Boolean Algebra Admits No Convergent Term Rewriting System

R. Socher-Ambrosius

SEKI Report SR-90-06

Boolean Algebra Admits No Convergent Term Rewriting System

Rolf Socher-Ambrosius
Fachbereich Informatik, Universität Kaiserslautern
Postfach 30 49, D-6750 Kaiserslautern, W.-Germany

This work was supported by the Deutsche Forschungsgemeinschaft, SFB 314.

Boolean Algebra Admits No Convergent Term Rewriting System

Rolf Socher-Ambrosius
Fachbereich Informatik, Universität Kaiserslautern
Postfach 3049, D-6750 Kaiserslautern, W.- Germany

Abstract: Although there exists a normal form for boolean algebra terms, the so called set of prime implicants, there does not exist a convergent term rewriting system for the theory of boolean algebra. The result seems well-known, but no formal proof exists as yet. In this paper a formal proof of this fact is given.

Keywords: Boolean Algebra, Term Rewriting, Automated Theorem Proving.

1 Introduction

The existence (and computability) of a normal form is a very pleasant property of mathematical structures. It guarantees the decidability of the word problem and it also overcomes the difficulty with choosing the simplicity criterion for simplification procedures. The existence of a normal form usually comes along with the existence of a convergent (i.e. terminating and confluent) term rewriting system. Knuth & Bendix (1970) introduced a method to construct convergent term rewriting systems from given equational theories. Their approach, however, fails for commutativity axioms, since equations like $fxy = fyx$ cannot be employed as reductions without violating the finite terminating condition. Equational term rewriting systems have been introduced by Lankford & Ballantyne (1977) and Peterson & Stickel (1981), in order to overcome the difficulties with such equations. Equational term rewriting systems are composed of a rewriting system R and an equational system E, which contains those axioms that cannot be used as rules. Equational rewriting can be seen as rewriting on the equivalence classes of terms modulo an equational system. Such equational rewriting systems modulo associativity and commutativity exist, for instance, for abelian groups or boolean rings. It is not, however, for

the theory of boolean algebra (BA). Since the 1950s it is known that boolean algebra admits a normal form, which is called the *set of prime implicants*. There exists, however, no term rewriting system that rewrites a given BA-term into this normal form. The set of prime implicants can only be produced on an algorithmic way. It was Quine (1952) and (1959), who first developed such an algorithm, and others (Slagle, Chang & Lee 1970, Tison 1969) followed. The non-existence of a convergent system for boolean algebra is well-known. It seems, however, that there does not exist any formal proof of this fact as yet. Sometimes it is argued that the minimal set of prime implicants is not unique for boolean algebra terms. This argument only provides some intuition, it lacks, however, the formal proof that clausal form could be the only possible normal form for boolean algebra terms. It is, for instance, not obvious that the boolean algebra equation

$$\neg(x \vee y) = \neg x \wedge \neg y$$

should be directed from left to right, which is required for the clausal form transformation.

Hullot (1980) and Peterson & Stickel (1981) report attempts to find a convergent system for BA using the Knuth-Bendix (1970) completion procedure, which failed to terminate in all experiments. Why then does there exist an algorithm for transforming BA-terms into normal form, but not a term rewriting system? The deeper reason seems to be the essential role that resolution plays in the algorithm. A resolution rewrite rule had to look like

$$(x \vee y) \wedge (\neg x \vee z) \rightarrow (x \vee y) \wedge (\neg x \vee z) \wedge (y \vee z)$$

Such a rule, however, obviously violates the condition of being noetherian. In the following we will give a formal proof that a convergent term rewiting system for boolean algebra cannot exist.

2 Boolean Algebra and Term Rewriting Systems

In the following we assume a term set $\mathcal{T} = \mathcal{T}(\mathcal{F},\mathcal{V})$ over a signature $\mathcal{F}$ and a variable set $\mathcal{V}$. For any object o, let $\mathcal{V}(o)$ denote the set of all variables occurring in o.

2.1 Definition (Equational System):

An **equational** system E is a set of termpairs s=t. This system generates an equality relation $=_E$ in the following way: We define a relation $\overset{1}{=}_E$ by $s \overset{1}{=}_E t$, iff there exists an occurrence u in s, an equation s'=t' or t'=s' in E, and a substitution σ, such that $s/u = s'\sigma$ and $t = s[u \rightarrow t'\sigma]$. The relation $=_E$ is defined

as the transitive, reflexive closure of $\overset{1}{=}_E$. It is clear that $=_E$ is an equivalence relation. The equivalence class of t modulo E will be denoted as $[t]_E$.

