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On von Neumann's inequality on the polydisc

Michael Hartz[1]

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Abstract

Given a d-tuple T of commuting contractions on Hilbert space and a polynomial p in d-variables, we seek upper bounds for the norm of the operator $p(T)$. Results of von Neumann and Andô show that if $d = 1$ or $d = 2$, the upper bound $\|p(T)\| \le \|p\|_\infty$, holds, where the supremum norm is taken over the polydisc $\mathbb{D}^d$. We show that for $d = 3$, there exists a universal constant C such that $\|p(T)\| \le C\|p\|_\infty$ for every homogeneous polynomial p. We also show that for general d and arbitrary polynomials, the norm $\|p(T)\|$ is dominated by a certain Besov-type norm of p.

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1 Introduction

A famous inequality of von Neumann [42] shows that if T is a contraction on a Hilbert space $\mathcal{H}$, then

$$\|p(T)\| \le \sup_{z\in\mathbb{D}} |p(z)|$$

for every polynomial $p \in \mathbb{C}[z]$. This inequality is the basis of an important connection between operator theory and complex analysis; see for instance [1, 38]. Andô [2] extended von Neumann's inequality to two variables. His inequality shows that if $T = (T_1, T_2)$ is a pair of commuting contractions on Hilbert space, then

$$\|p(T)\| \le \sup_{z\in\mathbb{D}^2} |p(z)|$$

for all polynomials $p \in \mathbb{C}[z_1, z_2]$. However, the corresponding inequality for three or more commuting contractions is false, as examples of Kaijser–Varopoulos [40] and Crabb–Davie [6] show. More background information can be found in the books [28, 32] and in [3].

✉ Michael Hartz
hartz@math.uni-sb.de

[1] Fachrichtung Mathematik, Universität des Saarlandes, 66123 Saarbrücken, Germany

Even though the counterexamples to von Neumann's inequality in three variables were discovered in the seventies, many questions surrounding this phenomenon were only answered recently or remain open. For instance, the smallest dimension of a Hilbert space on which there exist counterexamples in three variables was only determined a few years ago: there exist counterexamples in dimension four due to Holbrook [17], whereas Knese showed that the inequality holds in dimension three or less [21]; this relies on a result in complex geometry due to Kosiński [22]. See also [5] for a related counterexample in dimension three.

Remarkably, it is still not known if von Neumann's inequality for three commuting contractions holds up to a constant; see for instance [28, Chapter 5] and [32, Chapter 1] for a detailed discussion of this problem. Part of the difficulty comes from the lack of a convenient model for tuples of commuting contractions, unlike in the setting of operator tuples associated with the Euclidean ball; see [14]. See also [23] for a non-commutative approach.

To study whether von Neumann's inequality holds up to a constant, one defines $C(d,n) \in [1,\infty)$ to be the smallest constant such that

$$\|p(T)\| \le C(d,n) \sup_{z\in\mathbb{D}^d} |p(z)|$$

holds for every homogeneous polynomial $p \in \mathbb{C}[z_1,\dots,z_d]$ of degree n and every d-tuple T of commuting contractions on Hilbert space. By von Neumann's and Andô's inequalities, $C(1,n) = C(2,n) = 1$ for all $n \in \mathbb{N}$. Dixon [7] showed that for $n \ge 2$,

$$C(d,n) \le G_{\mathbb{C}} (3d)^{(n-2)/2} (2e)^n, \tag{1}$$

where $G_{\mathbb{C}}$ is the complex Grothendieck constant, which satisfies $G_{\mathbb{C}} < \frac{3}{2}$; see [12]. He also proved that for fixed n as $d \to \infty$, this estimate is not too far from optimal. Explicitly, up to a constant depending on n, he established a lower bound for $C(d,n)$ of the form $d^{\frac{1}{2}\lfloor (n-1)/2 \rfloor}$. See also [11] for some recent work on determining the value of $\lim_{d\to\infty} C(d,2)$. However, as Dixon already remarked, the asymptotic behavior of $C(d,n)$ as $d \to \infty$ does not directly bear on the question of whether von Neumann's inequality for d commuting contractions holds up to a constant. Indeed, for this question, the behavior of $C(d,n)$ for fixed d as $n \to \infty$ is relevant.

For fixed d, Dixon's upper bound (1) is exponential in the degree n. However, it is easy to obtain upper bounds for $C(d,n)$ that are polynomial in n. For instance, if $p(z) = \sum_\alpha \widehat{p}(\alpha) z^\alpha$ is homogeneous of degree n, then for any d-tuple of commuting contractions, the Cauchy–Schwarz inequality shows that

$$\begin{aligned} \|p(T)\| \le \sum_\alpha |\widehat{p}(\alpha)| &\le \binom{d+n-1}{d-1}^{1/2} \Big(\sum_\alpha |\widehat{p}(\alpha)|^2 \Big)^{1/2} \\ &\le \binom{d+n-1}{d-1}^{1/2} \sup_{z\in\mathbb{D}^d} |p(z)|. \end{aligned}$$

Here, the binomial coefficient is the dimension of the space of homogeneous polynomials in d variables of degree n. This gives the upper bound $C(d,n) \lesssim_d (n+1)^{(d-1)/2}$. Here and in the sequel, we write $f \lesssim_d g$ to mean that there exists a constant C depending only on d such that $f \le Cg$.

Our first main result gives an upper bound that is polylogarithmic in the degree n and in particular yields that $C(3,n)$ is uniformly bounded in n.

Theorem 1.1 *Let $d \ge 3$. Then for all $n \ge 1$,*

$$C(d,n) \lesssim_d (\log(n+1))^{d-3}.$$

In particular, $\sup_n C(3,n) < \infty$.

This result will be proved in Corollary 3.5. The proof yields the crude explicit upper bound $C(3,n) \le 121$, see Remark 3.6, but no attempt was made to optimize the numerical bound.

It is important to keep in mind that $\sup_n C(3,n) < \infty$ does not imply that von Neumann's inequality for three commuting contractions holds up to a constant, as $C(3,n)$ is only defined using homogeneous polynomials. Nonetheless, the known counterexamples to von Neumann's inequality in three variables all use homogeneous polynomials, and show in particular that $C(3,2) > 1$. (The best known lower abound appears to be $C(3,2) \ge \frac{1}{3}\sqrt{\frac{35+13\sqrt{13}}{6}} \approx 1.23$; see [9, Proposition 6.1].)

In [32, Chapter 4], Pisier gives an exposition of work of Daher, which studies von Neumann's inequality for tuples of commuting $N \times N$ matrices; see [32, Corollary 4.21]. In particular, modifying arguments of Bourgain [4], this work shows that von Neumann's inequality holds up to a factor of $(\log(N+1))^d$ in this context. Such modifications of Bourgain's arguments can also be used to establish the upper bound $C(d,n) \lesssim_d (\log(n+1))^d$.

Given a not necessarily homogeneous polynomial $p \in \mathbb{C}[z_1,\dots,z_d]$, the Schur–Agler norm of p is defined to be

$$\|p\|_{\mathrm{SA}} = \sup\{\|p(T)\|\},$$

where the supremum is taken over all d-tuples T of commuting contractions on Hilbert space. It is natural to seek function theoretic upper bounds for the Schur–Agler norm. To this end, recall that if $f : \mathbb{D}^d \to \mathbb{C}$ is holomorphic, the radial derivative of f is $(Rf)(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}(z)$. Let also write $f_r(z) = f(rz)$ and use $\|\cdot\|_\infty$ to denote the supremum norm on $\mathbb{D}^d$. We now have the following Besov-type upper bound for the Schur–Agler norm.

Theorem 1.2 *Let $d \ge 3$. Then for all $p \in \mathbb{C}[z_1,\dots,z_d]$,*

$$\|p\|_{\mathrm{SA}} \lesssim_d |p(0)| + \int_0^1 \|(Rp)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^{d-3} dr.$$

This result will be proved in Corollary 3.7.

The idea to use Besov norms in the context of functional calculi already appeared in the seminal work of Peller [30] on polynomially bounded operators. Indeed, this article takes inspiration from Peller's work and subsequent works, for instance of Vitse [41] and Schwenninger [35].

Our methods also yield some results about the Schur–Agler norm with constant 1. It was shown by Knese [20] that for certain rational inner functions $f : \mathbb{D}^3 \to \mathbb{C}$, one can find a monomial q such that $\|qf\|_{\mathrm{SA}} = \|f\|_\infty$. (See the disussion preceding Corollary 3.7 for the definition of Schur–Agler norm of a holomorphic function.) We will show an asymptotic version of Knese's theorem for polynomials. Grinshpan, Kaliuzhnyi-Verbovetskyi and Woerdeman proved that there exist polynomials $p \in \mathbb{C}[z_1, z_2, z_3]$ such that $\|z_3 p\|_{\mathrm{SA}} < \|p\|_{\mathrm{SA}}$; see [10, Theorem 2.3]. They also asked if for every polynomial p satisfying $\|p\|_\infty < \|p\|_{\mathrm{SA}}$, there exists a monomial q such that $\|qp\|_{\mathrm{SA}} < \|p\|_{\mathrm{SA}}$; see [10, Problem 2.4]. The following result answers this question in the affirmative; it also gives another proof of the existence of a polynomial $p \in \mathbb{C}[z_1, z_2, z_3]$ with $\|z_3 p\|_{\mathrm{SA}} < \|p\|_{\mathrm{SA}}$.

Theorem 1.3 *Let $d \geq 3$ and let $p \in \mathbb{C}[z_1, \ldots, z_d]$. Then*

$$\lim_{m \to \infty} \|(z_3 \cdot \ldots \cdot z_d)^m p\|_{\mathrm{SA}} = \|p\|_\infty.$$

This result will be proved in Theorem 3.2.

