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RESEARCH ARTICLE

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A finite-dimensional counterexample for Arveson's hyperrigidity conjecture

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Abstract

We construct an operator system generated by four operators that is not hyperrigid, although all restrictions of irreducible representations have the unique extension property.

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1 | INTRODUCTION

Let S be a separable operator system and $C^*(S)$ a C^*-algebra generated by S. Arveson conjectured in [4] that every representation of $C^*(S)$, restricted to S, has a unique extension to a unital completely positive map if and only if every irreducible representation of $C^*(S)$, restricted to S, has a unique unital completely positive extension — a conjecture known as Arveson's hyperrigidity conjecture. For a detailed background, we recommend [5].

It has been shown by Bilich and Dor-On in [5] that the conjecture fails for an infinite-dimensional operator system generated by an operator algebra in a non-commutative C^*-algebra. However, a critical part of that proof relies on the infinite dimensionality of the operator system. In this paper, we construct a finite-dimensional counterexample for Arveson's hyperrigidity conjecture in Theorem 2.5.

We use the main ideas of [5], with the only significant difference being in the proof of the unique extension property for the irreducible representations of $C^*(S)$. For this, we employ a technique found in [2], which can be used to show that a certain projection is below a family of positive multiplication operators, and that pure states have a maximal irreducible dilation [6].

2 | A FINITE-DIMENSIONAL COUNTEREXAMPLE

Let $\mathbb{B}_4$ be the Euclidean open unit ball in $\mathbb{R}^4$, and let S_3 be its boundary, the sphere in $\mathbb{R}^4$. Let $A(\overline{\mathbb{B}_4})$ denote the continuous affine functions on the closed unit ball, and let $t_i \in A(\overline{\mathbb{B}_4})$, $i = 1, 2, 3, 4$, be the projection onto the ith component. Note that $A(\overline{\mathbb{B}_4})$ is spanned by $1, t_1, t_2, t_3, t_4$.

Define $P : L^2(S_3) \to \mathbb{C}$, $g \mapsto \langle g, 1\rangle$. For $i = 1, 2, 3, 4$ and $c > 0$, we define the operators

$$T_i = M_{t_i} \oplus 0,\ T_{1,c} = \begin{pmatrix} M_{t_1} & cP^* \\ cP & 0 \end{pmatrix},\ \tilde{T}_{1,c} = \begin{pmatrix} M_{t_1} & cP^* \\ 0 & 0 \end{pmatrix}$$

on $L^2(S_3) \oplus \mathbb{C}$. Here, $L^2(S_3)$ is equipped with the unique rotation-invariant probability measure m on S_3. Define

$$S_c = \operatorname{span}\{I, \tilde{T}_{1,c}, T_2, T_3, T_4\}.$$

In Theorem 2.5, we show that S_c is not hyperrigid, but the restrictions of all irreducible representations of $C^*(S_c)$ are boundary representations.

A crucial part of the proof is to show that the *joint numerical range* of the operator tuple $(T_{1,c}, T_2, T_3, T_4)$, defined as

$$\mathcal{W}((T_{1,c}, T_2, T_3, T_4)) = \{(\langle T_{1,c}x, x\rangle, \langle T_2x, x\rangle, \langle T_3x, x\rangle, \langle T_4x, x\rangle);\ \|x\| = 1\},$$

is contained in $\mathbb{B}_4$. To prove this, we first need to show that $\mathcal{W}((T_1, T_2, T_3, T_4))$ is contained in $\mathbb{B}_4$.

Lemma 2.1. *It holds that:*

$$\mathcal{W}((T_1, T_2, T_3, T_4)) \subset \mathbb{B}_4.$$

Proof. Let $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$. Using the fact that $\sum_{i=1}^4 t_i^2 = 1$ on S_3, $T_i = T_i^*$, and applying the Cauchy–Schwarz inequality, we get:

$$\begin{aligned}\|(\langle T_1x, x\rangle, \langle T_2x, x\rangle, \langle T_3x, x\rangle, \langle T_4x, x\rangle)\|^2 &= \sum_{i=1}^4 |\langle T_ix, x\rangle|^2 \leqslant \sum_{i=1}^4 \|T_ix\|^2\|x\|^2 \\ &= \sum_{i=1}^4 \langle T_ix, T_ix\rangle = \sum_{i=1}^4 \langle T_i^2x, x\rangle \\ &\leqslant \|x\|^2 = 1.\end{aligned}$$