2.2 Definition (Equational Term Rewriting System):

A **term rewriting system** R (over $\mathcal{T}$) is a set of termpairs $l \to r$ (the so called **rules**), such that $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ (and $l, r \in \mathcal{T}$). A term t_1 **R-reduces** to a term t_2, written $t_1 \Rightarrow_R t_2$, iff there exists an occurrence u in t_1, a rule $l \to r$ in R, and a substitution σ, such that $t_1/u = l\sigma$ and $t_2 = t_1[u \leftarrow r\sigma]$.
A term t_1 **E,R-reduces** to t_2, written $t_1 \Rightarrow_{E,R} t_2$, iff there exist $t'_1 \in [t_1]$, $t'_2 \in [t_2]$ with $t'_1 \Rightarrow_R t'_2$.
$\overset{+}{\Rightarrow}_{E,R}$ denotes the transitive, $\overset{*}{\Rightarrow}_{E,R}$ denotes the reflexive transitive closure of $\Rightarrow_{E,R}$ and $=_{E,R}$ denotes the reflexive, symmetric, and transitive closure of $\Rightarrow_{E,R}$.
The pair (E,R) is called an equational term rewriting system (ETRS). It can be understood also as a rewriting system for $\mathcal{T}/=_E = \{[t] \mid t \in \mathcal{T}\}$.
(E,R) is **noetherian**, iff there is no infinite sequence of E,R-reductions from any term.
(E,R) is **confluent**, iff $t \overset{*}{\Rightarrow}_{E,R} t_1$ and $t \overset{*}{\Rightarrow}_{E,R} t_2$ implies the existence of a term t_3 with $t_1 \overset{*}{\Rightarrow}_{E,R} t_3$ and $t_2 \overset{*}{\Rightarrow}_{E,R} t_3$.
A noetherian and confluent system is called **convergent**.
A term t_1 is called **(E,R-)irreducible**, iff there is no term t_2 with $t_1 \Rightarrow_{E,R} t_2$, and (E,R-)reducible otherwise.
An irreducible term t is called a **normal form** for t_1, iff $t_1 \overset{*}{\Rightarrow}_{E,R} t$.
If (E,R) is convergent, then each term t has a normal form $t\!\downarrow$, and $s\!\downarrow =_E t\!\downarrow$ holds for each term s with $s =_{E,R} t$.

2.3 Definition:

Let $\mathfrak{R}$ be a convergent ETRS over $\mathcal{T}$. Then the noetherian partial ordering $>_{\mathfrak{R}}$ on $\mathcal{T}$ generated by $\mathfrak{R}$ is defined by $s>t$ iff $s \overset{+}{\Rightarrow}_{\mathfrak{R}} t$. In the following we shall usually drop the index $\mathfrak{R}$.

2.4 Lemma:

Let $\mathfrak{R}$ be a convergent system on $\mathcal{T}$ with $=_{\mathfrak{R}} \; = \; =_E$.

a) The ordering > generated by $\mathfrak{R}$ is compatible with substitutions, that is, $s>t$ implies $s\sigma > t\sigma$ for any $s,t \in \mathcal{T}$ and any substitution σ.
b) Let $s,t \in \mathcal{T}$. If $s =_E t$ and t is $\mathfrak{R}$-irreducible, then $s>t$ holds.

Proof: Obvious. ∎

In the following let AC be the equational system

$AC = \{x \vee y = y \vee x, x \wedge y = y \wedge x, x \vee (y \vee z) = (x \vee y) \vee z, x \wedge (y \wedge z) = (x \wedge y) \wedge z\}$.
and ACD the system $AC \cup \{x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\}$.

2.5 Definition (Boolean Algebra):

A **boolean algebra** is an algebra $(B,\wedge,\vee,\neg)$ with the binary operators $\wedge,\vee$ and the unary operator $\neg$, which satisfies:

a) $(B,\wedge,\vee)$ is a distributive lattice, that is for all $a,b \in B$:

$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
$(a \vee b) \wedge b = b$	$(a \wedge b) \vee b = b$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

b) $(a \wedge \neg a) \vee b = b$ $\qquad (a \vee \neg a) \wedge b = b$

The axioms of boolean algebra imply the following well-known properties of the operators $\vee,\wedge$, and $\neg$:

2.6 Lemma:

Let $(B,\wedge,\vee,\neg)$ be a boolean algebra. Then there are $0,1 \in B$, such that for all $a,b \in B$:

$a \vee \neg a = 1$	$a \wedge \neg a = 0$
$0 \vee a = a$	$1 \wedge a = a$
$1 \vee a = 1$	$0 \wedge a = 0$
$a \vee a = a$	$a \wedge a = a$
$\neg(a \vee b) = \neg a \wedge \neg b$	$\neg(a \wedge b) = \neg a \vee \neg b$
$\neg\neg a = a$	

■

2.7 Lemma

Let $(B,\wedge,\vee,\neg)$ be a boolean algebra, and let $x_1,\ldots,x_n \in B$ with $x_1 \wedge \ldots \wedge x_n = 1$. Then $x_i = 1$ holds for all $i \in \{1,\ldots,n\}$.

Proof: Let $x_1 \wedge \ldots \wedge x_n = 1$. Then $x_i = (x_1 \wedge \ldots \wedge x_n) \vee x_i = 1 \vee x_i = 1$ holds for each $i \in \{1,\ldots,n\}$. ■

In the following we shall consider exclusively the term set $\mathcal{T} = \mathcal{T}(F_B, \mathcal{V})$, where F_B is the signature $(\wedge,\vee,\neg)$ of boolean algebra.