On our way to establishing Theorems 1.1 and 1.2, we first study a version of the one-variable von Neumann inequality, but for polynomials with operator coefficients satisfying a commutativity hypothesis; see Section 2 for precise details. The idea to use one-variable polynomials with operator coefficients to study the scalar von Neumann inequality in several variables already appears in work of Daher; see [32, Chapter 4]. In Theorem 2.8, which may be of independent interest, we establish fairly precise upper and lower bounds for von Neumann's inequality with operator coefficients. However, we will see in Propositions 3.9 and 3.10 that the operators yielding logarithmic lower bounds in von Neumann's inequality with operator coefficients satisfy von Neumann's inequality for d-tuples up to a constant.

The remainder of this article is organized as follows. Section 2 contains the material on von Neumann's inequality with operator coefficients. Section 3 deals with von Neumann's inequality on the polydisc. Moreover, in the appendix, we collect a few basic facts about analytic Besov spaces.

2 Polynomials with operator coefficients

In this section, we consider operator-valued polynomials of the form

$$p(z) = \sum_{k=m}^{n} A_k z^k,$$

where each $A_k \in B(\mathcal{H})$. We call such polynomials (m, n)-band-limited. Given another operator $T \in B(\mathcal{H})$, we "evaluate" the polynomial at T as follows:

$$p(T) = \sum_{k=m}^{n} A_k T^k.$$

Notice that the product is not the tensor product, but composition of operators. (Using the tensor product, one obtains von Neumann's inequality with constant 1 by Sz.-Nagy's dilation theorem.)

We are mostly concerned with the case when each A_k commutes with T. Equivalently, $p(z)$ commutes with A_k for all $z \in \mathbb{D}$. We seek bounds on $\|p(T)\|$ in terms of $\sup_{z\in\mathbb{D}} \|p(z)\|$. In the case $m = 0$, this problem was already studied by Daher and Pisier; see [32, Chapter 4]. To get a feeling for the problem, we first consider the easier case without any assumption on commutation.

Throughout, we will use $\mathbb{T}$ to denote the unit circle, and we write

$$\int_{\mathbb{T}} f(z) \frac{|dz|}{2\pi} = \int_0^{2\pi} f(e^{it}) \frac{dt}{2\pi}.$$

Proposition 2.1 *Let $p(z) = \sum_{k=m}^{n} A_k z^k$ be a polynomial with operator coefficients and let $\|T\| \le 1$. Then*

$$\left\| \sum_{k=m}^{n} A_k T^k \right\| \le \sqrt{n-m+1} \sup_{z\in\mathbb{D}} \|p(z)\|.$$

Moreover, the factor $\sqrt{n-m+1}$ is best possible in the sense that for all $1 \le m \le n$, there exists a non-zero choice of $A_m, \ldots, A_n$ and T such that equality holds.

Proof We may normalize so that $\sup_{z\in\mathbb{D}} \|p(z)\| = 1$. Then

$$I \ge \int_{\mathbb{T}} p(z)p(z)^* \frac{|dz|}{2\pi} = \sum_{k=m}^{n} A_k A_k^*.$$

Hence the row $\begin{bmatrix} A_m & \cdots & A_n \end{bmatrix}$ is a contraction. Then

$$\sum_{k=m}^{n} A_k T^k = \begin{bmatrix} A_m & \cdots & A_n \end{bmatrix} \begin{bmatrix} T^m \\ T^{m+1} \\ \vdots \\ T^n \end{bmatrix}.$$

The column has norm at most $\sqrt{n-m+1}$, from which the upper bound follows.

To see that equality may hold, let $A_k = E_{1,k+1}$ (matrix units on ℓ^2) and let T be the unilateral shift. Then

$$\sum_{k=m}^{n} A_k T^k e_1 = \sum_{k=m}^{n} E_{1,k+1} e_{k+1} = (n-m+1)e_1,$$

hence $\|\sum_{k=m}^{n} A_k T^k\| \geq n-m+1$, but $p(z)p(z)^* = \sum_{k=m}^{n} E_{1,1}|z|^{2k}$, so that $\|p\|_\infty = \sqrt{n-m+1}$. □

Remark 2.2 The proof of the upper bound in fact applies to power bounded operators.

Recall that T is said to doubly commute with A if T commutes with A and A^*. In the doubly commuting case, one obtains the inequality with constant 1. It appears that this was first shown by Arveson and Parrott (unpublished) and Mlak [25].

Proposition 2.3 *Let* $p(z) = \sum_{k=0}^{n} A_k z^k$ *be a polynomial with operator coefficients and let* $T \in B(\mathcal{H})$ *with* $\|T\| \leq 1$. *If* T *doubly commutes with each* A_k, *then*

$$\Big\| \sum_{k=0}^{n} A_k T^k \Big\| \leq \sup_{z \in \mathbb{D}} \|p(z)\|.$$

Proof We use a small modification of a proof of von Neumann's inequality due to Heinz; see [15] and also [28, Exercise 2.15]. We may assume that $\|T\| < 1$. For $z \in \overline{\mathbb{D}}$, consider the Poisson-type kernel

$$P(z,T) = (1 - zT^*)^{-1} + (1 - \overline{z}T)^{-1} - I.$$

A simple computation shows that $P(z,T) \geq 0$ for all $z \in \overline{\mathbb{D}}$ and that

$$\sum_{k=0}^{n} A_k T^k = \int_{\mathbb{T}} p(z) P(z,T) \frac{|dz|}{2\pi} = \int_{\mathbb{T}} P(z,T)^{1/2} p(z) P(z,T)^{1/2} \frac{|dz|}{2\pi},$$

where the second equality follows from the doubly commuting assumption. Now, the map

$$\Phi : C(\mathbb{T}, B(\mathcal{H})) \to B(\mathcal{H}), \quad f \mapsto \int_{\mathbb{T}} P(z,T)^{1/2} f(z) P(z,T)^{1/2} \frac{|dz|}{2\pi}$$

is unital and completely positive, hence completely contractive; see for instance [28, Proposition 3.2]. In particular, $\|p(T)\| = \|\Phi(p)\| \leq \|p\|_\infty$. □

To deal with the singly commuting case, we require the following routine application of the Cauchy–Schwarz inequality.

Lemma 2.4 *Let X be a compact Hausdorff space, let μ be a Borel probability measure on X and let $K, L : X \to B(\mathcal{H})$ and let $f : X \to B(\mathcal{H})$ be norm continuous. Then*

$$\begin{aligned}&\left\| \int_X K(x) f(x) L(x)\, d\mu(x) \right\| \\ &\quad \leq \left\| \int_X K(x) K(x)^*\, d\mu(x) \right\|^{1/2} \left\| \int_X L(x)^* L(x)\, d\mu(x) \right\|^{1/2} \sup_{x \in X} \| f(x) \|.\end{aligned}$$

Proof Let $\xi, \eta \in \mathcal{H}$ be unit vectors and assume without loss of generality that $\sup_{x \in X} \| f(x) \| = 1$. Applying the Cauchy–Schwarz inequality to the positive semi-definite sesquilinear form on $C(X, \mathcal{H})$ defined by

$$(g, h) \mapsto \int_X \langle g(x), h(x) \rangle\, d\mu(x),$$

we find that

$$\begin{aligned}&\left| \int_X \langle K(x) f(x) L(x) \xi, \eta \rangle d\mu(x) \right|^2 \\ &\quad = \left| \int_X \langle f(x) L(x) \xi, K(x)^* \eta \rangle d\mu(x) \right|^2 \\ &\quad \leq \int_X \langle f(x)^* f(x) L(x) \xi, L(x) \xi \rangle\, d\mu(x) \int_X \langle K(x)^* \eta, K(x)^* \eta \rangle\, d\mu(x).\end{aligned}$$

Since $\sup_{x \in X} \| f(x) \| = 1$, we have $f(x)^* f(x) \leq I$ for all $x \in X$, so the first factor can be estimated by

$$\begin{aligned}\int_X \langle f(x)^* f(x) L(x) \xi, L(x) \xi \rangle\, d\mu(x) &\leq \int_X \langle L(x) \xi, L(x) \xi \rangle d\mu(x) \\ &\leq \left\| \int_X L(x)^* L(x) d\mu(x) \right\|.\end{aligned}$$

A similar estimate holds for the second factor, thus

$$\begin{aligned}&\left| \int_X \langle K(x) f(x) L(x) \xi, \eta \rangle d\mu(x) \right|^2 \\ &\quad \leq \left\| \int_X L(x)^* L(x) d\mu(x) \right\| \left\| \int_X K(x) K(x)^* d\mu(x) \right\|.\end{aligned}$$

The assertion now follows by taking the supremum over all unit vectors $\xi, \eta \in \mathcal{H}$. □

We now turn to the simply commuting case. Given natural numbers $0 \le m \le n$, let $K(m,n)$ be the smallest constant such that

$$\Big\| \sum_{k=m}^{n} A_k T^k \Big\| \le K(m,n) \sup_{z \in \mathbb{D}} \Big\| \sum_{k=m}^{n} A_k z^k \Big\|$$

holds for all operators $A_m, \dots, A_n \in B(\mathcal{H})$ and all $T \in B(\mathcal{H})$ satisfying $\|T\| \le 1$ and $TA_k = A_k T$ for all $k = m, m+1, \dots, n$. The arguments of Daher and Pisier show that $K(0,n) \simeq \log(n+1) + 1$; see [32, Chapter 4]. Here, we write $f \simeq g$ to mean $f \lesssim g$ and $g \lesssim f$. We will obtain fairly sharp estimates for $K(m,n)$ in general.

