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The topology of poker

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ABSTRACT

We introduce a topological invariant of games, based on homotopy theory, that measures their complexity. We examine it in the context of the "Texas Hold'em" variant of poker, and show that the invariant's value is at least 4. We deduce that evaluating the strength of a pair of cards in Texas Hold'em is an intricate problem, and that even the notion of who is bluffing against whom is ill-defined in some situations. The use of higher topological methods to study intransitivity of multi-player games seems new.

1. Introduction

In the popular "Texas Hold'em" variant of poker (see e.g. Ethier, 2010, Chapter 22), you and each of your opponents are dealt two cards, and five cards will be dealt to the table. The winner is the player making the best 5-card poker game out of their and the table's cards. Suppose you hold $J♣10♣$ and two other players respectively hold $2\diamond 2\heartsuit$ and $K♣2♣$. Who is the favourite? And what happens after one of the opponents folds?

Knowing the winning probabilities of a hand against another one is fundamental to any poker strategy, and are at the heart of von Neumann's analysis of poker (von Neumann, 1928). What we argue, however, is that winning probabilities *give at best partial information on the current game state*, and sometimes *are devoid of game-theoretic value*.

Let us pause to consider a much simpler game, "Rock, Paper, Scissors" (RPS). It would be absurd, in a televised retransmission of a RPS match, to display winning probabilities for each player, since by the game's symmetry each player wins against one play and loses against another. Win probabilities are routinely shown in televised poker, but we shall argue that the situation, there, is *far worse*, due both to the richness of the game and to the high number of players (typically, 8).

But first, the situation of RPS *does* occur in poker: for the pairs $J♣10♣$, $2\diamond 2\heartsuit$ and $K♣2♣$ we can check that the 1-on-1 winning chances are

$$w(J♣10♣, 2\diamond 2\heartsuit) = 54\%, \quad w(2\diamond 2\heartsuit, K♣2♣) = 63\%, \quad w(K♣2♣, J♣10♣) = 55\%.$$

This means that the *same player* may become the favourite or the underdog depending on which of its opponents folds.

To model such seemingly paradoxical phenomena, we introduce a topological invariant of a game, and apply computational tools to derive a non-trivial result for poker. We avoid any discussion on the exact winning probabilities to concentrate only on *who* wins: we therefore have a set X of player hands, and an *antisymmetric* relation $r \subseteq X \times X$, namely a relation such that for all $x, y \in X$ at

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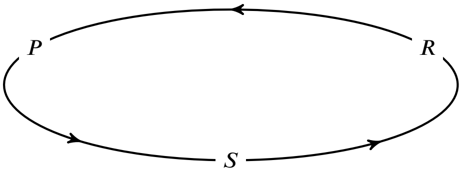
most one of $r(x,y), x = y, r(y,x)$ holds. An antisymmetric relation $r \subseteq X \times X$ is a partial order if and only if it is *transitive*: X does not contain elements R, P, S with $r(R,P), r(P,S), r(S,R)$.

This property can be interpreted topologically, by means of a *simplicial complex*, see §2. Start from a set X of points, and attach edges, triangles, etc.: a $(\#C - 1)$-simplex to each collection $C \subseteq X$ of points such that the restriction of r to $C \times C$ is transitive. The resulting simplicial complex is denoted by $\mathscr{K}_X$. This construction is classical when starting with a partial order: the theory of simplicial complexes and of partially ordered sets are essentially equivalent, see Björner (1995) for its applications to topology. However, we may also apply it to our relation r, and study the topology of $\mathscr{K}_X$. In the situation of “Rock, Paper, Scissors”, the resulting complex $\mathscr{K}_X$ is topologically a circle:



We posit that the complexity of a game is related to the topological complexity of $\mathscr{K}_X$. As a useful numerical invariant, we consider the maximal homological dimension of a subcomplex. This is the maximum $n \in \mathbb{N}$, over all $Y \subseteq X$, such that $H_n(\mathscr{K}_Y) \neq 0$; see §2 for details.

Definition 1. In this article, a *game with state set X* consists in a set X and an antisymmetric relation $r \subseteq X \times X$. We take the meaning of $r(x,y)$ to be “y beats x”. Equivalently, X is the set of strategies of a zero-sum two-player game with payoff function $p\colon X \times X \to \{-1, 1\}$, given by $p(x,y) = -1 \Leftrightarrow r(x,y)$.

The *intransitivity complex* of the game is the simplicial complex $\mathscr{K}_X$ with vertex set X and simplices $\{C \subseteq X : r \upharpoonright (C \times C) \text{ is transitive}\}$.

The *intransitivity dimension* of the game is the maximal homological dimension of induced subcomplexes of $\mathscr{K}_X$.

