Numerical Methods for Partial Differential Equations

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RESEARCH ARTICLE OPEN ACCESS

Homogeneous Multigrid for Hybrid Discretizations: Application to HHO Methods

Daniele A. Di Pietro[1] | Zhaonan Dong[2] | Guido Kanschat[3] | Pierre Matalon[4] | Andreas Rupp[5]

[1]IMAG, Univ Montpellier, CNRS, Montpellier, France | [2]Inria & CERMICS, ENPC, Institut Polytechnique de Paris, Paris, France | [3]Interdisciplinary Center for Scientific Computing (IWR), Heidelberg University, Heidelberg, Germany | [4]CMAP, CNRS, École Polytechnique, Institut Polytechnique de Paris, Palaiseau, France | [5]Department of Mathematics, Saarland University, Saarbrücken, Germany

Correspondence: Andreas Rupp (andreas.rupp@uni-saarland.de)

Received: 13 January 2025 | **Revised:** 22 May 2025 | **Accepted:** 7 July 2025

Funding: This work was supported by the European Research Council (Grant No. 101115663), the Deutsche Forschungsgemeinschaft (Grant No. EXC 2181/1-390900948), the Academy of Finland (Grant Nos. 350101, 354489, 359633, and 358944), and Business Finland (Grant No. 539/31/2023).

Keywords: geometric multigrid | homogeneous multigrid | hybrid high-order methods | skeleton methods

ABSTRACT

We prove the uniform convergence of the geometric multigrid V-cycle for hybrid high-order (HHO) and other discontinuous skeletal methods. Our results generalize previously established results for HDG methods, and our multigrid method uses standard smoothers and local solvers that are bounded, convergent, and consistent. We use a weak version of elliptic regularity in our proofs. Numerical experiments confirm our theoretical results.

MSC2020 Classification: 65F10, 65N30, 65N50

1 | Introduction

In the context of fast solvers for linear systems arising from hybrid discretizations of second-order elliptic equations, we propose in this work a generalized framework for the convergence analysis of geometric multigrid methods [1, 2].

Hybrid discretization methods have been part of the numerical analyst's toolbox to solve partial differential equations since the seventies, starting with hybridized versions of mixed methods such as the Raviart–Thomas (RT-H) [3] and the Brezzi–Douglas–Marini (BDM-H) [4] methods. They have gained growing interest in recent years with the outbreak of modern schemes such as the unifying framework of hybridizable discontinuous Galerkin (HDG) methods [5] and hybrid high-order (HHO) methods [6–8]. The list also includes, but is not limited to, compatible discrete operators (CDO) [9], mixed and hybrid finite volumes (MHFV) [10–12], Mimetic Finite Differences (MFD) [13], weak Galerkin [14], local discontinuous Galerkin-hybridizable (LDG-H) [15], and discontinuous Petrov–Galerkin (DPG) [16] methods. Hybrid discretization methods are characterized by the location of their degrees of freedom (DOFs), placed both within the mesh cells and on the faces. In this configuration, the discrete scheme is built so that the cell DOFs are only locally coupled, leaving the face DOFs in charge of the global coupling. Algebraically, this feature enables the local elimination of the cell unknowns from the arising linear system, resulting in a Schur complement of reduced size, where only the face unknowns remain. The mechanical engineering terminology refers to this elimination process as *static condensation*, and the resulting system is often called the (statically) *condensed* system. The terminology *trace* or *skeleton* system is also used. For an extended introduction to hybrid methods and hybridization, we refer to the preface of [6] and the first pages of [5].

This paper addresses the fast solution of the condensed systems arising from hybrid discretizations of second-order elliptic equations, focusing on multigrid methods. Various multigrid solvers and preconditioners have been designed over the past decade, focusing primarily on HDG [17–22] and HHO [23–26], but also covering specifically DPG [27, 28] and RT-H/BDM-H [29]. The above references focus on geometric multigrid methods, while a fully algebraic multigrid method has been considered in [30].

The main difficulty in designing a geometric multigrid algorithm for a trace system resides in the fact that the DOFs are supported by the mesh skeleton, which makes classical intergrid transfer operators designed for element-based DOFs unsuitable. While earlier approaches [17, 29, 31] recast trace functions into bulk functions to make use of a known efficient solver (typically, a standard piecewise linear continuous finite element multigrid solver), the more recent developments seem to converge toward the so-called notion of *homogeneous* multigrid, where the hybrid discretization is conserved at every level of the mesh hierarchy and "interskeleton" transfer operators are designed to communicate trace functions from one mesh skeleton to the other. To this end, multiple skeletal injection operators have already been proposed [19, 24].

In this paper, we build upon the work of [18, 19, 32, 33] on HDG methods to propose a generalized demonstration framework for the uniform convergence of *homogeneous* multigrid methods with V-cycle. The main motivation behind this generalization is to include the HHO methods in its application scope. Indeed, recent multigrid solvers for HHO methods [24–26] have experimentally shown their optimal behavior, but the supporting theory is still missing.

The main theoretical challenges of devising a multigrid theory for HHO methods over a multigrid theory for HDG methods lie in HHO's more abstract stabilization term, which requires a more abstract theory. That is, we need to address two main generalizations to cope with issues that did not come up in [19]. These are (i) the reduced smoothing property of HHO (assumption (LS1) in [19] does not hold and needs to be replaced by (HM1) with larger right-hand side) and (ii) the explicit expression of the HDG stabilizer cannot be used in the proofs resulting in more technical proofs while several intermediate results to prove multigrid convergence remain true (or very similar results hold). These issues will be addressed in the first theoretical part. In the second theory part of this paper, we then prove that the classical HHO method fulfills the assumptions of this new framework.

The present theory does not assume full elliptic regularity of the problem, thus allowing complex domains with re-entrant corners. However, although modern hybrid methods natively handle polyhedral elements, it is restricted to simplicial meshes since it relies on a compatibility condition with linear finite elements, cf. (HM4) in Section 3. The multigrid method is built in a standard fashion from an abstract injection operator: standard smoothers are used, and the restriction operator is chosen as the adjoint of the injection operator. A classical, symmetric V-cycle is used if the problem exhibits full elliptic regularity. If not, then a variable V-cycle is used, in which the number of (symmetric) smoothing steps increases as the level decreases in the hierarchy.

Despite our mesh restrictions, we expect this theory to lay the foundation for considering polyhedral meshes that can be decomposed into simplicial meshes for a vast group of hybrid methods. This goal has already been achieved for many other methods, such as mimetic finite differences [34], (non-hybridized) discontinuous Galerkin [35], which do not even require nested mesh sequences. Schwarz-based preconditioners have been derived under similar assumptions for the virtual finite elements in [36]. A related future research prospect lies in addressing multigrid methods on agglomerated meshes as it has been done for other finite element methods with the ability to cope with polyhedral meshes, such as discontinuous Galerkin [37] and virtual finite elements [38]. As this setting presents further challenges in our case, we leave it to future work. Indeed, since condensed systems arising from hybrid methods rely on face DOFs, it is required that faces be coarsened between levels [24, sec. 4.4.3], thus making suitable coarsening strategies more difficult to design.

The paper is organized as follows. In Section 2, we introduce notations and the model problem. Section 3 describes the abstract framework: (i) the hybrid method is described abstractly, in the form of approximation spaces, local linear operators, stabilization term and bilinear form; (ii) the injection operators used in the multigrid method and its analysis are introduced; (iii) the properties assumed from the hybrid method (HM1) to (HM8) and the injection operator (IA1) to (IA2) are listed. Section 4 describes the multigrid method and asserts the associated convergence results, whereas Section 5 carries out the convergence analysis. In Section 6, we verify our framework's assumptions for the standard HHO method, and analyze various injection operators in Section 7. Finally, Section 8 presents the numerical experiments supporting our theoretical results, realized with the HHO method on two- and three-dimensional test cases.

2 | Problem Formulation and Notation

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a polytopal bounded Lipschitz domain with boundary $\partial\Omega$. We approximate the unknown $u \in H_0^1(\Omega)$ satisfying

$$-\Delta u = f \qquad \text{in } \Omega \tag{2.1}$$

for some right-hand side f. We assume the problem to be regular in the sense that

$$\|u\|_{1+\alpha} \leq c\|f\|_{\alpha-1} \qquad \text{for some } \alpha \in (\frac{1}{2}, 1] \tag{2.2}$$

Here, $\|\cdot\|_\alpha$ for $\alpha \in \mathbb{R}_{\geq 0}$ denotes the induced norm of the Sobolev–Slobodeckij space $H^\alpha(\Omega)$. By $\|\cdot\|_{-\alpha}$ and $H^{-\alpha}(\Omega)$ we denote the dual norm and the dual space of $H_0^\alpha(\Omega)$ with respect to the extension of L^2 duality, respectively. We denote by $\|\cdot\|_X$ and $(\cdot,\cdot)_X$ the L^2-norm and L^2-scalar product with respect to the set $X \subset \Omega$, respectively. The same notation is also used for the inner product of $[L^2(X)]^d$ (the exact meaning can be inferred from the context) such that, for all $\boldsymbol{p}, \boldsymbol{q} \in [L^2(X)]^d$, $(p, q)_X := \int_X \boldsymbol{p} \cdot \boldsymbol{q}$, where $\cdot$ denotes the dot product. Without loss of generality, we assume that Ω has diameter 1.

Remark 2.1 (More general boundary conditions). The following approaches directly transfer to non-homogeneous Dirichlet boundary conditions since those only influence the right-hand sides of the numerical schemes. Adaptations are needed for other types of boundary conditions (e.g., Neumann or mixed).

3 | Discontinuous Skeletal Methods

We will render a general framework for all discontinuous skeletal methods covering HDG and HHO methods. To this end, we start with a family $(\mathcal{T}_\ell)_{\ell=0,\dots,L}$ of successively refined simplicial meshes and their corresponding face sets $(\mathcal{F}_\ell)_{\ell=0,\dots,L}$. For all level ℓ, we define $h_\ell = \max_{T\in\mathcal{T}_\ell} h_T$, where h_T denotes the diameter of T. The family $(\mathcal{T}_\ell)_{\ell=0,\dots,L}$ is regular (see [39, Def. 11.2] for a precise definition), which implies that mesh elements do not deteriorate, and all of its elements are geometrically conforming, which excludes hanging nodes. Additionally, we assume, for face F, either $F \subset \partial\Omega$ or $F \cap \partial\Omega$ comprises at most one point, and that refinement does not progress too fast, that is, there is $c_{\text{ref}} > 0$ such that

$$h_\ell \geq c_{\text{ref}} h_{\ell-1} \qquad \forall \ell \in \{1,\dots,L\} \tag{3.1}$$

Finally, we assume that our mesh is quasi-uniform implying that there is a constant c_{uni} such that $h_\ell \leq c_{\text{uni}} \min_{T\in\mathcal{T}_\ell} h_T$.

Next, we choose a finite-dimensional space $M_F \subset L^2(F)$ of functions living on face F for all $F \in \mathcal{F}_\ell$. In the numerical scheme, this space will be the test and trial space for the skeletal variable m_ℓ approximating the trace of the unknown u on the skeleton. Moreover, finite-dimensional approximation spaces $V_T \subset L^2(T)$ and $\boldsymbol{W}_T \subset L^2(T)^d$ are defined element-wise. The space V_T is the local test and trial space for the primary unknown u, while the space $\boldsymbol{W}_T$ is the local test and trial space for the dual unknown $\boldsymbol{q} = -\nabla u$. Global spaces are defined by concatenation via

$$M_\ell := \left\{ m \in L^2(\mathcal{F}_\ell) \,\middle|\, \begin{matrix} m_{|F} \in \mathcal{P}_p(F) & \forall F \in \mathcal{F}_\ell \\ m_{|F} = 0 & \forall F \subset \partial\Omega \end{matrix} \right\} \tag{3.2}$$

$$V_\ell := \left\{ v \in L^2(\Omega) \,\middle|\, v_{|T} \in V_T \quad \forall T \in \mathcal{T}_\ell \right\} \tag{3.3}$$

$$\boldsymbol{W}_\ell := \left\{ \boldsymbol{q} \in L^2(\Omega;\mathbb{R}^d) \,\middle|\, \boldsymbol{q}_{|T} \in \boldsymbol{W}_T \quad \forall T \in \mathcal{T}_\ell \right\} \tag{3.4}$$

where $\mathcal{P}_p(F)$ comprises polynomials of degree at most p over F. We define element-wise, abstract linear operators

$$\begin{aligned} \mathcal{U}_T &: M_\ell|_{\partial T} \to V_T, & m_\ell|_{\partial T} &\mapsto u_T, \\ \mathcal{V}_T &: L^2(T) \to V_T, & m_\ell|_{\partial T} &\mapsto u_T, \\ \mathcal{Q}_T &: M_\ell|_{\partial T} \to \boldsymbol{W}_T, & f &\mapsto \boldsymbol{q}_T \end{aligned}$$

The choice of these operators, called local solvers, influences the numerical schemes. Examples for $\mathcal{U}$ and $\mathcal{Q}$, which are relevant to the multigrid method, are provided in Remark 3.1. Global linear operators are constructed by concatenation of their element-local analogous:

$$\mathcal{U}_\ell : M_\ell \to V_\ell, \quad \mathcal{V}_\ell : L^2(\Omega) \to V_\ell, \quad \mathcal{Q}_\ell : M_\ell \to \boldsymbol{W}_\ell$$

One can show that $m_\ell \in M_\ell$ approximates the trace of u on the skeleton and that $u \approx \mathcal{U}_\ell m_\ell + \mathcal{V}_\ell f$ in the bulk of Ω if m_ℓ satisfies

$$a_\ell(m_\ell, \mu) = \int_\Omega f\, \mathcal{U}_\ell \mu \qquad \text{for all } \mu \in M_\ell \tag{3.5}$$

where the elliptic and continuous bilinear form a_ℓ has the form

$$a_\ell(m,\mu) = \int_\Omega \mathcal{Q}_\ell m \cdot \mathcal{Q}_\ell \mu + s_\ell(m,\mu) \tag{3.6}$$

The symmetric positive semidefinite s_ℓ is usually referred to as (condensed) *penalty* or *stabilizing term*.

