



Power in plurality voting games

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Abstract

Simple games in partition function form are used to model voting situations where a coalition being winning or losing might depend on the way players outside that coalition organize themselves. Such a game is called a plurality voting game if in every partition there is at least one winning coalition. In the present paper, we introduce an equal impact power index for this class of voting games and provide an axiomatic characterization. This power index is based on equal weight for every partition, equal weight for every winning coalition in a partition, and equal weight for each player in a winning coalition. Since some of the axioms we develop are conditioned on the power impact of losing coalitions becoming winning in a partition, our characterization heavily depends on a new result showing the existence of such elementary transitions between plurality voting games in terms of single embedded winning coalitions. The axioms restrict then the impact of such elementary transitions on the power of different types of players.

Keywords Axiomatization · Power index · Plurality voting game · Winning coalition

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1 Introduction

In a simple game in partition function form, a worth is assigned to every so-called embedded coalition being a pair consisting of a coalition and a partition that contains this coalition. The worth is one (respectively zero) if the corresponding coalition is winning (respectively losing) in the partition. Such a game is called a *plurality voting game* (cf. van den Brink et al. 2021) if in every partition, there is at least one coalition that wins in that partition. So, in this case, winning does not necessarily mean that the coalition has a majority and can pass a bill, but simply that it considers itself as leading in a given coalitional configuration as represented by the partition.

The need for modeling parliamentary situations as plurality voting games might arise when, after an election, not only the biggest party declares itself the winner but also a majority consisting of ideological opponents. Moreover, whether the biggest party is a winner or not might depend on whether the ideological opposing parties form a coalition or not.

The assumption that in every partition there is at least one winning coalition comes from the following two observations. First, after every parliamentary election, before coalitions are formed and we thus consider the partition into singletons, there is usually at least one party that claims to be winner. As mentioned in the previous paragraph, this can be the party that got the most number of votes, but not necessarily. It can also be the biggest party of an ideological majority, a current government party that expected to do much worse, a new party that got more votes than expected, and so on. Even when all parties got the same number of votes, there usually are parties that expected to get less votes and thus consider themselves a winner. In theory, it is possible that all parties feel that they lost the election, but this is very unlikely. In partitions with at least one non-singleton, parties form a block, usually with the intention to become a winning coalition even when the individual parties in the coalition are losing.

We assume plurality voting games to be *monotonic*, both with respect to a coalition as well as to a certain type of externalities regarding other coalitions. Specifically, we assume that (i) a winning coalition cannot become losing when it grows, and (ii) there are negative externalities of other coalitions growing in the sense that bigger outside coalitions give ‘more resistance’ and thus such coalitions becoming bigger cannot turn a losing coalition into a winning one. This reflects that in a finer partition, there is ‘less resistance’ against the winning coalition.

Whereas in van den Brink et al. (2021) the focus is mainly on the representability of these games by party weights, the current paper studies power distributions in plurality voting games. Specifically, we introduce and axiomatize a power index for this class of games. In the formulation of our axioms, we utilize the power impact of losing embedded coalitions becoming winning in a partition; that is, our axiomatization heavily depends on a new result describing elementary transitions between plurality voting games in terms of single embedded winning coalitions. Specifically, we show that in every plurality voting game with at least

one losing embedded coalition, there is always a losing embedded coalition that can be turned into a winning one without affecting the monotonicity of the game.

The rest of the paper is organized as follows. In the next section we introduce the basic ingredients of plurality voting games as a special class of simple games in partition function form. Section 3 starts with the formal definition of a power index and presents the mentioned useful result (Proposition 1) concerning transitions between plurality voting games in terms of single winning embedded coalitions. These transitions are correspondingly used as to formulate our five axioms uniquely characterizing the proposed power index (Theorem 1). We conclude with some final remarks in Sect. 4. The Appendix contains all proofs.

2 Setup

We consider a finite set N of players. Each non-empty subset is called a *coalition*. A collection π of coalitions is a *coalition structure* if π is a partition of N , i.e., if all coalitions in π are non-empty, pair-wise disjoint, and their union is N . We denote by $\mathcal{P}$ the set of all partitions (coalition structures) of N and slightly abuse notation by writing $(T_1, T_2, \dots, T_k)$ for the partition $\{T_1, T_2, \dots, T_k\}$. For $i \in N$ and $\pi \in \mathcal{P}$, the notation $\pi(i)$ stands for the coalition in π containing player i . A pair $(S; \pi)$ consisting of a non-empty coalition $S \subseteq N$ and a partition $\pi \in \mathcal{P}$ with $S \in \pi$ is called an *embedded coalition*. The set of all embedded coalitions is $E = \{(S; \pi) \in (2^N \setminus \{\emptyset\}) \times \mathcal{P} \mid S \in \pi\}$.

A *simple game in partition function form* is a pair (N, v) , where the partition function $v : E \rightarrow \{0, 1\}$ is such that $v(N; (N)) = 1$. An embedded coalition $(S; \pi) \in E$ is called *winning* in the game (N, v) if and only if $v(S; \pi) = 1$. Otherwise, it is called *losing*. We sometimes say that coalition S is winning in partition π when $(S; \pi)$ is a winning embedded coalition. The set of all winning embedded coalitions in the game v is denoted by $E_W(v)$, while $E_W^\pi(v) = \{S \in \pi : (S; \pi) \in E_W(v)\}$ stands for the set of all coalitions which are winning in π .