The motivation behind this choice of invariant is the following: on the one hand, it is natural to consider restrictions of the state set X to subsets, and to consider the restricted subgames. There could, for example, exist a dominating state $x_\infty \in X$ with $r(x, x_\infty)$ for all $x \in X \setminus \{x_\infty\}$, in which case $\mathscr{K}_X$ would be a cone with apex x_∞, and hence homologically trivial; this justifies the choice of an invariant that considers arbitrary subsets of X. There may nevertheless be complicated, intransitive games in which x_∞ does not appear. Homological dimension is a nice, numerical invariant whose non-vanishing in dimension d captures the idea that “something non-trivial happens in dimension d”.

We describe our main results, pertaining to general properties of the invariant $\mathscr{K}_X$ and its application to poker, in the next subsection. Section 2 recalls the basics of simplicial complexes and homology theory, while Section 3 details the intransitivity complex we introduce. Section 4 proves the results from §1.1, and Section 5 concludes with some perspectives. Further examples of $\mathscr{K}_X$, coming from intransitive dice are given in §5.3.

1.1. Main results

We first note that our invariant has full range:

Theorem 1. *For every $n \in \mathbb{N}$ there exists a game whose intransitivity dimension is n.*

The game arising from Theorem 1's proof is somewhat contrived, and we devote our attention to natural games, that already occurred in nature (or at least in the literature).

There is a natural generalization of “Rock, Paper, Scissors”, made popular by the television series “The big bang theory”, in which two additional characters, “Spock” and “Lizard” appear; see Lu et al. (2022). Mathematically, the *nth RPS game* has $X = \mathbb{Z}/(2n+1)\mathbb{Z}$, a cyclic group of odd order, and $r(x,y)$ if and only if $y - x \in \{1, \dots, n\}$. The special case $n = 1$ is the classical RPS game, and $n = 2$ is the “Rock, Paper, Scissors, Spock, Lizard” game.

Theorem 2. *In the nth RPS game, the complex $\mathscr{K}_n$ consists of a cycle of $(2n+1)$ simplices of dimension n, each sharing an $(n-1)$-face with its neighbour. The intransitivity dimension is* 1.

We then turn to an in-depth examination of poker, and more specifically the Texas Hold'em variant. Poker has a storied history in game theory, starting with the seminal works of Borel Borel (1938) and von Neumann von Neumann (1928), and is deeply investigated in the monumental von Neumann and Morgenstern (1944); see Ferguson and Ferguson (2003) for an account. Von Neumann and Morgenstern already consider poker with more than two players, but concentrate on these players forming coalitions of one group against another. Three-player poker is considered in Nash and Shapley (1950), and shown to be orders of magnitude more complex than two-player poker.

In contrast with these works in ideal settings, we consider actual poker, with 52 cards. We address the most fundamental problem, that of comparing hand strengths. Our problem can be phrased as follows. *Imagine that n pairs have been dealt at the table, but that the* *n players have not yet taken a seat. Which place should you select?* Naturally, you should select "the best hand", but as in RPS and other intransitive games there could *not* be a best hand among those available.

We consider the set $X = \{(i,j) : 1 \le i < j \le 52\}$ of pairs of cards, and the relation

$$r((i,j),(k,\ell)) \Longleftrightarrow \begin{cases} \text{averaging over the } \binom{52-4}{5} \text{ remaining cards,} \\ \text{pair } (k,\ell) \text{ wins with more than 50\% chance.} \end{cases}$$

We sometimes write '$(i,j) < (k,\ell)$' for $r((i,j),(k,\ell))$, without implying transitivity.

Our result is that the homotopy type of Texas Hold'em is quite rich; in other words, there are intricate card configurations in which every player could be winning against another one; thus even the concept of "bluff" (see e.g. Cassidy, 2015) needs to be revisited since it is impossible to define, at some moments, who is bluffing against whom:

Theorem 3. *The intransitivity dimension of Texas Hold'em is at least* 4.

More precisely, the simplicial complex $\mathscr{K}_X$ *contains* essential 4-*dimensional subcomplexes: subcomplexes that are not homotopically equivalent to* <4-*dimensional ones. In fact we shall exhibit* S^4 *as such a subcomplex of* $\mathscr{K}_X$.

If there were a 4-dimensional simplex in $\mathscr{K}_X$, it could be interpreted as follows: *there is a configuration with* 5 *players such that no hand is better than the others, but as soon as a player folds the remaining* 4 *are linearly ordered*. Loosely speaking, our result shows that *the same phenomenon occurs with coalitions instead of players*.

There is a long history of associating numbers to games; integers for Sprague (1936); Grundy (1939) or generalized (and in particular not necessarily ordered) numbers for Conway (2001). We view topological invariants as a new powerful tool supplementing these more classical invariants.