The choices of the spaces $M_F, V_T, \boldsymbol{W}_T$, the local solvers $\mathcal{U}_\ell, \mathcal{V}_\ell, \mathcal{Q}_\ell$ and the stabilizing term s_ℓ completely define a discontinuous skeletal method.

Remark 3.1 (Possible choices of hybrid methods).

- For the **LDG-H** method, we set $M_F = \mathcal{P}_p(F)$, $V_T = \mathcal{P}_p(T)$, $\boldsymbol{W}_T = \mathcal{P}_p^d(T)$, and $\tau_\ell > 0$. The operators $\mathcal{U}_T$ and $\mathcal{Q}_T$ map $m_{\partial T}$ to the element-wise solutions $u_T \in V_T$ and $\boldsymbol{q}_T \in \boldsymbol{W}_T$ of

$$\int_T \boldsymbol{q}_T \cdot \boldsymbol{p}_T - \int_T u_T \nabla \cdot \boldsymbol{p}_T = -\int_{\partial T} m_{\partial T} \boldsymbol{p}_T \cdot \boldsymbol{\nu} \tag{3.7a}$$

$$\int_{\partial T} (\boldsymbol{q}_T \cdot \boldsymbol{\nu} + \tau_\ell u_T) v_T - \int_T \boldsymbol{q}_T \cdot \nabla v_T = \tau_\ell \int_{\partial T} m_{\partial T} v_T \tag{3.7b}$$

with test functions $v_T \in V_T$ and $\boldsymbol{p}_T \in \boldsymbol{W}_T$, and $\boldsymbol{\nu}$ the outward normal vector to ∂T. In this sense, $\mathcal{U}_T : m_{\partial T} \mapsto u_T$, $\mathcal{Q}_T : m_{\partial T} \mapsto \boldsymbol{q}_T$ for all $T \in \mathcal{T}_\ell$.
Finally, the bilinear form a_ℓ is defined as

$$a_\ell(m_\ell,\mu) = \int_\Omega \mathcal{Q}_\ell m_\ell \cdot \mathcal{Q}_\ell \mu + \underbrace{\sum_{T\in\mathcal{T}_\ell} \tau_\ell \int_{\partial T} (\mathcal{U}_\ell m_\ell - m_\ell)(\mathcal{U}_\ell \mu - \mu)}_{=s_\ell(m_\ell,\mu)}$$

for some parameter $\tau_\ell > 0$.

- For the **RT-H** and **BDM-H** methods, one uses the framework of the LDG-H method replacing the local bulk spaces by the RT-H and BDM-H spaces with $\tau_\ell = 0$.

- For the **HHO** method, we set $M_F = \mathcal{P}_p(F)$, $V_T = \mathcal{P}_p(T)$, and $\boldsymbol{W}_T = \nabla\mathcal{P}_{p+1}(T)$. For $(u_T, m_{\partial T}) \in V_T \times M_{\partial T}$, we define $\boldsymbol{q}_T(u_T, m_{\partial T})$ as the element-wise solution of

$$\int_T \boldsymbol{q}_T(u_T, m_{\partial T}) \cdot \boldsymbol{p}_T - \int_T u_T \nabla \cdot \boldsymbol{p}_T = -\int_{\partial T} m_{\partial T} \boldsymbol{p}_T \cdot \boldsymbol{\nu} \tag{3.8}$$

for all $\boldsymbol{p}_T \in \boldsymbol{W}_T$. Note that this relation is the same as (3.7a) except for the different choice of $\boldsymbol{W}_T$. In HHO terminology, $\boldsymbol{q}_T(u_T, m_{\partial T})$ corresponds to $-\nabla\theta_T^{p+1}(u_T, m_{\partial T})$, where θ_T^{p+1} denotes the so-called local higher order reconstruction operator, defined by (3.8) and the closure condition

$$\int_T \theta_T^{p+1}(u_T, m_{\partial T}) = \int_{\partial T} u_T \tag{3.9}$$

In a second step, given a local bilinear stabilizer $\underline{s}_T((u_T, m_{\partial T}), (v_T, \mu))$, we introduce the local bilinear form

$$\underline{a}_T((u_T, m_{\partial T}), (v_T, \mu)) = \int_T \boldsymbol{q}_T(u_T, m_{\partial T}) \cdot \boldsymbol{q}_T(v_T, \mu) + \underline{s}_T((u_T, m_{\partial T}), (v_T, \mu)) \tag{3.10}$$

Consider the following local problems:

(i) For $m_{\partial T} \in M_{\partial T}$, find $u_T^1 \in V_T$ such that

$$\underline{a}_T((u_T^1, 0), (v_T, 0)) = -\underline{a}_T((0, m_{\partial T}), (v_T, 0)) \qquad \forall v_T \in V_T \tag{3.11}$$

(ii) For $f \in L^2(\Omega)$, find $u_T^2 \in V_T$ such that

$$\underline{a}_T((u_T^2, 0), (v_T, 0)) = \int_T f v_T \qquad \forall v_T \in V_T \tag{3.12}$$

We define $\mathcal{U}_T : m_{\partial T} \mapsto u_T^1$ solution of (3.11), $\mathcal{V}_T : f \mapsto u_T^2$ solution of (3.12), and we set element-by-element

$$\mathcal{Q}_T : m_{\partial T} \mapsto \boldsymbol{q}_T(\mathcal{U}_T m_{\partial T}, m_{\partial T}) \tag{3.13}$$

Finally, the global bilinear form is defined as

$$\begin{aligned} a_\ell(m_\ell, \mu) &=: \sum_{T\in\mathcal{T}_\ell} \underline{a}_T((\mathcal{U}_T m_{\partial T}, m_{\partial T}), (\mathcal{U}_T \mu, \mu)) \\ &= \int_\Omega \mathcal{Q}_\ell m_\ell \cdot \mathcal{Q}_\ell \mu + \underbrace{\sum_{T\in\mathcal{T}_\ell} \underline{s}_T((\mathcal{U}_\ell m_\ell, m_\ell), (\mathcal{U}_\ell \mu, \mu))}_{=s_\ell(m_\ell,\mu)} \end{aligned} \tag{3.14}$$

3.1 | Operators for the Multigrid Method and Analysis

For functions $\rho, \mu \in L^2(\mathcal{F}_\ell)$, we set

$$\langle \rho, \mu \rangle_\ell = \sum_{T\in\mathcal{T}} \frac{|T|}{|\partial T|} \int_{\partial T} \rho\mu$$

Importantly, $\langle \cdot, \cdot \rangle_\ell$ defines a scalar product on M_ℓ, and the norm of this scalar product

$$\|\cdot\|_\ell := \langle \cdot, \cdot \rangle_\ell^{\frac{1}{2}}$$

scales like the L^2-norm in the bulk of the domain. Moreover, the scalar product can readily be applied to a combination of bulk and skeleton functions: For example, $\langle \mu, v \rangle_\ell$ can readily be evaluated for $\mu \in M_\ell$ and $v \in V_\ell$ with the understanding that the trace of v on ∂T is used inside the integral over ∂T. We relate the bilinear forms a_ℓ and $\langle \cdot, \cdot \rangle_\ell$ using the operator A_ℓ, which is defined via

$$\langle A_\ell \rho, \mu \rangle_\ell := a_\ell(\rho, \mu) \qquad \forall \rho, \mu \in M_\ell \tag{3.15}$$

The injection operator $I_\ell : M_{\ell-1} \to M_\ell$ remains abstract at this level. Its properties securing our analytical findings are listed in Section 3.3. Possible realizations of injection operators can be found in [19]. The multigrid operator $B_\ell : M_\ell \to M_\ell$ for preconditioning A_ℓ will be defined in Section 4.2.

The operators $P^{\langle\cdot\rangle}{}_{\ell-1}$ and $P^a{}_{\ell-1}$, which are defined via

$$P^{\langle\cdot\rangle}{}_{\ell-1} : M_\ell \to M_{\ell-1}, \quad \langle P^{\langle\cdot\rangle}{}_{\ell-1}\rho, \mu \rangle_{\ell-1} = \langle \rho, I_\ell \mu \rangle_\ell \quad \forall \mu \in M_{\ell-1} \tag{3.16}$$

$$P^a{}_{\ell-1} : M_\ell \to M_{\ell-1}, \quad a_{\ell-1}(P^a{}_{\ell-1}\rho, \mu) = a_\ell(\rho, I_\ell \mu) \quad \forall \mu \in M_{\ell-1} \tag{3.17}$$

replace the L^2-orthogonal (for short, L^2) and the Ritz projections of conforming methods, respectively. While the former (or a discrete variation of it) is relevant for implementing multigrid methods, the latter is key to the analysis. We also introduce the L^2-projections

$$\begin{aligned} \Pi^\partial{}_\ell &: H_0^1(\Omega) \to M_\ell, & \langle \Pi^\partial{}_\ell v, \mu \rangle_\ell &= \langle v, \mu \rangle_\ell & \forall \mu \in M_\ell, \\ \Pi_\ell^{\mathrm{d}} &: H^1(\Omega) \to V_\ell, & (\Pi_\ell^{\mathrm{d}} v, w)_\Omega &= (v, w)_\Omega & \forall w \in V_\ell \end{aligned}$$

3.2 | Assumptions on Discontinuous Skeletal Methods

Here and in the following, $\lesssim$ means smaller than or equal to, up to a constant independent of the mesh size h_ℓ and the multigrid level ℓ. We also write $A \simeq B$ as a shortcut for "$A \lesssim B$ and $B \lesssim A$". We assume that the numerical hybrid method, characterized in Section 3, satisfies the following conditions for any $\mu \in M_\ell$:

- The trace of the bulk unknown approximates the skeletal unknown:

$$\|\mathcal{U}_\ell \mu - \mu\|_\ell \lesssim h_\ell \|\mu\|_{a_\ell} \tag{HM1}$$

  where $\|\cdot\|_{a_\ell}$ denotes the norm induced by a_ℓ on M_ℓ, that is,

$$\|\cdot\|_{a_\ell} := a_\ell(\cdot, \cdot)^{\frac{1}{2}} \tag{3.18}$$

- The operators $\mathcal{Q}_\ell \mu$ and $\mathcal{U}_\ell \mu$ are continuous:

$$\|\mathcal{Q}_\ell \mu\|_\Omega \lesssim h_\ell^{-1} \|\mu\|_\ell \quad \text{and} \quad \|\mathcal{U}_\ell \mu\|_\Omega \lesssim \|\mu\|_\ell \tag{HM2}$$

- The quantity $\mathcal{Q}_\ell \mu$ approximates $-\nabla_\ell \mathcal{U}_\ell \mu$, where ∇_ℓ denotes the broken gradient:

$$\|\mathcal{Q}_\ell \mu + \nabla_\ell \mathcal{U}_\ell \mu\|_\Omega \lesssim h_\ell^{-1} \|\mathcal{U}_\ell \mu - \mu\|_\ell \tag{HM3}$$

- Consistency with the standard linear finite element method: If $w \in \overline{V}^{\mathrm{c}}{}_\ell$, we have

$$\mathcal{U}_\ell \gamma_\ell w = w \qquad \text{and} \qquad \mathcal{Q}_\ell \gamma_\ell w = -\nabla w \tag{HM4}$$

  where γ_ℓ is the trace operator to the skeleton $\mathcal{F}_\ell$ and

$$\overline{V}^{\mathrm{c}}{}_\ell := \left\{ v \in H_0^1(\Omega) \;\middle|\; v_{|T} \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_\ell \right\}$$

- Convergence of the skeletal unknown to the traces of the analytical solution. That is, if m_ℓ is the skeletal function of the hybrid approximation to $u \in H^{1+\alpha}(\Omega)$, we have

$$\|m_\ell - \Pi^\partial{}_\ell u\|_{a_\ell} \lesssim h_\ell^\alpha \|u\|_{\alpha+1} \tag{HM5}$$

- The usual bounds on the eigenvalues of the condensed discretization matrix hold:

$$\|\mu\|_\ell^2 \lesssim a_\ell(\mu, \mu) \lesssim h_\ell^{-2} \|\mu\|_\ell^2 \tag{HM6}$$

- If $\rho = \gamma_\ell w$ for $w \in \overline{V}^{\mathrm{c}}{}_\ell$, the stabilization satisfies

$$s_\ell(\rho, \mu) = 0 \tag{HM7}$$

- The global bilinear form is bounded by an HDG-type norm:

$$a_\ell(\mu,\mu) \lesssim \|\mathcal{Q}\mu\|_\Omega^2 + \sum_{T\in\mathcal{T}_\ell} \frac{1}{h_T}\|\mathcal{U}\mu - \mu\|_{\partial T}^2 \tag{HM8}$$

Remark 3.2 (HDG methods). For the LDG-H (with $\tau_\ell h_\ell \lesssim 1$), the RT-H, and the BDM-H methods, (HM7) follows directly from (HM4), and (HM8) follows directly from the definition of the bilinear form a_ℓ. Thus, one can use almost identical (but slightly simpler) assumptions to prove multigrid convergence for the HDG, BDM-H, and RT-H methods. (HM2) to (HM6) have been used to prove the convergence of multigrid methods for HDG, while (HM1), (HM7), and (HM8) are novel assumptions that allow to generalize the convergence theory to HHO.