Furthermore, the Cauchy–Schwarz inequality implies that if equality holds, there must exist some $i \in \{1, 2, 3, 4\}$ and a $0 \neq \lambda \in \mathbb{C}$ such that $T_ix = \lambda x$. However, M_{t_i} has no eigenvalues, so we conclude that:

$$\sum_{i=1}^4 (\langle T_ix, x\rangle)^2 < 1,$$

which completes the proof. □

Lemma 2.2. *Let* $0 < c < 1/2$*. Then, the following holds:*

$$\mathcal{W}((T_{1,c}, T_2, T_3, T_4)) \subset \mathbb{B}_4.$$

Proof. Let $0 < c \leqslant 1/2$, let $\tilde{S}_c$ be the operator system generated by $T_{1,c}, T_2, T_3, T_4$, and let $f = \alpha + \beta t_1 + \gamma t_2 + \delta t_e + \epsilon t_4 \in A(\overline{\mathbb{B}_4})$. Define a map $\Phi : A(\overline{\mathbb{B}_4}) \to \tilde{S}_c$ by

$$\Phi(f) = \alpha I + \beta T_1 + \gamma T_2 + \delta T_3 + \epsilon T_4 = \begin{pmatrix} M_f & c\beta P^* \\ c\beta P & \alpha \end{pmatrix}.$$

It is clear that this map is well defined, bijective, and that its inverse is positive. The next step is to show that Φ is positive. Suppose $f \geqslant 0$ and notice that this implies $\alpha \geqslant 0$ and $\beta, \gamma, \delta, \epsilon \in \mathbb{R}$. If $\alpha = 0$, then $f = 0$, and there is nothing to show. Thus assume $\alpha > 0$. Then:

$$\Phi(f) \geqslant 0$$

if and only if the Schur complement

$$M_f - c^2\beta^2\alpha^{-1}P^*P$$

is positive (see, e.g., [1, Lemma 7.2.7]). Therefore, the positivity of $\Phi(f)$ is equivalent to

$$c^2\beta^2 P^*P \leqslant \alpha M_f.$$

Evaluating f in (1,0,0,0) and $(-1, 0, 0, 0)$ shows $|\beta| \leqslant \alpha$, and since there is nothing to show for $\beta = 0$, it suffices to check that

$$c^2 P^*P \leqslant |\beta|^{-1} M_f.$$

Recall that m is the unique rotation-invariant probability measure on S_3. Let $g \in L^2(S_3)$, and write $\tilde{f} = f/|\beta|$. Following the idea of [2], we get

$$\begin{aligned}
\langle P^*Pg, g\rangle &= \left| \int_{S_3} g\, dm \right|^2 \\
&\leqslant \left(\int_{S_3} \tilde{f}^{1/2} |g| \tilde{f}^{-1/2} dm \right)^2 \\
&\leqslant \left(\int_{S_3} \tilde{f} |g|^2 dm \right) \left(\int_{S_3} \tilde{f}^{-1} dm \right) \\
&= \langle M_{\tilde{f}} g, g\rangle \left(\int_{S_3} \tilde{f}^{-1} dm \right).
\end{aligned}$$

Note that $\tilde{f}(z) = |\beta|^{-1}(\alpha + \langle z, \omega\rangle_{\mathbb{R}^4})$, where $\omega = (\beta, \gamma, \delta, \epsilon)$, and $\alpha \geqslant \|\omega\|$ since $f \geqslant 0$. Let $U \in \mathcal{B}(\mathbb{R}^4)$ be an orthogonal matrix such that $U^*\omega = (\|\omega\|, 0, 0, 0)$. Then

$$\int_{S_3} \tilde{f}^{-1} dm = \int_{S_3} \frac{|\beta|}{\alpha + \langle z, \omega\rangle} dm(z) = \int_{S_3} \frac{|\beta|}{\alpha + \langle U(z), \omega\rangle} dm(z)$$
$$= \int_{S_3} \frac{|\beta|}{\alpha + \|\omega\| t_1} dm \leqslant \int_{S_3} \frac{|\beta|}{\|\omega\|} \frac{1}{1+t_1} dm \leqslant \int_{S_3} \frac{1}{1+t_1} dm$$

and

$$\int_{S_3} \frac{1}{1+t_1} dm = (2\pi^2)^{-1} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\sin^2(x)\sin(y)}{1+\cos(x)} dz dy dx = 2.$$

Therefore, $\Phi(f) \geqslant 0$ if $c \leqslant 1/2$. Thus, we have shown that if ϕ is a positive state on $\tilde{S}_c$, then $\phi \circ \Phi$ is a positive state on $A(\overline{\mathbb{B}_4})$, and therefore

$$(\phi(T_1), \phi(T_2), \phi(T_3), \phi(T_4)) \in \overline{\mathbb{B}_4}.$$

In particular, for $0 < c \leqslant 1/2$, we obtain

$$\mathcal{W}((T_{1,c}, T_2, T_3, T_4)) \subset \overline{\mathbb{B}_4}. \tag{1}$$