Let H^2 denote the classical Hardy space on the disc and let $\mathrm{Han} \subset H^2$ be the space of symbols of bounded Hankel operators $H^2 \to \overline{H^2}$. Thus, if $b \in \mathrm{Han}$, then we obtain a bounded Hankel operator $H_b : H^2 \to \overline{H^2}$ satisfying

$$\langle H_b f \overline{g} \rangle_{\overline{H^2}} = \langle fg, b \rangle_{H^2}$$

for all polynomials f, g. We equip Han with the norm $\|b\|_{\mathrm{Han}} = \|H_b\|$. Nehari's theorem ([26], see also [31, Theorem 1.1]) shows that $\mathrm{Han} \cong L^\infty / \overline{H^\infty_0}$ is the dual space of H^1 with respect to the Cauchy pairing. It is known that Han can be identified with BMOA, but we do not require this. More background on Hankel operators can be found in [31].

Given a function $h \in H^1$, we write $h(z) = \sum_{k=0}^\infty \widehat{h}(k) z^k$ for the Taylor series of h.

Proposition 2.5 *We have*

$$\begin{aligned} K(m,n) &= \inf\{\|h\|_{H^1} : \widehat{h}(k) = 1 \textit{ for } m \le k \le n\} \\ &= \sup\{|q(1)| : \|q\|_{\mathrm{Han}} \le 1 \textit{ and } \operatorname{supp} \widehat{q} \subset [m,n]\}. \end{aligned}$$

Proof The second equality follows from duality. Indeed, let

$$M = \{h \in H^1 : \widehat{h}(k) = 0 \text{ for } m \le k \le n\}.$$

Then the annihilator of M in Han is $\{q \in \mathrm{Han} : \operatorname{supp} \widehat{q} \subset [m,n]\}$. So if $f \in H^1$ is any function with $\widehat{f}(k) = 1$ for $m \le k \le n$, then by the Hahn–Banach theorem,

$$\begin{aligned} \inf\{\|h\|_{H^1} : \widehat{h}(k) = 1 \text{ for } m \le k \le n\} &= \operatorname{dist}(f, M) \\ &= \sup\{|\langle f, q\rangle| : \|q\|_{\mathrm{Han}} \le 1 \text{ and } \operatorname{supp} \widehat{q} \subset [m,n]\} \\ &= \sup\{|q(1)| : \|q\|_{\mathrm{Han}} \le 1 \text{ and } \operatorname{supp} \widehat{q} \subset [m,n]\}. \end{aligned}$$

Next, we prove that $K(m,n)$ is bounded above by the infimum. To this end, we use a factorization argument, which already appears in [32, Proposition 4.16] and [43, III.F.18]; see also [13] for extensions to semigroups on Banach spaces. Let $p(z) = \sum_{k=m}^n A_k z^k$ be an operator-valued polynomial with $\sup_{z \in \mathbb{D}} \|p(z)\| \le 1$ and let $T \in$

$B(\mathcal{H})$ be a contraction that commutes with all A_k. Let $h \in H^1$ satisfy $\widehat{h}(k) = 1$ for $m \leq k \leq n$. We have to show that

$$\|p(T)\| \leq \|h\|_{H^1}. \tag{2}$$

By replacing T with rT for $r < 1$, we may assume that $\sigma(T) \subset \mathbb{D}$. There exist $f, g \in H^2$ so that $h = fg$ and $\|f\|_{H^2}\|g\|_{H^2} = \|h\|_{H^1}$. Thus, by the commutation hypothesis,

$$p(T) = \int_{\mathbb{T}} p(z)h(\overline{z}T)\,\frac{|dz|}{2\pi} = \int_{\mathbb{T}} f(\overline{z}T)p(z)g(\overline{z}T)\,\frac{|dz|}{2\pi}.$$

By Lemma 2.4, it follows that

$$\|p(T)\| \leq \left\| \int_{\mathbb{T}} f(\overline{z}T)f(\overline{z}T)^*\,\frac{|dz|}{2\pi} \right\|^{1/2} \left\| \int_{\mathbb{T}} g(\overline{z}T)^* g(\overline{z}T)\,\frac{|dz|}{2\pi} \right\|^{1/2}.$$

Using orthogonality, we find that

$$\int_{\mathbb{T}} f(\overline{z}T)f(\overline{z}T)^*\,\frac{|dz|}{2\pi} = \sum_{k=0}^{\infty} |\widehat{f}(k)|^2 T^k (T^*)^k,$$

hence

$$\left\| \int_{\mathbb{T}} f(\overline{z}T)f(\overline{z}T)^*\,\frac{|dz|}{2\pi} \right\|^{1/2} \leq \|f\|_{H^2}.$$

Similarly,

$$\left\| \int_{\mathbb{T}} g(\overline{z}T)^* g(\overline{z}T)\,\frac{|dz|}{2\pi} \right\|^{1/2} \leq \|g\|_{H^2}.$$

Since $\|f\|_{H^2}\|g\|_{H^2} = \|h\|_{H^1}$, the upper bound (2) follows.

Finally, we show that $K(m, n)$ is bounded below by the supremum. To this end, we use Foguel–Hankel operators; see [28, Chapter 10] for background. For $q \in \text{Han}$, let $H_q : H^2 \to \overline{H^2}$ be the Hankel operator satisfying $\langle H_q f, \overline{g}\rangle = \langle fg, q\rangle$ for polynomials f, g. Let ζ denote the independent variable on $\mathbb{D}$, and let $M_\zeta : H^2 \to H^2$ be the shift. Then

$$H_q M_\zeta = M_{\overline{\zeta}}^* H_q. \tag{3}$$

Let

$$T = \begin{bmatrix} M_\zeta & 0 \\ 0 & M_{\overline{\zeta}}^* \end{bmatrix} \in B(H^2 \oplus \overline{H^2}).$$

If $q \in \text{Han}$ with $\operatorname{supp} \widehat{q} \subset [m, n]$, define

$$A_k = \begin{bmatrix} 0 & 0 \\ H_{\widehat{q}(k)\zeta^k} & 0 \end{bmatrix}.$$

Then (3) shows that T commutes with each A_k. Let $p(z) = \sum_{k=m}^n A_k z^k$. Then

$$\|p(z)\| = \Big\| \sum_{k=m}^{n} \widehat{q}(k)(\zeta \overline{z})^k \Big\|_{\mathrm{Han}} \leq \|q\|_{\mathrm{Han}}$$

for all $z \in \mathbb{T}$ by rotation invariance of Han, and hence for all $z \in \overline{\mathbb{D}}$ by the maximum principle. So if $\|q\|_{\mathrm{Han}} \leq 1$, then

$$K(m,n) \geq \Big\| \sum_{k=m}^{n} A_k T^k \Big\| = \Big\| \sum_{k=m}^{n} H_{\widehat{q}(k)\zeta^k} M_{\zeta^k} \Big\| = \|H_{q(1)}\| = |q(1)|.$$

This proves the lower bound for $K(m,n)$. □

Remark 2.6 (a) The proof of the upper bound for $K(m,n)$ in fact works more generally for power bounded operators T. Thus, using the result of Proposition 2.5, we find that if T is power bounded and A_k are operators commuting with T, then

$$\Big\| \sum_{k=m}^{n} A_k T^k \Big\| \leq \sup_n \|T^n\|^2 K(m,n) \sup_{z\in\mathbb{D}} \Big\| \sum_{k=m}^{n} A_k z^k \Big\|.$$

So in contrast to the scalar von Neumann inequality, contractions do not yield a qualitatively better estimate than power bounded operators.

(b) In the context of the classical inequalities of von Neumann and Andô, it is natural to consider matrices of polynomials because of the connections to dilation theory, see for instance [28, Chapter 7] and [32, Chapter 4]. In the present setting, one can similarly consider $r \times r$ matrices $[p_{ij}]$, where each entry p_{ij} is an operator-valued polynomial of the form

$$p_{ij}(z) = \sum_{k=m}^{n} A_k^{(ij)} z^k,$$

and each $A_k^{(ij)} \in B(\mathcal{H})$. Such a matrix can be evaluated entry-wise at an operator T that commutes with all coefficients $A_k^{(ij)}$. Considering such matrices of arbitrary size r, one defines a completely bounded version $K_{\mathrm{cb}}(m,n)$ of the constant $K(m,n)$. However, this setting is actually not more general, and we have $K_{\mathrm{cb}}(m,n) = K(m,n)$. Indeed, given an $r \times r$ matrix $[p_{ij}]$ as above and a contraction $T \in B(\mathcal{H})$ commuting with all coefficients $A_k^{(ij)}$, let $E_{ij} \in M_r(\mathbb{C})$ be the usual matrix units and define

$$q(z) = \sum_{i,j=1}^{r} \sum_{k=m}^{n} (A_k^{(ij)} \otimes E_{ij}) z^k.$$

Then q is a polynomial with coefficients in $B(\mathcal{H}) \otimes M_r(\mathbb{C})$, which we identify with $B(\mathcal{H}^r)$, and $\sup_{z\in\mathbb{D}} \|q(z)\| = \sup_{z\in\mathbb{D}} \|[p_{ij}(z)]\|$. The contraction $T \otimes I$ commutes with the coefficients of q, and

$$\|[p_{ij}(T)]\| = \|q(T \otimes I)\|.$$

This shows that $K_{\mathrm{cb}}(m,n) = K(m,n)$.

With the help of the last result, we can now get quantitative estimates for $K(m,n)$. One might build a function h in Proposition 2.5 with the help of de la Vallée-Poussin kernels and use known L^1-estimates of de la Vallée-Poussin kernels; see [24, 36, 37]. Unfortunately, these estimates do not appear to be completely sufficient for our needs.