Notably, assumption (HM6) typically requires mesh regularity, which might limit the method's flexibility. Moreover, we emphasize that (HM1) has been weakened compared with our previous work, complicating the analysis below. Moreover, the more general stabilization (when compared with HDG methods) needs to be controlled in our multigrid analysis, which is realized in (HM7) and (HM8).

3.3 | Assumptions on Injection Operators

Our analytical findings will rely on the following conditions to hold on all mesh levels ℓ:

1. Boundedness:

$$\|I_\ell \rho\|_\ell \lesssim \|\rho\|_{\ell-1} \qquad \forall \rho \in M_{\ell-1} \tag{IA1}$$

2. Conformity with linear finite elements:

$$I_\ell \gamma_{\ell-1} w = \gamma_\ell w \qquad \forall w \in \overline{V}^{\mathrm{c}}_{\ell-1} \tag{IA2}$$

3. Locality: there is a parameter $\overline{\sigma} > 0$ such that for any $T \in \mathcal{T}_{\ell-1}$

$$(I_\ell \mu)|_{\partial T} = (I_\ell \rho)|_{\partial T} \tag{IA3}$$

   for all $\mu, \rho \in M_{\ell-1}$ with $\mu|_{B(x_T,\overline{\sigma}h_{\ell-1})} = \rho|_{B(x_T,\overline{\sigma}h_{\ell-1})}$ and $B(x_T, \overline{\sigma}h_{\ell-1})$ denoting the ball with radius $\overline{\sigma}h_{\ell-1}$ around the barycenter of T called x_T.

(The second condition can be interpreted as conformity with overall continuous, piecewise linear finite element spaces, which are nested.) This way, we do not restrict ourselves to one specific injection operator. All injection operators in [19] satisfy the desired properties. An immediate consequence of (HM4) and (IA2) is

Lemma 3.3 (Quasi-orthogonality). *Let (HM4), (HM7), and (IA2) hold. Then we have, for all $\mu \in M_\ell$ and all $w \in \overline{V}^{\mathrm{c}}_{\ell-1}$,*

$$(\mathcal{Q}_\ell \mu - \mathcal{Q}_{\ell-1} P^a{}_{\ell-1}\mu, \nabla w)_\Omega = 0$$

Proof. Let $w \in \overline{V}^{\mathrm{c}}_{\ell-1}$ and set $\rho := \gamma_{\ell-1} w$. We have

$$\mathcal{Q}_{\ell-1}\rho = \mathcal{Q}_{\ell-1}\gamma_{\ell-1} w \overset{\text{(HM4)}}{=} -\nabla w \overset{\text{(HM4)}}{=} \mathcal{Q}_\ell \gamma_\ell w \\ \overset{\text{(IA2)}}{=} \mathcal{Q}_\ell I_\ell \gamma_{\ell-1} w = \mathcal{Q}_\ell I_\ell \rho \tag{3.19}$$

where we have additionally used the embedding $\overline{V}^{\mathrm{c}}_{\ell-1} \subset \overline{V}^{\mathrm{c}}_\ell$ in the third equality. The definitions of a_ℓ, $a_{\ell-1}$, and (HM7) yield

$$a_{\ell-1}(P^a{}_{\ell-1}\mu, \rho) \\ = (\mathcal{Q}_{\ell-1} P^a{}_{\ell-1}\mu, \mathcal{Q}_{\ell-1}\rho)_\Omega \overset{(3.19)}{=} -(\mathcal{Q}_{\ell-1} P^a{}_{\ell-1}\mu, \nabla w)_\Omega, \\ a_\ell(\mu, I_\ell \rho) = (\mathcal{Q}_\ell \mu, \mathcal{Q}_\ell I_\ell \rho)_\Omega \overset{(3.19)}{=} -(\mathcal{Q}_\ell \mu, \nabla w)_\Omega$$

The two lines are equal by definition (3.17) of the projection operator $P^a{}_{\ell-1}$. Thus, taking the difference gives the result. □

4 | Multigrid Algorithm and Abstract Convergence Results

If (2.1) has full elliptic regularity, which means that (2.2) holds for $\alpha = 1$, we use a standard (symmetric) V-cycle multigrid method. Otherwise, we use a variable V-cycle multigrid method to solve the system of linear equations arising from (3.5).

We follow the lines of [19] and avoid a discussion about good smoothing operators. That is, we allow our smoother in smoothing step i

$$R^i_\ell : M_\ell \to M_\ell$$

to fit the criteria of [40], which allow pointwise Jacobi and Gauss–Seidel methods. Next, we present the multigrid method as in [41]. Afterwards, we present the abstract convergence results of [42].

4.1 | Multigrid Algorithm

We recursively define the multigrid operator of the refinement level ℓ

$$B_\ell : M_\ell \ni \mu \mapsto B_\ell \mu \in M_\ell$$

with $n_\ell \in \mathbb{N} \setminus \{0\}$ smoothing steps on level ℓ: Let $B_0 = A_0^{-1}$ be the exact inverse of A_ℓ on the coarsest level. For $\ell > 0$, set $x^0 = 0 \in M_\ell$.

1. Perform n_ℓ smoothing steps

$$x^i = x^{i-1} + R^i_\ell(\mu - A_\ell x^{i-1})$$

2. Perform recursive multigrid step

$$q = B_{\ell-1} P^{\langle\cdot\rangle}{}_{\ell-1}(\mu - A_\ell x^{n_\ell})$$

   and set $y^0 = x^{n_\ell} + I_\ell q$.

3. Perform n_ℓ smoothing steps

$$y^i = y^{i-1} + R^{i+n_\ell}_\ell(\mu - A_\ell y^{i-1})$$

4. Set $B_\ell \mu = y^{n_\ell}$.

If $n_\ell = n$ independent of the level, we obtain the standard V-cycle. The variable V-cycle is characterized by

$$\rho_1 n_\ell \le n_{\ell-1} \le \rho_2 n_\ell \tag{4.1}$$

which needs to hold for all $\ell \le L$ and uniform constants $1 < \rho_1 \le \rho_2$, see [41].

4.2 | Main Convergence Results

We use the results obtained in [41, 42]. These publications show convergence under three abstract assumptions, which are assumed to hold for all level ℓ. To illustrate these assumptions, let λ_ℓ^A be the largest eigenvalue of A_ℓ, and set

$$K_\ell := \left(\mathbb{1} - (\mathbb{1} - R_\ell A_\ell)(\mathbb{1} - R'_\ell A_\ell)\right) A_\ell^{-1}$$

where $\mathbb{1}$ is the identity matrix and R'_ℓ is the transpose of R_ℓ. The aforementioned, abstract assumptions claim existence of constants $C_1, C_2, C_3 > 0$ independent of the mesh level ℓ and of the function $\mu \in M_\ell$ satisfying:

- Regularity of approximation:

$$|a_\ell(\mu - I_\ell P^a{}_{\ell-1}\mu, \mu)| \le C_1 \left(\frac{\|A_\ell \mu\|_\ell^2}{\lambda_\ell^A} \right)^\alpha a_\ell(\mu,\mu)^{1-\alpha} \quad \text{(A1)}$$

  If $\alpha = 1$ in (2.2), (A1) simplifies to

$$|a_\ell(\mu - I_\ell P^a{}_{\ell-1}\mu, \mu)| \le C_1 \frac{\|A_\ell \mu\|_\ell^2}{\lambda_\ell^A} \quad \text{(A1')}$$

- Boundedness of the composition $I_\ell \circ P^a{}_{\ell-1} : M_\ell \to M_\ell$ of the injection and Ritz quasi-projection operators:

$$\|\mu - I_\ell P^a{}_{\ell-1}\mu\|_{a_\ell} \le C_2 \|\mu\|_{a_\ell} \quad \text{(A2)}$$

- Smoothing hypothesis:

$$\frac{\|\mu\|_\ell^2}{\lambda_\ell^A} \le C_3 \langle K_\ell \mu, \mu\rangle_\ell \quad \text{(A3)}$$

Theorem 4.1. *Let (2.2) hold with $\alpha = 1$, as well as (A1'), (A2), and (A3). Then, for the standard V-cycle, for all $\ell \ge 0$, and for all $\mu \in M_\ell$*

$$|a_\ell(\mu - B_\ell A_\ell \mu, \mu)| \le \delta a_\ell(\mu,\mu)$$

where

$$\delta = \frac{C_1 C_3}{n - C_1 C_3} \qquad \text{with} \qquad n > 2C_1C_3$$

Proof. This result is [42, Thm. 3.1]. □

Theorem 4.2. *Let (2.2) hold for $\alpha \in \left(\frac{1}{2}, 1\right]$, and further assume (A1), (A2), (A3), and that (4.1) holds with $\rho_1, \rho_2 > 1$. Then, for all $\ell \ge 0$ and all $\mu \in M_\ell$, it holds*

$$\eta_0 a_\ell(\mu,\mu) \le a_\ell(B_\ell A_\ell \mu, \mu) \le \eta_1 a_\ell(\mu,\mu)$$

with

$$\eta_0 \ge \frac{n_\ell^\alpha}{M + n_\ell^\alpha}, \qquad \eta_1 \le \frac{M + n_\ell^\alpha}{n_\ell^\alpha}$$

where the constant $M > 0$ is independent of ℓ. Hence, the condition number of $B_\ell A_\ell$ does not depend on ℓ.

Proof. This result is obtained combining [41, Thm. 6] with the relation [41, (A.4) ⇔ (3.4)]. □

5 | Convergence Analysis

We follow the lines of [19] and extend their results to our more general framework. Thus, the theorems and lemmas in the following sections will be very similar to the ones in [19]. If Lu et al. [19] have not used any HDG-specific arguments in their proofs, we also accept the results to hold in our framework and cite them. Otherwise, we will redo the respective proofs.

5.1 | Energy Boundedness of the Injection and Proof of (A2)

In the first half of this section, we prove the energy bounds

$$\|I_\ell P^a{}_{\ell-1}\mu\|_{a_\ell}^2 \lesssim \|P^a{}_{\ell-1}\mu\|_{a_{\ell-1}}^2 \lesssim \|\mu\|_{a_\ell}^2 \qquad \forall \mu \in M_\ell$$

which are the key properties needed to prove (A2) (see (5.14) in the final proof of this section). The key step is proving that $\|I_\ell \mu\|_{a_\ell} \lesssim \|\mu\|_{a_{\ell-1}}$ holds for all $\mu \in M_{\ell-1}$ (i.e., the first inequality holds). Proving the second inequality is then a rather straightforward task and can be done in a few lines, cf. (5.13). Thus, the main target of this section can be reduced to proving the first inequality, which is rigorously stated in (5.7).

To this end, we use the averaging linear interpolation

$$\begin{aligned} &I^{\text{avg}}{}_\ell : V_\ell \to \overline{V}^{\text{c}}{}_\ell \\ &I^{\text{avg}}{}_\ell u(\boldsymbol{x}_{\boldsymbol{a}}) := \{\{u\}\}_{\boldsymbol{a}} \text{ for any internal mesh vertex } \boldsymbol{a} \end{aligned} \quad (5.1)$$

Here, $\boldsymbol{x}_{\boldsymbol{a}}$ is the point that corresponds to vertex $\boldsymbol{a}$, and $\{\{u\}\}_{\boldsymbol{a}}$ describes the arithmetic mean of all values that u attains in $\boldsymbol{a}$. For $\boldsymbol{x} \in \partial\Omega$, we set $I^{\text{avg}}{}_\ell u(\boldsymbol{x}) := 0$.

This averaging operator allows us to go from the coarse mesh (level $\ell - 1$) to the fine mesh (level ℓ) in way that increases the $\|\cdot\|_{a_\ell}$ norm in a controllable way. Thus, this (auxiliary) averaging operator can be used as a stepping stone to prove the boundedndess of the injection operator I_ℓ. This idea is solidified by the properties in the following lemma.

