Arch. Math. 121 (2023), 437–447

0003-889X/23/040437-11
published online August 18, 2023
https://doi.org/10.1007/s00013-023-01908-4

Archiv der Mathematik

A small remark on Bernstein's theorem

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Abstract. We investigate splitting-type variational problems with some linear growth conditions. For balanced solutions of the associated Euler–Lagrange equation, we receive a result analogous to Bernstein's theorem on non-parametric minimal surfaces. Without assumptions of this type, Bernstein's theorem cannot be carried over to the splitting case, which follows from an elementary counterexample. We also include some modifications of our main theorem.

Mathematics Subject Classification. 49Q20, 49Q05, 53A10, 35J20.

Keywords. Bernstein's theorem, Non-parametric minimal surfaces, Two-dimensional variational problems, Splitting-type functionals.

1. Introduction. A famous theorem of Bernstein (see [1]) states that a smooth solution $u = u(x), x = (x_1, x_2)$, of the non-parametric minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{1}$$

defined on the whole plane must be an affine function. Letting $f_0(P) := \sqrt{1+|P|^2}, P \in \mathbb{R}^2$, the validity of (1) on some domain $\Omega \subset \mathbb{R}^2$ just expresses the fact that u is a solution of the Euler–Lagrange equation associated to the area functional

$$J_0[u,\Omega] := \int_\Omega f_0(\nabla u)\,\mathrm{d}x. \tag{2}$$

For a general overview on minimal surfaces, variational integrals with linear growth, and for a careful analysis of Bernstein's theorem, the reader is referred for instance to [4–6,8,11,12] and the references quoted therein.

We ask the following question: does Bernstein's theorem extend to the case when the area integrand $f_0(\nabla u)$ is replaced by the energy density $\sqrt{1+(\partial_1 u)^2}+$

$\sqrt{1+(\partial_2 u)^2}$ being also of linear growth with respect to $|\nabla u|$ but without any obvious geometric meaning?

We like to mention that in the case of superlinear growth it is a familiar question to study problems with anisotropic behaviour, see, e.g., the counterexample of [7] or the discussion of anisotropic stiff materials in [9], while there are only few contributions to linear growth energies of splitting type, see [10]. An extension to the mixed linear-superlinear splitting type case is given in [2].

Here we let for $P = (p_1, p_2) \in \mathbb{R}^2$,

$$f(P) := f_1(p_1) + f_2(p_2) \tag{3}$$

with functions $f_i \in C^2(\mathbb{R}), i = 1, 2$, satisfying

$$0 < f_i''(t) \le C_i(1+t^2)^{-\frac{\mu_i}{2}}, t \in \mathbb{R}, \tag{4}$$

for numbers $C_i > 0$ and with exponents

$$\mu_i > 1. \tag{5}$$

Note that (4) implies the strict convexity of f and on account of (5), the density f is of linear growth in the sense that

$$|f_i(t)| \le a|t| + b, \quad t \in \mathbb{R},$$

for some constants $a, b > 0$. For a discussion of the properties of densities f satisfying (3)–(5), we refer to [10]. We then replace (1) by the equation

$$\operatorname{div}(Df(\nabla u)) = 0 \tag{6}$$

and observe that the non-affine function $(\alpha, \beta, \gamma, \delta \in \mathbb{R})$

$$w(x_1, x_2) := \alpha x_1 x_2 + \beta x_1 + \gamma x_2 + \delta \tag{7}$$

is an entire solution of Eq. (6), in other words: the classical version of Bernstein's theorem does not extend to the splitting case. The behaviour of the function w defined in (7) is characterized in

Definition 1.1. A function $u \in C^1(\mathbb{R}^2), u = u(x_1, x_2)$, is called unbalanced if and only if both of the following conditions hold:

$$\limsup_{|x|\to\infty} \frac{|\partial_1 u(x)|}{1+|\partial_2 u(x)|} = \infty, \tag{8}$$

$$\limsup_{|x|\to\infty} \frac{|\partial_2 u(x)|}{1+|\partial_1 u(x)|} = \infty. \tag{9}$$

Otherwise we say that u is of balanced form.

