\alpha)\\
&\qquad+\left(17\mu_1^3+(27\dot{\mu}_1+15)\,\mu_1^2+\left(18\dot{\mu}_1^2+22\dot{\mu}_1\right)\mu_1-\dot{\mu}_1^2\right)\cos^2(\alpha)\\
&\qquad+\left(-3\mu_1^3+(-4\dot{\mu}_1-5)\,\mu_1^2+\left(-3\dot{\mu}_1^2-7\dot{\mu}_1\right)\mu_1\right)\cos(\alpha)+4\mu_1\,(\dot{\mu}_1+\mu_1)\,(-1+\mu_1+\dot{\mu}_1)\bigg)\\
&\qquad\times S_1\,(\dot{\mu}_1+\mu_1)\Bigg)\\
&\quad\times\frac{16\sigma^8\cos(\alpha)\,\ddot{\mu}_1}{(\dot{\mu}_1+\mu_1)^5\,(\mu_1-1)^7\,(\cos(\alpha)-1)\,(\cos(\alpha)+1)^2\,(-1+2\cos(\alpha))^2\,\mu_1^2}\\
&-\frac{32\sigma^7\ddot{\mu}_1}{(\mu_1-1)^5\,(\dot{\mu}_1+\mu_1)^3}\\
&-\Big(\left(-21\mu_1^3+(-11\dot{\mu}_1-11)\,\mu_1^2+\left(2\dot{\mu}_1^2-37\dot{\mu}_1\right)\mu_1-18\dot{\mu}_1^2\right)\cos^4(\alpha)\\
&\quad+\left(21\mu_1^3+(11\dot{\mu}_1+11)\,\mu_1^2+\left(-2\dot{\mu}_1^2+37\dot{\mu}_1\right)\mu_1+18\dot{\mu}_1^2\right)\cos^2(\alpha)\\
&\quad-8\mu_1^3+(-20\dot{\mu}_1+8)\,\mu_1^2+\left(-4\dot{\mu}_1^2+12\dot{\mu}_1\right)\mu_1-4\dot{\mu}_1^2\Big)\\
&\quad\times\frac{\sigma^7S_1\ddot{\mu}_1}{\sin(\alpha)^2\cos(\alpha)^2\,(\dot{\mu}_1+\mu_1)^4\,\mu_1^2\,(\mu_1-1)^5}\\
&-\bigg(\Big(\Big(6\,(\mu_1+1)\left(\mu_1^3+\left(\tfrac{\dot{\mu}_1}{3}+\tfrac{5}{3}\right)\mu_1^2+\left(\dot{\mu}_1+\tfrac{4}{3}\right)\mu_1+\tfrac{2\dot{\mu}_1}{3}\right)\mu_1\,(\dot{\mu}_1+\mu_1)\sin(\alpha)+\mu_1^7+(2\dot{\mu}_1-1)\,\mu_1^6\\
&\qquad+\left(\dot{\mu}_1^2+9\dot{\mu}_1-21\right)\mu_1^5+\left(12\dot{\mu}_1^2-30\dot{\mu}_1-18\right)\mu_1^4+\left(2\dot{\mu}_1^3-9\dot{\mu}_1^2-27\dot{\mu}_1-10\right)\mu_1^3\\
&\qquad+\left(-8\dot{\mu}_1^2-20\dot{\mu}_1+1\right)\mu_1^2+\left(-10\dot{\mu}_1^2+2\dot{\mu}_1\right)\mu_1+\dot{\mu}_1^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times 8\left(\dot{\mu}_1+\mu_1\right)^2\cos^6(\alpha)\\
&-\Big(\Big(\mu_1^7+\left(4\dot{\mu}_1+\tfrac{9}{2}\right)\mu_1^6+\left(5\dot{\mu}_1^2-\dot{\mu}_1+23\right)\mu_1^5+\left(2\dot{\mu}_1^3-\tfrac{23}{2}\dot{\mu}_1^2+35\dot{\mu}_1+\tfrac{41}{2}\right)\mu_1^4\\
&\quad+\left(-2\dot{\mu}_1^3+10\dot{\mu}_1^2+31\dot{\mu}_1+12\right)\mu_1^3+\left(\tfrac{23}{2}\dot{\mu}_1^2+23\dot{\mu}_1-1\right)\mu_1^2+\left(15\dot{\mu}_1^2-2\dot{\mu}_1\right)\mu_1-\dot{\mu}_1^2\Big)\\
&\quad\times\left(\dot{\mu}_1+\mu_1\right)\sin(\alpha)\\
&\quad+\tfrac{5\mu_1^8}{2}+\left(-2+9\dot{\mu}_1\right)\mu_1^7+\left(-\tfrac{41}{2}+11\dot{\mu}_1^2+8\dot{\mu}_1\right)\mu_1^6+\left(-\tfrac{39}{2}-48\dot{\mu}_1+5\dot{\mu}_1^3+\tfrac{59}{2}\dot{\mu}_1^2\right)\mu_1^5\\
&\quad+\left(-\tfrac{101}{2}\dot{\mu}_1-\tfrac{67}{2}\dot{\mu}_1^2-9+\tfrac{1}{2}\dot{\mu}_1^4+22\dot{\mu}_1^3\right)\mu_1^4+\left(-\tfrac{85}{2}\dot{\mu}_1^2-\tfrac{59}{2}\dot{\mu}_1-6\dot{\mu}_1^3+\tfrac{1}{2}+\tfrac{5}{2}\dot{\mu}_1^4\right)\mu_1^3\\
&\quad+\left(-\tfrac{21}{2}\dot{\mu}_1^3-\tfrac{77}{2}\dot{\mu}_1^2+2\dot{\mu}_1\right)\mu_1^2+\left(-19\dot{\mu}_1^3+\tfrac{7}{2}\dot{\mu}_1^2\right)\mu_1+2\dot{\mu}_1^3\Big)\\
&\quad\times 8\left(\dot{\mu}_1+\mu_1\right)\cos^4(\alpha)\\
&+\Big(\Big(\mu_1^7+\left(4\dot{\mu}_1-\tfrac{8}{3}\right)\mu_1^6+\left(5\dot{\mu}_1^2-\tfrac{29}{3}\dot{\mu}_1+3\right)\mu_1^5+\left(2\dot{\mu}_1^3-\tfrac{37}{3}\dot{\mu}_1^2+\tfrac{35}{6}\dot{\mu}_1-\tfrac{7}{3}\right)\mu_1^4\\
&\quad+\left(-2\dot{\mu}_1^3+\tfrac{7}{6}\dot{\mu}_1^2-\tfrac{29}{6}\dot{\mu}_1+2\right)\mu_1^3+\left(-\tfrac{11}{6}\dot{\mu}_1^2+\tfrac{22}{3}\dot{\mu}_1-1\right)\mu_1^2+\left(\tfrac{26}{3}\dot{\mu}_1^2-2\dot{\mu}_1\right)\mu_1-\dot{\mu}_1^2\Big)\\
&\quad\times\left(\dot{\mu}_1+\mu_1\right)\sin(\alpha)\\
&\quad+\mu_1^8+\left(-\tfrac{8}{3}+\tfrac{9\dot{\mu}_1}{2}\right)\mu_1^7+\left(3+\tfrac{13}{2}\dot{\mu}_1^2-\tfrac{49}{6}\dot{\mu}_1\right)\mu_1^6+\left(-\tfrac{7}{3}+\tfrac{29}{3}\dot{\mu}_1+\tfrac{7}{2}\dot{\mu}_1^3-\tfrac{29}{6}\dot{\mu}_1^2\right)\mu_1^5\\
&\quad+\left(-\tfrac{97}{12}\dot{\mu}_1+\tfrac{31}{3}\dot{\mu}_1^2+2+\tfrac{1}{2}\dot{\mu}_1^4+\tfrac{1}{6}\dot{\mu}_1^3\right)\mu_1^4+\left(-\tfrac{39}{4}\dot{\mu}_1^2+\tfrac{31}{6}\dot{\mu}_1+\tfrac{11}{3}\dot{\mu}_1^3-1-\tfrac{1}{2}\dot{\mu}_1^4\right)\mu_1^3\\
&\quad+\left(-\tfrac{17}{6}\dot{\mu}_1^3-\tfrac{5}{4}\dot{\mu}_1^2-\tfrac{29}{12}\dot{\mu}_1\right)\mu_1^2+\left(-\tfrac{67}{12}\dot{\mu}_1^3-\tfrac{2}{3}\dot{\mu}_1^2\right)\mu_1+\tfrac{3\dot{\mu}_1^3}{4}\Big)\\
&\quad\times 12\left(\dot{\mu}_1+\mu_1\right)\cos^2(\alpha)\\
&+\Big(\left(\mu_1^2+\left(2\dot{\mu}_1-1\right)\mu_1-\dot{\mu}_1\right)\left(\dot{\mu}_1+\mu_1\right)\sin(\alpha)+\mu_1^3+\left(-1+\tfrac{5\dot{\mu}_1}{2}\right)\mu_1^2+\left(\tfrac{1}{2}\dot{\mu}_1^2-\tfrac{3}{2}\dot{\mu}_1\right)\mu_1+\tfrac{\dot{\mu}_1^2}{2}\Big)\\
&\quad\times\left(3\dot{\mu}_1+\mu_1\right)\left(\mu_1-1\right)\dot{\mu}_1\Big)\\
&\times 16\\
&+\Big(\Big(3\left(\mu_1+1\right)\Big(\mu_1^4+\left(\tfrac{5\dot{\mu}_1}{6}+\tfrac{5}{3}\right)\mu_1^3+\left(\tfrac{11\dot{\mu}_1}{6}+\tfrac{4}{3}\right)\mu_1^2+\left(\tfrac{1}{3}\dot{\mu}_1^2+\tfrac{4}{3}\dot{\mu}_1\right)\mu_1+\tfrac{\dot{\mu}_1^2}{6}\Big)\sin(\alpha)\\
&\quad+\mu_1^6+\left(2\dot{\mu}_1-\tfrac{5}{2}\right)\mu_1^5+\left(\dot{\mu}_1^2-7\right)\mu_1^4+\left(\dot{\mu}_1^2-2\dot{\mu}_1-\tfrac{25}{2}\right)\mu_1^3+\left(3\dot{\mu}_1^2-16\dot{\mu}_1-3\right)\mu_1^2\\
&\quad+\left(-\tfrac{7}{2}\dot{\mu}_1^2-4\dot{\mu}_1\right)\mu_1-\dot{\mu}_1^2\Big)\\
&\quad\times\left(\dot{\mu}_1+\mu_1\right)^2\cos^6(\alpha)\\
&-\Big(\Big(\tfrac{17\mu_1^6}{6}+\left(\tfrac{16}{3}+\tfrac{23\dot{\mu}_1}{4}\right)\mu_1^5+\left(\tfrac{55}{6}+\tfrac{22}{3}\dot{\mu}_1+\tfrac{37}{12}\dot{\mu}_1^2\right)\mu_1^4+\left(\tfrac{8}{3}+\tfrac{5}{6}\dot{\mu}_1^3-\tfrac{5}{3}\dot{\mu}_1^2+\tfrac{251}{12}\dot{\mu}_1\right)\mu_1^3\\
&\quad+\left(-\tfrac{10}{3}\dot{\mu}_1^3+\tfrac{46}{3}\dot{\mu}_1^2+6\dot{\mu}_1\right)\mu_1^2+\tfrac{13\left(\dot{\mu}_1+\frac{5}{3}\right)\dot{\mu}_1^2\mu_1}{4}+\tfrac{17\dot{\mu}_1^3}{12}\Big)\\
&\quad\times\sin(\alpha)\\
&\quad+\mu_1^7+\left(3\dot{\mu}_1-\tfrac{4}{3}\right)\mu_1^6+\left(3\dot{\mu}_1^2+\tfrac{7}{12}\dot{\mu}_1-\tfrac{17}{3}\right)\mu_1^5+\left(\dot{\mu}_1^3+\tfrac{47}{12}\dot{\mu}_1^2-\tfrac{47}{6}\dot{\mu}_1-\tfrac{23}{3}\right)\mu_1^4\\
&\quad+\left(\tfrac{2}{3}\dot{\mu}_1^3+\tfrac{17}{6}\dot{\mu}_1^2-\tfrac{217}{12}\dot{\mu}_1-\tfrac{7}{3}\right)\mu_1^3+\left(5\dot{\mu}_1^3-\tfrac{91}{6}\dot{\mu}_1^2-6\dot{\mu}_1\right)\mu_1^2+\left(-\tfrac{41}{12}\dot{\mu}_1^3-\tfrac{71}{12}\dot{\mu}_1^2\right)\mu_1-\tfrac{9\dot{\mu}_1^3}{4}\Big)\\
&\quad\times\tfrac{3}{2}\left(\dot{\mu}_1+\mu_1\right)\cos^4(\alpha)\\
&+\Big(\Big(\mu_1^7+\left(\tfrac{75\dot{\mu}_1}{16}-3\right)\mu_1^6+\left(\tfrac{7}{2}-\tfrac{63}{4}\dot{\mu}_1+\tfrac{121}{16}\dot{\mu}_1^2\right)\mu_1^5+\left(-\tfrac{3}{2}+\tfrac{241}{16}\dot{\mu}_1-\tfrac{509}{16}\dot{\mu}_1^2+\tfrac{109}{16}\dot{\mu}_1^3\right)\mu_1^4\\