Remark 2.7 Here are some simple ways to get quantitative estimates on $K(m,n)$. These are already sufficient for our main applications.

(1) For $m = 0$, we may use as in [32, Corollary 4.17] the function $h(z) = \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$. This function is a shifted version of the Dirichlet kernel, so $\|h\|_{H^1}$ is comparable to $\log(n+1)+1$. Thus,

$$K(0,n) \lesssim \log(n+1) + 1.$$

(2) For general $n \geq m \geq 0$, we can use the shifted Fejér-type kernels W_j whose Fourier coefficients are the triangular-shaped function supported in $(2^{j-1}, 2^{j+1})$ with peak at 2^j; see the appendix for more information. Let $h = \sum_{j=a}^{b} W_j$. Then $\widehat{h}(k) = 1$ for $2^a \leq k \leq 2^b$ (and $\widehat{h}(k) = 1$ for $0 \leq k \leq 2^b$ in case $a = 0$) and $\|h\|_{H^1} \leq \frac{3}{2}(b-a+1)$. (A slightly different function h with these properties was already contructed by Haase; see [13, Lemma A.2].) By Proposition 2.5, this yields

$$K(m,n) \lesssim 1 + \log\Big(\frac{n+1}{m+1}\Big).$$

(3) Let

$$f(z) = \frac{1}{m+1}\sum_{k=0}^{m} z^k, \quad g(z) = \sum_{k=0}^{n} z^k$$

and $h = fg$. Then $\widehat{h}(k) = 1$ for $m \leq k \leq n$, and

$$\|h\|_{H^1}^2 \leq \|f\|_{H^2}^2 \|g\|_{H^2}^2 = \frac{n+1}{m+1}.$$

This yields

$$K(m,n) \leq \Big(\frac{n+1}{m+1}\Big)^{1/2}.$$

Observe that estimate (2) is better than estimate (3) when the ratio $\frac{n+1}{m+1}$ is large, while estimate (3) is better when $\frac{n+1}{m+1}$ is close to 1 because of the implied constants in estimate (2). Even though the estimates in Remark 2.7 are sufficient for our applications, it seems worthwhile to determine the behavior of $K(m,n)$ more precisely and in particular establish an estimate for $K(m,n)$ that is good in both regimes. This is done in the following result.

Theorem 2.8 *We have*

$$\max\Big(1, \frac{1}{\pi}\log\Big(\frac{n+2}{m+1}\Big)\Big) \le K(m,n) \le \frac{1}{\pi}\log\Big(\frac{n+1}{m+1}\Big) + \min\Big(\frac{n+1}{m+1}, 2\Big).$$

Proof Upper bound: We use Proposition 2.5 and construct a function $h \in H^1$ with $\widehat{h}(k) = 1$ for $m \le k \le n$ whose norm is dominated by the right-hand side in the statement of the theorem. The construction below already appears in work of Haase, see the proof of [13, Lemma A.2]. But in order to obtain the stated bound, we need to estimate somewhat more carefully.

Define holomorphic functions on the disc by

$$\begin{aligned} f(z) &= \sum_{j=0}^{m-1} \frac{j+1}{m+1} z^j + \sum_{j=m}^{\infty} z^j = \frac{d}{dz}\Big(\frac{1-z^{m+1}}{(m+1)(1-z)}\Big) + \frac{z^m}{1-z} \\ &= \frac{1-z^{m+1}}{(m+1)(1-z)^2} \end{aligned}$$

and

$$u(z) = \frac{(1-z^{m+1})^{1/2}}{(m+1)^{1/2}(1-z)}$$

and

$$g(z) = \sum_{j=0}^{n} \widehat{u}(j) z^j.$$

Finally, we set $h = g^2$. Since g agrees with u to order n and $u^2 = f$, we see that h agrees with f to order n. In particular, $\widehat{h}(k) = 1$ for $m \le k \le n$.

It remains to estimate $\|h\|_{H^1}$. To this end, notice that

$$\|h\|_{H^1} = \|g\|_{H^2}^2 = \sum_{j=0}^{n} |\widehat{u}(j)|^2. \tag{4}$$

To compute the Taylor coefficients of u, we use the binomial series to obtain

$$(m+1)^{\frac{1}{2}} u(z) = \left(\sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}}{k} z^{k(m+1)}\right)\left(\sum_{k=0}^{\infty} z^k\right).$$

Expanding the product and writing $l = \lfloor j/(m+1) \rfloor$, we see that for all $j \ge 0$, the Taylor coefficients are given by

$$(m+1)^{1/2}\widehat{u}(j) = \sum_{\nu=0}^{l} (-1)^{\nu} \binom{\frac{1}{2}}{\nu} = (-1)^l \binom{-\frac{1}{2}}{l}. \tag{5}$$

Here, the last identity can be seen by expanding both sides of $(1-z)^{-1/2} = \frac{(1-z)^{1/2}}{1-z}$ into a binomial series and comparing coefficients. Expanding the binomial coefficient $\binom{-1/2}{l}$ and rearranging (5) yields

$$\widehat{u}(j) = (m+1)^{-1/2} 4^{-l} \binom{2l}{l}, \quad \text{where } l = \lfloor j/(m+1) \rfloor. \tag{6}$$

In words, the Taylor coefficients of u are given by the sequence on the right, and each element is repeated $m+1$ times.

Now, write $(n+1) = k(m+1) + r$ with natural numbers k, r with $r < m+1$ (hence $k = \lfloor (n+1)/(m+1) \rfloor$). Thus, from (4) and (6), we find that

$$\|h\|_{H^1} = \sum_{l=0}^{k-1} \left(4^{-l}\binom{2l}{l}\right)^2 + \frac{r}{m+1}\left(4^{-k}\binom{2k}{k}\right)^2.$$

To estimate the central binomial coefficients, we use Stirling's formula in the form

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} \le n! \le \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}},$$

see for example [34], to obtain the estimate

$$4^{-l}\binom{2l}{l} \le \frac{1}{\sqrt{\pi l}}$$

for $l \ge 1$. Thus,

$$\|h\|_{H^1} \le 1 + \sum_{l=1}^{k-1} \frac{1}{\pi l} + \frac{r}{\pi(m+1)k}. \tag{7}$$

Comparing the sum with an integral, we find that

$$\sum_{l=1}^{k-1} \frac{1}{l} \le \log(k) + \min(k-1, 1)$$

for $k \ge 1$. Moreover, recalling that $(n+1) = k(m+1) + r$, we see that

$$\frac{r}{(m+1)k} = \frac{\frac{n+1}{m+1} - \lfloor \frac{n+1}{m+1} \rfloor}{\lfloor \frac{n+1}{m+1} \rfloor} \le \min\left(\frac{n+1}{m+1} - 1, 1\right).$$

It therefore follows from (7) that

$$\|h\|_{H^1} \le 1 + \frac{1}{\pi}\log\left(\frac{n+1}{m+1}\right) + \frac{2}{\pi}\min\left(\frac{n+1}{m+1} - 1, 1\right).$$

The stated upper bound follows from this inequality.

Lower bound: The lower bound $K(m, n) \geq 1$ is trivial. Let

$$q(z) = \sum_{k=m}^{n} \frac{1}{k+1} z^k.$$

Then

$$q(1) = \sum_{k=m}^{n} \frac{1}{k+1} \geq \int_m^{n+1} \frac{1}{t+1}\, dt = \log\left(\frac{n+2}{m+1}\right).$$

To estimate $\|q\|_{\mathrm{Han}}$, observe that with respect to the standard bases of H^2 and $\overline{H^2}$, the Hankel operator H_q is the Hankel matrix corresponding to the sequence $(\widehat{q}(k))_{k=0}^{\infty}$. Since q has non-negative Taylor coefficients, it follows that $\|q\|_{\mathrm{Han}}$ is bounded above by the norm of the Hilbert matrix, which is equal to π. Thus, the lower bound follows from Proposition 2.5. □

As a consequence, we obtain a Besov-type functional calculus. Background on Besov spaces can be found in the appendix. If $f : \mathbb{D} \to B(\mathcal{H})$ is an operator-valued holomorphic function, define

$$\|f\|_{B^0_{\infty,1}} = \|f(0)\| + \int_0^1 \|f_r'\|_\infty \, dr.$$

Here, $f_r'(z) = f'(rz)$ and $\|g\|_\infty = \sup_{z \in \mathbb{D}} \|g(z)\|$.

Corollary 2.9 *Let $f : \mathbb{D} \to B(\mathcal{H})$ be an operator-valued analytic function with Taylor series*

$$f(z) = \sum_{k=0}^{\infty} A_k z^k.$$

Let $T \in B(\mathcal{H})$ be a contraction that commutes with each A_k. If $\|f\|_{B^0_{\infty,1}} < \infty$, then

$$f(T) = \lim_{r \nearrow 1} f(rT) = \lim_{r \nearrow 1} \sum_{n=0}^{\infty} A_k r^k T^k$$

exists, and

$$\|f(T)\| \lesssim \|f\|_{B^0_{\infty,1}}.$$

Proof Suppose initially that f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, so that the sum $\sum_{k=0}^{\infty} A_k T^k$ converges. We use the Fejér-type kernels W_n, see the appendix. Let $p_n = f * W_n$. Since $\sum_{n=0}^{\infty} \widehat{W}_n(k) = 1$ for all k and since f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, we may interchange the order of summation to find that

$$\sum_{n=0}^{\infty} p_n(T) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_k \widehat{W}_n(k) T^k = \sum_{k=0}^{\infty} A_k T^k = f(T).$$

For $n \geq 1$, the polynomial p_n is $(2^{n-1}, 2^{n+1})$-band-limited, so by Theorem 2.8 (or simply Remark 2.7 (2)),

$$\|p_n(T)\| \leq K(2^{n-1}, 2^{n+1})\|p_n\|_\infty \lesssim \|p_n\|_\infty.$$

Such an estimate also holds for $n = 0$. It follows that

$$\|f(T)\| \leq \sum_{n=0}^{\infty} \|p_n(T)\| \lesssim \sum_{n=0}^{\infty} \|f * W_n\|_\infty.$$

By Proposition A.4, the last expression is comparable to $\|f\|_{B^0_{\infty,1}}$.

