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ORIGINAL ARTICLE

Spatial Dynamics and Solitary Hydroelastic Surface Waves

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Abstract

This paper presents an existence theory for solitary waves at the interface between a thin ice sheet (modelled using the Cosserat theory of hyperelastic shells) and an ideal fluid (of finite depth and in irrotational motion). The theory takes the form of a review of the Kirchgässner reduction to a finite-dimensional Hamiltonian system, highlighting the refinements in the theory over the years and presenting some novel aspects including the use of a higher-order Legendre transformation to formulate the problem as a spatial Hamiltonian system, and a Riesz basis for the phase space to complete the analogy with a dynamical system. The reduced system is to leading order given by the focussing cubic nonlinear Schrödinger equation, agreeing with the result of formal weakly nonlinear theory (which is included for completeness). We give a precise proof of the persistence of two of its homoclinic solutions as solutions to the unapproximated reduced system which correspond to symmetric hydroeleastic solitary waves.

Keywords Solitary waves · Hydroelastic waves · Nonlinear Schrödinger equation · Centre-manifold reduction

1 Introduction

1.1 The Main Result

In this article, we examine the propagation of solitary waves on the surface of an ocean under ice, regarding the water as a perfect fluid in irrotational flow and the ice sheet as an elastic shell which bends with the surface without stretching and without friction or cavitation between it and the fluid beneath. For this purpose we consider the model

Dedicated to Jean-Marc Vanden-Broeck on the occasion of his seventieth birthday.

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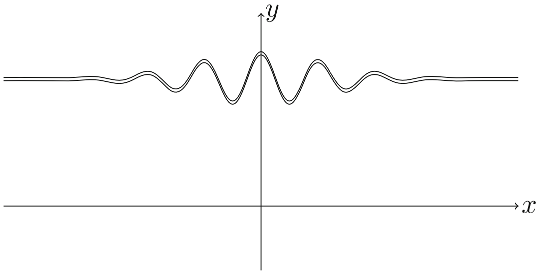

Fig. 1 An ice sheet on the free surface of a two-dimensional perfect fluid

derived by Plotnikov and Toland [33] using the Euler equations for inviscid fluid flow and the Cosserat theory of hyperelastic shells (Fig. 1).

We suppose that the fluid occupies the region bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by the free surface $\{y = h + \eta(x, t)\}$, where h is the depth of the water in its undisturbed state. *Travelling waves* move in the x-direction with constant speed c and without change of shape, so that $\eta(x, t) = \eta(x - ct)$, and *solitary waves* are localised travelling waves, so that $\eta(x - ct) \to 0$ as $x - ct \to \pm\infty$. In terms of an Eulerian velocity potential ϕ, the governing equations for the hydrodynamic problem in dimensionless coordinates and in a coordinate system moving with the wave are

$$\phi_{xx} + \phi_{yy} = 0, \qquad 0 < y < 1 + \eta \tag{1.1}$$

with boundary conditions

$$\phi_y\big|_{y=0} = 0, \tag{1.2}$$

$$\phi_y - \eta_x\phi_x + \eta_x\big|_{y=1+\eta} = 0, \tag{1.3}$$

$$\begin{aligned} &-\phi_x + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + \alpha\eta \\ &\quad + \beta\left(\frac{1}{(1+\eta_x^2)^{1/2}}\left[\frac{1}{(1+\eta_x^2)^{1/2}}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)_x\right]_x + \frac{1}{2}\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3\right)\Bigg|_{y=1+\eta} = 0 \end{aligned} \tag{1.4}$$

and asymptotic conditions $\eta \to 0$, $(\phi_x, \phi_y) \to (0, 0)$ as $x \to \pm\infty$ (see Guyenne and Parau [16]). The dimensionless parameters α and β are given by

$$\alpha = \frac{gh}{c^2}, \qquad \beta = \frac{D}{\rho h^3 c^2},$$

where D is the coefficient of flexural rigidity of the ice sheet, g is the acceleration due to gravity, c is the wave speed and ρ is the constant water density.

This formulation is unfavourable because of the variable fluid domain. It is, therefore, convenient to introduce the change of variable

$$\tilde{y} = \frac{y}{1 + \eta(x)}, \qquad \Phi(x, \tilde{y}) = \phi(x, y), \tag{1.5}$$

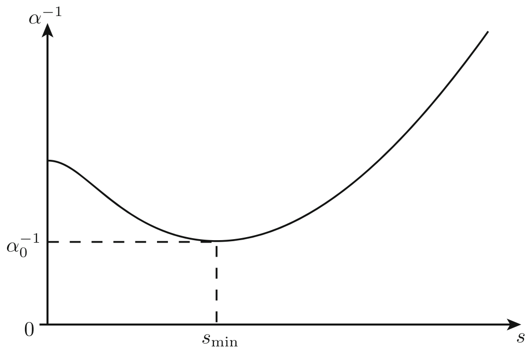

Fig. 2 The linear dispersion relation for a fixed β_0

which maps the variable fluid domain $\{(x, y) : x \in \mathbb{R},\ y \in (0, 1 + \eta(x))\}$ to the fixed strip $\mathbb{R} \times (0, 1)$. Dropping the tildes for notational simplicity, one obtains the transformed equations

$$\Phi_{xx} + \Phi_{yy}\frac{1 + y^2\eta_x^2}{(1+\eta)^2} - 2\Phi_{xy}\frac{y\eta_x}{1+\eta} - \Phi_y\frac{y\eta_{xx}}{1+\eta} + 2\Phi_y\frac{y\eta_x^2}{(1+\eta)^2} = 0, \qquad 0 < y < 1, \tag{1.6}$$

$$\Phi_y\big|_{y=0} = 0, \tag{1.7}$$

$$\frac{\Phi_y}{1+\eta} + \eta_x - \eta_x\Big(\Phi_x - \Phi_y\frac{y\eta_x}{1+\eta}\Big)\Big|_{y=1} = 0, \tag{1.8}$$

$$\begin{aligned} &-\Big(\Phi_x - \Phi_y\frac{y\eta_x}{1+\eta}\Big) + \frac{1}{2}\Big(\Big(\Phi_x - \Phi_y\frac{y\eta_x}{1+\eta}\Big)^2 + \Big(\frac{\Phi_y}{1+\eta}\Big)^2\Big) + \alpha\eta \\ &\quad + \beta\Big(\frac{1}{(1+\eta_x^2)^{1/2}}\Big[\frac{1}{(1+\eta_x^2)^{1/2}}\Big(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\Big)_x\Big]_x + \frac{1}{2}\Big(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\Big)^3\Big)\Big|_{y=1} = 0 \end{aligned} \tag{1.9}$$

with asymptotic conditions $\eta \to 0$, $(\Phi_x, \Phi_y) \to (0, 0)$ as $x \to \pm\infty$.

Let us briefly review the (formal) classical weakly nonlinear theory as it applies to this problem. Figure 2 shows the linear dispersion relation

$$\alpha + \beta s^4 = s\coth(s)$$

for a sinusoidal wave train with wave number s. For each fixed value β_0 of β, the dispersion curve has a unique minimum at $(s, \alpha^{-1}) = (s_{\min}, \alpha_0^{-1})$; the relationship between β_0, α_0 and $s = s_{\min}$ can be expressed in the form

$$\beta_0(s) = \frac{1}{4s^3}\coth(s) - \frac{1}{4s^2}\operatorname{cosech}^2(s), \qquad \alpha_0(s) = \frac{3s}{4}\coth(s) + \frac{s^2}{4}\operatorname{cosech}^2(s),$$

which defines a curve C in the (β,α)-plane parametrised by $s\in(0,\infty)$. Setting $\alpha=\alpha_0+\delta^2$, $\beta=\beta_0$, and substituting the modulational *Ansatz*

$$\eta(x)=\delta\big(A_1(\delta x)\mathrm{e}^{\mathrm{i}sx}+\bar{A}_1(\delta x)\mathrm{e}^{-\mathrm{i}sx}\big)\\ +\delta^2\big(A_2(\delta x)\mathrm{e}^{2\mathrm{i}sx}+\bar{A}_2(\delta x)\mathrm{e}^{-2\mathrm{i}sx}+A_0(\delta x)\big)+\cdots$$

into Eqs. (1.6)–(1.9), one finds that to leading order A_1 satisfies the nonlinear Schrödinger equation

$$A_1-b_1A_{1XX}-b_2|A_1|^2A_1=0, \tag{1.10}$$

where $X=\delta x$ (see Appendix A for details of the derivation and formulae for the coefficients b_1 and b_2). One finds that b_1 is positive for all values of s, and there exists a critical value $s^\star$ (numerically $s^\star\approx 177.33$) such that $b_2>0$ for $s<s^\star$ and $b_2<0$ for $s>s^\star$.

Suppose that $b_2>0$, that is, choose s sufficiently small, or equivalently β_0 sufficiently large (corresponding to sufficiently shallow water in physical variables). Equation (1.10) admits the family

$$A_1(X)=\left(\frac{2}{b_2}\right)^{1/2}\operatorname{sech}\left(\frac{X}{b_1^{1/2}}\right)\mathrm{e}^{\mathrm{i}\theta},\qquad \theta\in[0,2\pi)$$

of homoclinic solutions (solutions which decay to zero as $x\to\pm\infty$), which correspond to the solitary waves

$$\eta(x)=2\delta\left(\frac{2}{b_2}\right)^{1/2}\operatorname{sech}\left(\frac{\delta x}{b_1^{1/2}}\right)\cos(sx+\theta)+O(\delta^2).$$

These waves take the form of periodic wave trains modulated by exponentially decaying envelopes; the wave with $\theta=0$ is a symmetric wave of elevation, while the wave with $\theta=\pi$ is a symmetric wave of depression (see Fig. 3). In this article, we confirm the predictions of the weakly nonlinear theory and prove the following theorem.

Theorem 1.1 *Choose $s\in(0,s^\star)$ and let (β_0,α_0) denote the point on the curve C with this parameter value. For each sufficiently small value of $\delta>0$ and $\nu\in(0,1)$, the hydroelastic problem (1.1)–(1.4) with $\beta=\beta_0$ and $\alpha=\alpha_0+\delta^2$ admits two geometrically distinct, symmetric solitary-wave solutions $(\eta^\pm,\phi^\pm)$ which satisfy the estimate*

$$\eta^\pm(x)=\pm 2\delta\left(\frac{2}{b_2}\right)^{1/2}\operatorname{sech}\left(\frac{\delta x}{b_1^{1/2}}\right)\cos(sx)+O(\delta^2\mathrm{e}^{-\nu b_1^{-1/2}\delta|x|})$$

uniformly over $x\in\mathbb{R}$.

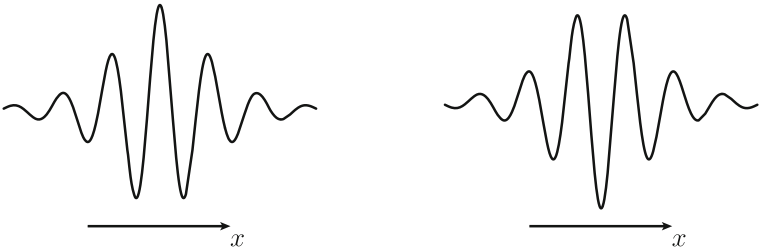

Fig. 3 Symmetric envelope solitary waves (with scaled amplitudes and wavelengths)

1.2 Spatial Dynamics and the Kirchgässner Reduction

We prove Theorem 1.1 using the *Kirchgässner reduction*: the hydrodynamic problem is formulated as a spatial Hamiltonian system and reduced to a locally equivalent Hamiltonian system with finitely many degrees of freedom; homoclinic solutions of the reduced system correspond to solitary waves. The method was introduced by Kirchgässner [21] and has been used for many problems in fluid mechanics, in particular for water waves (see Dias and Iooss [6] for a review), and more recently for water waves with vorticity (Groves and Wahlén [14, 15], Kozlov et al. [22], Kozlov and Lokharu [23]) and ferrofluids (Groves et al. [11], Groves and Nilsson [12]). In this paper, we review the method as it applies to hydroelastic solitary waves, presenting various refinements and new features.

Our starting point in Sect. 2 is the observation that the Eqs. (1.1)–(1.4) follow from the formal variational principle

$$\delta \int_{\mathbb{R}} \left\{ \int_0^{1+\eta(x)} \left(-\phi_x + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) \right) \mathrm{d}y + \tfrac{1}{2}\alpha\eta^2 + \tfrac{1}{2}\beta \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\} \mathrm{d}x = 0, \tag{1.11}$$

in which the variations are taken over η and ϕ (a modified version of the classical variational principle introduced by Luke [25]); this observation is confirmed by the calculation

$$\begin{aligned} &\delta \int_{\mathbb{R}} \left\{ \int_0^{1+\eta(x)} \left(-\phi_x + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) \right) \mathrm{d}y + \tfrac{1}{2}\alpha\eta^2 + \tfrac{1}{2}\beta \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}} \right\} \mathrm{d}x \\ &= \int_{\mathbb{R}} \Bigg\{ -\int_0^{1+\eta(x)} (\phi_{xx}+\phi_{yy})\dot{\phi}\,\mathrm{d}y + \left((-\eta_x\phi_x + \phi_y + \eta_x)\dot{\phi}\right)\Big|_{y=1+\eta} - (\phi_y\dot{\phi})\Big|_{y=0} \\ &\qquad + \Bigg(\left(-\phi_x + \tfrac{1}{2}(\phi_x^2+\phi_y^2)\right)\Big|_{y=1+\eta} + \alpha\eta + \tfrac{1}{2}\beta\left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right)^3 \\ &\qquad + \beta \frac{1}{(1+\eta_x^2)^{1/2}} \left[\frac{1}{(1+\eta_x^2)^{1/2}} \left(\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \right)_x \right]_x \Bigg) \dot{\eta} \Bigg\} \mathrm{d}x, \end{aligned}$$

where the formal first variations of η and ϕ are denoted by respectively $\dot{\eta}$ and $\dot{\phi}$ and we have used integration by parts and Green's integral formula.

We proceed using the change of variable (1.5), which transforms (1.11) into the new variational principle

$$\delta \int L(\eta, \eta_x, \eta_{xx}, \Phi, \Phi_x)\, \mathrm{d}x = 0$$

with Lagrangian

$$\begin{aligned} &L(\eta, \eta_x, \eta_{xx}, \Phi, \Phi_x) \\ &:= \int_0^1 \left(-\left[\Phi_x - \Phi_y \frac{y\eta_x}{1+\eta}\right] + \frac{1}{2}\left[\Phi_x - \Phi_y \frac{y\eta_x}{1+\eta}\right]^2 + \frac{1}{2}\frac{\Phi_y^2}{(1+\eta)^2}\right)(1+\eta)\, \mathrm{d}y \\ &\qquad + \tfrac{1}{2}\alpha\eta^2 + \tfrac{1}{2}\beta \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}}; \end{aligned}$$

this variational principle recovers the transformed equations (1.6)–(1.9). The next step is to perform a (formal) Legendre transform to obtain a formulation of the hydrodynamic problem as a spatial Hamiltonian system (in which the variable x plays the role of 'time'). The presence of second-order derivatives in the Lagrangian, however, necessitates the use of a higher-order Legendre transform (see Lanczos [24, Appendix I]), by means of which obtain the Hamiltonian system

$$\eta_x = \frac{\delta H}{\delta \omega}, \quad \rho_x = \frac{\delta H}{\delta \xi}, \quad \omega_x = -\frac{\delta H}{\delta \eta}, \quad \xi_x = -\frac{\delta H}{\delta \rho}, \quad \Phi_x = \frac{\delta H}{\delta \Psi}, \quad \Psi_x = -\frac{\delta H}{\delta \Phi} \tag{1.12}$$

with variables η, Φ and

$$\rho = \eta_x, \qquad \omega = \frac{\delta L}{\delta \eta_x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta L}{\delta \eta_{xx}}\right), \qquad \xi = \frac{\delta L}{\delta \eta_{xx}}, \qquad \Psi = \frac{\delta L}{\delta \Phi_x};$$

these equations are accompanied by the boundary conditions

$$-\Phi_y + y\rho\Psi\big|_{y=0,1} = 0, \tag{1.13}$$

which emerge when computing the variational derivatives.