Let $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$ and define

$$z_1 = (\langle T_1 x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle),$$
$$z_2 = (\langle T_{1,1/2} x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle).$$

Then, $z_1 \in \mathbb{B}_4$ by Lemma 2.1 and $z_2 \in \overline{\mathbb{B}_4}$ by Equation 1. Thus, for $0 < c < 1/2$, we have

$$(\langle T_{1,c} x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle) = (1-2c) z_1 + 2c z_2 \in \mathbb{B}_4. \qquad \square$$

Lemma 2.3. *Let* $0 < c < 1$ *and* $x \in L^2(S_3) \oplus \mathbb{C}$ *with* $\|x\| = 1$. *Then the following holds:*

$$\|(\langle \tilde{T}_{1,c} x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle)\| < 1.$$

Proof. Let $x = (y, a) \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$ and define

$$z = (\langle \tilde{T}_{1,c} x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle).$$

Assume, for contradiction, that $\|z\| \geqslant 1$. The equation

$$|\langle \tilde{T}_{1,c} x, x\rangle| = |\langle M_{t_1} y, y\rangle + ca\langle 1, y\rangle| \leqslant |\langle M_{t_1} y, y\rangle| + |ca\langle 1, y\rangle|$$

shows, on one hand, that $\langle 1, y\rangle \neq 0$, and on the other, that for

$$\tilde{x} = \begin{cases} \left(y, |a| \frac{\langle y,1\rangle}{|\langle y,1\rangle|}\right) & \text{if } \langle M_{t_1} y, y\rangle \geqslant 0 \\ \left(y, -|a| \frac{\langle y,1\rangle}{|\langle y,1\rangle|}\right) & \text{else} \end{cases}$$

and

$$\tilde{z} = (\langle \tilde{T}_{1,c}\tilde{x}, \tilde{x}\rangle, \langle T_2\tilde{x}, \tilde{x}\rangle, \langle T_3\tilde{x}, \tilde{x}\rangle, \langle T_4\tilde{x}, \tilde{x}\rangle),$$

we have $\|\tilde{x}\| = \|x\| = 1$, $\langle \tilde{T}_{1,c}\tilde{x}, \tilde{x}\rangle \in \mathbb{R}$ and $1 \leqslant \|z\| \leqslant \|\tilde{z}\|$. Thus, applying the identity $\mathrm{Re}(\tilde{T}_{1,c}) = T_{1,c/2}$, we conclude that

$$\tilde{z} = (\langle T_{1,c/2}\tilde{x}, \tilde{x}\rangle, \langle T_2\tilde{x}, \tilde{x}\rangle, \langle T_3\tilde{x}, \tilde{x}\rangle, \langle T_4\tilde{x}, \tilde{x}\rangle),$$

which lies within the unit ball $\mathbb{B}_4$ by Lemma 2.2, thus leading to a contradiction. Hence, $\|z\| < 1$. □

Lemma 2.4. *Let* $0 < c < 1$. *Then the compact operators* $\mathcal{K}(L^2(S_3) \oplus \mathbb{C})$ *are contained in* $C^*(S_c)$.

Proof. The proof is essentially the same as in [5, Lemma 3.2]. We start by noting that

$$\tilde{T}_{1,c}\tilde{T}_{1,c}^* + \sum_{i=2}^{4} T_i^* T_i - I = \begin{pmatrix} c^2 P^*P & 0 \\ 0 & -PP^* \end{pmatrix} \in C^*(S_c).$$

This implies

$$0 \oplus PP^* = \frac{1}{c^{-2}+1}\left(c^{-2}\begin{pmatrix} c^2 P^*P & 0 \\ 0 & -PP^* \end{pmatrix}^2 - \begin{pmatrix} c^2 P^*P & 0 \\ 0 & -PP^* \end{pmatrix}\right) \in C^*(S_c),$$

and

$$id \oplus 0 = I - 0 \oplus PP^* \in C^*(S_c).$$

Therefore,

$$P^*P \oplus 0 = \frac{1}{c^2}(id \oplus 0)(\tilde{T}_{1,c}\tilde{T}_{1,c}^* + \sum_{i=2}^{4} T_i^* T_i - I) \in C^*(S_c).$$

Additionally, $M_{t_1} \oplus 0 = \tilde{T}_{1,c}(id \oplus 0) \in C^*(S_c)$ and

$$\begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix} = \tilde{T}_{1,c}^* - M_{t_1} \oplus 0 \in C^*(S_c).$$