&\quad+\left(-\tfrac{113}{4}\dot{\mu}_1^3+\tfrac{193}{8}\dot{\mu}_1^2-\tfrac{7}{2}\dot{\mu}_1+\tfrac{55}{16}\dot{\mu}_1^4\right)\mu_1^3-\tfrac{155\left(\dot{\mu}_1^2-\frac{51}{31}\dot{\mu}_1-\frac{2}{31}\right)\dot{\mu}_1^2\mu_1^2}{16}+\left(5\dot{\mu}_1^3+\tfrac{23}{8}\dot{\mu}_1^4\right)\mu_1+\tfrac{23\dot{\mu}_1^4}{8}\Big)\\
&\quad\times\sin(\alpha)\\
&\quad+\mu_1^7+\left(-3+\tfrac{23\dot{\mu}_1}{4}\right)\mu_1^6+\left(\tfrac{7}{2}-\tfrac{187}{16}\dot{\mu}_1+\tfrac{43}{4}\dot{\mu}_1^2\right)\mu_1^5+\left(-\tfrac{3}{2}-\tfrac{91}{8}\dot{\mu}_1^2+\tfrac{47}{4}\dot{\mu}_1+\tfrac{21}{4}\dot{\mu}_1^3\right)\mu_1^4\\
&\quad+\left(\tfrac{19}{16}\dot{\mu}_1^3+\tfrac{151}{16}\dot{\mu}_1^2-\tfrac{85}{16}\dot{\mu}_1-\tfrac{5}{4}\dot{\mu}_1^4\right)\mu_1^3+\tfrac{35\left(\dot{\mu}_1^2+\frac{6}{35}\dot{\mu}_1-\frac{19}{10}\right)\dot{\mu}_1^2\mu_1^2}{8}+\tfrac{\dot{\mu}_1^3\left(\dot{\mu}_1-123\right)\mu_1}{16}-\tfrac{59\dot{\mu}_1^4}{16}\Big)\\
&\quad\times\cos^2(\alpha)\\
&-\Big(\Big(\left(-\tfrac{2}{5}+\dot{\mu}_1\right)\mu_1^3+\left(3\dot{\mu}_1^2-\tfrac{6}{5}\dot{\mu}_1+\tfrac{2}{5}\right)\mu_1^2+\left(-\tfrac{4}{5}\dot{\mu}_1^2-\tfrac{1}{5}\dot{\mu}_1\right)\mu_1-\tfrac{13\dot{\mu}_1^2}{5}\Big)\sin(\alpha)\\
&\quad+\left(-\dot{\mu}_1-\tfrac{2}{5}\right)\mu_1^3+\left(-3\dot{\mu}_1^2-\dot{\mu}_1+\tfrac{2}{5}\right)\mu_1^2+\left(-\tfrac{1}{5}\dot{\mu}_1^2+\tfrac{8}{5}\dot{\mu}_1\right)\mu_1+\tfrac{14\dot{\mu}_1^2}{5}\Big)\\
&\quad\times\tfrac{5}{32}\left(3\dot{\mu}_1+\mu_1\right)\left(\mu_1-1\right)\dot{\mu}_1\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times 256\,(\dot\mu_1+\mu_1)\,S_1\Big) \\
&\times \frac{\sigma^8 \sin(\alpha)}{\mu_1^3\,(\dot\mu_1+\mu_1)^5\cos^2(\alpha)\,(4\cos^2(\alpha)\sin(\alpha)-\sin(\alpha)-1)\,(\mu_1-1)^8} \\
&+\Bigg(\Bigg(4\,(2\mu_1+\dot\mu_1)\,(\mu_1+1)^2\,\mu_1\,(\dot\mu_1+\mu_1)^3\cos^7(\alpha) \\
&\quad+\Big(\mu_1^6+(3\dot\mu_1-10)\,\mu_1^5+\left(3\dot\mu_1^2-14\dot\mu_1-14\right)\mu_1^4+\left(\dot\mu_1^3-4\dot\mu_1^2-19\dot\mu_1-10\right)\mu_1^3 \\
&\quad+\left(-4\dot\mu_1^2-20\dot\mu_1+1\right)\mu_1^2+\left(-10\dot\mu_1^2+2\dot\mu_1\right)\mu_1+\dot\mu_1^2\Big) \\
&\quad\times(\dot\mu_1+\mu_1)^2\cos^6(\alpha) \\
&\quad-\Big(\mu_1^6+(3\dot\mu_1+12)\,\mu_1^5+\left(3\dot\mu_1^2+19\dot\mu_1+30\right)\mu_1^4 \\
&\quad+\left(\dot\mu_1^3+11\dot\mu_1^2+47\dot\mu_1+12\right)\mu_1^3+\left(18\dot\mu_1^2+13\dot\mu_1+1\right)\mu_1^2+\left(-3\dot\mu_1^2+2\dot\mu_1\right)\mu_1+\dot\mu_1^2\Big) \\
&\quad\times(\dot\mu_1+\mu_1)^2\cos^5(\alpha) \\
&\quad-\Big(15\mu_1^7+(60\dot\mu_1-126)\,\mu_1^6+\left(90\dot\mu_1^2-321\dot\mu_1-162\right)\mu_1^5+\left(60\dot\mu_1^3-288\dot\mu_1^2-378\dot\mu_1-126\right)\mu_1^4 \\
&\quad+\left(15\dot\mu_1^4-93\dot\mu_1^3-264\dot\mu_1^2-363\dot\mu_1+15\right)\mu_1^3+\left(-54\dot\mu_1^3-309\dot\mu_1^2+42\dot\mu_1\right)\mu_1^2 \\
&\quad+\left(-66\dot\mu_1^3+33\dot\mu_1^2\right)\mu_1+6\dot\mu_1^3\Big) \\
&\quad\times\tfrac{1}{6}\,(\dot\mu_1+\mu_1)\cos^4(\alpha) \\
&\quad+\Big(\mu_1^6+(3\dot\mu_1+6)\,\mu_1^5+\left(3\dot\mu_1^2+12\dot\mu_1+18\right)\mu_1^4+\left(\dot\mu_1^3+12\dot\mu_1^2+29\dot\mu_1+6\right)\mu_1^3 \\
&\quad+\left(12\dot\mu_1^2+2\dot\mu_1+1\right)\mu_1^2+\left(-10\dot\mu_1^2+2\dot\mu_1\right)\mu_1+\dot\mu_1^2\Big) \\
&\quad\times\tfrac{1}{2}\,(\dot\mu_1+\mu_1)^2\cos^3(\alpha) \\
&\quad+\Big(\mu_1^7+(4\dot\mu_1-18)\,\mu_1^6+\left(6\dot\mu_1^2-47\dot\mu_1-30\right)\mu_1^5+\left(4\dot\mu_1^3-48\dot\mu_1^2-\tfrac{293}{4}\dot\mu_1-18\right)\mu_1^4 \\
&\quad+\left(\dot\mu_1^4-19\dot\mu_1^3-\tfrac{225}{4}\dot\mu_1^2-\tfrac{93}{2}\dot\mu_1+1\right)\mu_1^3+\left(-\tfrac{27}{2}\dot\mu_1^3-\tfrac{119}{4}\dot\mu_1^2+\tfrac{11}{4}\dot\mu_1\right)\mu_1^2+\left(-\tfrac{3}{4}\dot\mu_1^3+2\dot\mu_1^2\right)\mu_1+\tfrac{\dot\mu_1^3}{4}\Big) \\
&\quad\times\tfrac{1}{2}\,(\dot\mu_1+\mu_1)\cos^2(\alpha) \\
&\quad+\Big(\mu_1^7+(4\dot\mu_1+2)\,\mu_1^6+\left(6\dot\mu_1^2+4\dot\mu_1+10\right)\mu_1^5+\left(4\dot\mu_1^3+4\dot\mu_1^2+\tfrac{107}{4}\dot\mu_1+2\right)\mu_1^4 \\
&\quad+\left(\dot\mu_1^4+2\dot\mu_1^3+\tfrac{95}{4}\dot\mu_1^2+\tfrac{5}{2}\dot\mu_1+1\right)\mu_1^3+\left(\tfrac{13}{2}\dot\mu_1^3-\tfrac{7}{4}\dot\mu_1^2+\tfrac{11}{4}\dot\mu_1\right)\mu_1^2+\left(-\tfrac{7}{4}\dot\mu_1^3+2\dot\mu_1^2\right)\mu_1+\tfrac{\dot\mu_1^3}{4}\Big) \\
&\quad\times\tfrac{1}{2}\,(\dot\mu_1+\mu_1)\cos(\alpha) \\
&\quad+\mu_1^8+(-2+5\dot\mu_1)\,\mu_1^7+\left(10\dot\mu_1^2-8\dot\mu_1+2\right)\mu_1^6+\left(-2+\tfrac{17}{2}\dot\mu_1+10\dot\mu_1^3-12\dot\mu_1^2\right)\mu_1^5 \\
&\quad+\left(-9\dot\mu_1+\tfrac{209}{16}\dot\mu_1^2+1+5\dot\mu_1^4-8\dot\mu_1^3\right)\mu_1^4 \\
&\quad+\dot\mu_1\left(-\tfrac{101}{8}\dot\mu_1+\tfrac{7}{2}+\tfrac{63}{8}\dot\mu_1^2-2\dot\mu_1^3+\dot\mu_1^4\right)\mu_1^3+\left(-\tfrac{23}{4}\dot\mu_1^3+\tfrac{57}{16}\dot\mu_1^2+\tfrac{25}{16}\dot\mu_1^4\right)\mu_1^2 \\
&\quad+\left(-\tfrac{5}{8}\dot\mu_1^4+\tfrac{7}{8}\dot\mu_1^3\right)\mu_1+\tfrac{\dot\mu_1^4}{16}\Bigg) \\
&\quad\times 128 \\
&+\Big(128\,(\mu_1+1)^2\left(\mu_1^2+\mu_1\dot\mu_1+\tfrac{1}{8}\dot\mu_1^2\right)(\dot\mu_1+\mu_1)^2\cos^7(\alpha) \\
&\quad-32\left(3\mu_1^4+10\mu_1^3+\left(-2\dot\mu_1^2+12\dot\mu_1+3\right)\mu_1^2+\left(2\dot\mu_1^2+4\dot\mu_1\right)\mu_1+\dot\mu_1^2\right)(\dot\mu_1+\mu_1)^2\cos^6(\alpha) \\
&\quad-4\,(\dot\mu_1+\mu_1)\left(64\mu_1^5+(152\dot\mu_1+96)\,\mu_1^4+\left(131\dot\mu_1^2+176\dot\mu_1+64\right)\mu_1^3+\left(35\dot\mu_1^3+86\dot\mu_1^2+120\dot\mu_1\right)\mu_1^2\right. \\
&\quad\left.+\left(6\dot\mu_1^3+43\dot\mu_1^2\right)\mu_1-5\dot\mu_1^3\right)\cos^5(\alpha) \\
&\quad+4\left(44\mu_1^5+(44\dot\mu_1+168)\,\mu_1^4+\left(-15\dot\mu_1^2+372\dot\mu_1+44\right)\mu_1^3+\left(-27\dot\mu_1^3+234\dot\mu_1^2+96\dot\mu_1\right)\mu_1^2\right. \\
&\quad\left.+\left(42\dot\mu_1^3+49\dot\mu_1^2\right)\mu_1-3\dot\mu_1^3\right)(\dot\mu_1+\mu_1)\cos^4(\alpha) \\
&\quad+\left(80\mu_1^6+(304\dot\mu_1+96)\,\mu_1^5+\left(476\dot\mu_1^2+256\dot\mu_1+80\right)\mu_1^4+\left(352\dot\mu_1^3+216\dot\mu_1^2+208\dot\mu_1\right)\mu_1^3\right. \\
&\quad\left.+\left(116\dot\mu_1^4+16\dot\mu_1^3+124\dot\mu_1^2\right)\mu_1^2+\left(-72\dot\mu_1^4-16\dot\mu_1^3\right)\mu_1+4\dot\mu_1^4\right)\cos^3(\alpha) \\
&\quad+\left(-112\mu_1^6+(-314\dot\mu_1-288)\,\mu_1^5+\left(-348\dot\mu_1^2-892\dot\mu_1-112\right)\mu_1^4+\left(-174\dot\mu_1^3-962\dot\mu_1^2-330\dot\mu_1\right)\mu_1^3\right. \\
&\quad\left.+\left(-44\dot\mu_1^4-396\dot\mu_1^3-274\dot\mu_1^2\right)\mu_1^2+\left(-6\dot\mu_1^4-38\dot\mu_1^3\right)\mu_1+2\dot\mu_1^4\right)\cos^2(\alpha) \\
&\quad+\left(48\mu_1^6+(174\dot\mu_1+32)\,\mu_1^5+\left(249\dot\mu_1^2+68\dot\mu_1+48\right)\mu_1^4+\left(160\dot\mu_1^3+44\dot\mu_1^2+142\dot\mu_1\right)\mu_1^3\right. \\
&\quad\left.+\left(41\dot\mu_1^4+8\dot\mu_1^3+123\dot\mu_1^2\right)\mu_1^2+\left(-8\dot\mu_1^4+24\dot\mu_1^3\right)\mu_1-\dot\mu_1^4\right)\cos(\alpha) \\
&\quad+8\mu_1\,(\dot\mu_1+\mu_1)\,(-1+\mu_1+\dot\mu_1)\left(4\mu_1^3+(11\dot\mu_1-4)\,\mu_1^2+\left(5\dot\mu_1^2-7\dot\mu_1\right)\mu_1-\dot\mu_1^2\right)\Big)
\end{aligned}
$$