For ease of notation, we shall use the following convention: For any $t \in \mathcal{T}$, we define the dual term $\bar{t}$, which is obtained from t by simultaneously replacing each occurrence of $\vee$ by $\wedge$ and vice versa, and each occurrence of 0 by 1, and vice versa.

In the following equality will tacitly be understood to be equality modulo AC. Equality modulo BA will be denoted by $\equiv$, and terms which are equal

under BA, will also be called *equivalent*. We will use the customary notion of *literals, clauses* and a *conjunctive normal form* (CNF). A term t is called a literal, iff it is either of the form a, or of the form $\neg a$, with a being a constant or a variable. The term t is a *clause*, if $t = s_1 \vee \ldots \vee s_n$, with pairwise distinct literals s_i. A term t is called a *CNF-term*, if $t = s_1 \wedge \ldots \wedge s_n$, where the s_i are pairwise distinct clauses. A term with topsymbol $\vee$ is also called a *disjunction*, a term with topsymbol $\wedge$ a *conjunction*, and a term with topsymbol $\neg$ a *negation*.

2.8 Lemma:

There is no convergent system (ACD,R) such that $=_{ACD,R}$ coincides with $\equiv$.

Proof: Let $\mathfrak{R}$=(ACD,R) be a convergent system with $=_{\mathfrak{R}} = \equiv$, and let > be the partial order generated by $\mathfrak{R}$. First we remark that from $x \wedge x \equiv x$, and $x \vee (x \wedge y) \equiv x$ follows $x \wedge x > x$, and $x \vee (x \wedge y) > x$ for any $x,y \in \mathcal{V}$, since the term x is irreducible.

Consider the term $t = x \vee (y \wedge z)$. We have

$$t =_{ACD} (x \vee y) \wedge (x \vee z) =_{ACD} (x \wedge x) \vee (x \wedge z) \vee (y \wedge x) \vee (y \wedge z) > x \vee (y \wedge z) = t,$$

which is a contradiction. ∎

2.9 Theorem:

There exists no convergent ETRS (AC,R) such that $=_{AC,R}$ coincides with $\equiv$.

Note that we deal exclusively with term rewriting systems over the fixed signature F_B. There exists, for instance, a convergent system over the extended signature $(\wedge,\vee,\neg,+,*,0,1)$, see Hsiang (1985).

In order to prove the theorem above, we first provide some lemmata. For the remainder of the paper, we shall assume that there exists a convergent system $\mathfrak{R} = (AC,R)$ for BA. Let > be the noetherian ordering associated with $\mathfrak{R}$.

2.10 Lemma:

The following relations hold:

$$(x \vee y) \wedge y > y$$
$$\neg x \vee x > 1$$
$$x \vee x > x$$
$$x \vee 0 > x$$
$$x \vee 1 > 1$$
$$\neg\neg x > x$$
$$(x \vee y) \wedge (\neg x \vee y) > y$$

Proof: For each line, the two terms are equivalent according to definition 2.5 and lemma 2.6. Furthermore, each right hand side is obviously irreducible, hence the assertion follows from lemma 2.4.b. ■

The proof of our main theorem proceeds essentially by considering a particular term t, and proving that all terms $t' \equiv t$ are reducible. The following lemmata will provide two important techniques to prove a term t reducible, which are used heavily in the sequel. The first states that the normal form of a symmetric term must be symmetric.

If t is a term containing the (distinct) symbols p,q, and $t(p,q) = t(q,p)$, then the term t is called *symmetric* in (p,q). t is called *semi-symmetric* in (p,q), iff $t(p,q) \equiv t(q,p)$.

2.11 Lemma (Symmetry Lemma):

Let $x,y \in \mathcal{V}$ with $x \neq y$, and let $t=t(x,y)$ be irreducible. If t is semi-symmetric in (x,y), then t is even symmetric in (x,y).

Proof: Assume $t(x,y) \neq t(y,x)$. Then we have $t(x,y) > t(y,x)$, since the latter is irreducible. But then, according to 2.4.a also $t(x,y)\sigma > t(y,x)\sigma$ for $\sigma = \{x \to y;\ y \to x\}$, which implies $t(y,x) > t(x,y)$, a contradiction. ■

The symmetry lemma can also be stated as follows: If the term t is symmetric in (x,y), then $t\downarrow$ is also symmetric in (x,y).

The next "subterm lemma" shows that a term t is reducible, if a subterm of t can be replaced by a shorter term, without changing the original term's value.

2.12 Lemma (Subterm Lemma):

Let $t = s_1 \wedge \ldots \wedge s_n$, with $n \geq 1$, and let $\sigma = \{x \to t_0\}$ be a substitution with $x \in \mathcal{V}(t)$ and $x \notin \mathcal{V}(t_0)$. If $s_1\sigma \not\equiv s_1$, and $s_1\sigma \wedge s_2 \wedge \ldots \wedge s_n \equiv t$, then t is reducible.