If f merely satisfies $\|f\|_{B^0_{\infty,1}} < \infty$, then by definition and the dominated convergence theorem, $\|f - f_r\|_{B^0_{\infty,1}} \to 0$ as $r \to 1$. From this and the first paragraph, it easily follows that the net $(f_r(T))$ is Cauchy in $B(\mathcal{H})$, hence $\lim_{r \to 1} f_r(T)$ exists in $B(\mathcal{H})$ and satisfies the desired norm estimate. □

3 von Neumann's inequality on the polydisc

In this section, we use the results on von Neumann's inequality with operator coefficients to study von Neumann's inequality for commuting contractions. The basic idea is very simple: we plug in operators successively and use the inequality with operator coefficients in each step. This approach already appeared in the work of Daher; see [32, Chapter 4].

As a first application, we establish an upper bound for the Schur–Agler norms of polynomials that is polylogarithmic in the degree of the polynomial. Supremum norms of d-variable functions are understood to be taken over the polydisc $\mathbb{D}^d$.

Proposition 3.1 *If $d \geq 2$ and if $p \in \mathbb{C}[z_1, \ldots, z_d]$ is a polynomial of degree $n \geq 1$, then*

$$\|p\|_{\mathrm{SA}} \lesssim_d (\log(n+1))^{d-2}\|p\|_\infty.$$

Proof The proof is by induction on d. If $d = 2$, then Andô's theorem shows that $\|p\|_{\mathrm{SA}} = \|p\|_\infty$. Let $d \geq 3$, and suppose that the result has been shown for $d - 1$. We write

$$p(z) = \sum_{k=0}^{n} p_k(z_1, \ldots, z_{d-1}) z_d^k,$$

for polynomials $p_k \in \mathbb{C}[z_1, \ldots, z_{d-1}]$ of degree at most n. Let T be a d-tuple of commuting contractions. By the inductive hypothesis, we have

$$\|p(T_1, \ldots, T_{d-1}, z)\| \lesssim_d (\log(n+1))^{d-3}\|p\|_\infty$$

for all $z \in \mathbb{D}$, so by Theorem 2.8 (or simply Remark 2.7 (1)), we have

$$\|p(T)\| \leq K(0, n) \sup_{z \in \mathbb{D}} \|p(T_1, \ldots, T_{d-1}, z)\| \lesssim_d (\log(n+1))^{d-2}\|p\|_\infty.$$

□

A similar technique as in the last proof can be used to establish Theorem 1.3, which we restate here.

Theorem 3.2 *Let $d \geq 3$ and let $p \in \mathbb{C}[z_1, \dots, z_d]$. Then*

$$\lim_{m\to\infty} \|(z_3 \cdot \dots \cdot z_d)^m p\|_{SA} = \|p\|_\infty.$$

Proof It is clear that

$$\|(z_3 \cdot \dots \cdot z_d)^m p\|_{SA} \geq \|(z_3 \cdot \dots \cdot z_d)^m p\|_\infty = \|p\|_\infty$$

for all $m \in \mathbb{N}$.

To establish the converse direction, we prove that for all $d \geq 2$ and all polynomials $p \in \mathbb{C}[z_1, \dots, z_d]$ of degree at most n, the estimate

$$\|(z_3 \cdot \dots \cdot z_d)^m p\|_{SA} \leq K(m, m+n)^{d-2} \|p\|_\infty \tag{8}$$

holds. Here, the (empty) product on the left is understood as 1 if $d = 2$. Assuming this estimate for a moment, we obtain the remaining inequality from Theorem 2.8 (or simply Part (3) of Remark 2.7), which shows that

$$\limsup_{m\to\infty} K(m, m+n) \leq 1.$$

The proof of (8) is similar to that of Proposition 3.1 and proceeds by induction on d. The case $d = 2$ follows from Andô's theorem. Let $d \geq 3$ and suppose that (8) has been shown for $d - 1$. Let p be a polynomial in d variables of degree at most n, which we write as

$$p(z) = \sum_{k=0}^{n} p_k(z_1, \dots, z_{d-1}) z_d^k,$$

where each $p_k \in \mathbb{C}[z_1, \dots, z_{d-1}]$ has degree at most n. Let T be a d-tuple of commuting contractions. Then,

$$\begin{aligned}
&\|((z_3 \cdot \dots \cdot z_d)^m p)(T)\| \\
&= \left\| (T_3 \cdot \dots \cdot T_{d-1})^m \sum_{k=0}^{n} p_k(T_1, \dots, T_{d-1}) T_d^{k+m} \right\| \\
&\leq K(m, m+n) \sup_{z_d \in \mathbb{D}} \left\| (T_3 \cdot \dots \cdot T_{d-1})^m \sum_{k=0}^{n} p_k(T_1, \dots, T_{d-1}) z_d^{k+m} \right\|.
\end{aligned}$$

For each $z_d \in \mathbb{D}$, the inductive hypothesis implies that

$$\left\| (T_3 \cdot \dots \cdot T_{d-1})^m \sum_{k=0}^{n} p_k(T_1, \dots, T_{d-1}) z_d^{k+m} \right\|$$

$$\leq K(m, m+n)^{d-3} \sup_{z' \in \mathbb{D}^{d-1}} |z_d^m p(z', z_d)|$$
$$\leq K(m, m+n)^{d-3} \|p\|_\infty.$$

Combining both inequalities yields (8) for d. □

If $p \in \mathbb{C}[z_1, \dots, z_d]$ is a polynomial, say

$$p(z) = \sum_\alpha \widehat{p}(\alpha) z^\alpha,$$

then we say that p is (m, n)-band-limited if $\widehat{p}(\alpha) = 0$ whenever $|\alpha| < m$ or $|\alpha| > n$. We say that p is (m, n)-band-limited with respect to z_j if $\widehat{p}(\alpha) = 0$ whenever $\alpha_j < m$ or $\alpha_j > n$. In other words, if we fix all variables but the jth one, then p is (m, n)-band-limited as a polynomial in z_j.

The following splitting lemma is crucial for establishing the upper bound of the Schur–Agler norm of homogeneous polynomials.

Lemma 3.3 *Let $p \in \mathbb{C}[z_1, \dots, z_d]$ be an (m, n)-band-limited polynomial. Then there exist polynomials $p_1, \dots, p_d$ such that*

(1) $p = \sum_{j=1}^d p_j$,
(2) *each p_j is $(\lfloor \frac{m}{2d} \rfloor, n)$-band-limited with respect to z_j, and*
(3) $\|p_j\|_\infty \lesssim_d \|p\|_\infty$ *for each j.*

Proof We first informally describe the basic idea. If $\widehat{p}(\alpha) \neq 0$, then $\alpha_j \geq \frac{m}{d}$ for some j, as p is (m, n)-band-limited. So we should assign the monomial $\widehat{p}(\alpha) z^\alpha$ to the polynomial p_j. In this way, one obtains a splitting that satisfies Conditions (1) and (2), even with $\frac{m}{d}$ in place of $\frac{m}{2d}$. However, to maintain supremum norm control, we need to smoothen the cut-off. This will be achieved with the help of de la Vallée-Poussin kernels.

We now come to the actual proof. By replacing m with $\lfloor \frac{m}{2d} \rfloor 2d$, we may assume that $\frac{m}{2d}$ is an integer. For an integer $k \geq 2$, let V_k be the real-valued trigonometric polynomial of one variable whose non-negative Fourier coefficients are the trapezoid-shaped function supported in $(\frac{m}{2d}, kn)$ that is identically one on $[\frac{m}{d}, n]$ and affine on $[\frac{m}{2d}, \frac{m}{d}]$ and on $[n, kn]$.

If $q \in \mathbb{C}[z_1, \dots, z_d]$ is any polynomial, we write

$$(q *_j V_k)(z) = \int_{\mathbb{T}} q(z_1, \dots, z_{j-1}, z_j \overline{w}, z_{j+1}, \dots, z_d) V_k(w) \frac{|dw|}{2\pi}$$
$$= \sum_\alpha \widehat{q}(\alpha) \widehat{V}_k(\alpha_j) z^\alpha.$$

If q has degree at most n, then $q *_j V_k$ is $(\frac{m}{2d}, n)$-band-limited with respect to z_j and independent of k. Since $\|V_k\|_{L^1} \leq 3 + \frac{k+1}{k-1}$ (see Lemma A.1 in the appendix), we have $\|q *_j V_k\|_\infty \leq 4\|q\|_\infty$.

Recursively, we define $p_1 = p *_1 V$ and

$$p_j = (p - p_1 - \ldots - p_{j-1}) *_j V_k$$

for $j = 2, 3, \ldots d$. Properties (2) and (3) are then clear. To show Property (1), let $q = p - p_1 - \ldots - p_{d-1}$. A simple induction argument shows that

$$\widehat{q}(\alpha) = \widehat{p}(\alpha) \prod_{j=1}^{d-1} (1 - \widehat{V_k}(\alpha_j))$$

for all multi-indices α. Thus, if α is a multi-index with $\alpha_k \geq \frac{m}{d}$ for some $1 \leq k \leq d-1$, then $\widehat{q}(\alpha) = 0$. On the other hand, q is (m, n)-band limited, so it follows that $\widehat{q}(\alpha) = 0$ if $\alpha_d < \frac{m}{d}$. Consequently,

$$p_d = q *_d V_k = q,$$

which gives (1). □

The following result will imply both results mentioned in the introduction fairly easily.