Equations (1.12), (1.13) are reversible, that is invariant under the transformation $(\eta, \omega, \rho, \xi, \Phi, \Psi)(x) \mapsto (\eta, -\omega, -\rho, \xi, -\Phi, \Psi)(-x)$; this symmetry is inherited from (1.6)–(1.9), which are invariant under $(\eta(x), \Phi(x, y)) \mapsto (\eta(-x), \Phi(-x, y))$. They are also invariant under the transformation $\Phi \mapsto \Phi + c$ for any constant c. To eliminate this symmetry, one replaces (Φ, Ψ) with new variables $(\bar{\Phi}, \Phi_0, \bar{\Psi}, \Psi_0)$, where $\bar{\Phi} = \Phi - \Phi_0$, $\bar{\Psi} = \Psi - \Psi_0$ and $\Phi_0 = \int_0^1 \Phi\, \mathrm{d}y$, $\Psi_0 = \int_0^1 \Psi\, \mathrm{d}y$, thus obtaining a new canonical Hamiltonian system with Hamiltonian

$$\begin{aligned} \bar{H}(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi}, \Phi_0, \Psi_0) &= H(\eta, \omega, \rho, \xi, \bar{\Phi} + \Phi_0, \bar{\Psi} + \Psi_0) \\ &= H(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi} + \Psi_0) \end{aligned}$$

and additional constraints $\int_0^1 \bar{\Phi}\,\mathrm{d}y = 0$, $\int_0^1 \bar{\Psi}\,\mathrm{d}y = 0$. The variable Φ_0 is cyclic, so that its conjugate Ψ_0 is a conserved quantity; we proceed in standard fashion by setting $\Psi_0 = -1$, considering the equations for $(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi})$ and recovering Φ_0 by quadrature. The nonlinear boundary condition

$$-\bar{\Phi}_y + y\rho(\bar{\Psi} - 1)\big|_{y=1} = 0$$

necessitates a further change of variable, namely

$$\bar{\Gamma} = \bar{\Phi} - \rho \int_0^y s(\bar{\Psi}(s) - 1)\,\mathrm{d}s + \rho \int_0^1 \int_0^y s(\bar{\Psi}(s) - 1)\,\mathrm{d}s\,\mathrm{d}y,$$

in terms of which the boundary conditions take the simple, linear form $\bar{\Gamma}_y\big|_{y=0,1} = 0$.

The formulation of the hydrodynamic problem as a spatial Hamiltonian system is discussed rigorously in Sect. 2, where a precise definition of a Hamiltonian system is given and Hamilton's equations are derived. Full details of the changes of variable, which are performed explicitly, are also given; the result is a quasilinear evolution equation of the form

$$u_x = Lu + N^\varepsilon(u) \tag{1.14}$$

for the variable $u = (\eta, \rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi})$ in the phase space

$$X = \{(\eta, \rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^1(0, 1) \times \bar{L}^2(0, 1)\},$$

where the overline denotes the subspace of functions with zero mean value; the domain of the linear operator L is

$$\mathcal{D}(L) = \left\{(\eta, \rho, \omega, \xi, \bar{\Gamma}, \bar{\Psi}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^2(0, 1) \times \bar{H}^1(0, 1) : \bar{\Gamma}_y\big|_{y=0,1} = 0\right\}$$

and the nonlinear term on the right-hand side of (1.14), which satisfies $N^\varepsilon(u) = O(\|(\varepsilon, u)\|\|u\|)$, maps a neighbourhood of the origin in $\mathbb{R}^2 \times \mathcal{D}(L)$ analytically into X. Here we have written $\alpha = \alpha_0 + \varepsilon_1$ and $\beta = \beta_0 + \varepsilon_2$, where α_0 and β_0 are fixed, and the superscript ε denotes the dependence upon this parameter.

In Sect. 3 we show that the spectrum of L is discrete. By reducing the spectral problem to a non self-adjoint Sturm–Liouville problem, we show that a complex number λ is an eigenvalue of L if and only if

$$\alpha_0 + \lambda^4\beta_0 = \lambda \cot(\lambda) \tag{1.15}$$

and deduce that $\sigma(L)$ consists of

(a) a countably infinite family $\{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ of simple real eigenvalues, where $\{\lambda_k\}_{k=1}^\infty$ are the positive real solutions of equation (1.15), so that $\lambda_k \in (k\pi, (k+1)\pi)$ for $k = 1, 2, \ldots$ and

$$\lambda_k^2 = k^2\pi^2 + \frac{2}{\beta_0} + o\left(\frac{1}{k}\right)$$

for large k, and $\lambda_{-k} = -\lambda_k$ for $k = 1, 2, \ldots,$

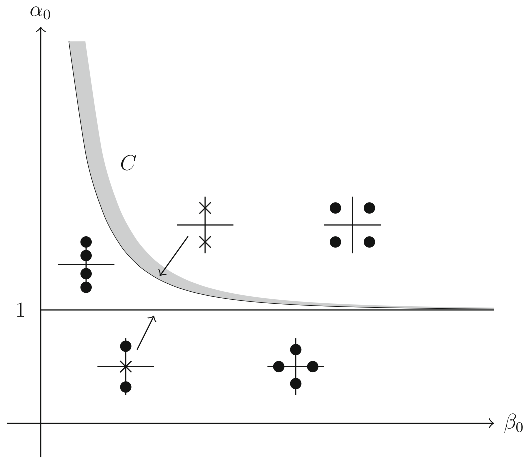

Fig. 4 The shaded region indicates the parameter regime in which homoclinic bifurcation is detected; dots and crosses denote, respectively, simple and algebraically double, geometrically simple eigenvalues

(b) four additional eigenvalues (counted according to multiplicity) which are shown in Fig. 4. Note in particular that a *Hamiltonian–Hopf bifurcation* occurs at each point $(\beta_0(s), \alpha_0(s))$ of the curve C: two pairs of purely imaginary eigenvalues become complex by colliding at the points $\pm \mathrm{i}s$ on the imaginary axis.

Remarkably, we can treat (1.14) as a dynamical system with countably infinitely many coordinates by showing that L is a *Riesz spectral operator*, that is its generalised eigenvectors form a Riesz basis for X (a Schauder basis obtained by an isomorphism from an orthonormal basis). In particular, at a point $(\beta_0(s), \alpha_0(s))$ of the curve C (a 'Hamiltonian–Hopf point') we can write

$$X = \left\{ u = Ae + Bf + \bar{A}\bar{e} + \bar{B}\bar{f} + \sum_{k \in \mathbb{Z}\setminus\{0\}} \beta_k e_{\lambda_k} : A, B \in \mathbb{C},\ \{\beta_k\} \in \ell^2 \right\},$$

where e, f and e_{λ_k} are suitably normalised generalised eigenvectors with $(L - \mathrm{i}sI)e = 0$, $(L - \mathrm{i}sI)f = e$ and $(L - \lambda_k I)e_{\lambda_k} = 0$. In the above notation,

$$Lu = (\mathrm{i}sA + B)e + \mathrm{i}sBf + (-\mathrm{i}s\bar{A} + \bar{B})\bar{e} - \mathrm{i}s\bar{B}\bar{f} + \sum_{k \in \mathbb{Z}\setminus\{0\}} \lambda_k \beta_k e_{\lambda_k}$$

and $u \in \mathcal{D}(L)$ whenever $\{\lambda_k \beta_k\} \in \ell^2$.

Homoclinic solutions of (1.12) are of particular interest since they correspond to solitary waves. We detect them using centre-manifold reduction (see Mielke [28, 29] for the version of the reduction theorem used here). Denoting the central and hyperbolic subspaces of X at a Hamiltonian-Hopf point by

$$X_1 = \{u_1 = Ae + Bf + \bar{A}\bar{e} + \bar{B}\bar{f} : A, B \in \mathbb{C}\},$$
$$X_2 = \left\{u_2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} \beta_k e_{\lambda_k} : \{\beta_k\} \in \ell^2\right\},$$

one finds that all small, globally bounded solutions to (1.14) lie on a *centre manifold* of the form $\{u_2 = r(u_1; \varepsilon)\}$, where the *reduction function* $r : X_1 \to \mathcal{D}(L)$ is $O(\|(\varepsilon, u)\|\|u\|)$. The flow on the centre manifold is governed by the reduced system

$$u_{1x} = Lu_1 + N^{\varepsilon}(u_1 + r(u_1; \varepsilon)), \tag{1.16}$$

which is itself a reversible Hamiltonian system (with two degrees of freedom). One of the key requirements in Mielke's theorem is that the operator $L_2 = L|_{\mathcal{X}_2}$ has L^p*-maximal regularity* in the sense that the differential equation

$$\partial_x u_2 = L_2 u_2 + h$$

admits a unique solution $u_2 \in W^{1,p}(\mathbb{R}, X_2) \cap L^p(\mathbb{R}, \mathcal{D}(L_2))$ for each $h \in L^p(\mathbb{R}, X_2)$ and $p > 1$. In fact L^p-maximal regularity for some $p > 1$ implies L^p-maximal regularity for all $p > 1$ (see Mielke [27]), and an operator has L^2-maximal regularity if and only if it is bisectorial (see Arendt and Duelli [1, Theorem 2.4]); the theorem is usually stated with bisectorality as a hypothesis. (Mielke's theorem actually requires maximal regularity in exponentially weighted spaces, a property which is implied by L^p-maximal regularity; see Mielke [27, Lemma 2.3]). In Sect. 4 we, however, demonstrate directly that a Riesz spectral operator with no imaginary eigenvalues has L^2-maximal regularity, and stipulate L^2-maximal regularity as a hypothesis in Mielke's theorem. This approach is more direct than that taken in the above references to the Kirchgässner reduction, in which central and hyperbolic subspaces of a suitable phase space are defined by Dunford integrals and the bisectorality condition is verified by a priori estimates.

Writing $(\varepsilon_1, \varepsilon_2) = (\mu, 0)$, so that positive values of μ correspond to points on the 'complex' side of C (the shaded region in Fig. 4), one finds after a Darboux and normal-form transformation that the reduced equation (1.16) can be formulated as the Hamiltonian system

$$A_x = \frac{\partial \tilde{H}^{\mu}}{\partial \bar{B}}, \qquad B_x = -\frac{\partial \tilde{H}^{\mu}}{\partial \bar{A}}, \tag{1.17}$$

$$\tilde{H}^{\mu}(A, B, \bar{A}, \bar{B}) = \mathrm{i}s(A\bar{B} - \bar{A}B) + |B|^2 + \tilde{H}^{\mu}_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B))$$
$$+O(|(A, B)|^2|(\mu, A, B)|^{n_0}),$$

where $\tilde{H}^{\mu}_{\mathrm{NF}}(A, B, \bar{A}, \bar{B})$ is a real polynomial function of its arguments which satisfies

$$\tilde{H}^{\mu}_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), \mu) = O(|(A, B)|^2|(\mu, A, B)|);$$

it contains the terms of order 3, ..., n_0+1 in the Taylor expansion of $\tilde{H}^\mu(A, B, \tilde{A}, \tilde{B})$. Equation (1.17) inherit the reversibility of (1.12): they are invariant under the transformation $(A, B)(x) \mapsto (\bar{A}, -\bar{B})(-x)$. Neglecting the remainder term in the Hamiltonian and introducing the scaled variables

$$A(x) = \delta e^{isx} \tilde{A}(X), \qquad B(x) = \delta^2 e^{isx} \tilde{B}(X), \qquad X = \delta x,$$

where $\delta = \mu^2$, confirms that the system is at leading order equivalent to the nonlinear Schrödinger equation

$$\tilde{A}_{XX} = -c_1 \tilde{A} - d_1 \tilde{A} |\tilde{A}|^2,$$

where c_1 and d_1 are the coefficients of respectively $\mu|A|^2$ and $|A|^4$ in the Taylor expansion of $\tilde{H}^\mu_{\mathrm{NF}}$. We compute these coefficients explicitly in Appendix B and find that

$$c_1 = -\frac{1}{b_1}, \qquad d_1 = \frac{\sinh^2(s)}{2\tau_1} \frac{b_2}{b_1},$$

where b_1, b_2 are the coefficients in Eq. (1.10) and $\tau_1 > 0$ is defined in Eq. (4.4).

A rigorous analysis of (1.17) is given in Sect. 5. Returning to real coordinates q, $p \in \mathbb{R}^2$ given by $A = \frac{1}{\sqrt{2}}(q_1 + iq_2)$, $B = \frac{1}{\sqrt{2}}(p_1 + ip_2)$, eliminating p and introducing the scaled variables

$$q(x) = \delta R_{sx} Q(X), \qquad X = \delta x,$$

where $\delta^2 = -c_1\mu$ and R_θ is the matrix representing a rotation through the angle θ, transforms (1.17) into

$$Q_{XX} = Q - CQ|Q|^2 + T_1^\delta(Q, Q_X) + R_{-sX/\delta} T_2^\delta(R_{sX/\delta} Q, R_{sX/\delta} Q_X, R_{sX/\delta} Q_{XX}), \tag{1.18}$$

where $C = -d_1/c_1$ and

$$T_1^\delta(Q, Q_X) = O(\delta|(Q, Q_X)|), \qquad T_2^\delta(Q, Q_X, Q_{XX}) = O(\delta^{n_0-2}|(Q, Q_X, Q_{XX})|).$$

Equation (1.18) is invariant under the transformation $X \mapsto -X$, $(Q_1(X), Q_2(X)) \mapsto (Q_1(-X), -Q_2(-X))$ and in the limit $\delta = 0$ has the explicit solution

$$Q(X) = \begin{pmatrix} h(X) \\ 0 \end{pmatrix}, \qquad h(X) = \left(\frac{2}{C}\right)^{1/2} \operatorname{sech}(X),$$

which is nondegenerate in the class of symmetric functions (see Sect. 5 for a precise statement of this result). This fact allows one to prove the following theorem with an implicit-function theorem argument.

Theorem 1.2 *For each* $\nu \in (0, 1)$ *and each sufficiently small value of* $\delta > 0$ *Eq.* (1.18) *has two homoclinic solutions* $Q^{\delta\pm}$ *which are symmetric, that is invariant under the transformation* $(Q_1(X), Q_2(X)) \mapsto (Q_1(-X), -Q_2(-X))$*, and satisfy the estimate*

$$Q^{\delta\pm}(X) = \pm \begin{pmatrix} h(X) \\ 0 \end{pmatrix} + O(\delta \mathrm{e}^{-\nu|X|})$$

for all $X \in \mathbb{R}$.

Finally, let us briefly mention some related work in the literature. Buffoni and Groves [4] show that (1.17) has an infinite number of geometrically distinct homoclinic solutions which generically resemble multiple copies of one of the 'primary' homoclinic solutions found here. In the present context, this result yields the existence of an infinite family of 'multi-pulse' hydroelastic solitary waves. A variational existence theory for hydroelastic solitary waves in the present parameter regime has been given by Groves et al. [10], while the Kirchgässner reduction (without the Hamiltonian framework) has also been applied to alternative models in which the ice sheet is modelled as a thin Euler–Bernoulli elastic plate (Parau and Dias [32]) and a Kirchhoff–Love elastic plate with non-zero thickness and inertial effects (Ilichev [17], Ilichev and Tomashpolskii [18]). There are also several numerical studies of hydroelastic solitary waves in deep water (Gao et al. [8], Guyenne and Parau [16], Milewski et al. [30]), and an alternative approach to centre-manifold reduction has been given by Chen et al. [5].

2 Formulation as a Spatial Hamiltonian System

In this section, we formulate the hydrodynamic problem as a spatial Hamiltonian system. Starting with a variational principle for the 'flattened' hydrodynamic problem (1.6)–(1.9), we perform a formal Legendre transform to detect its spatial Hamiltonian structure, the correctness of which is confirmed *a posteriori*.