Proof: Assume that t is irreducible. Let $s_1' = (s_1\sigma)\downarrow$, and let $t' = s_1' \wedge s_2 \wedge \ldots \wedge s_n$. Then, since $s_1\sigma \not\equiv s_1$, and $t' \equiv t$, we have $t' > t$. In particular, we have

$$t'\sigma > t\sigma,$$

which implies

$$s_1'\sigma \wedge s_2\sigma \wedge \ldots \wedge s_n\sigma > s_1\sigma \wedge s_2\sigma \wedge \ldots \wedge s_n\sigma,$$

and, since $s_1\sigma > s_1' = s_1'\sigma$, we have

$$s_1'\sigma \wedge s_2\sigma \wedge \ldots \wedge s_n\sigma > s_1'\sigma \wedge s_2\sigma \wedge \ldots \wedge s_n\sigma,$$

which is a contradiction. ■

It should be noted that the assertion of the subterm lemma also holds for a disjunction $t = s_1 \vee \ldots \vee s_n$.

2.13 Example:

Let $t = (x \vee y) \wedge \neg x$. We show that t is reducible. Let $\sigma = \{x \to 0\}$. First it is easy to see that $t \equiv y \wedge \neg x$, and $y = y\sigma \not\geq (x \vee y)$. If t were irreducible, then we had

$$y \wedge \neg x > (x \vee y) \wedge \neg x$$

hence

$$y \wedge \neg 0 = (y \wedge \neg x)\sigma > ((x \vee y) \wedge \neg x)\sigma = (0 \vee y) \wedge \neg 0 > y \wedge \neg 0$$

which is a contradiction.

2.14 Lemma:

Let t be a term with $\mathcal{V}(t) = \{x_1, \ldots, x_n\}$. Then there is a unique CNF-term $\tilde{t} = \tilde{c}_1 \wedge \ldots \wedge \tilde{c}_m$, where each $\tilde{c}_i$ is a clause containing all x_j's, and $\tilde{t} \equiv t$. The term $\tilde{t}$ is called the *standardized CNF* of t. Each $\tilde{c}_i$ is called a *standard clause* of t. The notion of a *standardized DNF* is defined analogously.

Proof: See, for instance, Rudeanu (1974). ∎

2.15 Example:

Let $t = (\neg x \vee y) \wedge (\neg x \vee \neg z)$. Then $\tilde{t} = (\neg x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee \neg z)$ is the standardized CNF of t.

2.16 Lemma:

If $t = t_1 \wedge \ldots \wedge t_n$, then for each $i \in \{1, \ldots, n\}$, there are standard clauses $\tilde{c}_{i1}, \ldots, \tilde{c}_{ik_i}$, with

$$t_i \equiv \tilde{c}_{i1} \wedge \ldots \wedge \tilde{c}_{ik_i}.$$

Moreover,

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \tilde{c}_{ij} = \{\tilde{c}_1, \ldots, \tilde{c}_n\}.$$ ∎

2.17 Lemma:

Let $t = x \vee y$. Then either $t\!\downarrow\, = t$, or $t\!\downarrow\, = \neg(\neg x \wedge \neg y)$.

Proof: Obvious. ∎

2.18 Lemma:

Let $t = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Then $t\!\downarrow\, \in \{t_1, \ldots, t_8\}$, where

$t_1 = (x\wedge y) \vee (y\wedge z) \vee (z\wedge x)$,
$t_2 = \neg(\neg y\vee\neg z) \vee \neg(\neg x\vee\neg z) \vee \neg(\neg y\vee\neg x)$,
$t_3 = (x\vee y) \wedge (y\vee z) \wedge (z\vee x)$,
$t_4 = \neg(\neg y\wedge\neg z) \wedge \neg(\neg x\wedge\neg z) \wedge \neg(\neg y\wedge\neg x)$
$t_5 = \neg[\neg(y\vee z) \vee \neg(x\vee z) \vee \neg(y\vee x)]$,
$t_6 = \neg[(\neg y\wedge\neg z) \vee (\neg x\wedge\neg z) \vee (\neg y\wedge\neg x)]$,
$t_7 = \neg[(\neg y\vee\neg z) \wedge (\neg x\vee\neg z) \wedge (\neg y\vee\neg x)]$,
$t_8 = \neg[\neg(y\wedge z) \wedge \neg(x\wedge z) \wedge \neg(y\wedge x)]$.

Proof:

a) Let $t\downarrow = s_1\vee\ldots\vee s_n$, and let $\tilde{t}$ be the standardized DNF of t. Then $\tilde{t} = d_1\vee d_2\vee d_3\vee d_4$, with

$d_1 = x\wedge y\wedge z$, $d_2 = \neg x\wedge y\wedge z$, $d_3 = x\wedge\neg y\wedge z$, $d_4 = x\wedge y\wedge\neg z$.