Lemma 5.1. *Assuming (HM1) and (HM3), we have, for all ℓ, that*

$$\|\nabla I^{\text{avg}}{}_\ell \mathcal{U}_\ell \mu\|_\Omega \lesssim \|\mu\|_{a_\ell}, \qquad \forall \mu \in M_\ell \quad (5.2)$$

$$\|\mathcal{U}_\ell \lambda - I^{\text{avg}}{}_\ell \mathcal{U}_\ell \lambda\|_\Omega \lesssim h_\ell \|\mu\|_{a_\ell} \qquad \forall \lambda \in M_\ell \quad (5.3)$$

$$\|\mu - \gamma_\ell I^{\text{avg}}{}_\ell \mathcal{U}_\ell \mu\|_\ell \lesssim h_\ell \|\mu\|_{a_\ell} \qquad \forall \mu \in M_\ell \quad (5.4)$$

Proof. This can be shown analogously to [19, Lem. 5.2 and Lem. 5.3]. However, their (LS1) needs to be replaced by our version of (HM1), which changes the right-hand sides in (5.2) and (5.3). □

Now, we can transport these boundedness arguments to the injection operator in the below lemma. Notice the similarities between (5.2) and (5.5), and between (5.3) and (5.6).

Lemma 5.2. *Let (IA1), (IA2), (HM1) to (HM4) hold. We have*

$$\|\mathcal{Q}_\ell I_\ell \mu\|_\Omega \lesssim \|\mu\|_{a_{\ell-1}} \quad (5.5)$$

$$\|\mathcal{U}_{\ell-1}\mu - \mathcal{U}_\ell I_\ell \mu\|_\Omega \lesssim h_{\ell-1}\|\mu\|_{a_{\ell-1}} \lesssim h_\ell \|\mu\|_{a_{\ell-1}} \quad (5.6)$$

for all $\mu \in M_{\ell-1}$. If, additionally, (HM8) holds, we have

$$\|I_\ell \mu\|_{a_\ell} \lesssim \|\mu\|_{a_{\ell-1}} \qquad \forall \mu \in M_{\ell-1} \tag{5.7}$$

$$\|P^a{}_{\ell-1}\mu\|_{a_{\ell-1}} \lesssim \|\mu\|_{a_\ell} \qquad \forall \mu \in M_\ell \tag{5.8}$$

To prove this lemma, we use another operator to compare with: the lifting operator $S_\ell : M_\ell \to V_{\text{disc}}$ (with $V_{\text{disc}} \subset C(\Omega) \cap L^2(\Omega)$ being a suitable discrete space). This operator serves as an intermediate: We transport the boundedness properties of the averaging operator to the lifting operator, and use the lifting operator to deduce boundedness of the injection operator (one might, loosely speaking, think of a triangle inequality-like argument). It is inspired by [43]. It is also used in [44, Lem. A.3] for the two-dimensional case and in [45, Def. 5.46] for three-dimensional settings. For any $\mu \in M_\ell$

$$(S_\ell \mu, v)_T = (\mathcal{U}_\ell \mu, v)_T \qquad \forall v \in \mathcal{P}_p(T),\ \forall T \in \mathcal{T}_\ell \tag{5.9a}$$

$$\langle S_\ell \mu, \eta\rangle_F = \langle \mu, \eta\rangle_F \qquad \forall \eta \in \mathcal{P}_{p+1}(F),\ \forall F \in \mathcal{F}_\ell \tag{5.9b}$$

$$S_\ell \mu(\boldsymbol{a}) = \{\{\mu\}\}_{\boldsymbol{a}} \qquad \forall \boldsymbol{a} \text{ vertex in } \mathcal{T}_\ell,\ \boldsymbol{a} \notin \partial\Omega \tag{5.9c}$$

$$S_\ell \mu(\boldsymbol{a}) = 0 \qquad \forall \boldsymbol{a} \text{ vertex in } \mathcal{T}_\ell,\ \boldsymbol{a} \in \partial\Omega \tag{5.9d}$$

in two dimensions. In three spatial dimensions, we add the constraints

$$\langle S_\ell \mu, \eta\rangle_\Gamma = \langle \{\{\mu\}\}_\Gamma, \eta\rangle_\Gamma \qquad \forall \eta \in \mathcal{P}_{p+2}(\Gamma) \tag{5.9e}$$

Here, $\{\{m\}\}_\Gamma$ is the average taken over all cells adjacent to Γ. For a precise definition of V_{disc} and a proof of the fact that S_ℓ is well-defined, please refer to [44, Lem. A.3] and [45, Def. 5.46]. This construction extends to dimensions higher than three. Moreover, we have

Lemma 5.3 (Properties of $S_\ell \lambda$). *Under assumptions (HM1)–(HM4), we have*

$$\|S_\ell \mu\|_\Omega \cong \|\mu\|_\ell \qquad \forall \lambda \in M_\ell, \qquad \text{(norm equivalence)} \tag{5.10}$$

$$S_\ell \gamma_\ell w = w \qquad \forall w \in \overline{V}^{c}{}_\ell, \qquad \text{(lifting identity)} \tag{5.11}$$

$$|S_\ell \mu|_{1,\Omega} \lesssim \|\mu\|_{a_\ell} \qquad \forall \lambda \in M_\ell, \qquad \text{(lifting bound)} \tag{5.12}$$

where $|\cdot|_{1,\Omega}$ denotes the $H^1(\Omega)$-seminorm.

Proof. The proof can be performed similarly to the proof of [19, Lem. 5.5] with our modified version of (HM1) when compared with their (LS1). □

With all preparations ready, we can now prove Lemma 5.2 with the key property (5.7):

Proof of Lemma 5.2. Inequalities (5.5) and (5.6) can be obtained as in [19, Lem. 5.1]. Let us obtain inequality (5.7). To this end, we observe that

$$\|I_\ell \mu\|^2_{a_\ell} \overset{\text{(HM8)}}{\lesssim} \|\mathcal{Q}_\ell I_\ell \mu\|^2_\Omega + \sum_{T\in\mathcal{T}_\ell} \frac{1}{h_T} \|\mathcal{U}_\ell I_\ell \mu - \mathcal{U}_{\ell-1}\mu\|^2_{\partial T} + \sum_{T\in\mathcal{T}_\ell} \frac{1}{h_T} \|\mathcal{U}_{\ell-1}\mu - I_\ell \mu\|^2_{\partial T}$$

where (5.5) allows us to bound $\|\mathcal{Q}_\ell I_\ell \mu\|^2_\Omega$, (5.6) allows us to bound

$$\sum_{T\in\mathcal{T}_\ell} h_T^{-1} \|\mathcal{U}_\ell I_\ell \mu - \mathcal{U}_{\ell-1}\mu\|^2_{\partial T} \lesssim h_\ell^{-2} \|\mathcal{U}_\ell I_\ell \mu - \mathcal{U}_{\ell-1}\mu\|^2_\Omega$$

Assumption (IA3) implies that $\|\mathcal{U}_{\ell-1}\mu - I_\ell \mu\|_{\partial T}$ depends only on $\mu|_{B(x_T,\overline{\sigma}h)}$. Thus, we use the face-wise L^2 orthogonal projection $\pi : L^2(\tilde{B}(x_T,\overline{\sigma}h) \cap \mathcal{F}) \to M_{\tilde{B}(x_T,\overline{\sigma}h)\cap \mathcal{F}}$ to define the operator

$$\mathcal{G} : H^1(\tilde{B}(x_T,\overline{\sigma}h)) \ni v \mapsto \mathcal{U}_{\ell-1}\pi v - I_\ell \pi v \in M_\ell|_{\partial T} \subset L^2(\partial T)$$

where $\tilde{B}(x_T,\overline{\sigma}h) = B(x_T,\overline{\sigma}h) \cap \Omega$. Operator $\mathcal{G}$ is continuous (because of (IA1) and (HM2)) and vanishes if v is an overall continuous, element-wise linear polynomial, c.f. (IA2). Thus, the standard scaling argument, where $B(x_T,h)$ is mapped to the unity ball $B(0,1)$ can be applied. This allows us to deduce that

$$\sum_{T\in\mathcal{T}_\ell} \frac{1}{h_T} \|\mathcal{U}_{\ell-1}\pi v - I_\ell \pi v\|^2_{\partial T} \lesssim |v|^2_{H^1(\Omega)}$$

Finally, setting $v = S_{\ell-1}\mu$ with the lifting operator $S_{\ell-1}$ as defined in (5.9) yields the result (in conjunction with (5.12)).

Relation (5.8) results from

$$\begin{aligned}\|P^a{}_{\ell-1}\mu\|^2_{a_{\ell-1}} &\overset{(3.18)}{=} a_{\ell-1}(P^a{}_{\ell-1}\mu, P^a{}_{\ell-1}\mu) \overset{(3.17)}{=} a_\ell(\mu, I_\ell P^a{}_{\ell-1}\mu) \\ &\le \|\mu\|_{a_\ell} \|I_\ell P^a{}_{\ell-1}\mu\|_{a_\ell} \overset{(5.7)}{\lesssim} \|\mu\|_{a_\ell} \|P^a{}_{\ell-1}\mu\|_{a_{\ell-1}}\end{aligned} \tag{5.13}$$

where we have used the Cauchy–Schwarz inequality to pass to the second line.

Lemma 5.4. *Under the assumptions of Lemma 5.2, (A2) holds.*

Proof. Using first the linearity and symmetry of a_ℓ and then the definition (3.17) of the Ritz quasi-projector $P^a{}_{\ell-1}$, we obtain, for all $\mu \in M_\ell$,

$$\begin{aligned}&a_\ell(\mu - I_\ell P^a{}_{\ell-1}\mu, \mu - I_\ell P^a{}_{\ell-1}\mu) \\ &\quad= a_\ell(\mu,\mu) - 2a_\ell(\mu, I_\ell P^a{}_{\ell-1}\mu) + a_\ell(I_\ell P^a{}_{\ell-1}\mu, I_\ell P^a{}_{\ell-1}\mu) \\ &\quad\le \|\mu\|^2_{a_\ell} \underbrace{-2a_{\ell-1}(P^a{}_{\ell-1}\mu, P^a{}_{\ell-1}\mu)}_{\le 0} + \|I_\ell P^a{}_{\ell-1}\mu\|^2_{a_\ell}\end{aligned}$$

To conclude, we write for the rightmost term

$$\|I_\ell P^a{}_{\ell-1}\mu\|^2_{a_\ell} \overset{(5.7)}{\lesssim} \|P^a{}_{\ell-1}\mu\|^2_{a_{\ell-1}} \overset{(5.8)}{\lesssim} \|\mu\|^2_{a_\ell} \tag{5.14}$$

□

5.2 | Proofs of (A1) and (A1')

A classical approach for multigrid proofs consists is considering $A_\ell \mu$ (cf. (3.15)) as a function in $L^2(\Omega)$ and using it as right-hand side in an auxiliary problem. However, this is not possible for discontinuous skeletal methods, since $A_\ell \mu \in M_\ell$ is defined only on the mesh skeleton and not on the whole domain Ω. To this end, we use the lifting operator S_ℓ. That is, we define the auxiliary right-hand side $f_\mu \in V_{\text{disc}}$ as the unique solution of

$$(f_\mu, S_\ell \eta) = \langle A_\ell \mu, \eta\rangle_\ell = a_\ell(\mu,\eta) \qquad \forall \eta \in M_\ell \tag{5.15}$$

and its perturbed skeletal approximation $\tilde{\mu} \in M_\ell$ with

$$a_\ell(\tilde{\mu}, \eta) = (f_\mu, \mathcal{U}_\ell \eta) \qquad \forall \eta \in M_\ell \tag{5.16}$$

Notably, both μ and $\tilde{\mu}$ approximate the solution of the continuous problem

$$(\nabla \tilde{u}, \nabla v) = (f_\mu, v) \qquad \forall v \in H_0^1(\Omega) \tag{5.17}$$

Thus, they are close to one another, which is formalized in the following lemma:

Lemma 5.5. *Let (HM1) to (HM4) hold. Then we have, for all $\mu \in M_\ell$,*

$$\|\mu - \tilde{\mu}\|_{a_\ell} \lesssim h_\ell \|f_\mu\|_\Omega, \qquad \text{and} \qquad \|f_\mu\|_\Omega \lesssim \|A_\ell \mu\|_\ell \tag{5.18}$$

The key trick in the proof of (A1′) and (A1) is the observation that, on one hand, functions from $\overline{V}^c_{\ell-1}$ are not changed by the injection/projection operators or the local solvers (as implied by the quasi orthogonality in Lemma 3.3), while, on the other hand, this function space has approximation properties. Thus, we can introduce an arbitrary $\overline{V}^c_{\ell-1}$ in a scalar product resembling that in Lemma 3.3 and hope to exploit the approximation properties described in Lemma 5.6, whose right-hand side can be controlled by the means of Lemma 5.7.