Remark 1.2. Condition (8) for example means that there exists a sequence of points $x_n \in \mathbb{R}^2$ such that $|x_n| \to \infty$ and for which

$$\lim_{n\to\infty} \frac{|\partial_1 u(x_n)|}{1+|\partial_2 u(x_n)|} = \infty.$$

Remark 1.3. If for instance (8) is violated, no such sequence exists. Thus we can find constants $R, M > 0$ such that $|\partial_1 u(x)| \le M(1 + |\partial_2 u(x)|)$ for all $|x| \ge R$. Since u is of class $C^1(\mathbb{R}^2)$, this just shows

$$|\partial_1 u(x)| \le m|\partial_2 u(x)| + M, \quad x \in \mathbb{R}^2, \tag{10}$$

with suitable new constants $m, M > 0$.

Now we can state the appropriate version of Bernstein's theorem in the above setting:

Theorem 1.4. *Let (3)–(5) hold and let $u \in C^2(\mathbb{R}^2)$ denote a solution of (6) on the entire plane. Then u is an affine function or of unbalanced type.*

Remark 1.5. We do not know if (7) is the only entire unbalanced solution of (6).

Before proving Theorem 1.4, we formulate some related results: in Theorem 1.7 below, we can slightly improve the result of Theorem 1.4 by adjusting the notation introduced in Definition 1.1 and by taking care of the growth rates of the second derivatives f_i'' (compare (4)).

Definition 1.6. Let $\mu := (\mu_1, \mu_2)$ with numbers $\mu_i > 1$, $i = 1, 2$. A function $u \in C^1(\mathbb{R}^2)$ is called μ-balanced if we can find a positive constant c and a number $\rho > 0$ such that at least one of the following inequalities holds:

$$|\partial_1 u| \le c(|\partial_2 u|^{\rho\mu_2} + 1), \tag{11}$$
$$|\partial_2 u| \le c(|\partial_1 u|^{\rho\mu_1} + 1), \tag{12}$$

where in case (11) we require $\rho \in (1/\mu_2, 1)$, whereas in case (12) $\rho \in (1/\mu_1, 1)$ must hold.

Note that for example (11) is a weaker condition in comparison to (10).

The extension of Theorem 1.4 reads as follows:

Theorem 1.7. *Let (3)–(5) hold and let $u \in C^2(\mathbb{R}^2)$ denote an entire solution of (6). If the function u is μ-balanced, then it must be affine.*

In Theorem 1.8, we suppose that $|\partial_1 u|$ is controlled in x_2-direction and from this, we derive a smallness condition in x_1-direction—at least for a suitable sequence satisfying $|x_1| \to \infty$. The idea of proving Theorem 1.8 again is of Bernstein-type in the sense that the proof follows the ideas of Theorem 1.4 combined with a splitting structure of the test functions.

Theorem 1.8. *Let (3)–(5) hold and let $u \in C^2(\mathbb{R}^2)$ denote a solution of (6) on the entire plane. Suppose that there exist real numbers $\kappa_1 > 0$, $0 \le \kappa_2 < 1$ such that*

$$\mu_1 > 1 + \frac{1}{\kappa_1}\frac{\kappa_2}{1 - \kappa_2} \tag{13}$$

and such that, with a constant $k > 0$,

$$\sup \frac{|\partial_1 u(x_1, x_2)|}{|x_2|^{\kappa_2}} \le k. \tag{14}$$

Then we have

$$\liminf \frac{|\partial_1 u(x_1, x_2)|}{|x_1|^{\kappa_1}} = 0. \tag{15}$$

More precisely, by (13), we choose ρ such that

$$\frac{\kappa_2}{1-\kappa_2} < \rho < \kappa_1(\mu_1 - 1). \tag{16}$$

Then the sup *in (14) is taken in the set*

$$M_{2,\rho} := \Big\{(x_1, x_2) \colon |x_1| \le 2|x_2|^{1/(1+\rho)},\, 1 \le |x_2|\Big\}$$

and the lim inf *in (15) is taken with respect to sequences*

$$(x_1^{(n)}, x_2^{(n)}) \in M_{1,\rho} := \big\{(x_1, x_2) \colon |x_2| \le 2|x_1|^{1+\rho},\, 1 \le |x_1|\big\}$$

such that $|x_1^{(n)}| \to \infty$.