This game form allows to model externalities of coalition formation. For instance, it can be that a coalition contained in two partitions π and π' is winning in π but losing in π' . Since the player set N is fixed, we often write a simple game in partition function form (N, v) by its partition function v . We use the following notion of inclusion, borrowed from Alonso-Meijide et al. (2017): For $(S'; \pi'), (S; \pi) \in E$, we say that $(S'; \pi')$ is *weakly included* in $(S; \pi)$, denoted by $(S'; \pi') \subseteq (S; \pi)$, if (i) $S' \subseteq S$, and (ii) for each $T \in \pi \setminus \{S\}$, there exists $T' \in \pi'$ with $T \subseteq T'$. A game v is then defined as *monotonic* if $(S'; \pi'), (S; \pi) \in E$ with $(S'; \pi') \subseteq (S; \pi)$ implies $v(S'; \pi') \leq v(S; \pi)$. This monotonicity notion reflects (i) a nonnegative effect when a coalition grows, and (ii) an idea of negative externalities when players outside a coalition form larger coalitions. In particular, it implies that when a coalition is winning in a partition, then it is winning in every finer partition that contains this coalition. In other words, the idea expressed here is that in a finer partition there is ‘less resistance’ against the winning coalition. Clearly, a winning coalition can become losing in a coarser

partition since other players forming coalitions might give a ‘stronger resistance’ against the winning coalition, or make the winning coalition more likely to ‘break down’.

We call a simple game in partition function form v a *plurality voting game* if (i) it is monotonic, and (ii) for each $\pi \in \mathcal{P}$ we have $v(S; \pi) = 1$ for at least one $S \in \pi$. We assume that the player set N is fixed and of size¹ $|N| = n \geq 3$, and identify a plurality voting game by its partition function. The set of all plurality voting games on the player set N is denoted by $\mathcal{G}^N$.

We close this section by the following example of a plurality voting game.

Example 1 Let $N = \{1, 2, 3\}$ and $v \in \mathcal{G}^{\{1,2,3\}}$ be as defined below where the underlined coalitions are winning in the corresponding partition.

$$\begin{array}{c} v \\ \underline{123} \\ \underline{12}, 3 \\ 13, \underline{2} \\ \underline{23}, \underline{1} \\ \underline{1}, \underline{2}, 3 \end{array}$$

The monotonicity of the game v requires, for example, (i) player 1 to be winning in the partition into singletons since he is winning against the coalition consisting of players 2 and 3, and (ii) the coalition consisting of players 1 and 3 to be winning against player 2 since player 1 is winning in the partition into singletons. Observe additionally that, although both players 1 and 2 are winning in the latter partition, only one of them is winning against a two-player coalition.

3 Axioms on power indices and characterization result

A *power index* for plurality voting games is a mapping $f : \mathcal{G}^N \rightarrow \mathbb{R}_+^n$ satisfying $\sum_{i \in N} f_i(v) = 1$ for each $v \in \mathcal{G}^N$. We interpret the real number $f_i(v) \in [0, 1]$ as the power of player i in the game v .

The axioms we introduce in this section concern implications when the only difference between two plurality voting games is a single winning embedded coalition. Hence, the first question we have to answer is if for every plurality voting game that has at least one losing embedded coalition, there is a losing embedded coalition such that turning it into a winning one, we still have a plurality voting game; specifically, monotonicity requires that the new winning coalition is also winning against ‘less resistance’. Our first result whose proof is relegated to the Appendix, answers this question in the positive.

¹ Notice that, when $n = 2$, whether an embedded coalition is winning or losing in a partition is in a trivial way independent of how the rest of the players are organized. This is the reason for considering only plurality games with at least three players.

Proposition 1 Let $v \in \mathcal{G}^N$ be such that $|E_W(v)| < |E|$. Then there exist $(S; \pi) \in E \setminus E_W(v)$ and $v' \in \mathcal{G}^N$ such that $E_W(v') = E_W(v) \cup \{(S; \pi)\}$

In other words, starting from any plurality voting game in which not all embedded coalitions are winning, there is always a path of games leading to the unique game in which all embedded coalitions are winning; along such a path, each next game differs from its direct predecessor only by one winning embedded coalition.

Example 2 Considering Example 1, one can make player 2 winning against the coalition consisting of players 1 and 3, as well as player 3 in the partition into singletons. However, player 3 cannot be made winning against the coalition of 1 and 2, as monotonicity would require this player to be winning in the partition into singletons as well.

Let us now introduce the requirements we impose on a power index by using the following additional notation. For $(S; \pi) \in E$, any two games $v, v' \in \mathcal{G}^N$ with $E_W(v') = E_W(v) \cup \{(S; \pi)\}$, and any $T \subseteq N$, we set $\Delta_i^f(v, v') = f_i(v') - f_i(v)$ and $\Delta_T^f(v, v') = \sum_{i \in T} \Delta_i^f(v, v')$. That is, $\Delta_i^f(v, v')$ displays the change in the power of player $i \in N$ (as measured by f) when a single embedded coalition which is losing in v becomes winning in v' . Correspondingly, $\Delta_T^f(v, v')$ stands for the change in the power of coalition T . For $S, T \in \pi$, we finally set $\Delta_{ST}^f(v, v') = \Delta_S^f(v, v') - \Delta_T^f(v, v')$ saying how far apart are the power change in S and the power change in T when S becomes the new winning coalition in the partition π .

We are ready now to present our axioms.

Unanimity (U): For all $v \in \mathcal{G}^N$: $E_W(v) = E$ implies $f_i(v) = f_j(v)$ for all $i, j \in N$.

Internal Impact (II): For all $v, v' \in \mathcal{G}^N$: $E_W(v') = E_W(v) \cup \{(S; \pi)\}$ implies $\Delta_i^f(v, v') = \Delta_j^f(v, v')$ for all $i, j \in T \in E_W^\pi(v')$.

External Impact (EI): For all $v, v' \in \mathcal{G}^N$: $E_W(v') = E_W(v) \cup \{(S; \pi)\}$ implies $\Delta_Q^f(v, v') = \Delta_R^f(v, v')$ for all $Q, R \in E_W^\pi(v)$.