Lemma 5.6 (Reconstruction approximation). *Let (IA1), (IA2), and (HM1)–(HM5) hold. Assume further that the model problem admits the regularity estimate (2.2). Then, for all $\mu \in M_\ell$, there exists an auxiliary function $\overline{u} \in \overline{V}^c_{\ell-1}$ such that*

$$\|\mathcal{Q}_\ell \mu + \nabla \overline{u}\|_\Omega + \|\mathcal{Q}_{\ell-1} P^a{}_{\ell-1} \mu + \nabla \overline{u}\|_\Omega \lesssim h_\ell^\alpha \|f_\mu\|_{\alpha-1} \tag{5.19}$$

Proof. This is [19, Lem. 5.8]. □

Lemma 5.7. *Let (HM1) to (HM4), (HM6), and (HM7) hold. Then,*

$$\|f_\mu\|_{-1} \lesssim \|\mu\|_{a_\ell}$$

Proof. By the definition of negative norms and properties of sup,

$$\|f_\mu\|_{-1} \le \sup_{\psi \in H_0^1(\Omega)} \frac{(f_\mu, \psi - S_\ell \gamma_\ell \overline{\Pi}^c{}_\ell \psi)}{|\psi|_{1,\Omega}} + \sup_{\psi \in H_0^1(\Omega)} \frac{(f_\mu, S_\ell \gamma_\ell \overline{\Pi}^c{}_\ell \psi)}{|\psi|_{1,\Omega}} \tag{5.20}$$

Here, $\overline{\Pi}^c{}_\ell$ is a (quasi-)interpolator onto $\overline{V}^c{}_\ell$ that satisfies

$$|\overline{\Pi}^c{}_\ell v|_{1,\Omega} \lesssim |v|_{1,\Omega} \qquad \forall v \in H^1(\Omega) \tag{5.21}$$

$$\|v - \overline{\Pi}^c{}_\ell v\|_\Omega \lesssim h_\ell^{1-k+\alpha} |v|_{\alpha+1}, \qquad \forall v \in H^{\alpha+1}(\Omega),\ k = 0, 1 \tag{5.22}$$

An example is given in [46]. We continue writing

$$\|\psi - S_\ell \gamma_\ell \overline{\Pi}^c{}_\ell \psi\|_\Omega \overset{(5.11)}{=} \|\psi - \overline{\Pi}^c{}_\ell \psi\|_\Omega \overset{(5.21)}{\lesssim} h_\ell |\psi|_{1,\Omega} \tag{5.23}$$

and observe that

$$(f_\mu, S_\ell \gamma_\ell \overline{\Pi}^c{}_\ell \psi)_\Omega \overset{(5.15)}{=} a_\ell(\mu, \gamma_\ell \overline{\Pi}^c{}_\ell \psi)$$

which immediately yields, using a Cauchy–Schwarz inequality,

$$|(f_\mu, S_\ell \gamma_\ell \overline{\Pi}^c{}_\ell \psi)_\Omega| \le \|\mu\|_{a_\ell} \|\gamma_\ell \overline{\Pi}^c{}_\ell \psi\|_{a_\ell} = \|\mu\|_{a_\ell} |\overline{\Pi}^c{}_\ell \psi|_{1,\Omega} \lesssim \|\mu\|_{a_\ell} |\psi|_{1,\Omega} \tag{5.24}$$

Applying a Cauchy–Schwarz inequality to the numerator of the first supremum in (5.20) followed by (5.23) and using (5.24) to estimate the numerator of the second supremum, we get, after simplification,

$$\|f_\mu\|_{-1} \le h_\ell \|f_\mu\|_\Omega + \|\mu\|_{a_\ell} \overset{(5.18)}{\lesssim} h_\ell \|A_\ell \mu\|_\ell + \|\mu\|_{a_\ell} \lesssim \|\mu\|_{a_\ell}$$

where the last inequality is the rightmost inequality of (HM6). □

With these preliminary results at hand, we can formulate a generalization of [18, Theo. 4.1] and [19, Theo. 5.10]:

Theorem 5.8. *If (2.2) holds with $\alpha \in \left(\frac{1}{2}, 1\right]$, and if the assumptions (IA1), (IA2), and (HM1)–(HM8) hold, then (A1) is satisfied.*

Remark 5.9 (Full elliptic regularity). Theorem 5.8 implies (A1′) if $\alpha = 1$.

Proof. We use the definitions (3.6) of a_ℓ and (3.17) of $P^a{}_{\ell-1}$ to obtain

$$\begin{aligned} &a_\ell(\mu - I_\ell P^a{}_{\ell-1}\mu, \mu) \\ &\quad = a_\ell(\mu,\mu) - a_{\ell-1}(P^a{}_{\ell-1}\mu, P^a{}_{\ell-1}\mu) \\ &\quad = \underbrace{(\mathcal{Q}_\ell\mu, \mathcal{Q}_\ell\mu)_\Omega - (\mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu, \mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu)_\Omega}_{\mathfrak{T}_1} \\ &\qquad + \underbrace{s_\ell(\mu,\mu) - s_{\ell-1}(P^a{}_{\ell-1}\mu, P^a{}_{\ell-1}\mu)}_{\mathfrak{T}_2}, \end{aligned}$$

and analyze the respective contributions $\mathfrak{T}_1$ and $\mathfrak{T}_2$ separately.

First, binomial factorization yields

$$\mathfrak{T}_1 = (\mathcal{Q}_\ell\mu + \mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu, \mathcal{Q}_\ell\mu - \mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu)_\Omega$$

Let now $w \in \overline{V}^c{}_{\ell-1}$. Invoking the quasi-orthogonality property stated in Lemma 3.3 to insert $2\nabla w$ into the first argument of the L^2-product, and adding and subtracting ∇w to the second argument, we obtain

$$\begin{aligned} \mathfrak{T}_1 &= \big((\mathcal{Q}_\ell\mu + \nabla w) + (\mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu + \nabla w), (\mathcal{Q}_\ell\mu + \nabla w) \\ &\quad - (\nabla w + \mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu)\big)_\Omega \\ &\le \big(\|\mathcal{Q}_\ell + \nabla w\|_\Omega + \|\mathcal{Q}_{\ell-1}P^a{}_{\ell-1}\mu + \nabla w\|_\Omega\big)^2 \overset{(5.19)}{\lesssim} h_\ell^{2\alpha}\|f_\mu\|^2_{\alpha-1} \end{aligned} \tag{5.25}$$

where we have used Cauchy–Schwarz and triangle inequalities to pass to the second line.

Second, we obtain a similar estimate for $\mathfrak{T}_2$. To this end, we detail the treatment of the first summand of $\mathfrak{T}_2$, the second summand can be treated analogously. For any $w \in \overline{V}^c{}_\ell$, it holds

$$\begin{aligned} s_\ell(\mu,\mu) &\overset{(\text{HM7})}{=} s_\ell(\mu - \gamma_\ell w, \mu - \gamma_\ell w) \le a_\ell(\mu - \gamma_\ell w, \mu - \gamma_\ell w) \\ &= \|\mu - \gamma_\ell w\|^2_{a_\ell} \end{aligned}$$

Using the triangle inequality and Lemma 5.5, we have that

$$\|\mu - \gamma_\ell w\|_{a_\ell} \le \|\mu - \tilde{\mu}\|_{a_\ell} + \|\tilde{\mu} - \gamma_\ell w\|_{a_\ell} \\ \le h_\ell \|f_\mu\|_\Omega + \|\tilde{\mu} - \gamma_\ell w\|_{a_\ell}$$

Defining w as the L^2 orthogonal projection of $\tilde{u}$ into $\overline{V}^{c}{}_\ell$ yields the result with (HM5) as we can write

$$\|\tilde{\mu} - \gamma_\ell w\|_{a_\ell} \le \|\tilde{\mu} - \Pi^\partial{}_\ell \tilde{u}\|_{a_\ell} + \|\Pi^\partial{}_\ell \tilde{u} - \gamma_\ell w\|_{a_\ell}$$

where the first summand is bounded via (HM5) and the second summand can be bounded using (HM6) and standard approximation properties. Proceeding similarly for the second summand in $\mathfrak{T}_2$, we infer

$$\mathfrak{T}_2 \lesssim h_\ell^{2\alpha} \|f_\mu\|_{\alpha-1}^2 \tag{5.26}$$

Using (5.25) and (5.26) to estimate the right-hand side of (5.26) and invoking Sobolev interpolation provides us with

$$|a_\ell(\mu - I_\ell P^a{}_{\ell-1}\mu, \mu)| \lesssim h_\ell^{2\alpha} \|f_\mu\|_{\alpha-1}^2 \lesssim h_\ell^{2\alpha} \|f_\mu\|_{-1}^{2(1-\alpha)} \|f_\mu\|_\Omega^{2\alpha}$$

and Lemmas 5.5 and 5.7 yield

$$|a_\ell(\mu - I_\ell P^a{}_{\ell-1}\mu, \mu)| \lesssim h_\ell^{2\alpha} \|A_\ell \mu\|_\ell^{2\alpha} \|\mu\|_{a_\ell}^{2(1-\alpha)}$$

This is the result, since (HM6) implies that $\underline{\lambda}_\ell^A \lesssim h_\ell^{-2}$. □

6 | Verification of Assumptions for HHO

6.1 | Preliminaries

For all $T \in \mathcal{T}_\ell$, we introduce the set $\mathcal{F}_T$ collecting the faces of T and denote *a polynomial space on* T by $\mathcal{P}(T)$.

6.1.1 | General Properties of Used Norms and Function Spaces

Let $T \in \mathcal{T}_\ell$, $F \in \mathcal{F}_T$. We recall the following results:

- The *discrete trace inequality* (see, e.g., [6, Lem. 1.32]):

$$\|v\|_F \lesssim h_T^{-\frac{1}{2}} \|v\|_T \qquad \forall v \in \mathcal{P}(T) \tag{6.1}$$

- The *discrete inverse inequality* (see, e.g., [6, Lem. 1.28]):

$$\|\nabla v\|_T \lesssim h_T^{-1} \|v\|_T \qquad \forall v \in \mathcal{P}(T) \tag{6.2}$$

- The *Poincaré–Friedrichs inequality* (see, e.g., [39, Lem. 3.30]):

$$\|\rho\|_T \lesssim h_T \|\nabla \rho\|_T + h_T^{\frac{1}{2}} \|\rho\|_{\partial T} \qquad \forall \rho \in H^1(T) \tag{6.3}$$

- The norm equivalence

$$\|\mu\|_\ell^2 \simeq \sum_{F \in \mathcal{F}_\ell} h_F \|\mu\|_F^2 \qquad \forall \mu \in M_\ell \tag{6.4}$$

  Thus, the square root of the expression in the right-hand side induces a global norm on M_ℓ.

6.1.2 | Properties of HHO

In this part, we enlist some properties of HHO that will be used to show (HM1) to (HM8).

Recalling the hybrid bilinear form $\underline{a}_\ell$ introduced in (3.10), we consider the hybrid, discrete problem: Find $(u_\ell, m_\ell) \in V_\ell \times M_\ell$ such that

$$\underline{a}_\ell((u_\ell, m_\ell), (v, \mu)) = \int_\Omega f v \qquad \forall (v, \mu) \in V_\ell \times M_\ell \tag{6.5}$$

The pair $\underline{u}_\ell := (\mathcal{U}_\ell m_\ell + \mathcal{V}_\ell f, m_\ell) \in V_\ell \times M_\ell$ is solution of (6.5) if and only if $m_\ell \in M_\ell$ is solution of the condensed problem (3.5) (see [47, Prop. 4]). In particular, $\mathcal{U}_\ell m_\ell + \mathcal{V}_\ell f$ approximates the exact solution u of the continuous problem (2.1).

As in [6, 2.1.2 (2.7), 2.2.2 (2.35)], we define

$$|(v, \mu)|_{\underline{1,\ell}}^2 := \sum_{T \in \mathcal{T}_\ell} |(v, \mu)|_{\underline{1,T}}^2 = \sum_{T \in \mathcal{T}_\ell} \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v - \mu\|_F^2 \tag{6.6a}$$

$$\|(v, \mu)\|_{\underline{1,\ell}}^2 := \sum_{T \in \mathcal{T}_\ell} \|(v, \mu)\|_{\underline{1,T}}^2 = \sum_{T \in \mathcal{T}_\ell} \left(\|\nabla v\|_T^2 + |(v, \mu)|_{\underline{1,T}}^2 \right) \tag{6.6b}$$

as the H^1-like seminorm on the hybrid space $V_\ell \times M_\ell$. Moreover, we assume that the properties of [6, Assumption 2.4] for the stabilization bilinear form $\underline{s}_T$ hold true. Then, we have:

- (Boundedness and stability of $\underline{a}_T$ [48, Lem. 2.6])

$$\|(v, \mu)\|_{\underline{1,T}}^2 \lesssim \underline{a}_T((v, \mu), (v, \mu)) \lesssim \|(v, \mu)\|_{\underline{1,T}}^2 \tag{6.7}$$

- (Energy error estimate [48, Lem. 2.8, Lem. 2.9])

$$\|\boldsymbol{q}\underline{u}_\ell - \boldsymbol{q}(\Pi_\ell^{\mathrm{d}} u, \Pi^\partial{}_\ell u)\|_{\underline{a}_\ell} \lesssim h_\ell^\alpha \|u\|_{\alpha+1} \tag{6.8}$$

  where $\alpha > \frac{1}{2}$ and $\|\cdot\|_{\underline{a}_\ell}$ is the norm induced by the bilinear form $\underline{a}_\ell$.