1.2. Acknowledgments

The calculations have made heavy use of the computer algebra program `Oscar` (The OSCAR Team, 2025), as well as the poker hand evaluator `PokerHandEvaluator.jl`.

2. Simplicial complexes

This section is a very brief introduction to the algebraic topology necessary for the definition of our invariant. It may be skipped by experts in topology, as well as those readers content with the informal definition above.

2.1. Abstract simplicial complexes

An *abstract simplicial complex* is a collection Δ of subsets of some set S, subject to the axiom: for all $X \in \Delta$ and all $Y \subseteq X$ one has $Y \in \Delta$. Every $X \in \Delta$ is called a *simplex*, and its *dimension* is $\#X - 1$.

This notion generalizes that of graphs; indeed, if all elements of Δ have cardinality at most 2, then each $\{x,y\} \in \Delta$ represents an edge connecting x with y, and Δ carries precisely the information of an undirected, simple graph.

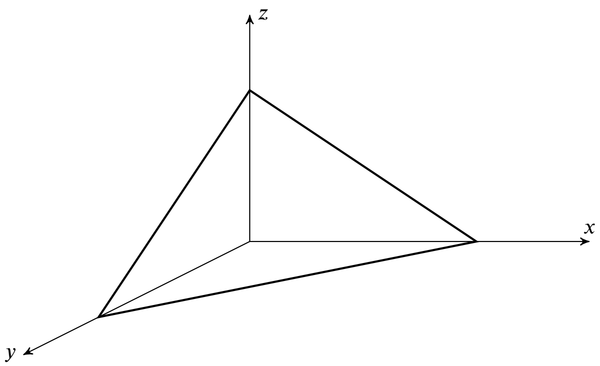
To every simplex $X \in \Delta$ one may associate its *geometric realization* $|X|$, which is a standard simplex in $\mathbb{R}^S$: by definition,

$$|X| = \{x \in [0,1]^S : \sum_{s\in S} x_s = 1,\ x_s > 0 \Rightarrow s \in X\}.$$

Then the *geometric realization* of Δ is the topological space

$$|\Delta| = \bigcup_{X\in\Delta} |X| = \left\{x \in [0,1]^S : \sum_{s\in S} x_s = 1,\ \{s : x_s > 0\} \in \Delta\right\}.$$

For example, $S = \{1,2,3\}$ and $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is the complex associated with RPS, and its geometric realization is a triangle without its interior—and therefore homeomorphic to a circle:



A *subcomplex* of a simplicial complex Δ is simply a subset of Δ that is also a simplicial complex, namely that contains subsets of its elements. The geometric realization of a subcomplex is naturally contained in the geometric realization of Δ. An *induced subcomplex* on a subset $S' \subset S$ is a maximal subcomplex subject to the condition that its simplices are subsets of S'; it may directly be defined as

$$\Delta_{S'} = \{X \in \Delta : X \subseteq S'\} = \{X \cap S' : X \in \Delta\}.$$

We insist that being an induced subcomplex is a quite restrictive notion; for example, the triangle above is a subcomplex of the full simplex $\mathscr{P}(\{1,2,3\})$, but is not an induced subcomplex; indeed the only induced subcomplexes of a simplex are simplices themselves.

A fundamental construction is the *join*. Given two simplicial complexes Δ, Δ' on sets S, S' respectively, their *join* is the simplicial complex

$$\Delta \star \Delta' = \{X \sqcup Y : X \in \Delta, Y \in \Delta'\} \subseteq \mathscr{P}(S \sqcup S').$$

If X, Y are respectively subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, their geometric join is the union of all lines between $X \times (0,\dots,0) \times 0$ and $(0,\dots,0) \times Y \times 1$ in $\mathbb{R}^{m+n+1}$; and the geometric realization of a join of simplicial complexes is the geometric join of their realizations. For example, the join of two simplices of dimension m, n is a simplex of dimension $m+n+1$, and the join of two spheres S^m, S^n is $S^m \star S^n \cong S^{m+n+1}$.

2.2. Homology

Let Δ be an abstract simplicial complex, and for every n let C_n be the real vector space with basis the set of n-dimensional simplices in Δ (so $C_n \cong \mathbb{R}^{d(n)}$ where $d(n)$ is the number of n-dimensional simplices in Δ). Assume that the set S is ordered, and that elements of simplices are always written in ascending order. The *boundary* of an n-dimensional simplex $X = \{s_0, \dots, s_n\}$ is by definition the linear combination

$$\partial X = \sum_{i=0}^{n} (-1)^i \cdot \{s_0, \dots, \widehat{s_i}, \dots, s_n\}$$

where the notation $\widehat{\dots}$ indicates a term that has been removed. Thus $\partial X \in C_{n-1}$ and extending the boundary by linearity gives a linear map $\partial = \partial_n : C_n \to C_{n-1}$.