Null Impact (NI): For all $v, v' \in \mathcal{G}^N$: $E_W(v') = E_W(v) \cup \{(S; \pi)\}$ implies $\Delta_i^f(v, v') = 0$ for all $i \in H \in \pi \setminus E_W^\pi(v')$.

Power Difference (PD): For all $v, v' \in \mathcal{G}^N$: $E_W(v') = E_W(v) \cup \{(S; \pi)\}$ implies $\sum_{T \in E_W^\pi(v)} \Delta_{ST}^f(v, v') = \frac{1}{|\mathcal{P}|}$.

Unanimity requires equal power in case all embedded coalitions are winning in the corresponding game and thus, it can be seen as a weak symmetry axiom.

Internal Impact requires that a losing embedded coalition becoming winning (i) has the same impact on the powers of the players in that winning coalition and (ii) for each other winning coalition in the corresponding partition (i.e., the partition that contains this new winning coalition) it also has the same impact on the power of the players inside each such winning coalition. Part (i) can be seen as some kind of Myerson (1977a) fairness applied to this game model.² Part (ii) of this axiom

² Myerson (1977a)'s fairness is introduced for communication graph games and requires that breaking a link between two players in a communication graph has the same impact on the payoff of these two

extends this idea also to players in other winning coalitions in the corresponding partition since also from the perspective of each of them the situation changed in a 'symmetric' way.

External Impact requires that the impact of a losing embedded coalition becoming winning is the same for each other winning coalition in the corresponding coalition in the sense that the sum of the powers of all players in each such winning coalition changes by the same amount. This can also be seen as a kind of fairness as above, but then applied on the coalition level.

Null Impact is a rather strong axiom that requires that a losing embedded coalition becoming winning has no effect on the powers of the players in losing coalitions in the corresponding partition. Although using a similar argument as the fairness criteria mentioned above it seems reasonable that within each such losing coalition the changes in power are the same, requiring the effect to be zero is an extreme case. However, if we consider the partition as a coalition structure (cf. Aumann and Drèze 1974 and Owen 1977) isolated from the rest, then a 'null' agent is powerless whatever is the configuration of winning coalitions in the coalition structure.

Finally, Power Difference is a balance axiom in the style of the collusion neutrality axioms for TU-games in Haller (1994) and Malawski (2002). They speak about a pairwise power difference axiom, where a certain change in the game (collusion of players) does not change the sum of the payoffs of the two players involved. We consider changes in the 'game' in the sense of losing embedded coalitions becoming winning in their partitions. A similar requirement as that of collusion neutrality would require that the sum of the powers of the new winning coalition and each other winning coalition in that partition would be zero. In our context this is, however, a very strong requirement. Therefore, we modify this requirement in two ways: first, we weaken this idea by requiring the sum of these power differences to be constant and second, this constant is not zero, but $\frac{1}{|P|}$. This reflects an equal importance of the partitions in the following sense. Since a power index is non-negative and the individual powers add up to one, it seems reasonable that the maximal power switch that can be obtained by any change in the game is 1. Then, making a minimal change (i.e., changing the status of only one coalition) in one partition being equal to $\frac{1}{|P|}$ could indeed be seen as reflecting equal importance of the partitions. (This is an extreme case, and we get back to this in the Conclusion).

For $i \in N$ and $v \in \mathcal{G}^N$, let $\mathcal{P}_i^v = \{\pi \in \mathcal{P} : \pi(i) \in E_W^\pi(v)\}$ be the set of all partitions π , where player i belongs to a winning coalition in π . We define the *equal impact power index* f^* such that, for each $v \in \mathcal{G}^N$ and $i \in N$,

$$f_i^*(v) = \frac{1}{|P|} \sum_{\pi \in \mathcal{P}_i^v} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|}. \quad (1)$$

Footnote 2 (continued)

players. Notice that we apply fairness to a coalition that might contain more than two players, and in this respect our axiom looks more similar to the fairness in Algaba et al. (2001) applied with respect to union stable systems where a link (called support) might contain more than two players.

This index can be seen as an application of the principle of insufficient reason in the sense that the allocation of power is based on assigning equal shares. Specifically, f^* follows a ‘three-step’ procedure: (i) the assumption of equal weight/importance for every partition results in the allocation of $\frac{1}{|\mathcal{P}|}$ over the players in every partition; (ii) in any partition, this number is equally allocated over the winning coalitions in the partition (recall that in a plurality voting game there is at least one winning coalition in every partition); and (iii) the power assigned to a winning coalition in a partition is equally allocated over the players in that winning coalition.

Another way for measuring power in simple partition function form games can be generated by applying the principle of insufficient reason only with respect to minimal winning coalitions. As shown in Alonso-Mejide et al. (2017) in the context where there can be partitions without any winning coalition, such an application results in a ‘two-step’ procedure of first sharing the full power equally among all minimal winning coalitions and second allocating the power assigned to each minimal winning coalition equally over the players in that coalition.

In order to show that f^* is a power index, fix $v \in \mathcal{G}^N$ and observe that, due to $E_W^{\{N\}}(v) = \{N\}$, the lowest value $f_i^*(v)$ can take for $i \in N$ is when player i belongs to a winning coalition only in the partition (N) ; hence, in such a case, we have $f_i^*(v) = \frac{1}{n \cdot |\mathcal{P}|} > 0$.

On the other hand,

$$\begin{aligned} \sum_{i \in N} f_i^*(v) &= \sum_{i \in N} \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^v} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} \\ &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} \sum_{i \in N, \pi \in \mathcal{P}_i^v} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} \\ &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} \sum_{S \in E_W^\pi(v)} \frac{1}{|E_W^\pi(v)|} \\ &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} 1 = \frac{1}{|\mathcal{P}|} \cdot |\mathcal{P}| = 1 \end{aligned}$$

and thus, f^* is indeed a power index.

We have the following characterization result.