The 'flattened' hydrodynamic problem follows from the variational principle

$$\delta \int L(\eta, \eta_x, \eta_{xx}, \Phi, \Phi_x)\, \mathrm{d}x = 0$$

with Lagrangian

$$\begin{aligned} L&(\eta, \eta_x, \eta_{xx}, \Phi, \Phi_x) \\ &= \int_0^1 \left(-\left[\Phi_x - \Phi_y \frac{y\eta_x}{1+\eta}\right] + \frac{1}{2}\left[\Phi_x - \Phi_y \frac{y\eta_x}{1+\eta}\right]^2 + \frac{1}{2}\frac{\Phi_y^2}{(1+\eta)^2}\right)(1+\eta)\, \mathrm{d}y \\ &\qquad + \tfrac{1}{2}\alpha\eta^2 + \tfrac{1}{2}\beta \frac{\eta_{xx}^2}{(1+\eta_x^2)^{5/2}}. \end{aligned}$$

We perform a formal Legendre transformation (see Lanczos [24, Appendix I]) by defining

$$\begin{aligned}
\rho &= \eta_x, \\
\omega &= \frac{\delta L}{\delta \eta_x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta L}{\delta \eta_{xx}}\right) \\
&= \int_0^1 \left(y\Phi_y - \left[\Phi_x - \Phi_y \frac{y\eta_x}{1+\eta}\right] y\Phi_y \right) \mathrm{d}y + \tfrac{5}{2}\beta \frac{\eta_x \eta_{xx}^2}{(1+\eta_x^2)^{7/2}} - \beta \frac{\eta_{xxx}}{(1+\eta_x^2)^{5/2}}, \\
\xi &= \frac{\delta L}{\delta \eta_{xx}} = \beta \frac{\eta_{xx}}{(1+\eta_x^2)^{5/2}}, \\
\Psi &= \frac{\delta L}{\delta \Phi_x} = -(1+\eta) + \left(\Phi_x - \Phi_y \frac{y\eta_x}{1+\eta}\right)(1+\eta)
\end{aligned}$$

and defining the Hamiltonian function by

$$\begin{aligned}
H(\eta, \rho, \omega, \xi, \Phi, \Psi) &= \omega \eta_x + \xi \eta_{xx} + \int_0^1 \Psi \Phi_x \, \mathrm{d}y - L(\eta, \rho, \omega, \xi, \Phi, \Psi) \\
&= \omega\rho - \tfrac{1}{2}\alpha\eta^2 + \frac{\xi^2}{2\beta}(1+\rho^2)^{5/2} + \tfrac{1}{2}(1+\eta) \\
&\quad + \int_0^1 \left(\frac{1}{2(1+\eta)}(\Psi^2 - \Phi_y^2) + \Psi + \frac{\rho y \Phi_y \Psi}{1+\eta} \right) \mathrm{d}y.
\end{aligned}$$

Writing $\alpha = \alpha_0 + \varepsilon_1$ and $\beta = \beta_0 + \varepsilon_2$, where α_0 and β_0 are fixed, we find that Hamilton's equations are given explicitly by

$$\eta_x = \frac{\delta H^\varepsilon}{\delta \omega} = \rho, \tag{2.1}$$

$$\rho_x = \frac{\delta H^\varepsilon}{\delta \xi} = \frac{(1+\rho^2)^{5/2}}{\beta_0 + \varepsilon_2}\xi, \tag{2.2}$$

$$\omega_x = -\frac{\delta H^\varepsilon}{\delta \eta} = \frac{1}{(1+\eta)^2} \int_0^1 \left(\tfrac{1}{2}(\Psi^2 - \Phi_y^2) + \rho y \Phi_y \Psi \right) \mathrm{d}y + (\alpha_0 + \varepsilon_1)\eta - \tfrac{1}{2}, \tag{2.3}$$

$$\xi_x = -\frac{\delta H^\varepsilon}{\delta \rho} = -\omega - \frac{5}{2}\frac{\rho}{\beta_0 + \varepsilon_2}\xi^2(1+\rho^2)^{3/2} - \frac{1}{1+\eta}\int_0^1 y\Phi_y\Psi \, \mathrm{d}y, \tag{2.4}$$

$$\Phi_x = \frac{\delta H^\varepsilon}{\delta \Psi} = \frac{\Psi + \eta}{1+\eta} + \frac{\rho y \Phi_y}{1+\eta}, \tag{2.5}$$

$$\Psi_x = -\frac{\delta H^\varepsilon}{\delta \Phi} = \frac{1}{1+\eta}(-\Phi_y + \rho y \Psi)_y, \tag{2.6}$$

where the superscript denotes the dependence upon $\varepsilon = (\varepsilon_1, \varepsilon_2)$, with boundary conditions

$$-\Phi_y + y\rho\Psi\big|_{y=0,1} = 0,$$

which emerge from the integration by parts used to compute (2.6). A straightforward calculation shows that the η- and Φ-components of any solution to these equations satisfy (1.6)–(1.9).

Note that Eqs. (2.1)–(2.6) are reversible, that is invariant under the transformation $(\eta, \omega, \rho, \xi, \Phi, \Psi)(x) \mapsto S(\eta, \omega, \rho, \xi, \Phi, \Psi)(-x)$, where the *reverser* is defined by

$$S(\eta, \omega, \rho, \xi, \Phi, \Psi) = (\eta, -\omega, -\rho, \xi, -\Phi, \Psi).$$

They are also invariant under the transformation $\Phi \mapsto \Phi + c$ for any constant c. To eliminate this symmetry it is convenient to replace (Φ, Ψ) with new variables $(\bar{\Phi}, \Phi_0, \bar{\Psi}, \Psi_0)$, where $\bar{\Phi} = \Phi - \Phi_0$, $\bar{\Psi} = \Psi - \Psi_0$ and

$$\Phi_0 = \int_0^1 \Phi \, \mathrm{d}y, \qquad \Psi_0 = \int_0^1 \Psi \, \mathrm{d}y.$$

This transformation leads to a new canonical Hamiltonian system with Hamiltonian

$$\begin{aligned} \bar{H}(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi}, \Phi_0, \Psi_0) &= H(\eta, \omega, \rho, \xi, \bar{\Phi} + \Phi_0, \bar{\Psi} + \Psi_0) \\ &= H(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi} + \Psi_0) \end{aligned}$$

and additional constraints

$$\int_0^1 \bar{\Phi} \, \mathrm{d}y = 0, \qquad \int_0^1 \bar{\Psi} \, \mathrm{d}y = 0.$$

Observe that Φ_0 is a cyclic variable whose conjugate Ψ_0 is a conserved quantity; we proceed in standard fashion by setting $\Psi_0 = -1$, considering the equations for $(\eta, \omega, \rho, \xi, \bar{\Phi}, \bar{\Psi})$ and recovering Φ_0 by quadrature. Dropping the bars for notational simplicity, one finds that Hamilton's equations for the reduced system are

$$\eta_x = \rho, \tag{2.7}$$

$$\rho_x = \frac{(1+\rho^2)^{5/2}}{\beta_0 + \varepsilon_2} \xi, \tag{2.8}$$

$$\omega_x = \frac{1}{(1+\eta)^2} \int_0^1 \left(\tfrac{1}{2}((\Psi - 1)^2 - \Phi_y^2) + \rho y \Phi_y (\Psi - 1) \right) \mathrm{d}y - \tfrac{1}{2} + (\alpha_0 + \varepsilon_1)\eta, \tag{2.9}$$

$$\xi_x = -\omega - \frac{5}{2} \frac{\rho}{\beta_0 + \varepsilon_2} \xi^2 (1+\rho^2)^{3/2} - \frac{1}{1+\eta} \int_0^1 y \Phi_y (\Psi - 1) \, \mathrm{d}y, \tag{2.10}$$

$$\Phi_x = \frac{\Psi}{1+\eta} + \frac{\rho}{1+\eta} \left(y\Phi_y - \int_0^1 y \Phi_y \, \mathrm{d}y \right), \tag{2.11}$$

$$\Psi_x = \frac{1}{1+\eta} (-\Phi_y + \rho y (\Psi - 1))_y, \tag{2.12}$$

with constraints

$$\int_0^1 \Phi \, \mathrm{d}y = 0, \qquad \int_0^1 \Psi \, \mathrm{d}y = 0 \tag{2.13}$$

and boundary conditions

$$-\Phi_y + y\rho(\Psi - 1)\big|_{y=0,1} = 0. \tag{2.14}$$

To make this construction rigorous we recall the differential-geometric definitions of a Hamiltonian system and Hamilton's equations for its associated vector field (see Groves and Toland [13, §1.4]).

Definition 2.1 A Hamiltonian system consists of a triple (M, Ω, H), where M is a manifold, $\Omega : TM \times TM \to \mathbb{R}$ is a closed, weakly nondegenerate bilinear form (the *symplectic 2-form*) and the *Hamiltonian* $H : M \to \mathbb{R}$ is a smooth function. Its Hamiltonian vector field v_H with domain $\mathcal{D}(v_H) \subseteq M$ is defined as follows. The point $m \in M$ belongs to $\mathcal{D}(v_H)$ with $v_H|_m := w \in TM|_m$ if and only if

$$\Omega|_m(w, v) = \mathbf{d}H|_m(v)$$

for all tangent vectors $v \in TM|_m$. Hamilton's equations for (M, Ω, H) are the differential equations

$$\dot{u} = v_H|_u$$

which determine the trajectories $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, \mathcal{D}(v_H))$ of its Hamiltonian vector field.

Let

$$X = \{(\eta, \rho, \omega, \xi, \Phi, \Psi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^1(0, 1) \times \bar{L}^2(0, 1)\},$$

where the overline denotes the subspace of functions with zero mean value, and define the manifold

$$M = \{(\eta, \rho, \omega, \xi, \Phi, \Psi) \in X : \eta > -1\}.$$

The 2-form Ω on M defined by

$$\begin{aligned}\Omega|_m\big((\eta_1, \rho_1, \omega_1, \xi_1, \Phi_1, \Psi_1), (\eta_2, \rho_2, \omega_2, \xi_2, \Phi_2, \Psi_2)\big) \\ = \int_0^1 (\Psi_2\Phi_1 - \Phi_2\Psi_1) \, \mathrm{d}y + \omega_2\eta_1 + \xi_2\rho_1 - \eta_2\omega_1 - \rho_2\xi_1\end{aligned}$$

is skew-symmetric, closed (since it is constant) and weakly nondegenerate at each point of M. The triple $(M, \Omega, H^\varepsilon)$ is, therefore, a Hamiltonian system in the sense of Definition 2.1.

Theorem 2.2 *Consider the Hamiltonian system* $(M, \Omega, H^\varepsilon)$*. The domain of the corresponding Hamiltonian vector field* v_{H^ε} *is*

$$\mathcal{D}(v_{H^\varepsilon}) = \Big\{(\eta, \rho, \omega, \xi, \Phi, \Psi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \bar{H}^2(0,1) \times \bar{H}^1(0,1) : \\ \eta > -1,\ \Phi_y - y\rho(\Psi - 1)\big|_{y=0,1} = 0\Big\},$$

upon which it is given by the right-hand sides of equations (2.7)–(2.12).

Proof Let $\bar{v}|_m = (\bar{\eta}, \bar{\rho}, \bar{\omega}, \bar{\xi}, \bar{\Phi}, \bar{\Psi}) \in TM|_m$, where $m = (\eta, \rho, \omega, \xi, \Phi, \Psi) \in M$. The point m lies in $\mathcal{D}(v_{H^\varepsilon})$ with $v_{H^\varepsilon}|_m = \bar{v}|_m$ if and only if

$$\Omega|_m(\bar{v}|_m, v_1|_m) = \mathbf{d}H^\varepsilon|_m(v_1|_m),$$

that is

$$\begin{aligned}
&\omega_1\bar{\eta} + \xi_1\bar{\rho} - \eta_1\bar{\omega} - \rho_1\bar{\xi} + \int_0^1 (\Psi_1\bar{\Phi} - \Phi_1\bar{\Psi}\,\mathrm{d}y \\
&= \left(-(\alpha_0 + \varepsilon_1)\eta + \tfrac{1}{2} - \frac{1}{2(1+\eta)^2}\int_0^1 \big((\Psi-1)^2 - \Phi_y^2\big)\,\mathrm{d}y \right.\\
&\qquad \left. - \frac{1}{(1+\eta)^2}\int_0^1 \rho y \Phi_y(\Psi - 1)\,\mathrm{d}y\right)\eta_1 \\
&\quad + \left(\omega + \frac{5}{2}\frac{\xi^2}{\beta_0 + \varepsilon_2}\rho(1+\rho^2)^{3/2} + \int_0^1 \frac{y\Phi_y(\Psi-1)}{1+\eta}\,\mathrm{d}y\right)\rho_1 \\
&\quad + \rho\omega_1 + \frac{\xi}{\beta_0 + \varepsilon_2}(1+\rho^2)^{5/2}\xi_1 + \frac{1}{1+\eta}\int_0^1 (-\Phi_y + \rho y(\Psi - 1))\Phi_{1y}\,\mathrm{d}y \\
&\quad + \frac{1}{1+\eta}\int_0^1 (\Psi + \eta + \rho y \Phi_y)\Psi_1\,\mathrm{d}y
\end{aligned} \tag{2.15}$$

for all $\bar{v}_1|_m = (\eta_1, \rho_1, \omega_1, \xi_1, \Phi_1, \Psi_1) \in TM|_m$.

The four particular choices $(\eta_1, \rho_1, \xi_1, \Phi_1, \Psi_1) = (0,0,0,0,0)$, $(\eta_1, \rho_1, \omega_1, \Phi_1, \Psi_1) = (0,0,0,0,0)$, $(\rho_1, \omega_1, \xi_1, \Phi_1, \Psi_1) = (0,0,0,0,0)$ and $(\eta_1, \omega_1, \xi_1, \Phi_1, \Psi_1) = (0,0,0,0,0)$ yield, respectively,

$$\begin{aligned}
\bar{\eta} &= \rho, \\
\bar{\rho} &= (1+\rho^2)^{5/2}\frac{\xi}{\beta_0 + \varepsilon_2}, \\
\bar{\omega} &= \frac{1}{(1+\eta)^2}\int_0^1 \Big(\tfrac{1}{2}((\Psi-1)^2 - \Phi_y^2) + \rho y \Phi_y(\Psi - 1)\Big)\,\mathrm{d}y - \tfrac{1}{2} + (\alpha_0 + \varepsilon_1)\eta, \\
\bar{\xi} &= -\omega - \frac{5}{2}\frac{\rho}{\beta_0 + \varepsilon_2}\xi^2(1+\rho^2)^{3/2} - \frac{1}{1+\eta}\int_0^1 y\Phi_y(\Psi - 1)\,\mathrm{d}y,
\end{aligned}$$

and with these expressions for $\bar{\omega}$, $\bar{\eta}$, $\bar{\rho}$ and $\bar{\xi}$ equation (2.15) becomes

$$\int_0^1 \big(\Psi_1\bar{\Phi} - \Phi_1\bar{\Psi}\big)\,\mathrm{d}y = \frac{1}{1+\eta}\int_0^1 \big((-\Phi_y + \rho y(\Psi-1))\Phi_{1y} + (\Psi+\eta+\rho y\Phi_y)\Psi_1\big)\,\mathrm{d}y.$$

Choosing $\tilde{\Phi} \in H^1(0,1)$, $\tilde{\Psi} \in L^2(0,1)$ and setting $\Phi_1 = 0$, $\Psi_1 = \tilde{\Psi} - \int_0^1 \tilde{\Psi}\,\mathrm{d}y$ and $\Phi_1 = \tilde{\Phi} - \int_0^1 \tilde{\Phi}\,\mathrm{d}y$, $\Psi_1 = 0$, we thus find that

$$\int_0^1 \tilde{\Psi}\left(\frac{\Psi}{1+\eta} + \frac{\rho}{1+\eta}\left(y\Phi_y - \int_0^1 y\Phi_y\,\mathrm{d}y\right) - \bar{\Phi}\right)\mathrm{d}y = 0$$

for all $\tilde{\Psi} \in L^2(0,1)$, and in particular for all $\tilde{\Psi} \in C_0^\infty(0,1)$, which implies that

$$\bar{\Phi} = \frac{\Psi}{1+\eta} + \frac{\rho}{1+\eta}\left(y\Phi_y - \int_0^1 y\Phi_y\,\mathrm{d}y\right) \in H^1(0,1), \tag{2.16}$$

and

$$\int_0^1 \left(\tilde{\Phi}\bar{\Psi} + \tilde{\Phi}_y\left(-\frac{\Phi_y}{1+\eta} + \frac{\rho y(\Psi-1)}{1+\eta}\right)\right)\mathrm{d}y = 0 \tag{2.17}$$

for all $\tilde{\Phi} \in H^1(0,1)$, and in particular for all $\tilde{\Phi} \in C_0^\infty(0,1)$, which implies that

$$\bar{\Psi} = \frac{1}{1+\eta}(-\Phi_y + \rho y(\Psi-1))_y \in L^2(0,1) \tag{2.18}$$

in the weak sense. It follows from (2.16) and (2.18) that $\Phi_y \in H^1(0,1)$ and $\Psi_y \in L^2(0,1)$, so that $\Phi \in H^2(0,1)$ and $\Psi \in H^1(0,1)$.