Our final Bernstein-type result is given in Theorem 1.9. Here a formulation in terms of the densities f_i is presented without requiring an upper bound for the second derivatives f_i'' in terms of some negative powers (see (4)).

Theorem 1.9. *Suppose that $f_i \in C^2(\mathbb{R})$, $i = 1, 2$, satisfies $f_i''(t) > 0$ for all $t \in \mathbb{R}$ and $f_i' \in L^\infty(\mathbb{R})$. Let $u \in C^2(\mathbb{R}^2)$ denote an entire solution of (6), i.e., it holds*

$$0 = \operatorname{div}(Df(\nabla u)) = f_1''(\partial_1 u)\partial_{11}u + f_2''(\partial_2 u)\partial_{22}u \quad \textit{on } \mathbb{R}^2.$$

If

$$\Theta := \frac{f_2''(\partial_2 u)}{f_1''(\partial_1 u)} \in L^\infty(\mathbb{R}^2) \quad \textit{or} \quad \frac{1}{\Theta} \in L^\infty(\mathbb{R}^2),$$

then u is an affine function.

In the next section, we prove our main Theorem 1.4 while in Section 3 the variants mentioned above are established.

2. Proof of Theorem 1.4. Our arguments make essential use of a Caccioppoli-type inequality involving negative exponents. This result was first introduced in [10]. We refer to the presentation given in [3, Section 6], where Proposition 6.1 applies to the situation at hand. Let us assume that the conditions (3)–(5) hold and that u is an entire solution of equation (6) being not necessarily of balanced type.

Lemma 2.1 (see [3, Prop. 6.1]). *Fix $l \in \mathbb{N}$ and suppose that $\eta \in C_0^\infty(\Omega)$, $0 \le \eta \le 1$, where Ω is a domain in $\mathbb{R}^2$. Then the inequality*

$$\int_\Omega D^2 f(\nabla u)(\nabla \partial_i u, \nabla \partial_i u)\eta^{2l}\Gamma_i^\alpha \, \mathrm{d}x \le \int_\Omega D^2 f(\nabla u)(\nabla \eta, \nabla \eta)\eta^{2l-2}\Gamma_i^{\alpha+1} \, \mathrm{d}x, \quad \Gamma_i := 1 + |\partial_i u|^2, \tag{17}$$

holds for any $\alpha > -1/2$ and for any fixed $i = 1, 2$.

Here and in what follows, the letter c denotes finite positive constants whose value may vary from line to line but being independent of the radius.

Assume next that the solution u is balanced and without loss of generality let u satisfy (10). In order to show that u is affine, we return to inequality (17), choose $i = 1$ and fix some function $\eta \in C_0^\infty(B_{2R}(0))$ according to $\eta \equiv 1$ on $B_R(0)$, $|\nabla \eta| \le c/R$. Then (17) yields for any exponent $\alpha \in (-1/2, \infty)$ and with the choice $l = 1$ ($B_r := B_r(0)$, $r > 0$),

$$\begin{aligned}
&\int_{B_{2R}} D^2 f(\nabla u)(\nabla \partial_1 u, \nabla \partial_1 u)\eta^2 \Gamma_1^\alpha \, \mathrm{d}x \\
&\le c \int_{B_{2R}} D^2 f(\nabla u)(\nabla \eta, \nabla \eta)\Gamma_1^{\alpha+1} \, \mathrm{d}x \\
&\overset{(3)}{=} c \int_{B_{2R}-B_R} \left(f_1''(\partial_1 u)|\partial_1 \eta|^2 + f_2''(\partial_2 u)|\partial_2 \eta|^2 \right) \Gamma_1^{\alpha+1} \, \mathrm{d}x \\
&\le cR^{-2} \left(\int_{B_{2R}-B_R} f_1''(\partial_1 u)\Gamma_1^{\alpha+1} \, \mathrm{d}x + \int_{B_{2R}-B_R} f_2''(\partial_2 u)\Gamma_1^{\alpha+1} \, \mathrm{d}x \right) \\
&\overset{(4)}{\le} cR^{-2} \left(\int_{B_{2R}-B_R} \Gamma_1^{\alpha+1-\frac{\mu_1}{2}} \, \mathrm{d}x + \int_{B_{2R}-B_R} \Gamma_2^{-\frac{\mu_2}{2}} \Gamma_1^{\alpha+1} \, \mathrm{d}x \right).
\end{aligned} \tag{18}$$