Theorem 1 *A power index f satisfies U, II, EI, NI, and PD if and only if $f = f^*$*

The proof of Theorem 1 is relegated to the Appendix.

4 Conclusion

The study of plurality voting games combines ideas from the analysis of simple games (cf. Shapley 1962) and insights from the general literature on partition function form games initiated by Thrall (1962) and Thrall and Lucas (1963).³ Specifically, the present paper contributes to the strand of literature devoted to power indices in simple games and to values for games in partition function form. Within this strand of literature, the focus is predominantly on extending the Shapley value to games with externalities (e.g., Myerson 1977b; Albizuri et al. 2005; Macho-Stadler et al. 2007; McQuillin 2009; Dutta et al. 2010; Grabisch and Funaki 2012). It is thus not surprising that the proposed power indices in this context are based on the marginal contributions of the players. Specifically, the classes of games on which power indices have been defined and axiomatically characterized include the class of games where in every partition there can be at most one winning coalition (cf. Bolger 1986) or the class of games where there might be no winning coalitions in some partitions (cf. Alonso-Meijide et al. 2017; Álvarez-Mozos et al. 2017). In contrast to Bolger (1986) and Álvarez-Mozos et al. (2017), but similar to Alonso-Meijide et al. (2017), the axioms we utilize in the present paper are not based on players' movements from one coalition to another within a partition but rather consider the direct impact on power of losing embedded coalitions becoming winning in a partition. The formulation of such axioms is based on a new result showing that in every plurality voting game with at least one losing embedded coalition, there is always a losing embedded coalition that can be turned into a winning one without affecting the monotonicity of the game.

We remark that our axioms also can be applied to the class of simple games in partition function form (with possibly partitions with no winning coalition) where the requirement of Power Difference should be applied only when the original game (before turning a losing coalition into a winning one) has at least one winning coalition. Redefining a power index such that the sum of powers always equals the fraction of partitions that have at least one winning coalition, these axioms could be used for characterizing the power index that has the same formula as in Equation (1).

An interesting question for future research, along with the study of the logical independence of the axioms, is to provide axiomatizations of power indices for so-called *decisive* plurality voting games, being games such that every partition contains exactly one winning coalition. This is challenging since we cannot just turn one losing coalition into a winning one without destroying the decisiveness of the game. So, we can look for axioms where the replacement of the winning coalition in one partition by another coalition in the same partition will have enough bite.

Another question for future research is to weaken some of the axioms and characterize the class of power indices that we obtain in this way. Two candidates are the following. Null Impact can be weakened by requiring that turning one losing coalition into a winning coalition in a partition has the same impact on the power of

³ We refer the reader to Lucas and Marcelli (1978) for a study of general properties of partition function form games and to Koczy (2018) for a detailed literature survey.

losing coalitions in that partition. Also, we might consider weakening Power Difference by requiring the total power shift (right hand side of the equation) to be any fraction between 0 and $\frac{1}{|P|}$. In Sect. 3 we already explained why $\frac{1}{|P|}$ can be considered the maximal power shift as a consequence of turning one losing coalition into a winning one. Obviously, 0 can be considered a lower bound if the new winning coalition is assumed to be benefitting at least as much as the other coalitions in the partition of becoming winning. Besides considering these two weakenings separately, we can consider both weakenings at the same time.

Appendix

This section contains the proofs of Proposition 1 and Theorem 1. In what follows, we introduce some additional notation. For partition $\pi \in \mathcal{P}$ and set of players $S \subset N$, we denote by $\pi_S = \{T \cap S \mid T \in \pi : T \cap S \neq \emptyset\}$ the partition of S induced by π . Further, we will often write $\{T_1, \dots, T_k, \pi_S\}$ for the partition $\pi = \{T_1, \dots, T_k, S_1, \dots, S_p\}$ if $\pi_S = \{S_1, \dots, S_p\}$. The *atomistic partition* $\pi^a \in \mathcal{P}$ is defined by $\pi^a(i) = \{i\}$ for each $i \in N$. Consequently, $(T_1, T_2, \dots, T_k, \pi^a)$ stands for the partition of N consisting of the coalitions T_1 to T_k and all players in $N \setminus (T_1 \cup T_2 \cup \dots \cup T_k)$ being singletons. When, for instance, $k = 1$ and $T_1 = \{i, j, \ell\}$, we slightly abuse notation and write $(ij\ell, \pi^a)$ instead of $(\{i, j, \ell\}, \pi^a)$.

Proof of Proposition 1

We start by introducing the notion of a useful embedded coalition and show in Lemma 1 the existence of at least one such coalition in a game $v \in \mathcal{G}^N$ for the case when $|E_W^{\pi^a}(v)| \leq n - 1$ (i.e., when there is at least one losing singleton coalition in the atomistic partition π^a). In Lemma 4 we show this for the case when $|E_W^{\pi^a}(v)| = n$ (i.e., when all singleton coalitions are winning in π^a). Such coalitions for the mentioned cases are then used in Lemma 2 and in Lemma 5 as to determine the corresponding elementary transitions between two plurality voting games and complete the proof.

For $v \in \mathcal{G}^N$ with $|E_W^{\pi^a}(v)| \leq n - 1$, we call an embedded coalition $(T; (T, \pi^a))$ *s-useful* if the following two conditions hold:

- (1) $v(T; (T, \pi^a)) = 0$, and
- (2) $v(T \cup \{j\}; (T \cup \{j\}, \pi^a)) = 1$ for each $j \in N \setminus T$.

That is, T is a losing coalition when all other players are singletons and it becomes winning in the corresponding new partition when a player from $N \setminus T$ joins T .⁴ For $v \in \mathcal{G}^N$, $L_s^k(v)$ is the set of all s -useful embedded coalitions $(T; (T, \pi_-^a))$ with $|T| = k$. Notice that $k \in \{1, 2, \dots, n-1\}$, the reason being that N is winning in the unique partition containing it (due to $v \in \mathcal{G}^N$) and thus, the embedded coalition $(N; (N))$ with $|N| = n$ cannot be s -useful.