According to 2.16, each s_i is equivalent to a disjunction of d_j's. Moreover, $t\downarrow$ must be symmetric in (x,y), in (y,z), and in (x,z), and thus there are only the following cases: Either $t\downarrow = s_1\vee s_2$, with $s_1 \cong d_1$, and $s_2 \cong d_2\vee d_3\vee d_4$, or $t\downarrow = s_1\vee s_2\vee s_3$, with the following possibilities:

$s_1 \cong d_1\vee d_2$, $s_2 \cong d_1\vee d_3$, $s_3 \cong d_1\vee d_4$,

$s_1 \cong d_1\vee d_2\vee d_3$, $s_2 \cong d_1\vee d_3\vee d_4$, $s_3 \cong d_1\vee d_2\vee d_4$.

Let $t\downarrow = s_1\vee s_2$ with $s_1 \cong d_1$, and $s_2 \cong d_2\vee d_3\vee d_4$, and let $\sigma=\{z\rightarrow 1\}$. Then $s_1\sigma \not\cong s_1$. We show that $s_1\sigma\vee s_2 \cong s_1\vee s_2$: We have

$s_1\sigma\vee s_2 \cong (x\wedge y\wedge z) \vee (x\wedge y\wedge\neg z) \vee d_2 \vee d_3 \cong$

$(x\wedge y) \vee d_2 \vee d_3 \cong (x\wedge y) \vee (x\wedge y\wedge\neg z) \vee d_2 \vee d_3 \cong s_1\vee s_2$

Hence the subterm lemma implies that $s_1\vee s_2$ is reducible.

Let $t\downarrow = s_1\vee s_2\vee s_3$. If $s_1 \cong d_1\vee d_2 \cong y\wedge z$, $s_2 \cong d_1\vee d_3 \cong x\wedge z$, $s_3 \cong d_1\vee d_4 \cong y\wedge x$, then we have either $s_1 = y\wedge z$, $s_2 = x\wedge z$, $s_3 = x\wedge y$, and $t\downarrow = t_1$, or $s_1 = \neg(\neg y\vee\neg z)$, $s_2 = \neg(\neg x\vee\neg z)$, $s_3 = \neg(\neg y\vee\neg x)$, and $t\downarrow = t_2$.

If $s_1 \cong d_1\vee d_2\vee d_3 \cong (x\vee y)\wedge z$, $s_2 \cong d_1\vee d_3\vee d_4 \cong x\wedge(y\vee z)$, $s_3 \cong d_1\vee d_2\vee d_4 \cong y\wedge(x\vee z)$, then let $\tau=\{x\rightarrow 0\}$. It is easy to see that

$s_1\tau\vee s_2\vee s_3 \cong s_1\vee s_2\vee s_3$,

and $s_1\tau \not\cong s_1$. Hence the subterm lemma implies that $s_1\vee s_2 \vee s_3$ is reducible.

b) Let $t\downarrow = s_1\wedge\ldots\wedge s_n$. Analogously to a) it can be shown that $t\downarrow \in \{t_3,t_4\}$ in this case.

c) Let $t\downarrow = \neg t'$, with $t' = s_1\vee\ldots\vee s_n$. Then $t\downarrow \cong \neg s_1\wedge\ldots\wedge\neg s_n$. Let $\tilde{t}$ be the standardized CNF of t. Then $\tilde{t} = c_1\wedge c_2\wedge c_3\wedge c_4$, with

$c_1 = x\vee y\vee z$, $z_2 = \neg x\vee y\vee z$, $z_3 = x\vee\neg y\vee z$, $z_4 = x\vee y\vee\neg z$.

Then each $\neg s_i$ is equivalent to a conjunction of c_j's, and analogously to part a) it can be shown that either $t\downarrow$ is reducible according to the subterm lemma, or $t\downarrow \in \{t_5,t_6\}$. The case where $t' = s_1\wedge\ldots\wedge s_n$ is treated analogously. ∎

2.19 Lemma:

If the terms $x\vee(y\wedge z)$ and $x\wedge(y\vee z)$ are both irreducible, then $\mathfrak{R}$ is not convergent.

Proof: The assumption of the lemma implies $(x\vee y)\wedge(x\vee z) > x\vee(y\wedge z)$, $(x\wedge y)\vee(x\wedge y) > x\wedge(y\vee z)$, and, in particular, since both $y\wedge z$ and $y\vee z$ are irreducible, $\neg(\neg y\wedge\neg z) > y\vee z$, and $\neg(\neg y\vee\neg z) > y\wedge z$. This proves all terms $t_1,\ldots,t_8$ of the previous lemma to be reducible, hence $\mathfrak{R}$ cannot be confluent. ∎

Hence it will be assumed in the following that one of the terms $x\vee(y\wedge z)$ and $x\wedge(y\vee z)$ is reducible. It is sufficient to assume the term $x\vee(y\wedge z)$ to be reducible, the alternative case admitting an analogical proof. In particular, this assumption implies that each disjunct s_i of an irreducible term $t = s_1\vee\ldots\vee s_n$ is either a negation or an atom.

2.20 Lemma:

Either the term $x\vee y$ or the term $x\wedge y$ is reducible.

Proof: We consider the term $t = (\neg x\vee y)\wedge(\neg y\vee x)\wedge(x\vee z)$. Since t is semi-symmetric in (x,y), but not symmetric, t must be reducible.

a) Let $t\!\downarrow\, = s_1\wedge\ldots\wedge s_n$, where the s_i are not conjunctions.