Theorem 3.4 *Let $d \geq 3$ and let $p \in \mathbb{C}[z_1, \ldots, z_d]$ be (m, n)-band limited with $n \geq 1$. Then*

$$\|p\|_{\mathrm{SA}} \lesssim_d \left(\log\left(\frac{n+1}{m+1}\right) + 1\right)(\log(n+1))^{d-3} \|p\|_\infty.$$

Proof By Lemma 3.3, we may split $p = \sum_{j=1}^d p_j$, where each p_j is $(\lfloor \frac{m}{2d} \rfloor, n)$-band-limited with respect to z_j and $\|p_j\|_\infty \lesssim_d \|p\|_\infty$. Write

$$p_j(z) = \sum_{k=\lfloor \frac{m}{2d} \rfloor}^{n} q_{kj}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d) z_j^k,$$

where each q_{kj} is a polynomial of degree at most n. Let T be a d-tuple of commuting contractions. By Proposition 3.1, we have

$$\|p_j(T_1, \ldots, T_{j-1}, z, T_{j+1}, \ldots, T_d)\| \lesssim_d (\log(n+1))^{d-3} \|p_j\|_\infty$$

for all $z \in \mathbb{D}$, so by Theorem 2.8 (or simply Remark 2.7 (2)), we have

$$\begin{aligned}
\|p_j(T)\| &\lesssim_d K\left(\lfloor \frac{m}{2d} \rfloor, n\right)(\log(n+1))^{d-3} \|p_j\|_\infty \\
&\lesssim_d \left(\log\left(\frac{n+1}{\lfloor \frac{m}{2d} \rfloor + 1}\right) + 1\right)(\log(n+1))^{d-3} \|p_j\|_\infty \\
&\lesssim_d \left(\log\left(\frac{n+1}{m+1}\right) + 1\right)(\log(n+1))^{d-3} \|p_j\|_\infty.
\end{aligned}$$

Recalling that $\|p_j\|_\infty \lesssim_d \|p\|_\infty$, and that $p = \sum_{j=1}^d p_j$, we find that

$$\|p(T)\| \lesssim_d \left(\log\left(\frac{n+1}{m+1}\right) + 1\right)(\log(n+1))^{d-3}\|p\|_\infty,$$

as desired. □

The desired upper bound for $C(d, n)$ follows immediately by taking $m = n$ above.

Corollary 3.5 *Let* $d \geq 3$. *Then for all* $n \geq 1$,

$$C(d, n) \lesssim_d (\log(n+1))^{d-3}.$$

In particular, $\sup_n C(3, n) < \infty$.

Remark 3.6 (a) The proof of Corollary 3.5 does yield a crude explicit estimate for $C(3, n)$. Let $p \in \mathbb{C}[z_1, z_2, z_3]$ be a homogeneous polynomial of degree n with $\|p\|_\infty \leq 1$. The proof of Lemma 3.3 shows that the splitting $p = p_1 + p_2 + p_3$ obeys the bounds $\|p_1\|_\infty \leq 4$, $\|p_2\|_\infty \leq (4+1)4 = 20$ and $\|p_3\|_\infty \leq 1 + 4 + 20 = 25$. Moreover, by Remark 2.7 (3), we have $K(\lfloor \frac{n}{6} \rfloor, n) \leq \sqrt{6}$. Thus, the proof of Theorem 3.4 yields $C(3, n) \leq \sqrt{6}(4 + 20 + 25) = 49\sqrt{6} \leq 121$. Undoubtedly, this estimate can be significantly improved, but no attempt was made to do so.

(b) Once again, one can define a completely bounded version $C_{\text{cb}}(d, n)$ of the constant $C(d, n)$ by considering matrices of homogeneous polynomials in place of ordinary homogeneous polynomials. The arguments above also yield, for each $d \geq 3$ and $n \geq 1$, the estimate

$$C_{\text{cb}}(d, n) \lesssim_d (\log(n+1))^{d-3}.$$

Indeed, by Remark 2.6 (b), the upper bound in Theorem 2.8 also holds in the completely bounded setting, and Proposition 3.1 and Lemma 3.3 extend to matrices of polynomials with essentially the same proofs. Hence the argument in Theorem 3.4 extends as well.

The estimate in Theorem 3.4 can be converted into a Besov norm upper bound for the Schur–Agler norm. If $d \geq 3$ and $f : \mathbb{D}^d \to \mathbb{C}$ is holomorphic, let $(Rf)(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}$ be the radial derivative. Let us define

$$\|f\|_d = |f(0)| + \int_0^1 \|(Rf)_r\|_\infty \left(\log\left(\frac{1}{1-r}\right)\right)^{d-3} dr.$$

If $f : \mathbb{D}^d \to \mathbb{C}$ is holomorphic, we define the Schur–Agler norm by $\|f\|_{\text{SA}} = \sup\{\|f(T)\|\}$, where the supremum is taken over all commuting d-tuples of strict contractions, and $f(T)$ is (for instance) defined with the help of power series. We say that f belongs to the Schur–Agler algebra if $\|f\|_{\text{SA}} < \infty$.

Corollary 3.7 *Let $d \geq 3$. If $f : \mathbb{D}^d \to \mathbb{C}$ is holomorphic and $\|f\|_d < \infty$, then f belongs to the Schur-Agler algebra and*

$$\|f\|_{\mathrm{SA}} \lesssim_d \|f\|_d.$$

Proof We use the decomposition $f = \sum_{n=0}^{\infty} f * W_n$, which converges uniformly on compact subsets of $\mathbb{D}^d$; see the appendix. Let T be a commuting tuple of strict contractions. Note that $f * W_n$ is $(2^{n-1}, 2^{n+1})$-band-limited for $n \geq 1$, so by Theorem 3.4, we find that

$$\|f(T)\| \leq \sum_{n=0}^{\infty} \|(f * W_n)(T)\| \lesssim_d \sum_{n=0}^{\infty} (n+1)^{d-3} \|(f * W_n)\|_{\infty}.$$

By Corollary A.5, the right-hand side is comparable to $\|f\|_d$, which gives the result. □

The bounds in von Neumann's inequality with operator coefficients used in the proof of Theorem 3.4 were essentially sharp; see Theorem 2.8. One might therefore try to establish sharpness of Theorem 3.4 in a similar way. Recall that the lower bound in von Neumann's inequality with operator coefficients was achieved by Foguel–Hankel operators of the form

$$\begin{bmatrix} M_\zeta & 0 \\ 0 & M_{\overline{\zeta}}^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix},$$

where M_ζ is the unilateral shift on H^2 and $H : H^2 \to \overline{H^2}$ is a Hankel operator, i.e. $H M_\zeta = M_{\overline{\zeta}}^* H$. It therefore seems natural to try to construct counterexamples for von Neumann's inequality in d-variables using operators of this type. We will show that some natural operator tuples built in this way in fact satisfy von Neumann's inequality with constant 1. We require the following standard lemma.

Lemma 3.8 *Let $V \in B(\mathcal{H})$ be an isometry, let $W \in B(\mathcal{K})$ be a co-isometry, let $H \in B(\mathcal{H}, \mathcal{K})$ and let $r \in [0, \infty)$. Then the operator*

$$\begin{bmatrix} rV & 0 \\ H & rW \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$$

is a contraction if and only if $r^2 + \|H\| \leq 1$.

Proof Let T denote the operator in the statement. Since V is an isometry, it is clear that $r \leq 1$ is necessary. If $r = 1$, then $\|T\| \leq 1$ if and only if $H = 0$. Thus, we may assume that $r \in (0, 1)$. We let I denote the identity operator on $\mathcal{H} \oplus \mathcal{K}$. Observe that T is a contraction if and only if $I - T^*T \geq 0$. Now,

$$I - T^*T = \begin{bmatrix} 1 - r^2 - H^*H & -rH^*W \\ -rW^*H & 1 - r^2 W^*W \end{bmatrix}.$$

Taking Schur complements, we find that this operator is positive if and only if

$$1 - r^2 - H^*H - r^2 H^* W(1 - r^2 W^* W)^{-1} W^* H \geq 0.$$

Since $WW^* = 1$, the left-hand side equals

$$1 - r^2 - (1 + r^2(1 - r^2)^{-1})H^*H = (1 - r^2)^{-1}((1 - r^2)^2 - H^*H).$$

Thus, $\|T\| \leq 1$ if and only if $\|H\| \leq 1 - r^2$. □

We can now show the announced result about tuples of Foguel–Hankel type. Recall that the Hankel operator $H_q : H^2 \to \overline{H^2}$ is defined by $\langle H_q f, g \rangle = \langle fg, q \rangle$ for polynomials p, q. Equivalently, $H_q f = P_{\overline{H^2}}(\overline{q} f)$, where the product is taken in L^2. We let $\mathrm{Han} \subset H^2$ denote the space of symbols of bounded Hankel operators.