$$\times\, 8\left(\dot\mu_1+\mu_1\right)S_1\Big)$$

$$\times\,\frac{\cos(\alpha)\,\sigma^8}{\sin^2(\alpha)\left(3\cos(\alpha)-4\cos^3(\alpha)-1\right)\left(\mu_1-1\right)^8\left(\dot\mu_1+\mu_1\right)^5\mu_1^3}$$

$$-\Bigg(\Bigg(\left(48\mu_1^3+54\dot\mu_1\mu_1^2+\left(-12\dot\mu_1+48\right)\mu_1+6\dot\mu_1\right)\left(\dot\mu_1+\mu_1\right)\left(\cos^2(\alpha)\right)$$

$$-36\mu_1^4+\left(-84\dot\mu_1+24\right)\mu_1^3+\left(-45\dot\mu_1^2+20\dot\mu_1-36\right)\mu_1^2+\left(6\dot\mu_1^2-32\dot\mu_1\right)\mu_1-3\dot\mu_1^2\Bigg)S_1(\mu_1+\dot\mu_1)$$

$$+\Bigg(6\left(2\mu_1^5+4\mu_1^4\dot\mu_1+\left(2\dot\mu_1^2+12\right)\mu_1^3+13\dot\mu_1\mu_1^2+2\left(-\dot\mu_1+1\right)\mu_1+\dot\mu_1\right)\left(\dot\mu_1+\mu_1\right)\cos^2(\alpha)$$

$$-12\mu_1^6+\left(-36\dot\mu_1+12\right)\mu_1^5+\left(-36\dot\mu_1^2+16\dot\mu_1-48\right)\mu_1^4+\left(-12\dot\mu_1^3+4\dot\mu_1^2-106\dot\mu_1+12\right)\mu_1^3$$

$$+\left(-55\dot\mu_1^2+16\dot\mu_1-12\right)\mu_1^2+\left(6\dot\mu_1^2-10\dot\mu_1\right)\mu_1-3\dot\mu_1^2\Bigg)\Bigg)$$

$$\times\,\frac{8\cos^2(\alpha)\,\sigma^8}{3\left(\dot\mu_1+\mu_1\right)^3\mu_1^3\left(\mu_1-1\right)^8}$$

$$-\Bigg(8\left(\dot\mu_1+\mu_1\right)^2\left(12\mu_1^4+\left(18\dot\mu_1+12\right)\mu_1^3+\left(9\dot\mu_1^2+4\dot\mu_1\right)\mu_1^2+\left(-6\dot\mu_1^2+8\dot\mu_1\right)\mu_1+3\dot\mu_1^2\right)\cos^6(\alpha)$$

$$+\Bigg(258\mu_1^6+\left(1119\dot\mu_1-336\right)\mu_1^5+\left(2067\dot\mu_1^2-1406\dot\mu_1+198\right)\mu_1^4+\left(1641\dot\mu_1^3-2254\dot\mu_1^2+641\dot\mu_1\right)\mu_1^3$$