Lemma 1 *Let $v \in \mathcal{G}^N$ with $|E_W^{\pi^a}(v)| \leq n-1$. Then there exists $k \in \{1, 2, \dots, n-1\}$ such that $L_s^k(v) \neq \emptyset$*

Proof Observe first that $|E_W^{\pi^a}(v)| = n-1$ implies $L_s^1(v) \neq \emptyset$. As to see it, let $\{i\}$ be the unique losing coalition in π^a . By $v \in \mathcal{G}^N$ and $v(j; (j, \pi_-^a)) = 1$ for each $j \in N \setminus \{i\}$, $v(ij; (ij, \pi_-^a)) = 1$ holds for each $j \in N \setminus \{i\}$ and thus, $L_s^1(v) = \{(i; \pi^a)\}$ follows.

Suppose now that $|E_W^{\pi^a}(v)| < n-1$ and $L_s^k(v) = \emptyset$ holds for each $k \in \{1, 2, \dots, n-2\}$. We show that $L_s^{n-1}(v) \neq \emptyset$. For this, fix $i \in N$ such that $v(i; (i, \pi_-^a)) = 0$ and notice that $L_s^1(v) = \emptyset$ implies the existence of $k_i \in N \setminus \{i\}$ such that $v(ik_i; (ik_i, \pi_-^a)) = 0$.⁵ By $L_s^2(v) = \emptyset$, there exists $\ell_i \in N \setminus \{i, k_i\}$ such that $v(ik_i\ell_i; (ik_i\ell_i, \pi_-^a)) = 0$. Repeating the argument for each $k \in \{3, \dots, n-2\}$ leads to the existence of some $j \in N$ such that $v(N \setminus \{j\}; (N \setminus \{j\}, \{j\})) = 0$. By $v \in \mathcal{G}^N$, $v(N; (N)) = 1$ follows. We have then $(N \setminus \{j\}; (N \setminus \{j\}, \{j\})) \in L_s^{n-1}(v)$. $\square$

Lemma 2 *Let $v \in \mathcal{G}^N$ be such that $|E_W^{\pi^a}(v)| \leq n-1$. Then there exist $(S; \pi) \in E \setminus E_W(v)$ and $v' \in \mathcal{G}^N$ such that $E_W(v') = E_W(v) \cup \{(S; \pi)\}$*

Proof Let $v \in \mathcal{G}^N$ be as above. By Lemma 1, there exists $k \in \{1, 2, \dots, n-1\}$ such that $L_s^k(v) \neq \emptyset$. Let k^* be the smallest integer such that $L_s^{k^*}(v) \neq \emptyset$. Fix an embedded coalition $(T; (T, \pi_-^a)) \in L_s^{k^*}(v)$. Let us show that $E_W(v') = E_W(v) \cup \{(T; (T, \pi_-^a))\}$ implies $v' \in \mathcal{G}^N$.

Suppose that this is not the case. Then there exist two embedded coalitions $(R'; \pi')$ and $(R; \pi)$ with $(R'; \pi') \subseteq (R; \pi)$ such that $v'(R'; \pi') > v'(R; \pi)$ holds. The latter inequality implies that, in the game v' , R' is winning in π' and R is losing in π . It follows from the way in which v' was constructed that $E_W(v) \subset E_W(v')$, and thus $v'(R; \pi) = 0$ implies $v(R; \pi) = 0$. By $v \in \mathcal{G}^N$ and $(R'; \pi') \subseteq (R; \pi)$, $v(R'; \pi') = 0$ follows. Since the only losing embedded coalition in v which is winning in v' is $(T; \pi_-^a)$, it must be that $(R'; \pi') = (T; (T, \pi_-^a))$.

⁴ We note that s -useful embedded coalitions are exactly those that are maximal elements of chains in the partial order induced by the weak inclusion $\subseteq$ over the set of losing coalitions embedded in partitions with at most one non-singleton.

⁵ Notice that $n = 3$ would imply $v(N; (N)) = 1$ and thus, $(ik_i; (ik_i, \ell)) \in L_s^2(v)$ with ℓ being the third player in the game v would directly follow.

Observe that $(T; (T, \pi_-^a)) = (R'; \pi') \subseteq (R; \pi) = (R; (R, \pi_{N \setminus R}))$ implies $T \subseteq R$ and $\pi_{N \setminus R} = \pi_{N \setminus R}^a$. We now show that in each of the two cases $T = R$ and $T \subset R$ we reach a contradiction.

- (1) $T = R$ implies $(R'; \pi') = (R; \pi)$ which is in contradiction to $v'(R'; \pi') > v'(R; \pi)$.
- (2) $T \subset R$ leads to $v'(T \cup \{i\}; (T \cup \{i\}, \pi_-^a)) = v(T \cup \{i\}; (T \cup \{i\}, \pi_-^a)) = 1$ for each $i \in R \setminus T$ by construction and $(T; (T, \pi_-^a)) \in L_s^{k^*}(v)$. Repeatedly applying the fact that $v \in \mathcal{G}^N$, we get $v'(R; (R, \pi_-^a)) = 1$. By $(R; \pi) = (R; (R, \pi_-^a))$, we have a contradiction to $v'(R'; \pi') = 1$ and $v'(R'; \pi') > v'(R; \pi)$.

□

We have the following helpful result.