Proposition 3.9 *Let* $q_1, \ldots, q_d \in \mathrm{Han}$ *and let* $r_1, \ldots, r_d \in [0, 1]$. *For* $j = 1, \ldots, d$, *let*

$$T_j = \begin{bmatrix} r_j M_\zeta & 0 \\ H_{q_j} & r_j M_{\overline{\zeta}}^* \end{bmatrix},$$

and assume that each T_j *is a contraction. Then the* T_j *commute and for every* $p \in \mathbb{C}[z_1, \ldots, z_d]$, *we have*

$$\|p(T_1, \ldots, T_d)\| \leq \|p\|_\infty.$$

Proof The T_j commute because of the relation $H_q M_\zeta = M_{\overline{\zeta}}^* H_q$ for every $q \in \mathrm{Han}$. Let $U : L^2 \to L^2$, $(Uf)(\zeta) = \zeta f(\zeta)$ denote the bilateral shift. Lemma 3.8 shows that $\|H_{q_j}\| \leq 1 - r_j^2$ for each j. By Nehari's theorem ([26], see also [31, Theorem 1.1]), there exist $h_j \in L^\infty(\mathbb{T})$ such that $\|h_j\|_\infty \leq 1 - r_j^2$ and such that $H_{q_j} f = P_{\overline{H^2}}(h_j f)$ for all $f \in H^2$. Let $U_j : L^2 \to L^2$, $U_j(g) = h_j g$. Then

$$N_j = \begin{bmatrix} r_j U & 0 \\ U_j & r_j U \end{bmatrix} \in B(L^2 \oplus L^2)$$

are commuting contractions by Lemma 3.8. They dilate $T_1, \ldots, T_d$ in the sense that

$$p(T_1, \ldots, T_d) = P_{H^2 \oplus \overline{H^2}} p(N_1, \ldots, N_d)\big|_{H^2 \oplus \overline{H^2}}$$

for every polynomial $p \in \mathbb{C}[z_1, \ldots, z_d]$. This can for instance be seen by noting that the equality holds for $p = z_j$ and that $H^2 \oplus \overline{H^2} = (H^2 \oplus L^2) \ominus (0 \oplus \overline{H^2}^\perp)$ is semi-invariant under $N_1, \ldots, N_j$. The entries of N_j are multiplication operators on L^2, so

$$\begin{aligned} \|p(T_1, \ldots, T_d)\| &\leq \|p(N_1, \ldots, N_d)\| \\ &\leq \sup_{\zeta \in \mathbb{T}} \left\| p\left(\begin{bmatrix} r_1\zeta & 0 \\ h_1(\zeta) & r_1\zeta \end{bmatrix}, \ldots, \begin{bmatrix} r_d\zeta & 0 \\ h_d(\zeta) & r_d\zeta \end{bmatrix} \right) \right\|. \end{aligned}$$

Since von Neumann's inequality holds for commuting 2×2 contractions (see [8] or [16]), this last quantity is dominated by $\|p\|_\infty$. □

More generally, one might consider Hankel operators on the polydisc. The following result shows in particular that also in this case, von Neumann's inequality holds, at least up to a constant.

Proposition 3.10 *Let $V_1, \dots, V_d \in B(\mathcal{H})$ be commuting isometries and let $W_1, \dots, W_d \in B(\mathcal{K})$ be commuting co-isometries. Let $H_1, \dots, H_d \in B(\mathcal{H}, \mathcal{K})$ be operators satisfying $H_i V_j = W_j H_i$ for $i, j = 1, \dots, d$. Let $r_1, \dots, r_d \in [0, 1]$ and assume that the commuting operators*

$$T_j = \begin{bmatrix} r_j V_j & 0 \\ H_j & r_j W_j \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$$

are contractions. Then

$$\|p(T_1, \dots, T_d)\| \le (d+1)\|p\|_\infty$$

for all $p \in \mathbb{C}[z_1, \dots, z_d]$.

Proof Let $p \in \mathbb{C}[z_1, \dots, z_d]$. We claim that

$$p(T_1, \dots, T_d) = \begin{bmatrix} p(r_1 V_1, \dots, r_d V_d) & 0 \\ \sum_{j=1}^d H_j \frac{\partial p}{\partial z_j}(r_1 V_1, \dots, r_d V_d) & p(r_1 W_1, \dots, r_d W_d) \end{bmatrix}. \quad (9)$$

Indeed, a simple induction argument using the intertwining relations $H_i V_j = W_j H_i$ shows that this formula holds for all monomials, hence it holds for all polynomials by linearity. It is a result of Itô [18], see also [28, Theorem 5.1], that commuting isometries extend to commuting unitaries, hence $\|p(r_1 V_1, \dots, r_d V_d)\| \le \|p\|_\infty$ and similarly $\|p(r_1 W_1, \dots, r_d W_d)\| \le \|p\|_\infty$. Applying the same result to $\frac{\partial p}{\partial z_j}$ and then using the classical Schwarz–Pick lemma, we find that

$$\begin{aligned} \left\| \frac{\partial p}{\partial z_j}(r_1 V_1, \dots, r_d V_d) \right\| &\le \sup \left\{ \left| \frac{\partial p}{\partial z_j}(\zeta_1, \dots, \zeta_d) \right| : |\zeta_j| \le r_j \right\} \\ &\le \frac{1}{1 - r_j^2} \|p\|_\infty, \end{aligned}$$

provided that $r_j < 1$. Lemma 3.8 shows that $\|H_j\| \le 1 - r_j^2$ for each j; whence

$$\left\| \sum_{j=1}^d H_j \frac{\partial p}{\partial z_j}(r_1 V_1, \dots, r_d V_d) \right\| \le d\|p\|_\infty.$$

Since the norms of the diagonal entries of (9) are bounded by $\|p\|_\infty$, the desired estimate follows from the triangle inequality. □

Appendix: Besov spaces

In this appendix, we collect a few results about Besov spaces of analytic functions. Results of this type are in principle well known, see for instance [27, Section 3.1], [30, Section 2] for analytic Besov spaces and [29, 39] for more general Besov spaces. However, we require vector-valued and weighted versions of the standard results, for which we do not have a reference. Thus, we provide the proofs.

We will make use of several integral kernels on $\mathbb{T}$. For integers $n \geq 1$, the Fejér kernel F_n is the real-valued trigonometric polynomial whose Fourier coefficients are the triangular-shaped function supported in $(-n, n)$ that is 1 at 0 and affine on $[-n, 0]$ and on $[0, n]$. Explicitly

$$F_n(z) = \sum_{|j| \leq n} \Big(1 - \frac{|j|}{n}\Big) z^j.$$

Next, for integers $0 \leq k < l \leq m < n$, the de la Vallée-Poussin-type kernel $V_{k,l,m,n}$ is defined to be the real-valued trigonometric polynomial whose non-negative Fourier coefficients are the trapezoid-shaped function supported in (k, n) that is identically 1 on $[l, m]$ and affine on $[k, l]$ and $[m, n]$. Explicitly,

$$V_{k,l,m,n}(z) = \sum_{k \leq |j| < l} \frac{|j| - k}{l - k} z^j + \sum_{l \leq |j| \leq m} z^j + \sum_{m < |j| \leq n} \frac{n - |j|}{n - m} z^j.$$

Finally, we will also use closely related holomorphic kernels W_n, defined by demanding the Fourier coefficients of W_n are the triangular-shaped function supported in $(2^{n-1}, 2^{n+1})$ that takes the value 1 at 2^n and is affine on $[2^{n-1}, 2^n]$ and $[2^n, 2^{n+1}]$. Explicitly,

$$W_n(z) = \sum_{j=2^{n-1}}^{2^n} \Big(\frac{j}{2^{n-1}} - 1\Big) z^j + \sum_{j=2^n+1}^{2^{n+1}} \Big(2 - \frac{j}{2^n}\Big) z^j.$$

We also set $W_0(z) = 1 + z$.

We recall the following standard and well-known estimates for these kernels.

Lemma A.1 *For all integers $0 \leq k < l \leq m < n$, the following estimate holds:*

$$\|V_{k,l,m,n}\|_{L^1} \leq \frac{n + m}{n - m} + \frac{l + k}{l - k}.$$

Moreover, for all integers $n \geq 0$,

$$\|W_n\|_{L^1} \leq \frac{3}{2}.$$

Proof It is well known that the Fejér kernel satisfies the estimate $\|F_n\|_{L^1} \leq 1$. Let

$$G_{m,n} = \frac{n}{n - m} F_n - \frac{m}{n - m} F_m.$$

Then the Fourier coefficients of $G_{m,n}$ are the trapezoid-shaped function supported in $(-n, n)$ that is identically 1 on $[-m, m]$ and affine on $[-n, -m]$ and $[m, n]$. Thus,

$$V_{k,l,m,n} = G_{m,n} - G_{k,l},$$

so the estimate for $V_{k,l,m,n}$ follows from the triangle inequality.

Similarly, $W_n = z^{2^n} F_{2^{n-1}} + \frac{1}{2} z^{3 \cdot 2^{n-1}} F_{2^{n-1}}$ for $n \geq 1$, which yields the estimate for W_n; the case $n = 0$ is an elementary computation. □

Let X be a Banach space. Given a holomorphic function $f : \mathbb{D} \to X$, we write $\|f\|_\infty = \sup_{z \in \mathbb{D}} \|f(z)\|_X$, $f_r(z) = f(rz)$ and $f_r'(z) = f'(rz)$. We also write the Taylor series of f as

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n,$$

where $\widehat{f}(n) \in X$.

We require the following standard inequalities, which include versions of Bernstein's inequality; see for instance [19, Section I.8] and [33].

Lemma A.2 *Let X be a Banach space and let $f : \mathbb{D} \to X$ be holomorphic. Let $n \in \mathbb{N}$ and $0 < r < 1$.*

(a) If $\operatorname{supp} \widehat{f} \subset [0, n]$, then $\|f\|_\infty \leq r^{-n} \|f_r\|_\infty$ and $\|f'\|_\infty \lesssim n \|f\|_\infty$.
(b) If $\operatorname{supp} \widehat{f} \subset [n, \infty]$, then $\|f_r\|_\infty \leq r^n \|f\|_\infty$ and $n \|f\|_\infty \lesssim \|f'\|_\infty$.