If $n\geq 3$, let a be an arbitrary constant and let $\sigma = \{x\to a, y\to a, z\to\neg a\}$. We have $t > t\!\downarrow$, and in particular $t\sigma > t\!\downarrow\!\sigma$, where $t\sigma = (\neg a\vee a)\wedge(\neg a\vee a)\wedge(a\vee\neg a)$, and $t\!\downarrow\!\sigma = s_1\sigma\wedge\ldots\wedge s_n\sigma$. From $t\sigma \equiv 1$ follows $t\!\downarrow\!\sigma \equiv 1$, and hence $s_i\sigma \equiv 1$, for each $i\in\{1,\ldots,n\}$. Hence $s_i\sigma > 1$, and, since $s_i\sigma$ is composed solely of the literals a and $\neg a$, the last step of this derivation must be of the form $a\vee\neg a \Rightarrow 1$. Thus we have the reduction $(\neg a\vee a)\wedge(\neg a\vee a)\wedge(a\vee\neg a) \Rightarrow^{+}_{\mathfrak{R}} (\neg a\vee a)\wedge\ldots\wedge(a\vee\neg a)$, where the second term has $n\geq 3$ conjuncts, which obviously contradicts the finite termination property of $\mathfrak{R}$.

Now let n=2, that is $t\!\downarrow\, = s_1\wedge s_2$. Let $\tilde{t}$ be the standardized CNF of t. Then $\tilde{t} = c_1\wedge\ldots\wedge c_5$, with

$$c_1 = \neg x\vee y\vee z,\ c_2 = x\vee\neg y\vee z,\ c_3 = \neg x\vee y\vee\neg z,\ c_4 = x\vee\neg y\vee\neg z,\ c_5 = x\vee y\vee z.$$

We distinguish two cases:

Case 1: s_1 is symmetric in (x,y). Then s_2 is also symmetric in (x,y), since $t\!\downarrow$ is. From lemma 2.16 follows that s_1 and s_2 are equivalent to conjunctions of the c_i. Taking into account the symmetry property, there remain the following possibilities:

$s_1 \equiv c_1\wedge c_2$, $s_2 \equiv c_3\wedge c_4\wedge c_5$,

$s_1 \equiv c_3\wedge c_4$, or $s_1 \equiv c_3\wedge c_4\wedge c_5$, and $s_2 \equiv c_1\wedge c_2\wedge c_5$,

$s_1 \equiv c_1\wedge c_2\wedge c_3\wedge c_4$, $s_2 \equiv c_5$, $s_2 \equiv c_1\wedge c_2\wedge c_5$, or $s_2 \equiv c_3\wedge c_4\wedge c_5$.

In the first line, let $\sigma=\{z\to 0\}$. We have $s_1\sigma\wedge s_2 \equiv t$, and $s_1 \not\equiv s_1\sigma$. From the subterm lemma follows that $s_1\wedge s_2$ is reducible.
In the second line, let $\tau=\{z\to 1\}$. We have $s_1\tau\wedge s_2 \equiv t$, and $s_1 \not\equiv s_1\sigma$. From the subterm lemma follows that $s_1\wedge s_2$ is reducible.
In the third line, let $\varphi=\{x\to y\}$. We obtain in all three cases $s_1\wedge s_2\varphi \equiv t$, and $s_2 \not\equiv s_2\varphi$, and from the subterm lemma follows that $s_1\wedge s_2$ is reducible.

Case 2: s_1 is not symmetric in (x,y). Then $s_1 = s_2\{x\to y;\ y\to x\}$, and for each c_i occurring in s_1, $c_i\{x\to y;\ y\to x\}$ must occur in s_2. Hence both s_1 and s_2 must consist of at least 3 c_i's, and both contain c_5. We have the following possibilities:

$s_1 \equiv c_1\wedge c_3\wedge c_5$, $s_2 \equiv c_2\wedge c_4\wedge c_5$,

$s_1 \equiv c_1\wedge c_4\wedge c_5$, $s_2 \equiv c_2\wedge c_3\wedge c_5$,

$s_1 \equiv c_1\wedge c_2\wedge c_3\wedge c_5$, $s_2 \equiv c_1\wedge c_2\wedge c_4\wedge c_5$,

$s_1 \equiv c_2\wedge c_3\wedge c_4\wedge c_5$, $s_2 \equiv c_1\wedge c_3\wedge c_4\wedge c_5$.

In the first, third, and fourth line, let $\sigma=\{z\to 1\}$. In either case, we have $s_1\sigma\wedge s_2 \equiv t$, and $s_1 \not\equiv s_1\sigma$, hence $s_1\wedge s_2$ must be reducible according to the subterm lemma.
In the second line, we have $s_1 \equiv (y\vee z)\wedge(x\vee\neg y\vee\neg z)$, and $s_2 \equiv (x\vee z)\wedge(\neg x\vee y\vee\neg z)$. Let $\tau=\{z\to\neg x\}$. Since $s_1\tau\wedge s_2 \equiv t$, and $s_1 \not\equiv s_1\tau$, $s_1\wedge s_2$ must be reducible according to the subterm lemma.

b) Let $t\downarrow = s_1\vee\ldots\vee s_n$. Let $\tilde{t}$ be the standardized DNF of t. Then $\tilde{t} = c_1\vee c_2\vee c_3$, with

$d_1 = \neg x\wedge\neg y\wedge z$, $d_2 = x\wedge y\wedge z$, $d_3 = x\wedge y\wedge\neg z$.