Lemma 3 *If $v \in \mathcal{G}^N$ is such that, for each $D \subseteq N$, all coalitions in (D, π_-^a) are winning in v , then $E_W(v) = E$*

Proof Suppose not, i.e., there is an embedded coalition $(S; \pi)$ such that $S \neq N$ and $v(S; \pi) = 0$. By the definition of a plurality voting game, $v(T; \pi) = 1$ for some $T \in \pi$. Take $i \in S$ and notice that, by the monotonicity of v , $v(\{i\}; (N \setminus \{i\}, \{i\})) = 0$ follows. Thus, we have a contradiction to $\{i\}$ being a winning coalition in $(N \setminus \{i\}, \{i\})$. □

Consider $v \in \mathcal{G}^N$ with $|E_W^{\pi^a}(v)| = n$ and fix $D \subset N$. An embedded coalition $(T; (D, T, \pi_-^a))$ is D -useful if the following two conditions hold:

- (1) $v(T; (D, T, \pi_-^a)) = 0$, and
- (2) $v(T \cup \{j\}; (D, T \cup \{j\}, \pi_-^a)) = 1$ for each $j \in N \setminus (D \cup T)$

In other words, T is a losing coalition when all players from $N \setminus (D \cup T)$ are singletons and it becomes winning in the corresponding new partition when a player from $N \setminus (D \cup T)$ joins T .⁶ For $v \in \mathcal{G}^N$, we collect in the set $L_D^m(v)$ all D -useful embedded coalitions $(T; (D, T, \pi_-^a))$ with $|T| = m$. Notice that $m \in \{1, 2, \dots, n - |D|\}$, the reason being that the largest coalition that can be losing in a partition containing the coalition D is $N \setminus D$ (of size $n - |D|$).

Lemma 4 *Let $v \in \mathcal{G}^N$ be such that $|E_W^{\pi^a}(v)| = n$ and $E_W(v) \neq E$. Then there exist $D \subset N$ and $m \in \{1, 2, \dots, n - |D|\}$ such that $L_D^m(v) \neq \emptyset$*

Proof Observe that, by $v \in \mathcal{G}^N$ and $|E_W^{\pi^a}(v)| = n$, $v(T; (T, \pi_-^a)) = 1$ holds for each $T \subset N$. By $E_W(v) \neq E$ and Lemma 3, there exist $D \subset N$ and $i \in N$ such that

⁶ Similar to our remark in footnote 4, D -useful embedded coalitions can be seen as maximal elements of chains in a corresponding partial order.

$(i; (D, i, \pi_-^a))$ is a losing embedded coalition in v (and, of course, $(D; (D, \pi_-^a))$ is a winning embedded coalition in v).

If $\{i\}$ is the only losing singleton coalition in (D, π_-^a) , then $L_D^1(v) \neq \emptyset$. As to see it, notice that by $v \in \mathcal{G}^N$, $v(ij; (D, ij, \pi_-^a)) = 1$ holds for each $j \in N \setminus (D \cup \{i\})$ and thus, $L_D^1(v) = \{(i; (D, i, \pi_-^a))\}$ follows.

Suppose now that there are at least two losing singleton coalitions in (D, π_-^a) and $L_D^m(v) = \emptyset$ holds for each $m \in \{1, 2, \dots, n - |D| - 2\}$. We show that either $L_D^{n-|D|-1}(v) \neq \emptyset$ or $L_D^{n-|D|}(v) \neq \emptyset$. For this, fix $i \in N$ such that $v(i; (D, i, \pi_-^a)) = 0$ and notice that $L_D^1(v) = \emptyset$ implies the existence of $k_i \in N \setminus (D \cup \{i\})$ such that $v(ik_i; (D, ik_i, \pi_-^a)) = 0$. By $L_D^2(v) = \emptyset$, there exists $\ell_i \in N \setminus (D \cup \{i, k_i\})$ such that $v(ik_i\ell_i; (D, ik_i\ell_i, \pi_-^a)) = 0$. Repeating the argument for each $m \in \{3, \dots, n - |D| - 2\}$ leads to the existence of some $j \in N$ such that $v(N \setminus (D \cup \{j\}); (D, N \setminus (D \cup \{j\}), \{j\})) = 0$. If $v(N \setminus D; (D, N \setminus D)) = 1$, then $(N \setminus (D \cup \{j\}); (D, N \setminus (D \cup \{j\}), \{j\})) \in L_D^{n-|D|-1}(v)$ follows. Otherwise, if $v(N \setminus D; (D, N \setminus D)) = 0$, then $(N \setminus D; (D, N \setminus D)) \in L_D^{n-|D|}(v)$ trivially holds due to $N \setminus (D \cup (N \setminus D)) = \emptyset$ and the definition of a D -useful embedded coalition. $\square$

Lemma 5 *Let $v \in \mathcal{G}^N$ be such that $|E_W^{\pi^a}(v)| = n$ and $E_W(v) \neq E$. Then there exist $(S; \pi) \in E \setminus E_W(v)$ and $v' \in \mathcal{G}^N$ such that $E_W(v') = E_W(v) \cup \{(S; \pi)\}$*

Proof Let $v \in \mathcal{G}^N$ be as above. By Lemma 4, there exist $D \subset N$ and $m \in \{1, 2, \dots, n - |D|\}$ such that $L_D^m(v) \neq \emptyset$. Let m^* be the smallest integer such that $L_D^{m^*}(v) \neq \emptyset$. Fix an embedded coalition $(T; (D, T, \pi_-^a)) \in L_D^{m^*}(v)$. Let us show that $E_W(v') = E_W(v) \cup \{(T; (D, T, \pi_-^a))\}$ implies $v' \in \mathcal{G}^N$.