By a fundamental calculation, *boundaries have no boundary*, that is, $\partial_{n-1}(\partial_n X) = 0$. We have therefore $\operatorname{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$, and we define the *homology groups*

$$H_n(\Delta) := \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}.$$

The homology groups are real vector spaces; the *Betti numbers* of Δ are the dimensions $\beta_n(\Delta) = \dim H_n(\Delta)$ of these vector spaces, and serve as fundamental numerical invariants of Δ. Note that the field $\mathbb{R}$ may be replaced by any other ring $\Bbbk$ (for example $\mathbb{Z}$), leading to *homology with coefficients* $H_n(\Delta; \Bbbk)$.

Continuing again with the example of the triangle above, we have $\partial\{1,2\} = \{1\} - \{2\}$ and $\partial\{1\} = 0$, etc., so

$$H_1(\text{Triangle}) = \frac{\mathbb{R} \cdot \{1\} + \mathbb{R} \cdot \{2\} + \mathbb{R} \cdot \{3\}}{\mathbb{R} \cdot (\{1\} - \{2\}) + \mathbb{R} \cdot (\{1\} - \{3\}) + \mathbb{R} \cdot (\{2\} - \{3\})} \cong \mathbb{R}.$$

We recall that a homotopy between two maps $h_0, h_1 : X \to Y$ is a smooth deformation of one into the other; the maps are then called homotopic. Two spaces X, Y are *homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps respectively of X and Y; and that a space X is *contractible* if it is homotopy equivalent to a point.

The homology groups may be defined for arbitrary topological spaces, and are homeomorphism invariants—actually, even homotopy invariants. Thus $H_1(\text{Triangle}) \cong \mathbb{R}$ implies that the triangle is not homotopy equivalent to a segment. These groups carry useful information; for example, $H_0(X) \cong \mathbb{R}$ if and only if X is connected, and more generally $H_0(X) \cong \mathbb{R}^d$ if X has d connected components. If Δ is 1-dimensional, namely represents a graph, then $H_1(\Delta) \cong \mathbb{R}^e$ where e measures the number of independent cycles in the graph.

Two simplicial complexes may be different as complexes, yet define homeomorphic spaces; this is the case, for example, for a triangle and a square. Homotopy equivalence is a yet coarser relation between spaces; for example, a circle S^1 and a Möbius band M are homotopy equivalent: the map $S^1 \to M$ is the inclusion of S^1 as a core curve of the Möbius band, and the map $M \to S^1$ is the retraction to the core curve. If two spaces X, Y are homotopy equivalent, then their homology groups are isomorphic.

The *homological dimension* of Δ is the maximal n such that $H_n(\Delta)$ is non-trivial. In particular, homotopy equivalent spaces have same homological dimension, and if X is contractible then its homological dimension is 0.

For example, if Δ is homeomorphic to the n-dimensional sphere S^n, then $H_0(\Delta) \cong H_n(\Delta) \cong \mathbb{R}$, the other groups being trivial. Spheres are basic examples of spaces, in that every space may be obtained, up to homotopy, by repeatedly attaching spheres. Here the non-triviality of homology in dimension n is at the heart of numerous results, e.g. Chichilnisky's proof of Arrow's theorem (Baryshnikov and Arrow, 2023).

3. The intransitivity complex

Let $r \subseteq X \times X$ be a relation on a set X. We assume that r is *antisymmetric*: for all $x, y \in X$ at most one of $r(x,y), x = y, r(y,x)$ holds. If exactly one of the three alternatives above holds, then r is called a *tournament*, the terminology coming from the results in an all-play-all tournament. An antisymmetric relation $r \subseteq X \times X$ is a partial order if and only if it is *transitive*: X does not contain elements R, P, S with $r(R, P), r(P, S), r(S, R)$; and it is a total order if and only if it is a transitive tournament.

Consider the simplicial complex $\mathscr{K}_X$ with vertex set X and simplices the ordered subsets of (X, r), namely $C \subseteq X$ is a simplex if and only if the restriction of r to $C \times C$ is transitive. It is clear that this construction produces a simplicial complex, since a subset of a transitive relation is still transitive.

This construction is classical when starting with a partial order; then the complex $\mathscr{K}_X$ is *flag*, meaning that whenever all the faces of a simplex belong to $\mathscr{K}_X$, then the simplex itself also belongs to $\mathscr{K}_X$. We apply it to the possibly non-transitive relation r, and study the topology of $\mathscr{K}_X$. For example, if r is *dominated* in the sense that for all $x, y \in X$ there exists z with $r(x, z)$ and $r(y, z)$, then $\mathscr{K}_X$ is contractible.