- (Cell unknown L^2-error estimate [48, Lem. 2.11])
  If $p \ge 1$, $\alpha > \frac{1}{2}$, then

$$\|\mathcal{U}_\ell m_\ell + \mathcal{V}_\ell f - \Pi_\ell^{\mathrm{d}} u\|_\Omega \lesssim h_\ell^{\alpha+\delta} \|u\|_{\alpha+1} \tag{6.9}$$

  where $\delta := \min\{\alpha, 1\}$.

- (Face unknown L^2-error estimate [48, Lem. 2.9])
  If $p \ge 1$, then

$$\|m_\ell - \Pi^\partial{}_\ell u\|_\ell \lesssim h_\ell^{\alpha+1} \|u\|_{\alpha+1} \tag{6.10}$$

Now, let us introduce the seminorms $\|\cdot\|_{1,\ell}$ and $|\cdot|_{1,\ell}$ on the skeletal space M_ℓ, which are based on the seminorms $\|\cdot\|_{\underline{1,\ell}}$ and $|\cdot|_{\underline{1,\ell}}$ defined in (6.6):

$$\|\mu\|_{1,\ell} := \|(\mathcal{U}_\ell \mu, \mu)\|_{\underline{1,\ell}} \quad \text{and} \quad |\mu|_{1,\ell} := |(\mathcal{U}_\ell \mu, \mu)|_{\underline{1,\ell}} \tag{6.11}$$

Similarly, the *condensed* bilinear form a_ℓ introduced in (3.14) is built from the *hybrid* bilinear form $\underline{a}_\ell$ (3.10), in which the generic cell unknown variable is also recovered from the skeletal variable through $\mathcal{U}_\ell$, that is,

$$a_\ell(m, \mu) = \underline{a}_\ell((\mathcal{U}_\ell m, m), (\mathcal{U}_\ell \mu, \mu))$$

In the same fashion, we also have

$$s_\ell(m, \mu) = \underline{s}_\ell((\mathcal{U}_\ell m, m), (\mathcal{U}_\ell \mu, \mu))$$

Thus, useful properties of a_ℓ and s_ℓ derive in a natural way from those of $\underline{a}_\ell$ and $\underline{s}_\ell$. In particular, we shall use in this work the following results:

- (Boundedness and stability of a_ℓ)

$$\|\mu\|_{1,\ell}^2 \lesssim a_\ell(\mu, \mu) \lesssim \|\mu\|_{1,\ell}^2 \tag{6.12}$$

- Since $\underline{s}_\ell$ is symmetric positive semidefinite [6, Assumption 2.4], so is s_ℓ.

Lemma 6.1. *The norm $\|\cdot\|_\ell$ is weaker than $\|\cdot\|_{1,\ell}$, that is,*

$$\|\mu\|_\ell \lesssim \|\mu\|_{1,\ell} \qquad \forall \mu \in M_\ell \tag{6.13}$$

Proof. Starting from (6.4), the triangle inequality is used via the insertion of $0 = \mathcal{U}\mu - \mathcal{U}\mu$ into the norms in the sum, yielding

$$\begin{aligned}\|\mu\|_\ell^2 &\lesssim \sum_{F\in\mathcal{F}} h_F\left(\|\mathcal{U}_\ell\mu - \mu\|_F^2 + \|\mathcal{U}_\ell\mu\|_F^2\right)\\ &\overset{6.1}{\lesssim} \sum_{F\in\mathcal{F}} h_F\|\mathcal{U}_\ell\mu - \mu\|_F^2 + \sum_{T\in\mathcal{T}_\ell}\|\mathcal{U}_\ell\mu\|_T^2\\ &\lesssim \sum_{F\in\mathcal{F}} h_F\|\mathcal{U}_\ell\mu - \mu\|_F^2 + \|(\mathcal{U}_\ell\mu, \mu)\|_{\underline{1,\ell}}^2\end{aligned} \tag{6.14}$$

where the last passage follows from the discrete Poincaré inequality [6, Lem. 2.15] applied to $(\mathcal{U}_\ell\mu, \mu)$ (notice that μ vanishes on $\partial\Omega$). Hence, recalling (6.11),

$$\|\mu\|_\ell^2 \lesssim \sum_{F\in\mathcal{F}} h_F\|\mathcal{U}_\ell\mu - \mu\|_F^2 + \|\nabla_\ell\mathcal{U}_\ell\mu\|_\Omega^2 + |\mu|_{1,\ell}^2 \tag{6.15}$$

The fact that $h_F \lesssim h_F^{-1}$ is a consequence of the fact that we assumed that Ω has diameter 1 without loss of generality so that $h_F \lesssim 1$. Hence, we can go on writing

$$\|\mu\|_\ell^2 \lesssim \|\nabla_\ell\mathcal{U}_\ell\mu\|_\Omega^2 + |\mu|_{1,\ell}^2 = \|\mu\|_{1,\ell}^2$$

and the result follows by taking the square root. □

6.2 | Verification of (HM1)

We write

$$\begin{aligned}&\|\mathcal{U}_\ell\mu - \mu\|_\ell^2\\ &\quad\lesssim \sum_{T\in\mathcal{T}_\ell}\sum_{F\in\mathcal{F}_T} h_F\|\mathcal{U}_\ell\mu - \mu\|_F^2\\ &\quad\le h_\ell^2\sum_{T\in\mathcal{T}_\ell}\sum_{F\in\mathcal{F}_T} h_F^{-1}\|\mathcal{U}_\ell\mu - \mu\|_F^2\\ &\quad\overset{(6.6b)}{\le} h_\ell^2\|(\mathcal{U}_\ell\mu, \mu)\|_{\underline{1,h}}^2 \overset{(6.7)}{\lesssim} h_\ell^2\|(\mathcal{U}_\ell\mu, \mu)\|_{\underline{a}_\ell}^2 \overset{(3.18)}{=} h_\ell^2\|\mu\|_{a_\ell}^2\end{aligned}$$

where the first inequality follows from mesh regularity implying $h_T \lesssim h_F$. Taking the square root of the above inequality proves (HM1).

6.3 | Verification of (HM2)

Let us start with the second inequality in (HM2) on an arbitrary element $T \in \mathcal{T}_\ell$. Recalling the definition (3.11) of $\mathcal{U}_\ell$ and choosing $v_T = u_T^1 = \mathcal{U}_T\mu$ and $m_{\partial T} = \mu$, we have, denoting by $\|\cdot\|_{\underline{a}_T}$ the seminorm induced by $\underline{a}_T$,

$$\begin{aligned}\|(\mathcal{U}_T\mu, 0)\|_{\underline{a}_T}^2 &= \underline{a}_T((\mathcal{U}_T\mu, 0), (\mathcal{U}_T\mu, 0))\\ &\overset{(3.11)}{=} -\underline{a}_T((0, \mu), (\mathcal{U}_T\mu, 0)) \le \|(0, \mu)\|_{\underline{a}_T}\ \|(\mathcal{U}_T\mu, 0)\|_{\underline{a}_T}\end{aligned}$$

where the conclusion follows from the Cauchy–Schwarz inequality. Hence,

$$\|(\mathcal{U}_T\mu, 0)\|_{\underline{a}_T} \le \|(0, \mu)\|_{\underline{a}_T} \tag{6.16}$$

We write

$$\begin{aligned}\|(\mathcal{U}_T\mu, 0)\|_{\underline{1,T}}^2 &\overset{(6.7)}{\lesssim} \|(\mathcal{U}_T\mu, 0)\|_{\underline{a}_T}^2 \overset{(6.16)}{\le} \|(0, \mu)\|_{\underline{a}_T}^2\\ &\overset{(6.7)}{\lesssim} \|(0, \mu)\|_{\underline{1,T}}^2 \lesssim h_T^{-1}\|\mu\|_{\partial T}^2\end{aligned} \tag{6.17}$$

where the last inequality follows recalling the definition (6.6b) of $\|\cdot\|_{\underline{1,T}}$ and noticing that $h_F^{-1} \lesssim h_T^{-1}$ for all $F \in \mathcal{F}_T$ by mesh regularity. Next, using the Poincaré–Friedrichs inequality (6.3) and definition (6.6b) of the $\|(\cdot,\cdot)\|_{\underline{1,T}}$-norm along with the fact that $h_T^{-1} \lesssim h_F^{-1}$ for all $F \in \mathcal{F}_T$ by mesh regularity, we have

$$\begin{aligned}\|\mathcal{U}_T\mu\|_T^2 &\lesssim h_T^2\|\nabla(\mathcal{U}_T\mu)\|_T^2 + h_T\|\mathcal{U}_T\mu\|_{\partial T}^2 \lesssim h_T^2\|(\mathcal{U}_T\mu, 0)\|_{\underline{1,T}}^2\\ &\overset{(6.17)}{\lesssim} h_T\|\mu\|_{\partial T}^2\end{aligned} \tag{6.18}$$

The second inequality in (HM2) is derived by using (6.18) and summing over all elements.

Second, consider the first inequality on an arbitrary element $T \in \mathcal{T}_\ell$. Recalling the definition (3.8) of $\boldsymbol{q}_T(\cdot,\cdot)$ with $(u_T, m_{\partial T}) = (\mathcal{U}_T\mu, \mu)$ and choosing $\boldsymbol{p}_T = \boldsymbol{q}_T(\mathcal{U}_T\mu, \mu) \overset{(3.13)}{=} \mathcal{Q}_T\mu$, $u_T = \mathcal{U}_T\mu$, and $m_{\partial T} = \mu$, we have

$$\int_T \mathcal{Q}_T\mu\cdot\mathcal{Q}_T\mu = \int_T \mathcal{U}_T\mu\,(\nabla\cdot\mathcal{Q}_T\mu) - \int_{\partial T}\mu\,(\mathcal{Q}_T\mu\cdot\boldsymbol{\nu})$$

Estimating the right-hand side of the above expression with Cauchy–Schwarz inequalities followed by the discrete inverse inequality (6.2) for the first term, the discrete trace inequality (6.1) for the second term, we have after simplifying and raising to the square,

$$\|\mathcal{Q}_T\mu\|_T^2 \lesssim h_T^{-2}\|\mathcal{U}_T\mu\|_T^2 + h_T^{-1}\|\mu\|_{\partial T}^2 \overset{(6.18)}{\le} h_T^{-1}\|\mu\|_{\partial T}^2 \tag{6.19}$$

Finally, the first inequality in (HM2) is derived by using (6.19), summing over all elements, and using the mesh quasi-uniformity assumption to write $\|\mathcal{Q}_\ell\mu\|_\Omega \lesssim h_\ell^{-1}\|\mu\|_\ell$.

6.4 | Verification of (HM3)

Plugging the definition (3.13) of $\mathcal{Q}_T$ into (3.8) we have,

$$\int_T \mathcal{Q}_T\mu \cdot \boldsymbol{p}_T - \int_T \mathcal{U}_T\mu\,(\nabla \cdot \boldsymbol{p}_T) = -\int_{\partial T} \mu\,(\boldsymbol{p}_T \cdot \boldsymbol{\nu}) \qquad \forall \boldsymbol{p}_T \in \boldsymbol{W}_T$$

Then, we integrate by parts the second term of the left-hand side and rearrange to infer

$$\int_T (\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu) \cdot \boldsymbol{p}_T = \int_{\partial T} (\mathcal{U}_T\mu - \mu)\boldsymbol{p}_T \cdot \boldsymbol{\nu} \qquad \forall \boldsymbol{p}_T \in \boldsymbol{W}_T$$

Now, after noticing that $\nabla\mathcal{U}_T\mu \in \nabla\mathcal{P}_p(T) \subset \nabla\mathcal{P}_{p+1}(T) = \boldsymbol{W}_T$, we can specify this relation for $\boldsymbol{p}_T = \mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu$, which gives

$$\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_T^2 = \int_{\partial T} (\mathcal{U}_T\mu - \mu)\,(\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu) \cdot \boldsymbol{\nu}$$

Using the Cauchy–Schwarz inequality on the right-hand side, we have

$$\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_T \le \|\mathcal{U}_T\mu - \mu\|_{\partial T}\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_{\partial T}$$

At this point, we refer to the discrete trace inequality (6.1), which we use component by component to bound the last term and obtain

$$\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_T \lesssim h_T^{-\frac{1}{2}}\|\mathcal{U}_T\mu - \mu\|_{\partial T}\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_T$$

Simplifying and squaring, we get

$$\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_T^2 \lesssim h_T^{-1}\|\mathcal{U}_T\mu - \mu\|_{\partial T}^2$$

Summing over all elements and using mesh quasi-uniformity yields

$$\begin{aligned} &\|\mathcal{Q}_T\mu + \nabla\mathcal{U}_T\mu\|_\Omega^2 \\ &\quad\lesssim \sum_{T\in\mathcal{T}_\ell} h_T^{-1}\|\mathcal{U}_T\mu - \mu\|_{\partial T}^2 \lesssim \sum_{T\in\mathcal{T}_\ell} h_T^{-2} \sum_{F\in\mathcal{F}_T} h_F\|\mathcal{U}_T\mu - \mu\|_F^2 \\ &\quad\lesssim h_\ell^{-2}\sum_{T\in\mathcal{T}_\ell}\sum_{F\in\mathcal{F}_T} h_F\|\mathcal{U}_T\mu - \mu\|_F^2 \lesssim h_\ell^{-1}\|\mathcal{U}_T\mu - \mu\|_\ell^2 \end{aligned}$$

where we have used the fact that $h_T^{-1}h_F \lesssim 1$ for all $T \in \mathcal{T}_\ell$ and all $F \in \mathcal{F}_T$ by mesh regularity in the second inequality, the mesh quasi-uniformity assumption to write $h_T^{-2} \lesssim h_\ell^{-2}$ in the third inequality, and (6.4) to conclude