Suppose this is not the case. Then there exist two embedded coalitions $(R'; \pi')$ and $(R; \pi)$ with $(R'; \pi') \subseteq (R; \pi)$ such that $v'(R'; \pi') > v'(R; \pi)$ holds. The latter inequality implies that, in the game v' , R' is winning in π' and R is losing in π . It follows from the way in which v' was constructed that $v'(R; \pi) = 0$ implies $v(R; \pi) = 0$ since $E_W(v) \subset E_W(v')$. By $v \in \mathcal{G}^N$ and $(R'; \pi') \subseteq (R; \pi)$, $v(R'; \pi') = 0$ follows. Since the only losing embedded coalition in v which is winning in v' is by construction $(T; (D, T, \pi_-^a))$, we conclude that $(R'; \pi') = (T; (D, T, \pi_-^a))$ should be the case. Observe that $(T; (D, T, \pi_-^a)) = (R'; \pi') \subseteq (R; \pi) = (R; (R, \pi_{N \setminus R}))$ implies $T \subseteq R$ and $\pi_{N \setminus R} \in \{\pi_{N \setminus R}^a, (D, \pi_{N \setminus (R \cup D)}^a)\}$. There are three possible cases to be considered.

- (1) $\pi_{N \setminus R} = \pi_{N \setminus R}^a$. As explained above, $v'(R; \pi) = 0$ implies $v(R; \pi) = 0$; that is, we have $v(R; (R, \pi_-^a)) = 0$. However, by $v \in \mathcal{G}^N$ and $|E_W^{\pi^a}(v)| = n$, $v(R; (R, \pi_-^a)) = 1$ should hold, a contradiction.
- (2) $T = R$ and $\pi_{N \setminus R} = (D, \pi_{N \setminus (R \cup D)}^a)$. In this case $(R'; \pi') = (R; \pi)$ follows, a direct contradiction to $v'(R'; \pi') > v'(R; \pi)$.
- (3) $T \subset R$ and $\pi_{N \setminus R} = (D, \pi_{N \setminus (R \cup D)}^a)$. By construction, $v'(T \cup \{i\}; (T \cup \{i\}, D, \pi_-^a)) = v(T \cup \{i\}; (T \cup \{i\}, D, \pi_-^a)) = 1$ holds for each

$i \in R \setminus T$. Repeatedly applying the fact that $v \in \mathcal{G}^N$, we get $v'(R; (R, D, \pi_-^a)) = 1$. By $(R; \pi) = (R; (R, D, \pi_-^a))$, we have a contradiction to $v'(R'; \pi') = 1$ and $v'(R'; \pi') > v'(R; \pi)$.

□

Proof of Theorem 1

We show first that the power index f^* as defined in Sect. 3 (see (1)) satisfies our five axioms.

Unanimity Let $v \in \mathcal{G}^N$ be such that $E_W(v) = E$. Notice that in such a case and for each $i \in N$, we have $\mathcal{P}_i^v = \mathcal{P}$. Clearly then, the number of times each player belongs to a coalition of a given size in a partition is the same. In other words, U is satisfied with $f_i^*(v) = \frac{1}{n}$ for each $i \in N$ (due to $\sum_{i \in N} f_i^*(v) = 1$).

In order to show that f^* also satisfies the other four axioms, suppose the games $v, v' \in \mathcal{G}^N$ are such that $E_W(v') = E_W(v) \cup (S; \pi^*)$.

Internal Impact For each $i \in T \in E_W^{\pi^*}(v)$ we have

$$f_i^*(v) = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^v} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^v \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} + \frac{1}{|\mathcal{P}|} \frac{1}{|E_W^{\pi^*}(v)| \cdot |T|}$$

and

$$\begin{aligned} f_i^*(v') &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^{v'}} \frac{1}{|E_W^\pi(v')| \cdot |\pi(i)|} = \frac{1}{|\mathcal{P}|} \left(\sum_{\pi \in \mathcal{P}_i^v \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} + \frac{1}{|E_W^{\pi^*}(v')| \cdot |T|} \right) \\ &= \frac{1}{|\mathcal{P}|} \left(\sum_{\pi \in \mathcal{P}_i^v \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} + \frac{1}{(|E_W^{\pi^*}(v)| + 1) \cdot |T|} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \Delta_i^{f^*}(v, v') &= f_i^*(v') - f_i^*(v) \\ &= \frac{1}{|\mathcal{P}|} \left(\frac{1}{(|E_W^{\pi^*}(v)| + 1) \cdot |T|} - \frac{1}{|E_W^{\pi^*}(v)| \cdot |T|} \right). \end{aligned} \quad (2)$$

We have further for each $i \in S$ that

$$f_i^*(v) = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^v} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^v \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|}$$

and

$$\begin{aligned}
f_i^*(v') &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^*} \frac{1}{|E_W^\pi(v')| \cdot |\pi(i)|} \\
&= \frac{1}{|\mathcal{P}|} \left(\sum_{\pi \in \mathcal{P}_i^* \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} + \frac{1}{|E_W^{\pi^*}(v')| \cdot |S|} \right) \\
&= \frac{1}{|\mathcal{P}|} \left(\sum_{\pi \in \mathcal{P}_i^* \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} + \frac{1}{(|E_W^{\pi^*}(v)| + 1) \cdot |S|} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta_i^{f^*}(v, v') &= f_i^*(v') - f_i^*(v) \\
&= \frac{1}{|\mathcal{P}| \cdot |S| \cdot (|E_W^{\pi^*}(v)| + 1)}.
\end{aligned} \tag{3}$$

We conclude from (2) and (3) that $\Delta_i^{f^*}(v, v')$ is independent of i for each player i belonging to a winning coalition in the partition π^* and thus, II is satisfied.

External Impact Take $T \in E_W^{\pi^*}(v)$ and notice that in view of (2) we have

$$\begin{aligned}
\Delta_T^{f^*}(v, v') &= \sum_{i \in T} \Delta_i^{f^*}(v, v') \\
&= \frac{|T|}{|\mathcal{P}|} \left(\frac{1}{(|E_W^{\pi^*}(v)| + 1) \cdot |T|} - \frac{1}{|E_W^{\pi^*}(v)| \cdot |T|} \right) \\
&= \frac{|T|}{|\mathcal{P}|} \cdot \frac{|E_W^{\pi^*}(v)| - |E_W^{\pi^*}(v)| - 1}{(|E_W^{\pi^*}(v)| + 1) \cdot |E_W^{\pi^*}(v)| \cdot |T|} \\
&= \frac{-1}{|\mathcal{P}| \cdot (|E_W^{\pi^*}(v)| + 1) \cdot |E_W^{\pi^*}(v)|},
\end{aligned}$$

i.e., $\Delta_T^{f^*}(v, v')$ is independent of T . We conclude that EI is satisfied.