Finally, integrating the second term in (2.17) by parts and using (2.18), we find that

$$\left[\Phi_1\left(-\frac{\Phi_y}{1+\eta} + \frac{\rho y(\Psi-1)}{1+\eta}\right)\right]_0^1 = 0$$

for all $\Phi_1 \in C^\infty[0,1]$, so that

$$-\frac{\Phi_y}{1+\eta} + \frac{\rho y(\Psi-1)}{1+\eta}\bigg|_{y=0,1} = 0.$$

□

One cannot work directly with (2.7)–(2.12) because of the nonlinear boundary condition at $y = 1$ in the domain of the Hamiltonian vector field v_{H^ε}. We overcome this difficulty using the change of variable $(\eta, \rho, \omega, \xi, \Phi, \Psi) \mapsto (\eta, \rho, \omega, \xi, \Gamma, \Psi)$, where

$$\Gamma = \Phi - \rho\int_0^y s(\Psi(s)-1)\,\mathrm{d}s + \rho\int_0^1\int_0^y s(\Psi(s)-1)\,\mathrm{d}s\,\mathrm{d}y,$$

which is a smooth diffeomorphism $X \to X$ and $M \to M$ with inverse

$$\Phi = \Gamma + \rho \int_0^y s(\Psi(s) - 1)\,\mathrm{d}s - \rho \int_0^1 \int_0^y s(\Psi(s) - 1)\,\mathrm{d}s\,\mathrm{d}y.$$

This change of variable transforms equations (2.7)–(2.12) into

$$\eta_x = \rho, \tag{2.19}$$

$$\rho_x = \frac{(1+\rho^2)^{5/2}}{\beta_0 + \varepsilon_2}\xi, \tag{2.20}$$

$$\begin{aligned}\omega_x = \frac{1}{(1+\eta)^2} \int_0^1 \Big\{ &\tfrac{1}{2}(\Psi - 1)^2 - \tfrac{1}{2}\big(\Gamma_y + \rho y(\Psi - 1)\big)^2\big) \\ &+ \rho y \Gamma_y (\Psi - 1) + \rho^2 y^2 (\Psi - 1)^2 \Big\}\,\mathrm{d}y \\ &- \tfrac{1}{2} + (\alpha_0 + \varepsilon_1)\eta,\end{aligned} \tag{2.21}$$

$$\xi_x = -\omega - \frac{5}{2}\frac{\rho}{\beta_0 + \varepsilon_2}\xi^2(1+\rho^2)^{3/2} - \frac{1}{1+\eta}\int_0^1 y(\Gamma_y + \rho y(\Psi - 1))(\Psi - 1)\,\mathrm{d}y, \tag{2.22}$$

$$\begin{aligned}\Gamma_x = \frac{\Psi}{1+\eta} &+ \frac{\rho y(\Gamma_y + \rho y(\Psi - 1))}{1+\eta} - \frac{\rho}{1+\eta}\int_0^1 y(\Gamma_y + \rho y(\Psi - 1))\,\mathrm{d}y \\ &- \frac{(1+\rho^2)^{5/2}}{\beta_0 + \varepsilon_2}\xi \int_0^y s(\Psi(s) - 1)\,\mathrm{d}s \\ &+ \frac{(1+\rho^2)^{5/2}}{\beta_0 + \varepsilon_2}\xi \int_0^1 \int_0^y s(\Psi(s) - 1)\,\mathrm{d}s\,\mathrm{d}y \\ &+ \rho \int_0^y \frac{s}{1+\eta}\Gamma_{yy}\,\mathrm{d}s - \rho \int_0^1 \int_0^y \frac{s}{1+\eta}\Gamma_{yy}\,\mathrm{d}s\,\mathrm{d}y,\end{aligned} \tag{2.23}$$

$$\Psi_x = -\frac{1}{1+\eta}\Gamma_{yy} \tag{2.24}$$

and the boundary conditions (2.14) into

$$\Gamma_y\big|_{y=0,1} = 0. \tag{2.25}$$

Equations (2.19)–(2.24) are Hamilton's equations for the Hamiltonian system $(M, \Upsilon, \hat{H}^\varepsilon)$, where

$$\begin{aligned}\hat{H}^\varepsilon(\eta, \rho, \omega, \xi, \Gamma, \Psi) = \omega\rho - \tfrac{1}{2}(\alpha_0 + \varepsilon_1)\eta^2 + \frac{\xi^2}{2\beta_0 + \varepsilon_2}(1+\rho^2)^{5/2} + \tfrac{1}{2}(\eta - 1) \\ + \int_0^1 \Big\{ \frac{1}{2(1+\eta)}\big((\Psi - 1)^2 - (\Gamma_y + \rho y(\Psi - 1))^2\big) \\ + \frac{1}{1+\eta}\big(\rho y \Gamma_y(\Psi - 1) + \rho^2 y^2 (\Psi - 1)^2\big)\Big\}\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}\Upsilon|_{(\eta,\rho,\omega,\xi,\Gamma,\Psi)}&\big((\tilde{\eta}_1,\tilde{\rho}_1,\tilde{\omega}_1,\tilde{\xi}_1,\tilde{\Gamma}_1,\tilde{\Psi}_1),(\tilde{\eta}_2,\tilde{\rho}_2,\tilde{\omega}_2,\tilde{\xi}_2,\tilde{\Gamma}_2,\tilde{\Psi}_2)\big)\\ &=\int_0^1\left\{\tilde{\Psi}_2\left(\tilde{\Gamma}_1+\tilde{\rho}_1\int_0^y s\Psi(s)\,\mathrm{d}s+\rho\int_0^y s\tilde{\Psi}_1(s)\,\mathrm{d}s\right)-\tfrac{1}{2}\tilde{\rho}_1 y^2\tilde{\Psi}_2\right.\\ &\qquad\left.-\tilde{\Psi}_1\left(\tilde{\Gamma}_2+\tilde{\rho}_2\int_0^y s\Psi(s)\,\mathrm{d}s+\rho\int_0^y s\tilde{\Psi}_2(s)\,\mathrm{d}s\right)+\tfrac{1}{2}\tilde{\rho}_2 y^2\tilde{\Psi}_1\right\}\mathrm{d}y\\ &\qquad+\tilde{\omega}_2\tilde{\eta}_1+\tilde{\xi}_2\tilde{\rho}_1-\tilde{\eta}_2\tilde{\omega}_1-\tilde{\rho}_2\tilde{\xi}_1;\end{aligned}$$

furthermore,

$$\mathcal{D}(v_{\hat{H}^\varepsilon})=\left\{(\eta,\rho,\omega,\xi,\Gamma,\Psi)\in M:\Gamma_y\big|_{y=0,1}=0\right\}.$$

We write (2.19)–(2.24) as

$$u_x=Lu+N^\varepsilon(u),$$

in which $L=\mathrm{d}v_{\hat{H}^0}[0]$, so that

$$L\begin{pmatrix}\eta\\ \rho\\ \omega\\ \xi\\ \Gamma\\ \Psi\end{pmatrix}=\begin{pmatrix}\rho\\ \frac{1}{\beta_0}\xi\\ (\alpha_0-1)\eta\\ -\omega-\frac{1}{3}\rho+\int_0^1 y\Gamma_y\,\mathrm{d}y\\ \frac{1}{2\beta_0}(y^2-\frac{1}{3})\xi+\Psi\\ -\Gamma_{yy}\end{pmatrix}$$

with

$$\mathcal{D}(L)=\left\{(\eta,\rho,\omega,\xi,\Gamma,\Psi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\bar{H}^2(0,1)\times\bar{H}^1(0,1):\Gamma_y\big|_{y=0,1}=0\right\}.$$

3 Spectral Analysis

In this section, we examine the spectrum of the linear operator $L:\mathcal{D}(L)\subseteq X\to X$ in detail. Our first result is obtained by a straightforward calculation.

Proposition 3.1 *A complex number λ is an eigenvalue of L if and only if*

$$\alpha_0+\lambda^4\beta_0=\lambda\cot(\lambda);\tag{3.1}$$

its eigenspace is one-dimensional and spanned by, respectively,

$$e_\lambda = \begin{pmatrix} \frac{1}{\lambda}\sin(\lambda) \\ \sin(\lambda) \\ \frac{1}{\lambda^2}(\alpha_0 - 1)\sin(\lambda) \\ \frac{1}{\lambda^2}\cos(\lambda) - \frac{1}{\lambda^3}\alpha_0\sin(\lambda) \\ \frac{1}{\lambda}\cos(\lambda y) - \frac{1}{\lambda^2}\sin(\lambda) + \frac{1}{2}(y^2 - \frac{1}{3})\sin(\lambda) \\ \cos(\lambda y) - \frac{1}{\lambda}\sin(\lambda) \end{pmatrix}, \qquad e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for $\lambda \neq 0$ *and* $\lambda = 0$ *(which arises only for* $\alpha_0 = 1$*). All eigenvalues are also algebraically simple, with the exception of the zero eigenvalue at* $\alpha_0 = 1$ *and the purely imaginary eigenvalues* $\pm \mathrm{i}s$ *at the point* $(\beta_0(s), \alpha_0(s))$ *of the curve*

$$C = \left\{(\beta_0(s), \alpha_0(s)) = \left(\frac{1}{4s^3}\coth(s) - \frac{1}{4s^2\sinh^2(s)}, \frac{3s}{4}\coth(s) + \frac{s^2}{4\sinh^2(s)}\right) : s \in (0, \infty)\right\}$$

in the parameter plane which are algebraically double.

The following lemma gives more precise information on the point spectrum of L.

Lemma 3.2 *Choose* $(\beta_0, \alpha_0) \in C$. *The point spectrum of* L *consists of a countably infinite family* $\{\lambda_k\}_{k \in \mathbb{Z}\setminus\{0\}}$ *of simple real eigenvalues, where* $\{\lambda_k\}_{k=1}^{\infty}$ *are the positive real solutions of equation* (3.1) *and* $\lambda_{-k} = -\lambda_k$ *for* $k = 1, 2, \ldots$ *together with*

(a) *two plus–minus pairs of simple purely imaginary eigenvalues if* $\alpha_0 > 1$ *and* (β_0, α_0) *lies to the left of the curve* C *in the parameter plane,*
(b) *a plus–minus pair of algebraically double purely imaginary eigenvalues* $\pm \mathrm{i}s$ *if* (β_0, α_0) *is the point with parameter value* s *on the curve* C,
(c) *a plus–minus quartet of genuinely complex eigenvalues if* $\alpha_0 > 1$ *and* (β_0, α_0) *lies to the right of the curve* C *in the parameter plane,*
(d) *a plus–minus pair of simple purely imaginary eigenvalues and an algebraically double zero eigenvalue if* $\alpha_0 = 1$,
(e) *an additional plus–minus pair of simple real eigenvalues and a plus–minus pair of simple purely imaginary eigenvalues if* $\alpha_0 < 1$.

Furthermore, $\lambda_k \in (k\pi, (k+1)\pi)$ *for* $k = 1, 2, \ldots$ *and*

$$\lambda_k^2 = k^2\pi^2 + \frac{2}{\beta_0} + o\left(\frac{1}{k}\right)$$

for large k.

Proof Observe that λ solves (3.1) if and only if $\nu = \lambda^2$ is an eigenvalue of the non-self-adjoint Sturm–Liouville problem

$$-v_{yy} = \nu v, \tag{3.2}$$

$$\frac{v_y(1)}{v(1)} = \alpha_0 + \beta_0\nu^2, \tag{3.3}$$

$$v(0) = 0. \tag{3.4}$$

This problem has a countable number of (not necessarily real) eigenvalues $\{\nu_n\}_{n\in\mathbb{N}_0}$, which repeated according to algebraic multiplicity and listed according in increasing absolute value, are given asymptotically for large n by

$$\nu_n = (n-1)^2\pi^2 + \frac{2}{\beta_0} + o\left(\frac{1}{n}\right) \tag{3.5}$$

(see Binding et al. [3, Theorem 2.2]). The real eigenvalues of the spectral problem (3.2)–(3.4) correspond to the intersections in the (ν, s) plane of the parabola $s = \alpha_0 + \beta_0\nu^2$ and the curve $s = B(\nu)$, where $B(\nu) = \sqrt{\nu}\cot\sqrt{\nu}$. The function $B(\nu)$ has poles exactly at the *Dirichlet eigenvalues*

$$\nu_n^{\mathrm{D}} = (n+1)^2\pi^2, \qquad n \in \mathbb{N}_0 \tag{3.6}$$

of the self-adjoint problem in which (3.3) is replaced by $v(1) = 0$; it is strictly decreasing from $+\infty$ to $-\infty$ in each interval $(-\infty, \nu_0^{\mathrm{D}})$ and $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}_0$. It follows that (3.2)–(3.4) has at least one real eigenvalue in each interval $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}_0$ (see Fig. 5).

Comparing (3.5) with (3.6) and using the above geometrical characterisation of the real eigenvalues, one concludes that

(1) each interval $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}$ contains a simple real eigenvalue;
(2) there are precisely two additional eigenvalues (counted according to algebraic multiplicity) in the form of either
   (a) a complex-conjugate pair (with non-vanishing imaginary part) whose absolute value is less than ν_0^{D} (Fig. 5**a**),
   (b) one negative, algebraically double eigenvalue (Fig. 5**b**),
   (c) two simple real eigenvalues to the left of ν_0^{D}, at least one of which is negative (Fig. 5**c–e**).

The solutions λ of (3.1) are recovered from the above analysis by the formula $\nu = \lambda^2$, so that in particular they occur in plus-minus pairs. Clearly, (3.1) has a real solution in each interval $((\nu_n^{\mathrm{D}})^{1/2}, (\nu_{n+1}^{\mathrm{D}})^{1/2})$ and $(-(\nu_{n+1}^{\mathrm{D}})^{1/2}, -(\nu_n^{\mathrm{D}})^{1/2})$, $n \in \mathbb{N}_0$ (see point (1) above), and it follows from point (2) that there are four additional solutions (counted according to multiplicity). The results in Proposition 3.1 and the fact that $B(0) = 1$ show that these four solutions are described by precisely one of the statements (a)–(e) (according to which of the scenarios in Fig. 5 occurs).

The asymptotic formula for λ_k follows by writing $k = n + 1$. □

According to this lemma the purely imaginary eigenvalues of L appear in pairs $\pm\mathrm{i}s$ satisfying the dispersion relation

$$\alpha_0 + s^4\beta_0 = s\coth(s). \tag{3.7}$$

Fig. 4 shows the dependence of these eigenvalues upon β_0 and α_0. At each point of $\{\alpha_0 = 1\}$, two real eigenvalues become purely imaginary by colliding at the origin, while at each point of the curve C two pairs of purely imaginary eigenvalues become

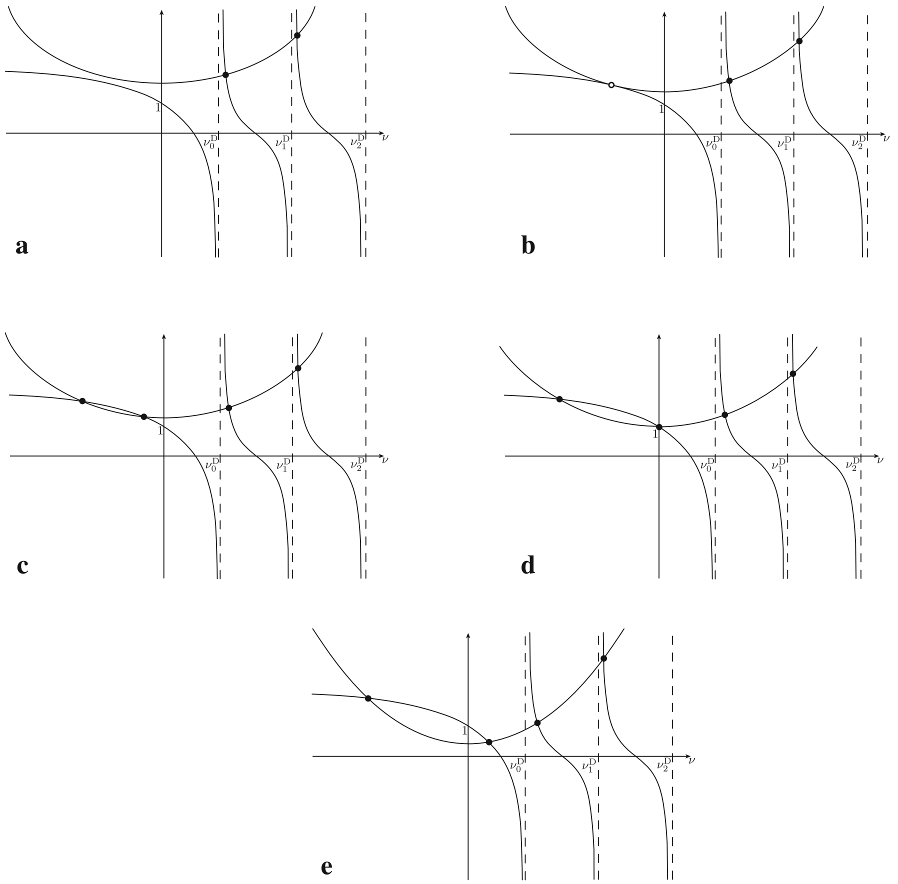

Fig. 5 Geometric characterisation of the eigenvalues ν_n as the points of intersection of the curve $s = B(\nu)$ with the parabola $s = \alpha_0 + \beta_0 \nu^2$; one real eigenvalue lies in each interval $(\nu_n^{\mathrm{D}}, \nu_{n+1}^{\mathrm{D}})$, $n \in \mathbb{N}_0$. **a** Two additional complex eigenvalues; **b** one additional algebraically double negative eigenvalue; **c**–**e** two additional real eigenvalues

complex by colliding at non-zero points $\pm \mathrm{i}s$ on the imaginary axis. For later reference, we record the formulae

$$e = \begin{pmatrix} \sinh(s) \\ \mathrm{i}s \sinh(s) \\ -\mathrm{i}\cosh(s) + \frac{\mathrm{i}}{s}\sinh(s) + \mathrm{i}\beta_0 s^3 \sinh(s) \\ -\beta_0 s^2 \sinh(s) \\ -\mathrm{i}\cosh(sy) + \frac{\mathrm{i}}{s}\sinh(s) - \frac{1}{2}\mathrm{i}s(y^2 - \frac{1}{3})\sinh(s) \\ s\cosh(sy) - \sinh(s) \end{pmatrix},$$

$$f = \begin{pmatrix} -\mathrm{i}\cosh(s) \\ \sinh(s) + s\cosh(s) \\ -\beta_0 s^2 \sinh(s) - \frac{1}{s^2}\sinh(s) - \frac{\alpha_0}{s}\cosh(s) + \frac{2}{s}\cosh(s) \\ \mathrm{i}\beta_0 s^2 \cosh(s) + 2\mathrm{i}\beta_0 s \sinh(s) \\ -y\sinh(sy) - \frac{1}{s^2}\sinh(s) + \frac{1}{s}\cosh(s) + \frac{1}{2}(y^2 - \frac{1}{3})(s\cosh(s) + \sinh(s)) \\ -\mathrm{i}sy\sinh(sy) - \mathrm{i}\cosh(sy) + \mathrm{i}\cosh(s) \end{pmatrix}$$

for an eigenvector e and generalised eigenvector f with eigenvalue $\mathrm{i}s$ when $(\beta_0, \alpha_0) \in C$ (the corresponding formulae for the zero eigenvalue at $\alpha_0 = 1$ are $e_0 = (1, 0, 0, 0, 0, 0)^{\mathrm{T}}$, $f_0 = (0, 1, -\frac{1}{3}, 0, 0, 0)^{\mathrm{T}}$).