Recall (5) and choose α according to

$$\alpha \in \left(-1/2, \min\left\{ -1 + \frac{\mu_1}{2}, -1 + \frac{\mu_2}{2} \right\} \right). \tag{19}$$

Here we note that – depending on the values of μ_1 and μ_2 – actually a negative exponent α can occur. It follows from (10) that

$$\begin{aligned}
&cR^{-2} \left(\int_{B_{2R}-B_R} \Gamma_1^{\alpha+1-\frac{\mu_1}{2}} \, \mathrm{d}x + \int_{B_{2R}-B_R} \Gamma_2^{-\frac{\mu_2}{2}} \Gamma_1^{\alpha+1} \, \mathrm{d}x \right) \\
&\le cR^{-2} \int_{B_{2R}-B_R} c \, \mathrm{d}x \le c < \infty,
\end{aligned} \tag{20}$$

recalling that c is independent of R. Combining (20) and (18), it is obvious that (by passing to the limit $R \to \infty$)

$$\int_{\mathbb{R}^2} D^2 f(\nabla u)(\nabla \partial_1 u, \nabla \partial_1 u)\Gamma_1^{\alpha}\,\mathrm{d}x < \infty \tag{21}$$

for α satisfying (19).

As in the proof of [3, Proposition 6.1] from (with $l = 1$) and by applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} &\int_{B_{2R}} D^2 f(\nabla u)(\nabla \partial_1 u, \nabla \partial_1 u)\eta^2\Gamma_1^{\alpha}\,\mathrm{d}x \\ &\quad\leq \left| \int_{B_{2R}} D^2 f(\nabla u)(\nabla \partial_1 u, \nabla \eta^2)\partial_1 u \Gamma_1^{\alpha}\,\mathrm{d}x \right| \\ &\quad\leq c \left[\int_{\operatorname{spt}\nabla\eta} D^2 f(\nabla u)(\nabla \partial_1 u, \nabla \partial_1 u)\eta^2\Gamma_1^{\alpha}\,\mathrm{d}x \right]^{\frac{1}{2}} \\ &\qquad \left[\int_{\operatorname{spt}\nabla\eta} D^2 f(\nabla u)(\nabla \eta, \nabla \eta)\Gamma_1^{\alpha+1}\,\mathrm{d}x \right]^{\frac{1}{2}}. \end{aligned} \tag{22}$$

The second integral on the right-hand side is bounded on account of our previous calculations. Because of the validity of (21), the limit of the first integral for $R \to \infty$ is 0. Thus (22) implies

$$\nabla \partial_1 u \equiv 0. \tag{23}$$

In particular, (23) guarantees the existence of a number $a \in \mathbb{R}$ such that

$$\partial_1 u \equiv a. \tag{24}$$

From (24), we obtain

$$u(x_1, x_2) - u(0, x_2) = \int_0^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2)\,\mathrm{d}t = \int_0^{x_1} a\,\mathrm{d}t = a\,x_1,$$

implying

$$u(x_1, x_2) = u(0, x_2) + a\,x_1. \tag{25}$$

Considering (24) again, equation (6) reduces to

$$\partial_2 \left(f_2'(\partial_2 u)\right) = 0. \tag{26}$$

We set $\varphi(t) := u(0, t)$ and interpret the PDE (26) as the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f_2'(\varphi'(t))\right) = 0 \tag{27}$$