6.5 | Verification of (HM4)

We show (HM4) for a generic element $T \in \mathcal{T}_\ell$. If the identities hold there, they will hold on all elements. We must show that HHO reproduces the $w \in \overline{V}^{\mathrm{c}}_\ell$. We take $w_{\partial T} = \gamma_\ell w$.

$$\int_T \boldsymbol{q}_T(w, w_{\partial T}) \cdot \boldsymbol{p}_T = \int_T w\nabla \cdot \boldsymbol{p}_T - \sum_{F\in\mathcal{F}_T}\int_F w_{\partial T}\boldsymbol{p}_T \cdot \boldsymbol{\nu}$$

Integrating by parts the first term on the right-hand side, it holds that

$$\int_T w\nabla \cdot \boldsymbol{p}_T = -\int_T \nabla w \cdot \boldsymbol{p}_T + \int_{\partial T} w_{\partial T}\boldsymbol{p}_T \cdot \boldsymbol{\nu}$$

Plugging that yields

$$\int_T \boldsymbol{q}_T(w, w_{\partial T}) \cdot \boldsymbol{p}_T = -\int_T \nabla w \cdot \boldsymbol{p}_T \qquad \forall \boldsymbol{p}_T \in \boldsymbol{W}_T$$

which shows that

$$\boldsymbol{q}_T(w, w_{\partial T}) = -\nabla w \tag{6.20}$$

Given the definition (3.11) of $\mathcal{U}_T$, we need to show that

$$\underline{a}_T((w, w_{\partial T}), (v_T, 0)) = 0 \qquad \forall v_T \in V_T$$

We have

$$\begin{aligned} &\underline{a}_T(w, w_{\partial T}, (v_T, 0)) \\ &\quad= \int_T \boldsymbol{q}_T(w, w_{\partial T}) \cdot \boldsymbol{q}_T(v_T, 0) + \underbrace{\underline{s}_T((w, w_{\partial T}), (v_T, 0))}_{0} \\ &\quad= -\int_T \nabla w \cdot \boldsymbol{q}_T(v_T, 0) \end{aligned}$$

where the stabilization term vanishes due to the polynomial consistency of $\underline{s}_T$ [6, Assumption 2.4 (S3)], and we have used (6.20) for the last equality. For the remaining term, (3.8) gives

$$\int_T \boldsymbol{q}_T(v_T, 0) \cdot \boldsymbol{p}_T - \int_T v_T\nabla \cdot \boldsymbol{p}_T = 0 \qquad \forall \boldsymbol{p}_T \in \boldsymbol{W}_T$$

which we specialize to $\boldsymbol{p}_T = -\nabla w$ to infer

$$-\int_T \boldsymbol{q}_T(v_T, 0) \cdot \nabla w + \int_T v_T\Delta w = 0$$

As $w \in \overline{V}^{\mathrm{c}}_\ell$, $\Delta w = 0$, which concludes the proof of $\mathcal{U}\gamma_\ell w = w$.

Remark 6.2 (Generalization to $p = 0$). Equation (HM4) is one reason why we assume that $p \ge 1$ throughout this manuscript. However, this assumption can be relaxed by inserting the L^2-orthogonal projector at several locations in the above argument. This observation motivates that our multigrid framework can be extended to the case $p = 0$ if one can find ways to make the following criterion work, which might work only under additional constraints, cf. Remark 8.1.

6.6 | Verification of (HM5)

This is an extension of Theorem 2.27 in [48] by Sobolev interpolation. Following Lemma 2.9, the left-hand side of (HM5) is bounded by the consistency error estimates. The (HM5) is derived using Theorem 2.10 in [48].

6.7 | Verification of (HM6)

Combining some preliminary results, we have that

$$\|\mu\|_\ell^2 \overset{(6.13)}{\lesssim} \|\mu\|_{1,\ell}^2 \overset{(6.12)}{\lesssim} a_\ell(\mu, \mu) \overset{(6.12)}{\lesssim} \|\mu\|_{1,\ell}^2$$

Thus, we only need to prove that

$$\|\mu\|_{1,\ell} \lesssim h_\ell^{-1}\|\mu\|_\ell \tag{6.21}$$

Inserting $0 = \mathcal{Q}_\ell \mu - \mathcal{Q}_\ell \mu$ into the gradient term of (6.11) and using the triangle inequality gives

$$\begin{aligned}\|\mu\|_{1,\ell}^2 &\lesssim \|\nabla_\ell \mathcal{U}_\ell \mu + \mathcal{Q}_\ell \mu\|_\Omega^2 + \|\mathcal{Q}_\ell \mu\|_\Omega^2 + |\mu|_{1,\ell} \\ &\overset{\text{(HM3)}}{\lesssim} h_\ell^{-2}\|\mathcal{U}_\ell \mu - \mu\|_\ell^2 + \|\mathcal{Q}_\ell \mu\|_\Omega^2 + |\mu|_{1,\ell}\end{aligned} \tag{6.22}$$

To make $\|\cdot\|_\ell$ appear in $|\mu|_{1,\ell}$, we observe that, by mesh quasi-uniformity,

$$|\mu|_{1,\ell} \simeq h_\ell^{-2} \sum_{T\in\mathcal{T}_\ell} \sum_{F\in\mathcal{F}_T} h_\ell \|\mathcal{U}_\ell \mu - \mu\|_F^2 \overset{(6.4)}{\simeq} h_\ell^{-2}\|\mathcal{U}_\ell \mu - \mu\|_\ell^2 \tag{6.23}$$

Plugging (6.23) into (6.22) gives

$$\|\mu\|_{1,\ell}^2 \lesssim h_\ell^{-2}\|\mathcal{U}_\ell \mu - \mu\|_\ell^2 + \|\mathcal{Q}_\ell \mu\|_\Omega^2 \overset{\text{(HM1)}}{\lesssim} \|\mu\|_{a_\ell}^2 \tag{6.24}$$

Finally, (6.21) follows from (HM2).

6.8 | Verification of (HM7)

This equality corresponds to [6, Assumption 2.4 (S3)].

6.9 | Verification of (HM8)

This is relation (6.12).

7 | Injection Operators for HHO

Consider two successive levels ℓ (fine) and $\ell-1$ (coarse). Given the coarse faces $\mathcal{F}_{\ell-1}$, the mesh nestedness allows us to decompose $\mathcal{F}_\ell$ as the disjoint union $\hat{\mathcal{F}}_\ell \cup \mathring{\mathcal{F}}_\ell$, where

$$\hat{\mathcal{F}}_\ell := \{F \in \mathcal{F}_\ell | \exists F_{\ell-1} \in \mathcal{F}_{\ell-1} \text{ s.t. } F \subset F_{\ell-1}\}, \qquad \mathring{\mathcal{F}}_\ell := \mathcal{F}_\ell \setminus \hat{\mathcal{F}}_\ell$$

In the following, we introduce three injection operators used in [24], denoted by I_ℓ^i, $i \in \{1,2,3\}$. The first one is defined as

$$(I_\ell^1 \mu)_{|F} := \begin{cases} \mu_{|F} & \text{if } F \in \hat{\mathcal{F}}_\ell, \\ (\mathcal{U}_{\ell-1}\mu)_{|F} & \text{otherwise} \end{cases} \tag{7.1}$$

This first operator exploits the nestedness of the meshes to straightforwardly transfer values from coarse faces to their embedded fine faces. Regarding the fine faces that are *not* geometrically included in the coarse skeleton (i.e., $\mathring{\mathcal{F}}_\ell$), we make use of the local solver: $\mathcal{U}_{\ell-1}$ builds a bulk function in the coarse cells, whose traces provide admissible approximations on the fine faces.

Instead of the straight injection for the embedded faces, an alternative is to also use the bulk functions reconstructed from the local solver. As $\mathcal{U}_{\ell-1}$ yields a discontinuous polynomial, and the faces of $\hat{\mathcal{F}}_\ell$ are located at the interface of two coarse cells, we propose to take the average of the respective traces, that is,

$$(I_\ell^2 \mu)_{|F} := \{\{\mathcal{U}_{\ell-1}\mu\}\}_F \tag{7.2}$$

Remark that this formula also holds for $F \in \mathring{\mathcal{F}}_\ell$, as the average trace of a continuous bulk function on both sides of a face reduces to its regular trace. This injection operator comes with advantages. First, it allows the mesh nestedness condition to be relaxed, paving the way to generalized multigrid methods on non-nested meshes, as developed in [25]. Additionally, a more complex formula can handle difficulties occurring at cell interfaces. Typically, using an adequate weighted average formula can yield robust convergence in the presence of large jumps in the diffusion coefficient; see [24].

Finally, the third injection operator leverages the higher order reconstruction operator, a salient feature of the HHO methods:

$$(I_\ell^3 \mu)_{|F} := \pi_\ell^p\{\{\theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\mu, \mu)\}\}_F \tag{7.3}$$

where $\pi_\ell^p : L^2(\mathcal{F}_\ell) \to M_\ell$ denotes the L^2-orthogonal projector onto the skeletal polynomial space of degree p. I_ℓ^3 is based on the same principle of the averaged trace as I_ℓ^2, except that the higher order reconstruction operator $\theta_{\ell-1}^{p+1}$ enables the gain of one extra polynomial degree in the approximation of the coarse error. Then, after computing the average trace, the polynomial degree is lowered back to its original value by applying the L^2-orthogonal projector onto the lower order space.

While the first injection operator is local with respect to each coarse element, the later injection operators use information from their neighboring elements, increasing their domain of dependence. Thus, the first injection operator can be implemented more efficiently than the latter ones. On the contrary, the evaluation of the HHO reconstruction produces negligible cost (solving a local system of equations per cell) if the degree of parallelization is high.

To fit these injection operators into the framework, we demonstrate

Lemma 7.1. *Under the assumptions (HM2) and (HM4), the injection operators I_ℓ^i, $i \in \{1,2,3\}$ verify (IA1)–(IA2).*

Proof. Regarding I_ℓ^1, we refer to [19, Lem. 3.2], as they investigate the same injection operator: I_ℓ^1 is their third operator.

Let us consider I_ℓ^2: condition (IA2) follows directly from (HM4) recalling that we assume here $p \geq 1$ (cf. Remark 8.1 below). The discrete trace inequality (6.1) along with mesh regularity, that is, $\frac{|T|}{|\partial T|}\|\mu\|_{\partial T}^2 \lesssim \|\mu\|_T^2$ for all $T \in \mathcal{T}_\ell$, gives

$$\|\mathcal{U}_{\ell-1}\mu\|_\ell \lesssim \|\mathcal{U}_{\ell-1}\mu\|_\Omega \lesssim \|\mu\|_{\ell-1}$$

where the second inequality is (HM2). (IA1) now follows from the triangle inequality $\|\{\{\mathcal{U}_{\ell-1}\mu\}\}_F\|_\ell \leq \|\mathcal{U}_{\ell-1}\mu\|_\ell$.

Let us consider (IA2) for I_ℓ^3. On one hand, (HM4) says that

$$-\mathcal{Q}_{\ell-1}\gamma_{\ell-1}w = \nabla_\ell \theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\gamma_{\ell-1}w, \gamma_{\ell-1}w) = \nabla w$$

for $w \in \overline{V}^c$. This, in turn, implies that

$$\theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\gamma_{\ell-1}w, \gamma_{\ell-1}w) = w + c$$

for some constant $c \in \mathbb{R}$. On the other hand, the high-order reconstruction guarantees (cf. (3.9)) that the mean value of

$\theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\gamma_{\ell-1}w, \gamma_{\ell-1}w)$ equals the mean values of $\mathcal{U}_{\ell-1}\gamma_{\ell-1}w = w$ (through (HM4)), which implies that $c = 0$.