$$+\left(459\dot\mu_1^4-1394\dot\mu_1^3+745\dot\mu_1^2\right)\mu_1^2+\left(-258\dot\mu_1^4+287\dot\mu_1^3\right)\mu_1+9\dot\mu_1^4\Bigg)\cos^4(\alpha)$$

$$+\Big(-354\mu_1^6+\left(-1455\dot\mu_1+240\right)\mu_1^5+\left(-2523\dot\mu_1^2+1182\dot\mu_1-198\right)\mu_1^4+\left(-1929\dot\mu_1^3+2142\dot\mu_1^2-705\dot\mu_1\right)\mu_1^3$$

$$+\left(-531\dot\mu_1^4+1458\dot\mu_1^3-897\dot\mu_1^2\right)\mu_1^2+\left(306\dot\mu_1^4-399\dot\mu_1^3\right)\mu_1-33\dot\mu_1^4\Big)\cos^2(\alpha)$$

$$+6\left(\mu_1-1\right)\dot\mu_1\left(3\dot\mu_1+\mu_1\right)\left(4\mu_1^3+\left(11\dot\mu_1-4\right)\mu_1^2+\left(5\dot\mu_1^2-7\dot\mu_1\right)\mu_1-\dot\mu_1^2\right)\Bigg)$$

$$\times\,\frac{\sigma^7S_1}{3\cos(\alpha)^2\left(\mu_1-1\right)^6\mu_1^3\left(\dot\mu_1+\mu_1\right)^4\sin(\alpha)^2}$$

$$+\Bigg((6\mu_1^2+(2\dot\mu_1+6)\mu_1+7\dot\mu_1)(\dot\mu_1+\mu_1)\tfrac{8}{3}\cos^2(\alpha)+33\mu_1^4+(99\dot\mu_1-40)\mu_1^3+(99\dot\mu_1^2-112\dot\mu_1+59)\mu_1^2$$

$$+(33\dot\mu_1^3-40\dot\mu_1^2+98\dot\mu_1)\mu_1+7\dot\mu_1^2\Bigg)$$

$$\times\,\frac{2\sigma^7}{\mu_1\left(\mu_1-1\right)^6\left(\dot\mu_1+\mu_1\right)^3}+o(1)$$

and

$$\frac{c_3}{2c_3^1c_1^1}=-\frac{S_1\,\sigma^7\ddot\mu_1}{\left(\mu_1-1\right)^4\mu_1\left(\mu_1+\dot\mu_1\right)^3}+\frac{2S_1\,\sigma^7\ddot\mu_1^2}{\left(\mu_1-1\right)^4\mu_1\left(\mu_1+\dot\mu_1\right)^4}$$

$$-\Big(2\sin(\alpha)\,a_1+2\cos^2(\alpha)-5\sin(\alpha)+2a_1-5\Big)\frac{32S_1\sigma^8\ddot\mu_1^2}{\left(\mu_1-1\right)^6\mu_1\left(a_1-2\right)\left(a_1+2\right)^2\left(a_1-1\right)^2\left(\mu_1+\dot\mu_1\right)^4}$$

$$+\frac{32\sigma^8a_1\ddot\mu_1^2}{\left(\mu_1-1\right)^6\mu_1\left(a_1-2\right)\left(a_1+2\right)^2\left(a_1-1\right)^2\left(\mu_1+\dot\mu_1\right)^5}-\frac{32\sigma^8S_1\left(-1+\sin(\alpha)\right)\ddot\mu_1^2}{\left(a_2+2\right)^2\left(a_2-2\right)\left(\mu_1-1\right)^6\left(\mu_1+\dot\mu_1\right)^4\mu_1}$$

$$+\frac{32a_2\,\sigma^8\ddot\mu_1^2}{\left(a_2+2\right)^2\left(a_2-2\right)\left(\mu_1-1\right)^6\left(a_2-1\right)^2\left(\mu_1+\dot\mu_1\right)^5\mu_1}-\frac{16\sigma^7S_1\ddot\mu_1^2}{\left(\mu_1-1\right)^4\mu_1\left(\mu_1+\dot\mu_1\right)^4\left(-1+\sin(\alpha)\right)a_1^2}$$

$$-\frac{32\sigma^7\ddot\mu_1}{3\left(\mu_1+\dot\mu_1\right)^3\left(\mu_1-1\right)^5}+\Big(6\dot\mu_1^2\mu_1+15\dot\mu_1\,\mu_1^2-15\mu_1^3-6\dot\mu_1^2-47\dot\mu_1\mu_1-17\mu_1^2\Big)\frac{\sigma^7S_1\ddot\mu_1}{3\left(\mu_1-1\right)^5\mu_1^2\left(\mu_1+\dot\mu_1\right)^4}$$

$$-\Bigg(-\left(a_1+1\right)\left(\mu_1+\dot\mu_1\right)\cos^4(\alpha)\,\frac{\mu_1}{8}$$

$$+\Bigg(\left(\mu_1+\dot\mu_1\right)\Big(\left(a_1-\tfrac{3}{2}\right)\mu_1^2+\Big(\left(\tfrac{a_1}{2}-\tfrac{1}{2}\right)\dot\mu_1+\tfrac{17a_1}{16}-\tfrac{7}{16}\Big)\mu_1+\tfrac{(a_1-2)\dot\mu_1}{2}\Big)\sin(\alpha)+\left(-\tfrac{7}{4}+\tfrac{a_1}{4}\right)\mu_1^3$$

$$+\Big(-2\dot\mu_1+\tfrac{11}{16}+\tfrac{11a_1}{16}\Big)\mu_1^2-\Big(\left(a_1-\tfrac{3}{5}\right)\dot\mu_1-\tfrac{19a_1}{20}+\tfrac{13}{20}\Big)\tfrac{5}{4}\dot\mu_1\mu_1+\tfrac{3}{2}\left(a_1-\tfrac{5}{3}\right)\dot\mu_1^2\Bigg)\cos^2(\alpha)$$

$$+\Bigg((-\tfrac{3}{4}+\tfrac{a_1}{8})\mu_1^3+((-3+\tfrac{5}{8}a_1)\dot\mu_1+\tfrac{3}{4}-\tfrac{a_1}{8})\mu_1^2+2\dot\mu_1((a_1-\tfrac{15}{8})\dot\mu_1-\tfrac{a_1}{4}+\tfrac{9}{8})\mu_1-\tfrac{15}{8}\dot\mu_1^2\left(a_1-\tfrac{8}{5}\right)\Bigg)\sin(\alpha)$$

$$+(-\tfrac{3}{4}-\tfrac{7a_1}{8})\mu_1^3+((-\tfrac{11a_1}{8}-3)\dot\mu_1+\tfrac{3}{4}+\tfrac{7a_1}{8})\mu_1^2+((-\tfrac{15}{4}+a_1)\dot\mu_1+\tfrac{9}{4}+\tfrac{a_1}{2})\dot\mu_1\mu_1-\tfrac{15}{8}\dot\mu_1^2(a_1-\tfrac{8}{5})\Bigg)$$

$$\times \frac{256 S_1\,\sigma^8 \ddot{\mu}_1}{(\mu_1+\dot{\mu}_1)^4\,(a_1-1)^2\,(a_1+2)^2\,(a_1-2)\,\mu_1^2\,(\mu_1-1)^7\,(a_1+1)}$$

$$-\Bigg(\Bigg(\Big((a_1-1)\,\mu_1^3+\big((a_1-1)\,\dot{\mu}_1-\tfrac{3}{2}+a_1\big)\,\mu_1^2+\big(\tfrac{1}{2}+(a_1-2)\,\dot{\mu}_1\big)\,\mu_1+\tfrac{\dot{\mu}_1}{2}\Big)\sin(\alpha)$$

$$+\big(\tfrac{a_1}{2}-\tfrac{1}{2}\big)\,\mu_1^3+\big(\big(\tfrac{a_1}{2}-\tfrac{1}{2}\big)\,\dot{\mu}_1-\tfrac{7}{4}+\tfrac{a_1}{4}\big)\,\mu_1^2+\big(-3\dot{\mu}_1+\tfrac{5}{4}+\tfrac{a_1}{4}\big)\,\mu_1+\tfrac{\dot{\mu}_1(a_1+5)}{4}\Bigg)\,(\mu_1+\dot{\mu}_1)\cos^2(\alpha)$$

$$+\Bigg(\big(-\tfrac{3}{4}+\tfrac{a_1}{8}\big)\,\mu_1^3+\Big(\Big(\tfrac{3a_1}{8}-\tfrac{7}{4}\Big)\,\dot{\mu}_1+\tfrac{3}{4}-\tfrac{a_1}{8}\Big)\,\mu_1^2+\tfrac{\dot{\mu}_1(\dot{\mu}_1 a_1-a_1+4)\mu_1}{4}-\tfrac{\dot{\mu}_1^2(6+a_1)}{8}\Bigg)\sin(\alpha)$$

$$+\Big(-\tfrac{3}{4}-\tfrac{7a_1}{8}\Big)\,\mu_1^3+\Big(\Big(-\tfrac{19a_1}{8}-\tfrac{7}{4}\Big)\,\dot{\mu}_1+\tfrac{3}{4}+\tfrac{7a_1}{8}\Big)\,\mu_1^2$$

$$-\dot{\mu}_1\,\big(\dot{\mu}_1 a_1-\tfrac{3}{2}a_1-1\big)\,\mu_1+\tfrac{\dot{\mu}_1^2(-6+a_1)}{8}\Bigg)$$

$$\times \frac{256\sigma^8 \ddot{\mu}_1}{(\mu_1+\dot{\mu}_1)^5\,(a_1-1)^2\,(a_1+2)^2\,(a_1-2)\,\mu_1^2\,(\mu_1-1)^7\,(a_1+1)}$$

$$+\Bigg(\Big((\mu_1+\dot{\mu}_1)\,\Big(\big(a_2-\tfrac{3}{2}\big)\,\mu_1^2+\big(\big(-\tfrac{1}{2}+\tfrac{a_2}{2}\big)\,\dot{\mu}_1-\tfrac{1}{2}+a_2\big)\,\mu_1+\tfrac{(a_2-2)\dot{\mu}_1}{2}\Big)\cos^2(\alpha)+\big(\tfrac{a_2}{8}-\tfrac{3}{4}\big)\,\mu_1^3$$