implying

$$f_2'(\varphi'(t))) = \mathrm{const}.$$

Since f_2' is strictly monotonically increasing, this just means

$$\varphi'(t) = b, \quad t \in \mathbb{R},$$

for some real number b, which consequently gives

$$u(x_1, x_2) = a\, x_1 + b\, x_2 + c, \quad a, b, c \in \mathbb{R}, \tag{28}$$

completing our proof. □

3. Remaining proofs. Ad Theorem 1.7. Let the assumptions of Theorem 1.7 hold and assume without loss of generality that we have inequality (11) from Definition 1.6. Consider the mixed term in the last line of (18) and note that on account of (11), we may estimate

$$\Gamma_2^{-\frac{\mu_2}{2}} \Gamma_1^{\alpha+1} \le c \Gamma_2^{-\frac{\mu_2}{2}(1-2\rho(\alpha+1))}. \tag{29}$$

The validity of $\rho < 1$ allows us to choose α sufficiently close to $-1/2$ such that $1 - 2\rho(\alpha+1) > 0$ which again yields (21) and allows us to proceed as before giving our claim.

□

Ad Theorem 1.8. Suppose by contradiction that there exists a real number $\hat{c} > 0$ such that, with respect to the set $M_{1,\rho}$,

$$\hat{c} \le \liminf_{|x_1| \to \infty} \frac{|\partial_1 u(x_1, x_2)|}{|x_1|^{\kappa_1}}. \tag{30}$$

For intervals $I_1, I_2 \subset \mathbb{R}$ we let

$$S_{I_1;I_2} := \{x \in \mathbb{R}^2 : |x_1| \in I_1, |x_2| \in I_2\}.$$

We fix $0 < R_1 < R_2$ and consider

$$\begin{gathered} \eta \in C_0^\infty(S_{[0,2R_1);[0,2R_2)}), \quad 0 \le \eta \le 1, \quad \eta \equiv 1 \quad \text{on} \quad S_{[0,2R_1);[0,2R_2)}\,, \\ \operatorname{spt}\partial_1\eta \subset S_{(R_1,2R_1);[0,2R_2)}\,, \quad \operatorname{spt}\partial_2\eta \subset S_{[0,2R_1);(R_2,2R_2)}\,, \qquad (31) \\ |\partial_1\eta| \le c/R_1\,, \quad |\partial_2\eta| \le c/R_2. \qquad (32) \end{gathered}$$

Exactly as in (18), one obtains, using (31) and (32),

$$\begin{aligned} &\int_{S_{[0,2R_1);[0,2R_2)}} D^2 f(\nabla u)(\nabla\partial_1 u, \nabla\partial_1 u)\eta^2 \Gamma_1^\alpha \,\mathrm{d}x \\ &\le \int_{S_{[0,2R_1);[0,2R_2)}} D^2 f(\nabla u)(\nabla\eta, \nabla\eta)\eta^2 \Gamma_1^{1+\alpha} \,\mathrm{d}x \\ &\le \frac{c}{R_1^2} \int_{S_{(R_1,2R_1);[0,2R_2)}} \Gamma_1^{\alpha+1-\frac{\mu_1}{2}} \,\mathrm{d}x + \frac{c}{R_2^2} \int_{S_{[0,2R_1);(R_2,2R_2)}} \Gamma_2^{-\frac{\mu_2}{2}} \Gamma_1^{\alpha+1} \,\mathrm{d}x. \end{aligned} \tag{33}$$

By definition, we have

$$|S_{[0,2R_1);(R_2,2R_2)}| \le c R_1 R_2.$$

Moreover, our assumption $\kappa_2 < 1$ implies that α can be chosen such that, in the case $\kappa_1 > 0$,

$$-\frac{1}{2} < \alpha < \frac{1}{2\kappa_2} - 1. \tag{34}$$

In the case $\kappa_2 = 0$, we do not need an additional condition. We apply assumption (14), which leads to

$$\begin{aligned} \frac{1}{R_2^2} \int\limits_{S_{[0,2R_1);(R_2,2R_2)}} \Gamma_2^{-\frac{\mu_2}{2}} \Gamma_1^{\alpha+1} \,\mathrm{d}x &\le \frac{c}{R_2^2} \int\limits_{S_{[0,2R_1);(R_2,2R_2)}} |x_2|^{2\kappa_2(\alpha+1)} \,\mathrm{d}x \\ &\le c R_1 R_2^{2\kappa_2(\alpha+1)-1}. \end{aligned} \tag{35}$$