Lemma 3.3 *The operator L is regular, that is its spectrum consists entirely of isolated eigenvalues of finite algebraic multiplicity.*

Proof Since $\mathcal{D}(L)$ is compactly embedded in X it suffices to show that $\rho(L)$ is non-empty, so that L has compact resolvent (Kato [20, Theorem III.6.29]). In the case $\alpha_0 \neq 1$, a direct calculation shows that L is invertible with

$$L^{-1}\begin{pmatrix} \eta \\ \rho \\ \omega \\ \xi \\ \Gamma \\ \Psi \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\alpha_0 - 1}\omega \\ \eta \\ -\frac{1}{3}\eta - \xi - \displaystyle\int_0^1 y \int_0^y \Psi(t)\,\mathrm{d}t\,\mathrm{d}y \\ \beta_0 \rho \\ -\displaystyle\int_0^y \int_0^s \Psi(t)\,\mathrm{d}t\,\mathrm{d}s + \int_0^1 \int_0^y \int_0^s \Psi(t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}y \\ \Gamma - \frac{1}{2}(y^2 - \frac{1}{3})\rho \end{pmatrix}.$$

To deal with the case $\alpha_0 = 1$ note that $L|_{\alpha_0=1}$ is a compact perturbation of $L|_{\alpha_0=\frac{1}{2}}$, so that the essential spectrum of these two operators (the set of λ for which $(\lambda I - L)$ is not Fredholm with index zero) is identical (see Schechter [34]). It follows that the spectrum of $L|_{\alpha_0=1}$ consists of the solution set of (3.1); in particular, its resolvent set is non-empty. □

Finally, we show that the set of generalised eigenvectors of L form a Schauder basis for X, which is henceforth replaced by its complexification. In particular, we show that this set is a *Riesz basis*, that is a basis obtained from an orthonormal basis by an isomorphism (see Gohberg and Krein [9, §VI.2]); note that we use the Dirichlet norm for the space $\bar{H}^1(0, 1)$.

Proposition 3.4 *The set*

$$\mathcal{A} = \left\{ \begin{pmatrix} (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\}_{k \in \mathbb{Z}\setminus\{0\}}$$

is an orthonormal basis for $\bar{H}^1(0, 1) \times \bar{L}^2(0, 1)$.

Proof Note that $\{\sqrt{2}\cos(k\pi y)\}_{k=1}^{\infty}$, $\{\sqrt{2}(k\pi)^{-1}\cos(k\pi y)\}_{k=1}^{\infty}$ are orthonormal bases for, respectively, $\bar{L}^2(0,1)$ and $\bar{H}^1(0,1)$. It, therefore, follows from

$$\mathrm{sp}\left\{\begin{pmatrix}(k\pi)^{-1}\cos(k\pi y)\\ \cos(k\pi y)\end{pmatrix}\right\}_{k\in\mathbb{Z}\setminus\{0\}} = \mathrm{sp}\left\{\begin{pmatrix}\sqrt{2}(k\pi)^{-1}\cos(k\pi y)\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \sqrt{2}\cos(k\pi y)\end{pmatrix}\right\}_{k=1}^{\infty}$$

in $\bar{H}^1(0,1)\times\bar{L}^2(0,1)$ that $\mathcal{A}$ is complete, and it is evidently orthonormal. □

Corollary 3.5 *Let P be the spectral projection onto the four-dimensional subspace of X corresponding to the eigenvalues shown in Fig. 4, and let $\{e_1, e_2, e_3, e_4\}$ be a basis for $P[X]$ consisting of generalised eigenvectors of L. The set*

$$\{e_1, e_2, e_3, e_4\}\cup\{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}, \qquad f_k = \begin{pmatrix}0\\0\\0\\0\\(k\pi)^{-1}\cos(k\pi y)\\ \cos(k\pi y)\end{pmatrix}$$

is a Riesz basis for X.

Proof Let $\{g_1, g_2, g_3, g_4\}$ denote the usual basis for the subset $\mathbb{C}^4\times\{(0,0)\}$ of X, and note that $\{g_1, g_2, g_3, g_4\}\cup\{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is an orthonormal basis for X. Let $\pi : X\to\mathbb{C}^4$ denote the projection $(\eta,\rho,\omega,\xi,\Phi,\Psi)\mapsto(\eta,\rho,\omega,\xi)$, and note that $\{\pi e_1, \pi e_2, \pi e_3, \pi e_4\}$ spans $\mathbb{C}^4$.

The formula $S(\eta,\rho,\omega,\xi,\Phi,\Psi) = (T(\eta,\rho,\omega,\xi),(\Phi,\Psi))$, where $T(\eta,\rho,\omega,\xi)$ is the coordinate vector of (η,ρ,ω,ξ) with respect to the basis $\{\pi e_1, \pi e_2, \pi e_3, \pi e_4\}$ for $\mathbb{C}^4$, defines an isomorphism $X\to X$ with

$$S[\{g_1, g_2, g_3, g_4\}\cup\{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}] = \{e_1, e_2, e_3, e_4\}\cup\{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}.$$

It follows that $\{e_1, e_2, e_3, e_4\}\cup\{f_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis for X. □

Theorem 3.6 *The set $\{e_1, e_2, e_3, e_4\}\cup\{e_{\lambda_k}\}_{k\in\mathbb{Z}\setminus\{0\}}$ is a Riesz basis for X.*

Proof We first note that the set $\{e_1, e_2, e_3, e_4\}\cup\{e_{\lambda_k}\}_{k\in\mathbb{Z}\setminus\{0\}}$ is ω-linearly independent since it is the union of bases for the generalised eigenspaces of a regular operator (see Gohberg and Krein [9, p. 329]).

Choose $\mu^\star\in(0,\lambda_1)$. The function $h:(0,\infty)\to\mathcal{X}$ defined by

$$h(\mu) = \begin{pmatrix}\frac{1}{\mu}\sin(\mu)\\ \sin(\mu)\\ \frac{1}{\mu^2}(\alpha_0-1)\sin(\mu)\\ \frac{1}{\mu^2}\cos(\mu)-\frac{1}{\mu^3}\alpha_0\sin(\mu)\\ \frac{1}{\mu}\cos(\mu y)-\frac{1}{\mu^2}\sin(\mu)+\frac{1}{2}(y^2-\frac{1}{3})\sin(\mu)\\ \cos(\mu y)-\frac{1}{\mu}\sin(\mu)\end{pmatrix}$$

satisfies

$$\|h(\mu_1) - h(\mu_2)\| \leq \sup_{\mu \in [\mu^\star, \infty)} \|h'(\mu)\| |\mu_1 - \mu_2| \lesssim |\mu_1 - \mu_2|$$

for all $\mu_1, \mu_2 \in (\mu^\star, \infty)$. With $\mu_1 = \lambda_k$ and $\mu_2 = k\pi$ this calculation shows in particular that

$$\begin{aligned}
\left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| &= \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ 0 \\ 0 \end{pmatrix} \right\| \\
&\leq \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| + \frac{1}{k^2\pi^2} \\
&\lesssim |\lambda_k - k\pi| + \frac{1}{k^2\pi^2} \\
&= O(\tfrac{1}{k})
\end{aligned}$$

as $k \to \infty$, and similarly

$$\left\| e_{-\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ -(k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| = \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\| = O(\tfrac{1}{k})$$

as $k \to \infty$. Hence,

$$\sum_{j=1}^{4} \|e_j - e_j\|^2 + \sum_{k \in \mathbb{Z}\setminus\{0\}} \left\| e_{\lambda_k} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ (k\pi)^{-2}\cos(k\pi) \\ (k\pi)^{-1}\cos(k\pi y) \\ \cos(k\pi y) \end{pmatrix} \right\|^2 < \infty$$

and the conclusion now follows by Bari's theorem (Gohberg and Krein [9, Theorem VI.2.3]). □

Let $\{e^1, e^2, e^3, e^4\} \cup \{e^{\lambda_k}\}_{k \in \mathbb{Z}\setminus\{0\}}$ be the dual Riesz basis to $\{e_1, e_2, e_3, e_4\} \cup \{e_{\lambda_k}\}_{k \in \mathbb{Z}\setminus\{0\}}$ (see Gohberg and Krein [9, §VI.1-2]), so that

$$P = \sum_{i=1}^{4} \langle \cdot, e^i \rangle e_i, \qquad (I - P) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \langle \cdot, e^{\lambda_k} \rangle e_{\lambda_k},$$

and define $X_2 = (I - P)X$, $L_2 = L|_{X_2}$ (with $\mathcal{D}(L_2) = \mathcal{D}(L) \cap X_2$). Note that

$$X_2 = \left\{ \sum_{k \in \mathbb{Z}\setminus\{0\}} \beta_k e_{\lambda_k} : \{\beta_k\} \in \ell^2 \right\}, \qquad \mathcal{D}(L_2) = \left\{ \sum_{k \in \mathbb{Z}\setminus\{0\}} \beta_k e_{\lambda_k} : \{\lambda_k \beta_k\} \in \ell^2 \right\}.$$

We conclude this section with a maximal regularity result for $\tilde{L}$ which is used in Sect. 4 below.

Lemma 3.7 *The operator $L_2 : \mathcal{D}(L_2) \subseteq X_2 \to X_2$ has L^2-maximal regularity in the sense that the differential equation*

$$\dot{w} = L_2 w + h$$

admits a unique solution $w \in H^1(\mathbb{R}, X_2) \cap L^2(\mathbb{R}, \mathcal{D}(L_2))$ for each $h \in L^2(\mathbb{R}, X_2)$.

Proof Writing

$$w = \sum_{k \in \mathbb{Z}\setminus\{0\}} w_k e_{\lambda_k}, \qquad h = \sum_{k \in \mathbb{Z}\setminus\{0\}} h_k e_{\lambda_k}$$

(where $w_k = \langle w, e^{\lambda_k} \rangle$, $h_k = \langle h, e^{\lambda_k} \rangle$), we find that

$$\dot{w}_k = \lambda_k w_k + h_k, \tag{3.8}$$

which is solved by

$$w_k(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} h_k(s) \mathrm{e}^{\lambda_k (t-s)} \, \mathrm{d}s, & k < 0, \\ \displaystyle -\int_{t}^{\infty} h_k(s) \mathrm{e}^{\lambda_k (t-s)} \, \mathrm{d}s, & k > 0. \end{cases}$$

Note that

$$\|w_k\|_{L^2(\mathbb{R},\mathbb{R})} \leq \frac{1}{\lambda_k} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}$$

because

$$\begin{aligned}\|w_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 &= \int_{-\infty}^{\infty} \left| \int_t^{\infty} h_k(s) \mathrm{e}^{\lambda_k(t-s)}\, \mathrm{d}s \right|^2 \mathrm{d}t \\ &\leq \int_{-\infty}^{\infty} \int_t^{\infty} \mathrm{e}^{\lambda_k(t-s)}\, \mathrm{d}s \int_t^{\infty} \mathrm{e}^{\lambda_k(t-s)} |h_k(s)|^2\, \mathrm{d}s\, \mathrm{d}t \\ &= \frac{1}{\lambda_k} \int_{-\infty}^{\infty} \int_t^{\infty} \mathrm{e}^{\lambda_k(t-s)} |h_k(s)|^2\, \mathrm{d}s\, \mathrm{d}t \\ &= \frac{1}{\lambda_k} \int_{-\infty}^{\infty} \int_{-\infty}^{s} \mathrm{e}^{\lambda_k(t-s)}\, \mathrm{d}t\, |h_k(s)|^2\, \mathrm{d}s \\ &= \frac{1}{\lambda_k^2} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}^2\end{aligned}$$

for $k > 0$ with a similar calculation for $k < 0$. It follows that

$$\begin{aligned}\|w\|_{L^2(\mathbb{R},X_2)}^2 &= \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}\setminus\{0\}} |w_k(t)|^2\, \mathrm{d}t \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \|w_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \\ &\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \\ &= \|h\|_{L^2(\mathbb{R},X_2)}^2\end{aligned}$$

and similarly

$$\begin{aligned}\|L_2 w\|_{L^2(\mathbb{R},X_2)}^2 &= \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}\setminus\{0\}} \lambda_k^2 |w_k(t)|^2\, \mathrm{d}t \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \lambda_k^2 \|w_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \\ &\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \|h_k\|_{L^2(\mathbb{R},\mathbb{R})}^2 \\ &= \|h\|_{L^2(\mathbb{R},X_2)}^2,\end{aligned}$$

so that w, $L_2 w \in L^2(\mathbb{R}, X)$. Equation (3.8) shows that w is differentiable, satisfies $\dot{w} \in L^2(\mathbb{R}, X)$ and solves the given differential equation.

The uniqueness of the solution follows by noting that Eq. (3.8) has no nontrivial solution in $L^2(\mathbb{R}, \mathbb{R})$ when $h_k = 0$. □

4 Centre-Manifold Reduction

Our strategy in finding solutions to Hamilton's equations (2.19)–(2.24) for $(M, \Upsilon, H^\varepsilon)$ consists in applying a reduction principle which asserts that it is locally equivalent to a finite-dimensional Hamiltonian system. The key result is the following theorem due to Mielke [28, 29].

Theorem 4.1 *Consider the differential equation*

$$\dot{u} = \mathcal{L}u + \mathcal{N}(u; \lambda), \tag{4.1}$$

which represents Hamilton's equations for the reversible Hamiltonian system $(M, \Omega^\lambda, H^\lambda)$*. Here* u *belongs to a Hilbert space* $\mathcal{X}$, $\lambda \in \mathbb{R}^l$ *is a parameter and* $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ *is a densely defined, closed linear operator. Regarding* $\mathcal{D}(\mathcal{L})$ *as a Hilbert space equipped with the graph norm, suppose that* 0 *is an equilibrium point of* (4.1) *when* $\lambda = 0$ *and that*

(H1) *The part of the spectrum* $\sigma(\mathcal{L})$ *of* $\mathcal{L}$ *which lies on the imaginary axis of a finite number of eigenvalues of finite multiplicity and is separated from the rest of* $\sigma(\mathcal{L})$ *in the sense of Kato, so that* $\mathcal{X}$ *admits the decomposition* $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$*, where* $\mathcal{X}_1 = \mathcal{P}(\mathcal{X})$, $\mathcal{X}_2 = (I - \mathcal{P})(\mathcal{X})$ *are the centre and hyperbolic subspaces of* $\mathcal{L}$ *defined by the spectral projection* $\mathcal{P}$ *corresponding the purely imaginary part of* $\sigma(\mathcal{L})$.

(H2) *The operator* $\mathcal{L}_2 = \mathcal{L}|_{\mathcal{X}_2}$ *has* L^2*-maximal regularity in the sense that the differential equation*

$$\dot{u}_2 = \mathcal{L}_2 u_2 + h$$

admits a unique solution $u_2 \in H^1(\mathbb{R}, \mathcal{X}_2) \cap L^2(\mathbb{R}, \mathcal{D}(\mathcal{L}_2))$ *for each* $h \in L^2(\mathbb{R}, \mathcal{X}_2)$.

(H3) *There exist a natural number* k *and neighbourhoods* $\Lambda \subset \mathbb{R}^l$ *of* 0 *and* $U \subset \mathcal{D}(\mathcal{L})$ *of* 0 *such that* $\mathcal{N}$ *is* $(k+1)$ *times continuously differentiable on* $U \times \Lambda$*, its derivatives are bounded and uniformly continuous on* $U \times \Lambda$ *and* $\mathcal{N}(0, 0) = 0$, $\mathrm{d}_1\mathcal{N}[0, 0] = 0$.