$$+\Big(\Big(-3+\tfrac{5a_2}{8}\Big)\,\dot{\mu}_1+\tfrac{3}{4}-\tfrac{a_2}{8}\Big)\,\mu_1^2+2\dot{\mu}_1\,\big(\big(a_2-\tfrac{15}{8}\big)\,\dot{\mu}_1-\tfrac{a_2}{4}+\tfrac{9}{8}\big)\,\mu_1-\tfrac{15\dot{\mu}_1^2\left(a_2-\frac{8}{5}\right)}{8}\Big)\sin(\alpha)$$

$$+\Bigg(\big(\tfrac{7}{4}-\tfrac{a_2}{4}\big)\,\mu_1^3+\Big(2\dot{\mu}_1-\tfrac{3}{4}-\tfrac{3a_2}{4}\Big)\,\mu_1^2+\tfrac{5\dot{\mu}_1\left(a_2-\frac{3}{5}\right)(\dot{\mu}_1-1)\mu_1}{4}-\tfrac{3\left(a_2-\frac{5}{3}\right)\dot{\mu}_1^2}{2}\Bigg)\cos^2(\alpha)$$

$$+\Big(\tfrac{3}{4}+\tfrac{7a_2}{8}\Big)\,\mu_1^3+\Big(\Big(\tfrac{11a_2}{8}+3\Big)\,\dot{\mu}_1-\tfrac{3}{4}-\tfrac{7a_2}{8}\Big)\,\mu_1^2-\dot{\mu}_1\,\big(\big(a_2-\tfrac{15}{4}\big)\,\dot{\mu}_1+\tfrac{a_2}{2}+\tfrac{9}{4}\big)\,\mu_1+\tfrac{15}{8}\dot{\mu}_1^2\,\big(a_2-\tfrac{8}{5}\big)\Bigg)$$

$$\times \frac{256 S_1\,\sigma^8 \ddot{\mu}_1}{(\mu_1+\dot{\mu}_1)^4\,(a_2-1)^2\,(\mu_1-1)^7\,(a_2+1)\,(a_2-2)\,(a_2+2)^2\,\mu_1^2}$$

$$+\Bigg((\mu_1+\dot{\mu}_1)\,\Big(\Big((a_2-1)\,\mu_1^3+\big((a_2-1)\,\dot{\mu}_1-\tfrac{3}{2}+a_2\big)\,\mu_1^2+\big(\tfrac{1}{2}+(a_2-2)\,\dot{\mu}_1\big)\,\mu_1+\tfrac{\dot{\mu}_1}{2}\Big)\sin(\alpha)$$

$$+\big(-\tfrac{a_2}{2}+\tfrac{1}{2}\big)\,\mu_1^3+\big(\big(-\tfrac{a_2}{2}+\tfrac{1}{2}\big)\,\dot{\mu}_1+\tfrac{7}{4}-\tfrac{a_2}{4}\big)\,\mu_1^2+\big(3\dot{\mu}_1-\tfrac{5}{4}-\tfrac{a_2}{4}\big)\,\mu_1-\tfrac{\dot{\mu}_1(a_2+5)}{4}\Big)\cos^2(\alpha)$$

$$+\Bigg(\big(\tfrac{a_2}{8}-\tfrac{3}{4}\big)\,\mu_1^3+\Big(\Big(\tfrac{3a_2}{8}-\tfrac{7}{4}\Big)\,\dot{\mu}_1+\tfrac{3}{4}-\tfrac{a_2}{8}\Big)\,\mu_1^2+\tfrac{\dot{\mu}_1(\dot{\mu}_1 a_2-a_2+4)\mu_1}{4}-\tfrac{\dot{\mu}_1^2(a_2+6)}{8}\Bigg)\sin(\alpha)$$

$$+\Big(\tfrac{3}{4}+\tfrac{7a_2}{8}\Big)\,\mu_1^3+\Big(\Big(\tfrac{19a_2}{8}+\tfrac{7}{4}\Big)\,\dot{\mu}_1-\tfrac{3}{4}-\tfrac{7a_2}{8}\Big)\,\mu_1^2+\dot{\mu}_1\,\big(\dot{\mu}_1 a_2-1-\tfrac{3}{2}a_2\big)\,\mu_1-\tfrac{\dot{\mu}_1^2(a_2-6)}{8}\Bigg)$$

$$\times \frac{256\sigma^8 \ddot{\mu}_1}{\mu_1^2\,(a_2+2)^2\,(a_2-2)\,(a_2+1)\,(\mu_1-1)^7\,(a_2-1)^2\,(\mu_1+\dot{\mu}_1)^5}$$

$$-\Big(-1+3\sin(\alpha)\Big)\frac{8\sigma^6\ddot{\mu}_1}{3\,(\mu_1-1)^3\,(\mu_1+\dot{\mu}_1)^3\,(-1+\sin(\alpha))}$$

$$-\Big((\mu_1+\dot{\mu}_1)\,\Big(\mu_1^2-\frac{2}{3}\mu_1+\dot{\mu}_1\Big)\sin^2(\alpha)-\tfrac{2}{3}\mu_1\,(\mu_1+\dot{\mu}_1)\sin(\alpha)+\big(\mu_1^2+(4\dot{\mu}_1-2)\,\mu_1+\dot{\mu}_1^2-4\dot{\mu}_1\big)\,\mu_1\Big)$$

$$\times \frac{32 S_1\,\sigma^7\ddot{\mu}_1}{(\mu_1-1)^5\,\mu_1^2\,(\mu_1+\dot{\mu}_1)^4\,a_1^2\,(-1+\sin(\alpha))}$$

$$+\Big(\cos^2(\alpha)+2\sin(\alpha)+2\Big)\frac{32\sigma^7\ddot{\mu}_1}{3\,(\mu_1-1)^5\,(\mu_1+\dot{\mu}_1)^3\,a_1^2\,(-1+\sin(\alpha))}$$

$$+\Bigg(\Big(\mu_1^3+\Big(\tfrac{11\dot{\mu}_1}{4}-2\Big)\,\mu_1^2+\big(\tfrac{3}{2}\dot{\mu}_1^2-\tfrac{7}{6}\dot{\mu}_1+1\big)\,\mu_1+\tfrac{5\dot{\mu}_1}{12}\Big)\,\mu_1\cos^2(\alpha)$$

$$-5\mu_1^4+\Big(-\tfrac{83\dot{\mu}_1}{8}+4\Big)\,\mu_1^3+\big(-\tfrac{39}{8}\dot{\mu}_1^2+\tfrac{19}{4}\dot{\mu}_1-5\big)\,\mu_1^2+\big(-\tfrac{1}{4}\dot{\mu}_1^2-\tfrac{27}{8}\dot{\mu}_1\big)\,\mu_1+\tfrac{\dot{\mu}_1^2}{8}\Bigg)$$

$$\times \frac{16 S_1\,\sigma^8}{(\mu_1+\dot{\mu}_1)^2\,\mu_1^3\,(\mu_1-1)^8}$$

$$+\Bigg(\mu_1(\mu_1^5+(3\dot{\mu}_1-2)\,\mu_1^4+\big(3\dot{\mu}_1^2-\tfrac{8}{3}\dot{\mu}_1+2\big)\,\mu_1^3+\big(\dot{\mu}_1^3-2+\tfrac{31}{6}\dot{\mu}_1-\tfrac{2}{3}\dot{\mu}_1^2\big)\,\mu_1^2+\big(\tfrac{8}{3}\dot{\mu}_1^2-\tfrac{5}{3}\dot{\mu}_1+1\big)\,\mu_1+\tfrac{\dot{\mu}_1}{6})\cos^2(\alpha)$$

$$-\tfrac{7\mu_1^6}{2}+\Big(-\tfrac{21\dot{\mu}_1}{2}+4\Big)\,\mu_1^5+\big(8\dot{\mu}_1-\tfrac{21}{2}\dot{\mu}_1^2-13\big)\,\mu_1^4+\big(4\dot{\mu}_1^2+4-\tfrac{99}{4}\dot{\mu}_1-\tfrac{7}{2}\dot{\mu}_1^3\big)\,\mu_1^3+\big(-\tfrac{43}{4}\dot{\mu}_1^2+\tfrac{11}{2}\dot{\mu}_1-\tfrac{7}{2}\big)\,\mu_1^2$$

$$+\big(-\tfrac{1}{2}\dot{\mu}_1^2-\tfrac{9}{4}\dot{\mu}_1\big)\,\mu_1+\tfrac{\dot{\mu}_1^2}{4}\Bigg)$$