Obviously, $n\leq 3$, since otherwise one s_i, say s_n, would be redundant, that is $t\downarrow \equiv s_1\vee\ldots\vee s_{n-1}$, which obviously contradicts the irreducibility of $t\downarrow$. If $n=3$, then $t\downarrow = s_1\vee s_2\vee s_3$, with $s_i \equiv d_i$. But then $s_2\vee s_3 \equiv x\wedge y \equiv \neg(\neg x\vee\neg y)$, hence $s_2\vee s_3$ is reducible.
Thus we have $t\downarrow = s_1\vee s_2$, where both s_1 and s_2 are negations, with the following possibilities:

$s_1 \equiv d_1$, $s_1 \equiv d_1\vee d_3$, or $s_1 \equiv d_1\vee d_2$, and $s_2 \equiv d_2\vee d_3$,

$s_1 \equiv d_3$, or $s_1 \equiv d_1\vee d_3$, and $s_2 \equiv d_1\vee d_2$,

$s_1 \equiv d_2$, $s_2 \equiv d_1\vee d_3$,

In the first line, $s_2 \equiv d_2\vee d_3 \equiv x\wedge y \equiv \neg(\neg x\vee\neg y)$ holds. One of the last two terms is irreducible, hence $s_2 = x\wedge y$, or $s_2 = \neg(\neg x\vee\neg y)$. But s_2 is a negation, hence $t\downarrow = s_1\vee\neg(\neg x\vee\neg y)$, from which follows that $\neg(\neg x\vee\neg y)$ is irreducible and thus $x\wedge y$ is reducible.
In both the second and the third line, let $\sigma=\{z\to 1\}$. Then $s_1\sigma \vee s_2 \equiv t$, and from the subterm lemma follows that $s_1\wedge s_2$ is reducible.

c) Let $t\downarrow = \neg s$. Then either $t\downarrow = \neg(s_1 \vee \ldots \vee s_n)$, which can be treated analogously to a), or $t\downarrow = \neg(s_1 \wedge \ldots \wedge s_n)$. In this case we obtain, similarly to b), $t\downarrow = \neg(s_1' \wedge s_2')$, with $s_1' \equiv d_1'$, or $s_1' \equiv d_1' \wedge d_2'$, or $s_1' \equiv d_1' \wedge d_3'$ and $s_2' \equiv d_2' \wedge d_3'$, where

$$d_1' = x \vee y \vee \neg z,\ d_2' = \neg x \vee \neg y \vee \neg z,\ d_3' = \neg x \vee \neg y \vee z.$$

First of all, $t\downarrow = \neg(s_1' \wedge s_2')$ implies that $\neg(x \wedge y)$ is irreducible, hence $\neg x \vee \neg y$ is reducible. We have $s_2' \equiv d_2' \wedge d_3' \equiv \neg x \vee \neg y$, and since s_2' is irreducible, $s_2' = \neg(x \wedge y)$. Now $t\downarrow = \neg(s_1' \wedge \neg(x \wedge y))$ implies that $\neg(x \wedge \neg y)$ is irreducible, hence $\neg x \vee y$ is reducible. Assume that s_1' is a disjunction, say $s_1' = u_1 \vee \ldots \vee u_m$. Then each u_j must be an atom, since both $x \vee (y \wedge z)$ and $x \vee \neg y$ are reducible. But it is easy to see that there is no disjunction of the atoms x, y, and z can be equivalent to one of the terms d_1', $d_1' \wedge d_2'$, or $d_1' \wedge d_3'$. Hence s_1' must be of the form $s_1' = \neg u$, which implies that $t\downarrow = \neg(\neg u \wedge \neg(x \wedge y))$ is irreducible. Hence also $\neg(\neg x \wedge \neg y)$ is irreducible, which implies that $x \vee y$ is reducible. ∎

2.21 Lemma:

Either the terms $x \vee y$ and $\neg(x \wedge y) \wedge \neg(x \wedge z)$ are both reducible, or the terms $x \wedge y$ and $\neg(x \vee y) \vee \neg(x \vee z)$ are both reducible.