Proof By composing f with continuous linear functionals on X, it suffices to consider the case $X = \mathbb{C}$.

(a) The first inequality follows from the maximum principle, applied to the function $w \mapsto w^n f(\frac{z}{w})$ on the circle of radius $\frac{1}{r}$. The second inequality (with constant 1) is Bernstein's inequality. With a worse constant, it also follows from the first inequality, applied to f', and the Schwarz–Pick lemma by choosing $r = 1 - \frac{1}{n}$.

(b) Write $f = z^n g$. Then $\|f_r\|_\infty \leq r^n \|g\|_\infty = r^n \|f\|_\infty$, which is the first inequality. The second inequality is a special case of the reverse Bernstein inequality. It also can be seen from the first inequality as follows. Let $n \geq 1$, so $f(0) = 0$. We have

$$f(z) = f(0) + \int_0^1 f'(rz) z \, dr,$$

so

$$\|f\|_\infty \leq \int_0^1 \|f_r'\|_\infty \, dr \leq \|f'\|_\infty \int_0^1 r^{n-1} \, dr = \frac{1}{n} \|f'\|_\infty.$$

□

We also require the following basic asymptotic relation.

Lemma A.3 *Let $a \geq 0$. Then for all $N \in \mathbb{N}$, $N \geq 1$,*

$$N \int_0^1 r^N \Big(\log \Big(\frac{1}{1-r} \Big) \Big)^a \, dr \simeq_a (\log(N+1))^a.$$

Proof Lower bound: By monotonicity, we have

$$
\begin{aligned}
N\int_0^1 r^N\Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a dr &\geq N\int_{1-\frac{1}{N}}^1 r^N\Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a dr\\
&\geq \Big(1-\frac{1}{N}\Big)^N(\log(N))^a\\
&\gtrsim (\log(N))^a.
\end{aligned}
$$

Upper bound: We break up the domain of integration into two intervals, namely $[0, 1-\frac{1}{N}]$ and $[1-\frac{1}{N}, 1]$. For the first integral, we estimate

$$
N\int_0^{1-\frac{1}{N}} r^N\Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a dr \leq N\int_0^{1-\frac{1}{N}} r^N(\log(N))^a\,dr \leq (\log(N))^a.
$$

For the second integral, we use the substitution $s = N(r-1)+1$ to compute

$$
\begin{aligned}
N\int_{1-\frac{1}{N}}^1 r^N\Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a dr &\leq N\int_{1-\frac{1}{N}}^1 \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a dr\\
&= \int_0^1 \Big(\log\Big(\frac{N}{1-s}\Big)\Big)^a ds\\
&\lesssim_a (\log(N))^a + \int_0^1 \Big(\log\Big(\frac{1}{1-s}\Big)\Big)^a ds.
\end{aligned}
$$

The second summand is a constant only depending on a, which gives the upper bound. □

If $f : \mathbb{D} \to X$ is holomorphic with Taylor series $f(z) = \sum_{k=0}^\infty \widehat{f}(k)z^n$, where $\widehat{f}(k) \in X$, define

$$
(f * W_n)(z) = \sum_{k=0}^\infty \widehat{f}(k)\widehat{W}_n(k)z^k,
$$

which is in fact a finite sum. Since $\sum_{n=0}^\infty \widehat{W}_n(k) = 1$ for all k, it is easy to check that

$$
\sum_{n=0}^\infty f * W_n = f,
$$

where the convergence is uniform on compact subsets of $\mathbb{D}$. We also write $(Rf)(z) = zf'(z)$.

We can now establish a dyadic description of a Besov-type norm.

Proposition A.4 *Let $a \geq 0$ and $f : \mathbb{D} \to X$ be holomorphic. Then*

$$
\|f(0)\| + \int_0^1 \|f_r'\|_\infty\Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a dr
$$

$$\simeq \|f(0)\| + \int_0^1 \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr$$
$$\simeq_a \sum_{n=0}^{\infty} (n+1)^a \|f * W_n\|_\infty.$$

Proof Note that $\|(Rf)_r\|_\infty = r\|f_r'\|_\infty$, and that $\|f_r'\|_\infty$ is increasing in r by the maximum principle. So the integrals over $[0, 1]$ are comparable to the respective integrals over $[\frac{1}{2}, 1]$. Hence the first two quantities are comparable.

Next, we obtain an upper bound for the second quantity in terms of the third. It is clear that $\|f(0)\| \le \|f * W_0\|_\infty$. By monotone convergence, we have

$$\int_0^1 \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \le \sum_{n=0}^{\infty} \int_0^1 \|(Rf)_r * W_n\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr.$$

Applying Lemma A.2 (b) and then (a), we find that for $n \ge 1$,

$$\|(Rf)_r * W_n\|_\infty \le r^{2^{n-1}} \|(Rf) * W_n\|_\infty \lesssim r^{2^{n-1}} 2^{n+1} \|f * W_n\|_\infty.$$

Integration and Lemma A.3 yield

$$\int_0^1 \|(Rf)_r * W_n\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr$$
$$\lesssim 2^{n+1} \int_0^1 r^{2^{n-1}} \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \|f * W_n\|_\infty$$
$$\lesssim_a (n+1)^a \|f * W_n\|_\infty,$$

which is also valid for $n = 0$. Thus,

$$\int_0^1 \|(Rf)_r\|_\infty \, dr \lesssim \sum_{n=0}^{\infty} (n+1)^a \|f * W_n\|_\infty.$$

For the lower bound, we use the fact that $\|g * W_n\|_\infty \le \frac{3}{2}\|g\|_\infty$ as $\|W_n\|_{L^1} \le \frac{3}{2}$ to obtain

$$\int_0^1 \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr$$
$$\ge \sum_{n=1}^{\infty} \int_{1-2^{-n}}^{1-2^{-n-1}} \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr$$
$$\gtrsim \sum_{n=1}^{\infty} \int_{1-2^{-n}}^{1-2^{-n-1}} \|(Rf)_r * W_n\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr.$$

Using Lemma A.2 (a) and then (b), we estimate

$$\|(Rf)_r * W_n\|_\infty \geq r^{2^{n+1}} \|(Rf) * W_n\|_\infty \gtrsim r^{2^{n+1}} 2^{n-1} \|f * W_n\|_\infty$$

for $n \geq 1$. It follows that

$$\begin{aligned}
&\int_{1-2^{-n}}^{1-2^{-n-1}} \|(Rf)_r * W_n\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \\
&\gtrsim \|f * W_n\|_\infty 2^{n-1} \int_{1-2^{-n}}^{1-2^{-n-1}} r^{2^{n+1}} \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \\
&\gtrsim_a \|f * W_n\|_\infty \, n^a.
\end{aligned}$$

Here, the last estimate is obtained by bounding below the integrand by a constant. Hence,

$$\int_0^1 \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \gtrsim_a \sum_{n=1}^\infty (n+1)^a \|f * W_n\|_\infty.$$

To deal with the summand for $n = 0$, we use a simple integration and the maximum principle to see that

$$\begin{aligned}
\|f\|_\infty \leq \|f(0)\| + \int_0^1 \|f_r'\|_\infty \, dr &\lesssim \|f(0)\| + \int_{1-\frac{1}{e}}^1 \|(Rf)_r\|_\infty \, dr \\
&\leq \|f(0)\| + \int_0^1 \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr.
\end{aligned}$$

Since $\|f * W_0\|_\infty \lesssim \|f\|_\infty$, the result follows. □

If $f : \mathbb{D}^d \to \mathbb{C}$ is holomorphic, we let $(Rf)(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}$ be the radial derivative. We also set

$$(f * W_n)(z) = \sum_\alpha \widehat{f}(\alpha) W_n(|\alpha|) z^\alpha.$$

Corollary A.5 *Let $a \geq 0$ and $f : \mathbb{D}^d \to \mathbb{C}$ be holomorphic. Then*

$$|f(0)| + \int_0^1 \|(Rf)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \simeq_a \sum_{n=0}^\infty (n+1)^a \|f * W_n\|_\infty.$$

Proof Define $g : \mathbb{D} \to C(\mathbb{T}^d)$ by

$$g(w)(z) = f(wz) \quad (z \in \mathbb{T}^d, w \in \mathbb{D}).$$

It is easy to check that g is weakly holomorphic, hence holomorphic. Thus, it follows from Proposition A.4 that

$$\|g(0)\| + \int_0^1 \|(Rg)_r\|_\infty \Big(\log\Big(\frac{1}{1-r}\Big)\Big)^a \, dr \simeq_a \sum_{n=0}^\infty (n+1)^a \|g * W_n\|_\infty .$$

It remains to compute both sides. Clearly, $\|g(0)\| = |f(0)|$. Moreover,

$$(Rg)(w)(z) = w \frac{d}{dw} f(wz) = w \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}(wz) = (Rf)(wz),$$

so

$$\|(Rg)_r\|_\infty = \sup_{w \in \mathbb{D}} \sup_{z \in \mathbb{T}^d} |(Rg)(rw)(z)| = \sup_{z \in \mathbb{D}^d} |(Rf)(rz)| = \|(Rf)_r\|_\infty .$$

On the other hand,

$$g(w)(z) = \sum_\alpha \widehat{f}(\alpha)(wz)^\alpha = \sum_{k=0}^\infty \sum_{|\alpha|=k} \widehat{f}(\alpha) z^\alpha w^k$$

so

$$(g * W_n)(w)(z) = \sum_{k=0}^\infty \sum_{|\alpha|=k} \widehat{f}(\alpha) \widehat{W}_n(k)(zw)^\alpha = (f * W_n)(zw)$$

and thus

$$\|g * W_n\|_\infty = \|f * W_n\|_\infty .$$

This completes the proof. □

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