$$\times \frac{8\sigma^8}{\mu_1^3\,(\mu_1+\dot{\mu}_1)^3\,(\mu_1-1)^8}$$

$$
\begin{aligned}
&+\Big(51\dot{\mu}_1^4\mu_1^2+234\dot{\mu}_1^3\mu_1^3+387\dot{\mu}_1^2\mu_1^4+264\dot{\mu}_1\,\mu_1^5+72\mu_1^6-6\dot{\mu}_1^4\mu_1-52\dot{\mu}_1^3\mu_1^2-118\dot{\mu}_1^2\mu_1^3-48\dot{\mu}_1\,\mu_1^4+3\dot{\mu}_1^4+34\dot{\mu}_1^3\mu_1\\
&\quad+115\dot{\mu}_1^2\mu_1^2+96\dot{\mu}_1\,\mu_1^3+24\mu_1^4\Big)\\
&\quad\times\frac{2\sigma^7 S_1}{3\left(\mu_1+\dot{\mu}_1\right)^4\mu_1^3\left(\mu_1-1\right)^6}\\
&+\left(3\dot{\mu}_1\,\mu_1^3+\mu_1^4+\left(3\dot{\mu}_1^2-2\dot{\mu}_1+3\right)\mu_1^2+\left(\dot{\mu}_1^3-\tfrac{2}{3}\dot{\mu}_1^2+6\dot{\mu}_1\right)\mu_1+\tfrac{5\dot{\mu}_1^2}{3}\right)\frac{16\sigma^7}{\left(\mu_1+\dot{\mu}_1\right)^3\mu_1\left(\mu_1-1\right)^6}\\
&-\Bigg(\tfrac{1}{2}\left(\mu_1+1\right)\left(\mu_1+\dot{\mu}_1\right)^2\left(\mu_1+\tfrac{\dot{\mu}_1}{2}\right)\mu_1\cos^6\left(\alpha\right)\\
&\quad+\Bigg(\bigg(-4\mu_1^5+\left(-8\dot{\mu}_1-9-\tfrac{a_1}{2}\right)\mu_1^4+\left(-\tfrac{9\dot{\mu}_1^2}{2}+\left(-\tfrac{147}{8}-\tfrac{9a_1}{8}\right)\dot{\mu}_1-\tfrac{7}{2}\right)\mu_1^3\\
&\qquad\quad-\left(\dot{\mu}_1^2+\left(\tfrac{5a_1}{4}+\tfrac{79}{4}\right)\dot{\mu}_1-\tfrac{a_1}{4}+\tfrac{57}{4}\right)\tfrac{\dot{\mu}_1\,\mu_1^2}{2}-\dot{\mu}_1^2\left(\dot{\mu}_1-\tfrac{a_1}{8}+\tfrac{37}{8}\right)\mu_1-\tfrac{\dot{\mu}_1^3}{2}\bigg)\sin(\alpha)\\
&\qquad+\left(a_1-1\right)\mu_1^5+\left(\left(-a_1+1\right)\dot{\mu}_1+\tfrac{41a_1}{8}-8\right)\mu_1^4+\left(\left(-\tfrac{17a_1}{4}+\tfrac{33}{4}\right)\dot{\mu}_1^2+\left(\tfrac{187a_1}{16}-\tfrac{151}{8}\right)\dot{\mu}_1+\tfrac{13a_1}{8}\right)\mu_1^3\\
&\qquad-\left(\left(-\tfrac{17}{9}+a_1\right)\dot{\mu}_1^2+\left(-\tfrac{133a_1}{36}+\tfrac{31}{6}\right)\dot{\mu}_1-\tfrac{67a_1}{36}+\tfrac{1}{2}\right)\tfrac{9\dot{\mu}_1\,\mu_1^2}{4}\\
&\qquad+\left(\left(a_1-1\right)\dot{\mu}_1+\tfrac{49a_1}{24}-\tfrac{53}{12}\right)\dot{\mu}_1^2\tfrac{3\mu_1}{2}+\left(a_1-\tfrac{11}{3}\right)\tfrac{3\dot{\mu}_1^3}{4}\Bigg)\left(\mu_1+\dot{\mu}_1\right)\cos^4\left(\alpha\right)\\
&\quad+\Bigg(\bigg(\left(-a_1-7\right)\mu_1^6+\left(\left(-5a_1-36\right)\dot{\mu}_1+14+\tfrac{25a_1}{4}\right)\mu_1^5\\
&\qquad\quad+\left(\left(-\tfrac{281}{4}-\tfrac{57a_1}{8}\right)\dot{\mu}_1^2+\left(55+\tfrac{351a_1}{16}\right)\dot{\mu}_1-\tfrac{5a_1}{4}-7\right)\mu_1^4\\
&\qquad\quad-\left(\left(a_1+\tfrac{226}{21}\right)\dot{\mu}_1^2+\left(-\tfrac{103a_1}{21}-\tfrac{290}{21}\right)\dot{\mu}_1+\tfrac{25a_1}{28}+\tfrac{76}{21}\right)\tfrac{21\dot{\mu}_1\mu_1^3}{4}\\
&\qquad\quad-\left(\left(a_1+\tfrac{106}{17}\right)\dot{\mu}_1^2+\left(-\tfrac{13a_1}{2}-\tfrac{280}{17}\right)\dot{\mu}_1+\tfrac{61a_1}{17}+\tfrac{74}{17}\right)\tfrac{17\dot{\mu}_1^2\mu_1^2}{8}\\
&\qquad\quad+\left(\left(a_1-\tfrac{2}{15}\right)\dot{\mu}_1-\tfrac{39a_1}{20}+2\right)\tfrac{15\dot{\mu}_1^3\mu_1}{4}-\left(a_1-\tfrac{54}{25}\right)\tfrac{25\dot{\mu}_1^4}{8}\bigg)\sin(\alpha)\\
&\qquad+\left(-8a_1-11\right)\mu_1^6+\left(\left(-\tfrac{223}{4}-\tfrac{141a_1}{4}\right)\dot{\mu}_1+\tfrac{47a_1}{4}+22\right)\mu_1^5\\
&\qquad+\left(\left(-105-\tfrac{449a_1}{8}\right)\dot{\mu}_1^2+\left(\tfrac{173}{2}+\tfrac{745a_1}{16}\right)\dot{\mu}_1-\tfrac{31a_1}{4}-11\right)\mu_1^4\\
&\qquad-\left(\left(a_1+\tfrac{325}{166}\right)\dot{\mu}_1^2+\left(-\tfrac{121a_1}{83}-\tfrac{457}{166}\right)\dot{\mu}_1+\tfrac{377a_1}{664}+\tfrac{123}{166}\right)\tfrac{83\dot{\mu}_1\mu_1^3}{2}\\
&\qquad-\left(\left(a_1+\tfrac{128}{109}\right)\dot{\mu}_1^2+\left(-\tfrac{571a_1}{218}-\tfrac{428}{109}\right)\dot{\mu}_1+\tfrac{195a_1}{109}+\tfrac{162}{109}\right)\tfrac{109\dot{\mu}_1^2\mu_1^2}{8}\\
&\qquad+12\dot{\mu}_1^3\left(\left(a_1-\tfrac{25}{48}\right)\dot{\mu}_1-\tfrac{215a_1}{192}+\tfrac{23}{48}\right)\mu_1-\tfrac{47}{8}\left(a_1-\tfrac{90}{47}\right)\dot{\mu}_1^4\Bigg)\cos^2\left(\alpha\right)\\
&\quad+\left(\left(a_1+2\right)\mu_1^3+\left(\left(\tfrac{7a_1}{2}+4\right)\dot{\mu}_1-a_1-2\right)\mu_1^2+\left(\left(-2+\tfrac{7a_1}{2}\right)\dot{\mu}_1^2+\left(-\tfrac{5a_1}{2}-2\right)\dot{\mu}_1\right)\mu_1-\tfrac{5}{2}\dot{\mu}_1^2\left(a_1-\tfrac{8}{5}\right)\right)\\
&\qquad\times\dot{\mu}_1\left(\mu_1+3\dot{\mu}_1\right)\left(\mu_1-1\right)\tfrac{3}{4}\left(1+\sin\left(\alpha\right)\right)\Bigg)\\
&\quad\times\frac{256S_1\,\sigma^8}{\left(\mu_1+\dot{\mu}_1\right)^4\left(a_1-1\right)^2\left(a_1+2\right)^2\left(a_1-2\right)\left(\mu_1-1\right)^8\mu_1^3a_1\left(a_1+1\right)}\\
&+\Bigg(-\left(\mu_1+1\right)\left(\mu_1+\dot{\mu}_1\right)^3\mu_1^2\cos^6\left(\alpha\right)\\
&\quad+\bigg(\left(\mu_1+\dot{\mu}_1\right)\mu_1\left(\left(a_1+19\right)\mu_1^3+\left(30+\left(a_1+11\right)\dot{\mu}_1\right)\mu_1^2+\left(16\dot{\mu}_1+a_1+17\right)\mu_1+8\dot{\mu}_1\right)\tfrac{\sin(\alpha)}{2}+\left(a_1-1\right)\mu_1^6\\
&\qquad+\left(\left(-3+3a_1\right)\dot{\mu}_1-\tfrac{9a_1}{2}+9\right)\mu_1^5+\left(\left(-3+3a_1\right)\dot{\mu}_1^2+\left(14-5a_1\right)\dot{\mu}_1-\tfrac{29a_1}{4}+4\right)\mu_1^4\\
&\qquad+\left(\dot{\mu}_1^3\left(a_1-1\right)+\left(-3-\tfrac{a_1}{2}\right)\dot{\mu}_1^2+\left(5-\tfrac{33a_1}{4}\right)\dot{\mu}_1+7-\tfrac{23a_1}{4}\right)\mu_1^3+\left(\left(15-\tfrac{55a_1}{4}\right)\dot{\mu}_1+a_1-1\right)\mu_1^2\\
&\qquad+\left(\left(-8a_1+16\right)\dot{\mu}_1^2+\left(2a_1-2\right)\dot{\mu}_1\right)\mu_1+\left(a_1-1\right)\dot{\mu}_1^2\bigg)\\
&\qquad\times\left(\mu_1+\dot{\mu}_1\right)^2\cos^4\left(\alpha\right)\\
&\quad+\Bigg(\bigg(\left(a_1+\tfrac{7}{2}\right)\mu_1^7+\left(\left(4a_1+14\right)\dot{\mu}_1-\tfrac{13a_1}{4}-7\right)\mu_1^6+\left(\left(21+6a_1\right)\dot{\mu}_1^2+\left(-\tfrac{43a_1}{4}-\tfrac{39}{2}\right)\dot{\mu}_1+\tfrac{a_1}{4}+7\right)\mu_1^5\\
&\qquad\quad+\left(\left(4a_1+14\right)\dot{\mu}_1^3+\left(-\tfrac{55a_1}{4}-12\right)\dot{\mu}_1^2+\left(2a_1+24\right)\dot{\mu}_1-3a_1-7\right)\mu_1^4\\
&\qquad\quad+\left(\left(a_1+\tfrac{7}{2}\right)\dot{\mu}_1^4+\left(-\tfrac{25a_1}{4}+\tfrac{1}{2}\right)\dot{\mu}_1^3+\left(\tfrac{11a_1}{4}+29\right)\dot{\mu}_1^2+\left(-8a_1-\tfrac{57}{2}\right)\dot{\mu}_1+a_1+\tfrac{7}{2}\right)\mu_1^3
\end{aligned}
$$