Proof: According to the previous lemma, either $x \vee y$ or $x \wedge y$ is reducible.
Case 1: $x \vee y$ is reducible. Consider the term $t = (\neg x \vee y) \wedge (\neg y \vee x) \wedge (\neg x \vee \neg z)$. Since t is semi-symmetric in (x,y), but not symmetric, t must be reducible. Since $x \vee y$ is reducible, $t\downarrow$ cannot be a disjunction. Hence we have either $t\downarrow = s_1 \wedge \ldots \wedge s_n$ or $t\downarrow = \neg s$. The first case is treated analogously to case a) of the previous lemma. In the case, where $t\downarrow = \neg s$, we have $t\downarrow = \neg(s_1' \wedge s_2')$, with $s_1' \equiv d_1'$, or $s_1' \equiv d_1' \wedge d_2'$, or $s_1' \equiv d_1' \wedge d_3'$ and $s_2' \equiv d_2' \wedge d_3'$, where

$$d_1' = \neg x \vee \neg y \vee z,\ d_2' = x \vee y \vee z,\ d_3' = x \vee y \vee \neg z.$$

Analogouly to case c) of the previous lemma, we obtain $s_2' = \neg(\neg x \wedge \neg y)$, hence from $t\downarrow = \neg(s_1' \wedge s_2')$ follows that the term $t_0 := \neg(x \wedge \neg(\neg x \wedge \neg y))$ is irreducible, which in turn implies that $t_1 := \neg(x \wedge y) \wedge \neg(x \wedge z)$, which is equivalent to t_0, is reducible.
Case 2: $x \wedge y$ is reducible. Consider the term $t = (x \vee y \vee z) \wedge (\neg x \vee \neg y)$. Since $x \wedge y$ is reducible, t is also reducible, and, moreover, $t\downarrow$ cannot be a conjunction. Hence we have either $t\downarrow = s_1 \vee \ldots \vee s_n$ or $t\downarrow = \neg s$. The first case is treated analogously to case a) of the previous lemma. In the case, where $t\downarrow = \neg s$, we have $t\downarrow = \neg(s_1' \vee s_2')$, with $s_1' \equiv d_1'$, or $s_1' \equiv d_1' \vee d_2'$, or $s_1' \equiv d_1' \vee d_3'$ and $s_2' \equiv d_2' \vee d_3'$, where

$$d_1' = \neg x \wedge \neg y \wedge \neg z,\ d_2' = x \wedge y \wedge z,\ d_3' = x \wedge y \wedge \neg z.$$

Analogouly to case c) of the previous lemma, we obtain $s_2' = \neg(\neg x \vee \neg y)$, hence from $t{\downarrow} = \neg(s_1' \wedge s_2')$ follows that the term $t_0 := \neg(x \vee \neg(\neg x \vee \neg y))$ is irreducible, which in turn implies that $t_1 := \neg(x \vee y) \vee \neg(x \vee z)$, which is equivalent to t_0, is reducible. ■

2.22 Corollary:

$\mathfrak{R}$ is not confluent.

Proof: We consider again the term $t = (x \vee y) \wedge (y \vee z) \wedge (z \vee x) \equiv (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ of lemma 2.16.

Case 1: The terms $x \vee y$ and $\neg(x \wedge y) \wedge \neg(x \wedge z)$ are both reducible. The reducibility of $x \vee y$ excludes t_1, t_2, t_3, t_5, t_6, and t_7 of lemma 2.16 from being irreducible, and the reducibility of $\neg(x \wedge y) \wedge \neg(x \wedge z)$ excludes both t_4 and t_8 from being irreducible.

Case 2: The terms $x \wedge y$ and $\neg(x \vee y) \vee \neg(x \vee z)$ are both reducible. The reducibility of $x \wedge y$ excludes t_1, t_3, t_4, t_6, t_7, and t_8 of lemma 2.16 from being irreducible, and the reducibility of $\neg(x \vee y) \vee \neg(x \vee z)$ excludes both t_2 and t_5 from being irreducible. ■

This corollary provides the proof of our main theorem 2.1.

References

Hullot, J.M. (1980). A Catalogue of Canonical Term Rewriting Systems. Report CSL-113, SRI International, Menlo Park, CA.

Hsiang, J. (1985). Refutational Theorem Proving using Term-rewriting Systems. *Artificial Intelligence* **25**, 255 - 300.

Knuth, D.E.; Bendix, P.B. (1970). Simple Word Problems in Universal Algebra, in: J. Leech (Ed.): *Computational Problems in Universal Algebra*, Pergamon Press.

Lankford, D.S. & Ballantyne, A.M. (1977). Decision procedures for simple equational theories with a commutative axiom: Complete sets of commutative reductions. Tech. Report, Mathematics Dept., University of Texas, Austin, Texas.

Peterson, G.E. & Stickel, M.E. (1981). Complete Sets of Reductions for Some Equational Theories. *Journal of the ACM*, **28**, 233 - 264.

Quine, W.V. (1952). The Problem of Simplifying Truth Functions. *American Math. Monthly*, **59**, 521 - 531.

Quine, W.V. (1959). On Cores and Prime Implicants of Truth Functions. *Am. Math. Monthly*, **66**, 755 - 760.

Slagle, J.R.; Chang, C.L. & Lee, R.C.T. (1970). A New Algorithm for Generating Prime Implicants. *IEEE Trans. on Comp.* **19/4**, 304 - 310.

Tison, P. (1967). Generalized Consensus Theory and Application to the Minimization of Boolean Functions. *IEEE Trans. on Comp.* **16/4**, 446 - 456.