Further multiplying by $M_{t_i} \oplus 0$, $i = 1, 2, 3, 4$, from the left and right to $P^*P \oplus 0$ and $\begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}$ shows

$$\begin{pmatrix} p\langle \cdot, q\rangle & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \langle \cdot, q\rangle & 0 \end{pmatrix} \in C^*(S_c)$$

for every $p, q \in \mathbb{C}[t_1, t_2, t_3, t_4]$. Finally, since the polynomials are dense in $L^2(S_3)$, any compact operator can be approximated by elements of $C^*(S_c)$, completing the proof. □

The previous lemma places us in a setting similar to the discussion following [5, Lemma 3.2]. Since $(T_{1,c}, T_2, T_3, T_4)$ is a compact perturbation of $(M_{t_1} \oplus 1, M_{t_2} \oplus 1, M_{t_3} \oplus 1, M_{t_4} \oplus 1)$, by the previous lemma, we have the following split short exact sequence:

$$0 \to \mathcal{K}(L^2(S_3 \oplus \mathbb{C})) \to C^*(S_c) \to C(S_3) \to 0.$$

Consequently, $C^*(S_c)$ is a type I C^*-algebra by [7, Theorem 1], and the only irreducible representations of $C^*(S_c)$ are given by the identity representation and the evaluations e_z, defined by

$$C^*(S_c) \to \mathbb{C}, (T_{1,c}, T_2, T_3, T_4) \mapsto z$$

for $z \in S_3$, see [3, Theorem 1.3.4] and [3, p. 20, Corollary 2].

Additionally, the proof of the following theorem uses the facts that a state on S_c is pure if and only if it is an extreme point of the state space $S(S_c)$, and that the restriction of a unital $*$-homomorphism is maximal if and only if it has the unique extension property.

Theorem 2.5. *Let $0 < c < 1$. The operator system S_c is not hyperrigid. However, the restrictions of all irreducible representations of $C^*(S_c)$ to S_c have the unique extension property.*

Proof. To show that S_c is not hyperrigid, we begin by considering the $*$-homomorphisms

$$\pi : C^*(S_c) \to \mathcal{B}(L^2(S_3)), (\tilde{T}_{1,c}, T_2, T_3, T_4) \mapsto (M_{t_1}, M_{t_2}, M_{t_3}, M_{t_4})$$

and

$$\Phi : C^*(S_c) \to \mathcal{B}(L^2(S_3)), A \mapsto P_{L^2(S_3)} A|_{L^2(S_3)}.$$

Clearly, $\pi|_{S_c} = \phi|_{S_c}$, but $\pi \neq \phi$ because the range of π is commutative, whereas the range of Φ contains all compact operators by Lemma 2.4. Hence, $\pi|_{S_c}$ does not have the unique extension property, and thus S_c is not hyperirigid.

It remains to show that the irreducible representations of $C^*(S_c)$ are boundary representations. By Arveson's boundary theorem (see [3]), we observe that since

$$1 = \| \sum_{i=1}^{4} T_i^* T_i \| < \| \tilde{T}_{1,c}^* \tilde{T}_{1,c} + \sum_{i=2}^{4} T_i^* T_i \|,$$

the identity representation of $C^*(S_c)$ is a boundary representation. Thus, it remains to verify that the point evaluations e_z, restricted to S_c, are maximal for all $z \in S_3$.

We begin by showing that

$$\|(\phi(\tilde{T}_{1,c}), \phi(T_2), \phi(T_3), \phi(T_4))\| \leqslant 1 \tag{2}$$

for every $\phi \in S(S_c)$ and that

$$\|(\phi(\tilde{T}_{1,c}), \phi(T_2), \phi(T_3), \phi(T_4))\| < 1 \tag{3}$$

for every pure state ϕ that is not maximal.

Let $\phi \in S(S_c)$. If ϕ is pure and maximal, we have

$$\|(\phi(\tilde{T}_{1,c}), \phi(T_2), \phi(T_3), \phi(T_4))\| = 1,$$

since the only representations of $C^*(S_c)$ with image in $\mathbb{C}$ are given by the point evaluations. If ϕ is pure and not maximal, then by [6, Theorem 2.4], ϕ dilates non-trivially to a maximal irreducible unital completely positive map, which must be the identity representation, since this is the only irreducible representation besides the point evaluations. Thus, there exists $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$ such that $\phi(\cdot) = \langle \cdot x, x \rangle$. Since $c < 1$, we can apply Lemma 2.3 to obtain Equation 3, and since the convex hull of the extreme points of $S(S_c)$ is $S(S_c)$, by Carathéodory's theorem, we also obtain Equation 2.

It follows from Equation 2 that the restrictions of the maps e_z to S_c are extreme points of $S(S_c)$. By Equation 3, these maps must also be maximal, completing the proof. □

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