Under these hypotheses, there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ *of* 0 *and* $\tilde{U}_1 \subset U \cap \mathcal{X}_1$, $\tilde{U}_2 \subset U \cap \mathcal{X}_2$ *of* 0 *and a reduction function* $r : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2$ *with the following properties. The reduction function* r *is* k *times continuously differentiable on* $\tilde{U}_1 \times \tilde{\Lambda}$ *and* $r(0; 0) = 0$, $\mathrm{d}_1 r[0; 0] = 0$*. The graph* $\tilde{M}^\lambda = \{u_1 + r(u_1; \lambda) \in \mathcal{X}_1 \oplus \mathcal{X}_2 : u_1 \in \tilde{U}_1\}$ *is a Hamiltonian centre manifold for* (4.1)*, so that*

(i) $\tilde{M}^\lambda$ *is a locally invariant manifold of* (4.1)*: through every point in* $\tilde{M}^\lambda$*, there passes a unique solution of* (4.1) *that remains on* $\tilde{M}^\lambda$ *as long as it remains in* $\tilde{U}_1 \times \tilde{U}_2$.

(ii) *Every bounded solution* $u(x)$, $x \in \mathbb{R}$ *of* (4.1) *that satisfies* $(u_1(x), u_2(x)) \in \tilde{U}_1 \times \tilde{U}_2$ *lies completely in* $\tilde{M}^\lambda$.

(iii) *Every solution* $u_1 : (x_1, x_2) \to \tilde{U}_1$ *of the reduced equation*

$$\dot{u}_1 = \mathcal{L}u_1 + \tilde{\mathcal{N}}^\lambda(u_1), \tag{4.2}$$

where $\tilde{\mathcal{N}}^\lambda(u_1) = \mathcal{P}\mathcal{N}(u_1 + r(u_1;\lambda);\lambda)$*, generates a solution*

$$u(x) = u_1(x) + r(u_1(x);\lambda)$$

of the full equation (4.1).

(iv) $\tilde{M}^\lambda$ *is a symplectic submanifold of* M *and the flow determined by the Hamiltonian system* $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$*, where the tilde denotes restriction to* $\tilde{M}^\lambda$*, coincides with the flow on* $\tilde{M}^\lambda$ *determined by* $(M, \Omega^\lambda, H^\lambda)$*. The reduced equation* (4.2) *is reversible and represents Hamilton's equations for* $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$.

Remarks 4.2 (i) We find that

$$\begin{aligned}\tilde{H}^\lambda(u_1) &= H^\lambda(u_1 + r(u_1;\lambda)),\\ \tilde{\Omega}^\lambda|_{u_1}(v_1, v_2) &= \Omega^0|_0(v_1 + \mathrm{d}_1 r[u_1;\lambda](v_1), v_2 + \mathrm{d}r[u_1;\lambda](v_2))\\ &= \Omega^0|_0(v_1, v_2) + O(|(\lambda, u_1)|)\end{aligned} \tag{4.3}$$

as $(\lambda, u_1) \to 0$. Using a parameter-dependent version of Darboux's theorem (e.g. see Buffoni and Groves [4]), we may assume that the remainder term in (4.3) vanishes identically.

(ii) Substituting $u = u_1 + r(u_1;\lambda)$ into (4.1) and eliminating $\dot{u}_1$ using (4.2) leads to the equation

$$\mathcal{L}r(u_1;\lambda) - \mathrm{d}_1 r[u_1;\lambda](\mathcal{L}u_1) = \tilde{\mathcal{N}}^\lambda(u_1) + \mathrm{d}_1 r[u_1;\lambda](\tilde{\mathcal{N}}^\lambda(u_1)) - \mathcal{N}(u_1 + r(u_1;\lambda);\lambda),$$

which can be used to recursively determine the terms in the Taylor series of $r(u_1;\lambda)$ and $\mathcal{N}^\lambda(u_1)$.

We proceed by choosing $(\beta_0(s), \alpha_0(s)) \in C$, setting $(\varepsilon_1, \varepsilon_2) = (\mu, 0)$, and applying Theorem 4.1 to $(M, \Upsilon, H^\varepsilon)$. Hypothesis (H3) is clearly satisfied for any natural number k, and we henceforth refer to functions which are continuously differentiable an arbitrary, but fixed number of times as 'smooth'. The spectral theory in Sect. 3 shows that (H1), (H2) are also satisfied; indeed, the (complexified) four-dimensional centre subspace of L is spanned by the generalised eigenvectors

$$E = \tau_1^{-1/2} e, \quad \bar{E} = \tau_1^{-1/2} \bar{e}, \quad F = \tau_1^{-1/2}\Big(f - \frac{\mathrm{i}\tau_2}{2\tau_1} e\Big), \quad \bar{F} = \tau_1^{-1/2}\Big(\bar{f} + \frac{\mathrm{i}\tau_2}{2\tau_1}\bar{e}\Big),$$

where

$$\tau_1 = -s\coth(s) + \frac{3\sinh(s)\cosh(s)}{2s} - \frac{1}{2} > 0, \tag{4.4}$$

$$\tau_2 = -\frac{\sinh(2s)}{2s^2} + \frac{4s}{3} - \frac{1}{2s} - \frac{3\cosh(2s)}{2s} + \frac{s + \sinh(2s)}{\sinh^2(s)},$$

so that the centre and hyperbolic subspaces of L are, respectively,

$$X_1 = \{AE + BF + \bar{A}E + \bar{B}F : A, B \in \mathbb{C}\}, \qquad X_2 = \left\{ \sum_{k \in \mathbb{Z}\setminus\{0\}} \beta_k f_k : \{\beta_k\} \in \ell^2 \right\}.$$

The vectors are normalised such that $(L - \mathrm{i}sI)E = 0$, $(L - \mathrm{i}sI)F = E$ with $SE = \bar{E}$, $SF = -\bar{F}$, and

$$\Upsilon|_0(E, \bar{F}) = \Upsilon|_0(\bar{E}, F) = 1, \qquad \Upsilon|_0(\bar{F}, E) = \Upsilon|_0(F, \bar{E}) = -1$$

and the symplectic product of any other combination of the vectors E, F, $\bar{E}$, $\bar{F}$ is zero (so that $\{E, F, \bar{E}, \bar{F}\}$ is a symplectic basis for the centre subspace of L). Writing

$$u_1 = AE + BF + \bar{A}\bar{E} + \bar{B}\bar{F},$$

we, therefore, find that A, B are canonical coordinates for the reduced Hamiltonian system (see Remark 4.2(i)), which can therefore be written as

$$A_x = \frac{\partial \tilde{H}^\mu}{\partial \bar{B}}, \qquad B_x = -\frac{\partial \tilde{H}^\mu}{\partial \bar{A}}$$

(with a slight abuse of notation we abbreviate $\tilde{H}^\varepsilon|_{(\varepsilon_1,\varepsilon_2)=(\mu,0)}$ to $\tilde{H}^\mu$); this system is reversible with reverser $S : (A, B) \to (\bar{A}, -\bar{B})$. Note that the quadratic, parameter-independent part of the Hamiltonian is

$$H_2^0(A, B, \bar{A}, \bar{B}) = \mathrm{i}s(A\bar{B} - \bar{A}B) + |B|^2,$$

so that in coordinates

$$L \begin{pmatrix} A \\ B \\ \bar{A} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} \mathrm{i}s & 1 & 0 & 0 \\ 0 & \mathrm{i}s & 0 & 0 \\ 0 & 0 & -\mathrm{i}s & 1 \\ 0 & 0 & 0 & -\mathrm{i}s \end{pmatrix} \begin{pmatrix} A \\ B \\ \bar{A} \\ \bar{B} \end{pmatrix}.$$

The next step is to use a normal-form transform to simplify the Hamiltonian. For this purpose we use the following result due to Elphick [7].

Lemma 4.3 *Let $n_0 \geq 2$. There exists a near-identity, canonical change of variables which transforms the Hamiltonian to*

$$\mathrm{i}s(A\bar{B} - \bar{A}B) + |B|^2 + H_{\mathrm{NF}}^\mu(A, B, \bar{A}, \bar{B}) + O(|(A, B)|^2 |(\mu, A, B)|^{n_0}),$$

where the complexification of H_{NF}^μ lies in $\ker \mathcal{L}_{L^*}$, *and $\mathcal{L}_{M^*} : \mathbb{C}[Z] \to \mathbb{C}[Z]$ is defined by*

$$(\mathcal{L}_{M^*} p)(Z) = M^* Z \cdot \nabla p(Z)$$

for $M \in \mathbb{C}^{4\times 4}$, where the coefficients of the polynomials in the complex polynomial rings depend upon μ and the gradient is taken with respect to $Z = (A, B, \bar{A}, \bar{B})$.

We proceed by characterising $\ker \mathcal{L}_{L^*}$ using the following lemma, the statements in which are obtained from results by Murdock [31, Lemma 3.4.8], Malonza [26, Lemma 4, Theorem 9] and Billera et al. [2, Section 4], respectively. Corollary 4.5 takes into account that H^μ_{NF} is real valued.

Lemma 4.4 *Let $S = \mathrm{diag}(\mathrm{i}s, \mathrm{i}s, -\mathrm{i}s, -\mathrm{i}s)$ and $N = L - S$.*

(i) *The kernel of $\mathcal{L}_{L^*} : \mathbb{C}[Z] \to \mathbb{C}[Z]$ is given by* $\ker \mathcal{L}_{L^*} = \ker \mathcal{L}_{N^*} \cap \ker \mathcal{L}_{S^*}$.
(ii) *The kernel of $\mathcal{L}_{N^*}$ is given by* $\ker \mathcal{L}_{N^*} = \mathbb{C}[A, \bar{A}, A\bar{B} - \bar{A}B]$.
(iii) *The kernel of $\mathcal{L}_{S^*}$ is given by* $\ker \mathcal{L}_{S^*} = \mathbb{C}[A\bar{A}, A\bar{B}, B\bar{A}, B\bar{B}]$.

Corollary 4.5 *The kernel of $\mathcal{L}_{L^*} : \mathbb{C}[Z] \to \mathbb{C}[Z]$ is given by $\mathbb{C}[|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B)]$ and $H^\mu_{\mathrm{NF}} \in \mathbb{R}[|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B)]$.*

Writing the transformed reduced system as

$$u_{1x} = Lu_1 + P^\mu(u_1),$$

where

$$\begin{aligned} u_1 &= AE + BF + \bar{A}\bar{E} + \bar{B}\bar{F}, \\ P^\mu(u_1) &= \partial_{\bar{B}}\tilde{H}^\mu(A, B, \bar{A}, \bar{B})E - \partial_{\bar{A}}\tilde{H}^\mu(A, B, \bar{A}, \bar{B})F \\ &\quad + \partial_B\tilde{H}^\mu(A, B, \bar{A}, \bar{B})\bar{E} - \partial_A\tilde{H}^\mu(A, B, \bar{A}, \bar{B})\bar{F}, \end{aligned}$$

we can compute the Taylor series of $r(u_1; \mu)$ and $\mathcal{N}^\mu(u_1)$, and hence $H^\mu(A, B, \bar{A}, \bar{B})$, recursively using the equation

$$Lr(u_1; \mu) - \mathrm{d}_1 r[u_1; \mu](Lu_1) = P^\mu(u_1) + \mathrm{d}_1 r[u_1; \mu](P^\mu(u_1)) - N^\mu(u_1 + r(u_1; \mu)) \tag{4.5}$$

(see Remark 4.2(ii)), where with a slight abuse of notation we have applied the near-identity normal-form transformation to the reduction function. Corollary 4.5 states that there are real constants c_1, c_2, d_1, d_2, d_3 such that

$$\begin{aligned} &\tilde{H}^1_2(A, B, \bar{A}, \bar{B}, 0) = c_1|A|^2 + c_2\mathrm{i}(A\bar{B} - \bar{A}B), \\ &\tilde{H}^0_3(A, B, \bar{A}, \bar{B}, 0) = 0, \\ &\tilde{H}^0_4(A, B, \bar{A}, \bar{B}, 0) = d_1|A|^4 + d_2\mathrm{i}(A\bar{B} - \bar{A}B)|A|^2 - d_3(A\bar{B} - \bar{A}B)^2, \end{aligned}$$

where $\mu^j\tilde{H}^j_k(A, B, \bar{A}, \bar{B})$ denotes the part of the Taylor expansion of $\tilde{H}^\mu(A, B, \bar{A}, \bar{B})$ which is homogeneous of order j in μ and k in $(A, B, \bar{A}, \bar{B})$. The coefficients c_1 and d_1, whose values are required in Sect. 5 below, are computed in Appendix B; we find that $c_1 < 0$ and there exists a critical value $s^\star$ of s such that $d_1 > 0$ for $s < s^\star$, which we now assume.

5 Homoclinic Solutions

In this section, we examine the reduced Hamiltonian system

$$\begin{aligned} A_x &= \partial_{\bar{B}} \tilde{H}^{\mu}(A, B, \bar{A}, \bar{B}) \\ &= \mathrm{i}sA + B + \partial_{\bar{B}} \tilde{H}^{\mu}_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B, \mu) + \underline{O}(|(A, B)||(\mu, A, B)|^{n_0}), \end{aligned} \tag{5.1}$$

$$\begin{aligned} B_x &= -\partial_{\bar{A}} \tilde{H}^{\mu}(A, B, \bar{A}, \bar{B}) \\ &= \mathrm{i}sB - \partial_{\bar{A}} \tilde{H}^{\mu}_{\mathrm{NF}}(|A|^2, \mathrm{i}(A\bar{B} - \bar{A}B), \mu) + \underline{O}(|(A, B)||(\mu, A, B)|^{n_0}), \end{aligned} \tag{5.2}$$

where the underscore indicates that the order-of-magnitude estimate remains valid when formally differentiated with respect to (A, B). The truncated system without the remainder terms was examined in detail by Iooss and Pérouème [19], who also studied the ‘persistence’ of certain solutions as solutions to the full system. Here, we present an alternative, functional-analytic proof of the existence of two reversible homoclinic solutions to (5.1), (5.2).

We begin by returning to real coordinates $q = (q_1, q_2)^{\mathrm{T}}$, $p = (p_1, p_2)^{\mathrm{T}}$ given by

$$A = \frac{1}{\sqrt{2}}(q_1 + \mathrm{i}q_2), \qquad B = \frac{1}{\sqrt{2}}(p_1 + \mathrm{i}p_2)$$

and, hence, obtaining the real Hamiltonian system

$$q_x = \frac{\partial \tilde{H}^{\mu}}{\partial p} = p + sR_{\frac{\pi}{2}}q + \overbrace{\partial_2 \tilde{H}^{\mu}_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q)R_{\frac{\pi}{2}}q}^{:= P_1^{\mu}(q, p)} + R_1^{\mu}(q, p), \tag{5.3}$$

$$p_x = -\frac{\partial \tilde{H}^{\mu}}{\partial p} = sR_{\frac{\pi}{2}}p \underbrace{-\partial_1 \tilde{H}^{\mu}_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q)q + \partial_2 \tilde{H}^{\mu}_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q)R_{\frac{\pi}{2}}p}_{:= P_2^{\mu}(q, p)} + R_2^{\mu}(q, p), \tag{5.4}$$

in which

$$\tilde{H}^{\mu}(q, p) = \frac{1}{2}|p|^2 + sp\cdot R_{\frac{\pi}{2}}q + \tilde{H}^{\mu}_{\mathrm{NF}}(\tfrac{1}{2}|q|^2, p\cdot R_{\frac{\pi}{2}}q, \mu) + O(|(q, p)|^2|(\mu, q, p)|^{n_0}),$$

so that $P_1^{\mu}(q, p)$, $P_2^{\mu}(q, p)$ are polynomials in μ, q and p and

$$R_1^{\mu}(q, p),\ R_2^{\mu}(q, p) = \underline{O}(|(q, p)|(\mu, q, p)|^{n_0}).$$

Note that this system is reversible with reverser $S : (q_1, p_1, q_2, p_2) \mapsto (q_1, -p_1, -q_2, p_2)$ and that

$$R_\theta P_1^{\mu}(q, p) = P_1^{\mu}(R_\theta q, R_\theta p), \qquad R_\theta P_2^{\mu}(q, p) = P_2^{\mu}(R_\theta q, R_\theta p)$$

for all $\theta \in [0, 2\pi)$, where R_θ is the matrix representing a rotation through the angle θ.