Next, we prove (IA1) for I_ℓ^3: let us consider a coarse element T. We have that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_{\ell-1}} \sum_{\overline{T}\in\mathcal{T}_\ell}^{\overline{T}\subset T} \sqrt{\frac{|\overline{T}|}{|\partial\overline{T}|}} \|\pi_\ell^p \theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\mu|_T, \mu|_{\partial\overline{T}})\|_{\partial T} \\
&\lesssim \sum_{T\in\mathcal{T}_{\ell-1}} \|\theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\mu|_T, \mu|_{\partial T})\|_T \\
&\leq \sum_{T\in\mathcal{T}_{\ell-1}} \|\theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\mu|_T, \mu|_{\partial T}) - \mathcal{U}_{\ell-1}\mu|_T\|_T + \|\mathcal{U}_{\ell-1}\mu|_T\|_\Omega \\
&\lesssim \sum_{T\in\mathcal{T}_{\ell-1}} h_T \| \underbrace{\nabla\theta_{\ell-1}^{p+1}(\mathcal{U}_{\ell-1}\mu|_T, \mu|_{\partial T})}_{=-Q\mu} \\
&\quad - \nabla\mathcal{U}_{\ell-1}\mu|_T\|_T + \|\mathcal{U}_{\ell-1}\mu|_T\|_\Omega \\
&\lesssim h_T\|\mu\|_{a_{\ell-1}} + \|\mathcal{U}_{\ell-1}\mu|_T\|_\Omega \lesssim \|\mu\|_{\ell-1}
\end{aligned}$$

Here, the first inequality uses the boundedness of the L^2-orthogonal projector along with discrete trace inequalities. The second passage follows inserting $\pm\mathcal{U}_{\ell-1}\mu$ into the norm and using a triangle inequality. The third inequality is obtained using a local Poincaré–Wirtinger inequality. The fourth passage is obtained using (HM3) and (HM1). The last inequality uses (HM6) and (HM2). □

8 | Numerical Experiments

8.1 | Experimental Setup

The numerical tests reported in this section are performed on the Poisson problem (2.1), on two- and three-dimensional domains. Namely, the fully elliptic problem shall be tested on the unit square and cube, and the low-regularity problem shall be tested on an L-shaped domain. The problem is discretized using the standard HHO method as described in Section 3, with the classical stabilizing term [6, Ex. 2.7 (2.22)]. Tests will span the polynomial degrees $p = 1, 2, 3$. Modal polynomial bases are used in cells and on faces to assemble the method (specifically, we use L^2-orthogonal Legendre bases).

Remark 8.1 (The lowest order case). The HHO method with $p = 0$ is not considered in the numerical tests as it does not verify the error estimates (6.9) and (6.10). Consequently, (HM5) does not hold and the multigrid method might not be uniformly convergent. This limitation does not appear to be just theoretical: it has been observed in practice in [24, 25], where suboptimal convergence is reported for $p = 0$ while the method exhibits optimal convergence for the higher orders, with no variation in any other parameter.

The multigrid method is constructed as described in Section 4.1. Section 7 describes the injection operators we consider. The V-cycle is carried out using pointwise Gauss–Seidel smoothing iterations, arranged so that both the smoothing step and the multigrid iteration remain symmetric. Specifically, for V(1,1), the pre-smoothing iteration is performed in the forward order and the post-smoothing iteration in the backward order. For V(2,2), the pre- and post-smoothing procedures involve a single forward iteration followed by a single backward iteration. The grid hierarchy is built by successive refinements of an initial simplicial mesh, and the condensed problem is assembled at every level. The coarsest system is solved with the Cholesky factorization. The stopping criterion relies on the backward error $\|\mathbf{r}\|_2/\|\mathbf{b}\|_2$, where $\mathbf{r}$ denotes the residual of the algebraic system, $\mathbf{b}$ the right-hand side, and $\|\cdot\|_2$ the standard Euclidean norm applied to the vector space of coordinates. In all experiments, convergence is considered to be reached once $\|\mathbf{r}\|_2/\|\mathbf{b}\|_2 < 10^{-6}$ is satisfied.

8.2 | Numerical Tests

8.2.1 | Full Regularity: The Unit Square

The Poisson problem is set up for the manufactured solution $u : (x, y) \mapsto \sin(4\pi x)\sin(4\pi y)$. The unit square is discretized by a hierarchy of 7 structured triangular meshes. For each problem solved on this hierarchy, the number of face unknowns is given in Table 1. The largest problem size we consider involves three million unknowns. Considering the injection operator I_ℓ^1, we notice that the V(1,1) cycle diverges regardless of the value of p. Increasing the number of smoothing steps restores convergence. With the V(2,2) cycle, the method converges for $p \in \{1, 2\}$ and still diverges for $p = 3$. The numbers of iterations are reported in Table 2 (the symbol ∞ indicating divergence or a number of iterations > 100). Their mild increase as the number of levels grows indicates asymptotic optimality. Table 3 now considers the injection operator I_ℓ^2. In V(1,1), contrary to the same cycle with I_ℓ^1, the method now converges for $p \in \{1, 2\}$, indicating better robustness of I_ℓ^2 compared with I_ℓ^1. The number of iterations of the V(2,2) cycle illustrates the uniform convergence of the method for all values of p. Finally, using the injection operator I_ℓ^3, the results of Table 4 show that the method converges uniformly with both cycles and for all values of p, making this injection operator the most robust amongst those considered here.

Remark 8.2 (Preconditioning). For the cases where the multigrid solver diverges, good performance may nonetheless be

TABLE 1 | For the square domain, number of face unknowns at each level and for each value of p.

Levels	3	4	5	6	7
$p = 1$	6016	24,320	97,792	392,192	1,570,816
$p = 2$	9024	36,480	146,688	588,288	2,356,224
$p = 3$	12,032	48,640	195,584	784,384	3,141,632

TABLE 2 | In the square domain, number of V(2,2) iterations with the injection operator I_ℓ^1.

Levels	3	4	5	6	7
$p = 1$	13	14	14	15	15
$p = 2$	36	38	38	39	39
$p = 3$	∞	∞	∞	∞	∞

TABLE 3 | In the square domain, number of iterations with the injection operator I_ℓ^2.

	V(1,1)					V(2,2)				
Levels	**3**	**4**	**5**	**6**	**7**	**3**	**4**	**5**	**6**	**7**
$p = 1$	24	25	25	26	26	13	13	14	14	14
$p = 2$	20	22	25	26	27	10	10	11	11	11
$p = 3$	∞	∞	∞	∞	∞	13	13	13	14	14

TABLE 4 | In the square domain, number of iterations with the injection operator I_ℓ^3.

	V(1,1)					V(2,2)				
Levels	**3**	**4**	**5**	**6**	**7**	**3**	**4**	**5**	**6**	**7**
$p = 1$	18	18	19	19	20	10	10	11	11	11
$p = 2$	17	17	17	17	18	9	10	10	10	10
$p = 3$	20	21	21	21	21	11	11	11	11	11

TABLE 5 | In the square domain, number conjugate gradient iterations, preconditioned with one V(1,1) multigrid cycle, using the injection operator I_ℓ^1.

Levels	**3**	**4**	**5**	**6**	**7**
$p = 1$	18	19	20	21	21
$p = 2$	20	22	24	25	26
$p = 3$	21	23	25	26	27

achieved if it is used as a preconditioner. For instance, consider the multigrid solver with V(1,1) cycle and the injection operator I_ℓ^1. When used as a solver, the method diverges for all values of p. On the other hand, used as a preconditioner for the conjugate gradient, the method only exhibits a mild dependency on the number of unknowns, as shown in Table 5.

8.2.2 | Low Regularity: The L-Shaped Domain

The computational domain is now the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [0,-1])$. The source function of the Poisson problem is set to zero. The Dirichlet boundary condition follows the manufactured exact solution $u : (r, \varphi) \mapsto r^{2/3} \sin(\frac{2}{3}\varphi)$, where (r, φ) represent polar coordinates of domain points. An unstructured Delaunay triangulation builds the coarsest mesh, and a classical refinement technique by edge bisection is used to build finer meshes successively. Although the theory would require the variable cycle, fixed cycles exhibit uniform convergence in practice. Therefore, the test results presented here are obtained with classical V(1,1) and V(2,2) cycles. Similarly to the preceding test case, the problem sizes are indicated in Table 6. The results, presented in Tables 7, 8, and 9 are also qualitatively similar. With the least robust injection operator, I_ℓ^1, the V(1,1) cycle diverges for all values of p, and the V(2,2) cycle is uniformly convergent only for $p = 1$ (cf. Table 7). With I_ℓ^2, V(1,1) provides a uniformly convergent solver up to $p = 2$, while V(2,2) converges uniformly for all

TABLE 6 | For the L-shaped domain, number of face unknowns at each level and for each value of p.

Levels	**3**	**4**	**5**	**6**	**7**
$p = 1$	4480	18,176	73,216	293,888	1,177,600
$p = 2$	6720	27,264	109,824	440,832	1,766,400
$p = 3$	8960	36,352	146,432	587,776	2,355,200

TABLE 7 | In the L-shaped domain, number of V(2,2) iterations with the injection operator I_ℓ^1.

Levels	**3**	**4**	**5**	**6**	**7**
$p = 1$	15	12	14	15	15
$p = 2$	∞	∞	∞	∞	∞
$p = 3$	∞	∞	∞	∞	∞

TABLE 8 | In the L-shaped domain, number of iterations with the injection operator I_ℓ^2.

	V(1,1)					V(2,2)				
Levels	**3**	**4**	**5**	**6**	**7**	**3**	**4**	**5**	**6**	**7**
$p = 1$	20	20	20	20	20	11	11	11	11	11
$p = 2$	17	17	17	17	17	9	9	9	9	9
$p = 3$	∞	∞	∞	∞	∞	11	11	11	11	11

$p \in \{1, 2, 3\}$ (cf. Table 8). Finally, using I_ℓ^3, Table 9 shows that the convergence is optimal for both cycles and all values of p. Based on these experiments, one can conclude that the convergence of the multigrid solver does not seem to suffer from the lower regularity of the solution. However, if we compare the number of iterations obtained with I_ℓ^2 to those obtained with I_ℓ^3, we remark that, robustness aside, I_ℓ^3 does not generally yield a faster convergence (see especially V(2,2)). Recall that I_ℓ^3 is obtained from I_ℓ^2 by inserting the higher order reconstruction operator. Since it is known that sufficient regularity is required for higher orders to have beneficial effects, the low regularity of the solution might explain this lack of noticeable improvement. Looking back at the fully regular problem on the square, one sees that the higher order reconstruction did improve convergence in that case (Table 3 vs. Table 4).

8.2.3 | 3D Test Case: The Cubic Domain

The unit cube is discretized by a Cartesian grid, where each element is decomposed into six geometrically similar tetrahedra, following [49, fig. 9]. Repeating the same procedure on successive levels of embedded Cartesian grids ensures that the subsequent tetrahedral grids are also embedded. Five levels are built, and the considered problem sizes are given in Table 10. The manufactured solution is $u : (x, y) \mapsto \sin(4\pi x) \sin(4\pi y) \sin(4\pi z)$. This problem is more challenging, so we restrict ourselves to the most efficient injection operator I_ℓ^3 and the V(2,2) cycle (V(1,1) diverges). The convergence results presented in Table 11, while still exhibiting asymptotically optimal convergence, also show higher numbers of iterations than in 2D. More numerical tests

TABLE 9 | In the L-shaped domain, number of iterations with the injection operator I_ℓ^3.

	V(1,1)					V(2,2)				
Levels	**3**	**4**	**5**	**6**	**7**	**3**	**4**	**5**	**6**	**7**
$p = 1$	16	16	16	16	16	9	9	10	11	11
$p = 2$	17	17	17	17	17	8	9	9	9	9
$p = 3$	20	20	20	20	20	10	10	10	10	10

TABLE 10 | For the cubic domain, number of face unknowns at each level and for each value of p.

Levels	3	4	5
$p = 1$	17,280	142,848	1,161,216
$p = 2$	34,560	285,696	2,322,432
$p = 3$	57,600	476,160	3,870,720

TABLE 11 | In the cubic domain, number of V(2,2) iterations with the injection operator I_ℓ^3.

Levels	3	4	5
$p = 1$	18	23	25
$p = 2$	23	24	22
$p = 3$	22	23	23

are available in [24], where it is shown that better convergence can be achieved for comparable cost through simple parameter tunings such as the use of cycles with post-smoothing only, non-alternating directions in the Gauss–Seidel sweeps, or blockwise Gauss–Seidel smoothers.

8.3 | Conclusions

We have constructed and rigorously analyzed a homogeneous multigrid method for HHO methods, which does not internally change the discretization scheme. To this end, we generalized the homogeneous multigrid framework for HDG methods and verified the resulting, more general assumptions for HHO methods. We verified our analytical findings numerically using several injection operators.

There are three important generalizations to our approach that we would like to address in future work: First, we would like to extend our theory to polyhedral meshes. Second, we would like to extend our multigrid framework to work as a preconditioner. Third, we want to extend this work to piece-wise constant approximations as detailed in Remark 6.2.

Acknowledgments

The work of D. Di Pietro was partially funded by the European Union (ERC Synergy, NEMESIS, project number 101115663). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

G. Kanschat has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2181/1- 390900948 (the Heidelberg STRUCTURES Excellence Cluster).

A. Rupp has been supported by the Academy of Finland's grant number 350101 *Mathematical models and numerical methods for water management in soils*, grant number 354489 *Uncertainty quantification for PDEs on hypergraphs*, grant number 359633 *Localized orthogonal decomposition for high-order, hybrid finite elements*, Business Finland's project number 539/31/2023 *3D-Cure: 3D printing for personalized medicine and customized drug delivery*, and the Finnish *Flagship of advanced mathematics for sensing, imaging and modelling*, decision number 358944.

The authors thank Ulrich Rüde (FAU Erlangen-Nuremberg) for his support. Open Access funding enabled and organized by Projekt DEAL.

Data Availability Statement

The experiments presented in this study can be reproduced using the provided resources. The code is open-source and available at https://github.com/pmatalon/fhhos4. Specific command lines required to reproduce the experiments are accessible on the author's personal webpage at https://pmatalon.github.io/publication/2023-hho-hdg-demo-framework.

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