$$+\Big(11\dot\mu_1^3+\Big(-\tfrac{79}{2}-\tfrac{5a_1}{2}\Big)\dot\mu_1^2+\Big(10+\tfrac{5a_1}{2}\Big)\dot\mu_1\Big)\mu_1^2+\Big(\big(a_1-\tfrac{34}{7}\big)\dot\mu_1+\tfrac{2a_1}{7}+\tfrac{17}{7}\Big)\tfrac{7\dot\mu_1^2\mu_1}{2}-\tfrac{\dot\mu_1^3(a_1-4)}{2}\Big)$$

$$\times\sin(\alpha)$$

$$+\Big(\tfrac{7a_1}{2}+\tfrac{11}{2}\Big)\mu_1^7+\Big((14a_1+22)\dot\mu_1-\tfrac{23a_1}{4}-11\Big)\mu_1^6$$

$$+\Big((21a_1+33)\dot\mu_1^2+\Big(-\tfrac{77a_1}{4}-\tfrac{63}{2}\Big)\dot\mu_1+\tfrac{35a_1}{4}+11\Big)\mu_1^5$$

$$+\Big((14a_1+22)\dot\mu_1^3+\Big(-\tfrac{89a_1}{4}-24\Big)\dot\mu_1^2+\Big(\tfrac{111a_1}{4}+\tfrac{151}{4}\Big)\dot\mu_1-6a_1-11\Big)\mu_1^4$$

$$+\Big(\Big(\tfrac{7a_1}{2}+\tfrac{11}{2}\Big)\dot\mu_1^4+\Big(-\tfrac{35a_1}{4}-\tfrac{7}{2}\Big)\dot\mu_1^3+\big(29a_1+\tfrac{183}{4}\big)\dot\mu_1^2+\Big(-\tfrac{43a_1}{2}-44\Big)\dot\mu_1+\tfrac{7a_1}{2}+\tfrac{11}{2}\Big)\mu_1^3$$

$$+\Big(\big(a_1+\tfrac{7}{3}\big)\dot\mu_1^2+\Big(-\tfrac{71a_1}{30}-\tfrac{229}{30}\Big)\dot\mu_1+\tfrac{37a_1}{30}+\tfrac{21}{10}\Big)\tfrac{15\dot\mu_1\mu_1^2}{2}$$

$$+\tfrac{1}{4}\left((a_1-91)\dot\mu_1+22a_1+54\right)\dot\mu_1^2\mu_1-\tfrac{\dot\mu_1^3}{4}(a_1-13)\Big)$$

$$\times 2(\mu_1+\dot\mu_1)\cos^2(\alpha)$$

$$-\Big((a_1+2)\mu_1^3+\Big(\Big(\tfrac{8a_1}{3}+\tfrac{17}{3}\Big)\dot\mu_1-a_1-2\Big)\mu_1^2+\Big((a_1+3)\dot\mu_1-\tfrac{5a_1}{3}-\tfrac{11}{3}\Big)\dot\mu_1\mu_1-\dot\mu_1^2\Big)$$

$$\times\tfrac{3}{2}(1+\sin(\alpha))(\mu_1+3\dot\mu_1)\dot\mu_1(\mu_1-1)\Big)$$

$$\times\frac{128\sigma^8}{(\mu_1+\dot\mu_1)^5(a_1-1)^2(a_1+2)^2(a_1-2)(\mu_1-1)^8\mu_1^3a_1(a_1+1)}$$

$$-\Bigg(\Bigg(4(\mu_1+1)^2\big(\dot\mu_1\mu_1+\mu_1^2+\tfrac{1}{8}\dot\mu_1^2\big)(\mu_1+\dot\mu_1)\sin(\alpha)+(a_2-1)\mu_1^5-(a_2-1)(\dot\mu_1-6)\mu_1^4$$

$$+\Big(\Big(-\tfrac{17a_2}{4}+\tfrac{33}{4}\Big)\dot\mu_1^2+(-14+14a_2)\dot\mu_1+a_2-1\Big)\mu_1^3$$

$$-\tfrac{9}{4}\dot\mu_1\Big(\big(-\tfrac{17}{9}+a_2\big)\dot\mu_1^2+\Big(-\tfrac{38a_2}{9}+\tfrac{38}{9}\Big)\dot\mu_1-\tfrac{4a_2}{3}+\tfrac{4}{3}\Big)\mu_1^2$$

$$+\tfrac{3}{2}\dot\mu_1^2\Big((a_2-1)\dot\mu_1+\tfrac{11a_2}{6}-\tfrac{9}{2}\Big)\mu_1+\tfrac{3}{4}\big(a_2-\tfrac{11}{3}\big)\dot\mu_1^3\Big)$$

$$\times(\mu_1+\dot\mu_1)\cos^4(\alpha)$$

$$+\Big(\Big((a_2+7)\mu_1^6+((5a_2+36)\dot\mu_1-6a_2-14)\mu_1^5+\Big(\Big(\tfrac{57a_2}{8}+\tfrac{281}{4}\Big)\dot\mu_1^2+(-21a_2-55)\dot\mu_1+a_2+7\Big)\mu_1^4$$

$$+\Big(\Big(\tfrac{21a_2}{4}+\tfrac{113}{2}\Big)\dot\mu_1^3+\Big(-\tfrac{99a_2}{4}-\tfrac{145}{2}\Big)\dot\mu_1^2+(4a_2+19)\dot\mu_1\Big)\mu_1^3$$

$$+\Big(\big(a_2+\tfrac{106}{17}\big)\dot\mu_1^2+\Big(-\tfrac{108a_2}{17}-\tfrac{280}{17}\Big)\dot\mu_1+\tfrac{57a_2}{17}+\tfrac{74}{17}\Big)\tfrac{17\dot\mu_1^2\mu_1^2}{8}$$

$$-\tfrac{15}{4}\dot\mu_1^3\Big(\big(a_2-\tfrac{2}{15}\big)\dot\mu_1-\tfrac{29a_2}{15}+2\Big)\mu_1+\tfrac{25}{8}\dot\mu_1^4\big(a_2-\tfrac{54}{25}\big)\Big)$$

$$\times\sin(\alpha)$$

$$+(-8a_2-11)\mu_1^6+\Big(\Big(-\tfrac{141a_2}{4}-\tfrac{223}{4}\Big)\dot\mu_1+12a_2+22\Big)\mu_1^5$$

$$+\Big(\Big(-\tfrac{449a_2}{8}-105\Big)\dot\mu_1^2+\Big(\tfrac{95a_2}{2}+\tfrac{173}{2}\Big)\dot\mu_1-8a_2-11\Big)\mu_1^4$$

$$-\tfrac{83}{2}\dot\mu_1\Big(\big(a_2+\tfrac{325}{166}\big)\dot\mu_1^2+\Big(-\tfrac{123a_2}{83}-\tfrac{457}{166}\Big)\dot\mu_1+\tfrac{97a_2}{166}+\tfrac{123}{166}\Big)\mu_1^3$$

$$+\Big(\Big(-16-\tfrac{109a_2}{8}\Big)\dot\mu_1^4+\big(\tfrac{107}{2}+36a_2\big)\dot\mu_1^3+\Big(-\tfrac{81}{4}-\tfrac{199a_2}{8}\Big)\dot\mu_1^2\Big)\mu_1^2$$

$$+12\dot\mu_1^3\Big(\big(a_2-\tfrac{25}{48}\big)\dot\mu_1-\tfrac{9a_2}{8}+\tfrac{23}{48}\Big)\mu_1-\tfrac{47}{8}\dot\mu_1^4\big(a_2-\tfrac{90}{47}\big)\Big)$$

$$\times\cos^2(\alpha)$$

$$-\Bigg((a_2+2)\mu_1^3+\Big(\Big(\tfrac{7a_2}{2}+4\Big)\dot\mu_1-a_2-2\Big)\mu_1^2+\Big(\Big(\tfrac{7a_2}{2}-2\Big)\dot\mu_1^2+\Big(-\tfrac{5a_2}{2}-2\Big)\dot\mu_1\Big)\mu_1-\frac{5\dot\mu_1^2\big(a_2-\frac{8}{5}\big)}{2}\Bigg)$$

$$\times(\mu_1+3\dot\mu_1)\tfrac{3}{4}\dot\mu_1(-1+\sin(\alpha))(\mu_1-1)\Bigg)$$

$$\times\frac{256S_1\sigma^8}{\mu_1^3a_2(\mu_1+\dot\mu_1)^4(a_2-1)^2(\mu_1-1)^8(a_2+1)(a_2-2)(a_2+2)^2}$$

$$+\Bigg(\Bigg(-8(\mu_1+1)^2(\mu_1+\dot\mu_1)\mu_1\Big(\mu_1+\tfrac{\dot\mu_1}{2}\Big)\sin(\alpha)+(a_2-1)\mu_1^6+3(a_2-1)(\dot\mu_1-2)\mu_1^5$$

$$+3\big(\dot\mu_1^2-\tfrac{8}{3}\dot\mu_1-2\big)(a_2-1)\mu_1^4+\big((a_2-1)\dot\mu_1^3+(-2a_2-6)\dot\mu_1^2+(-7a_2+7)\dot\mu_1-6a_2+6\big)\mu_1^3$$

$$+((-14a_2+14)\dot\mu_1+a_2-1)\mu_1^2+\big((-8a_2+16)\dot\mu_1^2+(2a_2-2)\dot\mu_1\big)\mu_1+(a_2-1)\dot\mu_1^2\Big)$$