The next step is to recast equations (5.3), (5.4) as a single second-order equation. Writing

$$p = q_x - sR_{\frac{\pi}{2}}q + v,$$

we find from equation (5.3) that

$$v + P_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v) + R_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v) = 0, \tag{5.5}$$

and using the implicit-function theorem, we now construct a solution of (5.5) of the form

$$v = v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q) + v_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q),$$

where v_1^μ solves the truncated equation with $R_1^\mu = 0$ and takes the particular form

$$v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q) = w_1^\mu(|q|^2, R_{\frac{\pi}{2}}q \cdot (q_x - sR_{\frac{\pi}{2}}q))R_{\frac{\pi}{2}}q. \tag{5.6}$$

Note that w_1^μ necessarily solves

$$w_1 + \partial_2\tilde{H}_{\mathrm{NF}}^\mu(\tfrac{1}{2}|q|^2, w_1|q|^2 + R_{\frac{\pi}{2}}q \cdot (q_x - sR_{\frac{\pi}{2}}q)) = 0, \tag{5.7}$$

while v_2^μ necessarily solves

$$\begin{aligned} &v_2 + P_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q) + v_2) \\ &\quad - P_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q)) \\ &\quad + R_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q) + v_2) = 0. \end{aligned} \tag{5.8}$$

Proposition 5.1

(i) *Equation* (5.7) *has a unique solution* $w_1 = w_1^\mu(|q|^2, R_{\frac{\pi}{2}}q \cdot (q_x - sR_{\frac{\pi}{2}}q))$ *which depends analytically upon* μ, $|q|^2$ *and* $R_{\frac{\pi}{2}}q\cdot(q_x - sR_{\frac{\pi}{2}}q)$ *and satisfies* $w_1^0(0, 0) = 0$. *The function* v_1^μ *defined by* (5.6) *satisfies*

$$v_1^\mu + P_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu) = 0$$

and

$$R_\theta v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q) = v_1^\mu(R_\theta q, R_\theta(q_x - sR_{\frac{\pi}{2}}q))$$

for all $\theta \in [0, 2\pi)$.

(ii) *Equation* (5.8) *has a unique solution* $v_2 = v_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q)$ *which depends smoothly upon* μ, q *and* $q_x - sR_{\frac{\pi}{2}}q$ *and satisfies*

$$v_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q) = \underline{O}(|(q, q_x - sR_{\frac{\pi}{2}}q)||(\mu, q, q_x - sR_{\frac{\pi}{2}}q)|^{n_0}).$$

Substituting

$$p = q_x - sR_{\frac{\pi}{2}}q + v_1^\mu + v_2^\mu$$

into Eq. (5.4), where we have omitted the arguments of v_1^μ, v_2^μ for notational simplicity, shows that

$$(\partial_x - sR_{\frac{\pi}{2}})^2 q = -(\partial_x - sR_{\frac{\pi}{2}})(v_1^\mu + v_2^\mu) + \tilde{P}^\mu(q, q_x - sR_{\frac{\pi}{2}}q) + \tilde{R}^\mu(q, q_x - sR_{\frac{\pi}{2}}q),$$

in which

$$\begin{aligned}
\tilde{P}^\mu(q, q_x - sR_{\frac{\pi}{2}}q) &= P_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu),\\
\tilde{R}^\mu(q, q_x - sR_{\frac{\pi}{2}}q) &= P_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu + v_2^\mu) - P_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu)\\
&\qquad + R_2^\mu(q, q_x - sR_{\frac{\pi}{2}}q + v_1^\mu + v_2^\mu).
\end{aligned}$$

It follows that

$$\begin{aligned}
(\partial_x - sR_{\frac{\pi}{2}})^2 q = &-\partial_1 v_1^\mu(q_x - sR_{\frac{\pi}{2}}q) - \partial_2 v_1^\mu(\partial_x - sR_{\frac{\pi}{2}})^2 q\\
&+ \tilde{P}^\mu(q, q_x - sR_{\frac{\pi}{2}}q) - \partial_1 v_2^\mu(q_x - sR_{\frac{\pi}{2}}q)\\
&- \partial_2 v_2^\mu(\partial_x - sR_{\frac{\pi}{2}})^2 q - \partial_1 v_2^\mu sR_{\frac{\pi}{2}}q\\
&- \partial_2 v_2^\mu sR_{\frac{\pi}{2}}(q_x - sR_{\frac{\pi}{2}}q) + sR_{\frac{\pi}{2}}v_2^\mu + \tilde{R}^\mu(q, q_x - sR_{\frac{\pi}{2}}q), \quad (5.9)
\end{aligned}$$

where $\partial_j v_k^\mu$ is the matrix $\mathrm{d}_j v_k^\mu[q, q_x - sR_{\frac{\pi}{2}}q]$ and we have used the calculation

$$\begin{aligned}
&(\partial_x - sR_{\frac{\pi}{2}})v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q)\\
&\quad= (\partial_x - sR_{\frac{\pi}{2}})R_{sx}v_1^\mu(R_{-sx}q, R_{-sx}(q_x - sR_{\frac{\pi}{2}}q))\\
&\quad= R_{sx}\partial_x v_1^\mu(R_{-sx}q, R_{-sx}(q_x - sR_{\frac{\pi}{2}}q))\\
&\quad= R_{sx}\partial_1 v_1^\mu(R_{-sx}q, R_{-sx}(q_x - sR_{\frac{\pi}{2}}q))\partial_x(R_{-sx}q)\\
&\qquad+ R_{sx}\partial_2 v_1^\mu(R_{-sx}q, R_{-sx}(q_x - sR_{\frac{\pi}{2}}q))\partial_x(R_{-sx}(q_x - sR_{\frac{\pi}{2}}q))\\
&\quad= R_{sx}\partial_1 v_1^\mu(R_{-sx}q, (q_x - sR_{\frac{\pi}{2}}q))R_{-sx}(q_x - sR_{\frac{\pi}{2}}q)\\
&\qquad+ R_{sx}\partial_2 v_1^\mu(R_{-sx}q, (q_x - sR_{\frac{\pi}{2}}q))R_{-sx}(\partial_x - sR_{\frac{\pi}{2}})^2 q\\
&\quad= \partial_1 v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q)(q_x - sR_{\frac{\pi}{2}}q) + \partial_2 v_1^\mu(q, q_x - sR_{\frac{\pi}{2}}q)(\partial_x - sR_{\frac{\pi}{2}})^2 q.
\end{aligned}$$

Introducing the scaled variables

$$q(x) = \delta R_{sx} Q(X), \qquad X = \delta x,$$

where $\delta^2 = -c_1\mu$, so that

$$q_x - sR_{\frac{\pi}{2}} q = \delta^2 R_{sx} Q_X(X), \qquad (\partial_x - sR_{\frac{\pi}{2}})^2 q = \delta^3 R_{sx} Q_{XX}(X),$$

transforms equation (5.9) into

$$Q_{XX} = Q - CQ|Q|^2 + T_1^\delta(Q, Q_X) + R_{-sX/\delta} T_2^\delta(R_{sX/\delta}Q, R_{sX/\delta}Q_X, R_{sX/\delta}Q_{XX}), \tag{5.10}$$

where $C = -d_1/c_1$ and

$$T_1^\delta(Q, Q_X) = O(\delta|(Q, Q_X)|), \qquad T_2^\delta(Q, Q_X, Q_{XX}) = O(\delta^{n_0-2}|(Q, Q_X, Q_{XX})|).$$

Remark 5.2 The various changes of variable preserve the reversibility symmetry, so that equation (5.10) is invariant under the transformation $X \mapsto -X$, $(Q_1, Q_2) \mapsto (Q_1, -Q_2)$.

Before proving the existence of homoclinic solutions to (5.10) we define the function spaces with which we work and refer to some functional-analytic results which are used in the proof (see Kirchgässner [21, Proposition 5.1]).

Definition 5.3 Suppose that $k \in \mathbb{N}_0$ and $\nu \geq 0$. Define

$$C_\nu^k(\mathbb{R}) = \{f \in C^k(\mathbb{R})\colon \|f\|_{k,\nu} < \infty\}, \qquad \|f\|_{k,\nu} := \sup_{t\in\mathbb{R}} \sum_{j=0}^{k} |f^{(j)}(t)|\mathrm{e}^{\nu|t|}$$

and their subspaces

$$C_{\nu,\mathrm{e}}^k = \{f \in C_\nu^k(\mathbb{R})\colon f(-t) = f(t),\ t \in \mathbb{R}\},$$
$$C_{\nu,\mathrm{o}}^k = \{f \in C_\nu^k(\mathbb{R})\colon f(-t) = -f(t),\ t \in \mathbb{R}\}.$$

In the case $k = 0$ we just write $C_\nu(\mathbb{R})$, $C_{\nu,\mathrm{e}}(\mathbb{R})$ and $C_{\nu,\mathrm{o}}(\mathbb{R})$.

Proposition 5.4

(i) *The formula*

$$K\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_{1XX} - z_1 \\ z_{2XX} - z_2 \end{pmatrix}$$

defines a bounded linear operator $C_\nu^2(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ *and* $C_{\nu,\mathrm{e}}^2(\mathbb{R}) \times C_{\nu,\mathrm{o}}^2(\mathbb{R}) \to C_{\nu,\mathrm{e}}(\mathbb{R}) \times C_{\nu,\mathrm{o}}(\mathbb{R})$ *for each* $\nu \geq 0$.

(ii) *For* $0 \leq \nu < 1$ *the operator* $K: C_\nu^2(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ *is invertible with bounded inverse given by*

$$(K^{-1}f)(t) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|t-s|} f(s)\,\mathrm{d}s,$$

where the integration is taken componentwise.

(iii) *Suppose that* $C > 0$, $h \in C_1(\mathbb{R})$ *and* $0 \leq \nu < 1$. *The formula*

$$K_h z = K^{-1}\begin{pmatrix} -3Ch^2 z_1 \\ -Ch^2 z_2 \end{pmatrix}$$

defines a bounded linear operator $C_0(\mathbb{R})^2 \to C_\nu^2(\mathbb{R})^2$ *and a compact operator* $C_\nu(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$.

Theorem 5.5 *For each* $\nu \in (0, 1)$ *and each sufficiently small value of* $\delta > 0$ *equation* (5.10) *has two homoclinic solutions* $Q^{\delta\pm}$ *which are symmetric, that is invariant under the transformation* $(Q_1(X), Q_2(X)) \mapsto (Q_1(-X), -Q_2(-X))$, *and satisfy the estimate*

$$Q^{\delta\pm}(X) = \pm\begin{pmatrix} h(X) \\ 0 \end{pmatrix} + O(\delta \mathrm{e}^{-\nu|X|})$$

for all $X \in \mathbb{R}$.

Proof For $\delta = 0$ equation (5.10) has the family

$$\left\{(Q_1, Q_2)^{\mathrm{T}} = R_\theta(h(X_0 + \cdot), 0)^{\mathrm{T}} : \theta \in [0, 2\pi),\ X_0 \in \mathbb{R}\right\}$$

of homoclinic solutions, where

$$h(X) = \left(\frac{2}{C}\right)^{1/2} \mathrm{sech}(X).$$

Two of these solutions, namely those with $(\theta, X_0) = (0, 0)$ and $(\theta, X_0) = (\pi, 0)$, which we denote by respectively Q^+ and Q^-, are symmetric. We seek a solution of (5.10) in the form of a perturbation of Q^+ by writing

$$Q_1 = h + z_1, \qquad Q_2 = z_2,$$

so that $z = (z_1, z_2)^{\mathrm{T}}$ satisfies

$$z_{1XX} - z_1 = -3Ch^2 z_1 + r_1^\delta(z_1, z_2, z_{1X}, z_{2X}, z_{1XX}, z_{2XX}, X), \tag{5.11}$$

$$z_{2XX} - z_2 = -Ch^2 z_2 + r_2^\delta(z_1, z_2, z_{1X}, z_{2X}, z_{1XX}, z_{2XX}, X) \tag{5.12}$$

with the obvious definitions of r_1^δ and r_2^δ. We study the system (5.11), (5.12) in the space $C_\nu^2(\mathbb{R})^2$ with fixed $\nu \in (0, 1)$ and, with a slight abuse of notation, consider the nonlinearity $r^\delta = (r_1^\delta, r_2^\delta)^{\mathrm{T}}$ as a mapping $C_\nu^2(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ and $C_{\nu,\mathrm{e}}^2(\mathbb{R}) \times C_{\nu,\mathrm{o}}^2(\mathbb{R}) \to C_{\nu,\mathrm{e}}(\mathbb{R}) \times C_{\nu,\mathrm{o}}(\mathbb{R})$ with

$$\|r^\delta(z_1, z_2)\|_{0,\nu} = O(\delta) + \underline{O}_1(\|(z_1, z_2)\|_{2,\nu}^2).$$

In terms of the operators K and K_h defined in Proposition 5.4 equations (5.11), (5.12) can thus be written as

$$z = K_h z + K^{-1} r^\delta(z). \tag{5.13}$$

The eigenvalue problem

$$K_h z = z$$

is equivalent to the decoupled system

$$z_{1XX} = z_1 - 3Ch^2 z_1, \tag{5.14}$$

$$z_{2XX} = z_2 - Ch^2 z_2 \tag{5.15}$$

of ordinary differential equations. Let

$$z_1^1(X) = \operatorname{sech}(X)\tanh(X), \qquad z_2^1(X) = \operatorname{sech}(X),$$

$$z_1^2(X) = \operatorname{sech}(X)(-3 + \cosh^2(X) + 3X\tanh(X)), \quad z_2^2(X) = \operatorname{sech}(X)(2X + \sinh(2X)),$$

so that $\{z_1^1, z_1^2\}$ and $\{z_2^1, z_2^2\}$ are fundamental solution sets for, respectively, (5.14) and (5.15). Since z_1^1, z_2^1 are bounded while z_1^2, z_2^2 are unbounded, we conclude that all bounded solutions of equation (5.14) are multiples of $z_1^1 = -h_X$ and all bounded solutions of equation (5.15) are multiples of $z_2^1 = (2/C)^{-1/2}h$. The eigenspace of $K_h\colon C_\nu(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$ corresponding to the eigenvalue 1 is, therefore,

$$\mathrm{sp}\left\{\begin{pmatrix} h_X \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ h \end{pmatrix}\right\},$$

which lies in $C_{\nu,\mathrm{o}}(\mathbb{R}) \times C_{\nu,\mathrm{e}}(\mathbb{R})$. This calculation shows that 1 is not an eigenvalue of $K_h|_{C_{\nu,\mathrm{e}}(\mathbb{R})\times C_{\nu,\mathrm{o}}(\mathbb{R})}$ and since K_h is a compact operator $C_\nu(\mathbb{R})^2 \to C_\nu(\mathbb{R})^2$, one concludes that the spectrum of $K_h|_{C_{\nu,\mathrm{e}}(\mathbb{R})\times C_{\nu,\mathrm{o}}(\mathbb{R})}$ consists only of eigenvalues, so that 1 lies in the resolvent set of $K_h|_{C_{\nu,\mathrm{e}}(\mathbb{R})\times C_{\nu,\mathrm{o}}(\mathbb{R})}$. It follows that

$$I - K_h\colon C_{\nu,\mathrm{e}}(\mathbb{R}) \times C_{\nu,\mathrm{o}}(\mathbb{R}) \to C_{\nu,\mathrm{e}}^2(\mathbb{R}) \times C_{\nu,\mathrm{o}}^2(\mathbb{R})$$

is invertible. We can, therefore, solve equation (5.13) for sufficiently small values of $\delta > 0$ using the implicit-function theorem; the solution $z_\star^\delta$ satisfies $\|z_\star^\delta\|_{2,\nu} = O(\delta)$.