Let us consider the first integral on the right-hand side of (33) recalling that $\alpha + 1 - \mu_1/2 < 0$. Assumption (30) implies

$$\begin{aligned} \frac{1}{R_1^2} \int\limits_{S_{(R_1,2R_1);[0,2R_2)}} \Gamma_1^{\alpha+1-\frac{\mu_1}{2}} \,\mathrm{d}x &\le \frac{c}{R_1^2} \int\limits_{(R_1,2R_1);[0,2R_2)} |x_1|^{2\kappa_1(\alpha+1-\frac{\mu_1}{2})} \,\mathrm{d}x \\ &\le c R_2 R_1^{-\kappa_1(\mu_1 - 2(\alpha+1))-1}, \end{aligned} \tag{36}$$

where we suppose that $\mu_1 > 2(\alpha+1)$ by choosing α sufficiently close to $-1/2$. If we further suppose that

$$R_2 = R_1^{1+\rho} \quad \text{with a positive real number} \quad \rho < \kappa_1(\mu_1 - 1), \tag{37}$$

then by decreasing α, if necessary, still satisfying $\alpha > -1/2$, we obtain from (36),

$$\frac{1}{R_1^2} \int\limits_{S_{(R_1,2R_1);[0,2R_2)}} \Gamma_1^{\alpha+1-\frac{\mu_1}{2}} \,\mathrm{d}x \to 0 \quad \text{as} \quad R_1 \to \infty. \tag{38}$$

Using $R_2 = R_1^{1+\rho}$ (recall (37)), we return to (35) recalling that, by the choice (34), we have $2\kappa_2(\alpha+1) - 1 < 0$. We calculate

$$R_1 R_1^{(1+\rho)(2\kappa_2(\alpha+1)-1)} = R_1^{2\kappa_2(\alpha+1)+\rho(2\kappa_2(\alpha+1)-1)}. \tag{39}$$

If we suppose that

$$\kappa_2 + \rho(\kappa_2 - 1) < 0, \tag{40}$$

then we may choose $\alpha > -1/2$ sufficiently small such that the exponent on the right-hand side of (39) is negative, hence together with (35), we have

$$\frac{1}{R_2^2} \int\limits_{S_{[0,2R_1);(R_2,2R_2)}} \Gamma_2^{-\frac{\mu_2}{2}} \Gamma_1^{\alpha+1} \,\mathrm{d}x \to 0 \quad \text{as} \quad R_1 \to \infty. \tag{41}$$

By (33), (38), and (41), it follows that

$$\int\limits_{S_{[0,2R_1);[0,2R_2)}} D^2 f(\nabla u)(\nabla \partial_1 u, \nabla \partial_1 u) \eta^2 \Gamma_1^{\alpha} \,\mathrm{d}x \to 0 \quad \text{as} \quad R_1 \to \infty, \tag{42}$$

provided that we have (37) and (40), i.e., provided that we have (16) which is a consequence of (13). Hence we have (42) which exactly as in the proof of

Theorem 1.4 shows that u has to be an affine function and this contradicts (30) which in turn proves Theorem 1.8.

□

ad Theorem 1.9. Without loss of generality, we suppose that $\Theta \in L^\infty(\mathbb{R}^2)$ and that $u \in C^3(\mathbb{R}^2)$, f_1, $f_2 \in C^3(\mathbb{R})$. Otherwise we argue in a weak sense. Let

$$w_i := f_i'(\partial_i u), \quad i = 1, 2.$$

Then we have

$$\partial_1 w_1 + \partial_2 w_2 = 0 \quad \text{on } \mathbb{R}^2, \tag{43}$$

hence

$$\partial_{11} w_1 + \partial_1 \partial_2 w_2 = \partial_{11} w_1 + \partial_2 \partial_1 w_2 = 0 \quad \text{on } \mathbb{R}^2. \tag{44}$$