$$\times(\mu_1+\dot\mu_1)^2\cos^4(\alpha)$$

$$
\begin{aligned}
&- \Bigg(\Bigg(\left(a_2+\frac72\right)\mu_1^7 + ((4a_2+14)\dot\mu_1 - 3a_2 - 7)\mu_1^6 + \left((21+6a_2)\dot\mu_1^2 + \left(-10a_2-\frac{39}{2}\right)\dot\mu_1 + 7\right)\mu_1^5 \\
&\quad + \left((4a_2+14)\dot\mu_1^3 + (-13a_2-12)\dot\mu_1^2 + \left(\frac{3a_2}{2}+24\right)\dot\mu_1 - 3a_2 - 7\right)\mu_1^4 \\
&\quad + \left(\left(a_2+\frac72\right)\dot\mu_1^4 + \left(-6a_2+\frac12\right)\dot\mu_1^3 + \left(\frac{5a_2}{2}+29\right)\dot\mu_1^2 + \left(-8a_2-\frac{57}{2}\right)\dot\mu_1 + a_2 + \frac72\right)\mu_1^3 \\
&\quad + \left(11\dot\mu_1^3 + \left(-\frac{79}{2}-\frac{5a_2}{2}\right)\dot\mu_1^2 + \left(10+\frac{5a_2}{2}\right)\dot\mu_1\right)\mu_1^2 \\
&\quad + \frac72\left(\left(a_2-\frac{34}{7}\right)\dot\mu_1 + \frac{2a_2}{7}+\frac{17}{7}\right)\dot\mu_1^2\mu_1 - \frac{\dot\mu_1^3}{2}(a_2-4)\Bigg) \\
&\quad \times\sin(\alpha) \\
&\quad + \left(-\frac{7a_2}{2}-\frac{11}{2}\right)\mu_1^7 + ((-14a_2-22)\dot\mu_1 + 6a_2+11)\mu_1^6 \\
&\quad + \left((-33-21a_2)\dot\mu_1^2 + \left(20a_2+\frac{63}{2}\right)\dot\mu_1 - 9a_2 - 11\right)\mu_1^5 \\
&\quad + \left((-14a_2-22)\dot\mu_1^3 + (23a_2+24)\dot\mu_1^2 + \left(-\frac{113a_2}{4}-\frac{151}{4}\right)\dot\mu_1 + 6a_2+11\right)\mu_1^4 \\
&\quad + \left(\left(-\frac{7a_2}{2}-\frac{11}{2}\right)\dot\mu_1^4 + \left(9a_2+\frac72\right)\dot\mu_1^3 + \left(-\frac{117a_2}{4}-\frac{183}{4}\right)\dot\mu_1^2 + \left(\frac{43a_2}{2}+44\right)\dot\mu_1 - \frac{7a_2}{2}-\frac{11}{2}\right)\mu_1^3 \\
&\quad - \left(\left(a_2+\frac73\right)\dot\mu_1^2 + \left(-\frac{71a_2}{30}-\frac{229}{30}\right)\dot\mu_1 + \frac{37a_2}{30}+\frac{21}{10}\right)\frac{15\dot\mu_1\,\mu_1^2}{2} \\
&\quad - \frac14((a_2-91)\dot\mu_1 + 22a_2+54)\dot\mu_1^2\mu_1 + \frac{\dot\mu_1^3}{4}(a_2-13)\Bigg) \\
&\quad \times 2(\mu_1+\dot\mu_1)\cos^2(\alpha) \\
&\quad + \left((a_2+2)\mu_1^3 + \left(\left(\frac{8a_2}{3}+\frac{17}{3}\right)\dot\mu_1 - a_2 - 2\right)\mu_1^2 + \left((a_2+3)\dot\mu_1 - \frac{5a_2}{3}-\frac{11}{3}\right)\dot\mu_1\mu_1 - \dot\mu_1^2\right) \\
&\quad \times\frac32(\mu_1+3\dot\mu_1)(\mu_1-1)(-1+\sin(\alpha))\dot\mu_1\Bigg) \\
&\quad \times\frac{128\sigma^8}{\mu_1^3 a_2(\mu_1+\dot\mu_1)^5(a_2-1)^2(\mu_1-1)^8(a_2+1)(a_2-2)(a_2+2)^2} \\
&- \left((\mu_1^3+\dot\mu_1\sigma - 2\mu_1^2+\mu_1)\sin(\alpha) - \left(\mu_1+\frac{\dot\mu_1}{2}\right)\sigma\left(\cos^2(\alpha)+2\right)\right)\frac{4S_1\sigma^5}{\mu_1(\mu_1-1)^4(\mu_1+\dot\mu_1)} \\
&- \Bigg((\mu_1+\dot\mu_1)\left(\left(-\mu_1^2-\frac56\dot\mu_1\mu_1-\frac16\dot\mu_1\right)\sin(\alpha)+\mu_1^3+(-2+2\dot\mu_1)\mu_1^2+\left(\dot\mu_1^2-\frac56\dot\mu_1+2\right)\mu_1-\frac{7\dot\mu_1}{6}\right)\cos^2(\alpha) \\
&\quad + \left(-3\dot\mu_1\,\mu_1^3 - \mu_1^4 + \left(-3\dot\mu_1^2+\frac53\dot\mu_1\right)\mu_1^2 + \left(-\dot\mu_1^3-\frac13\dot\mu_1^2-\frac53\dot\mu_1\right)\mu_1 + \frac{\dot\mu_1^2}{3}\right)\sin(\alpha) \\
&\quad + \mu_1^4 + 3\dot\mu_1\,\mu_1^3 + \left(-\frac73\dot\mu_1+3\dot\mu_1^2\right)\mu_1^2 + \left(\frac73\dot\mu_1-\frac53\dot\mu_1^2+\dot\mu_1^3\right)\mu_1 + \frac{5\dot\mu_1^2}{3}\Bigg) \\
&\quad \times\frac{8\sigma^6}{(\mu_1+\dot\mu_1)^3(\mu_1-1)^4\mu_1(-1+\sin(\alpha))} \\
&- \Bigg((\mu_1+\dot\mu_1)^2\left(\mu_1^4+\left(\frac{3\dot\mu_1}{2}+1\right)\mu_1^3+\left(\frac34\dot\mu_1^2+\frac13\dot\mu_1\right)\mu_1^2+\left(-\frac12\dot\mu_1^2+\frac23\dot\mu_1\right)\mu_1+\frac{\dot\mu_1^2}{4}\right)\cos^4(\alpha) \\
&\quad + \Bigg(-\frac16(\mu_1+\dot\mu_1)\left((\dot\mu_1-6)\mu_1^3+\left(\frac32\dot\mu_1^2-4\dot\mu_1+2\right)\mu_1^2+(-\dot\mu_1^2-2\dot\mu_1)\mu_1+\frac{\dot\mu_1^2}{2}\right)\sin(\alpha)+\mu_1^5+\left(\frac{23\dot\mu_1}{6}-1\right)\mu_1^4 \\
&\quad + \left(\frac{25}{4}\dot\mu_1^2-\frac{14}{3}\dot\mu_1+\frac23\right)\mu_1^3 + \left(\frac{11}{4}\dot\mu_1^3-\frac{137}{18}\dot\mu_1^2+\frac73\dot\mu_1\right)\mu_1^2+\left(-\frac{13}{6}\dot\mu_1^3+\frac{109}{36}\dot\mu_1^2\right)\mu_1+\frac{\dot\mu_1^3}{4}\Bigg) \\
&\quad \times 6(\mu_1+\dot\mu_1)\cos^2(\alpha) \\
&\quad - \left(-\frac{\dot\mu_1}{3}(\mu_1+\dot\mu_1)\sin(\alpha)+\mu_1^3+\left(\frac{11\dot\mu_1}{4}-\frac43\right)\mu_1^2+\left(\frac54\dot\mu_1^2-\frac{25}{12}\dot\mu_1\right)\mu_1-\frac{\dot\mu_1^2}{4}\right)4(\mu_1+3\dot\mu_1)(\mu_1-1)\dot\mu_1\Bigg) \\
&\quad \times\frac{16S_1\,\sigma^7}{(\mu_1-1)^6\mu_1^3(\mu_1+\dot\mu_1)^4a_1^2(-1+\sin(\alpha))} \\
&+ \Bigg(-\left(\mu_1^2+\left(\frac{\dot\mu_1}{3}+1\right)\mu_1+\frac{7\dot\mu_1}{6}\right)(\mu_1+\dot\mu_1)\frac{\cos^4(\alpha)}{2} \\
&\quad + \Bigg((\mu_1+\dot\mu_1)\left(\mu_1^3+(2\dot\mu_1-3)\mu_1^2+(\dot\mu_1^2-2\dot\mu_1)\mu_1-\frac{3\dot\mu_1}{2}\right)\sin(\alpha)-2\mu_1^4+(-6\dot\mu_1+3)\mu_1^3 \\
&\quad + (-6\dot\mu_1^2+8\dot\mu_1-3)\mu_1^2+\left(-\frac{11}{2}\dot\mu_1+\frac{13}{3}\dot\mu_1^2-2\dot\mu_1^3\right)\mu_1-\frac{11\dot\mu_1^2}{6}\Bigg)\cos^2(\alpha) \\
&\quad - \frac23\dot\mu_1(1+\sin(\alpha))(\mu_1-1)(\mu_1+3\dot\mu_1)\Bigg) \\
&\quad \times\frac{32\sigma^7}{\mu_1(\mu_1-1)^6(\mu_1+\dot\mu_1)^3a_1^2(-1+\sin(\alpha))} + o(1),
\end{aligned}
$$

where

$$a_1^2 = 2 + 2\sin\alpha, \qquad a_2^2 = 2 - 2\sin\alpha, \qquad a_2^2 = 4 - a_1^2,$$

as $\beta_0 \to 0$.

The results of this calculation for constant μ are shown in Figure 3.8.

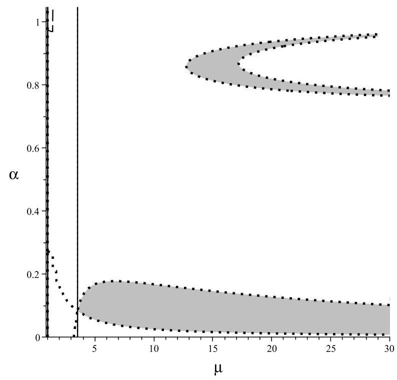


Figure 3.8:
The shaded areas of (μ, α)-parameter space (for constant μ and deep fluids) in which symmetric corner defects exist. The solid, dotted and dashed lines delineate the regions in which $c_1 < 0, d_1 \in (1, 4 + \sqrt{13})$ and $d_2 > -d_3$.

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