Simple game-theoretical properties may then be translated to topology. For example, if there is a maximal element in X in the sense of an element $x \in X$ with $r(y, x)$ for all $y \neq x$, then $\mathscr{K}_X$ is a cone with apex x.

There is a vast literature in game theory that makes highly non-trivial use of graph theory. We highlight in particular (Johnston et al., 2024), which considers the "best-response" graph in some games: given n players respectively with options (pure strategies) $X_1, \dots, X_n$, this is the graph with vertex set $X_1 \times \cdots \times X_n$ and an edge from $(x_1, \dots, x_i, \dots, x_n)$ to $(x_1, \dots, x_i', \dots, x_n)$ whenever player #i gains from switching his play from x_i to x_i' given that the other players play $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$). They study connectivity properties of this graph for "generic" games, and it is quite possible that the tools we propose yield further information on the nature of generic games.

The seminal paper by Nash (1950) also makes fundamental appeal to topology; there, the existence of an equilibrium point is proven by considering a best-response graph in a continuous context, and using convexity as a strong form of contractibility to deduce the existence of a fixed point.

As we mentioned, it is possible for $\mathscr{K}_X$ to be contractible even if for a rich relation r, e.g. if X contains a maximal element. This is our motivation to consider induced subcomplexes of $\mathscr{K}_X$, namely all simplicial complexes $\mathscr{K}_Y$ obtained by restricting the set X to a subset Y.

4. Proofs

We prove in this section the results announced in the introduction.

4.1. Theorem 1: arbitrary intransitivity dimension

Every edge orientation on a simple graph may be viewed as a game, by interpreting the vertex set as the options (pure strategies), and the oriented edge $x \to y$ as "option y wins against x". In particular, a *tournament* — an edge orientation of the complete graph — yields a game. We move freely between orientations on a simple graph and antisymmetric relations by writing $r(x, y)$ whenever there is an edge $x \to y$.

Given two games (X, r) and (X', r'), we may consider their *join* $X \star X'$: this is the game with options $X \sqcup X'$ and winning relation r'' given by

$$r''(x, y) \Leftrightarrow \begin{cases} x, y \in X \text{ and } r(x, y), \\ \text{or } x, y \in X' \text{ and } r'(x, y), \\ \text{or } x \in X, y \in X'. \end{cases}$$

In other words, all options in X' are better than those in X, and options are otherwise compared in their respective sets.

It is easy to see that the join of games is compatible with the construction of the intransitivity complex:

$$\mathscr{K}_{X \star X'} \cong \mathscr{K}_X \star \mathscr{K}_{X'}.$$

From this, we deduce that any join may appear as an intransitivity complex; for example, the sphere S^n, which is the n-fold iterated join of 0-dimensional spheres S^0; or the sphere S^{2n-1}, which is the n-fold iterated join of circles.

Consider, for concreteness, the game with $3n$ options $X = \{R, P, S\} \times \{1, \dots, n\}$, and the winning relation

$$r((R, i), (P, j)) \Leftrightarrow i < j, \quad r((P, i), (S, j)) \Leftrightarrow i < j, \quad r((S, i), (R, j)) \Leftrightarrow i < j.$$

Then $\mathscr{K}_X \cong S^{2n-1}$.

This game is of course highly contrived, and only serves as an illustration of the range of the construction. We shall see later that poker exhibits such phenomena.

We also highlight the difference between two notions of intransitivity for a game: on the one hand, it may exhibit a large number of "independent" intransitivity relations; this would appear as a *large dimension of a low-degree homology group* such as H_1. On the other hand, it may exhibit a complex "interlocking" of intransitivity relations; this would appear as a *non-trivial dimension of a high-degree homology group*.

Consider, in contrast with the above example, the game with $3n$ options $X = \{R, P, S\} \times \{1, \dots, n\}$, and the winning relation

$$\forall i : \quad r((R, i), (P, i)), \quad r((P, i), (S, i)), \quad r((S, i), (R, i)).$$

Then $\mathscr{K}_X \cong S^1 \sqcup \cdots \sqcup S^1$.