Returning to equation (5.10), we have found a symmetric solution $Q^{\delta+} = Q^+ + z_\star^\delta$ which satisfies the stated estimate. The second homoclinic solution $Q^{\delta-}$ is obtained from Q^- by the same procedure. □

Appendix A: Formal Derivation of the Nonlinear Schrödinger Equation

Writing $\beta = \beta_0$, $\alpha = \alpha_0 + \delta^2$ and substituting the formal asymptotic expansions

$$\eta(x) = \delta\eta_1(x, X) + \delta^2\eta_2(x, X) + \delta^3\eta_3(x, X) + \cdots,$$
$$\Psi(x, y) = \delta\Psi_1(x, X, y) + \delta^2\Psi_2(x, X, y) + \delta^3\Psi_3(x, X, y) + \cdots,$$

where $X = \delta x$, into Eqs. (1.6)–(1.9) yields the boundary-value problems

$$\Psi_{1xx} + \Psi_{1yy} = 0, \qquad 0 < y < 1, \tag{5.16}$$
$$\Psi_{1y}\big|_{y=0} = 0, \tag{5.17}$$
$$\Psi_{1y} + \eta_{1x}\big|_{y=1} = 0, \tag{5.18}$$
$$\alpha_0\eta_1 - \Psi_{1x} + \beta_0\eta_{1xxxx}\big|_{y=1} = 0 \tag{5.19}$$

for Ψ_1,

$$\Psi_{2xx} + \Psi_{2yy} + 2\Psi_{1xX} + 2\eta_1\Psi_{1xx} - 2y\eta_{1x}\Psi_{1xy} - y\Psi_{1y}\eta_{1xx} = 0, \qquad 0 < y < 1, \tag{5.20}$$
$$\Psi_{2y}\big|_{y=0} = 0, \tag{5.21}$$
$$\Psi_{2y} + \eta_{1X} + \eta_{2x} - \eta_{1x}\Psi_{1x} + \eta_{1x}\eta_1\big|_{y=1} = 0, \tag{5.22}$$
$$-\Psi_{2x} - \Psi_{1X} + \alpha_0\eta_2 + 4\beta_0\eta_{1xxxX} + \beta_0\eta_{2xxxx} + \Psi_{1y}\eta_{1x} + \tfrac{1}{2}\Psi_{1x}^2 + \tfrac{1}{2}\Psi_{1y}^2\big|_{y=1} = 0 \tag{5.23}$$

for Ψ_2 and

$$\begin{aligned}
&\Psi_{3xx} + \Psi_{3yy} + 2\Psi_{2xX} + 4\eta_1\Psi_{1xX} + 2\eta_1\Psi_{2xx} + 2\eta_2\Psi_{1xx} \\
&\quad - 2y\eta_{1x}\Psi_{1Xy} + \Psi_{1XX} - 2y\eta_{1x}\Psi_{2xy} - 2y\eta_{1X}\Psi_{1xy} \\
&\quad - 2y\eta_{2x}\Psi_{1xy} - 2y\eta_{1xX}\Psi_{1y} - y\eta_{2xx}\Psi_{1y} - y\eta_{1xx}\Psi_{2y} \\
&\quad + \eta_1^2\Psi_{1xx} + y^2\eta_{1x}^2\Psi_{1yy} - 2y\eta_1\eta_{1x}\Psi_{1xy} - y\eta_1\eta_{1xx}\Psi_{1y} \\
&\quad + 2y\eta_{1x}^2\Psi_{1y} = 0, \qquad 0 < y < 1,
\end{aligned} \tag{5.24}$$
$$\Psi_{3y}\big|_{y=0} = 0, \tag{5.25}$$
$$\begin{aligned}
&\Psi_{3y} + \eta_{2X} + \eta_{3x} - \eta_{1x}\Psi_{1X} - \eta_{1x}\Psi_{2x} - \eta_{2x}\Psi_{1x} - \eta_{1X}\Psi_{1x} \\
&\quad + \eta_1\eta_{1X} + \eta_{1x}\eta_2 + \eta_1\eta_{2x} - \eta_1\eta_{1x}\Psi_{1x} + y\eta_{1x}^2\Psi_{1y}\big|_{y=1} = 0,
\end{aligned} \tag{5.26}$$
$$\begin{aligned}
&-\Psi_{3x} - \Psi_{2X} + \alpha_0\eta_3 + 6\beta_0\eta_{1xxXX} + 4\beta_0\eta_{2xxxX} + \beta_0\eta_{3xxxx} \\
&\quad + \eta_{1x}\Psi_{2y} + \eta_{1X}\Psi_{1y} + \eta_{2x}\Psi_{1y} + \Psi_{1x}\Psi_{1X} + \Psi_{1x}\Psi_{2x} \\
&\quad + \Psi_{1y}\Psi_{2y} - \tfrac{5}{2}\beta_0\eta_{1x}^2\eta_{1xxxx} - \eta_1\eta_{1x}\Psi_{1y} - \eta_1\Psi_{1y}^2 \\
&\quad - \eta_{1x}\Psi_{1x}\Psi_{1y} + \eta_1 - 10\beta_0\eta_{1x}\eta_{1xx}\eta_{1xxx} - \tfrac{5}{2}\beta_0\eta_{1xx}^3\big|_{y=1} = 0
\end{aligned} \tag{5.27}$$

for Ψ_3. We proceed by making the modulational *Ansatz*

$$\begin{aligned}
\eta_1(x,X) &= A_1(X)\mathrm{e}^{\mathrm{i}sx} + \text{c.c.}\,,\\
\eta_2(x,X) &= A_2(X)\mathrm{e}^{2\mathrm{i}sx} + \text{c.c.} + A_0(X),\\
\eta_3(x,X) &= A_3(X)\mathrm{e}^{3\mathrm{i}sx} + A_4(X)\mathrm{e}^{2\mathrm{i}sx} + A_5(X)\mathrm{e}^{\mathrm{i}sx} + \text{c.c.} + A_6(X).
\end{aligned}$$

- From (5.16)–(5.18) it follows that

$$\begin{aligned}
\Psi_{1xx} + \Psi_{1yy} &= 0, && 0 < y < 1,\\
\Psi_{1y}\big|_{y=0} &= 0,\\
\Psi_{1y}\big|_{y=1} &= -\mathrm{i}sA_1\mathrm{e}^{\mathrm{i}sx} + \text{c.c.}\,,
\end{aligned}$$

the solution to which is

$$\Psi_1(x,X,y) = -\frac{\mathrm{i}\cosh(sy)}{\sinh(s)}A_1\mathrm{e}^{\mathrm{i}sx} + \text{c.c.} + g_1(X),$$

where g_1 is an arbitrary function of a single variable. The equation

$$(\alpha_0 + \beta_0 s^4)A_1 - \Psi_{1x}\big|_{y=1} = 0,$$

which follows from (5.19), then recovers the dispersion relation (3.7).
- From (5.20)–(5.22), it follows that

$$\begin{aligned}
\Psi_{2xx} + \Psi_{2yy} = {}& -2s\frac{\cosh(sy)}{\sinh(s)}A_{1X}\mathrm{e}^{\mathrm{i}sx}\\
&- \mathrm{i}s(3sy\sinh(sy) - 2\cosh(sy))A_1^2\mathrm{e}^{2\mathrm{i}sx} + \text{c.c.}, \qquad 0 < y < 1,\\
\Psi_{2y}\big|_{y=0} = {}& 0,\\
\Psi_{2y}\big|_{y=1} = {}& -A_{1X}\mathrm{e}^{\mathrm{i}sx} + \mathrm{i}s\left(\frac{s\cosh(sy)}{\sinh(s)}A_1^2 - A_1^2 - 2A_2\right)\mathrm{e}^{2\mathrm{i}sx},
\end{aligned}$$

the solution to which is

$$\begin{aligned}
\Psi_2(x,X,y) = {}& \left(\frac{\coth(s)}{\sinh(s)}\cosh(sy) - y\frac{\sinh(sy)}{\sinh(s)}\right)A_{1X}\mathrm{e}^{\mathrm{i}sx}\\
&+ \left(\mathrm{i}s\left(\frac{\coth(s)\cosh(2sy)}{\sinh(2s)} - y\frac{\sinh(sy)}{\sinh(s)}\right)A_1^2 + \frac{\mathrm{i}\cosh(2sy)}{\sinh(2s)}A_2\right)\mathrm{e}^{2\mathrm{i}sx}\\
&+ \text{c.c.} + g_2(X),
\end{aligned}$$

where g_2 is an arbitrary function of a single variable. Substituting the formulae for Ψ_1, Ψ_2 and the modulational *Ansatz* into (5.23), and equating the coefficients of $\mathrm{e}^{0\mathrm{i}sx}$, $\mathrm{e}^{\mathrm{i}sx}$, $\mathrm{e}^{2\mathrm{i}sx}$, we then find that

$$\begin{aligned} g_{1X} &= \frac{s^2}{\sinh^2(s)}|A_1|^2 + \alpha_0 A_0, \\ A_2 &= \frac{1}{2}\,\frac{(1-3\coth^2(s))s^2}{\alpha_0 + 16s^4\beta_0 - s(\coth(s) + (\coth(s))^{-1})}\,A_1^2, \\ \beta_0 &= \frac{1}{4s^3}\coth(s) - \frac{1}{4s^2}\operatorname{cosech}^2(s). \end{aligned} \tag{5.28}$$

Using the dispersion relation and the above formula for β_0, we find that

$$\alpha_0 = \frac{3s}{4}\coth(s) + \frac{s^2}{4}\operatorname{cosech}^2(s).$$

- Similarly, (5.24)–(5.26) yield a Poisson equation for Ψ_3 with boundary conditions at $y = 0$ and $y = 1$, the solution to which is

$$\begin{aligned} &\Psi_3(x, X, y) \\ &= \Bigg(\bigg(\frac{3}{2}\,\frac{\mathrm{i}s^2\cosh(sy)}{\sinh(s)} - \frac{2\mathrm{i}s^2\coth(2s)\cosh(s)\cosh(sy)}{\sinh^2(s)} - \frac{1}{2}\,\frac{\mathrm{i}s^2y^2\cosh(sy)}{\sinh(s)} \\ &\qquad + \mathrm{i}s^2y\frac{\sinh(2sy)}{\sinh^2(s)}\bigg)\bar{A}_1A_1^2 \\ &\quad + \bigg(\frac{(2\mathrm{i}s\coth(s) + \mathrm{i}s\tanh(s))\cosh(sy)}{\sinh(s)} + \frac{\mathrm{i}sy\sinh(sy)}{\sinh(s)} - \frac{2\mathrm{i}sy\sinh(2sy)}{\sinh(2s)}\bigg)\bar{A}_1A_2 \\ &\quad + \bigg(\frac{\mathrm{i}s\coth(s)\cosh(sy)}{\sinh(s)} - \frac{\mathrm{i}sy\sinh(sy)}{\sinh(s)}\bigg)A_0A_1 \\ &\quad + \bigg(\frac{\mathrm{i}y^2\cosh(sy)}{2\sinh(s)} + \frac{\mathrm{i}(2\coth^2(s)-1)\cosh(sy)}{2\sinh(s)} - \frac{\mathrm{i}y\coth(s)\sinh(sy)}{\sinh(s)}\bigg)A_{1XX} \\ &\quad - \frac{\mathrm{i}\cosh(sy)}{\sinh(s)}A_5 + \frac{\mathrm{i}\cosh(sy)}{\sinh(s)}A_1g_{1X}\Bigg)\mathrm{e}^{\mathrm{i}sx} + (\cdots)\mathrm{e}^{2\mathrm{i}sx} + (\cdots)\mathrm{e}^{3\mathrm{i}sx} + \mathrm{c.c} \\ &\quad + \frac{\sinh(sy)(sy\coth(s) + \coth(s) - y) - \cosh(sy)(sy^2 + \coth(s))}{\sinh(s)}\,\frac{\mathrm{d}}{\mathrm{d}X}|A_1|^2 \\ &\quad - \tfrac{1}{2}y^2 - \tfrac{1}{2}y^2g_{1XX} \end{aligned}$$

with

$$g_{1XX} - A_{0X} + 2s\coth(s)\frac{\mathrm{d}}{\mathrm{d}X}|A_1|^2 = 0.$$

By integrating this equation and substituting it into (5.28) we find that

$$A_0 = \bigg(\frac{s^2}{\alpha_0 - 1}(1 - \coth^2(s)) - \frac{2s\coth(s)}{\alpha_0 - 1}\bigg)|A_1|^2,$$

so that

$$g_{1X} = \left(\frac{s^2\alpha_0}{\alpha_0 - 1}(1 - \coth^2(s)) - \frac{2s\alpha_0}{\alpha_0 - 1}\coth(s) - s^2(1 - \coth^2(s)) \right) |A_1|^2.$$

Substituting the formulae for A_0, A_2, g_{1X}, Ψ_1, Ψ_2, Ψ_3 and the modulational Ansatz into (5.27), and equating coefficients of $\mathrm{e}^{\mathrm{i}sx}$, finally yields the nonlinear Schrödinger equation

$$\begin{aligned} &A_1 - (6\beta_0 s^2 - (1-\sigma^2)(1-s\sigma))A_{1XX} \\ &\quad + \left(\frac{-s^4(1-3\sigma^2)^2}{2(\alpha_0 + 16\beta_0 s^4 - s(\sigma + \sigma^{-1}))} + s^3(-5s^3\beta_0 + 4\sigma - 2\sigma^3) \right. \\ &\qquad \left. - \frac{s^4(1-\sigma^2)^2}{\alpha_0 - 1} + \frac{4s^3\sigma(1-\sigma^2)}{\alpha_0 - 1} - \frac{4\alpha_0 s^2\sigma^2}{\alpha_0 - 1} \right) |A_1|^2 A_1 = 0, \end{aligned}$$

where $\sigma = \coth(s)$.

Appendix B: Computation of the Normal-Form Coefficients

For this purpose we make use of the calculation

$$\Upsilon|_0(Lu, v) = H_2^0(u, v) = H_2^0(v, u) = \Upsilon|_0(Lv, u),$$

denote the parts of $H^\mu(w)$, $g^\mu(w)$ which are homogeneous of order m in μ and n in w by $\mu^m H_n^m(w)$, $\mu^m N_n^m(w)$ and the part of $r(u_1; \mu)$ which is homogeneous of order m in μ and n in u_1 by $r_n^m(u_1; \mu)$. With a slight abuse of notation we use the same symbols for the multilinear operators associated with these quantities.

Write

$$r_n^m(u_1; \mu) = \sum_{i+j+k+\ell=m} r_{ijk\ell}^n \mu^m A^i B^j \bar{A}^k \bar{B}^\ell$$

and consider the μA-component of (4.5), namely

$$(L - \mathrm{i}sI)r_{1000}^1 = c_2 \mathrm{i}E - c_1 F - N_1^1(E).$$

Taking the symplectic product of this equation with $\bar{E}$, we find that

$$c_1 = -\Upsilon|_0(r_{1000}^1, \underbrace{(L + \mathrm{i}sI)\bar{E}}_{=0}) + \Upsilon|_0(N_1^1(E), \bar{E}) = 2H_2^1(E, \bar{E}) = -\frac{\sinh^2(s)}{\tau_1}.$$

To compute d_1 we consider the $A^2\bar{A}$-component of (4.5), namely

$$(L - \mathrm{i}sI)r^0_{2010} = \mathrm{i}d_2 E - 2d_1 F - 3N^0_3(E, E, \bar{E}) \\ -2N^0_2(\bar{E}, r^0_{200000}) - 2N^0_2(E, r^0_{101000}),$$

and again take the symplectic product with $\bar{E}$, so that

$$2d_1 = -\Upsilon|_0(r^0_{2010}, \underbrace{(L + \mathrm{i}sI)\bar{E}}_{=0}) + 3\Upsilon|_0(N^0_3(E, E, \bar{E}), \bar{E}) \\ + 2\Upsilon|_0(N^0_2(\bar{E}, r^0_{2000}), \bar{E}) + 2\Upsilon|_0(N^0_2(E, r_{1010}), \bar{E}).$$

The functions r^0_{2000} and r^0_{1010} are obtained from the A^2- and $A\bar{A}$-components of (4.5), which are respectively

$$(K - 2\mathrm{i}sI)r^0_{2000} = -N^0_2(E, E), \\ Kr^0_{1010} = -2N^0_2(E, \bar{E})$$

(note that r^0_{101000} is determined up to addition of a multiple of F). Altogether we find that

$$d_1 = \frac{\sinh^4(s)}{2\tau_1^2}\left(\frac{s^4(1-3\sigma^2)^2}{2(\alpha_0 + 16\beta_0 s^4 - s(\sigma + \sigma^{-1})} - s^3(-5s^3\beta_0 + 4\sigma - 2\sigma^3) \right. \\ \left. + \frac{s^4(1-\sigma^2)^2}{\alpha_0 - 1} - \frac{4s^3\sigma(1-\sigma^2)}{\alpha_0 - 1} + \frac{4\alpha_0 s^2\sigma^2}{\alpha_0 - 1}\right),$$

where $\sigma = \coth(s)$.

For completeness, we record the formulae for $\tilde{r}^0_{101000}$ and $\tilde{r}^0_{200000}$, namely

$$\tilde{r}^0_{1010} = \begin{pmatrix} -s(\alpha_0 - 1)^{-1}(s + \sinh(2s)) \\ 0 \\ 0 \\ 0 \\ 0 \\ -s\sinh(2s) + 2s\sinh(s)(sy\sinh(sy) + \cosh(sy)) \end{pmatrix},$$

$$\tilde{r}^0_{2000} = \tilde{a}^0_{2000} \begin{pmatrix} \mathrm{i}\sinh(2s) \\ -2s\sinh(2s) \\ \frac{1}{2s}(\alpha_0 - 1)\sinh(2s) \\ -4\mathrm{i}s^2\beta_0\sinh(2s) \\ -(\frac{1}{2s} + (y^2 - \frac{1}{3})s)\sinh(2s) + \cosh(2sy) \\ 2\mathrm{i}s\cosh(2sy) - \mathrm{i}\sinh(2s) \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{1}{2}s\sinh(2s) \\ \mathrm{i}s^2\sinh(2s) \\ -\frac{\mathrm{i}}{8}(2\alpha_0 - 3)\sinh(2s) - \frac{\mathrm{i}}{s}\sinh^2(s) \\ -2\beta_0 s^3\sinh(2s) \\ \mathrm{i}\sinh(s)(-2sy\sinh(sy) + \cosh(sy)) \\ \mathrm{i}(1 + \frac{1}{2}s^2(y^2 - \frac{1}{3}))\sinh(2s) + \mathrm{i}(-\frac{3}{s} + \frac{1}{2}s(y^2 - \frac{1}{3}))\sinh^2(s) \\ s\sinh(s)(sy\sinh(sy) + \cosh(sy)) - \frac{1}{2}s\sinh(2s) \end{pmatrix},$$

where

$$\tilde{a}^0_{2000} = \frac{1}{2}\mathrm{i}\left(\frac{s^2(\cosh(2s) + 2)}{\sinh(2s)(\alpha_0 + 16\beta_0 s^4) - 2s\cosh(2s)} + s\right).$$

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