A direct calculation shows

$$\partial_1 w_2 = \partial_1 \Big(f_2'(\partial_2 u) \Big) = f_2''(\partial_2 u) \partial_1 \partial_2 u = \Theta f_1''(\partial_1 u) \partial_2 \partial_1 u = \Theta \partial_2 w_1$$

and the weak form of (44) reads as

$$\int_{\mathbb{R}^2} \begin{pmatrix} \partial_1 w_1 \\ \Theta \partial_2 w_1 \end{pmatrix} \cdot \nabla \varphi \, \mathrm{d}x = 0, \quad \varphi \in C_0^1(\mathbb{R}^2). \tag{45}$$

Inserting $\varphi = \eta^2 w_1$ with suitable $\eta \in C_0^1(B_{2R})$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_R, and $|\nabla \eta| \leq c/R$, we obtain

$$\begin{aligned} &\int_{B_{2R}} |\partial_1 w_1|^2 \eta^2 \, \mathrm{d}x + \int_{B_{2R}} \Theta |\partial_2 w_1|^2 \eta^2 \, \mathrm{d}x \\ &\quad = -2 \int_{B_{2R} - B_R} \eta \partial_1 w_1 \, \partial_1 \eta \, w_1 \, \mathrm{d}x - 2 \int_{B_{2R} - B_R} \Theta \, \eta \, \partial_2 w_1 \, \partial_2 \eta \, w_1 \, \mathrm{d}x. \end{aligned} \tag{46}$$

Applying Young's inequality and using w_1, $\Theta \in L^\infty(\mathbb{R}^2)$, we obtain that

$$\int_{\mathbb{R}^2} \Big(|\partial_1 w_1|^2 + \Theta |\partial_2 w_1|^2 \Big) \, \mathrm{d}x < \infty. \tag{47}$$

We then return to (46) and apply the inequality of Cauchy-Schwarz to obtain

$$\begin{aligned} &\int_{B_{2R}} |\partial_1 w_1|^2 \eta^2 \, \mathrm{d}x + \int_{B_{2R}} \Theta |\partial_2 w_1|^2 \eta^2 \, \mathrm{d}x \\ &\quad \leq 2 \Bigg[\int_{B_{2R} - B_R} \eta^2 |\partial_1 w_1|^2 \, \mathrm{d}x \Bigg]^{\frac{1}{2}} \Bigg[\int_{B_{2R} - B_R} |\partial_1 \eta|^2 w_1^2 \, \mathrm{d}x \Bigg]^{\frac{1}{2}} \\ &\qquad + 2 \Bigg[\int_{B_{2R} - B_R} \Theta \eta^2 |\partial_2 w_1|^2 \, \mathrm{d}x \Bigg]^{\frac{1}{2}} \Bigg[\int_{B_{2R} - B_R} \Theta |\partial_2 \eta|^2 w_1^2 \, \mathrm{d}x \Bigg]^{\frac{1}{2}}. \end{aligned} \tag{48}$$

On the right-hand side of (48), we observe that for both parts the first integral is vanishing when passing to the limit $R \to \infty$ since we have (47), while the remaining integrals stay uniformly bounded.

This gives $\partial_1 w_1 = 0$ and $\partial_2 w_1 = 0$ since we have $\Theta > 0$. Hence we obtain $w_1 \equiv c_1$ for some constant c_1. The monotonicity of f_1' then implies $\partial_1 u \equiv \tilde{c}_1$ for some different constant $\tilde{c}_1$.

By (43), we then also have $\partial_2 w_2 = 0$. Since we have already observed above that $\partial_1 w_2 = \Theta \partial_2 w_1$, we deduce $\partial_2 u \equiv \tilde{c}_2$ for some other real number $\tilde{c}_2$ and in conclusion u must be an affine function which completes the proof of Theorem 1.9.

□

Funding Open Access funding enabled and organized by Projekt DEAL.

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Received: 20 March 2023

Revised: 8 July 2023

Accepted: 28 July 2023