A referee kindly suggested the following example; the graph we consider is not directly the winning relationship, but rather a best-response graph. The game is "robber vs detective", where the robber hides a treasure in one of 100 locations, and the detective looks at a location; the payoff is $+1$ for the detective if she finds the treasure and -1 if she doesn't; the payoff to the robber is the opposite. Thus the space X is $\{1,\dots,100\}^2$, namely the position of the robber and the detective. The best-response graph has a "robber" edge from (x,x) to (y,x) for all $y \neq x$, and a "detective" edge from (x,y) to (x,x) for all $x \neq y$. In particular, there are no chains of length 2, so $\mathscr{K}_X$ is 1-dimensional—it is a simple graph. It is connected, so $H_0 \cong \mathbb{R}$, and a direct calculation shows $H_1 \cong \mathbb{R}^e$ for $e = \#\text{edges} - \#X = 99^2$. Thus H_1 grows quadratically in the number of locations while the homological degree remains constant $= 1$.

4.2. Theorem 2: generalized RPS games

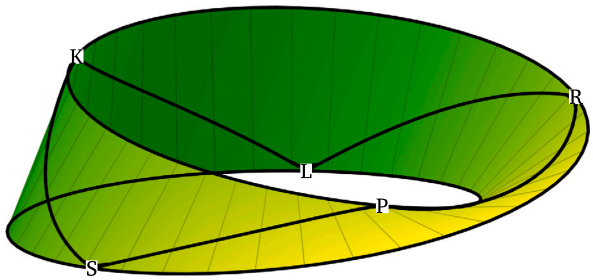
For illustration of the intransitivity complex, we consider a generalization of the famous "Rock, Paper, Scissors" (RPS) game to a set of $2n+1$ items $X = \mathbb{Z}/(2n+1)\mathbb{Z} = \{0,1,\dots,2n\}$ considered modulo $2n+1$, with the rule

$$r(x,y) \Leftrightarrow y - x \in \{1,\dots,n\}.$$

The special case $n=1$ is classical RPS, with $R=0, P=1, S=2$. The special case $n=2$ was invented by Kass and Bryla (2025), and corresponds to

$$\text{Rock} = 0,\quad \text{Spock} = 1,\quad \text{Paper} = 2,\quad \text{Lizard} = 3,\quad \text{Scissors} = 4.$$

Its associated complex $\mathscr{K}_{RPSSL}$ is a Möbius strip:



For general n, the complex consists of $2n+1$ copies of the n-simplex, on the sets $\{i, i+1, \dots, i+n\}$ for all $i \in \mathbb{Z}/(2n+1)\mathbb{Z}$. Two consecutive simplices, say the ith and $(i+1)$th, intersect on $\{i+1,\dots,i+n\}$, an $(n-1)$-simplex.

Even though the intransitivity complex $\mathscr{K}_{RPS(n)}$ is n-dimensional, its homological dimension is 1, and furthermore all its subcomplexes are homotopy equivalent to a point or a circle, so *the intransitivity dimension of $RPS(n)$ is* 1. Consider indeed any subset $S \subseteq \{0,\dots,2n\}$. If S is contained in a simplex $\{i, i+1, \dots, i+n\}$ for some i, then the induced subcomplex $(\mathscr{K}_{RPS(n)})_S$ on S is itself a simplex, and therefore homotopy equivalent to a point. Otherwise, list $S = \{s_1,\dots,s_k\}$ in increasing order, and map geometrically point s_k to $\exp(2\pi i k/(2n+1))$ along a circle, with simplices mapped by linear extension; this provides a homotopy from the induced subcomplex $(\mathscr{K}_{RPS(n)})_S$ to the circle S^1.

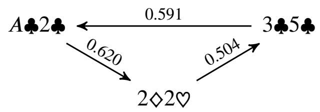
4.3. Theorem 3: Texas Hold'em

The proof of Theorem 3 is obtained through a quite complicated computer calculation.

Using the computer language Julia and its packages PlayingCards and PokerHandEvaluator, we computed the relation r. (We also independently re-implemented PokerHandEvaluator to make sure of its correctness.) The code is available on the Zenodo repository https://doi.org/10.5281/zenodo.7885276; the file `poker-data.jl` defines an array `CARDPAIRS` of size 1326 listing all pairs of cards, and the array `r` in the HDF5 dataset `poker-data.hdf5` has size $1326 \times 1326 \times 3$, in such a way that `r[i,j,1:3]=(w,t,l)` means that playing `CARDPAIRS[i]` against `CARDPAIRS[j]` results in `w` wins, `t` ties and `l` losses when considering all $\binom{48}{5}$ possible table cards; thus $\mathtt{w}+\mathtt{t}+\mathtt{l} = 1712304$.

We then explored subsets $Y \subseteq X$, computed the corresponding simplicial complex $\mathscr{K}_Y$ using the computer algebra package Oscar and its Polymake interface, and its homology.