Null Impact Take $i \in T \in \pi^* \setminus E_W(v')$. We have

$$f_i^*(v) = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^*} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|} = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^* \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|}$$

and

$$\begin{aligned}
 f_i^*(v') &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^{v'}} \frac{1}{|E_W^\pi(v')| \cdot |\pi(i)|} = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^{v'} \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v')| \cdot |\pi(i)|} \\
 &= \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}_i^{v'} \setminus \{\pi^*\}} \frac{1}{|E_W^\pi(v)| \cdot |\pi(i)|}.
 \end{aligned}$$

Hence,

$$\Delta_i^{f^*}(v, v') = f_i^*(v') - f_i^*(v) = 0$$

holds and thus, NI is satisfied.

Power Difference In view of (2) and (3), we have

$$\begin{aligned}
 \sum_{T \in E_W^{\pi^*}(v)} \Delta_{ST}^{f^*}(v, v') &= |E_W^{\pi^*}(v)| \cdot \Delta_S^{f^*}(v, v') - \sum_{T \in E_W^{\pi^*}(v)} \Delta_T^{f^*}(v, v') \\
 &= \frac{|S| \cdot |E_W^{\pi^*}(v)|}{|\mathcal{P}| \cdot |S| \cdot (|E_W^{\pi^*}(v)| + 1)} + \frac{|E_W^{\pi^*}(v)|}{|\mathcal{P}| \cdot (|E_W^{\pi^*}(v)| + 1) \cdot |E_W^{\pi^*}(v)|} \\
 &= \frac{1}{|\mathcal{P}| \cdot (|E_W^{\pi^*}(v)| + 1)} \cdot (|E_W^{\pi^*}(v)| + 1) = \frac{1}{|\mathcal{P}|}
 \end{aligned}$$

as required for the fulfillment of PD.

Suppose now that f satisfies the above axioms. To show that the power index is uniquely determined, consider $v \in \mathcal{G}^N$ and let us proceed by induction on the cardinality of the set $E_W(v)$ of winning embedded coalitions in v .

Initialization: Suppose $|E_W(v)| = |E|$. By U and the definition of a power index, $f_i(v) = \frac{1}{n}$ for each $i \in N$ follows.

Induction Hypothesis: Suppose that the power index is uniquely determined for each $v^* \in \mathcal{G}^N$ with $|E_W(v^*)| > |E_W(v)|$.

By Proposition 1, there exists $v' \in \mathcal{G}^N$ such that $E_W(v') = E_W(v) \cup \{(S; \pi)\}$ for some $(S; \pi) \in E \setminus E_W(v)$. Observe that, by $|E_W(v')| > |E_W(v)|$ and the Induction Hypothesis, the power vector $f(v')$ is uniquely determined. In what follows, we show that $f(v)$ is uniquely determined as well.

For this, and w.l.o.g., let $\pi = (S_1, \dots, S_{k-1}, S_k, S_{k+1}, \dots, S_K)$ with $v(S_\ell; \pi) = 1$ for each $\ell \in \{1, \dots, k-1\}$, $S_k = S$, and $v(S_\ell; \pi) = 0$ for each $\ell \in \{k+1, \dots, K\}$. Notice then that $f_i(v') - f_i(v) = 0$ holds for each $i \in \cup_{\ell=k+1}^K S_\ell$ due to NI. That is, the exact determination of any such $f_i(v)$ directly follows from the fact that $f_i(v')$ has already been fixed.

For each $T \subseteq N$, set $f_T(v) := \sum_{i \in T} f_i(v)$ and $f_T(v') := \sum_{i \in T} f_i(v')$. Notice further that, by PD we have

$$\begin{aligned}
& \frac{1}{|\mathcal{P}|} \\
&= \sum_{\ell=1}^{k-1} \left((f_S(v') - f_S(v)) - (f_{S_\ell}(v') - f_{S_\ell}(v)) \right) \\
&= (k-1) \cdot (f_S(v') - f_S(v)) - \sum_{\ell=1}^{k-1} (f_{S_\ell}(v') - f_{S_\ell}(v)) \\
&= (k-1) \cdot (f_S(v') - f_S(v)) - \sum_{\ell=1}^{k-1} f_{S_\ell}(v') + \sum_{\ell=1}^{k-1} f_{S_\ell}(v) \\
&= (k-1) \cdot (f_S(v') - f_S(v)) - \sum_{\ell=1}^{k-1} f_{S_\ell}(v') + \left(1 - f_S(v) - \sum_{\ell=k+1}^K f_{S_\ell}(v) \right),
\end{aligned}$$

where the last equality follows from the definition of a power index. Thus, we can exactly determine $f_S(v)$ due to (1) $f(v')$ being already fixed by the Induction Hypothesis, and (2) $f_{S_\ell}(v)$ being also fixed for each $\ell \in \{k+1, \dots, K\}$ by NI as shown above.

Observe further that axiom EI requires the difference $f_Q(v') - f_Q(v)$ to be the same for each $Q \in \{S_1, \dots, S_{k-1}\}$ and thus, $f_Q(v)$ is uniquely determined as well. Finally, II requires for each $Q \in \{S_1, \dots, S_{k-1}\}$ the power differences $f_i(v') - f_i(v)$ to be the same for each $i \in Q$. Since $f(v')$ is known by the Induction Hypothesis, we conclude that $f(v)$ is uniquely determined. $\square$

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