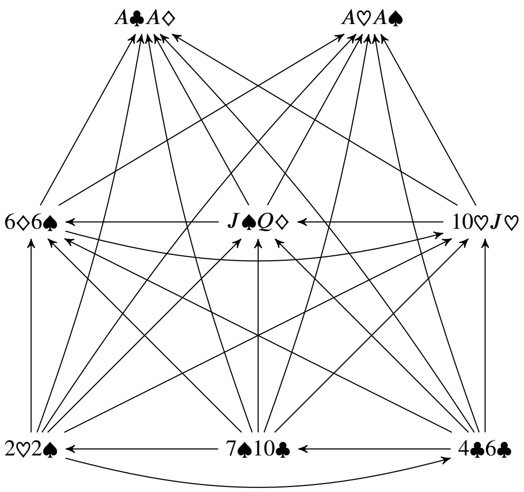
Let us begin with an example of an S^1 in $\mathscr{K}_X$. Consider the hands $A♣2♣$, $3♣5♣$, $2♢2♡$. Note that $3♣5♣$, known as "Carabas" in Russian, is a well-known tricky holding. The first wins against the second on average, because of the strength of the ace. The second wins against the third because of the possibilities of forming a flush. The third wins against the first because of the pair. These winning probabilities are respectively 0.591, 0.504, 0.620. We write these data in the following diagram:



It seemed useful to consider hands in which the cards are close, so we ordered the 52 cards by their value, kept each card independently with some probability p, and formed the pairs out these cards in increasing order. After a few thousand runs, our computer search came up with the subset

$$Y = \{A♣A◇,\ A♡A♠,\quad 6◇6♠,\ J♠Q◇,\ 10♡J♡,\quad 2♡2♠,\ 7♠10♣,\ 4♣6♣\}.$$

The relations between these pairs are given in the following diagram:



A direct calculation showed that all its homology groups are trivial except $H_4(\mathscr{K}_Y) = \mathbb{R}$.

In fact, this can be checked manually, assuming of course knowledge of the winning relation r. Indeed the first row defines an S^0, since these card holdings are incomparable; the second and the third row define S^1, with $6◇6♠ > J♠Q◇ > 10♡J♡ > 6◇6♠$ and $2♡2♠ > 7♠10♣ > 4♣6♣ > 2♡2♠$; and the complex $\mathscr{K}_Y$ is the join of these three subcomplexes, totally ordered as $\{A♣A◇, A♡A♠\} > \{6◇6♠, J♠Q◇, 10♡J♡\} > \{2♡2♠, 7♠10♣, 4♣6♣\}$, hence is homeomorphic to S^4.

5. Outlook

We have barely scratched the surface of the topological complexity of Texas Hold'em. In particular, it does not seem possible to compute the homotopy type, or even just the homology, of $\mathscr{K}_X$ with current technology.

Indeed, using the usual limit of 10 players per table, there are $\binom{52}{2,\dots,2} = 52!/2^{10}10!32! > 8 \cdot 10^{22}$ collections of pairs of hands to consider, and for each a homological calculation to perform. Additionally, each homological calculation is feasible in the cases we considered, but can become quite expensive, since in general computing homology groups is NP-hard (Adamaszek and Stacho, 2016). When considering such large datasets, researchers typically concentrate on H_1 and possibly H_2, while we are interested in higher-degree phenomena.

We could modify the game by taking into account the symmetries between the four suits. More precisely, there is a *quotient game* in which two hands are identified if they differ only by relabeling the suits (which is legitimate since the suits play no role in hand evaluation). There would then be $\binom{13}{2} + 13^2$ hands to consider, corresponding to the suited and offsuit possibilities, rather than $\binom{52}{2}$ hands as we considered here. The results would not fundamentally change.

5.1. Revealing table cards one at a time

We have made the simplifying assumption that all table cards are unknown. In actual Texas Hold'em, there are more than one bidding round, and more cards are progressively revealed. As more cards are revealed, X shrinks to a subset X' because fewer cards may appear in our opponents' hands; it would be interesting to study the importance of the partial revealing of information, in the form of a failure of the inclusion $X' \hookrightarrow X$ to induce a simplicial map.

5.2. Sensitivity to data

As shown by the brief calculation above, the probabilities associated with edges in r are typically not microscopically away from 0.5. The closest one is $3x3y$ versus $Ax10x$ for two suits $x \neq y \in \{♣, ◇, ♡, ♠\}$, with winning probability 0.50007. The natural tool with which to explore the sensitivity of $\mathscr{K}_X$ is *persistent homology*: for a probability $p \in [0.5, 1]$, consider the relation r_p in which $r_p((i,j),(k,\ell))$ means that the probability that (k,ℓ) wins is at least p, and the associated simplicial complex $\mathscr{K}_{X,p}$. We have for each $0.5 \le p < q \le 1$ inclusion maps $\mathscr{K}_{Y,q} \hookrightarrow \mathscr{K}_{Y,p}$; what are their relative homologies, as $Y \subseteq X$ and p,q vary?

5.3. Intransitive dice

The reader may be familiar with "intransitive dice", which are sets of dice such that the expected winning probability from one dice against another do not form a transitive relation. For example, three dice A,B,C with faces labeled respectively by $3,3,3,3,3,6$ and $2,2,2,5,5,5$ and $1,4,4,4,4,4$, each one beats the next one, so $\mathscr{K}_{\{A,B,C\}}$ is a triangle.

A set $X = \{A,B,C,D,E\}$ of five intransitive dice was discovered by James Grime: their number of dots are respectively

A	2	2	2	7	7	7
B	1	1	6	6	6	6
C	0	5	5	5	5	5
D	4	4	4	4	4	9
E	3	3	3	3	8	8,

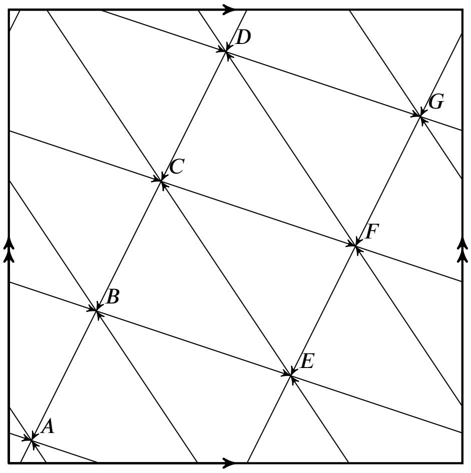
and a direct calculation shows that $\mathscr{K}_X$ coincides with $\mathscr{K}_{RPSSL} = \mathscr{K}_{RPS(2)}$, namely it is a Möbius strip.

Oskar van Deventer has found a collection of seven dice $X = \{A,B,C,D,E,F,G\}$, whose number of dots are respectively

A	2	2	14	14	17	17
B	7	7	10	10	16	16
C	5	5	13	13	15	15
D	3	3	9	9	21	21
E	1	1	12	12	20	20
F	6	6	8	8	19	19
G	4	4	11	11	18	18,

for which a simple calculation shows that $\mathscr{K}_X$ is a 2-dimensional torus, consisting of triangles

$$\{A,B,D\},\{A,B,F\},\{A,C,D\},\{A,C,G\},\{A,E,F\},\{A,E,G\},\{B,C,E\},$$
$$\{B,C,G\},\{B,D,E\},\{B,F,G\},\{C,D,F\},\{C,E,F\},\{D,E,G\},\{D,F,G\}\;:$$



5.4. Other intransitive games

There is a substantial literature on "intransitive games", see Gardner, 2001, Chapters 22 and 23, however only considering 2 players. One of the very interesting ones is the "Penney game" (Guibas and Odlyzko, 1981). Some parameter $n \in \mathbb{N}$ is fixed. Every player chooses a binary sequence of length n. An infinite binary sequence is then drawn at random, one bit at a time. The first player whose sequence shows up wins.

It is well known that this game is not transitive; for $n = 3$, we have $011 > 110 > 100 > 001 > 011$, and the associated complex $\mathscr{K}_{\{0,1\}^3}$ is a bouquet of 3 circles. We computed the homology of the whole complex $\mathscr{K}_{\{0,1\}^n}$ for $n \le 6$, giving for the last case

$$H_*(\mathscr{K}_{\{0,1\}^6}) = (0,0,0,0,0,\mathbb{R}^{38},\mathbb{R}^{149},\mathbb{R}^{12}) \text{ for } 0 \le * \le 7.$$

There does not seem to be any obvious pattern to these numbers.

We mention interesting contributions by Poddiakov et al., see e.g. Poddiakov and Lebedev (2023), in which nested games are considered. In our language, this amounts to considering two games X,Y, and putting on $Z = X \times Y$ the lexicographic relation:

$r((x,y),(x',y'))$ if and only if $r(x,x')$ or ($x = x'$ and $r(y,y')$).

The associated simplicial complex $\mathscr{K}_Z$ has as simplices all $C \subseteq Z$ whose projection to X is a simplex in $\mathscr{K}_X$ and whose projection to Y is a simplex in $\mathscr{K}_Y$. This operation does not seem to have a pre-existing meaning in topology.

Poker is quite apart from these games in that it is a real-life game, with a large population of expert or professional players, in which it is essential to estimate with accuracy the winning chances relative to all the other participants. Our results show that these data must be considered globally, and that the one-on-one probabilities only serve as the carrier for a powerful, high-dimensional topological invariant.

Declaration of competing interest

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Data availability

Data will be made available on request.

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